Time-inconsistency of VaR and time-consistent alternatives

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Abstract

We show that VaR (value-at-risk) is not time-consistent and discuss examples where this can lead to dynamically inconsistent behavior. Then we propose two time-consistent alternatives to VaR. The first one is a composition of one-period VaR's. It is time-consistent but not coherent. The second one is a composition of average VaR's. It is a time-consistent coherent risk measure.

Keywords: Value-at-risk, Time-consistency, Composed value-at-risk, Composed average value-at-risk.

JEL Classification: D81, G11, G32

1 Introduction

VaR (value-at-risk) is currently one of the most widely used financial risk measures. In practice, it is used in different ways, and its exact definition can slightly vary from case to case. In the most narrow sense, VaR at level $\alpha \in (0,1)$ of a random variable X is just the negative of its right-hand α-quantile

$$q_+^\alpha(X) = \sup \{m \in \mathbb{R} : P[X < m] \leq \alpha\}$$

and can be written as

$$\mathrm{VaR}^\alpha(X) := -q_+^\alpha(X) = \inf \{m \in \mathbb{R} : P[X + m < 0] \leq \alpha\} \; .$$

The random variable X typically models the net worth, profit or return rate of a future financial position. For more details on VaR, see for instance, Duffie and Pan (1997), Jorion (2001), or McNeil et al. (2005). Part of VaR's popularity is certainly due to its simple definition and straightforward interpretation. But it is well known that it has the following drawbacks:

(D1) It is immediate from the definition that VaR at level α does not give any information about the magnitude of losses that occur with probability less than α.

(D2) It has been pointed out by Artzner et al. (1999) that VaR is not subadditive. Therefore, there exist situations where it behaves poorly under aggregation of positions.

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(D3) Since VaR is not convex, optimization problems with VaR constraints can be difficult to solve numerically; see for instance, Winker and Maringer (2007).

In this paper we show that VaR is not time-consistent, discuss some ramifications and propose ComVaR (composed value-at-risk) and ComAVaR (composed average value-at-risk) as alternatives. We start with a short review of dynamic risk measures and the notion of time-consistency. Then we give examples that show the time-inconsistency of VaR and how this can lead to dynamically inconsistent behavior. Finally, we discuss composed VaR and composed AVaR as time-consistent alternatives to VaR.

2 Dynamic risk measures and time-consistency

Consider a dynamic setup where time runs through the discrete set $\{0, 1, \ldots, T\}$ for some fixed natural number T. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, P)$ be a filtered probability space and denote by $L^0(\mathcal{F}_t)$ the vector space of all real-valued, $\mathcal{F}_t$-measurable random variables, where two random variables are identified if they coincide P-almost surely. We write $X \geq Y$ for random variables X and Y with the property $P[X \geq Y] = 1$. A dynamic risk measure on $L^0(\mathcal{F}_T)$ consists of a sequence of mappings

$$\rho_t : L^0(\mathcal{F}_T) \to L^0(\mathcal{F}_t)\,, \quad t = 0, \ldots, T-1\,,$$

where $\rho_t(X)$ is understood as an assessment of the downside risk of the position X conditional on the information available at time t. We work with the following notion of time-consistency:

Definition 2.1 *A dynamic risk measure $(\rho_t)_{t=0}^{T-1}$ on $L^0(\mathcal{F}_T)$ is time-consistent if for all $X, Y \in L^0(\mathcal{F}_T)$ and $t = 0, \ldots, T-2$, $\rho_{t+1}(X) \geq \rho_{t+1}(Y)$ implies $\rho_t(X) \geq \rho_t(Y)$.*

Similar or equivalent conditions have been studied for dynamic preferences; see for instance, Koopmans (1960), Kreps and Porteus (1978), Wang (2003), Epstein and Schneider (2003), Maccheroni et al (2006). In the context of risk measurement time-consistency has been investigated among others, by Riedel (2004), Barrieu and El Karoui (2004), Detlefsen and Scandolo (2005), Cheridito et al. (2006), Delbaen (2006), Weber (2006), Artzner et al. (2007).

If one is working with a time-inconsistent dynamic risk measure $(\rho_t)_{t=0}^{T-1}$, there exist $X, Y \in L^0(\mathcal{F}_T)$, $t \leq T-2$ and $A \in \mathcal{F}_t$ with $P[A] > 0$ such that $\rho_{t+1}(X) \geq \rho_{t+1}(Y)$ almost everywhere and $\rho_t(X) < \rho_t(Y)$ on the event A. This means that if A occurs, one considers the future payoff Y strictly riskier than X at time t although it is certain that this assessment will be reversed at time $t+1$. This can lead to inconsistent behavior. Specific situations where this occurs in relation with VaR are discussed in Examples 3.1 and 3.3 below.

3 VaR's with different time horizons are inconsistent

The natural definition of VaR of a position $X \in L^0(\mathcal{F}_T)$ at the level $\alpha \in (0, 1)$ at time t is

$$\mathrm{VaR}_t^\alpha(X) := \operatorname{ess\,inf}\left\{m \in L^0(\mathcal{F}_t) : P[X + m < 0 \mid \mathcal{F}_t] \leq \alpha\right\}\,,$$

where "ess inf" denotes the greatest lower bound of a family of random variables with respect to P-almost sure inequality (see e.g. Proposition VI.1.1 in Neveu, 1975) and $P[A \mid \mathcal{F}_t]$ is the conditional expectation $\mathrm{E}[1_A \mid \mathcal{F}_t]$. Alternatively, one can choose a regular conditional distribution Q_t^X of X with respect to $\mathcal{F}_t$ and define $Y \in L^0(\mathcal{F}_t)$ ω-wise by

$$Y(\omega) := -\sup\left\{m \in \mathbb{R} : Q_t^X(\omega, (-\infty, m)) \leq \alpha\right\} .$$

Then $Y = \mathrm{VaR}_t^\alpha(X)$ P-almost surely.

The following example shows that VaRs with different time horizons are not consistent.

Example 3.1 Fix an initial value $s_0 > 0$, a volatility $\sigma > 0$ and a constant $\nu \in \mathbb{R}$. Let Z_1, Z_2, be two independent standard normal random variables and define a stock price process $(S_t)_{t=0}^2$ by

$$S_0 := s_0 \quad \text{and} \quad S_t := s_0 \exp\left(\sigma \sum_{j=1}^{t} Z_j + \nu t\right), \quad t = 1, 2 .$$

Let $(\mathcal{F}_t)_{t=0}^2$ be the filtration generated by $(S_t)_{t=0}^2$. Set $R_1 := S_1/S_0$ and $R_2 := S_2/S_1$. Now choose an arbitrary probability level $\alpha \in (0,1)$. Then there exist constants a and $b < B$ such that

$$\alpha < P[R_2 \leq b] < P[R_2 \leq B] .$$

and

$$P[R_1 \leq a]P[R_2 \leq b] < \alpha < P[R_1 \leq a]P[R_2 \leq B] .$$

Consider the random payoffs

$$X := -C 1_E + d 1_{E^c} , \quad Y := -c 1_F + D 1_{F^c} ,$$

where C, c, D, d are constants such that $C > c > 0$ and $D > d > 0$, and E, F are the random events given by

$$E := \{R_1 \leq a, R_2 \leq b\} , \quad F := \{R_1 \leq a, R_2 \leq B\} .$$

One could choose for example, $s_0 = 1$, $\sigma = 0.1$ and $\nu = 0.06 - \sigma^2/2$ and take α to be equal to 0.05. Then possible values for a, b, B would be $a = 1$, $b = 0.95$, and $B = 1$. So X would be a bet against the event that the return rate $R_1 - 1$ is negative and $R_2 - 1$ is below -5%. Y would be a bet against the event that $R_1 - 1$ and $R_2 - 1$ are both negative. In any case, one has

$$\mathrm{VaR}_0^\alpha(X) = -d , \quad \mathrm{VaR}_0^\alpha(Y) = c ,$$

$$\mathrm{VaR}_1^\alpha(X) = \begin{cases} C & \text{if } R_1 \leq a \\ -d & \text{if } R_1 > a \end{cases} \quad \text{and} \quad \mathrm{VaR}_1^\alpha(Y) = \begin{cases} c & \text{if } R_1 \leq a \\ -D & \text{if } R_1 > a \end{cases} .$$

So

$$\mathrm{VaR}_1^\alpha(X) > \mathrm{VaR}_1^\alpha(Y) \quad \text{but} \quad \mathrm{VaR}_0^\alpha(X) < \mathrm{VaR}_0^\alpha(Y) , \tag{3.1}$$

and the dynamic risk measure $(\mathrm{VaR}_0^\alpha, \mathrm{VaR}_1^\alpha)$ is not time-consistent. This can lead to inconsistent behavior in situations like the following:

Consider a trader who wants to minimize VaR_t^α under the constraint $\mathrm{E}[\,.\,|\mathcal{F}_t] \geq m$ for some $m \in \mathbb{R}$. Choose D and d so large that $D > d \geq m$ and

$$-CP[R_2 \leq b] + dP[R_2 > b] \geq m , \quad -cP[R_2 \leq B] + DP[R_2 > B] \geq m .$$

Then

$$\mathrm{E}\left[X|\mathcal{F}_1\right] = \left\{ \begin{array}{cc} -CP[R_2 \le b] + dP[R_2 > b] & \text{if } R_1 \le a \\ d & \text{if } R_1 > a \end{array} \right\} \ge m\,,$$

$$\mathrm{E}\left[Y|\mathcal{F}_1\right] = \left\{ \begin{array}{cc} -cP[R_2 \le B] + DP[R_2 > B] & \text{if } R_1 \le a \\ D & \text{if } R_1 > a \end{array} \right\} \ge m\,,$$

and therefore also $\mathrm{E}[X] \ge m$ and $\mathrm{E}[Y] \ge m$. So X and Y satisfy the constraint at all times, and by (3.1), the trader prefers the future payoff X to Y at time 0 although it is certain that at time 1 s/he will regret this decision and would rather have Y than X.

Remark 3.2 Note that for Example 3.1 to work, it is not necessary that the returns R_1, R_2 be log-normal. It is enough if there exist constants a, b, B such that

$$\alpha < P[R_2 \le b \mid R_1 \le a] \le P[R_2 \le B \mid R_1 \le a]$$

and

$$P[R_1 \le a, R_2 \le b] < \alpha < P[R_1 \le a, R_2 \le B]\,.$$

This is for instance the case when R_1 and R_2 are continuously distributed and independent.

Also, the example can easily be adjusted to show that $(\mathrm{VaR}^{\alpha_0}_0, \mathrm{VaR}^{\alpha_1}_1)$ is not time-consistent for arbitrary $\alpha_0, \alpha_1 \in (0,1)$. One just has to choose the constants a, b, B such that

$$\alpha_1 < P[R_2 \le b \mid R_1 \le a] \le P[R_2 \le B \mid R_1 \le a]$$

and

$$P[R_1 \le a, R_2 \le b] < \alpha_0 < P[R_1 \le a, R_2 \le B]\,.$$

The following is a second example that shows how the use of VaR in a dynamic setup can lead to dynamically inconsistent behavior.

Example 3.3 Consider a trader who at time t tries to optimize the conditional preference functional

$$U_t(X) = \mathrm{E}[u(X)|\mathcal{F}_t] - \lambda\,\mathrm{VaR}^{\alpha}_t(X)\,, \tag{3.2}$$

where $\lambda > 0$, $\alpha \in (0,1)$ and $u : \mathbb{R} \to \mathbb{R}$ is a strictly increasing concave function with $u(0) = 0$. The idea is that the trader wants to optimize expected utility but is penalized for taking too much risk.

Specify a stock price process $(S_t)_{t=0}^2$ as in Example 3.1 and choose constants a, $b < B$ such that the following relations hold:

$$\alpha < P[R_2 \le b] < P[R_2 \le B]\,,$$

$$P[R_1 \le a]P[R_2 \le b] < \alpha < P[R_1 \le a]P[R_2 \le B]\,.$$

Then define

$$X := -C1_E + d1_{E^c}\,, \quad Y := -c1_F + D1_{F^c}$$

for

$$E := \{R_1 \le a, R_2 \le b\}\ , \quad F := \{R_1 \le a, R_2 \le B\}\ ,$$

and constants $C > c > 0$ and $D > d > 0$ such that

$$\begin{aligned} \frac{u(D)}{u(d)} &= \frac{P[R_2 > b]}{P[R_2 > B]}, \\ c &> \frac{u(D)P[F^c] - u(d)P[E^c]}{\lambda} - d, \\ \frac{u(-C)}{u(-c)} &= \frac{P[R_2 \le B]}{P[R_2 \le b]}. \end{aligned} \tag{3.3}$$

Then, at time $t = 0$ one has,

$$\begin{aligned} U_0(X) &= u(-C)P[E] + u(d)P[E^c] + \lambda d \\ &= u(-c)P[F] + u(d)P[E^c] + \lambda d \\ &> u(-c)P[F] + u(D)P[F^c] - \lambda c = U_0(Y). \end{aligned}$$

On the other hand, at time $t = 1$ on $\{R_1 \le a\}$,

$$\begin{aligned} U_1(X) &= u(-C)P[R_2 \le b] + u(d)P[R_2 > b] - \lambda C \\ &= u(-c)P[R_2 \le B] + u(D)P[R_2 > B] - \lambda C \\ &< u(-c)P[R_2 \le B] + u(D)P[R_2 > B] - \lambda c = U_1(Y), \end{aligned}$$

and on $\{R_1 > a\}$,

$$U_1(X) = u(d) + \lambda d < u(D) + \lambda D = U_1(Y).$$

Hence, again at time $t = 0$ the trader prefers the future payoff X to Y even to the extent that if s/he is owning Y, s/he is willing to pay money to swap Y with X. However, s/he knows that in every possible scenario at time 1, s/he will assess the situation differently and prefer Y to X.

Remark 3.4 The same problem as in Example 3.3 occurs for preference functionals of the form

$$U_t(X) = \mathrm{E}[u(X)|\mathcal{F}_t] - \lambda[\mathrm{VaR}_t^\alpha(X) - m]^+. \tag{3.4}$$

Here the trader is not penalized if s/he stays within the risk limit $m \ge 0$ but has to pay a linear penalty if the limit is exceeded.

To see that (3.4) is dynamically inconsistent, consider the same model as in Example 3.3. But choose the constant c so that instead of (3.3) it satisfies

$$c > \left\{ \frac{u(D)P[F^c] - u(d)P[E^c]}{\lambda} + m, m \right\}.$$

Then $(c - m)^+ = c - m$ and $(C - m)^+ = C - m$. So at time $t = 0$, we have

$$\begin{aligned} U_0(X) &= u(-C)P[E] + u(d)P[E^c] - \lambda(-d - m)^+ \\ &= u(-c)P[F] + u(d)P[E^c] \\ &> u(-c)P[F] + u(D)P[F^c] - \lambda(c - m) = U_0(Y). \end{aligned}$$

On the other hand, at time $t = 1$ on $\{R_1 \leq a\}$,

$$\begin{aligned} U_1(X) &= u(-C)P[R_2 \leq b] + u(d)P[R_2 > b] - \lambda(C - m) \\ &= u(-c)P[R_2 \leq B] + u(D)P[R_2 > B] - \lambda(C - m) \\ &< u(-c)P[R_2 \leq B] + u(D)P[R_2 > B] - \lambda(c - m) = U_1(Y)\,, \end{aligned}$$

and on $\{R_1 > a\}$,

$$U_1(X) = u(d) - \lambda(-d - m)^+ = u(d) < u(D) = u(D) - (-D - m)^+ = U_1(Y)\,.$$

Remark 3.5 VaR in a dynamic setup has also been discussed in other papers, like for instance, Basak and Shapiro (2001), Berkelaar et al. (2005), Leippold et al. (2006) and Cuoco et al. (2007). However, in the first two papers, VaR is only measured at time 0. In the other two, it is updated. But at time t, VaR of positions at time $t + \tau$ is measured for fixed $\tau > 0$. Also, in practice VaR is often calculated for a fixed time to maturity (for instance, one day) as opposed to a fixed maturity date. In all these instances the question of time-consistency does not arise.

4 Composed VaR

A possible way to make VaR time-consistent is to compose one-period VaR's over time. Fix $\alpha \in (0, 1)$ and start with $\mathrm{ComVaR}^\alpha_{T-1}(X) := \mathrm{VaR}^\alpha_{T-1}$. For $t \leq T - 2$, define recursively

$$\mathrm{ComVaR}^\alpha_t(X) := \mathrm{VaR}^\alpha_t \left(- \mathrm{ComVaR}^\alpha_{t+1}(X) \right)\,.$$

Then $(\mathrm{ComVaR}^\alpha_t)_{t=0}^{T-1}$ is time-consistent by construction. We emphasize that for $t \leq T - 2$, VaR^α_t and ComVaR^α_t are different risk measures on $L^0(\mathcal{F}_T)$. ComVaR^α_t still has the drawbacks (D1)–(D3). However, if VaR is used to measure the risk of positions which are, for instance, elliptically distributed, then (D1)–(D3) do not pose a problem; see for example, Theorem 6.8 in McNeil et al. (2005).

5 Composed AVaR

If positions are not elliptically distributed, there are situations where the use of a coherent or convex risk measure provides much better risk assessments than VaR, see Artzner et al. (1999) or Föllmer and Schied (2004). A popular static coherent risk measure is AVaR (average value-at-risk). At the level $\alpha \in (0, 1)$, it is defined by

$$\mathrm{AVaR}^\alpha(X) := \frac{1}{\alpha} \int_0^\alpha \mathrm{VaR}^u(X) du\,.$$

It is equal to the conditional value-at-risk introduced in Rockafellar and Uryasev (2000, 2002) and Pflug (2000). For continuously distributed X, it also coincides with tail conditional expectation,

$$\mathrm{TCE}^\alpha(X) := -\mathrm{E}\left[X | X \leq q^\alpha_+(X)\right] = \mathrm{E}\left[-X \mid -X \geq \mathrm{VaR}^\alpha(X)\right]\,.$$

Situations where AVaR behaves better than traditional risk measures are for example discussed in Alexander and Baptista (2004), Agarwal and Naik (2004), Merino and Nyfeler (2004). The dynamic extension of AVaR is given by

$$\mathrm{AVaR}^\alpha_t(X) := \frac{1}{\alpha} \int_0^\alpha \mathrm{VaR}^u_t(X) du\,.$$

It is shown in Artzner et al. (2007) that AVaR is not time-consistent. As for VaR, this can be corrected by composing one-period AVaR's: Define $\mathrm{ComAVaR}^{\alpha}_{T-1}(X) := \mathrm{AVaR}^{\alpha}_{T-1}(X)$ and then recursively,

$$\mathrm{ComAVaR}^{\alpha}_{t}(X) := \mathrm{AVaR}^{\alpha}_{t}\left(-\mathrm{ComAVaR}^{\alpha}_{t+1}(X)\right) \quad \text{for } t \leq T-2\,.$$

$(\mathrm{ComAVaR}^{\alpha}_{t})_{t=0}^{T-1}$ is time-consistent by construction, and it inherits all the coherency properties from AVaR^{α}.

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