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Sensitivity of the Black–Scholes Option Price to the Local Path Behavior of the Stochastic Process Modeling the Underlying Asset

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Abstract—We show that a change in the local path behavior of the stock price process in the Black–Scholes model can have a dramatic effect on option prices and hedging strategies.

Wir führen ein Zeitintervall τ in die Betrachtung ein, welches sehr klein sei gegen die beobachtbaren Zeitintervalle, aber doch so gross, dass die in zwei aufeinanderfolgenden Zeitintervallen τ von einem Teilchen ausgeführten Bewegungen als voneinander unabhängige Ereignisse aufzufassen sind.[2]

A. Einstein

1. INTRODUCTION

We consider a money market account and a stock that pays no dividends. All economic activity takes place in a time interval $[0,T]$ for some $T \in (0,\infty)$. The short-term rate for lending and borrowing money is the same and equal to a constant r, so that an amount m of money deposited in the money market account at time 0 grows like

$$me^{rt}, \qquad t \in [0,T]. \tag{1.1}$$

Short-selling the stock is allowed, and it is possible to buy and sell any fraction of stock shares. Moreover, there exist no transaction costs, and stock shares can be bought and sold at the same price. At time 0, we only know the actual stock price S_0. The price S_t at which we will be able to buy or sell stock shares at time $t \in (0,T]$ is a random variable. Samuelson [13] proposed modeling the evolution of a stock price as a geometric Brownian motion

$$Y_t = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right\}, \qquad t \in [0,T], \tag{1.2}$$

where μ and $\sigma > 0$ are constants and B is a Brownian motion. Black and Scholes [1] noticed that, if there exists a money market account where money grows according to (1.1) and a stock whose price behaves as in (1.2), the pay-off of a European call option on the stock can be replicated with a dynamic investment strategy. They concluded that, in equilibrium, the price of the option must be equal to the value of the replicating portfolio, and by solving a partial differential equation they found explicit formulas for the price and the replicating strategy of a European call option. This argument was made rigorous by Harrison and Pliska [7]. By using martingale techniques, they were

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[2]See [5, p. 556].

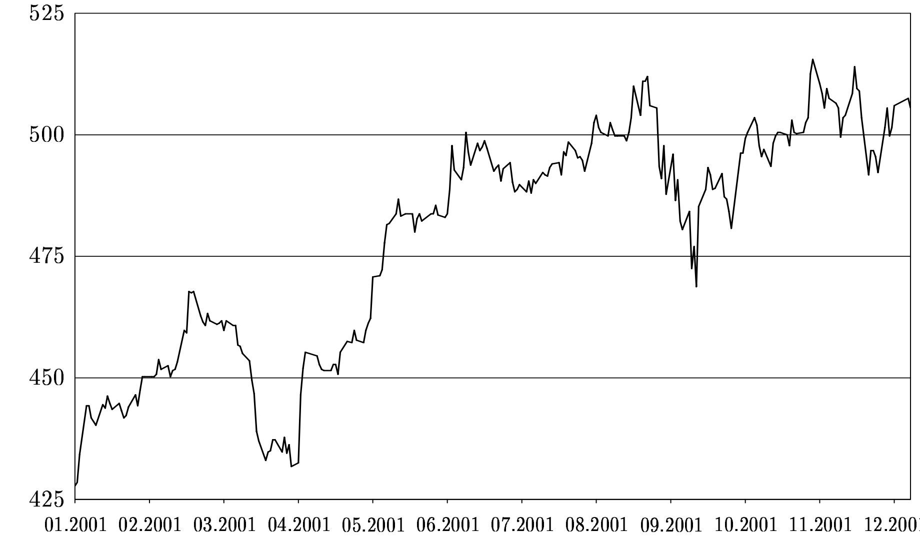

Fig. 1. Daily prices of a particular stock at the Swiss Exchange for the year 2001

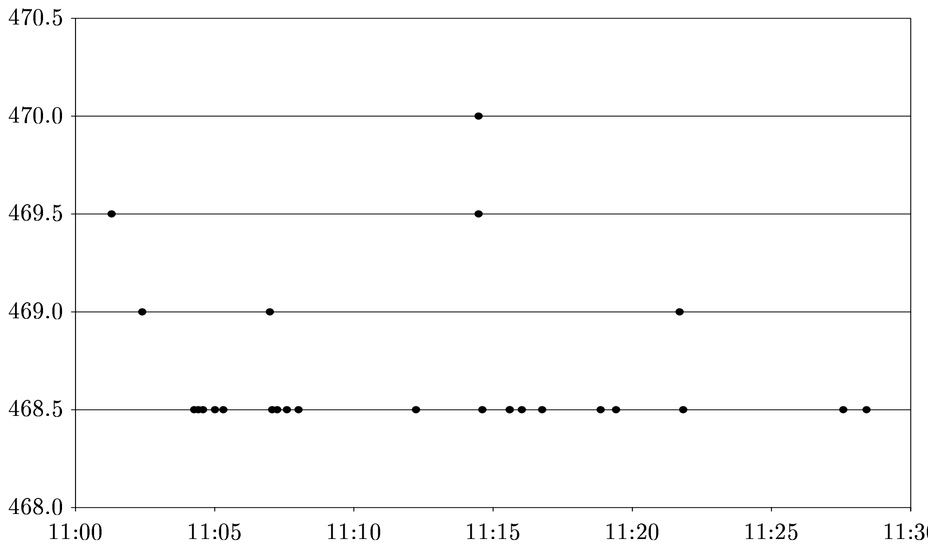

Fig. 2. Every transaction price of the same stock as in Fig. 1 between 11:00 and 11:30 on a day in May 2001

not only able to give an alternative derivation of the Black–Scholes formula but could also show that the Black–Scholes strategy is the one with the lowest initial portfolio value among all super-replicating strategies with nonnegative portfolio processes. Since then, both the analytic method of Black and Scholes [1] and the martingale method of Harrison and Pliska [7] have been extended in various directions: more general contingent claims have been treated, including contingent claims depending on several assets; more general stochastic processes have been used to model financial securities; and the effects of short-selling constraints, transaction costs, and unequally distributed information have been studied, as well as large investor problems and others. However, one point has received amazingly little attention: while most practitioners still find $Y_{t+\Delta} - Y_t$ a sufficiently good approximation to $S_{t+\Delta} - S_t$ if Δ is equal to a few days or longer, it is clear that the local path behavior of Y is completely different from that of S (see Figs. 1 and 2 for the evolution of the price of a particular stock over the period of one year and half an hour, respectively). Whereas stock shares are only traded at a discrete set of prices, a path of Y attains uncountably many different values. On the other hand, since the Black–Scholes hedging strategy adjusts its positions continuously in time, the Black–Scholes option price depends on the local path behavior of Y. The purpose of this paper is to show that the local path behavior of Y is in fact the property of the Black–Scholes model with the essential influence on option prices. To give evidence of this fact, we replace Brownian motion in the Black–Scholes model by a Gaussian process with a similar covariance structure but different local path behavior and discuss the effects on option pricing.

The paper is organized as follows. In Section 2, we give a critical review of the derivation of the Black–Scholes formula in the spirit of Harrison and Pliska [7] and fix the notation. In Section 3, we construct a class of Gaussian processes with covariance structure close to the one of a Brownian motion but with different local path behavior. In Section 4, we use these processes to study the effects of a local perturbation of the stock price process in the Black–Scholes model on option pricing. The last section contains some concluding remarks and a discussion of related work.

2. OPTION PRICING IN THE BLACK–SCHOLES MODEL

Let us assume that we can invest in a money market account where one unit of money grows like

$$e^{rt}, \qquad t \in [0,T],$$

and a stock whose price follows a stochastic process $(S_t)_{t\in[0,T]}$, and we want to hedge a European call option on the stock with maturity T and strike price K. The option has a time T random pay-off which is given by $(S_T - K)^+$. To avoid trivial arbitrage opportunities, its time 0 price has to lie in the interval $((S_0 - e^{-rT}K)^+, S_0)$. In the Black–Scholes model, $(S_t)_{t\in[0,T]}$ is approximated by a geometric Brownian motion (1.2) for appropriate constants μ and $\sigma > 0$ and a Brownian motion $(B_t)_{t\in[0,T]}$ on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Typically, if T is, for instance, one year and μ and σ are chosen carefully, $C_T = (Y_T - K)^+$ is a very good description of $(S_T - K)^+$. On the other hand, as discussed in the introduction, the local path behavior of Y is totally different from that of S. By C_0, we denote the minimal initial amount of money needed to superreplicate C_T with a dynamically adjusted portfolio consisting of money in the money market account and stock shares. It is clear that C_0 depends on the class of trading strategies that we take into consideration.

A trading strategy is a pair $\vartheta = (\vartheta^0, \vartheta^1)$ of stochastic processes $(\vartheta^0_t)_{t\in[0,T]}$ and $(\vartheta^1_t)_{t\in[0,T]}$. The process $\vartheta^0_t e^{rt}$ describes the money in the money market account at time t; and ϑ^1_t, the number of stock shares held at time t. Hence, the evolution of the portfolio value of a strategy ϑ is given by

$$V^\vartheta_t = \vartheta^0_t e^{rt} + \vartheta^1_t Y_t, \qquad t \in [0,T].$$

We set

$$\widetilde{Y}_t = e^{-rt} Y_t, \quad \widetilde{V}^\vartheta_t = e^{-rt} V^\vartheta_t, \quad t \in [0,T], \qquad \text{and} \qquad \widetilde{C}_T = e^{-rT} C_T.$$

Definition 2.1. Let ξ be a $[0,\infty]$-valued random variable with $\mathsf{P}[\xi > 0] > 0$. A trading strategy ϑ is a ξ-arbitrage if $\widetilde{V}^\vartheta_T = V^\vartheta_0 + \xi$. A trading strategy is an arbitrage if it is a ξ-arbitrage for some $[0,\infty]$-valued random variable ξ with $\mathsf{P}[\xi > 0] > 0$. A ξ-arbitrage is a strong arbitrage if there exists a constant $c > 0$ such that $\xi \geq c$.

It is clear that we must put certain restrictions on a trading strategy to give it an economic meaning. We set

$$\mathcal{N}^Y_T = \Big\{ N \subset \Omega\colon\ N \subset M \text{ for some } M \in \sigma((Y_t)_{t\in[0,T]}) \text{ with } \mathsf{P}[M] = 0 \Big\}$$

and

$$\mathcal{F}^Y_t = \sigma((Y_s)_{s\in[0,t]}, \mathcal{N}^Y_T), \qquad t \in [0,T].$$

Definition 2.2. Let $h \geq 0$. Then,

$$\mathbf{S}^h(\mathcal{F}^Y) := \Bigg\{ g_0 \mathbf{1}_{\{0\}} + \sum_{j=1}^{n-1} g_j \mathbf{1}_{(\tau_j, \tau_{j+1}]} \colon \; n \geq 2;\; 0 = \tau_1,\; \tau_1 + h \leq \tau_2, \dots, \tau_{n-1} + h \leq \tau_n = T; \\ \text{all } \tau_j\text{'s are } \mathcal{F}^Y\text{-stopping times; } g_0 \text{ is a constant, and} \\ \text{all other } g_j\text{'s are real, } \mathcal{F}^Y_{\tau_j}\text{-measurable random variables} \Bigg\}.$$

To avoid trading within time intervals over which Y is not a good description of S, one should determine an $h > 0$ such that $Y_{t+\Delta} - Y_t$ is a sufficiently good approximation to $S_{t+\Delta} - S_t$ for all $\Delta \geq h$ and $t \in [0, T - \Delta]$, and restrict trading strategies to the class

$$\Theta^h(\mathcal{F}^Y) := \left\{ \vartheta = (\vartheta^0, \vartheta^1) \colon \; \vartheta^0, \vartheta^1 \in \mathbf{S}^h(\mathcal{F}^Y) \right\}.$$

Since we do not want to add or withdraw money during the hedging, trading strategies should also satisfy the following self-financing condition:

Definition 2.3. Let $h \geq 0$ and $\vartheta = (\vartheta^0, \vartheta^1) \in \Theta^h(\mathcal{F}^Y)$. There exist stopping times

$$0 = \tau_1 \leq \tau_2 \leq \ldots \leq \tau_n = T$$

such that ϑ^0 and ϑ^1 can be written in the following form:

$$\vartheta^0 = f_0 \mathbf{1}_{\{0\}} + \sum_{j=1}^{n-1} f_j \mathbf{1}_{(\tau_j, \tau_{j+1}]}, \qquad \vartheta^1 = g_0 \mathbf{1}_{\{0\}} + \sum_{j=1}^{n-1} g_j \mathbf{1}_{(\tau_j, \tau_{j+1}]}. \tag{2.1}$$

For $h > 0$, we call ϑ self-financing if, for all $j = 1, \dots, n-1$,

$$(f_j - f_{j-1}) e^{r\tau_j} + (g_j - g_{j-1}) Y_{\tau_j} = 0. \tag{2.2}$$

For $h = 0$, we set $\tau_0 = -1$ and call ϑ self-financing if, for all $j = 1, \dots, n-1$, $k = 1, \dots, j$, and $l = 0, \dots, n - j - 1$,

$$\mathbf{1}_{\{\tau_{j-k} < \tau_{j-k+1} = \tau_{j+l} < \tau_{j+l+1}\}} \left\{ (f_{j+l} - f_{j-k}) e^{r\tau_j} + (g_{j+l} - g_{j-k}) Y_{\tau_j} \right\} = 0. \tag{2.3}$$

(Note that properties (2.2) and (2.3) are independent of the representation (2.1) of ϑ.) We denote also

$$\Theta^h_{\mathrm{sf}}(\mathcal{F}^Y) := \left\{ \vartheta \in \Theta^h(\mathcal{F}^Y) \colon \; \vartheta \text{ is self-financing} \right\}.$$

Proposition 2.4. *Let $\vartheta \in \Theta^0(\mathcal{F}^Y)$. Then, the following are equivalent:*

(i) *ϑ is self-financing for $(e^{rt}, Y_t)_{t \in [0,T]}$;*

(ii) *$V^\vartheta_t = V^\vartheta_0 + \int_0^t \vartheta^0_u r e^{ru} du + \int_0^t \vartheta^1_u dY_u$ for all $t \in [0, T]$;*

(iii) *ϑ is self-financing for $(1, \widetilde{Y}_t)_{t \in [0,T]}$;*

(iv) *$\widetilde{V}^\vartheta_t = V^\vartheta_0 + \int_0^t \vartheta^1_u d\widetilde{Y}_u$ for all $t \in [0, T]$.*

Proof. The proof is straightforward. For details, see [3, Proposition 2.6]. $\square$

It follows from Proposition 2.4 that, for all $\vartheta \in \Theta^0_{\mathrm{sf}}(\mathcal{F}^Y)$,

$$\vartheta^0_t = V^\vartheta_0 + \int_0^t \vartheta^1_u d\widetilde{Y}_u - \vartheta^1_t \widetilde{Y}_t, \qquad t \in [0, T]. \tag{2.4}$$

This shows that, for all $(x, \vartheta^1) \in \mathbb{R} \times \mathbf{S}^0(\mathcal{F}^Y)$, there exists a unique $\vartheta^0 \in \mathbf{S}^0(\mathcal{F}^Y)$ such that $\vartheta = (\vartheta^0, \vartheta^1) \in \Theta^0_{\mathrm{sf}}(\mathcal{F}^Y)$ and $V^\vartheta_0 = x$.

It can be shown that there exists no arbitrage in $\Theta^0_{\mathrm{sf}}(\mathcal{F}^Y)$. On the other hand, it is not possible to replicate C_T with a strategy from $\Theta^0_{\mathrm{sf}}(\mathcal{F}^Y)$. Moreover, the cheapest way to superreplicate C_T with a strategy from $\Theta^0_{\mathrm{sf}}(\mathcal{F}^Y)$ is to buy the stock. A cheaper alternative would consist in trying to approximate the option pay-off in some sense with a strategy from $\Theta^0_{\mathrm{sf}}(\mathcal{F}^Y)$. To give such an approximation an economic sense, the strategy should be in $\Theta^h_{\mathrm{sf}}(\mathcal{F}^Y)$ for an $h > 0$ so large that $Y_{t+\Delta} - Y_t$ is reasonably close to $S_{t+\Delta} - S_t$ for all $\Delta \geq h$ and $t \in [0, T-\Delta]$. However, if Y were also locally a good description of S, one could consider all strategies in $\Theta^0_{\mathrm{sf}}(\mathcal{F}^Y)$ as trading strategies, and this would lead naturally to the class

$$\Theta(\mathcal{F}^Y) := \left\{ \vartheta\colon\ \vartheta^0 \text{ and } \vartheta^1 \text{ are } \mathcal{F}^Y\text{-predictable}, \int_0^T |\vartheta^0_u|\,du < \infty \text{ and } \int_0^T (\vartheta^1_u)^2\,du < \infty \text{ almost surely} \right\}$$

because every $\vartheta \in \Theta(\mathcal{F}^Y)$ can be understood as a limit of strategies in $\Theta^0(\mathcal{F}^Y)$. To extend the self-financing property from $\Theta^0(\mathcal{F}^Y)$ to $\Theta(\mathcal{F}^Y)$, one can use the equivalence between (i) and (ii) of Proposition 2.4.

Definition 2.5. We call $\vartheta \in \Theta(\mathcal{F}^Y)$ self-financing if

$$V^\vartheta_t = V^\vartheta_0 + \int_0^t \vartheta^0_u r e^{ru}\,du + \int_0^t \vartheta^1_u\,dY_u \qquad \text{for all } t \in [0,T]$$

and denote

$$\Theta_{\mathrm{sf}}(\mathcal{F}^Y) := \left\{ \vartheta \in \Theta(\mathcal{F}^Y)\colon\ \vartheta \text{ is self-financing} \right\}.$$

The next proposition extends the equivalence between (ii) and (iv) of Proposition 2.4 from strategies in $\Theta^0(\mathcal{F}^Y)$ to strategies in $\Theta(\mathcal{F}^Y)$.

Proposition 2.6. *Let $\vartheta \in \Theta(\mathcal{F}^Y)$. Then, the following are equivalent*:

(i) $V^\vartheta_t = V^\vartheta_0 + \int_0^t \vartheta^0_u r e^{ru}\,du + \int_0^t \vartheta^1_u\,dY_u$ *for all* $t \in [0,T]$;

(ii) $\widetilde{V}^\vartheta_t = V^\vartheta_0 + \int_0^t \vartheta^1_u\,d\widetilde{Y}_u$ *for all* $t \in [0,T]$.

Proof. See the proof of Proposition 3.24 in [7]. □

It follows from Proposition 2.6 that (2.4) can be generalized to

$$\vartheta^0_t = V^\vartheta_0 + \int_0^t \vartheta^1_u\,d\widetilde{Y}_u - \vartheta^1_t \widetilde{Y}_t, \qquad t \in [0,T], \qquad \text{for all } \vartheta \in \Theta_{\mathrm{sf}}(\mathcal{F}^Y). \tag{2.5}$$

In contrast to $\Theta^0_{\mathrm{sf}}(\mathcal{F}^Y)$, there exist strong arbitrage strategies in $\Theta_{\mathrm{sf}}(\mathcal{F}^Y)$, for example, the well-known doubling strategies. It was noticed by Harrison and Pliska [7] that they can be ruled out by putting an admissibility condition on the trading strategies. We use the admissibility condition of Delbaen and Schachermayer [4]. It is more liberal than the one of Harrison and Pliska [7] but restrictive enough to exclude arbitrage.

Definition 2.7. Let $c \geq 0$. We call a $\vartheta \in \Theta_{\mathrm{sf}}(\mathcal{F}^Y)$ c-admissible if

$$\inf_{t\in[0,T]} \int_0^t \vartheta^1_u\,d\widetilde{Y}_u \geq -c.$$

We call ϑ admissible if it is c-admissible for some $c \geq 0$. Denote

$$\Theta_{\mathrm{sf,adm}}(\mathcal{F}^Y) := \left\{ \vartheta \in \Theta_{\mathrm{sf}}(\mathcal{F}^Y)\colon\ \vartheta \text{ is admissible} \right\}.$$

Next, we construct an equivalent probability measure. Let

$$\mathsf{P}^* = \exp\left\{-\left(\frac{\mu - r}{\sigma}\right) B_T - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T\right\} \cdot \mathsf{P}.$$

Then,

$$B_t^* = B_t + \left(\frac{\mu - r}{\sigma}\right) t, \qquad t \in [0, T],$$

is a Brownian motion on $(\Omega, (\mathcal{F}_t^Y)_{t\in[0,T]}, \mathsf{P}^*)$, and

$$\widetilde{Y}_t = S_0 \exp\left\{\left(\mu - r - \frac{\sigma^2}{2}\right) t + \sigma B_t\right\} = S_0 \exp\left\{\sigma B_t^* - \frac{\sigma^2}{2} t\right\}, \qquad t \in [0, T].$$

From the fact that a Brownian motion has the predictable representation property (see, e.g., [12, Ch. V]), it can be deduced that there exists a unique $\widehat{\vartheta} \in \Theta_{\mathrm{sf,adm}}(\mathcal{F}^Y)$ such that

$$\left(\int_0^t \widehat{\vartheta}_u^1 d\widetilde{Y}_u\right)_{t\in[0,T]}$$

is a square-integrable martingale under P^* and

$$\widetilde{V}_T^{\widehat{\vartheta}} = V_0^{\widehat{\vartheta}} + \int_0^T \widehat{\vartheta}_u^1 d\widetilde{Y}_u = \widetilde{C}_T.$$

On the other hand, for all $\vartheta \in \Theta_{\mathrm{sf,adm}}(\mathcal{F}^Y)$,

$$\widetilde{V}_t^{\vartheta} - V_0^{\vartheta} = \int_0^t \vartheta_u^1 d\widetilde{Y}_u, \qquad t \in [0, T],$$

is a local martingale under P^* that is uniformly bounded from below. Therefore, it is also a supermartingale. This implies that there exists no arbitrage in $\Theta_{\mathrm{sf,adm}}(\mathcal{F}^Y)$ and $V_0^{\widehat{\vartheta}} \le V_0^{\vartheta}$ for all $\vartheta \in \Theta_{\mathrm{sf,adm}}(\mathcal{F}^Y)$ with

$$\widetilde{V}_T^{\vartheta} = V_0^{\vartheta} + \int_0^T \vartheta_u^1 d\widetilde{Y}_u \ge \widetilde{C}_T.$$

Hence, $V_0^{\widehat{\vartheta}}$ is the minimal amount of initial wealth needed to superreplicate C_T with a strategy from $\Theta_{\mathrm{sf,adm}}(\mathcal{F}^Y)$. Since $(\widetilde{V}_t^{\widehat{\vartheta}})_{t\in[0,T]}$ is a P^*-martingale, $V_0^{\widehat{\vartheta}}$ can be calculated as follows:

$$V_0^{\widehat{\vartheta}} = \mathsf{E}^*[\widetilde{C}_T] = \mathrm{BS}_{r,K}(\sigma, S_0, T),$$

where

$$\mathrm{BS}_{r,K}(\sigma, x, z) = xN(d_1) - Ke^{-rz} N(d_2),$$

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^d \exp\left(-\frac{u^2}{2}\right) du,$$

$$d_1 = \frac{1}{\sigma\sqrt{z}}\left\{\log\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right) z\right\} \qquad \text{and} \qquad d_2 = d_1 - \sigma\sqrt{z}.$$

Moreover, it follows from the Markov property of Y that, for all $t \in [0,T]$,

$$V_t^{\widehat{\vartheta}} = e^{rt}\mathsf{E}^*\left[\widetilde{C}_T \mid \mathcal{F}_t^Y\right] = \mathsf{E}^*\left[e^{-r(T-t)}C_T \mid Y_t\right] = \mathrm{BS}_{r,K}(\sigma, Y_t, T-t).$$

Since $\mathrm{BS}_{r,K}(\sigma,x,z)$ is continuously differentiable in z and twice continuously differentiable in x, it can be derived from Itô's formula that

$$\widehat{\vartheta}_t^1 = \frac{\partial}{\partial x}\mathrm{BS}_{r,K}(\sigma, Y_t, T-t), \qquad t \in [0,T].$$

Then, $\widehat{\vartheta}^0$ can be obtained from $V_0^{\widehat{\vartheta}}$ and $\widehat{\vartheta}^1$ by (2.5).

3. ε-BROWNIAN MOTIONS

Definition 3.1. Let $\varepsilon > 0$. We call a stochastic process $(X_t)_{t\ge 0}$ an ε-Brownian motion if it is a centered Gaussian process such that, for all $t,s \ge 0$,

$$|\operatorname{cov}(X_t, X_s) - t \wedge s| \le \varepsilon.$$

In the rest of this section, we show how, for given $\varepsilon > 0$, ε-Brownian motions with arbitrary quadratic variations can be constructed. To this end, we need a two-sided Brownian motion. This is an almost surely continuous, centered Gaussian process $(W_t)_{t\in\mathbb{R}}$ with

$$\operatorname{cov}(W_t, W_s) = \frac{1}{2}(|t| + |s| - |t-s|), \qquad t,s \in \mathbb{R}.$$

It can be constructed by taking two independent one-sided Brownian motions $(W_t^1)_{t\ge 0}$ and $(W_t^2)_{t\ge 0}$ and setting

$$W_t := \begin{cases} W_t^1, & t \ge 0, \\ W_{-t}^2, & t < 0, \end{cases} \qquad t \in \mathbb{R}.$$

Moreover, we need real functions φ that satisfy

(R1) $\varphi\colon \mathbb{R} \to \mathbb{R}$ is measurable, $\varphi(x) = 0$ for all $x < 0$, and

(R2) for all $t \in \mathbb{R}$, $\int_{\mathbb{R}}[\varphi(t-u) - \varphi(-u)]^2 du < \infty$.

Such functions can be used to define, for all $t \in \mathbb{R}$,

$$X_t^\varphi := \int_{\mathbb{R}} [\varphi(t-u) - \varphi(-u)]dW_u$$

in the L^2-sense. It is clear that $(X_t^\varphi)_{t\in\mathbb{R}}$ is a centered Gaussian process with stationary increments, and, for all $t \in \mathbb{R}$, the variance of X_t^φ can be written as

$$\mathsf{D}\, X_t^\varphi = \int_{\mathbb{R}} [\varphi(t-u) - \varphi(-u)]^2 du.$$

Moreover,

$$\operatorname{cov}(X_t^\varphi, X_s^\varphi) = \frac{1}{2}\left[\mathsf{D}\, X_t^\varphi + \mathsf{D}\, X_s^\varphi - \mathsf{D}(X_t^\varphi - X_s^\varphi)\right] = \frac{1}{2}\left[\mathsf{D}\, X_{|t|}^\varphi + \mathsf{D}\, X_{|s|}^\varphi - \mathsf{D}\, X_{|t-s|}^\varphi\right], \qquad t,s \in \mathbb{R}.$$

We denote by $(\mathcal{F}_t^W)_{t\ge 0}$ the smallest filtration that satisfies the usual assumptions such that

$$\sigma(W_s\colon -\infty < s \le t) \subset \mathcal{F}_t^W, \qquad t \ge 0,$$

and by $(\mathcal{F}_t^{X^\varphi})_{t\geq 0}$ the smallest filtration that satisfies the usual assumptions such that

$$\sigma(X_s^\varphi : 0 \leq s \leq t) \subset \mathcal{F}_t^{X^\varphi}, \qquad t \geq 0.$$

Note that, if φ is of the form

$$\text{(R3)} \qquad \varphi(t) = \begin{cases} \varphi(0) + \int_0^t \psi(u)du, & t \geq 0, \\ 0, & t < 0, \end{cases} \qquad \text{for some } \psi \in L^2(\mathbb{R}^+),$$

then it also satisfies (R1) and (R2). Hence, $(X_t^\varphi)_{t\in\mathbb{R}}$ is well-defined, and it follows from the following proposition that $(X_t^\varphi)_{t\geq 0}$ is a continuous semimartingale on $(\Omega, (\mathcal{F}_t^W)_{t\geq 0}, \mathsf{P})$ and, therefore, by Stricker's theorem (see, e.g., [11, Theorem II.4]), also on $(\Omega, (\mathcal{F}_t^{X^\varphi})_{t\geq 0}, \mathsf{P})$.

Proposition 3.2. *If φ satisfies* (R3), *then, for all $t \geq 0$,*

$$X_t^\varphi = \varphi(0)W_t + \int_0^t \int_{-\infty}^s \psi(s-u)dW_u ds. \tag{3.1}$$

Proof.

$$\begin{aligned} X_t^\varphi &= \int_{-\infty}^t [\varphi(t-u) - \varphi(-u)]dW_u = \int_{-\infty}^0 [\varphi(t-u) - \varphi(-u)]dW_u + \int_0^t \varphi(t-u)dW_u \\ &= \int_{-\infty}^0 \int_0^t \psi(s-u)ds dW_u + \int_0^t \left[\int_u^t \psi(s-u)ds + \varphi(0) \right] dW_u. \end{aligned}$$

By the stochastic version of Fubini's theorem (see, e.g., [11, Theorem IV.46]), we can change the order of integration. Hence, the above equals

$$\int_0^t \int_{-\infty}^0 \psi(s-u)dW_u ds + \int_0^t \int_0^s \psi(s-u)dW_u ds + \varphi(0)W_t = \int_0^t \int_{-\infty}^s \psi(s-u)dW_u ds + \varphi(0)W_t;$$

i.e., (3.1) holds. □

It can be seen from Proposition 3.2 that, provided φ satisfies (R3), then $[X^\varphi, X^\varphi]_t = \varphi^2(0)t$ and $(X_t^\varphi)_{t\geq 0}$ is a finite variation process if $\varphi(0) = 0$. Furthermore,

Theorem 3.3. *If φ satisfies* (R3) *and* $\varphi(0) \neq 0$, *then*

$$\exp\left\{ -\int_0^t \int_{-\infty}^s \frac{\psi(s-u)}{\varphi(0)} dW_u dW_s - \frac{1}{2} \int_0^t \left(\int_{-\infty}^s \frac{\psi(s-u)}{\varphi(0)} dW_u \right)^2 ds \right\}, \qquad t \geq 0,$$

is a martingale on $(\Omega, (\mathcal{F}_t^W)_{t\geq 0}, \mathsf{P})$.

Proof. Since $\psi \in L^2(\mathbb{R}_+)$, it can be shown that there exists a sequence of real numbers $\{t_n\}_{n=0}^\infty$ with

$$0 = t_0 < \ldots < t_n \nearrow \infty,$$

such that

$$\mathsf{E}\left[\exp\left\{ \frac{1}{2} \int_{t_{n-1}}^{t_n} \left(\int_{-\infty}^s \frac{\psi(s-u)}{\varphi(0)} dW_u \right)^2 ds \right\} \right] < \infty \qquad \text{for all } n \geq 1.$$

Then, the theorem follows from [10, Corollary 3.5.14]. For a detailed proof, see [3, Theorem 3.6]. □

Corollary 3.4. *If φ satisfies* (R3) *and $\varphi(0) \neq 0$, then $(\frac{1}{\varphi(0)} X_t^{\varphi})_{t\in[0,T]}$ is a Brownian motion on $(\Omega, (\mathcal{F}_t^W)_{t\in[0,T]}, \mathsf{P}^{\varphi})$, where*

$$\mathsf{P}^{\varphi} = \exp\left\{ -\int_0^T \int_{-\infty}^s \frac{\psi(s-u)}{\varphi(0)} dW_u dW_s - \frac{1}{2} \int_0^T \left(\int_{-\infty}^s \frac{\psi(s-u)}{\varphi(0)} dW_u \right)^2 ds \right\} \cdot \mathsf{P}.$$

In particular, it is a Brownian motion on $(\Omega, (\mathcal{F}_t^{X^{\varphi}})_{t\in[0,T]}, \mathsf{P}^{\varphi})$.

Proof. The corollary follows from Theorem 3.3 and Girsanov's theorem. □

To obtain Gaussian processes with arbitrary quadratic variation and a covariance structure close to the one of a Brownian motion, we need, for every $v \geq 0$, a function φ of the form (R3) with $\varphi(0) = v$ and such that

$$\int_{-\infty}^t [\varphi(t-u) - \varphi(-u)]^2 du \simeq t, \qquad t \geq 0.$$

The simplest functions with this properties are the following:

For $v \geq 0$ and $d > 0$, define

$$\varphi^{v,d}(t) := \mathbf{1}_{\{0\leq t\leq d\}} \left(v + \frac{1-v}{d} t \right) + \mathbf{1}_{\{d<t\}}, \qquad t \in \mathbb{R}.$$

A calculation shows that

$$\mathsf{D}\, X_t^{\varphi^{v,d}} = \int_{-d}^t [\varphi^{v,d}(t-u) - \varphi^{v,d}(-u)]^2 du = \begin{cases} t + \dfrac{1}{3}(v-1)(2v+1)d, & t \geq d, \\ t + t(v-1)\left[v\left(1 - \dfrac{1}{3}\dfrac{t^2}{d^2}\right) + 1 - \dfrac{t}{d} + \dfrac{1}{3}\dfrac{t^2}{d^2} \right], & t < d. \end{cases}$$

It can be deduced from this that

$$\left| \mathsf{D}\, X_t^{\varphi^{v,d}} - t \right| \leq |v^2 - 1| d, \qquad t \geq 0.$$

This shows that, for given $\varepsilon > 0$ and $v \geq 0$, there exists a $d > 0$ such that, for all $t \geq 0$,

$$\left| \mathsf{D}\, X_t^{\varphi^{v,d}} - t \right| \leq \frac{2}{3}\varepsilon.$$

Therefore, for all $t, s \geq 0$,

$$\begin{aligned} \left| \operatorname{cov}\left(X_t^{\varphi^{v,d}}, X_s^{\varphi^{v,d}} \right) - t \wedge s \right| &= \frac{1}{2} \left| \mathsf{D}\, X_t^{\varphi^{v,d}} + \mathsf{D}\, X_s^{\varphi^{v,d}} - \mathsf{D}\, X_{|t-s|}^{\varphi^{v,d}} - t - s + |t-s| \right| \\ &\leq \frac{1}{2} \left(\left| \mathsf{D}\, X_t^{\varphi^{v,d}} - t \right| + \left| \mathsf{D}\, X_s^{\varphi^{v,d}} - s \right| + \left| \mathsf{D}\, X_{|t-s|}^{\varphi^{v,d}} - |t-s| \right| \right) \leq \varepsilon. \end{aligned}$$

Hence, for every $\varepsilon > 0$ and $v \geq 0$, we can find a $d > 0$ such that $(X_t^{\varphi^{v,d}})_{t\geq 0}$ is an almost surely continuous ε-Brownian motion with stationary increments.

Remarks. 1. For all $v \geq 0$ and $d > 0$, the function

$$u \mapsto \varphi^{v,d}(t-u) - \varphi^{v,d}(-u)$$

is so regular that, almost surely, the integral

$$X_t^{\varphi^{v,d}} = \int_{\mathbb{R}} [\varphi^{v,d}(t-u) - \varphi^{v,d}(-u)] dW_u$$

can, for all $t \in \mathbb{R}$, be understood as a limit of Riemann–Stieltjes sums, and, for all $v \geq 0$,

$$\sup_{t\in[0,T]} \left| W_t - X_t^{\varphi^{v,d}} \right| \overset{(d\searrow 0)}{\longrightarrow} 0 \qquad \text{almost surely.}$$

In particular, the laws of $(X_t^{\varphi^{v,d}})_{t\in[0,T]}$ converge weakly to the Wiener measure as $d \searrow 0$.

2. The functions $\varphi^{v,d}$ are of the form

$$\varphi^{v,d}(t) = v + \int_0^t \psi^{v,d}(u) du, \qquad t \geq 0,$$

where

$$\psi^{v,d}(t) = \mathbf{1}_{\{0 \leq t \leq d\}} \frac{1-v}{d}.$$

Hence, it follows from Proposition 3.2 that

$$X_t^{\varphi^{v,d}} = vW_t + (1-v) \int_0^t \frac{W_s - W_{s-d}}{d} ds, \qquad t \geq 0.$$

4. OPTION PRICING IN LOCALLY PERTURBED BLACK–SCHOLES MODELS

In this section, we assume that there exist constants r, μ, and $\sigma > 0$, such that one unit of money in the money market account grows like

$$e^{rt}, \qquad t \in [0,T],$$

and

$$Y_T = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) T + \sigma B_T\right)$$

is a good description of S_T. Let $\varepsilon > 0$ be very small. For every $v \geq 0$, there exists a $d(\varepsilon, v) > 0$ such that

$$X_t^{\varepsilon,v} := X_t^{\varphi^{v,d(\varepsilon,v)}}, \qquad t \geq 0,$$

is an ε-Brownian motion. We replace the geometric Brownian motion Y in the Black–Scholes model by

$$Y_t^{\varepsilon,v} = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma X_t^{\varepsilon,v}\right), \qquad t \in [0,T].$$

This has no strong influence on option pricing if only strategies in Θ_{sf}^h are allowed where h is so large that $Y_{t+\Delta}^{\varepsilon,v} - Y_t^{\varepsilon,v}$ is close to $Y_{t+\Delta} - Y_t$ for all $\Delta \geq h$ and $t \in [0, T-\Delta]$. However, if the class of trading strategies is $\Theta_{\mathrm{sf,adm}}$, a change in the local path behavior of Y can have a dramatic effect on option pricing. In this situation, there is a qualitative difference between the cases $v > 0$ and $v = 0$. For $v > 0$, the model has no arbitrage, and the European call option

$$(Y_T^{\varepsilon,v} - K)^+$$

can be replicated. On the other hand, $Y^{\varepsilon,0}$ is a finite variation process, and the model has strong arbitrage.

4.1. The case $v > 0$. It follows from Corollary 3.4 that, for all $v > 0$, there exists a probability measure $\mathsf{P}^{\varepsilon,v} \sim \mathsf{P}$ under which $(\frac{1}{v}X_t^{\varepsilon,v})_{t\in[0,T]}$ is a Brownian motion. Therefore, it follows as in Section 2 that there exists a unique $\widehat{\vartheta} \in \Theta_{\mathrm{sf,adm}}(\mathcal{F}^{Y^{\varepsilon,v}})$ such that

$$\left(\int_0^t \widehat{\vartheta}_u^1 d\widetilde{Y}_u^{\varepsilon,v}\right)_{t\in[0,T]}$$

is a square-integrable martingale under $\mathsf{P}^{\varepsilon,v}$ and

$$\widetilde{V}_0^{\widehat{\vartheta}} + \int_0^T \widehat{\vartheta}_u^1 d\widetilde{Y}_u^{\varepsilon,v} = \widetilde{C}_T.$$

Hence, the option's superreplication price is

$$V_0^{\widehat{\vartheta}} = e^{-rT}\mathsf{E}_{\mathsf{P}^{\varepsilon,v}}[(Y_T^{\varepsilon,v} - K)^+] = \mathrm{BS}_{r,K}(v\sigma, S_0, T), \tag{4.1}$$

and the minimal replicating strategy $\widehat{\vartheta}$ is given by

$$\widehat{\vartheta}_t^1 = \frac{\partial}{\partial x}\mathrm{BS}_{r,K}(v\sigma, Y_t^{\varepsilon,v}, T-t), \qquad t\in[0,T],$$

$$\widehat{\vartheta}_t^0 = V_0^{\widehat{\vartheta}} + \int_0^t \widehat{\vartheta}_u^1 d\widetilde{Y}_u^{\varepsilon,v} - \widehat{\vartheta}_t^1\widetilde{Y}_t^{\varepsilon,v}, \qquad t\in[0,T].$$

This shows that, if it is possible to trade continuously and without transaction costs, a local perturbation of Y can have a dramatic effect on the price and the replicating strategy of a European call option. In fact, since for all r and $K, S_0, T > 0$ the map

$$\sigma \mapsto \mathrm{BS}_{r,K}(\sigma, S_0, T)$$

is a bijection from $(0,\infty)$ to $((S_0 - e^{-rT}K)^+, S_0)$, there exists, for every $c \in ((S_0 - e^{-rT}K)^+, S_0)$, a v such that $\mathrm{BS}_{r,K}(v\sigma, S_0, T) = c$.

4.2. The case $v = 0$. The process $X^{\varepsilon,0}$ (and, hence, also $Y^{\varepsilon,0}$) is almost surely continuous and has finite variation. It was already noticed by Harrison *et al.* [8] that almost surely continuous, finite variation price processes have significant positive correlation between small successive increments, which can be exploited by investors who are able to trade continuously and without transaction costs. In fact, if $(Z_t)_{t\in[0,T]}$ is an almost surely continuous, finite variation process, it can be checked that, for all $c > 0$,

$$\vartheta_t = c\left(-\widetilde{Z}_t^2 + Z_0^2, 2(\widetilde{Z}_t - Z_0)\right), \qquad t\in[0,T],$$

is a self-financing strategy for $(e^{rt}, Z_t)_{t\in[0,T]}$, and

$$\widetilde{V}_t^{\vartheta} = c(\widetilde{Z}_t - Z_0)^2, \qquad t\in[0,T].$$

Hence, if $\widetilde{Z}$ is nontrivial, ϑ is a 0-admissible arbitrage in $\Theta_{\mathrm{sf,adm}}(\mathcal{F}^Z)$. Following the arguments in [3, Section 2.3], we can even construct a strong arbitrage in a smaller class than $\Theta_{\mathrm{sf,adm}}(\mathcal{F}^Z)$ if Z is an almost surely continuous, finite variation process that also satisfies condition (4.2) below, which is fulfilled by $Y^{\varepsilon,0}$. We need the following generalization of $\mathbf{S}^0$.

Definition 4.1.

$$\mathbf{aS}(\mathcal{F}^Z) := \Bigg\{ g_0\,\mathbf{1}_{\{0\}} + \sum_{j=1}^{\infty} g_j\,\mathbf{1}_{(\tau_j,\tau_{j+1}]} \colon \ 0=\tau_1 \le \tau_2 \le \ldots \le T;\ \text{all } \tau_j\text{'s are } \mathcal{F}^Z\text{-stopping times; } g_0 \text{ is a constant, and all other } g_j\text{'s are real, } \mathcal{F}^Z_{\tau_j}\text{-measurable random variables; } \mathsf{P}[\exists j \text{ such that } \tau_j = T] = 1 \Bigg\}$$

and

$$\Theta^{\mathbf{aS}}_{\mathrm{sf,adm}}(\mathcal{F}^Z) := \Big\{ \vartheta \in \Theta_{\mathrm{sf,adm}}(\mathcal{F}^Z) \colon\ \vartheta^0, \vartheta^1 \in \mathbf{aS}(\mathcal{F}^Z) \Big\}.$$

Note that there exist doubling strategies in $\Theta^{\mathbf{aS}}_{\mathrm{sf}}(\mathcal{F}^Z)$. Hence, the admissibility condition is important.

Theorem 4.2. *Let $(Z_t)_{t\in[0,T]}$ be an almost surely continuous, finite variation process such that*

$$\mathsf{P}\Big[\widetilde{Z}_{T(1-2^{-k})} = \widetilde{Z}_{T(1-2^{1-k})}\Big] = 0 \qquad \textit{for all } k \ge 1. \tag{4.2}$$

Then, for all $c_1 > 0$ and $c_2 > 0$, there exists a c_1-admissible c_2-arbitrage for $(e^{rt}, Z_t)_{t\in[0,T]}$ in $\Theta^{\mathbf{aS}}_{\mathrm{sf,adm}}(\mathcal{F}^Z)$.

Proof. Let $c_1, c_2 > 0$. There is no loss of generality in assuming $T = 1$. For all $k \ge 1$, we set

$$I_k = (a_k, a_{k+1}], \qquad \text{where} \quad a_k = 1 - 2^{1-k}.$$

By (4.2), there exist $\delta_k > 0$ such that

$$\mathsf{P}\Big[(\widetilde{Z}_{a_{k+1}} - \widetilde{Z}_{a_k})^2 < \delta_k\Big] < \frac{1}{2k}, \qquad k \ge 1.$$

Since $\widetilde{Z}$ is an almost surely continuous, finite variation process, for all $k \ge 1$, there exist deterministic times

$$a_k = t_0^k < \ldots < t^k_{n(k)} = a_{k+1}$$

such that

$$\mathsf{P}\left[\sup_{t\in I_k} \sum_{j=0}^{n(k)-1} \left(\widetilde{Z}_{t^k_{j+1}\wedge t} - \widetilde{Z}_{t^k_j\wedge t}\right)^2 \ge \frac{c_1\delta_k}{c_1 + 2^k(c_1+c_2)}\right] < \frac{1}{2k}. \tag{4.3}$$

Then,

$$\zeta_k = a_{k+1} \wedge \inf\left\{ t \in I_k \colon\ \sum_{j=0}^{n(k)-1} \left(\widetilde{Z}_{t^k_{j+1}\wedge t} - \widetilde{Z}_{t^k_j\wedge t}\right)^2 \ge \frac{c_1\delta_k}{c_1 + 2^k(c_1+c_2)} \right\} \tag{4.4}$$

is an $\mathcal{F}^Z$-stopping time, and (4.3) implies

$$\mathsf{P}[\zeta_k < a_{k+1}] < \frac{1}{2k}. \tag{4.5}$$

For all $k \ge 1$,

$$\beta^k = \frac{2}{2^k\delta_k}(c_1 + 2^k(c_1+c_2)) \sum_{j=1}^{n(k)-1} (\widetilde{Z}_{t^k_j} - \widetilde{Z}_{a_k})\,\mathbf{1}_{(t^k_j, t^k_{j+1}]}\,\mathbf{1}_{[0,\zeta_k]}$$

is in $\mathbf{S}^0(\mathcal{F}^Z)$, and an easy calculation shows that, for all $t \in I_k$,

$$\int_{a_k}^{t} \beta_u^k d\widetilde{Z}_u = \frac{1}{2^k \delta_k}(c_1 + 2^k(c_1 + c_2)) \left[(\widetilde{Z}_{t\wedge\zeta_k} - \widetilde{Z}_{a_k})^2 - \sum_{j=0}^{n(k)-1} \left(\widetilde{Z}_{t^k_{j+1}\wedge t\wedge\zeta_k} - \widetilde{Z}_{t^k_j\wedge t\wedge\zeta_k} \right)^2 \right].$$

It follows from (4.4) that

$$\sup_{t\in I_k} \sum_{j=0}^{n(k)-1} \left(\widetilde{Z}_{t^k_{j+1}\wedge t\wedge\zeta_k} - \widetilde{Z}_{t^k_j\wedge t\wedge\zeta_k} \right)^2 \le \frac{c_1\delta_k}{c_1 + 2^k(c_1 + c_2)}.$$

Hence,

$$\inf_{t\in I_k} \int_{a_k}^{t} \beta_u^k d\widetilde{Z}_u \ge -\frac{1}{2^k\delta_k}(c_1 + 2^k(c_1 + c_2)) \left(\frac{c_1\delta_k}{c_1 + 2^k(c_1 + c_2)} \right) = -\frac{c_1}{2^k}. \tag{4.6}$$

Moreover,

$$\begin{aligned} \mathsf{P}\left[\int_{I_k} \beta_u^k d\widetilde{Z}_u < c_1 + c_2 \right] &= \mathsf{P}\left[(\widetilde{Z}_{\zeta_k} - \widetilde{Z}_{a_k})^2 - \sum_{j=0}^{n(k)-1} \left(\widetilde{Z}_{t^k_{j+1}\wedge\zeta_k} - \widetilde{Z}_{t^k_j\wedge\zeta_k} \right)^2 < \frac{2^k\delta_k}{c_1 + 2^k(c_1 + c_2)}(c_1 + c_2) \right] \\ &\le \mathsf{P}\left[(\widetilde{Z}_{\zeta_k} - \widetilde{Z}_{a_k})^2 < \frac{2^k\delta_k}{c_1 + 2^k(c_1 + c_2)}(c_1 + c_2) + \frac{c_1\delta_k}{c_1 + 2^k(c_1 + c_2)} \right] \\ &= \mathsf{P}\left[(\widetilde{Z}_{\zeta_k} - \widetilde{Z}_{a_k})^2 < \delta_k \right] \\ &\le \mathsf{P}[\zeta_k < a_{k+1}] + \mathsf{P}\left[(\widetilde{Z}_{a_{k+1}} - \widetilde{Z}_{a_k})^2 < \delta_k \right] < \frac{1}{k}. \end{aligned} \tag{4.7}$$

Now, we set

$$\beta = \sum_{k=1}^{\infty} \beta^k \mathbf{1}_{I_k} \qquad \text{and} \qquad \zeta = \inf\left\{ t \in [0,1] \colon \int_0^t \beta_u d\widetilde{Z}_u \ge c_2 \right\}.$$

It follows from (4.6) that

$$\inf_{t\in[0,1)} \int_0^t \beta_u d\widetilde{Z}_u \ge -c_1,$$

and $\mathsf{P}[\zeta < 1] = 1$ because of (4.7). Hence, ϑ defined by

$$\vartheta^1 = \beta \mathbf{1}_{[0,\zeta]}$$

and

$$\vartheta^0_t = \int_0^t \vartheta^1_u d\widetilde{Z}_u - \vartheta^1_t \widetilde{Z}_t, \qquad t \in [0,1],$$

is a c_1-admissible c_2-arbitrage in $\Theta^{\mathbf{aS}}_{\mathrm{sf,adm}}(\mathcal{F}^Z)$. □

In a stock price model with strong arbitrage, it is possible to superreplicate a European call option without initial endowment in the following way. At time 0, one borrows money to buy one stock share. Then, one applies a strong arbitrage strategy to generate the amount of money needed to pay back the debts without selling the stock share. At time T, one owns a stock share and has no debts. This hedges the option.

5. CONCLUDING REMARKS

We have constructed centered Gaussian processes $X^{\varepsilon,v}$, $\varepsilon > 0$, $v \geq 0$, with

$$|\operatorname{cov}(X_t^{\varepsilon,v}, X_s^{\varepsilon,v}) - t \wedge s| \leq \varepsilon, \quad t, s \geq 0, \qquad \text{and} \qquad [X^{\varepsilon,v}, X^{\varepsilon,v}]_t = v^2 t, \quad t \geq 0.$$

Then, we have shown that the replacement of a Brownian motion in the Black–Scholes model by a process $X^{\varepsilon,v}$ can lead to totally different option prices if it is possible to trade continuously and without transaction costs. As Brownian motions, the processes $X^{\varepsilon,v}$ are almost surely continuous and have stationary increments. On the other hand, they do not have independent increments and are not self-similar. If continuous trading without transaction costs is possible, the superreplication price of a European call option in the model

$$\left(e^{rt}, S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma X_t^{\varepsilon,v}\right\}\right), \qquad t \in [0, T],$$

depends on v, which determines the quadratic variation of $X^{\varepsilon,v}$, but not on $\mathsf{D}\, X_T^{\varepsilon,v}$. This shows that the Black–Scholes option price depends crucially on the local path behavior of the stochastic process modeling the underlying asset, which is insofar problematic as the local path behavior of a geometric Brownian motion is totally different from the local path behavior of the evolution of real stock prices.

A similar result was obtained by Brigo and Mercurio [2]. For a given finite time-grid $\Gamma \subset [0, T]$, they constructed a class of processes $(Y_t^\nu)_{t \in [0,T]}$, $\nu \in (0, \infty)$, such that each process Y^ν has the same finite-dimensional distribution on Γ as the geometric Brownian motion Y given in (1.2), the same one-dimensional marginal distributions as Y for all $t \in [0, T]$, and a unique equivalent martingale measure. As in our case, the quadratic variation of the processes Y^ν in [2] can be very different from that of Y, and, for every constant

$$c \in \left((S_0 - e^{-rT} K)^+, S_0\right),$$

there exists a $\nu \in (0, \infty)$ such that the time zero price of a European call option with maturity T and strike price K on a stock modeled with Y^ν equals c. In contrast to our processes $Y^{\varepsilon,v}$, the processes Y^ν in [2] have exactly the same distribution as Y on the finite-time grid Γ. On the other hand, the log-processes $(\log Y_t^\nu)_{t \in [0,T]}$ do not have stationary increments whereas our log-processes $(\log Y_t^{\varepsilon,v})_{t \in [0,T]}$ do.

Another example in the same spirit can be found in Hubalek and Schachermayer [9]. In their paper, it is shown that, for a Black–Scholes model with given parameters, there exists, for every constant

$$c \in \left((S_0 - e^{-rT} K)^+, S_0\right),$$

a sequence of binomial models that converges weakly to the Black–Scholes model, and, at the same time, the prices of a European call option with maturity T and strike price K evaluated in the binomial models converge to c. Moreover, conditions are given that ensure the convergence of option prices when a sequence of asset price models converges weakly to a complete asset price model. In our case, it can be seen from formula (4.1) that the price of a European call option in the model

$$(e^{rt}, Y_t^{\varepsilon,v})_{t \in [0,T]}$$

is close to its price in the Black–Scholes model

$$(e^{rt}, Y_t)_{t \in [0,T]}$$

when the quadratic variation of $Y^{\varepsilon,v}$ is close to the one of Y. This is also in accordance with Theorem 3.1 of Gallus [6], which gives an estimate for the additional costs resulting from a wrong hedging strategy.

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