Problem Set 3

Due on March 24

1. Let L be of the form $L = YZ$, where Y is a Bernoulli random variable with mean $p \in (0,1)$ and Z an independent exponential random variable with mean $m > 0$.
(This is a simple example of a mixture distribution.)

   a) Calculate the mean and the variance of L.
   b) Derive the cdf of L.
   c) Does L have a density?
   d) Calculate $\mathrm{VaR}_\alpha(L)$.
   e) Calculate $\mathrm{ES}_\alpha(L)$.

2. Let L be a random variable taking values in an interval $I \subseteq \mathbb{R}$. Show that

$$\mathrm{VaR}_\alpha(f(L)) = f(\mathrm{VaR}_\alpha(L))$$

for each $\alpha \in (0,1)$ and every strictly increasing continuous function $f \colon I \to \mathbb{R}$.

3. Let L be a random variable with cdf $F_L(x) = 1 - x^{-\lambda}$, $x \geq 1$, for a parameter $\lambda > 0$. Calculate $\mathrm{VaR}_\alpha(L + \log(L))$.

4. Let $L = \mu + \sigma t_\nu$, where t_ν is a Student-t random variable with $\nu > 1$ degrees of freedom. Calculate $\mathrm{VaR}_\alpha(X)$ and $\mathrm{ES}_\alpha(X)$.

5. Let L be a random variable and $a \in \mathbb{R}$ a number such that $\mathbb{P}[L \geq a] > 0$. Show that

$$\mathbb{E}[L \mid L \geq a] = a + \frac{\int_a^\infty \mathbb{P}[L > x]dx}{\mathbb{P}[L \geq a]} = a + \frac{\int_a^\infty \mathbb{P}[L \geq x]dx}{\mathbb{P}[L \geq a]}.$$

Hint: One can write $\mathbb{E}[L \mid L \geq a] = a + \mathbb{E}[L - a \mid L \geq a] = a + \mathbb{E}\left[\int_0^\infty 1_{\{L-a>u\}}du \mid L \geq a\right]$.