Problem Set 2

Due on March 17

1. The density of a normally distributed random variable X with parameters $\mu \in \mathbb{R}$ and $\sigma^2 > 0$ is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

   a) Calculate the mean and variance of X.
   b) Calculate $\mathbb{E}\left[e^X\right]$.
   c) Calculate the conditional expectation $\mathbb{E}[X \mid X \geq \mu]$.

2. Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with cdf F_X.

   a) Show that $\mathbb{P}[X < x] = F_X(x-) := \lim_{y \uparrow x} F_X(y)$ for all $x \in \mathbb{R}$.

   A function $q\colon (0,1) \to \mathbb{R}$ is said to be a quantile function of X if

$$\mathbb{P}[X < q(u)] \leq u \leq \mathbb{P}[X \leq q(u)] \quad \text{for all } u \in (0,1).$$

   b) Show that every quantile function q of X is non-decreasing.

   The left-quantile function of X is given by

$$q_X^-(u) := \inf \{x \in \mathbb{R} : F_X(x) \geq u\}.$$

   c) Show that the "inf" can be replaced by a "min".
   d) Show that q_X^- is a quantile function of X.
   e) Show that there exists a countable set $N \subseteq (0,1)$ such that $q(u) = q_X^-(u)$ for every quantile function q of X and all $u \in (0,1) \setminus N$.
   f) Let q be a quantile function of X, $x \in \mathbb{R}$ and $u \in (0,1)$. Prove that
      (i) $u < F_X(x)$ implies $q(u) \leq x$, and
      (ii) $q(u) \leq x$ implies $u \leq F_X(x)$.

      This shows that for every $x \in \mathbb{R}$, one has

$$\{u \in (0,1) : u < F_X(x)\} \subseteq \{u \in (0,1) : q(u) \leq x\} \subseteq \{u \in (0,1) : u \leq F_X(x)\},$$

      from which one obtains

$$\eta \{u \in (0,1) : q(u) \leq x\} = F_X(x),$$

      where η is the Lebesgue measure on $(0,1)$. So q can be seen as a random variable on $(0,1)$, which under η, has the same distribution as X under $\mathbb{P}$.