Problem Set 11

Due on May 25

1. Consider a bivariate random vector $X = (X_1, X_2)$ with components

$$X_1 = \sqrt{W}(Z_1 + Z_2) \quad \text{and} \quad X_2 = \sqrt{W}(Z_1 - Z_2),$$

where Z_1 and Z_2 are independent $N(0,1)$-distributed and W is a non-negative random variable with cdf $F(x) = 1 - x^{-\beta}$, $x \geq 1$, $\beta > 0$, independent of (Z_1, Z_2).

   a) Show that X has a normal variance mixture distribution.
   b) Calculate the covariance matrix of X.
   c) Calculate the correlation matrix of X.

2. Let C be an Archimedean copula with generator ψ given by $\psi(x) = \mathbb{E}\, e^{-xV}$, where V is an exponentially distributed random variable with expectation 1. Calculate the probability $\mathbb{P}[U_1 > 1/2, U_2 > 1/2]$ for $(U_1, U_2) \sim C$.

3. Calculate the lower and upper tail dependence coefficients λ_l and λ_u of a two-dimensional copula C if ...

   a) C is a Clayton copula.
   b) C is a Gumbel copula.

4. Total annual operational loss in a unit of measure is usually modeled with a random variable of the form $L = \sum_{j=1}^{N} X_j$ for an $\mathbb{N}$-valued random variable N and a sequence $X_1, X_2, \ldots$ of i.i.d. $\mathbb{R}_+$-valued random variables that are independent of N.

   Assume N has a Poisson distribution with parameter $\lambda > 0$ (that is, $\mathbb{P}[N = n] = e^{-\lambda}\lambda^n/n!$) and all X_j are exponentially distributed with parameter $\mu > 0$ (that is, $\mathbb{P}[X_j > x] = e^{-\mu x}$). Calculate the expectation of L.