Prof. P. Cheridito September 2, 2021

Quantitative Risk Management

Important:

- Put your student card on the table
- Begin each problem on a new sheet of paper, and write your name on each sheet
- Only pen, paper and ten sides of summary are allowed

Please fill in the following table.

Last name	
First name	
Student number (if available)	

Please do <u>not</u> fill in the following table.

Question	**Points**	**Control**	**Maximum**
#1			
#2			
#3			
#4			
#5			
Total			50

Question 1 (10 Pts)
Let L be a random loss of the form $L = YZ$, where Y is a Bernoulli random variable with mean $p \in (0,1)$ and Z an independent random variable with cdf

$$F_Z(x) = \begin{cases} 1 - x^{-\beta} & \text{if} \quad x \geq 1 \\ 0 & \text{if} \quad x < 1 \end{cases}$$

for a parameter $\beta > 2$.

a) Compute the mean and the variance of L. (2 Pts)

b) Derive the cdf of L. (1 Pt)

c) Does L have a density? If yes, can you derive it? (1 Pt)

d) Compute $\mathrm{VaR}_\alpha(L)$ for $\alpha \in (0,1)$. (2 Pts)

e) Compute $\mathrm{ES}_\alpha(L)$ for $\alpha \in (0,1)$. (2 Pts)

f) For which $\alpha \in (0,1)$ is $\mathrm{AVaR}_\alpha(L)$ equal to $\mathrm{ES}_\alpha(L)$? (2 Pts)

Question 2 (10 Pts)

a) Consider a d-dimensional random vector $X = (X_1, \dots, X_d) \sim N_d(\mu, \Sigma)$ such that $X_1 \equiv 1$. Denote by $\mathcal{L}$ the set of random losses $\{v^T X : v \in \mathbb{R}^d\}$ and let $\alpha \in [1/2, 1)$. Which properties of a coherent risk measure does the mapping $\mathrm{VaR}_\alpha \colon \mathcal{L} \to \mathbb{R}$ have? Explain your answers. (5 Pts)

b) Assume d financial returns are described by the components of a d-dimensional random vector $X = (X_1, \dots, X_d)$ with an elliptical distribution such that $\mathbb{E}[X_i^2] < \infty$ for all $i = 1, \dots, d$. Let $v, w \in \mathbb{R}^d$ be two portfolio vectors such that $v^T \mu = w^T \mu$, where $\mu \in \mathbb{R}^d$ is the mean vector of X. Show that, for all $\alpha \in [1/2, 1)$, one has

$$\mathrm{ES}_\alpha\left(-v^T X\right) \leq \mathrm{ES}_\alpha\left(-w^T X\right) \quad \text{if and only if} \quad \mathrm{Var}\left(v^T X\right) \leq \mathrm{Var}\left(w^T X\right).$$

(5 Pts)

Question 3 (10 Pts)
Let X be a non-negative random variable with cdf

$$F(x) = 1 - \frac{1}{\sqrt{1+2x}}, \quad x \geq 0.$$

a) Does X have a density? If yes, can you derive it? (1 Pt)

b) Find all $k \in \{1, 2, \dots\}$ such that $\mathbb{E}[|X|^k] < \infty$? (1 Pt)

c) Does F belong to the maximum domain of attraction of a standard generalized extreme value distribution H_ξ? If yes, determine the shape parameter ξ and a pair of normalizing sequences. (3 Pts)

d) Calculate the excess distribution function $F_u(x) = \mathbb{P}[X - u \le x \mid X > u]$, $x \ge 0$, over a threshold $u > 0$. (2 Pts)

e) Does there exist a parameter $\xi \in \mathbb{R}$ and a function β such that

$$\lim_{u\to\infty} \sup_{x>0} |F_u(x) - G_{\xi,\beta(u)}(x)| = 0,$$

for a generalized Pareto distribution $G_{\xi,\beta(u)}$? If yes, for which ξ and $\beta(u)$ does this hold? (3 Pts)

Question 4 (10 Pts)
Let (X, Y) be a two-dimensional random vector with cdf

$$F_{X,Y}(x,y) = \frac{(\sqrt{1+2x} - 1)\,(1 - e^{-4y^2})}{\sqrt{1+2x} - \frac{1}{2}e^{-4y^2}}, \quad x, y \ge 0.$$

a) What are the marginal distributions of X and Y? (3 Pts)

b) Compute a copula C of (X, Y). Is it unique? (3 Pts)

c) Calculate the coefficient of upper tail dependence λ_u between X and Y. (2 Pts)

d) Calculate the coefficient of lower tail dependence λ_l between X and Y. (2 Pts)

Question 5 (10 Pts)

a) Why is subadditivity a desirable property of a risk measure? (2 Pts)

b) Why does one usually assume stationarity in time series modelling? (2 Pts)

c) How can a multivariate t-distribution be represented as a normal mixture distribution? (3 Pts)

d) Name advantages and disadvantages of elliptical distributions in financial modelling. (3 Pts)