Utility maximization under increasing risk aversion in one-period models

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Abstract: It has been shown at different levels of generality that under increasing risk aversion utility indifference sell prices of a contingent claim converge to the super-replication price and the shortfalls of utility maximizing hedging portfolios starting from the super-replication price tend to zero in L^1.

In this paper we give an example of a one-period financial model with bounded prices where utility optimal strategies and terminal wealths stay bounded but do not converge when the risk aversion is going to infinity. Then we give general results on the behavior of utility maximizing strategies and terminal wealths under increasing risk aversion in one-period models. Thereby, the concept of a balanced strategy turns out to play a crucial role.

Keywords: utility maximization; utility indifference price; balanced strategy; super-replication

JEL Classification: C60, G13

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1 Introduction

Consider two financial securities that can be traded at time 0 and $T > 0$. We assume that the price of the first security is always positive and use it as numéraire. The time

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0 price of the second security is a positive constant S_0 while at time T it is worth $S_T = S_0 + \Delta S$, for a random variable ΔS on a probability space $(\Omega, \mathcal{F}, P)$. A portfolio consisting of $\xi \in \mathbb{R}$ shares of the first and $\vartheta \in \mathbb{R}$ shares of the second security has a time 0 value of $v = \xi + \vartheta S_0$ and a time T value of $\xi + \vartheta S_T = v + \vartheta \Delta S$. In addition to the two tradable securities we consider a contingent claim whose time T payoff is given by a random variable B. Throughout the paper we assume that

(M) $\qquad P[\Delta S > 0] > 0 \quad \text{and} \quad P[\Delta S < 0] > 0,$

which guarantees the absence of arbitrage in the market composed of the two tradable securities and the non-emptiness of the set

$$\mathcal{Q} := \left\{ Q \text{ probability measure on } (\Omega, \mathcal{F}) \;\middle|\; \begin{array}{l} Q \text{ equivalent to } P \\ dQ/dP \text{ bounded} \\ \mathrm{E}_Q\left[\Delta S\right] = 0 \\ \mathrm{E}_Q\left[|B|\right] < \infty \end{array} \right\}$$

(see Section 1.2 in [5]). The super-replicating price of B is given by

$$c^*(B) := \inf \left\{ c \in \mathbb{R} : c + \vartheta \Delta S \geq B \text{ for some } \vartheta \in \mathbb{R} \right\} ,$$

where $\inf \emptyset = \infty$ and the inequality, like all equalities and inequalities in this paper, is understood in the P-almost sure sense. It can be shown (see e.g Section 1.3 in [5]) that

$$c^*(B) = \sup_{Q \in \mathcal{Q}} \mathrm{E}_Q\left[B\right] , \tag{1.1}$$

and if $c^*(B) < \infty$, then the set

$$\Theta^* := \left\{ \vartheta \in \mathbb{R} : c^*(B) + \vartheta \Delta S \geq B \right\}$$

is a non-empty, closed subset of $\mathbb{R}$.

However, a financial institution with time T liability B might only be willing to invest an amount $c < c^*(B)$ in a hedging portfolio for B. Then it is not possible to super-replicate B, and the optimal strategy depends on the institution's attitude towards risk. In this paper the optimality criterion for strategies ϑ is given by the expected utility

$$\mathrm{E}\left[U(v + \vartheta \Delta S - B)\right] ,$$

for a twice continuously differentiable utility function $U : \mathbb{R} \to \mathbb{R}$ that satisfies

(U1) $U' > 0$, $U'' < 0$, and $\lim_{x \to -\infty} U'(x) = \infty$

(U2) $U(v + \vartheta \Delta S - B) \in L^1(P)$ for all $\vartheta \in \mathbb{R}$

(U3) $U'(v + \vartheta \Delta S - B) \Delta S \in L^1(P)$ for all $\vartheta \in \mathbb{R}$.

It can easily be checked that for given v and B the conditions (M) and (U1)-(U3) guarantee the existence of a unique $\vartheta^{B,U,v} \in \mathbb{R}$ such that

$$\mathrm{E}\left[U'(v + \vartheta^{B,U,v} \Delta S - B) \Delta S\right] = 0\,, \tag{1.2}$$

and for all $\vartheta \neq \vartheta^{B,U,v}$,

$$\mathrm{E}\left[U(v+\vartheta\Delta S - B)\right] < \mathrm{E}\left[U(v+\vartheta^{B,U,v}\Delta S - B)\right] .$$

If we assume that (U1)-(U3) hold for all $v \in \mathbb{R}$ and also for 0 instead of B, then

$$x \mapsto \mathrm{E}\left[U(x+\vartheta\Delta S - B)\right]$$

is strictly increasing, and there exists a unique $c^{B,U,v} \in \mathbb{R}$ such that

$$\mathrm{E}\left[U(v+\vartheta^{0,U,v}\Delta S)\right] = \mathrm{E}\left[U(v + c^{B,U,v} + \vartheta^{B,U,v'}\Delta S - B)\right] , \tag{1.3}$$

where $v' = v + c^{B,U,v}$. $c^{B,U,v}$ is called utility indifference sell price of B (see e.g [8, 11, 4, 3, 1, 2, 7]), and obviously, $c^{B,U,v} \leq c^*(B)$.

For the exponential utility function $V_\alpha(x) := -\exp(-\alpha x)$ with Arrow-Pratt absolute risk aversion coefficient $-V''_\alpha(x)/V'_\alpha(x) = \alpha > 0$, equation (1.3) reduces to

$$\mathrm{E}\left[V_\alpha(\vartheta^{0,\alpha}\Delta S)\right] = \mathrm{E}\left[V_\alpha(c^{B,\alpha} + \vartheta^{B,\alpha}\Delta S - B)\right] ,$$

and the optimal strategies $\vartheta^{0,\alpha}$, $\vartheta^{B,\alpha}$ and the indifference price $c^{B,\alpha}$ do not depend on the initial wealth v. Moreover,

$$\vartheta^{0,\alpha} = \frac{1}{\alpha}\vartheta^{0,1} \quad \text{and} \quad \mathrm{E}\left[V_\alpha(\vartheta^{0,\alpha}\Delta S)\right] = \mathrm{E}\left[V_1(\vartheta^{0,1}\Delta S)\right] \quad \text{for all } \alpha > 0 .$$

Hence, for $X_{B,\alpha} = c^{B,\alpha} + \vartheta^{B,\alpha}\Delta S - B$, we get from Jensen's inequality,

$$\begin{aligned} & \mathrm{E}\left[X^-_{B,\alpha}\right] \leq \frac{1}{\alpha}\log \mathrm{E}\left[\exp(\alpha X^-_{B,\alpha})\right] \qquad (1.4) \\ \leq \; & \frac{1}{\alpha}\log\left(1+\mathrm{E}\left[\exp(-\alpha X_{B,\alpha})\right]\right) = \frac{1}{\alpha}\log\left(1 - \mathrm{E}\left[V_1(\vartheta^{0,1}\Delta S\right]\right) \to 0 \end{aligned}$$

for $\alpha \to \infty$. This implies that for all $Q \in \mathcal{Q}$,

$$\liminf_{\alpha\to\infty} c^{B,\alpha} - \mathrm{E}_Q\left[B\right] = \liminf_{\alpha\to\infty} \mathrm{E}_Q\left[X_{B,\alpha}\right] \geq 0 ,$$

which, by (1.1), shows that

$$c^{B,\alpha} \to c^*(B) \quad \text{for } \alpha \to \infty . \tag{1.5}$$

Under additional assumptions on S and B, the results (1.4) and (1.5) can also be proved in a continuous-time setup (see e.g [3, 1, 2]). However they do not give insight into the behavior of the optimal strategies $\vartheta^{B,\alpha}$ or terminal wealths $v + \vartheta^{B,\alpha}\Delta S - B$ as the risk aversion tends to infinity.

In this paper we study convergence questions for utility maximizing strategies $\vartheta^{B,U,v}$ under increasing risk aversion in one-period models. In our setup this is equivalent to studying the behavior of the optimal terminal wealths $v + \vartheta^{B,U,v}\Delta S - B$ corresponding to fixed v and B under increasing risk aversion. The structure of the paper is as follows: In Section 2, we give an example of a one-period model with bounded ΔS and B such that the optimal strategies $\vartheta^{B,\alpha}$ corresponding to B and exponential utility $V_\alpha(x) = -\exp(-\alpha x)$ stay bounded but do not converge when the absolute risk aversion α tends to infinity. In Section 3, we study the behavior of utility maximizing strategies and terminal wealths under increasing risk aversion in general one-period models. This naturally leads to the concept of a balanced strategy, which also helps clarifying the structure of the example in Section 2. More on balanced strategies and wealth processes can be found in the Ph.D. thesis [12]. For balanced strategies and wealth processes in multi-period models and connections to the optional decomposition see [6]. [9] and [10] contain results on the convergence of expected utility optimal trading strategies in continuous-time models.

2 Utility maximizing strategies and terminal wealths need not converge when the risk aversion is going to infinity

Let the probability space be of the form

$$\Omega = \bigcup_{n \in \mathbb{Z}\setminus\{0\}} \{\omega_n\} \ ,$$

$\mathcal{F}$ consists of all subsets of Ω, and the probability measure P is given by

$$P[\omega_n] = p_n \quad \text{and} \quad P[\omega_{-n}] = p_{-n}\,, \quad n \geq 1\,,$$

where

$$\begin{aligned} p_1 &:= p_{-1} := a\,, \\ p_n &:= p_{-(n-1)} 3^{3-3\cdot 2^{2n-3}}\,,\ n \geq 2\,, \qquad (2.1) \\ p_{-n} &:= p_n 3^{3-3\cdot 2^{2n-2}}\,,\ n \geq 2\,, \qquad (2.2) \end{aligned}$$

and the constant a is chosen such that $\sum_{n\geq 1} p_n + \sum_{n \geq 1} p_{-n} = 1$. Let S be given by $S_0 := 1$ and $S_1 := S_0 + \Delta S$, where

$$\Delta S(\omega_n) := 3^{-2n+2} \quad \text{and} \quad \Delta S(\omega_{-n}) := -3^{-2n+1}\,, \quad n \geq 1\,,$$

and the contingent claim B by

$$B(\omega_n) := 1 - \Delta S(\omega_n) \quad \text{and} \quad B(\omega_{-n}) := 1 + \Delta S(\omega_{-n}) \quad , n \geq 1\,.$$

This gives for $n \geq 1$,

$$v + \vartheta \Delta S(\omega_n) - B(\omega_n) = v - 1 + (\vartheta + 1)\Delta S(\omega_n) \quad \text{and}$$
$$v + \vartheta \Delta S(\omega_{-n}) - B(\omega_{-n}) = v - 1 + (\vartheta - 1)\Delta S(\omega_{-n})\,.$$

It can easily be seen that the super-replication price $c^*(B)$ is equal to 1, and starting with initial capital $c^*(B)$, all strategies $\vartheta \in [-1, 1]$ super-replicate B.

We consider the exponential utility functions $V_\alpha(x) = -e^{-\alpha x}$ for $\alpha > 0$. Then (1.4) and (1.5) hold true. On the other hand, we will show that as the risk aversion α tends to infinity, the utility maximizing strategies $\vartheta^{B,\alpha}$ and therefore the terminal wealths $v + \vartheta^{B,\alpha}\Delta S - B$ do not converge. Obviously, (U1)-(U3) are satisfied. Therefore, there exists for each fixed α, a unique strategy $\vartheta^{B,\alpha} \in \mathbb{R}$, independent of the initial wealth, that maximizes the function $\vartheta \mapsto \mathrm{E}\left[V_\alpha(\vartheta\Delta S - B)\right]$. By (1.2), it is the solution of the equation

$$\begin{aligned} &\sum_{n\geq 1} p_n \exp\left(-\alpha(3\vartheta^{B,\alpha} + 3)3^{-2n+1}\right) 3^{-2n+2} \\ = &\sum_{n\geq 1} p_{-n} \exp\left(-\alpha(3 - 3\vartheta^{B,\alpha})3^{-2n}\right) 3^{-2n+1}\,. \end{aligned}$$

We denote the left hand side of the above equality by $\mathrm{LHS}(\vartheta^{B,\alpha}, \alpha)$ and the right hand side by $\mathrm{RHS}(\vartheta^{B,\alpha}, \alpha)$. Note that $\mathrm{LHS}(\vartheta, \alpha)$ is decreasing and $\mathrm{RHS}(\vartheta, \alpha)$ increasing in ϑ.

In the following we will construct two sequences $\{\alpha_k\}_{k\geq 1}$ and $\{\beta_k\}_{k\geq 1}$ that converge to infinity such that

$$\mathrm{LHS}(-1/3, \alpha_k) \leq \mathrm{RHS}(-1/3, \alpha_k) \tag{2.3}$$

and

$$\mathrm{LHS}(1/3, \beta_k) \geq \mathrm{RHS}(1/3, \beta_k)\,. \tag{2.4}$$

This implies that $\vartheta^{B,\alpha_k} \leq -1/3$ and $\vartheta^{B,\beta_k} \geq 1/3$ and shows that $\vartheta^{B,\alpha}$ cannot converge as $\alpha \to \infty$.

We set

$$\alpha_k := \frac{1}{2} 3^{2k} \log(2 \cdot 3^{-2+3\cdot 2^{2k-2}}) \quad \text{and} \quad \beta_k := \frac{1}{2} 3^{2k+1} \log(2 \cdot 3^{-2+3\cdot 2^{2k-1}})\,.$$

These two sequences obviously are increasing and tend to ∞ for $k \to \infty$. Note that (2.1) and (2.2) are equivalent to

$$p_n = p_{-(n-1)} \frac{4}{3} \exp(-4\alpha_{n-1} 3^{-2n+2}) \tag{2.5}$$

and

$$p_{-n} = p_n \frac{4}{3} \exp(-4\beta_{n-1} 3^{-2n+1})\,.$$

We first show (2.3):

$$\mathrm{LHS}(-1/3,\alpha_k) \le \sum_{n=1}^{k} p_n \exp\left(-2\alpha_k 3^{-2n+1}\right) 3^{-2n+2} + \sum_{n\ge k+1} p_{k+1} 3^{-2n+2}\,. \tag{2.6}$$

For $n = 1,\dots,k$, we deduce from

$$\frac{1}{6}\exp\left(2\alpha_k 3^{-2n}\right) \ge \frac{1}{6}\exp\left(2\alpha_n 3^{-2n}\right) = \frac{1}{6}2\cdot 3^{-2+3\cdot 2^{2n-2}} = 3^{-3+3\cdot 2^{2n-2}} = \frac{p_n}{p_{-n}}$$

that

$$p_n \exp\left(-2\alpha_k 3^{-2n+1}\right) 3^{-2n+2} \le \frac{1}{2} p_{-n} \exp\left(-4\alpha_k 3^{-2n}\right) 3^{-2n+1}\,.$$

If we plug this into (2.6) and use for the second step (2.5), we get

$$\begin{aligned}
&\mathrm{LHS}(-1/3,\alpha_k) \le \frac{1}{2}\sum_{n=1}^{k} p_{-n}\exp\left(-4\alpha_k 3^{-2n}\right)3^{-2n+1} + p_{k+1}\frac{1}{8}3^{-2k+2}\\
=\ & \frac{1}{2}\sum_{n=1}^{k} p_{-n}\exp\left(-4\alpha_k 3^{-2n}\right)3^{-2n+1} + \frac{1}{2}p_{-k}\exp\left(-4\alpha_k 3^{-2k}\right)3^{-2k+1}\\
\le\ & \sum_{n\ge 1} p_{-n}\exp\left(-4\alpha_k 3^{-2n}\right)3^{-2n+1} = \mathrm{RHS}(-1/3,\alpha_k)\,,
\end{aligned}$$

which proves (2.3).

To see (2.4), note that for $n = 1,\dots,k$, we get from

$$\frac{1}{6}\exp\left(2\beta_k 3^{-2n-1}\right) \ge \frac{1}{6}\exp\left(2\beta_n 3^{-2n-1}\right) = 3^{-3+3\cdot 2^{2n-1}} = \frac{p_{-n}}{p_{n+1}}$$

that

$$p_{-n}\exp\left(-2\beta_k 3^{-2n}\right)3^{-2n+1} \le \frac{1}{2}p_{n+1}\exp\left(-4\beta_k 3^{-2n-1}\right)3^{-2n},.$$

It follows that

$$\begin{aligned}
&\mathrm{RHS}(1/3,\beta_k) \le \sum_{n=1}^{k} p_{-n}\exp\left(-2\beta_k 3^{-2n}\right)3^{-2n+1} + \sum_{n\ge k+1} p_{-(k+1)}3^{-2n+1}\\
=\ & \sum_{n=1}^{k} p_{-n}\exp\left(-2\beta_k 3^{-2n}\right)3^{-2n+1} + p_{-(k+1)}\frac{1}{8}3^{-2k+1}\\
\le\ & \frac{1}{2}\sum_{n=1}^{k} p_{n+1}\exp\left(-4\beta_k 3^{-2n-1}\right)3^{-2n} + \frac{1}{2}p_{k+1}\exp\left(-4\beta_k 3^{-2k-1}\right)3^{-2k}\\
\le\ & \sum_{n\ge 1} p_{n+1}\exp\left(-4\beta_k 3^{-2n-1}\right)3^{-2n}\\
=\ & \sum_{n\ge 2} p_n\exp\left(-4\beta_k 3^{-2n+1}\right)3^{-2n+2} \le \mathrm{LHS}(1/3,\beta_k)\,.
\end{aligned}$$

Although we have just shown that for $\alpha \to \infty$, the optimal strategies $\vartheta^{B,\alpha}$ cannot converge to a single point in $\mathbb{R}$, it follows from Theorem 3.7 below that

$$\operatorname{dist}(\vartheta^{B,\alpha}, [-1,1]) \to 0 \quad \text{as } \alpha \to \infty\,,$$

where for $x \in \mathbb{R}$ and a set $A \subset \mathbb{R}$, $\operatorname{dist}(x, A) := \inf\{|x-y| : y \in A\}$. This implies that

$$\operatorname{dist}(\vartheta^{B,\alpha_k}, [-1,-1/3]) \to 0 \quad \text{and} \quad \operatorname{dist}(\vartheta^{B,\beta_k}, [1/3,1]) \to 0 \quad \text{as } k \to \infty\,.$$

3 General results for one-period models

In this section, ΔS and B are random variables on a general probability space $(\Omega, \mathcal{F}, P)$ such that condition (M) is satisfied. We define the events

$$\begin{aligned} \Omega_+ &:= \{\omega \in \Omega :\ \triangle S(\omega) > 0\}\,, \\ \Omega_0 &:= \{\omega \in \Omega :\ \triangle S(\omega) = 0\}\,, \\ \Omega_- &:= \{\omega \in \Omega :\ \triangle S(\omega) < 0\}\,. \end{aligned}$$

and set $Z(\vartheta) := \vartheta \Delta S - B$. Moreover, we denote

$$z_+(\vartheta) := \operatorname{ess\,inf}_{\omega\in\Omega_+} Z(\vartheta)(\omega) \quad \text{and} \quad z_-(\vartheta) := \operatorname{ess\,inf}_{\omega\in\Omega_-} Z(\vartheta)(\omega)\,.$$

Note that for $\omega \in \Omega_+$, $Z(\vartheta)(\omega)$ is strictly increasing and affine in ϑ, and for $\omega \in \Omega_-$, $Z(\vartheta)(\omega)$ is strictly decreasing and affine in ϑ. Therefore, $z_+(.)$ is a right-continuous, increasing, concave function with $\lim_{\vartheta\to-\infty} z_+(\vartheta) = -\infty$, whereas $z_-(.)$ is a left-continuous, decreasing, concave function with $\lim_{\vartheta\to\infty} z_-(\vartheta) = -\infty$. It might happen that $z_+(\vartheta)$ or $z_-(\vartheta)$ take the value $-\infty$. However, if the probability space $(\Omega, \mathcal{F}, P)$ is finite, then $z_+(\vartheta)$ is a strictly increasing real-valued function, $z_-(\vartheta)$ is a strictly decreasing real-valued function, and there exists a unique $\bar\vartheta \in \mathbb{R}$ such that $z_+(\bar\vartheta) = z_-(\bar\vartheta)$. For general probability spaces we need the following definitions:

Definition 3.1

$$\begin{aligned} \Theta_+ &:= \{\vartheta \in \mathbb{R} :\ P[\{\omega \in \Omega_- :\ z_+(\vartheta) \ge Z(\vartheta)(\omega)\}] > 0\}\,, \\ \Theta_- &:= \{\vartheta \in \mathbb{R} :\ P[\{\omega \in \Omega_+ :\ z_-(\vartheta) \ge Z(\vartheta)(\omega)\}] > 0\}\,. \end{aligned}$$

By the above discussed properties of $z_+(\vartheta)$ and $z_-(\vartheta)$, Θ_+ is of the form (ϑ, ∞) or $[\vartheta, \infty)$ and Θ_+ is of the form $(-\infty, \vartheta)$ or $(-\infty, \vartheta]$, where ϑ might be $\pm\infty$ and $(-\infty,-\infty) := \emptyset =: (\infty, \infty)$. Let

$$A_+ := \{\vartheta :\ z_+(\vartheta) > z_-(\vartheta)\} \text{ and } A_- := \{\vartheta :\ z_+(\vartheta) < z_-(\vartheta)\}\,.$$

If the probability space $(\Omega, \mathcal{F}, P)$ is finite, the sets Θ_+ and Θ_- are equal to closure(A_+) and closure(A_-), respectively. For general $(\Omega, \mathcal{F}, P)$, the following inclusions are valid:

$$A_+ \subset \Theta_+ \subset \operatorname{closure}(A_+) \quad \text{and} \quad A_- \subset \Theta_- \subset \operatorname{closure}(A_-)\,.$$

Definition 3.2 *We call*

$$\Theta^{\mathrm{ba}} := \{\vartheta \in [-\infty, \infty] : \ \vartheta_- \le \vartheta \le \vartheta_+ \text{ for all } \vartheta_- \in \Theta_- \text{ and all } \vartheta_+ \in \Theta_+\}$$

the set of balanced *strategies.*

The set of balanced strategies Θ^{ba} is a non-empty subset of $[-\infty, \infty]$. Indeed, if it were empty, there would exist $\vartheta_+ \in \Theta_+$ and $\vartheta_- \in \Theta_-$ such that $\vartheta_- > \vartheta_+$, implying the existence of a $\vartheta \in A_- \cap A_+ = \emptyset$. Furthermore, Θ^{ba} is a closed interval, possibly equal to $\{-\infty\}$ or $\{\infty\}$. It does not have to be singleton. For instance, in the example of Section 2, Θ^{ba} is equal to $[-1, 1]$. However, if $(\Omega, \mathcal{F}, P)$ is finite, then $\Theta^{\mathrm{ba}} = \{\bar{\vartheta}\}$, where $\bar{\vartheta}$ is the unique real number such that $z_+(\bar{\vartheta}) = z_-(\bar{\vartheta})$. If B is a constant, then $\Theta^{\mathrm{ba}} = \{0\}$. To see this notice that $Z(0) = -B$. Therefore $z_+(0) = -B = z_-(0)$ and $0 \in \Theta_+$, $0 \in \Theta_-$, yielding $\Theta^{\mathrm{ba}} = \{0\}$.

Proposition 3.3 *Assume* (M) *and* $c^*(B) < \infty$. *Then* $\Theta^{\mathrm{ba}} \subset \Theta^*$.

Proof. If $c^*(B) < \infty$, then there exists a $\vartheta^* \in \mathbb{R}$ such that $\vartheta^* \Delta S - B \ge -c^*(B)$. In particular, $z_+(\vartheta^*), z_-(\vartheta^*) > -\infty$, from which it can be deduced that Θ_- and Θ_+ are non-empty. Hence, $\Theta^{\mathrm{ba}} \subset \mathbb{R}$, and to finish the proof it is enough to show that for each $\bar{\vartheta} \in \Theta^{\mathrm{ba}}$, $\operatorname{ess\,inf}_{\omega \in \Omega} Z(\bar{\vartheta})(\omega) \ge \operatorname{ess\,inf}_{\omega \in \Omega} Z(\vartheta)(\omega)$ for all $\vartheta \in \mathbb{R}$. Since $Z(\vartheta)(\omega) = -B(\omega)$ for all $\omega \in \Omega_0$ and $\vartheta \in \mathbb{R}$, it suffices to prove

$$z_+(\bar{\vartheta}) \wedge z_-(\bar{\vartheta}) \ge z_+(\vartheta) \wedge z_-(\vartheta), \quad \text{for all } \vartheta \in \mathbb{R}. \tag{3.1}$$

Let us consider the case $\bar{\vartheta} < \vartheta$. The case $\bar{\vartheta} > \vartheta$ works analogously. Since $\bar{\vartheta}$ is a balanced strategy and $\bar{\vartheta} < \vartheta$, $\vartheta \notin \Theta_-$ and thus $z_+(\vartheta) \ge z_-(\vartheta)$. Therefore, (3.1) simplifies to $z_+(\bar{\vartheta}) \wedge z_-(\bar{\vartheta}) \ge z_-(\vartheta)$, and since $z_-(\cdot)$ is decreasing, it is enough to show that

$$z_+(\bar{\vartheta}) \ge z_-(\vartheta)\,. \tag{3.2}$$

Given any $0 < \varepsilon < \vartheta - \bar{\vartheta}$, we have $\bar{\vartheta} + \varepsilon \notin \Theta_-$, and therefore

$$z_+(\bar{\vartheta} + \varepsilon) \ge z_-(\bar{\vartheta} + \varepsilon) \ge z_-(\vartheta)\,.$$

Thus, (3.2) follows by letting ε go to 0 because $z_+(\cdot)$ is right-continuous. $\square$

Example 3.4 *Assume* $\Omega = \{0, 1, 2, \dots\}$, $\mathcal{F}$ *consists of all subsets of* Ω *and* P *is given by* $P[n] = 2^{-(n+1)}$, $n \ge 0$. *Let* $\Delta S(0) = -1$, $\Delta S(n) = 1$, *for* $n \ge 1$, *and* $B(n) = n$ *for all* $n \ge 0$. *Then,* $c^*(B) = \infty$, *and* $\Theta^{\mathrm{ba}} = \{\infty\}$.

Example 3.5 *Assume* $\Omega = \{\omega_{-1}, \omega_0, \omega_1\}$, $\mathcal{F}$ *consists of all subsets of* Ω *and* P *gives positive mass to each element in* Ω. *Let* $\Delta S = (-1, 0, 1)$ *and* $B = (0, 1, 0)$. *Then* $c^*(B) = 1$, $\Theta^* = [-1, 1]$ *and* $\Theta^{\mathrm{ba}} = \{0\}$.

For a utility function $U : \mathbb{R} \to \mathbb{R}$ that satisfies (U1)-(U3) we set $r_U(x) := -U''(x)/U'(x) > 0$, and denote by $\vartheta^{B,U,v}$ the maximizer of $\mathrm{E}\left[U(v + Z(\vartheta))\right]$.

Lemma 3.6 *Assume* (M) *and fix* $v \in \mathbb{R}$. *Then for each* $\vartheta_+ \in \Theta_+$ *there exists a constant* $\gamma_+ > 0$*, such that for every function* U *that satisfies* (U1)-(U3)*,*

$$\vartheta^{B,U,v} - \vartheta_+ \leq \frac{\gamma_+}{r_+^{U,v}}\,, \quad \textit{where} \quad r_+^{U,v} = \inf\{r_U(x) : x \leq z_+(\vartheta_+) + v\}\,,$$

and for each $\vartheta_- \in \Theta_-$ *there exists a constant* $\gamma_- > 0$*, such that for every function* U *that satisfies* (U1)-(U3)*,*

$$\vartheta_- - \vartheta^{B,U,v} \leq \frac{\gamma_-}{r_-^{U,v}}\,, \quad \textit{where} \quad r_-^{U,v} = \inf\{r_U(x) : x \leq z_-(\vartheta_-) + v\}\,.$$

Proof. The first claim is obviously true if $\vartheta^{B,U,v} - \vartheta_+ \leq 0$. So let us assume $\vartheta^{B,U,v} - \vartheta_+ > 0$. Since $\vartheta_+ \in \Theta_+$, there exists a measurable set $\tilde{\Omega}_- \subset \Omega_-$ and an $\varepsilon > 0$ such that $P\left[\tilde{\Omega}_-\right] > 0$, $1_{\tilde{\Omega}_-}\left[z_+(\vartheta_+) - Z(\vartheta_+)\right] \geq 0$, and $1_{\tilde{\Omega}_-}\left[\Delta S + \varepsilon\right] \leq 0$. Hence,

$$\begin{aligned} 1_{\tilde{\Omega}_-}\left(z_+(\vartheta_+) - Z(\vartheta^{B,U,v})\right) &\geq 1_{\tilde{\Omega}_-}\left(Z(\vartheta_+) - Z(\vartheta^{B,U,v})\right) \\ &\geq 1_{\tilde{\Omega}_-}\varepsilon(\vartheta^{B,U,v} - \vartheta_+)\,. \end{aligned}$$

Note that $Z(\vartheta^{B,U,v})(\omega) \geq Z(\vartheta_+)(\omega) \geq z_+(\vartheta_+)$ for all $\omega \in \Omega_+$ and U' is decreasing. Thus by (1.2), $\vartheta^{B,U,v}$ satisfies

$$\begin{aligned} 0 &= \mathrm{E}\left[U'(v + Z(\vartheta^{B,U,v}))\Delta S\right] \\ &\leq \int_{\Omega_+} U'(v + z_+(\vartheta_+))\Delta S dP + \int_{\tilde{\Omega}_-} U'(v + Z(\vartheta^{B,U,v}))\Delta S dP \\ &= U'(v + z_+(\vartheta_+))\left[\int_{\Omega_+} \Delta S dP + \int_{\tilde{\Omega}_-} \frac{U'(v + Z(\vartheta^{B,U,v}))}{U'(v + z_+(\vartheta_+))}\Delta S dP\right] \end{aligned}$$

Note further that $U' > 0$ and $U'(a)/U'(b) = \exp\left(\int_a^b r_U(x)dx\right)$ for all $a < b$. Therefore,

$$\begin{aligned} \int_{\Omega_+} \Delta S dP &\geq \int_{\tilde{\Omega}_-} \frac{U'(v + Z(\vartheta^{B,U,v}))}{U'(v + z_+(\vartheta_+))}(-\Delta S)dP \\ &\geq \varepsilon \int_{\tilde{\Omega}_-} \exp\left(\int_{v+Z(\vartheta^{B,U,v})}^{v+z_+(\vartheta_+)} r_U(x)dx\right) dP \\ &\geq \varepsilon P\left[\tilde{\Omega}_-\right] \exp\left(\varepsilon(\vartheta^{B,U,v} - \vartheta_+)r_+^{U,v}\right)\,. \end{aligned}$$

This shows that

$$\vartheta^{B,U,v} - \vartheta_+ \leq \frac{1}{\varepsilon}\log\left(\frac{\int_{\Omega_+}\Delta S dP}{\varepsilon P\left[\tilde{\Omega}_-\right]}\right)\frac{1}{r_+^{U,v}}\,,$$

which proves the first claim. The second claim can be shown analogously. $\square$

Theorem 3.7 *Assume* (M) *and let* $(U_\alpha)_{\alpha>0}$ *be a family of utility functions satisfying* (U1)-(U3) *with corresponding risk aversions* $(r_\alpha)_{\alpha>0}$. *Let* ϑ^α *be the optimal strategy for the utility maximization problem*

$$\sup_\vartheta \mathrm{E}\left[U_\alpha(v + Z(\vartheta))\right]$$

and assume that

$$r^*_\alpha := \sup_{x\in\mathbb{R}} r_\alpha(x) \to \infty \quad \text{as } \alpha \to \infty\,.$$

Then the following hold:

a) *If* Θ_+ *and* Θ_- *are of the form*

$$\Theta_- = (-\infty, \vartheta_-] \quad \text{and} \quad \Theta_+ = [\vartheta_+, \infty) \quad \text{for } \vartheta_-, \vartheta_+ \in \mathbb{R}\,, \tag{3.3}$$

then there exists a constant $\gamma > 0$ *such that for all* $\alpha > 0$,

$$\mathrm{dist}(\vartheta^\alpha, \Theta^{\mathrm{ba}}) \leq \frac{\gamma}{r^*_\alpha}\,.$$

b) *If* $\Theta^{\mathrm{ba}} \cap \mathbb{R} \neq \emptyset$, *then*

$$\lim_{\alpha\to\infty} \mathrm{dist}(\vartheta^\alpha, \Theta^{\mathrm{ba}}) = 0\,.$$

c) *If* $\Theta^{\mathrm{ba}} = \{\infty\}$, *then* $\vartheta^\alpha \to \infty$ *as* $\alpha \to \infty$.
If $\Theta^{\mathrm{ba}} = \{-\infty\}$, *then* $\vartheta^\alpha \to -\infty$ *as* $\alpha \to \infty$.

Proof. If (3.3) holds, then $\Theta^{\mathrm{ba}} = [\vartheta_-, \vartheta_+]$, and a) follows directly from Lemma 3.6. To prove b) we let $\varepsilon > 0$. If $\Theta^{\mathrm{ba}} = \mathbb{R}$, there is nothing to prove. If $\sup(\Theta^{\mathrm{ba}}) < \infty$, then $\sup(\Theta^{\mathrm{ba}}) + \varepsilon/2 \in \Theta_+$. Hence, it follows from Lemma 3.6 that there exists a constant $\gamma_+ > 0$ such that $\vartheta^\alpha - (\sup(\Theta^{\mathrm{ba}}) + \varepsilon/2) \leq \gamma_+/r^*_\alpha$, which shows that there exists an $\alpha_+ > 0$ such that $\vartheta^\alpha - \sup(\Theta^{\mathrm{ba}}) \leq \varepsilon$ for all $\alpha \geq \alpha_+$. Analogously, it can be shown that there exists an $\alpha_- > 0$ such that $\inf(\Theta^{\mathrm{ba}}) - \vartheta^\alpha \leq \varepsilon$, for all $\alpha \geq \alpha_-$. This proves b). c) can be proved like b). $\square$

Remark 3.8 *We now are in a position to shed some more light on the structure of the example in Section 2. If only the states* ω_1 *and* ω_{-1} *are taken into account, the unique balanced strategy is* $-1/2$. *For* $\omega_1, \omega_{-1}, \omega_2$ *it is* $1/2$, *for* $\omega_1, \omega_{-1}, \omega_2, \omega_{-2}$ *again* $-1/2$ *and so on. Now, it is possible to choose the probabilities and the sequences* $\{\alpha_k\}_{k\geq1}$ *and* $\{\beta_k\}_{k\geq1}$ *in such a way that every strategy* ϑ^{B,α_k} *is so close to* $-1/2$ *that it is below* $-1/3$ *and every strategy* ϑ^{B,β_k} *is so close to* $1/2$ *that it is above* $1/3$.

Recall that in a one-period model on a finite probability space $(\Omega, \mathcal{F}, P)$ that satisfies (M) there exists a unique real number $\bar\vartheta$ such that $z_+(\bar\vartheta) = z_-(\bar\vartheta)$, $\Theta_+ = [\bar\vartheta, \infty)$, $\Theta_- = (-\infty, \bar\vartheta]$ and $\Theta^{\mathrm{ba}} = \left\{\bar\vartheta\right\}$. Hence, it follows from Theorem 3.7.a that there exists a constant $\gamma > 0$, such that for all $\alpha > 0$, $\left|\vartheta^\alpha - \bar\vartheta\right| \leq \gamma/r^*_\alpha$. The following proposition shows that for a general probability space convergence of the optimal strategies to the set of balanced strategies can be arbitrarily slow. In particular, it is not possible to obtain the result of Theorem 3.7.a under the assumptions of Theorem 3.7.b

Proposition 3.9 *Let $(x_k)_{k\geq 1}$ be a decreasing sequence of real numbers with $\lim_{k\to\infty} x_k = 0$. Then there exist bounded random variables ΔS and B such that* (M) *holds,* $\Theta^{\mathrm{ba}} = \{0\}$*, and for all $k \geq 1$, the optimal strategy $\vartheta^{B,k}$ corresponding to the utility function $V_k(x) = -\exp(-kx)$, satisfies $\vartheta^{B,k} > x_k$.*

Proof. Let $\Omega = \{1, 2, \dots\}$. Let $\mathcal{F}$ consist of all subsets of Ω and define P by

$$P\,[n] = 2^{-n}\,,\; n \geq 1\,.$$

Set

$$\Delta S(1) := 1 \quad \text{and} \quad \Delta S(n) := -1\,,\; n \geq 2\,.$$

To define B we first construct a strictly increasing sequence of natural numbers as follows:

$$\begin{aligned} n_0 &:= 1 \\ n_k &:= \inf\left\{ m \in \mathbb{N} \,:\, m \geq (1 + n_{k-1}) \vee \left(2 + \frac{2kx_k}{\log 2}\right)\right\}\,,\; k \geq 1 \end{aligned}$$

Now, we define B by $B(1) = 1 + 2x_1$ and

$$B(n) := B(1) - \frac{1}{k} - 2x_k\,,\ \text{if } n_{k-1} < n \leq n_k\,.$$

Note that $\{B(n)\}_{n\geq 2}$ is an increasing sequence of positive real numbers with $\lim_{n\to\infty} B(n) = B(1)$. It can easily be checked that $\Theta^{\mathrm{ba}} = \{0\}$ and that for all $k \geq 1$, the function $V_k(x) = -\exp(-kx)$ satisfies (U1)-(U3). By (1.2), for all $k \geq 1$, the k-optimal strategy $\vartheta^{B,k}$ satisfies

$$\sum_{n\geq 1} 2^{-n} \exp\left\{-k\left[\vartheta^{B,k}\Delta S(n) - B(n)\right]\right\} \Delta S(n) = 0\,.$$

Hence,

$$\begin{aligned} \exp\left(-2k\vartheta^{B,k}\right) &= \sum_{n\geq 2} 2^{1-n} \exp\left\{k[B(n) - B(1)]\right\} \\ = & \sum_{n=2}^{n_k} 2^{1-n} \exp\left\{k[B(n) - B(1)]\right\} + \sum_{n\geq n_k+1} 2^{1-n} \exp\left\{k[B(n) - B(1)]\right\} \\ < & \exp\left\{-1 - 2kx_k\right\} + 2^{1-n_k} \leq \exp\left\{-1 - 2kx_k\right\} + \frac{1}{2}\exp\left\{-2kx_k\right\} \\ < & \exp\left(-2kx_k\right)\,, \end{aligned}$$

which shows that $\vartheta^{B,k} > x_k$. □

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