Processes of Class Sigma, Last Passage Times and Drawdowns

Patrick Cheridito*
Princeton University
Princeton, NJ, USA

Ashkan Nikeghbali
University of Zurich
Zurich, Switzerland

Eckhard Platen
University of Technology
Sydney, Australia

March 2012

Abstract

We propose a general framework for studying last passage times, suprema and drawdowns of a large class of continuous-time stochastic processes. Our approach is based on processes of class Sigma and the more general concept of two processes, one of which moves only when the other is at the origin. After investigating certain transformations of such processes and their convergence properties, we provide three general representation results. The first allows to recover a process of class Sigma from its final value and the last time it visited the origin. In many situations this gives access to the distribution of the last time a stochastic process attains a certain level or is equal to its running maximum. It also leads to recently discovered formulas expressing option prices in terms of last passage times. Our second representation result is a stochastic integral representation that will allow us to price and hedge options on the running maximum of an underlying that are triggered when the underlying drops to a given level, or alternatively, when the drawdown or relative drawdown of the underlying attains a given height. The third representation gives conditional expectations of certain functionals of processes of class Sigma. It can be used to deduce the distributions of a variety of interesting random variables such as running maxima, drawdowns and maximum drawdowns of suitably stopped processes.

Keywords: Processes of class Sigma, last passage times, drawdowns, relative drawdowns, maximum drawdowns, options on running maxima.

1 Introduction

The aim of this paper is to develop a framework for studying various properties of continuous-time stochastic processes such as the behavior of last passage times, running maxima and drawdowns. As applications, we discuss the pricing and hedging of options depending on running maxima and drawdowns of an underlying process as well as distributions of running maxima, maximum drawdowns and maximum relative drawdowns of suitably stopped processes. Our approach is based on processes of class (Σ) and the more general concept of two processes of which one moves only when the other is at zero. Nonnegative local submartingales of class (Σ) were introduced by Yor [27] and further studied by Nikeghbali [14, 15]. They are related to relative martingales (see Azéma et al. [3, 1]). Here we extend the class (Σ) so that it also includes semimartingales which are not local submartingales, and we study the more general situation of a continuous finite variation process that does not move unless a given semimartingale is at the origin. In Section 2 we give precise definitions and provide preliminary results on convergence, positive parts and products of processes of class (Σ). Then we

*Supported by NSF Grant DMS-0642361

study a family of transformations that map a pair consisting of a semimartingale and a continuous finite variation process to a new semimartingale and that leave the class (Σ) invariant when applied to semimartingales of class (Σ) together with their finite variation parts. In Section 3 we prove three general representation results. The first gives conditions under which a process X of class (Σ) converges to a limit X_∞ and can be recovered from X_∞ and the last time L it visited zero. In situations where $\mathbb{E}[X_t]$ can be calculated, this gives access to the distribution of the random time L. It also allows to give simple proofs of recent results by Madan et al. [10] and Profeta et al. [23] expressing option prices in terms of last passage times of the underlying price processes. Our second representation result shows how to write certain functionals of a semimartingale as stochastic integrals, and the third one gives a formula for their conditional expectations. They have applications in the study of running maxima $\overline{Y}_t := \sup_{u\leq t} Y_u$ of continuous-time stochastic processes Y as well as drawdowns $DD_t := \overline{Y}_t - Y_t$, relative drawdowns $rDD_t := (\overline{Y}_t - Y_t)/\overline{Y}_t = 1 - Y_t/\overline{Y}_t$, maximum drawdowns $\overline{DD}_t$ and maximum relative drawdowns $\overline{rDD}_t$. In Section 4 we apply the stochastic integral representation of Section 3 to analyze options on running maxima that are triggered when the underlying first falls to a given level, or alternatively, when the drawdown or relative drawdown attains a certain height. We first discuss perpetual options and then consider options with finite time horizons. Our options are related to lookback options, the crash option introduced by Vecer [26] and the Russian option of Shepp and Shiryaev [25]. However, due to their particular form, our options are always replicable by dynamic trading even if the market is incomplete. Moreover, we will be able to derive closed form expressions for their prices and hedging strategies. In Section 5 we exploit our third representation result to calculate distributions of running maxima, maximum drawdowns and maximum relative drawdowns of processes Y which admit a continuous increasing function s such that $s(Y)$ is a local martingale. This includes local martingales and diffusion processes. Running maxima, drawdowns and maximum drawdowns have been studied by a number of authors and play an important role in various applications such as finance, flood control or change point detection. We extend a formula of Lehoczky [9] for the distribution of the running maximum of a process that is stopped when the drawdown hits a given level. Then we deduce the distributions of maximum drawdowns and maximum relative drawdowns of suitably stopped processes. In contrast to most of the existing literature, our methods do not need Markov assumptions.

2 Definitions and preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual assumptions. Equalities and inequalities between random variables are understood in the $\mathbb{P}$-almost sure sense. All stochastic processes will be indexed by $t \in \mathbb{R}_+$, and all semimartingales will be assumed to be càdlàg. In the whole paper X is a semimartingale with decomposition $X = N + A$, and L denotes the random time

$$L := \sup\{t \in \mathbb{R}_+ : X_t = 0\} \quad \text{with the convention } \sup\emptyset = 0.$$

$\overline{X}_t$ denotes the running supremum $\sup_{u\leq t} X_u$. We recall that X is said to be of class (D) if the family of random variables $\left\{|X_T| 1_{\{T<\infty\}} : T \text{ a stopping time}\right\}$ is uniformly integrable.

Definition 2.1 *By $\Sigma(X)$ we denote the set of all adapted continuous finite variation processes B starting at 0 and satisfying*

$$\int_0^t 1_{\{X_u \neq 0\}} dB_u = 0 \quad \textit{for all } t \in \mathbb{R}_+.$$

We say X is of class (Σ) if $A \in \Sigma(X)$. If X is of class (Σ) and (D), we say it is of class $(\Sigma\mathrm{D})$.

Note that $\Sigma(X)$ is a vector space and, in particular, contains 0. But if X is of class (Σ), it is a special semimartingale with canonical decomposition $X = N + A$, and it follows that the process $A \in \Sigma(X)$ making $X - A$ a local martingale is uniquely given by X. Every local martingale is of class (Σ), and all uniformly integrable martingales are of class (ΣD). Lemma 2.2 below shows that if a process X of class (Σ) is nonnegative, then A has to be increasing and X is a local submartingale. This case includes the absolute value $|M|$ of continuous local martingales M as well as drawdown processes $\overline{M} - M$ of local martingales whose running suprema $\overline{M}$ are continuous. It will also follow from Lemma 2.2 that for every constant $K \in \mathbb{R}$, the process $(K - M)^+$ is of class (Σ) if M is a local martingale with no positive jumps. Many other processes, such as suitably transformed diffusions or the Azéma submartingale in the filtration generated by the Brownian zeros, fall into the class (Σ). In Lemma 2.3 we prove that the product of processes of class (Σ) with vanishing quadratic covariation is again of class (Σ), and Lemma 2.4 shows that the class (Σ) is stable under transformations of the form $X \mapsto f(A)X$, where $f : \mathbb{R} \to \mathbb{R}$ is a locally bounded Borel function.

We start by studying positive and negative parts of processes of class (Σ), nonnegative processes of class (Σ) and the convergence of X_t for $t \to \infty$.

Lemma 2.2 *Assume X is of class (Σ). Then the following hold:*

(1) *X^+ and X^- are local submartingales.*

(2) *If X has no negative jumps, then X^+ is again of class (Σ). If X has no positive jumps, then X^- is of class (Σ).*

(3) *If X is nonnegative, then it is a local submartingale with $A_t = \sup_{u \le t}(-N_u) \vee 0$.*

(4) *If X is of class (ΣD), then N is a uniformly integrable martingale and A of integrable total variation; in particular, there exist integrable random variables $X_\infty, N_\infty, A_\infty$ such that $X_t \to X_\infty$, $N_t \to N_\infty$, $A_t \to A_\infty$ almost surely and in L^1.*

Proof. (1) Since A is continuous, one has $\int_0^t 1_{\{X_{u-} > 0\}} dA_u = \int_0^t 1_{\{X_u > 0\}} dA_u = 0$. So Tanaka's formula yields

$$X_t^+ = X_0^+ + \int_0^t 1_{\{X_{u-}>0\}} dX_u + V_t = X_0^+ + \int_0^t 1_{\{X_{u-}>0\}} dN_u + V_t \tag{2.1}$$

for the increasing finite variation process

$$V_t = \sum_{0<u\le t} 1_{\{X_{u-}\le 0\}} X_u^+ + \sum_{0<u\le t} 1_{\{X_{u-}>0\}} X_u^- + \frac{1}{2} l_t$$

and the local time l of X at 0 (see, for instance, Protter [24]). This shows that X^+ is a local submartingale. The same is true for X^- because $-X$ is also of class (Σ).

(2) If X has no negative jumps, (2.1) reduces to

$$X_t^+ = X_0^+ + \int_0^t 1_{\{X_{u-}>0\}} dN_u + \sum_{0<u\le t} 1_{\{X_{u-}\le 0\}} X_u^+ + \frac{1}{2} l_t. \tag{2.2}$$

$\int_0^t 1_{\{X_{u-}>0\}} dN_u$ is a local martingale, and the local time l is continuous and has the property $\int_0^t 1_{\{X_u \neq 0\}} dl_u = 0$, $t \in \mathbb{R}_+$. It remains to show that the process $Y_t = \sum_{0<u\le t} 1_{\{X_{u-}\le 0\}} X_u^+$ can

be decomposed into the sum of a local martingale and an adapted continuous increasing process C satisfying $C_0 = 0$ and

$$\int_0^t 1_{\{X_u^+ \neq 0\}} dC_u = 0 \quad \text{for all } t \geq 0. \tag{2.3}$$

Since N and $\int_0^t 1_{\{X_{u-} > 0\}} dN_u$ are local martingales and A is continuous, there exists a sequence of stopping times T_n, $n \in \mathbb{N}$, increasing to ∞ such that

$$\mathbb{E}\left[(X_{T_n})^+\right] = \mathbb{E}\left[(N_{T_n} + A_{T_n})^+\right] < \infty \quad \text{and} \quad \mathbb{E}\left[\int_0^{T_n} 1_{\{X_{u-} > 0\}} dN_u\right] = 0, \quad n \in \mathbb{N}.$$

So it follows from (2.2) that $\mathbb{E}[Y_{T_n}] \leq \mathbb{E}[(X_{T_n})^+] < \infty$ for all $n \in \mathbb{N}$. Hence, by Theorem VI.80 of Dellacherie and Meyer [5], there exists a right-continuous increasing predictable process C starting at 0 such that $Y - C$ is a local martingale. Since A is continuous, the jumps of X coincide with those of N. Due to the local martingale property of N and the fact that the jumps are positive, they have to occur at totally inaccessible stopping times. From Theorem VI.76 of Dellacherie and Meyer [5], one obtains $\mathbb{E}[\Delta C_T] = \mathbb{E}[\Delta Y_T] = 0$ for every predictable stopping time $T < \infty$, which shows that C must be continuous. Moreover, there exists a sequence of stopping times R_n, $n \in \mathbb{N}$, increasing to ∞ such that

$$\mathbb{E}\left[\int_0^{t \wedge R_n} 1_{\{X_{u-}^+ \neq 0\}} dC_u\right] = \mathbb{E}\left[\int_0^{t \wedge R_n} 1_{\{X_{u-}^+ \neq 0\}} dY_u\right] = \mathbb{E}\left[\sum_{0 < u \leq t \wedge R_n} 1_{\{X_{u-}^+ \neq 0\}} 1_{\{X_{u-} \leq 0\}} X_u^+\right] = 0$$

for all $n \in \mathbb{N}$. By monotone convergence one obtains

$$\mathbb{E}\left[\int_0^t 1_{\{X_{u-}^+ \neq 0\}} dC_u\right] = \mathbb{E}\left[\int_0^t 1_{\{X_{u-}^+ \neq 0\}} dY_u\right] = \mathbb{E}\left[\sum_{0 < u \leq t} 1_{\{X_{u-}^+ \neq 0\}} 1_{\{X_{u-} \leq 0\}} X_u^+\right] = 0.$$

This shows (2.3) and proves that X^+ is of class (Σ). That X^- is of class (Σ) if X has no positive jumps follows from the same arguments applied to $-X$.

(3) If X is nonnegative, it follows from (1) that it is a local submartingale. Hence, $A_t \geq A_u \geq -N_u \vee 0$ for all $t \geq u$, and therefore, $A_t \geq \sup_{u \leq t}(-N_u) \vee 0$. Now assume

$$\mathbb{P}\left[A_t > \sup_{u \leq t}(-N_u) \vee 0\right] > 0 \tag{2.4}$$

and introduce the random time $T = \sup\left\{s \leq t : A_s = \sup_{u \leq s}(-N_u) \vee 0\right\}$. Since A is continuous and $\sup_{u \leq s}(-N_u) \vee 0$ increasing, one has $A_T = \sup_{u \leq T}(-N_u) \vee 0$. Moreover, since $X_u > 0$ on the stochastic interval $\{(u, \omega) : T(\omega) < u \leq t\}$, it follows from $\int_0^t 1_{\{X_u \neq 0\}} dA_u = 0$ that $A_t = A_T$, a contradiction to (2.4). Hence, $A_t = \sup_{u \leq t}(-N_u) \vee 0$.

(4) If X is of class (D), then X^+ and X^- are submartingales of class (D). Therefore, both have a Doob–Meyer decomposition into the sum of a uniformly integrable martingale and a predictable increasing process of integrable total variation:

$$X_t^+ = N_t^1 + V_t^1, \quad X_t^- = N_t^2 + V_t^2.$$

Since the predictable finite variation part of a special semimartingale is unique, one must have $N_t = N_t^1 - N_t^2$ and $A_t = V_t^1 - V_t^2$. So N is a uniformly integrable martingale and A of integrable total variation. It follows that there exist integrable random variables $X_\infty, N_\infty, A_\infty$ such that $X_t \to X_\infty$, $N_t \to N_\infty$, $A_t \to A_\infty$ almost surely and in L^1. □

The next lemma shows that the product of processes of class (Σ) with vanishing quadratic covariations is again of class (Σ).

Lemma 2.3 *Let $X^1, \ldots, X^n$ be processes of class* (Σ) *such that $\left[X^i, X^j\right] \equiv 0$ for $i \neq j$. Then $\prod_{i=1}^n X^i$ is again of class* (Σ).

Proof. Since $[X^1, X^2]_t = 0$, integration by parts yields

$$X_t^1 X_t^2 = X_0^1 X_0^2 + \int_0^t X_{u-}^1 dN_u^2 + \int_0^t X_{u-}^2 dN_u^1 + \int_0^t X_u^1 dA_u^2 + \int_0^t X_u^2 dA_u^1.$$

$\int_0^t X_{u-}^1 dN_u^2 + \int_0^t X_{u-}^2 dN_u^1$ is a local martingale and $\int_0^t X_u^1 dA_u^2 + \int_0^t X_u^2 dA_u^1$ a continuous finite variation process starting at 0 which moves only when $X_t^1 = 0$ or $X_t^2 = 0$. Hence, $X^1 X^2$ is of class (Σ). If $n \geq 3$, then $\left[X^1 X^2, X^3\right]_t = 0$, and the lemma follows by induction. □

In the following lemma and the subsequent corollary we extend results of Nikeghbali [15] to our framework that will be needed later in the paper.

Lemma 2.4 *Let $B \in \Sigma(X)$ and let $f : \mathbb{R} \to \mathbb{R}$ be a locally bounded Borel function. Denote $F(x) = \int_0^x f(y)dy$. Then*

$$f(B_t)X_t = f(0)X_0 + \int_0^t f(B_u)d(X_u - B_u) + F(B_t) \quad \text{and} \quad F(B) \in \Sigma(f(B)X). \tag{2.5}$$

In particular, if X is of class (Σ), *$f(A)X$ is again of class* (Σ) *with decomposition*

$$f(A_t)X_t = f(0)X_0 + \int_0^t f(A_u)dN_u + F(A_t), \tag{2.6}$$

and if X is of class (Σ) *such that $f(A)X$ is of class* (D), *then $M_t = f(A_t)X_t - F(A_t)$ is a uniformly integrable martingale and therefore,*

$$f(A_T)X_T - F(A_T) = \mathbb{E}\left[M_\infty \mid \mathcal{F}_T\right] \quad \text{for every stopping time } T. \tag{2.7}$$

Proof. It can easily be checked that for $B \in \Sigma(X)$, $f(B_t)X_t$ is càdlàg. To show (2.5), we first assume that f is C^1. Then

$$f(B_t)X_t = f(0)X_0 + \int_0^t f(B_u)dX_u + \int_0^t X_u f'(B_u)dB_u.$$

But since $B \in \Sigma(X)$, the last integral vanishes. So

$$f(B_t)X_t = f(0)X_0 + \int_0^t f(B_u)d(X_u - B_u) + F(B_t). \tag{2.8}$$

It follows from the functional monotone class theorem that this equation extends to all bounded Borel functions f since the set of bounded C^1 functions is stable under multiplication and $\int_0^t f_n(B_u)d(X_u - B_u)$ converges to $\int_0^t f(B_u)d(X_u - B_u)$ in probability if f_n is a sequence of bounded Borel functions increasing pointwise to a bounded Borel function f. From there, one obtains (2.8) for locally bounded Borel functions f by localization with a sequence of stopping times. Clearly, one also has

$$\int_0^t 1_{\{f(B_u)X_u \neq 0\}} dF(B_u) = \int_0^t 1_{\{f(B_u)X_u \neq 0\}} f(B_u) dB_u = 0, \quad t \in \mathbb{R}_+,$$

for all locally bounded Borel functions f. So we have proved (2.5).

If X is of class (Σ), (2.5) becomes

$$f(A_t)X_t = f(0)X_0 + \int_0^t f(A_u)dN_u + F(A_t) \quad \text{and} \quad A \in \Sigma(f(A)X).$$

So it follows that $f(A)X$ is again of class (Σ).

Finally, if X is of class (Σ) and $f(A)X$ of class (D), one obtains from Lemma 2.2 that $M_t = f(A_t)X_t - F(A_t)$ is a uniformly integrable martingale. Formula (2.7) is then a consequence of Doob's optional stopping theorem. □

Remark 2.5 If M is a local martingale starting at $m \in \mathbb{R}$ with continuous running supremum $\overline{M}$, then $X = \overline{M} - M$ is of class (Σ) with decomposition $X_t = (m - M_t) + (\overline{M}_t - m)$. So one obtains from Lemma 2.4 that for every locally bounded Borel function $f : \mathbb{R} \to \mathbb{R}$ and $F(x) = \int_0^x f(y)dy$, the process

$$F(A_t) - f(A_t)X_t = F(\overline{M}_t - m) - f(\overline{M}_t - m)(\overline{M}_t - M_t) \tag{2.9}$$

is again a local martingale. This transformation was used by Azéma and Yor [2] in their solution of the Skorokhod embedding problem. Obloj [18] showed that if M is a continuous local martingale, then every local martingale starting at 0 which can be written as $H(M_t, \overline{M}_t)$ for a deterministic function H is of the form (2.9).

One can use Lemma 2.4 to calculate the probability that processes of the form $f(A_t)X_t$ stay below a given constant, which, without loss of generality, can be taken to be 1. This will prove useful in the study of drawdowns and relative drawdowns in Section 5.

Corollary 2.6 *Let X be a nonnegative process of class (Σ) with no positive jumps such that $A_\infty = \infty$, $f : \mathbb{R}_+ \to \mathbb{R}_+$ a Borel function and $T < \infty$ a stopping time. Then*

$$\begin{aligned} &\mathbb{P}[f(A_t)X_t < 1 \text{ for all } t \geq T \mid \mathcal{F}_T] = \mathbb{P}[f(A_t)X_t \leq 1 \text{ for all } t \geq T \mid \mathcal{F}_T] \\ &= (1 - f(A_T)X_T)^+ \exp\left(-\int_{A_T}^\infty f(x)dx\right). \end{aligned} \tag{2.10}$$

Moreover, in both cases

(1) *K is an $\mathcal{F}_T$-measurable random variable such that $K > A_T$ and $T_K = \inf\{t \in \mathbb{R}_+ : A_t \geq K\}$*

(2) *K is an $\mathcal{F}_T$-measurable random variable such that $K \geq A_T$ and $T_K = \inf\{t \in \mathbb{R}_+ : A_t > K\}$,*

one has

$$\begin{aligned} &\mathbb{P}[f(A_t)X_t < 1 \text{ for all } t \in [T, T_K] \mid \mathcal{F}_T] = \mathbb{P}[f(A_t)X_t \leq 1 \text{ for all } t \in [T, T_K] \mid \mathcal{F}_T] \\ &= (1 - f(A_T)X_T)^+ \exp\left(-\int_{A_T}^K f(x)dx\right). \end{aligned} \tag{2.11}$$

Proof. Let us first assume that f is bounded and $F(\infty) = \int_0^\infty f(y)dy < \infty$. Then one obtains from Lemma 2.4 that $Y_t = f(A_t)X_t$ is a nonnegative process of class (Σ) with no positive jumps. For a given stopping time $T < \infty$, denote $R = \inf\{t \geq T : Y_t \geq 1\}$. By (2.6), Y_t decomposes as

$$Y_t = f(0)X_0 + \int_0^t f(A_u)dN_u + F(A_t),$$

and

$$e^{F(A_t)}Y_t = f(0)X_0 + \int_0^t e^{F(A_u)}f(A_u)dN_u + e^{F(A_t)}.$$

is again of class (Σ). In particular, $e^{F(A_t)}(1 - Y_t)$ is a local martingale and

$$M_t = 1_{\{t \geq T\}}\left(e^{F(A_{R\wedge t})}(1 - Y_{t\wedge R}) - e^{F(A_T)}(1 - Y_T)\right)$$

a bounded martingale such that $M_0 = 0$ and $M_t \to M_\infty$ almost surely and in L^1 for an integrable random variable M_∞. Note that $M_\infty = 0$ on $\{T = R\}$ and $M_\infty = -e^{F(A_T)}(1-Y_T)$ on $\{T < R < \infty\}$. Moreover, since $A_\infty 1_{\{L<\infty\}} = A_L 1_{\{L<\infty\}}$ is real-valued, it follows from $A_\infty = \infty$ that $L = \infty$. Hence, there exists a sequence T_n, $n \in \mathbb{N}$, of stopping times increasing to ∞ almost surely such that $Y_{T_n} = 0$ for all $n \in \mathbb{N}$, and one obtains

$$M_\infty = \lim_{n\to\infty} e^{F(A_{T_n})}(1 - Y_{T_n}) - e^{F(A_T)}(1 - Y_T) = e^{F(\infty)} - e^{F(A_T)}(1 - Y_T)$$

almost everywhere on $\{R = \infty\}$. So $\mathbb{E}[M_\infty \mid \mathcal{F}_T] = 0$ yields

$$e^{F(A_T)}(1 - Y_T)^+ = \mathbb{P}[R = \infty \mid \mathcal{F}_T]e^{F(\infty)},$$

which is equivalent to

$$\mathbb{P}[f(A_t)X_t < 1 \text{ for all } t \geq T \mid \mathcal{F}_T] = (1 - f(A_T)X_T)^+ \exp\left(-\int_{A_T}^\infty f(x)dx\right). \tag{2.12}$$

The equality

$$\mathbb{P}[f(A_t)X_t \leq 1 \text{ for all } t \geq T \mid \mathcal{F}_T] = (1 - f(A_T)X_T)^+ \exp\left(-\int_{A_T}^\infty f(x)dx\right) \tag{2.13}$$

follows from the same argument applied to the stopping time $\tilde{R} = \inf\{t \geq T : Y_t > 1\}$. That (2.12) and (2.13) still hold for general Borel functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ can be seen by approximating f with $f^n = f \wedge n1_{[0,n]}$, $n \in \mathbb{N}$. Note that the functions f^n increase to f, and for every $x \geq 0$ there exists an $n_0 \in \mathbb{N}$ such that $f^n(x) = f(x)$ for all $n \geq n_0$. Therefore, one has

$$\bigcap_{n\in\mathbb{N}} \{f^n(A_t)X_t < 1 \text{ for all } t \geq T\} = \{f(A_t)X_t < 1 \text{ for all } t \geq T\}$$

as well as

$$\bigcap_{n\in\mathbb{N}} \{f^n(A_t)X_t \leq 1 \text{ for all } t \geq T\} = \{f(A_t)X_t \leq 1 \text{ for all } t \geq T\}.$$

In case (2) one obtains (2.11) from (2.10) simply by setting f equal to 0 on (K, ∞). In case (1), setting f equal to 0 on $[K, \infty)$ gives

$$\begin{aligned}&\mathbb{P}[f(A_t)X_t < 1 \text{ for all } t \in [T, T_K) \mid \mathcal{F}_T] = \mathbb{P}[f(A_t)X_t \leq 1 \text{ for all } t \in [T, T_K) \mid \mathcal{F}_T]\\ &= (1 - f(A_T)X_T)^+ \exp\left(-\int_{A_T}^K f(x)dx\right).\end{aligned}$$

But this is equivalent to (2.11) since $X_{T_K} = 0$. □

3 Representation results

3.1 Representations in terms of last passage times

The results in this subsection are inspired by a representation formula for relative martingales by Azéma and Yor [3] and recent formulas by Madan et al. [10] and Profeta et al. [23] relating prices of put options to last passage times. Some of our formulas involve conditional expectations of random variables which are conditionally integrable but not necessarily integrable. To cover this case, we define the conditional expectation of any random variable Y with respect to a sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$ as

$$\mathbb{E}\left[Y \mid \mathcal{G}\right] = \sup_{m \in \mathbb{Z}} \inf_{n \in \mathbb{Z}} \mathbb{E}\left[m \wedge (n \vee Y) \mid \mathcal{G}\right]. \tag{3.1}$$

Then

$$\mathbb{E}\left[YZ \mid \mathcal{G}\right] = Y\mathbb{E}\left[Z \mid \mathcal{G}\right]$$

for all $\mathcal{G}$-measurable random variables Y and integrable random variables Z.

Theorem 3.1 *Let X be a process of class (Σ) and $f : \mathbb{R} \to \mathbb{R}$ a Borel function. Then the following hold:*

(1) *If X is of class* (D), *then there exist integrable random variables X_∞, N_∞, A_∞ such that $X_t \to X_\infty$, $N_t \to N_\infty$, $A_t \to A_\infty$ almost surely as well as in L^1 and*

$$f(A_T)X_T = \mathbb{E}\left[f(A_\infty)X_\infty 1_{\{L \le T\}} \mid \mathcal{F}_T\right] \quad \text{for every stopping time } T. \tag{3.2}$$

(2) *If $q : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is a Borel function such that $q(A)X$ is of class* (D), *then there exist random variables X_∞, N_∞, A_∞ such that $X_t \to X_\infty$, $N_t \to N_\infty$, $A_t \to A_\infty$ almost everywhere on $\{L < \infty\}$, and*

$$f(A_T)X_T = \mathbb{E}\left[f(A_\infty)X_\infty 1_{\{L \le T\}} \mid \mathcal{F}_T\right] \quad \text{for all stopping times } T < \infty. \tag{3.3}$$

In particular, in both cases one has

$$X_T = \mathbb{E}\left[X_\infty 1_{\{L \le T\}} \mid \mathcal{F}_T\right] \quad \text{for all stopping times } T < \infty. \tag{3.4}$$

Proof. (1) If X is of class (ΣD), it follows from Lemma 2.2 that N is a uniformly integrable martingale and A of integrable total variation. So there exist integrable random variables X_∞, N_∞, A_∞ such that $X_t \to X_\infty$, $N_t \to N_\infty$, $A_t \to A_\infty$ almost surely as well as in L^1. For a given stopping time T, denote

$$d_T = \inf\{t > T : X_t = 0\} \quad \text{with the convention } \inf \emptyset = \infty.$$

Since $X_\infty 1_{\{L \le T\}} = X_{d_T}$ and $A_T = A_{d_T}$, it follows from Doob's optional stopping theorem that

$$\mathbb{E}\left[X_\infty 1_{\{L \le T\}} \mid \mathcal{F}_T\right] = \mathbb{E}\left[N_{d_T} + A_{d_T} \mid \mathcal{F}_T\right] = N_T + A_T = X_T.$$

Moreover, one has $A_\infty = A_T$ almost everywhere on $\{L \le T\}$, and therefore,

$$\mathbb{E}\left[f(A_\infty)X_\infty 1_{\{L \le T\}} \mid \mathcal{F}_T\right] = \mathbb{E}\left[f(A_T)X_\infty 1_{\{L \le T\}} \mid \mathcal{F}_T\right] = f(A_T)X_T.$$

(2) If there exists a Borel function $q : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ such that $q(A)X$ is of class (D), then $h(x) = |q(x)| \wedge 1$ is a bounded Borel function and $Y = h(A)X$ is still of class (D). By Lemma 2.4, it is also of class (Σ). So one obtains from (1) that $Y_t \to Y_\infty$ almost surely as well as in L^1 and

$$Y_T = \mathbb{E}\left[Y_\infty 1_{\{L \le T\}} \mid \mathcal{F}_T\right] \quad \text{for every stopping time } T.$$

Since $\int_0^t 1_{\{X_u \neq 0\}} dA_u = 0$ for all $t \in \mathbb{R}_+$, A_t converges to A_L almost everywhere on $\{L < \infty\}$. Hence, it follows from $h \neq 0$ that $X_t \to X_\infty = Y_\infty / h(A_L)$ and $N_t \to N_\infty = X_\infty - A_L$ almost everywhere on $\{L < \infty\}$. On $\{L = \infty\}$, set $X_\infty = N_\infty = A_\infty = 0$. If T is a stopping time satisfying $T < \infty$, then

$$f(A_T)X_T = \frac{f(A_T)}{h(A_T)} Y_T = \mathbb{E}\left[\frac{f(A_T)}{h(A_T)} Y_\infty 1_{\{L \le T\}} \mid \mathcal{F}_T\right] = \mathbb{E}\left[f(A_\infty) X_\infty 1_{\{L \le T\}} \mid \mathcal{F}_T\right].$$

□

Corollary 3.2 *Let X be a process of class (Σ) and $f : \mathbb{R} \to \mathbb{R}$ a Borel function. Assume that at least one of the following two conditions holds:*

(1) *N is a uniformly integrable martingale.*

(2) *X^- and N^+ are of class* (D).

Then there exist random variables X_∞, N_∞, A_∞ such that $X_t \to X_\infty$, $N_t \to N_\infty$, $A_t \to A_\infty$ almost everywhere on $\{L < \infty\}$ and

$$f(A_T)X_T = \mathbb{E}\left[f(A_\infty) X_\infty 1_{\{L \le T\}} \mid \mathcal{F}_T\right] \quad \textit{for all stopping times } T < \infty.$$

In particular,

$$X_T = \mathbb{E}\left[X_\infty 1_{\{L \le T\}} \mid \mathcal{F}_T\right] \quad \textit{for all stopping times } T < \infty.$$

Proof. In both cases $e^{-|A_t|} X_t$ is of class (D). So the corollary follows from part (2) of Theorem 3.1.□

Remark 3.3 If X satisfies the assumptions of part (2) of Theorem 3.1 or Corollary 3.2, then there exists a random variable X_∞ such that $X_t \to X_\infty$ almost everywhere on the set $\{L < \infty\}$, and one has

$$X_t = \mathbb{E}\left[X_\infty 1_{\{L \le t\}} \mid \mathcal{F}_t\right], \quad t \in \mathbb{R}_+. \tag{3.5}$$

In particular, the whole process X_t, $t \in \mathbb{R}_+$, can be recovered from X_∞ and L. This extends the representation of a uniformly integrable martingale as

$$M_t = \mathbb{E}\left[M_\infty \mid \mathcal{F}_t\right], \quad t \in \mathbb{R}_+;$$

see Najnudel and Nikeghbali [13] for related results. In the special case where $X_\infty = 1$ and the expectations $\mathbb{E}[X_t]$, $t \in \mathbb{R}_+$, can be calculated, one obtains from (3.5) all the probabilities $\mathbb{P}[L \le t]$, $t \in \mathbb{R}_+$, and hence, the distribution of L. We refer to Nikeghbali and Platen [16] for examples.

Remark 3.4 For representations of the form (3.2), (3.3) or (3.4) to hold, it is not sufficient that a process X of class (Σ) has an almost sure finite limit $\lim_{t \to \infty} X_t$. For example, $X_t = 1 - \exp(B_t - t/2)$ is of class (Σ) with $X_0 = 0$ and $\lim_{t \to \infty} X_t = 1$ almost surely. But $X_t = \mathbb{P}[L \le t \mid \mathcal{F}_t]$ cannot hold since there is a positive probability that X_t is negative and $\mathbb{P}[L \le t \mid \mathcal{F}_t]$ is always between 0 and 1.

Processes of class (Σ) that are not of class (D) but satisfy (3.3) and (3.4) can be constructed from strict local martingales as follows: Take a nonnegative continuous strict local martingale M starting at $m \in \mathbb{R}_+ \setminus \{0\}$ such that $\lim_{t\to\infty} M_t = 0$ almost surely (for instance, $M_t = \|W_t\|_2^{-1}$ for a three-dimensional Brownian motion W starting from a point $x \in \mathbb{R}^3 \setminus \{0\}$ and $\|.\|_2$ the Euclidean norm on $\mathbb{R}^3$). M is a supermartingale but not a martingale. So there exists $u \in \mathbb{R}_+$ such that $\mathbb{E}[M_u] < m$, and it follows from Lemma 2.1 and Proposition 2.3 of Elworthy et al. [7] that $\mathbb{E}\left[\overline{M}_t\right] = \infty$ for all $t \geq u$. Hence, $X = \overline{M} - M$ is a nonnegative process of class (Σ) with $\lim_{t\to\infty} X_t = \overline{M}_\infty$ almost surely and $\mathbb{E}[X_t] = \infty$ for all $t \geq u$. Clearly, X satisfies condition (2) of Corollary 3.2. So

$$f(A_T)X_T = \mathbb{E}\left[f(A_\infty)X_\infty 1_{\{L\leq T\}} \mid \mathcal{F}_T\right]$$

for every Borel function $f : \mathbb{R} \to \mathbb{R}$ and stopping time $T < \infty$, even though X_∞ is not integrable and the conditional expectation has to be understood in the sense of (3.1).

As a consequence of Lemma 2.2 and Theorem 3.1 one obtains the following formulas relating put option prices to last passage times:

Corollary 3.5 (Madan–Roynette–Yor [10])
Let K be a constant and M a local martingale with no positive jumps such that M^- is of class (D)*. Denote $g_K = \sup\{t \in \mathbb{R}_+ : M_t \geq K\}$. Then*

$$(K - M_T)^+ = \mathbb{E}\left[(K - M_\infty)^+ 1_{\{g_K\leq T\}} \mid \mathcal{F}_T\right] \tag{3.6}$$

for every stopping time T. In particular, if $M_\infty = m \in \mathbb{R}$, then

$$\mathbb{E}\left[(K - M_T)^+ \mid \mathcal{F}_t\right] = (K - m)^+ \mathbb{P}[g_K \leq T \mid \mathcal{F}_{T\wedge t}], \quad t \in \mathbb{R}_+. \tag{3.7}$$

Proof. $K - M$ is a local martingale with no negative jumps. So it follows from Lemma 2.2 that $(K - M)^+$ is a local submartingale of class (Σ). Since M^- is of class (D), $(K - M)^+$ is of class (D) too, and (3.6) follows from Theorem 3.1 by noting that $g_K = \sup\{t \in \mathbb{R}_+ : (K - M_t)^+ = 0\}$. Formula (3.7) follows from (3.6) by taking the conditional expectation with respect to $\mathcal{F}_t$. □

Remark 3.6 If K is a constant and M a local martingale with no negative jumps such that M^+ is of class (D), one can apply Corollary 3.5 to $-K$, $-M$ and $g_K = \sup\{t \in \mathbb{R}_+ : M_t \leq K\}$. This gives

$$(M_T - K)^+ = \mathbb{E}\left[(M_\infty - K)^+ 1_{\{g_K\leq T\}} \mid \mathcal{F}_T\right] \tag{3.8}$$

for all stopping times T. In particular, if $M_\infty = m \in \mathbb{R}$, one obtains

$$\mathbb{E}\left[(M_T - K)^+ \mid \mathcal{F}_t\right] = (m - K)^+ \mathbb{P}[g_K \leq T \mid \mathcal{F}_{T\wedge t}], \quad t \in \mathbb{R}_+. \tag{3.9}$$

However, if, for instance, $M_t = \exp(W_t - t/2)$ for a Brownian motion W, the assumptions of Corollary 3.5 are satisfied, but M^+ is not of class (D). So even though $M_\infty = 0$, formula (3.9) does not hold. Indeed, for $K \geq 0$ and $0 = t < T \in \mathbb{R}_+$, the right-hand side is zero, but $\mathbb{E}[(M_t - K)^+] > 0$. For a more detailed discussion of this case, we refer to Section 6 in Madan et al. [10].

The following extension of Corollary 3.5 has been proved by Profeta et al. [23] with methods from the theory of enlargement of filtrations. We can deduce it under slightly weaker assumptions from Lemma 2.3 and Theorem 3.1.

Corollary 3.7 (Profeta–Roynette–Yor [23])
Let $K^1, \dots, K^n$ be constants and $M^1, \dots, M^n$ local martingales that are bounded from below and have no positive jumps. Assume $[M^i, M^j] \equiv 0$ for $i \neq j$ and denote $g^i = \sup\left\{t \in \mathbb{R}_+ : M^i_t \geq K^i\right\}$. Then

$$\prod_{i=1}^n (K^i - M^i_T)^+ = \mathbb{E}\left[\prod_{i=1}^n (K^i - M^i_\infty)^+ 1_{\{g^i \leq T\}} \mid \mathcal{F}_T\right] \tag{3.10}$$

for every stopping time T. In particular, if $M^i_\infty = m^i \in \mathbb{R}$ for all $i = 1, \dots, n$, then

$$\mathbb{E}\left[\prod_{i=1}^n (K^i - M^i_T)^+ \mid \mathcal{F}_t\right] = \prod_{i=1}^n (K^i - m^i)^+ \mathbb{P}\left[\bigvee_{i=1}^n g^i \leq T \mid \mathcal{F}_{T \wedge t}\right], \quad t \in \mathbb{R}_+. \tag{3.11}$$

Proof. By Lemma 2.2, $X^i = (K^i - M^i)^+$ are local submartingales of class (Σ) such that $\left[X^i, X^j\right] \equiv 0$ for $i \neq j$. So we obtain from Lemma 2.3 that $\prod_{i=1}^n X^i$ is again of class (Σ), which, since all M^i are bounded from below, is bounded. Now (3.10) follows from Theorem 3.1, and (3.11) is a direct consequence of (3.10). □

3.2 Stochastic integral representations and conditional distributions

We now use Lemma 2.4 to derive stochastic integral representations for functionals of processes B in $\Sigma(X)$. In Section 4 they will be applied in situations where $f(B_t)X_t$ can be stopped with a stopping time R such that $f(B_R)X_R = 1$.

Proposition 3.8 *Assume X is a nonnegative semimartingale and B an increasing process in $\Sigma(X)$. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a Borel function for which there exists a sequence of numbers $a_n \in \mathbb{R}_+$, $n \in \mathbb{N}$, increasing to $a \in (0, \infty]$ such that $f 1_{[0,a_n]}$ is bounded for all n, and $f(x) = 0$ for $x \geq a$. Denote $F(x) = \int_0^x f(y)dy$ and suppose that $B_t < a$ for all $t \in \mathbb{R}_+$. Then for every Borel function $h : [0, a) \to \mathbb{R}$ satisfying*

$$\int_0^a |h(y)| e^{-F(y)} dF(y) < \infty, \tag{3.12}$$

one has

$$\begin{aligned} & h(B_t)f(B_t)X_t + h^F(B_t)(1 - f(B_t)X_t) \\ = \; & h(0)f(0)X_0 + h^F(0)(1 - f(0)X_0) + \int_0^t (h - h^F)(B_u) f(B_u) d(X_u - B_u) \end{aligned} \tag{3.13}$$

for all $t \in \mathbb{R}_+$, where

$$h^F(x) := e^{F(x)} \int_x^a h(y) e^{-F(y)} dF(y), \quad 0 \leq x < a.$$

Proof. Set $f^n := f 1_{[0,a_n]}$ and $F^n(x) := \int_0^x f^n(y)dy$, $n \in \mathbb{N}$. If h is bounded, the functions

$$h^n(x) := e^{F^n(x)} \int_x^a h(y) e^{-F^n(y)} dF^n(y), \quad 0 \leq x < a,$$

are bounded too, and $\Phi^n(x) := h^n(0) - h^n(x)$ can be written as $\Phi^n(x) = \int_0^t \varphi^n(y)dy$ for $\varphi^n := (h - h^n)f^n$. It follows from Lemma 2.4 that

$$\varphi^n(B_t)X_t - \Phi^n(B_t) = \varphi^n(0)X_0 + \int_0^t \varphi^n(B_u) d(X_u - B_u), \quad t \in \mathbb{R}_+,$$

which for $n \to \infty$ becomes

$$\varphi(B_t)X_t - \Phi(B_t) = \varphi(0)X_0 + \int_0^t \varphi(B_u)d(X_u - B_u), \quad t \in \mathbb{R}_+,$$

for

$$\Phi(x) := h^F(0) - h^F(x) \quad \text{and} \quad \varphi(x) := (h - h^F)f.$$

This shows (3.13) in the case where h is bounded. For Borel functions h satisfying the integrability condition (3.14) the formula follows by approximation. □

Corollary 3.9 *Assume X is a nonnegative semimartingale and B an increasing process in $\Sigma(X)$. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a Borel function for which there exists a sequence $a_n \in \mathbb{R}_+$, $n \in \mathbb{N}$, increasing to $a \in (0, \infty]$ such that $f 1_{[0,a_n]}$ is bounded for all $n \in \mathbb{N}$, and $f(x) = 0$ for $x \geq a$. Denote $F(x) = \int_0^x f(y)dy$ and suppose that $f(B_t)X_t \to 1$ almost surely. Then $L < \infty$, $B_L = B_\infty < a$ and $X_t \to 1/f(B_\infty) > 0$ almost surely. Moreover, for every Borel function $h : [0, a) \to \mathbb{R}$ satisfying*

$$\int_0^a |h(y)| e^{-F(y)} dF(y) < \infty, \tag{3.14}$$

one has

$$h(B_\infty) = h(B_t)f(B_t)X_t + h^F(B_t)(1 - f(B_t)X_t) + \int_t^\infty (h - h^F)(B_u)f(B_u)d(X_u - B_u), \quad t \in \mathbb{R}_+, \tag{3.15}$$

where

$$h^F(x) := e^{F(x)} \int_x^a h(y)e^{-F(y)} dF(y), \quad 0 \leq x < a.$$

Proof. Since $f(B_t)X_t \to 1$ almost surely, one has $L < \infty$, $B_L = B_\infty < a$ and $X_t \to 1/f(B_\infty)$ almost surely. Proposition 3.8 gives

$$\begin{aligned} &h(B_t)f(B_t)X_t + h^F(B_t)(1 - f(B_t)X_t) \\ &= h(0)f(0)X_0 + h^F(0)(1 - f(0)X_0) + \int_0^t (h - h^F)(B_u)f(B_u)d(X_u - B_u), \quad t \in \mathbb{R}_+, \end{aligned}$$

which, together with

$$h(B_t)f(B_t)X_t + h^F(B_t)(1 - f(B_t)X_t) \to h(B_\infty) \quad \text{almost surely},$$

implies (3.15). □

Theorem 3.10 *Let X be a nonnegative process of class* (Σ) *and $f : \mathbb{R}_+ \to \mathbb{R}_+$ a Borel function for which there exists a sequence a_n, $n \in \mathbb{N}$, increasing to $a \in (0, \infty]$ such that $f 1_{[0,a_n]}$ is bounded for all $n \in \mathbb{N}$, and $f(x) = 0$ for $x \geq a$. Denote $F(x) = \int_0^x f(y)dy$ and assume that the process $f(A)X$ is of class* (D) *and $f(A_t)X_t \to 1$ almost surely.*

(1) *If $F(a) < \infty$, then $A \equiv 0$ and $X \equiv 1/f(0) > 0$.*

(2) *If $F(a)=\infty$, then $L<\infty$, $A_L = A_\infty < a$ and $X_t \to 1/f(A_\infty) > 0$ almost surely. Moreover, for every stopping time T one has*

$$f(A_T)X_T = \mathbb{P}[L \le T \mid \mathcal{F}_T], \tag{3.16}$$

and for all Borel functions $h : [0,a) \to \mathbb{R}$ satisfying

$$\int_0^a |h(y)| e^{-F(y)} dF(y) < \infty, \tag{3.17}$$

$$\mathbb{E}\left[h(A_\infty) \mid \mathcal{F}_T\right] = h(A_T) f(A_T) X_T + h^F(A_T)(1 - f(A_T)X_T), \tag{3.18}$$

where

$$h^F(x) := e^{F(x)} \int_x^a h(y) e^{-F(y)} dF(y), \quad 0 \le x < a.$$

In particular, conditioned on $\mathcal{F}_T$, the distribution of A_∞ is given by

$$\mathbb{P}[A_\infty > x \mid \mathcal{F}_T] = 1_{\{A_T > x\}} + 1_{\{A_T \le x\}}(1 - f(A_T)X_T) e^{F(A_T) - F(x)}, \quad x \ge 0. \tag{3.19}$$

Proof. It follows from Corollary 3.9 that $L < \infty$, $A_L = A_\infty < a$ and $X_t \to 1/f(A_\infty) > 0$ almost surely. Now set $f^n := f \wedge n$ and $F^n(x) := \int_0^x f^n(y) dy$. By Lemma 2.4, the processes $f^n(A)X$ are of class (ΣD). So one obtains from Theorem 3.1 that

$$f^n(A_T)X_T = \mathbb{E}\left[f^n(A_\infty) X_\infty 1_{\{L^n \le T\}} \mid \mathcal{F}_T\right] \quad \text{for every stopping time } T, \tag{3.20}$$

where $L^n := \sup\{t \in \mathbb{R}_+ : f^n(A_t)X_t = 0\}$. But since $f^n(A_t)X_t \to f^n(A_L)/f(A_L) > 0$ almost surely, one has $L^n = L$, and (3.16) follows from (3.20) by letting n tend to ∞. It follows from Proposition 3.8 that

$$h(A_t)f(A_t)X_t + h^F(A_t)(1 - f(A_t)X_t) = h(0)f(0)X_0 + h^F(0)(1 - f(0)X_0) + \int_0^t (h - h^F)(A_u) f(A_u) dN_u.$$

If h is bounded, h^F is bounded too, and it follows from (3.16) that

$$h(A_t)f(A_t)X_t + h^F(A_t)(1 - f(A_t)X_t)$$

is a bounded martingale with limit $h(A_\infty)$. So (3.18) follows by optional stopping. If h is unbounded but satisfies the integrability condition (3.17), equation (3.18) follows by decomposing h into $h = h^+ - h^-$ and monotone approximation.

Formula (3.18) applied to $h \equiv 1$ gives $e^{F(A_t) - F(a)}(1 - f(A_t)X_t) = 0$ for all $t \in \mathbb{R}_+$. Hence, for $F(a) < \infty$, one must have $f(A_t)X_t = 1$ for all t, which shows (1). If $F(a) = \infty$, (3.18) is equivalent to (3.19). This completes the proof of (2). □

4 Options depending on running maxima, drawdowns and relative drawdowns

Let S and S^0 be semimartingales. S models the value of a financial asset such as a stock, stock index, interest rate contract or foreign currency cash investment. S^0 describes a second asset that we assume to be strictly positive and use as a reference unit. Often S^0 is chosen to be a money market account evolving like $\exp(\int_0^t r_u du)$, where r_t is the instantaneous risk-free interest rate. But it can be any self-financing portfolio of liquid assets as long as its value is strictly positive. For instance, in the benchmark approach of Platen [19] or Platen and Heath [20], S^0 is the growth optimal portfolio. The price of S expressed in reference units is $Y_t = S_t/S^0_t$. In the whole section we assume Y to be continuous and to start from a constant $y \in \mathbb{R}$.

4.1 Drawdown and relative drawdown

The drawdown process $DD_t := \overline{Y}_t - Y_t$ is a nonnegative semimartingale and $\overline{Y} - y$ an increasing process in $\Sigma(DD)$. If $y > 0$, the relative drawdown process $rDD_t := DD_t/\overline{Y}_t = 1 - Y_t/\overline{Y}_t$ is well-defined and again a nonnegative semimartingale. By Lemma 2.4, it can be written as

$$rDD_t = -\int_0^T \frac{dY_u}{\overline{Y}_u} + \log(\overline{Y}_t) - \log(y). \tag{4.1}$$

In the special case where Y is a local martingale, DD and rDD are both nonnegative local submartingales of class (Σ) with semimartingale decompositions $DD_t = (y - Y_t) + (\overline{Y}_t - y)$ and (4.1), respectively. The following result is a consequence of Proposition 3.8 and will be crucial for the derivation of prices and hedging strategies for the options discussed in the next two subsections.

Proposition 4.1 *Let $\lambda : [y, \infty) \to \mathbb{R}$ be a Borel function such that $1/(x - \lambda(x))$ is positive and locally bounded on $[y, \infty)$. Denote*

$$T_\lambda := \inf\left\{t \in \mathbb{R}_+ : Y_t = \lambda(\overline{Y}_t)\right\} \quad \text{and} \quad \Lambda(x) := \int_y^x \frac{dz}{z - \lambda(z)}, \quad x \geq y. \tag{4.2}$$

Then for all Borel functions $h : [y, \infty) \to \mathbb{R}$ satisfying

$$\int_y^\infty |h(z)| e^{-\Lambda(z)} d\Lambda(z) < \infty, \tag{4.3}$$

one has

$$h(\overline{Y}_t)\frac{\overline{Y}_t - Y_t}{\overline{Y}_t - \lambda(\overline{Y}_t)} + h^\Lambda(\overline{Y}_t)\frac{Y_t - \lambda(\overline{Y}_t)}{\overline{Y}_t - \lambda(\overline{Y}_t)} = h^\Lambda(y) + \int_0^t \frac{(h^\Lambda - h)(\overline{Y}_u)}{\overline{Y}_u - \lambda(\overline{Y}_u)} dY_u, \quad t \in \mathbb{R}_+, \tag{4.4}$$

where

$$h^\Lambda(x) := e^{\Lambda(x)} \int_x^\infty h(z) e^{-\Lambda(z)} d\Lambda(z), \quad x \geq y.$$

In particular, if $T_\lambda < \infty$, then

$$h(\overline{Y}_{T_\lambda}) = h(\overline{Y}_T)\frac{\overline{Y}_T - Y_T}{\overline{Y}_T - \lambda(\overline{Y}_T)} + h^\Lambda(\overline{Y}_T)\frac{Y_T - \lambda(\overline{Y}_T)}{\overline{Y}_T - \lambda(\overline{Y}_T)} + \int_T^{T_\lambda} \frac{(h^\Lambda - h)(\overline{Y}_u)}{\overline{Y}_u - \lambda(\overline{Y}_u)} dY_u \tag{4.5}$$

for every stopping time $T \leq T_\lambda$.

Proof. Formula (4.4) follows from Proposition 3.8 applied to $X = DD$, $B = \overline{Y} - y$, $a = \infty$,

$$f(x) = \frac{1}{x + y - \lambda(x + y)}$$

and $\tilde{h}(x) = h(x + y)$ instead of h. Equation (4.5) is a consequence of (4.4) since for $t = T_\lambda$, the left side of (4.4) becomes $h(\overline{Y}_{T_\lambda})$. □

4.2 Perpetual options

Let us first consider an option paying $h(\overline{Y}_{T_c})$ units of the benchmark portfolio at time T_c, where $h : [y, \infty) \to \mathbb{R}$ is a Borel function and T_c equals one of the following stopping times:

1. **Stop-loss trigger:** $T_c = \inf \{t \in \mathbb{R}_+ : Y_t = c\}$ for a constant $c \in (0, y)$
2. **Drawdown trigger:** $T_c = \inf \{t \in \mathbb{R}_+ : DD_t = c\}$ for a constant $c \in (0, y)$
3. **Relative drawdown trigger:** $T_c = \inf \{t \in \mathbb{R}_+ : rDD_t = c\}$ for a constant $c \in (0, 1)$.

(In the third case we assume $y > 0$ so that the relative drawdown rDD is well-defined.)

The monetary payoff of the option is $h(\overline{Y}_{T_c})S^0_{T_c}$. But in the special case where $S^0 \equiv 1$ (e.g., a money market account with zero interest), one has $Y = S$, and the option just pays $h(\overline{S}_{T_c})$ units of currency. If T_c is a stop-loss trigger, our option is similar to a lookback call paying $(\overline{S}_T - K)^+$ at some deterministic maturity T. In case T_c is a drawdown trigger, it is related to the crash option introduced by Vecer [26], which pays c dollars the first time when $\overline{S}_t - S_t$ exceeds c. In fact, in the special case where $S^0 \equiv 1$, our option is a generalization of the crash option. If T_c is a relative drawdown trigger, our option is related to a Russian option, which, in the case where the underlying follows a geometric Brownian motion, was shown to be optimally exercised at the first time when the relative drawdown hits a certain level c (see Shepp and Shiryaev [25]).

If one supposes $\liminf_{t\to\infty} Y_t = 0$ almost surely, then $T_c < \infty$ for all three specifications of T_c above. So the option will be triggered eventually, but in general T_c is not dominated by a constant $T^* < \infty$. Theoretically, an option like this requires counterparties with unlimited life spans, for instance, two financial institutions that are assumed to exist forever. Alternatively, one can understand such options as approximations or benchmark cases for options with finite maturity.

Note that the three stopping times above are of the form $T_\lambda = \inf \left\{t \in \mathbb{R}_+ : Y_t = \lambda(\overline{Y}_t)\right\}$ for

1. $\lambda \equiv c$,
2. $\lambda(x) = x - c$,
3. $\lambda(x) = x(1 - c)$,

respectively. In all three cases, $1/(x - \lambda(x))$ is positive and locally bounded on $[y, \infty)$. So Proposition 4.1 applies, and formula (4.5) shows that the option can be perfectly replicated by trading in S and S^0. Its price at any stopping time $T \le T_c$ is

$$\left(h(\overline{Y}_T)\frac{\overline{Y}_T - Y_T}{\overline{Y}_T - \lambda(\overline{Y}_T)} + h^\Lambda(\overline{Y}_T)\frac{Y_T - \lambda(\overline{Y}_T)}{\overline{Y}_T - \lambda(\overline{Y}_T)}\right) S^0_T \tag{4.6}$$

since this is the amount needed to replicate the payoff with a self-financing trading strategy. The replicating strategy consists of holding at each time $T \le u \le T_\lambda$

$$\frac{(h^\Lambda - h)(\overline{Y}_u)}{\overline{Y}_u - \lambda(\overline{Y}_u)} \quad \text{units of } S \tag{4.7}$$

and

$$\begin{aligned} &h(\overline{Y}_u)\frac{\overline{Y}_u - Y_u}{\overline{Y}_u - \lambda(\overline{Y}_u)} + h^\Lambda(\overline{Y}_u)\frac{Y_u - \lambda(\overline{Y}_u)}{\overline{Y}_u - \lambda(\overline{Y}_u)} - \frac{(h^\Lambda - h)(\overline{Y}_u)}{\overline{Y}_u - \lambda(\overline{Y}_u)}Y_u \\ &= \frac{h(\overline{Y}_u)\overline{Y}_u - h^\Lambda(\overline{Y}_u)\lambda(\overline{Y}_u)}{\overline{Y}_u - \lambda(\overline{Y}_u)} \quad \text{units of } S^0. \end{aligned} \tag{4.8}$$

For the three specifications of T_c above, Λ and h^Λ take the following forms:

1. $\Lambda(x) = \log(x-c) - \log(y-c), \qquad h^\Lambda(x) = (x-c)\int_x^\infty \frac{h(z)}{(z-c)^2} dz,$
2. $\Lambda(x) = (x-y)/c, \qquad h^\Lambda(x) = \frac{1}{c} e^{x/c} \int_x^\infty h(z) e^{-z/c} dz,$
3. $\Lambda(x) = \log(x/y)/c, \qquad h^\Lambda(x) = \frac{1}{c} x^{1/c} \int_x^\infty h(z) z^{-(1+c)/c} dz.$

So in all three cases the option is replicable, and there exist explicit expressions for the price and hedge of the option that do not depend on the particular form of S, S^0 or Y.

4.3 Options with finite maturity

We now consider the same options as in Subsection 4.2, except that there exists a deterministic maturity $T^* < \infty$ at which the option is settled if T_c has not occurred until then. More precisely, the option payoff occurs at time $T_c \wedge T^*$ and consists of

$$h(\overline{Y}_{T_c}) 1_{\{T_c \leq T^*\}} + H 1_{\{T_c > T^*\}} \tag{4.9}$$

units of S^0, where H is the settlement amount in the case that T_c does not happen until T^*. If one sets

$$H = h(\overline{Y}_{T^*}) \frac{\overline{Y}_{T^*} - Y_{T^*}}{\overline{Y}_{T^*} - \lambda(\overline{Y}_{T^*})} + h^\Lambda(\overline{Y}_{T^*}) \frac{Y_{T^*} - \lambda(\overline{Y}_{T^*})}{\overline{Y}_{T^*} - \lambda(\overline{Y}_{T^*})}, \tag{4.10}$$

one obtains from formula (4.4) that the option can be replicated by trading in S and S^0. Now this is true even if $\mathbb{P}[T_c = \infty] > 0$. So it is not necessary to assume $\liminf_{t \to \infty} Y_t = 0$ almost surely. It follows from the same arguments as in Subsection 4.2 that the price of the option at any stopping time $T \leq T_c \wedge T^*$ is given by (4.6) and the hedging strategy by (4.7)–(4.8). In particular, if H is specified as in (4.10), the option is still replicable, and the price and hedging strategy can be given in closed form. On the other hand, if H is different from (4.10), for instance, $H = 0$, it cannot be expected that the option is replicable by trading in S and S^0. This will depend on the particular form of Y. And even if it is replicable, the hedging strategy and therefore also the price of the option will depend on the form of Y.

In the following we take a closer look at two special cases.

Nonnegative payoff with zero settlement

If h is nonnegative, then so is the random variable (4.10) on the set $\{T_c > T^*\}$, and it follows that the option with payoff function h and settlement $H = 0$ is super-replicated by the hedging strategy (4.7)–(4.8).

Hedging without borrowing

If

$$h(z) z \geq h^\Lambda(z) \lambda(z) \quad \text{for all } z \geq y, \tag{4.11}$$

it follows from (4.8) that the hedge of the payoff (4.9) with H equal to (4.10) does not require taking short positions in S^0. So, provided that S is continuous and it is possible to lend money at zero interest or keep it in a safe, one can assume $S^0 \equiv 1$ and therefore $Y_t = S_t$.

Alternatively, if condition (4.11) holds, S is continuous and S^0 increasing (i.e., $S^0_t \geq S^0_u$ for $t \geq u$), one can use the hedging strategy corresponding to $(S, 1)$ and an option with cash payoff

$$h(\overline{S}_{T_c}) 1_{\{T_c \leq T^*\}} + H 1_{\{T_c > T^*\}} \quad \text{at time} \quad T_c \wedge T^* \tag{4.12}$$

with

$$H = h(\overline{S}_{T^*}) \frac{\overline{S}_{T^*} - S_{T^*}}{\overline{S}_{T^*} - \lambda(\overline{S}_{T^*})} + h^{\Lambda}(\overline{S}_{T^*}) \frac{S_{T^*} - \lambda(\overline{S}_{T^*})}{\overline{S}_{T^*} - \lambda(\overline{S}_{T^*})}$$

and T_c defined in terms of $Y = S$, on the pair (S, S^0). Then it is no longer self-financing, but it replicates (4.12) if the returns of the investments in S^0 are continuously withdrawn from the hedging portfolio. Indeed, for

$$\vartheta_t := \frac{(h^{\Lambda} - h)(\overline{S}_t)}{\overline{S}_t - \lambda(\overline{S}_t)} \quad \text{and} \quad V_t := h^{\Lambda}(S_0) + \int_0^t \vartheta_u dS_u,$$

one obtains from formula (4.4) that

$$V_{T_c \wedge T^*} = h(\overline{S}_{T_c}) 1_{\{T_c \leq T^*\}} + H 1_{\{T_c > T^*\}}$$

and from (4.11) together with (4.8) that $V_t - \vartheta_t S_t \geq 0$. So if one starts with initial capital V_0 and keeps ϑ_t shares of S and the cash amount $V_t - \vartheta_t S_t$ invested in S^0, one obtains a portfolio that replicates the payoff (4.12) at time $T_c \wedge T^*$ and, in addition, yields cumulative cash flows of

$$\int_0^t (V_u - \vartheta_u S_u) \frac{dS^0_u}{S^0_u}, \quad 0 \leq t \leq T_c \wedge T^*. \tag{4.13}$$

Equivalently, one could construct a self-financing investment strategy in (S, S^0) that super-replicates the option by reinvesting the cash flows (4.13) in S^0.

5 Distributions of maxima, maximum drawdowns and maximum relative drawdowns

In this section we first generalize a result of Lehoczky [9] on the distribution of the maximum of a stopped diffusion process. Then we calculate distributions of maximum drawdowns and maximum relative drawdowns of suitably stopped processes. Several authors have studied the distribution of the maximum drawdown of a Brownian motion (with or without drift) over a deterministic time interval $[0, T]$; see, for instance, Berger and Whitt [4], Douady et al. [6], Graversen and Shiryaev [8], Magdon-Ismail et al. [11]. Vecer [26] and Pospisil and Vecer [21, 22] calculated expectations of drawdown-related random variables with PDE and PIDE methods. With the techniques developed here we will be able to derive conditional distributions of maximum drawdowns of a wide class of continuous-time stochastic processes on intervals of the form $[T, R]$ for suitable stopping times T and R.

In the whole section, Y is a stochastic process starting from a constant $y \in \mathbb{R}$ and taking values in an interval $I \subset \mathbb{R}$. We assume that there exists a strictly increasing continuous function $s : I \to \mathbb{R}$ such that $M_t := s(Y_t)$ is a local martingale with continuous running maximum $\overline{M}$. Then the drawdown $\overline{M} - M$ of M is a nonnegative process of class (Σ) with semimartingale decomposition

$(m-M)+(\overline{M}-m)$, where $m := s(y)$. If Y already is a local martingale with continuous running maximum $\overline{Y}$, one can choose $s(x)=x$. Or if Y is a diffusion of the form

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t, \quad Y_0 = y \in I, \tag{5.1}$$

where W is a Brownian motion and $\mu, \sigma : I \to \mathbb{R}$ are deterministic functions such that

$$\gamma(x) = 2\int_y^x \frac{\mu(z)}{\sigma^2(z)}dz \quad \text{and} \quad \int_y^x e^{-\gamma(z)}dz \quad \text{are finite for all } x \in I, \tag{5.2}$$

s can be chosen as $s(x) = \alpha + \beta \int_y^x e^{-\gamma(z)}dz$ for arbitrary constants $\alpha \in \mathbb{R}$ and $\beta > 0$. For instance, if $Y_t = W_t + \mu t$ for $\mu \in \mathbb{R} \setminus \{0\}$, then $I = \mathbb{R}$ and one can take $s(x) = -\text{sign}(\mu)e^{-2\mu x}$. Or if Y is a Bessel process of dimension $\delta = 2(1-\nu) > 2$ starting at $y > 0$, one can choose $I = (0,\infty)$ and $s(x) = -x^{2\nu}$.

Let us now consider a stopping time of the form

$$T_\lambda := \inf\left\{t \in \mathbb{R}_+ : Y_t \le \lambda(\overline{Y}_t)\right\}$$

for a Borel function λ mapping the interval $[y,\infty) \cap I$ to the closure $\bar{I}$ of I in $[-\infty,\infty]$ such that $\lambda(x) < x$ for all $x \in [y, \sup I)$. We extend s continuously to $s : [y, \sup I] \to [-\infty, \infty]$ and denote

$$g_\lambda = \sup\left\{t \le T_\lambda : Y_t = \overline{Y}_t\right\} \quad \text{and} \quad \Lambda(x) = \int_y^x \frac{ds(z)}{s(z) - s \circ \lambda(z)}. \tag{5.3}$$

Then Λ is a well-defined increasing function from $[y, \sup I]$ to $[0,\infty]$. We start with a result on the distribution of the maximum attained by Y until time T_λ.

Proposition 5.1 *Let a_n, $n \in \mathbb{N}$, be an increasing sequence in $(y, \sup I)$ with limit $a \in (y, \sup I]$ and ε_n, $n \in \mathbb{N}$, a decreasing sequence in $(0,\infty)$ such that $\lambda(x) \le x - \varepsilon_n$ for $y \le x \le a_n$. Assume that*

$$\overline{Y}_{T_\lambda} < a \quad \textit{and} \quad Y_{T_\lambda} = \lambda(\overline{Y}_{T_\lambda}). \tag{5.4}$$

Then $\Lambda(a) = \infty$, $g_\lambda < T_\lambda$, $Y_{T_\lambda} < \overline{Y}_{g_\lambda} = \overline{Y}_{T_\lambda}$ and

$$\mathbb{P}[g_\lambda \le T \mid \mathcal{F}_T] = \frac{s(\overline{Y}_T) - s(Y_T)}{s(\overline{Y}_T) - s \circ \lambda(\overline{Y}_T)} \tag{5.5}$$

for every stopping time $T \le T_\lambda$. Moreover, for all Borel functions $h : [y,a) \to \mathbb{R}$ satisfying

$$\int_y^a |h(z)| e^{-\Lambda(z)} d\Lambda(z) < \infty,$$

one has

$$\mathbb{E}\left[h(\overline{Y}_{T_\lambda}) \mid \mathcal{F}_T\right] = \frac{h(\overline{Y}_T)[s(\overline{Y}_T) - s(Y_T)] + h^\Lambda(\overline{Y}_T)[s(Y_T) - s \circ \lambda(\overline{Y}_T)]}{s(\overline{Y}_T) - s \circ \lambda(\overline{Y}_T)}, \tag{5.6}$$

where

$$h^\Lambda(x) = e^{\Lambda(x)} \int_x^a h(z) e^{-\Lambda(z)} d\Lambda(z), \quad x \ge y.$$

In particular,

$$\mathbb{P}[\overline{Y}_{T_\lambda} > x \mid \mathcal{F}_T] = 1_{\{\overline{Y}_T > x\}} + 1_{\{\overline{Y}_T \le x\}} \frac{s(Y_T) - s \circ \lambda(\overline{Y}_T)}{s(\overline{Y}_T) - s \circ \lambda(\overline{Y}_T)} e^{\Lambda(\overline{Y}_T) - \Lambda(x)} \quad \textit{for } x \ge y. \tag{5.7}$$

Proof. $X_t = s(\overline{Y}_{t\wedge T_\lambda}) - s(Y_{t\wedge T_\lambda})$ is a nonnegative process of class (Σ) starting at 0 with decomposition $(s(y) - s(Y_{t\wedge T_\lambda})) + (s(\overline{Y}_{t\wedge T_\lambda}) - s(y))$. The function $f : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$f(x) := 1_{\{x < s(a) - s(y)\}} \frac{1}{x + s(y) - s \circ \lambda \circ s^{-1}(x + s(y))}$$

satisfies the assumptions of Theorem 3.10 with $\tilde{a}_n = s(a_n) - s(y)$ and $\tilde{a} = s(a) - s(y)$ instead of a_n and a. Condition (5.4) guarantees that the process

$$f(s(\overline{Y}_{t\wedge T_\lambda}) - s(y)) X_t = 1_{\{\overline{Y}_{t\wedge T_\lambda} < a\}} \frac{s(\overline{Y}_{t\wedge T_\lambda}) - s(Y_{t\wedge T_\lambda})}{s(\overline{Y}_{t\wedge T_\lambda}) - s \circ \lambda(\overline{Y}_{t\wedge T_\lambda})}$$

takes values in $[0,1]$ and converges to 1 almost surely. So it follows from Theorem 3.10 that $\Lambda(a) = \int_0^{\tilde{a}} f(x)dx = \infty$, $g_\lambda < T_\lambda$, $Y_{T_\lambda} < \overline{Y}_{g_\lambda} = \overline{Y}_{T_\lambda}$ and

$$\mathbb{P}[g_\lambda \le T \mid \mathcal{F}_T] = \frac{s(\overline{Y}_T) - s(Y_T)}{s(\overline{Y}_T) - s \circ \lambda(\overline{Y}_T)}$$

for all stopping times $T \le T_\lambda$. Formulas (5.6)–(5.7) follow from Theorem 3.10 applied to the function

$$\tilde{h}(x) = h(s^{-1}(x + s(y))), \quad 0 \le x < \tilde{a}.$$

□

Remark 5.2 If Y is a nonnegative local martingale starting at 1 such that $\overline{Y}$ is continuous and $Y_t \to 0$ almost surely, then formula (5.7) with $T = 0$, $s(x) = x$ and $\lambda \equiv 0$ yields that $1/\overline{Y}_\infty$ is uniformly distributed on the interval $(0,1)$. This is Doob's maximal identity, which was studied in depth by Mansuy and Yor [12] and Nikeghbali and Yor [17].

Remark 5.3 If Y is of the form $dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t$ for a Brownian motion W such that (5.2) holds, set $\gamma(x) = 2\int_y^x \mu(z)/\sigma^2(z)dz$ and $s(x) = \alpha + \beta \int_y^x e^{-\gamma(z)}dz$ for constants $\alpha \in \mathbb{R}$ and $\beta > 0$. If $\lambda : [y, \infty) \cap I \to \bar{I}$ is a Borel function satisfying the assumptions of Proposition 5.1 for some $a \le \infty$, the function Λ defined in (5.3) becomes

$$\Lambda(x) = \int_y^x \frac{e^{-\gamma(z)}dz}{\int_{\lambda(z)}^z e^{-\gamma(u)}du},$$

and one obtains from Proposition 5.1 that for all stopping times $T \le T_\lambda$,

$$\mathbb{P}[g_\lambda \le T \mid \mathcal{F}_T] = \frac{\int_{Y_T}^{\overline{Y}_T} e^{-\gamma(z)}dz}{\int_{\lambda(\overline{Y}_T)}^{\overline{Y}_T} e^{-\gamma(z)}dz}$$

and

$$\mathbb{P}[\overline{Y}_{T_\lambda} > x \mid \mathcal{F}_T] = 1_{\{\overline{Y}_T > x\}} + 1_{\{\overline{Y}_T \le x\}} \frac{\int_{\lambda(\overline{Y}_T)}^{Y_T} e^{-\gamma(z)}dz}{\int_{\lambda(\overline{Y}_T)}^{\overline{Y}_T} e^{-\gamma(z)}dz} \exp\left(-\int_{\overline{Y}_T}^x \frac{e^{-\gamma(z)}dz}{\int_{\lambda(z)}^z e^{-\gamma(u)}du}\right) \quad \text{for } x \ge y. \quad (5.8)$$

In the special case $\lambda \equiv c$, (5.8) reduces to

$$\mathbb{P}[\overline{Y}_{T_\lambda} > x \mid \mathcal{F}_T] = 1_{\{\overline{Y}_T > x\}} + 1_{\{\overline{Y}_T \le x\}} \frac{\int_c^{\overline{Y}_T} e^{-\gamma(z)} dz}{\int_c^x e^{-\gamma(z)} dz} \quad \text{for } x \ge y,$$

and for $\lambda(x) = x - c$ it becomes

$$\mathbb{P}[\overline{Y}_{T_\lambda} > x \mid \mathcal{F}_T] = 1_{\{\overline{Y}_T > x\}} + 1_{\{\overline{Y}_T \le x\}} \frac{\int_{\overline{Y}_T - c}^{Y_T} e^{-\gamma(z)} dz}{\int_{\overline{Y}_T - c}^{\overline{Y}_T} e^{-\gamma(z)} dz} \exp\left(- \int_{\overline{Y}_T}^x \frac{e^{-\gamma(z)} dz}{\int_{z-c}^z e^{-\gamma(u)} du} \right) \quad \text{for } x \ge y.$$

For $T = 0$, this gives

$$\mathbb{P}[\overline{Y}_{T_\lambda} > x] = \exp\left(- \int_y^x \frac{e^{-\gamma(z)} dz}{\int_{z-c}^z e^{-\gamma(u)} du} \right) \quad \text{for } x \ge y,$$

which (in the case $y = 0$) is formula (3) of Lehoczky [9].

If $y > 0$ and $\lambda(x) = x(1-c)$, one obtains

$$\mathbb{P}[\overline{Y}_{T_\lambda} > x \mid \mathcal{F}_T] = 1_{\{\overline{Y}_T > x\}} + 1_{\{\overline{Y}_T \le x\}} \frac{\int_{\overline{Y}_T(1-c)}^{Y_T} e^{-\gamma(z)} dz}{\int_{\overline{Y}_T(1-c)}^{\overline{Y}_T} e^{-\gamma(z)} dz} \exp\left(- \int_{\overline{Y}_T}^x \frac{e^{-\gamma(z)} dz}{\int_{z(1-c)}^z e^{-\gamma(u)} du} \right) \quad \text{for } x \ge y.$$

The following proposition gives sufficient conditions for assumption (5.4) to hold.

Proposition 5.4 *Assume Y is continuous and let $a \in (y, \sup I]$. Then both of the following conditions imply* (5.4)*:*

(1) $s(y) > 0$, $s(Y_t) \to 0$ *almost surely,* $Y_t < a$ *for all* t *and* $s \circ \lambda(x) \ge 0$ *for all* $x \in [y, a)$

(2) $s(\overline{Y}_\infty) = \Lambda(a) = \infty$

Proof. Under assumption (1) one has $\overline{Y}_\infty < a$ and

$$\frac{s(\overline{Y}_t) - s(Y_t)}{s(\overline{Y}_t) - s \circ \lambda(\overline{Y}_t)} \ge \frac{s(\overline{Y}_t) - s(Y_t)}{s(\overline{Y}_t)} \to 1 \text{ almost surely.}$$

It follows that $\overline{Y}_{T_\lambda} < a$ and $Y_{T_\lambda} = \lambda(\overline{Y}_{T_\lambda})$.

If condition (2) holds, then $\int_0^\infty f(x) dx = \infty$ for the function

$$f(x) = 1_{\{0 \le x < s(a) - s(y)\}} \frac{1}{x + s(y) - s \circ \lambda \circ s^{-1}(x + s(y))}.$$

Since $X_t = s(\overline{Y}_t) - s(Y_t)$ is a nonnegative continuous process of class (Σ) with decomposition $X_t = (s(y) - s(Y_t)) + (s(\overline{Y}_t) - s(y))$, it follows from Corollary 2.6 that $\inf\left\{t \in \mathbb{R}_+ : f(s(\overline{Y}_t) - s(y)) X_t \ge 1\right\} < \infty$. This implies $T_\lambda < \infty$, and (5.4) follows. □

In the following results we derive distributions of maximum drawdowns and maximum relative drawdowns. We denote by $DD_t = \overline{Y}_t - Y_t$ the drawdown of Y and, provided that $y > 0$, by $rDD_t = DD_t / \overline{Y}_t = 1 - Y_t / \overline{Y}_t$ the relative drawdown of Y.

Proposition 5.5 *Assume Y is continuous with $s(y) > 0$ and $s(Y_t) \to 0$ almost surely. Let $T < \infty$ be a stopping time and K an $\mathcal{F}_T$-measurable random variable such that $0 \le s(K) < s(Y_T)$. Denote $T_K = \inf\{t \ge T : Y_t = K\}$. Then one has for all $x \ge 0$*

$$\mathbb{P}\left[\sup_{t\in[T,T_K]\cap\mathbb{R}_+} DD_t > x \mid \mathcal{F}_T\right] = 1_{\{\overline{Y}_T - K > x\}} + 1_{\{\overline{Y}_T - K \le x\}} \frac{s(Y_T) - s(K)}{s(K+x) - s(K)}. \tag{5.9}$$

If, in addition, $y > 0$, then

$$\mathbb{P}\left[\sup_{t\in[T,T_K]\cap\mathbb{R}_+} rDD_t > x \mid \mathcal{F}_T\right] = 1_{\{1-K/\overline{Y}_T > x\}} + 1_{\{1-K/\overline{Y}_T \le x < 1\}} \frac{s(Y_T) - s(K)}{s(K/(1-x)) - s(K)}. \tag{5.10}$$

Proof. First assume that $T = 0$ and K is a constant. Then Y with $\lambda \equiv K$ and $a = \sup I$ fulfills condition (1) of Proposition 5.4. Indeed, it is part of the assumptions that $s(y) > 0$, $s(Y_t) \to 0$ almost surely and $s \circ \lambda(x) = s(K) \ge 0$ for all $x \in [y, a)$. To see that $Y_t < a$ for all t, denote $R = \inf\{t \in \mathbb{R}_+ : s(Y_t) = 0\}$ and notice that $M_{t\wedge R} = s(Y_{t\wedge R})$ is a nonnegative local martingale starting at $s(y) > 0$ and converging to zero almost surely. Therefore, it follows from Doob's maximal identity that $s(y)/s(\overline{Y}_R)$ is uniformly distributed on the interval $(0,1)$ (see Remark 5.2). In particular, $s(a) = \infty$, and hence, $Y_t < a$ for all t. It now follows from Proposition 5.4 that the conditions of Proposition 5.1 are fulfilled. Moreover,

$$\sup_{t\in[T,T_K]\cap\mathbb{R}_+} DD_t = \overline{Y}_{T_K} - K \quad \text{and} \quad \sup_{t\in[T,T_K]\cap\mathbb{R}_+} rDD_t = 1 - K/\overline{Y}_{T_K}.$$

So (5.9) and (5.10) can be deduced from formula (5.7). In the general case, the proposition follows by considering the process $\tilde{Y}_t = Y_{T+t}$ in the filtration $\tilde{\mathcal{F}}_t = \mathcal{F}_{T+t}$ and conditioning on $\mathcal{F}_T$. □

Proposition 5.6 *Assume Y is continuous and $s(\overline{Y}_\infty) = \infty$. Let $T < \infty$ be a stopping time and K a $[0,\infty]$-valued $\mathcal{F}_T$-measurable random variable such that $\overline{Y}_T < K \le \overline{Y}_\infty$. Denote $T_K = \inf\left\{t \ge T : \overline{Y}_t = K\right\}$. Then*

$$\mathbb{P}[Y_t \ge \lambda(\overline{Y}_t) \text{ for all } t \in [T, T_K] \cap \mathbb{R}_+ \mid \mathcal{F}_T] = \left(\frac{s(Y_T) - s\circ\lambda(\overline{Y}_T)}{s(\overline{Y}_T) - s\circ\lambda(\overline{Y}_T)}\right)^+ e^{\Lambda(\overline{Y}_T) - \Lambda(K)}. \tag{5.11}$$

Proof. $X_t = s(\overline{Y}_t) - s(Y_t)$ is a nonnegative process of class (Σ) with decomposition $(s(y) - s(Y_t)) + (s(\overline{Y}_t) - s(y))$ and

$$f(x) = \frac{1}{x + s(y) - s\circ\lambda\circ s^{-1}(x + s(y))}$$

a nonnegative Borel function from $[0, s(\sup I) - s(y))$ to $\mathbb{R}_+$. Since $Y_t \ge \lambda(\overline{Y}_t)$ is equivalent to $f(s(\overline{Y}_t) - s(y))X_t \le 1$ and $T_K = \inf\left\{t \ge T : s(\overline{Y}_t) - s(y) = s(K) - s(y)\right\}$, one obtains from Corollary 2.6 that

$$\begin{aligned}
&\mathbb{P}[Y_t \ge \lambda(\overline{Y}_t) \text{ for all } t \in [T, T_K] \cap \mathbb{R}_+ \mid \mathcal{F}_T] \\
=\;& \left(\frac{s(Y_T) - s\circ\lambda(\overline{Y}_T)}{s(\overline{Y}_T) - s\circ\lambda(\overline{Y}_T)}\right)^+ \exp\left(-\int_{s(\overline{Y}_T)-s(y)}^{s(K)-s(y)} f(x)dx\right) \\
=\;& \left(\frac{s(Y_T) - s\circ\lambda(\overline{Y}_T)}{s(\overline{Y}_T) - s\circ\lambda(\overline{Y}_T)}\right)^+ \exp\left(\Lambda(\overline{Y}_T) - \Lambda(K)\right).
\end{aligned}$$

□

Corollary 5.7 *If the assumptions of Proposition 5.6 hold, then*

$$\mathbb{P}\left[\sup_{t\in[T,T_K]\cap\mathbb{R}_+} DD_t \leq x \mid \mathcal{F}_T\right] = \left(\frac{s(Y_T)-s(\overline{Y}_T - x)}{s(\overline{Y}_T)-s(\overline{Y}_T - x)}\right)^+ \exp\left(-\int_{\overline{Y}_T}^{K} \frac{ds(z)}{s(z)-s(z-x)}\right) \tag{5.12}$$

for every constant $x > 0$ *such that* $y - x \in \bar{I}$. *If, in addition,* $y > 0$, *then*

$$\mathbb{P}\left[\sup_{t\in[T,T_K]\cap\mathbb{R}_+} rDD_t \leq x \mid \mathcal{F}_T\right] = \left(\frac{s(Y_T)-s([1-x]\overline{Y}_T)}{s(\overline{Y}_T)-s([1-x]\overline{Y}_T)}\right)^+ \exp\left(-\int_{\overline{Y}_T}^{K} \frac{ds(z)}{s(z)-s([1-x]z)}\right) \tag{5.13}$$

for each $x > 0$ *such that* $\inf\{(1-x)z : z \in [y, \sup I)\} \in \bar{I}$.

Proof. Formula (5.12) follows from Proposition 5.6 applied to the function $\lambda(z) = z - x$. The condition $y - x \in \bar{I}$ ensures that $\lambda([y,\infty) \cap I) \subset \bar{I}$. Formula (5.13) is obtained from Proposition 5.6 applied to the function $\lambda(z) = (1-x)z$. $\inf\{(1-x)z : z \in [y, \sup I)\} \in \bar{I}$ implies that $\lambda([y,\infty) \cap I) \subset \bar{I}$. □

Remark 5.8 In the case where Y is a local martingale, one can choose $s(x) = x$, and formulas (5.12)–(5.13) become

$$\mathbb{P}\left[\sup_{t\in[T,T_K]\cap\mathbb{R}_+} DD_t \leq x \mid \mathcal{F}_T\right] = 1_{\{x>0\}}\left(1 - \frac{DD_T}{x}\right)^+ \exp\left(\frac{\overline{Y}_T - K}{x}\right)$$

and

$$\mathbb{P}\left[\sup_{t\in[T,T_K]\cap\mathbb{R}_+} rDD_t \leq x \mid \mathcal{F}_T\right] = 1_{\{x>0\}}\left(1 - \frac{rDD_T}{x}\right)^+ \left(\frac{\overline{Y}_T}{K}\right)^{1/x}.$$

On the other hand, if Y is of the form $dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t$ such that (5.2) holds and

$$\int_y^{\overline{Y}_\infty} e^{-\gamma(y)}dy = \infty,$$

then for every $[0,\infty]$-valued $\mathcal{F}_T$-measurable random variable K satisfying $\overline{Y}_T < K \leq \overline{Y}_\infty$, one can denote $T_K = \inf\{t \geq T : Y_t = K\}$ and obtain from Corollary 5.7 that for all $x \geq 0$,

$$\mathbb{P}\left[\sup_{t\in[T,T_K]\cap\mathbb{R}_+} DD_t \leq x \mid \mathcal{F}_T\right] = \left(\frac{\int_{\overline{Y}_T - x}^{Y_T} e^{-\gamma(z)}dz}{\int_{\overline{Y}_T - x}^{\overline{Y}_T} e^{-\gamma(z)}dz}\right)^+ \exp\left(-\int_{\overline{Y}_T}^{K} \frac{e^{-\gamma(z)}dz}{\int_{z-x}^{z} e^{-\gamma(u)}du}\right)$$

for every constant $x > 0$ such that $y - x \in \bar{I}$. If in addition $y > 0$, then

$$\mathbb{P}\left[\sup_{t\in[T,T_K]\cap\mathbb{R}_+} rDD_t \leq x \mid \mathcal{F}_T\right] = \left(\frac{\int_{(1-x)\overline{Y}_T}^{Y_T} e^{-\gamma(z)}dz}{\int_{(1-x)\overline{Y}_T}^{\overline{Y}_T} e^{-\gamma(z)}dz}\right)^+ \exp\left(-\int_{\overline{Y}_T}^{K} \frac{e^{-\gamma(z)}dz}{\int_{(1-x)z}^{z} e^{-\gamma(u)}du}\right)$$

for each $x > 0$ such that $\inf\{(1-x)z : z \in [y, \sup I)\} \in \bar{I}$.

References

[1] J. Azéma, P.A. Meyer, M. Yor (1992). Martingales relatives. Sém. Proba. XXVI, Lecture Notes in Mathematics 1526, 307–321.

[2] J. Azéma, M. Yor (1979). Une solution simple au problème de Skorokhod. Sém. Proba. XIII, Lecture Notes in Mathematics 721, 90–115 and 625–633.

[3] J. Azéma, M. Yor (1992). Sur les zéros des martingales continues. Sém. Proba. XXVI, Lecture Notes in Mathematics 1526, 248–306.

[4] A.W. Berger, W. Whitt (1995). Maximum values in queuing processes. Prob. Eng. Inf. Sciences 9, 375–409.

[5] C. Dellacherie, P.A. Meyer (1980). Probabilités et Potentiel. Chapitres V à VIII. Théorie des Martingales. Revised Edition. Hermann, Paris.

[6] R. Douady, A. Shiryaev, M. Yor (2000). On probability characteristics of downfall in a standard Brownian motion. Theory Prob. Appl. 44, 29–38.

[7] K.D. Elworthy, X.M. Li, M. Yor (1999). The importance of strictly local martingales; applications to radial Ornstein-Uhlenbeck processes. Probab. Theory Related Fields 115(2), 325–355.

[8] S.E. Graversen, A. Shiryaev (2000). An extension of P. Levys distributional properties to the case of a Brownian motion with drift. Bernoulli 6(4), 615–620.

[9] J. Lehoczky (1977). Formulas for stopped diffusion processes with stopping times based on the maximum. Ann. Prob. 5, 601–608.

[10] D. Madan, B. Roynette, M. Yor (2008). Option prices as probabilities. Finance Research Letters 5(2), 79–87.

[11] M. Magdon-Ismail, A.F. Atiya, A. Pratap, Y.S. Abu-Mostafa (2004). On the maximum drawdown of a Brownion motion. J. Appl. Prob. 41, 147–161.

[12] R. Mansuy, M. Yor (2006). Random Times and Enlargement of Filtrations in a Brownian Setting. Lecture Notes in Mathematics 1873, Springer-Verlag, Berlin Heidelberg.

[13] J. Najnudel, A. Nikeghbali (2009). On some universal sigma finite measures and some extensions of Doob's optional stopping theorem. Math Arxiv Preprint 0906.1782.

[14] A. Nikeghbali (2006). Enlargements of filtrations and path decompositions at non stopping times. Prob. Theory Related Fields 136(4), 524–540.

[15] A. Nikeghbali (2006). A class of remarkable submartingales. Stochastic Processes and their Applications 116, 917–938.

[16] A. Nikeghbali, E. Platen (2008). On honest times in financial modeling. Math Arxiv Preprint 0808.2892.

[17] A. Nikeghbali, M. Yor (2006). Doob's maximal identity, multiplicative decompositions and enlargements of filtrations. Illinois J. Math. 50, 791–814.

[18] J. Obloj (2006). A complete characterization of local martingales which are functions of Brownian motion and its maximum. Bernoulli 12(6), 955–969.

[19] E. Platen (2006). A benchmark approach to finance. Math. Finance 16(1), 131–151.

[20] E. Platen, D. Heath (2006). A Benchmark Approach to Quantitative Finance. Springer Finance. Springer-Verlag, Berlin Heidelberg.

[21] L. Pospisil, J. Vecer (2008). Partial differential equation methods for the maximum drawdown. J. Comp. Finance 12(2), 59—76.

[22] L. Pospisil, J. Vecer (2010). Maximum drawdown of a jump-diffusion process and the correstponding partial integro-differential equations. GloriaMundi Preprint.

[23] C. Profeta, B. Roynette, M. Yor (2010). Option Prices as Probabilities: A New Look at Generalized Black-Scholes Formulae. Springer-Verlag, Berlin Heidelberg.

[24] P.E. Protter (2005). Stochastic Integration and Differential Equations. Second Edition, Version 2.1. Springer-Verlag Berlin Heidelberg.

[25] L. Shepp, A.N. Shiryaev (1993). The Russian option: reduced regret. Ann. Appl. Prob. 3(3), 631–640.

[26] J. Vecer (2008). Maximum drawdown and directional trading. Risk 3(3), 631–640.

[27] M. Yor (1979). Les inégalités de sous-martingales comme conséquences de la relation de domination. Stochastics 3(1), 1–15.