Prof. P. Cheridito August 18, 2020

Quantitative Risk Management

Important:

- Put your student card on the table
- Begin each problem on a new sheet of paper, and write your name on each sheet
- Only pen, paper and ten sides of summary are allowed

Please fill in the following table.

Last name	
First name	
Student number (if available)	

Please do <u>not</u> fill in the following table.

Question	Points	Control	Maximum
#1			12
#2			10
#3			10
#4			10
#5			8
Total			50

Question 1 (12 Pts)

a) Let X be a random variable with a standard Laplace distribution; that is, the cdf of X is

$$F(x) = \begin{cases} \frac{1}{2}\exp(x) & \text{if} \quad x \leq 0 \\ 1 - \frac{1}{2}\exp(-x) & \text{if} \quad x \geq 0. \end{cases}$$

Calculate $\mathrm{VaR}_\alpha(X)$ and $\mathrm{AVaR}_\alpha(X)$ for $\alpha \in [1/2, 1)$. (3 Pts)

b) Let X be a random variable such that $\mathbb{E}[|X|] < \infty$. Show that

$$\mathrm{AVaR}_\alpha(X) = \mathrm{VaR}_\alpha(X) + \frac{1}{1-\alpha}\mathbb{E}\left[(X - \mathrm{VaR}_\alpha(X))_+\right]$$

for all $\alpha \in (0, 1)$. (3 Pts)

c) Name one advantage of VaR over AVaR and one advantage of AVaR over VaR. (2 Pts)

d) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and consider the risk measure $\rho\colon L^1(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ given by

$$\rho(X) = \max\{\mathrm{AVaR}_{0.75}(X), \mathrm{VaR}_{0.95}(X)\}.$$

Which properties of a coherent risk measure does ρ have? Please, justify your answers. (4 Pts)

Question 2 (10 Pts)

a) Let $X_i \sim S_d(\psi_i)$, $i = 1, \ldots, n$, be independent random vectors and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. Show that $Z = \sum_{i=1}^n \alpha_i X_i$ is spherically distributed. (3 Pts)

b) Assume that the daily losses of an investment during the next t days are given by

$$(X_1, \ldots, X_t) \sim M_t(0, \Sigma, \widehat{F}_W)$$

for a non-negative random variable W and a $t \times t$-matrix $\Sigma = \sigma^2 P$, where $\sigma > 0$ is a constant and P a correlation matrix with $P_{ij} = \rho$ for all $i \neq j$. Show that there exists a function $f\colon \mathbb{N} \to \mathbb{R}$ such that

$$\mathrm{VaR}_\alpha(X_1 + \cdots + X_t) = f(t)\,\mathrm{VaR}_\alpha(X_1)$$

for all $\alpha \in (0, 1)$. Can you compute f explicitly? (4 Pts)

c) Let $X \sim E_d(\mu, \Sigma, \psi)$ and $Y \sim E_d(\nu, \Sigma, \varphi)$ be two independent random vectors. Is $Z = X + Y$ again elliptically distributed? If yes, derive $m \in \mathbb{R}^d$, $M \in \mathbb{R}^{d\times d}$ and $\xi\colon \mathbb{R}_+ \to \mathbb{R}$ such that $Z \sim E_d(m, M, \xi)$. If no, give a counterexample. (3 Pts)

Question 3 (10 Pts)

a) Let X be an $\mathrm{Exp}(\lambda)$-distributed random variable for a parameter $\lambda > 0$. Calculate the distribution function and the moments of $Y = \exp(X)$. (3 Pts)

b) Does Y have a density? If yes, can you compute it? (1 Pts)

c) Now, consider a two-dimensional random vector (X_1, X_2) such that $X_i \sim \text{Exp}(\lambda_i)$ for parameters $\lambda_i > 0$, $i = 1, 2$. Under which conditions does the linear correlation between $Y_1 = \exp(X_1)$ and $Y_2 = \exp(X_2)$ exist? (2 Pts)

d) Assume $\lambda_1 = 3$ and $\lambda_2 = 4$. What is the range of possible correlations between Y_1 and Y_2? (4 Pts)

Question 4 (10 Pts)

a) Let (X, Y) be a two-dimensional random vector with joint distribution function

$$F(x,y) = \frac{1}{\frac{x^\alpha}{x^\alpha - 1} + e^{-y}} \qquad x > 1,\ y \in \mathbb{R},\ \alpha > 0.$$

Compute the marginal distributions and the copula of (X, Y). (5 Pts)

b) Let $F \colon \mathbb{R} \to [0, 1]$ be a cdf satisfying

$$\lim_{x\to\infty} (1 - F(x))e^{\lambda x} = b$$

for constants $\lambda, b > 0$. Does F belong to the maximum domain of attraction of a standard extreme value distribution H_ξ? If yes, determine the shape parameter ξ and a pair of normalizing sequences. (5 Pts)

Question 5 (8 Pts)

a) Name different stylized facts of typical daily equity log-return series. (4 Pts)

b) Discuss and compare different methods of generating loss distributions of financial assets. (4 Pts)