Prof. P. Cheridito August 24, 2022

Quantitative Risk Management

Important:

- Put your student card on the table
- Begin each problem on a new sheet of paper, and write your name on each sheet
- Only pen, paper and ten sides of summary are allowed

Please fill in the following table.

Last name	
First name	
Student number (if available)	

Please do <u>not</u> fill in the following table.

Question	**Points**	**Control**	**Maximum**
#1			11
#2			10
#3			9
#4			10
#5			10
Total			50

Question 1 (11 Pts)

a) Let X be a random variable with cumulative distribution function

$$F(x) = \begin{cases} 0 & \text{if} \quad x < 1 \\ 1 - x^{-\theta} & \text{if} \quad 1 \le x < 3 \\ 1 - (1+x)^{-\theta} & \text{if} \quad x \ge 3 \end{cases}$$

for a parameter $\theta > 1$. Calculate $\mathrm{VaR}_\alpha(X)$ and $\mathrm{AVaR}_\alpha(X)$ for $\alpha \in (0,1)$. (4 Pts)

b) Let X_1, X_2 be independent and identically distributed random variables which take the values 100 with probability p and 0 with probability $1-p$. For which $\alpha \in (0,1)$ does one have $\mathrm{VaR}_\alpha(X_1 + X_2) > \mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2)$? Explain your answer. (3 Pts)

c) Consider two stocks with current values $S_t^1 = 500$ and $S_t^2 = 200$. The monthly log-returns of both stocks in % over the last 5 months are given in the following table:

lag k	5	4	3	2	1
log-return of S_1 at lag k in %	10	-5	-3	15	4
log-return of S_2 at lag k in %	12	-10	5	10	2

Use historical simulation to estimate the one-month VaR_α for $\alpha = 0.8$ of the linearized loss L_{t+1}^{Δ} of a portfolio consisting of one share of S^1 and two shares of S^2. (4 Pts)

Question 2 (10 Pts)

a) Consider two random variables $X \sim \text{Log-Norm}(\mu_1, \sigma_1^2)$ and $Y \sim \text{Log-Norm}(\mu_2, \sigma_2^2)$. Is it true that $XY \sim \text{Log-Norm}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$? Explain your answer. (2 Pts)

b) Suppose the losses $L_1, \ldots, L_d$ of d financial assets are described by the factor model

$$L_i = \rho_i Z_0 + \sqrt{1 - \rho_i^2} Z_i, \quad i = 1, \ldots, d,$$

where $Z_0, Z_1, \ldots, Z_d$ are iid $N(0, \sigma^2)$-random variables and $\rho_i \in (0,1)$. Determine the distribution of the random vector $L = (L_1, \ldots, L_d)$. (2 Pts)

c) Calculate $\mathrm{VaR}_\alpha(L_i)$, $i = 1, \ldots, d$, and $\mathrm{VaR}_\alpha\left(\frac{1}{d}\sum_{i=1}^d L_i\right)$. (3 Pts)

d) Does investing capital equally in the d financial assets decrease the portfolio's VaR compared to investing the whole capital in one asset? Explain your answer. (3 Pts)

Question 3 (9 Pts)
Let X be a random variable with cumulative distribution function

$$F(x) = \begin{cases} \dfrac{3x+1}{3x+2} & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0. \end{cases}$$

a) Does X have a density? If yes, can you derive it? (1 Pts)

b) Find all $k \in \mathbb{N} = \{1, 2, \dots\}$ such that $\mathbb{E}[|X|^k] < \infty$. (1 Pts)

c) Does F belong to $\mathrm{MDA}(H_\xi)$ for a standard generalized extreme value distribution H_ξ? If yes, what is ξ and what are the normalizing sequences? (3 Pts)

d) Calculate the excess distribution function $F_u(x) = \mathbb{P}[X - u \leq x \mid X > u]$, $x \geq 0$. (2 Pts)

e) Does there exist a parameter $\xi \in \mathbb{R}$ and a function β such that

$$\lim_{u \to \infty} \sup_{x > 0} |F_u(x) - G_{\xi, \beta(u)}(x)| = 0,$$

where $G_{\xi,\beta}$ denotes the cumulative distribution function of a generalized Pareto distribution? If yes, for which ξ and β does this hold? (2 Pts)

Question 4 (10 Pts)

a) Compute the upper tail dependence coefficient λ_u of the two-dimensional copula

$$C(u,v) = 1 - \Big((1-u)^\theta + (1-v)^\theta - (1-u)^\theta(1-v)^\theta\Big)^{1/\theta}, \quad u, v \in (0,1),$$

for $\theta \in \mathbb{R}\backslash\{0\}$. (2 Pts)

b) Let (X, Y) be a two-dimensional random vector with Exp(1)-marginals and copula $C(u, v)$ given in a). Does (X, Y) have a density? If yes, can you compute it? (3 Pts)

c) Let (X, Y) be a two-dimensional random vector with cumulative distribution function

$$F(x,y) = \frac{x^2}{\sqrt{x^4(1+e^{-y^3})^2 + 1 + 2x^2}}$$

defined on $\mathbb{R}^+ \times \mathbb{R}$. Compute the marginal distributions and the copula of (X, Y). (5 Pts)

Question 5 (10 Pts)

a) Describe how one can test univariate distributions with graphical tests. (3 Pts)

b) Describe what a p-factor model is. (3 Pts)

c) Name advantages and disadvantages of the multivariate normal distribution as a model for financial log-returns. (4 Pts)