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Regularizing Fractional Brownian Motion with a View towards Stock Price Modelling

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Abstract

There have been several attempts to remedy some of the shortcomings of the Samuelson model for stock price movements using fractional Brownian motion.

In the first part of this thesis we construct arbitrage strategies for two different models based on fractional Brownian motion and show how arbitrage can be ruled out by putting restrictions on the trading strategies. Since these models with the restricted trading strategies are incomplete, it is not clear how to price derivatives within them.

Alternatively, arbitrage can be excluded from fractional Brownian motion models by regularizing the local path behaviour of fractional Brownian motion. We introduce two different ways of regularizing fractional Brownian motion and discuss the pricing of a European call option in regularized fractional Samuelson models.

Contents

Chapter 1

Preliminaries

1.1 Notation

Throughout this thesis $(\Omega, \mathcal{A}, P)$ will be a probability space.

Let $I \subset \mathbb{R}$ be an interval and $(X_t)_{t\in I}$ a stochastic process. We call X continuous, right-continuous or càdlàg (continu à droit, limites à gauche) if all paths have the corresponding property. If almost all paths have the property, we call X a.s. continuous, a.s. right-continuous or a.s. càdlàg. We say X is stochastically right-continuous if for all $t \in I \setminus \{\sup I\}$, $\lim_{s\searrow t} X_s = X_t$ in probability.

By $\mathbb{F}^X$ we denote the filtration generated by X, i.e. $\mathbb{F}^X = \left(\mathcal{F}_t^X\right)_{t\in I}$, where $\mathcal{F}_t^X := \sigma\left(X_s \,:\, s \in I, s \leq t\right), t \in I$.

Let $T \in (0, \infty)$. We say that a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual assumptions if it is right-continuous, $\mathcal{F}_T$ is complete and $\mathcal{F}_0$ contains all null sets of $\mathcal{F}_T$. If $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is an arbitrary filtration, we denote by $\bar{\mathbb{F}} = \left(\bar{\mathcal{F}}_t\right)_{t\in[0,T]}$ the smallest filtration that contains $\mathbb{F}$ and satisfies the usual assumptions.

1.2 Fractional Brownian motion

Definition 1.1 *A fractional Brownian motion with Hurst parameter $H \in (0, 1]$, is a continuous, centred Gaussian process $\left(B_t^H\right)_{t\in\mathbb{R}}$ with*

$$\operatorname{Cov}\left(B_t^H, B_s^H\right) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \; t, s \in \mathbb{R}. \qquad (1.2.1)$$

These processes were first studied by Kolmogorov (1940) within a Hilbert space framework. For $H = 1$, fractional Brownian motion can be constructed as follows:

$$B_t^1 = t\xi \ , \ t \in \mathbb{R} \,, \tag{1.2.2}$$

where ξ is a standard normal random variable. For $H = \frac{1}{2}$, fractional Brownian motion is a two-sided Brownian motion. It can be constructed by taking two independent one-sided Brownian motions $\left(W_t^1\right)_{t\geq 0}$, $\left(W_t^2\right)_{t\geq 0}$ and setting

$$B_t^{\frac{1}{2}} = \begin{cases} W_t^1 & \text{if } t \geq 0 \\ W_{-t}^2 & \text{if } t < 0 \end{cases} .$$

For $H \in (0, 1)$, Mandelbrot and Van Ness (1968) gave the following construction of fractional Brownian motion:

$$B_t^H = c_H \int_{\mathbb{R}} [\varphi_H(t-s) - \varphi_H(-s)]\, dW_s \ , \ t \in \mathbb{R} \,, \tag{1.2.3}$$

where $(W_s)_{s\in\mathbb{R}}$ is a two-sided Brownian motion,

$$\varphi_H(x) = 1_{\{x\geq 0\}} x^{H-\frac{1}{2}} , \ x \in \mathbb{R} \,, \tag{1.2.4}$$

and c_H is a normalizing constant. If $H = \frac{1}{2}$, it is clear that for all $t \in \mathbb{R}$,

$$\int_{\mathbb{R}} [\varphi_H(t-s) - \varphi_H(-s)]\, dW_s = W_t \,.$$

For $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$, the integrals

$$\int_{\mathbb{R}} [\varphi_H(t-s) - \varphi_H(-s)]\, dW_s \ , \ t \in \mathbb{R}$$

can be understood as L^2-limits or almost sure-limits.
In order to define $\int_{\mathbb{R}} [\varphi_H(t-s) - \varphi_H(-s)]\, dW_s$ for every $t \in \mathbb{R}$ in the L^2-sense, we define for step-functions

$$f = \sum_{k=0}^{n-1} a_k 1_{(t_k, t_{k+1}]} \,,$$

where $a_0, \dots, a_{n-1} \in \mathbb{R}$ and $-\infty < t_0 < \dots < t_n < \infty$,

$$L^2\text{-}\int_{\mathbb{R}} f(s) dW_s := \sum_{k=0}^{n-1} a_k \left(W_{t_{k+1}} - W_{t_k}\right) \,.$$

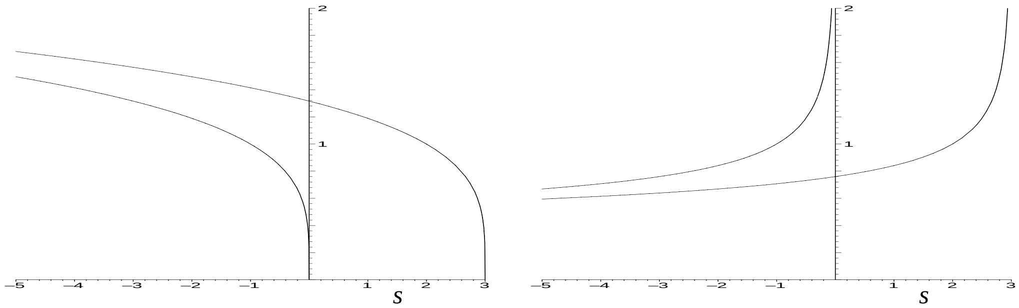

Figure 1.1: *Left: The functions* $\varphi_H(t-s)$ *and* $\varphi_H(-s)$ *for* $H=\frac{3}{4}$ *and* $t=3$. *Right: The functions* $\varphi_H(t-s)$ *and* $\varphi_H(-s)$ *for* $H=\frac{1}{4}$ *and* $t=3$.

Since the step-functions are dense in $L^2(\mathbb{R})$ and

$$\mathrm{E}\left[\left(L^2\text{-}\int_{\mathbb{R}} f(s)dW_s\right)^2\right]=\int_{\mathbb{R}} f^2(s)ds$$

for all step-functions f, L^2-$\int_{\mathbb{R}}$ can be extended continuously to a linear, norm-preserving mapping from $L^2(\mathbb{R})$ to $L^2(\Omega)$.

$$Y_t^H = L^2\text{-}\int_{\mathbb{R}} \left[\varphi_H(t-s)-\varphi_H(-s)\right] dW_s$$

is then for all $t\in\mathbb{R}$, an L^2-limit of linear combinations of random variables from $\{W_t\,:\,t\in\mathbb{R}\}$. Hence, $\left(Y_t^H\right)_{t\in\mathbb{R}}$ is a centred Gaussian process. It is easy to see that it has stationary increments. Furthermore, we obtain for $t\geq 0$,

$$\begin{aligned}\operatorname{Var}\left(Y_t^H\right) &= \int_{-\infty}^0 \left[(t-s)^{H-\frac{1}{2}}-(-s)^{H-\frac{1}{2}}\right]^2 ds + \int_0^t (t-s)^{2H-1}ds\\ &= t^{2H}\left(\int_0^\infty \left[(1+x)^{H-\frac{1}{2}}-x^{H-\frac{1}{2}}\right]^2 dx+\frac{1}{2H}\right).\end{aligned}$$

It follows that for all $t,s\in\mathbb{R}$,

$$\begin{aligned}\operatorname{Cov}\left(Y_t^H,Y_s^H\right) &= \frac{1}{2}\left[\operatorname{Var}\left(Y_t^H\right)+\operatorname{Var}\left(Y_s^H\right)-\operatorname{Var}\left(Y_t^H-Y_s^H\right)\right]\\ &= \frac{1}{2}\left[\operatorname{Var}\left(Y_t^H\right)+\operatorname{Var}\left(Y_s^H\right)-\operatorname{Var}\left(Y_{t-s}^H\right)\right]\\ &= \frac{1}{2}\left[\operatorname{Var}\left(Y_{|t|}^H\right)+\operatorname{Var}\left(Y_{|s|}^H\right)-\operatorname{Var}\left(Y_{|t-s|}^H\right)\right]\end{aligned}$$

$$= \left(\int_0^\infty \left[(1+x)^{H-\frac{1}{2}} - x^{H-\frac{1}{2}} \right]^2 dx + \frac{1}{2H} \right) \frac{1}{2} \left(|t|^2 + |s|^2 - |t-s|^2 \right) .$$

Hence, for

$$c_H = \left(\int_0^\infty \left[(1+x)^{H-\frac{1}{2}} - x^{H-\frac{1}{2}} \right]^2 + \frac{1}{2H} \right)^{-\frac{1}{2}} ,$$

$\left(c_H Y_t^H\right)_{t\in\mathbb{R}}$ is a centred Gaussian process with covariance (1.2.1) (In contrast to this, Mandelbrot and Van Ness (1968) chose to set $c_H = \Gamma(H + \frac{1}{2})^{-1}$). We could now apply the Kolmogorov-Čentsov Theorem (compare Theorem 2.2.8 of Karatzas and Shreve (1988)) to obtain a continuous modification of $\left(c_H Y_t^H\right)_{t\in\mathbb{R}}$. But we can also prove that $\left(c_H Y_t^H\right)_{t\in\mathbb{R}}$ has a continuous modification by showing that almost surely, $\int_{\mathbb{R}} [\varphi_H(t-s) - \varphi_H(-s)]\, dW_s$ can be understood as an improper Riemann-Stieltjes integral for all $t \in \mathbb{R}$. To prove this we need the following lemma, which follows from Theorem 2.21 of Wheeden and Zygmund (1977) and Remark 2 on page 23 of the same book.

Lemma 1.2 *Let $[a, b]$ be a finite interval, $f \in C[a, b]$ and $\phi \in C^1[a, b]$. Then the Riemann-Stieltjes integral* RS-$\int_a^b \phi(s) df(s)$ *exists and equals*

$$-\text{R-}\int_a^b f(s)\phi'(s)ds + f(b)\phi(b) - f(a)\phi(a) ,$$

where R-$\int_a^b f(s)\phi'(s)ds$ *is the Riemann integral.*

Proposition 1.3 *Let $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, and let $(W_t)_{t\in\mathbb{R}}$ be a two-sided Brownian motion. Then there exists a measurable $\tilde{\Omega} \subset \Omega$ with $P[\tilde{\Omega}] = 1$ such that for each $\omega \in \tilde{\Omega}$ the improper Riemann-Stieltjes integral*

$$\text{iRS-}\int_{\mathbb{R}} [\varphi_H(t-s) - \varphi_H(-s)]\, dW_s(\omega)$$

exists for all $t \in \mathbb{R}$ and is continuous in t. If we set

$$Z_t^H(\omega) := \begin{cases} \text{iRS-}\int_{\mathbb{R}} [\varphi_H(t-s) - \varphi_H(-s)]\, dW_s(\omega) & \text{if } \omega \in \tilde{\Omega} \\ 0 & \text{if } \omega \in \tilde{\Omega}^c \end{cases} ,$$

$\left(Z_t^H\right)_{t\in\mathbb{R}}$ is a continuous modification of $\left(Y_t^H\right)_{t\in\mathbb{R}}$. Hence, $\left(c_H Z_t^H\right)_{t\in\mathbb{R}}$ is a fractional Brownian motion.

Proof. It follows from the law of the iterated logarithm (see e.g. Theorem 2.9.24 of Karatzas and Shreve (1988)) that there exists a measurable set $\Omega_0 \subset \Omega$ with $P[\Omega_0] = 1$ such that for all $\omega \in \Omega_0$,

$$\lim_{t \to \infty} \frac{W_{-t}(\omega)}{\sqrt{t \log t}} = 0 \tag{1.2.5}$$

Furthermore, it follows from Theorem 2.9.25 of Karatzas and Shreve (1988) that for all $n \in \mathbb{N}$, there exists a measurable set $\Omega_n \subset \Omega$ with $P[\Omega_n] = 1$ such that for all $\omega \in \Omega_n$ and all $t \in [-n, n]$,

$$\lim_{s \to t} \frac{W_t(\omega) - W_s(\omega)}{\sqrt{|t-s| \log\left(\frac{1}{|t-s|}\right)}} = 0 \tag{1.2.6}$$

We set $\tilde{\Omega} = \bigcap_{n=0}^{\infty} \Omega_n$. It is clear that $P[\tilde{\Omega}] = 1$. We assume $t > 0$. For $t \leq 0$, the proof is analogous. Let us first treat the case $H \in \left(\frac{1}{2}, 1\right)$. It follows from Lemma 1.2 that for each $\omega \in \tilde{\Omega}$ and all $x \in (0, t)$,

$$\text{RS-}\int_0^x \varphi_H(t-s) dW_s(\omega)$$

$$= (H - \frac{1}{2})\text{R-}\int_0^x W_s(\omega)(t-s)^{H-\frac{3}{2}} ds + (t-x)^{H-\frac{1}{2}} W_x(\omega)$$

Since $\lim_{x \nearrow t} (t-x)^{H-\frac{1}{2}} W_x(\omega) = 0$ and the improper Riemann integral

$$\text{iR-}\int_0^t W_s(\omega)(t-s)^{H-\frac{3}{2}} ds = \lim_{x \nearrow t} \text{R-}\int_0^x W_s(\omega)(t-s)^{H-\frac{3}{2}} ds$$

exists, the improper Riemann-Stieltjes integral

$$\text{iRS-}\int_0^t \varphi_H(t-s) dW_s(\omega) = \lim_{x \nearrow t} \text{RS-}\int_0^x \varphi_H(t-s) dW_s(\omega)$$

exists too and equals

$$(H - \frac{1}{2})\text{iR-}\int_0^t W_s(\omega)(t-s)^{H-\frac{3}{2}} ds\,. \tag{1.2.7}$$

To show that (1.2.7) is continuous in t we set for $t > 0$,

$$f_\omega^t(s) = 1_{[0,t]}(s) W_s(\omega)(t-s)^{H-\frac{3}{2}}\,,\; s \in \mathbb{R}\,,$$

and observe that for all $T > 0$, the family $\left(f_\omega^t\right)_{t\in(0,T)}$ is uniformly integrable with respect to Lebesgue measure. Therefore, the t-continuity of (1.2.7) follows from a generalized version of Lebesgue's Dominated Convergence Theorem (see e.g. Theorem II.6.4.b of Shiryaev (1984)). Lemma 1.2 implies that for all $x > 0$,

$$\begin{aligned}
&\text{RS-}\int_{-x}^{-\frac{1}{x}} [\varphi_H(t-s) - \varphi_H(-s)]\, dW_s(\omega) \\
&= (H - \frac{1}{2})\text{R-}\int_{-x}^{-\frac{1}{x}} W_s(\omega)\left[(t-s)^{H-\frac{3}{2}} - (-s)^{H-\frac{3}{2}}\right] ds \\
&+ \left[(t+\frac{1}{x})^{H-\frac{1}{2}} - (\frac{1}{x})^{H-\frac{1}{2}}\right] W_{-\frac{1}{x}}(\omega) - \left[(t+x)^{H-\frac{1}{2}} - x^{H-\frac{1}{2}}\right] W_{-x}(\omega)
\end{aligned}$$

It follows from (1.2.6) that

$$\lim_{x\to\infty} \left[(t+\frac{1}{x})^{H-\frac{1}{2}} - (\frac{1}{x})^{H-\frac{1}{2}}\right] W_{-\frac{1}{x}}(\omega) = 0\,.$$

Moreover, for all $x > 0$,

$$\left|(t+x)^{H-\frac{1}{2}} - x^{H-\frac{1}{2}}\right| \le t(H-\frac{1}{2})x^{H-\frac{3}{2}}\,.$$

This together with (1.2.5) implies that

$$\lim_{x\to\infty} \left[(t+x)^{H-\frac{1}{2}} - x^{H-\frac{1}{2}}\right] W_{-x}(\omega) = 0\,.$$

Furthermore, it follows from (1.2.5) that the improper Riemann integral

$$\begin{aligned}
&\text{iR-}\int_{-\infty}^{0} W_s(\omega)\left[(t-s)^{H-\frac{3}{2}} - (-s)^{H-\frac{1}{2}}\right] ds \\
&= \lim_{x\to\infty} \text{R-}\int_{-x}^{-\frac{1}{x}} W_s(\omega)\left[(t-s)^{H-\frac{3}{2}} - (-s)^{H-\frac{3}{2}}\right] ds
\end{aligned}$$

exists. Hence, the improper Riemann-Stieltjes integral

$$\begin{aligned}
&\text{iRS-}\int_{-\infty}^{0} [\varphi_H(t-s) - \varphi_H(-s)]\, dW_s(\omega) \\
&= \lim_{x\to\infty} \text{RS-}\int_{-x}^{-\frac{1}{x}} [\varphi_H(t-s) - \varphi_H(-s)]\, dW_s(\omega)
\end{aligned}$$

exists too and equals

$$(H-\frac{1}{2})\text{iR-}\int_{-\infty}^{0} W_s(\omega)\left[(t-s)^{H-\frac{3}{2}}-(-s)^{H-\frac{3}{2}}\right]ds\,,$$

which is continuous in t by Lebegue's Dominated Convergence Theorem.

$$\text{iRS-}\int_{-\infty}^{t} [\varphi_H(t-s)-\varphi_H(-s)]\,dW_s(\omega)$$

can now be defined as

$$\text{iRS-}\int_{-\infty}^{0} [\varphi_H(t-s)-\varphi_H(-s)]\,dW_s(\omega)+\text{iRS-}\int_{0}^{t} \varphi_H(t-s)dW_s(\omega)\,.$$

It is continuous in t because

$$\text{iRS-}\int_{0}^{t} \varphi_H(t-s)dW_s(\omega) \quad\text{and}\quad \text{iRS-}\int_{-\infty}^{0} [\varphi_H(t-s)-\varphi_H(-s)]\,dW_s(\omega)$$

are.
Also for $H\in\left(0,\frac{1}{2}\right)$, we define

$$\text{iRS-}\int_{-\infty}^{t} [\varphi_H(t-s)-\varphi_H(-s)]\,dW_s(\omega)$$

as

$$\text{iRS-}\int_{-\infty}^{0} [\varphi_H(t-s)-\varphi_H(-s)]\,dW_s(\omega)+\text{iRS-}\int_{0}^{t} \varphi_H(t-s)dW_s(\omega)\,.$$

It can be deduced from Lemma 1.2 that for each $\omega\in\tilde{\Omega}$ and all $x\in(0,t)$,

$$\begin{aligned}
&\text{RS-}\int_{0}^{x} \varphi_H(t-s)dW_s(\omega)=\text{RS-}\int_{0}^{x}(t-s)^{H-\frac{1}{2}}d(W_s(\omega)-W_t(\omega))\\
=\quad &(H-\frac{1}{2})\text{R-}\int_{0}^{x}(W_s(\omega)-W_t(\omega))(t-s)^{H-\frac{3}{2}}ds\\
&+(t-x)^{H-\frac{1}{2}}(W_x(\omega)-W_t(\omega))+t^{H-\frac{1}{2}}W_t(\omega)\,.
\end{aligned}$$

It follows from (1.2.6) that

$$(t-x)^{H-\frac{1}{2}}(W_x(\omega)-W_t(\omega))\stackrel{(x\nearrow t)}{\longrightarrow}0\,,$$

and that the improper Riemann integral

$$\begin{aligned}
&\text{iR-}\int_0^t (W_s(\omega) - W_t(\omega))(t-s)^{H-\frac{3}{2}} ds \\
= \quad & \lim_{x \nearrow t} \text{R-}\int_0^x (W_s(\omega) - W_t(\omega))(t-s)^{H-\frac{3}{2}} ds
\end{aligned}$$

exists. Therefore the improper Riemann-Stieltjes integral

$$\text{iRS-}\int_0^t \varphi_H(t-s) dW_s(\omega) = \lim_{x \nearrow t} \text{RS-}\int_0^x \varphi_H(t-s) dW_s(\omega)$$

exists too and equals

$$(H - \frac{1}{2})\text{iR-}\int_0^t (W_s(\omega) - W_t(\omega))(t-s)^{H-\frac{3}{2}} ds + t^{H-\frac{1}{2}} W_t(\omega)\,,$$

That $t^{H-\frac{1}{2}} W_t(\omega)$ is continuous in t is clear. The t-continuity of

$$\text{iR-}\int_0^t (W_s(\omega) - W_t(\omega))(t-s)^{H-\frac{3}{2}} ds$$

can as before be derived from Theorem II.6.4.b of Shiryaev (1984). As in the case $H \in (\frac{1}{2}, 1)$, we have for all $x > 0$,

$$\begin{aligned}
&\text{RS-}\int_{-x}^{-\frac{1}{x}} [\varphi_H(t-s) - \varphi_H(-s)]\, dW_s(\omega) \\
= \quad & (H - \frac{1}{2})\text{R-}\int_{-x}^{-\frac{1}{x}} W_s(\omega) \left[(t-s)^{H-\frac{3}{2}} - (-s)^{H-\frac{3}{2}}\right] ds \\
&+ \left[(t + \frac{1}{x})^{H-\frac{1}{2}} - (\frac{1}{x})^{H-\frac{1}{2}}\right] W_{-\frac{1}{x}}(\omega) - \left[(t+x)^{H-\frac{1}{2}} - x^{H-\frac{1}{2}}\right] W_{-x}(\omega)
\end{aligned}$$

As before,

$$\lim_{x \to \infty} \left[(t + \frac{1}{x})^{H-\frac{1}{2}} - (\frac{1}{x})^{H-\frac{1}{2}}\right] W_{-\frac{1}{x}}(\omega) = 0\,,$$

$$\lim_{x \to \infty} \left[(t+x)^{H-\frac{1}{2}} - x^{H-\frac{1}{2}}\right] W_{-x}(\omega) = 0$$

and the improper Riemann integral

$$\text{iR-}\int_{-\infty}^0 W_s(\omega) \left[(t-s)^{H-\frac{3}{2}} - (-s)^{H-\frac{3}{2}}\right] ds$$

$$= \lim_{x\to\infty} \text{R-}\int_{-x}^{-\frac{1}{x}} W_s(\omega)\left[(t-s)^{H-\frac{3}{2}} - (-s)^{H-\frac{3}{2}}\right] ds$$

exists. Hence, the improper Riemann-Stieltjes integral

$$\begin{aligned} &\text{iRS-}\int_{-\infty}^{0} [\varphi_H(t-s) - \varphi_H(-s)]\, dW_s(\omega) \\ = \quad &\lim_{x\to\infty} \text{RS-}\int_{-x}^{-\frac{1}{x}} [\varphi_H(t-s) - \varphi_H(-s)]\, dW_s(\omega) \end{aligned}$$

exists too and equals

$$(H - \frac{1}{2})\text{iR-}\int_{-\infty}^{0} W_s(\omega)\left[(t-s)^{H-\frac{3}{2}} - (-s)^{H-\frac{3}{2}}\right] ds\,,$$

which is continuous in t by Lebegue's Dominated Convergence Theorem. To show that $\left(Z_t^H\right)_{t\in\mathbb{R}}$ is a modification of $\left(Y_t^H\right)_{t\in\mathbb{R}}$ we set for all $t \in \mathbb{R}$,

$$f^{H,t}(s) = \varphi_H(t-s) - \varphi_H(-s)\,,\ s \in \mathbb{R}\,,$$

and for all $n \in \mathbb{N}$ and $s \in \mathbb{R}$,

$$f_n^{H,t}(s) = \sum_{k=-n^2}^{n^2} f^{H,t}\left(\frac{k+\frac{1}{2}}{n}\right) 1_{(\frac{k}{n},\frac{k+1}{n}]}(s)\,.$$

Since $\lim_{n\to\infty} f_n^{H,t} = f^{H,t}$ in L^2, $Y_t^H = L^2\text{-}\lim_{n\to\infty}\int_{\mathbb{R}} f_n^{H,t}(s)dW_s$. At the same time

$$Z_t^H(\omega) = \lim_{n\to\infty}\int_{\mathbb{R}} f_n^{H,t}(s)dW_s(\omega)$$

for all $\omega \in \tilde{\Omega}$. Hence, for all $t \in \mathbb{R}$, Z_t^H is measurable and $Z_t^H = Y_t^H$ almost surely. □

It can be deduced from (1.2.1) that fractional Brownian motions divide into three different families. $B^{\frac{1}{2}}$ has independent increments. For $H \in (\frac{1}{2}, 1]$, the covariance between two increments over non-overlapping time-intervals is positive, for $H \in (0, \frac{1}{2})$ it is negative.

From the representations (1.2.2) and (1.2.3) it can be seen that fractional Brownian motion has stationary increments. Furthermore, it can easily be checked that B^H is stochastically self-similar with self-similarity parameter H, i.e. for all $a > 0$,

$$\left(a^H B_{\frac{t}{a}}^H\right)_{t\in\mathbb{R}} \quad \text{has the same distribution as} \quad \left(B_t^H\right)_{t\in\mathbb{R}}\,.$$

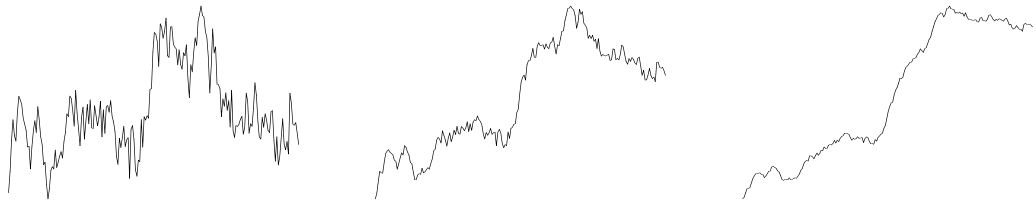

Figure 1.2: *Simulation of a typical path of fractional Brownian motion for H= 0.1, H=0.5 and H=0.8*

Let $(X_t)_{t\geq 0}$ be a stochastic process with stationary increments. We say that the increments of X exhibit long-range dependence if for all $h > 0$,

$$\sum_{n=1}^{\infty} \left|\mathrm{Cov}\left(X_h - X_0, X_{nh} - X_{(n-1)h}\right)\right| = \infty\,.$$

It can be derived from (1.2.1) that for $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ and fixed $h > 0$,

$$\lim_{t\to\infty} \frac{\mathrm{Cov}\left(B_h^H, B_{t+h}^H - B_t^H\right)}{t^{2(H-1)}} = H(2H-1)h^2\,.$$

This implies that the increments of $\left(B_t^H\right)_{t\geq 0}$ exhibit long-range dependence if and only if $H \in (\frac{1}{2}, 1)$.

In the following lemma we collect some facts about fractional Brownian motion that we will need throughout the thesis. They are already well-known.

Lemma 1.4 *Let B^H be a fractional Brownian motion for some $H \in (0, 1]$, and $T, p, q > 0$. Then:*

a) *For all* $\gamma < H$ *there exist a constant* δ *and an almost everywhere positive random variable* ξ *such that*
$P\left[\omega : \sup_{\substack{t,u\in[0,T];\\ 0<t-u<\xi(\omega)}} \frac{\left|B_t^H(\omega)-B_u^H(\omega)\right|}{(t-u)^\gamma} \leq \delta\right] = 1$

b) $n^{pH-1}\sum_{j=0}^{n-1}\left|B^H_{\frac{(j+1)}{n}T} - B^H_{\frac{j}{n}T}\right|^p \overset{(n\to\infty)}{\longrightarrow} \mathrm{E}\left[\left|B_T^H\right|^p\right]$ *in* L^1

c) $n^{pH-1-q}\sum_{j=0}^{n-1}\left|B^H_{\frac{(j+1)}{n}T} - B^H_{\frac{j}{n}T}\right|^p \overset{(n\to\infty)}{\longrightarrow} 0$ *in* L^1

d) $n^{pH-1+q}\sum_{j=0}^{n-1}\left|B^H_{\frac{(j+1)}{n}T} - B^H_{\frac{j}{n}T}\right|^p \overset{(n\to\infty)}{\longrightarrow} \infty$ *in probability,*
i.e. for all $L > 0$*, there exists an* n_0 *such that for all* $n \geq n_0$,
$P\left[n^{pH-1+q}\sum_{j=0}^{n-1}\left|B^H_{\frac{(j+1)}{n}T} - B^H_{\frac{j}{n}T}\right|^p < L\right] < \frac{1}{L}$

Proof. a) follows from the Kolmogorov-Čentsov Theorem (see e.g. Theorem 2.2.8 of Karatzas and Shreve (1988)).
To prove b) we recall that the sequence $\left(B^H_{(j+1)T} - B^H_{jT}\right)_{j=0}^{\infty}$ is stationary. Since it is Gaussian and

$$\mathrm{Cov}\left(B_T^H - B_0^H, B^H_{(j+1)T} - B^H_{jT}\right) \overset{(j\to\infty)}{\longrightarrow} 0\,,$$

it is also mixing. Hence, the Ergodic theorem (see e.g. Theorem V.3.3 of Shiryaev (1984)) implies

$$\frac{1}{n}\sum_{j=0}^{n-1}\left|B^H_{(j+1)T} - B^H_{jT})\right|^p \overset{(n\to\infty)}{\longrightarrow} \mathrm{E}\left[\left|B_T^H\right|^p\right] \text{ in } L^1\,. \tag{1.2.8}$$

On the other hand, it follows from the self-similarity of B^H that for all n,

$$n^{pH-1}\sum_{j=0}^{n-1}\left|B^H_{\frac{(j+1)}{n}T} - B^H_{\frac{j}{n}T}\right|^p$$

has the same distribution as

$$\frac{1}{n}\sum_{j=0}^{n-1}\left|B^H_{(j+1)T} - B^H_{jT})\right|^p\,.$$

This together with (1.2.8) implies b).
c) follows immediately from b).

To prove d) we choose $L > 0$. It follows from b) that there exists an $n_1 \in \mathbb{N}$ such that

$$P\left[\left|\mathrm{E}\left[\left|B_T^H\right|^p\right] - n^{pH-1}\sum_{j=0}^{n-1}\left|B_{\frac{(j+1)}{n}T}^H - B_{\frac{j}{n}T}^H\right|^p\right| > \frac{1}{2}\mathrm{E}\left[\left|B_T^H\right|^p\right]\right] < \frac{1}{L}$$

for all $n \geq n_1$. This implies that for all $n \geq n_1$,

$$P\left[n^{pH-1}\sum_{j=0}^{n-1}\left|B_{\frac{(j+1)}{n}T}^H - B_{\frac{j}{n}T}^H\right|^p < \frac{1}{2}\mathrm{E}\left[\left|B_T^H\right|^p\right]\right] < \frac{1}{L}$$

or, equivalently,

$$P\left[n^{pH-1+q}\sum_{j=0}^{n-1}\left|B_{\frac{(j+1)}{n}T}^H - B_{\frac{j}{n}T}^H\right|^p < n^q\frac{1}{2}\mathrm{E}\left[\left|B_T^H\right|^p\right]\right] < \frac{1}{L}\,.$$

This shows that there exists an $n_0 \in \mathbb{N}$ such that

$$P\left[n^{pH-1+q}\sum_{j=0}^{n-1}\left|B_{\frac{(j+1)}{n}T}^H - B_{\frac{j}{n}T}^H\right|^p < L\right] < \frac{1}{L} \quad \text{for all} \quad n \geq n_0\,,$$

and d) is proved. □

1.3 Weak semimartingales

The classical notion of a semimartingale stands at the end of a chain of generalizations of Brownian motion, each of which extended the class of stochastic processes that can play the role of the integrator in stochastic integration in the Itô-sense (see Itô (1944) for Itô's construction of the stochastic integral). It reached its final form in Doléans-Dade and Meyer (1970). In their paper a stochastic process (X_t) that is adapted to a filtration $\mathbb{F} = (\mathcal{F}_t)$ satisfying the usual assumptions is called an $\mathbb{F}$-semimartingale if it admits a decomposition of the form

$$X_t = X_0 + M_t + A_t\,, \tag{1.3.1}$$

where X_0 is an $\mathcal{F}_0$-measurable random variable, $M_0 = A_0 = 0$, M is an a.s. right-continuous local martingale with respect to $\mathbb{F}$ and A an a.s. right-continuous, $\mathbb{F}$-adapted finite variation process. Later it was found that if for

$T \in (0,\infty)$, a filtration $\mathbb{F} = (\mathcal{F})_{t\in[0,T]}$ satisfies the usual assumptions, an a.s. right-continuous, $\mathbb{F}$-adapted stochastic process $(X_t)_{t\in[0,T]}$ is of the form (1.3.1) if and only if X fulfils the following condition:

$$I_X(\beta(\mathbb{F})) \text{ is bounded in } L^0\,, \tag{1.3.2}$$

where

$$\beta(\mathbb{F}) = \Bigg\{ \sum_{j=0}^{n-1} g_j 1_{(t_j,t_{j+1}]} \,:\, n \in \mathbb{N},\ 0 \le t_0 < \cdots < t_n \le T,$$

$$\forall j,\ g_j \text{ is } \mathcal{F}_{t_j}\text{-measurable and } \left|g_j\right| \le 1 \text{ a.s. } \Bigg\} \tag{1.3.3}$$

and

$$I_X(\vartheta) = \sum_{j=0}^{n-1} g_j \left(X_{t_{j+1}} - X_{t_j}\right) \text{ for } \vartheta = \sum_{j=0}^{n-1} g_j 1_{(t_j,t_{j+1}]} \in \beta(\mathbb{F})\,.$$

This result is usually referred to as the Bichteler-Dellacherie theorem (see e.g. Section VIII.4 of Dellacherie and Meyer (1980) for a proof). For our purposes it is more convenient to work with condition (1.3.2) than with the decomposition property (1.3.1). If one does not require the process to be a.s. right-continuous and the filtration to satisfy the usual assumptions, one obtains a weaker form of the semimartingale property than the classical one.

Definition 1.5 *A stochastic process $(X_t)_{t\in[0,T]}$ is a weak semimartingale with respect to a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ if X is $\mathbb{F}$-adapted and satisfies (1.3.2).*

Let $(X_t)_{t\in[0,T]}$ be a stochastic process. If $\mathbb{F}^1 = \left(\mathcal{F}_t^1\right)_{t\in[0,T]}$ and $\mathbb{F}^2 = \left(\mathcal{F}_t^2\right)_{t\in[0,T]}$ are two filtrations with $\mathcal{F}_t^1 \subset \mathcal{F}_t^2$ for all $t \in [0,T]$, then $\beta\left(\mathbb{F}^1\right) \subset \beta\left(\mathbb{F}^2\right)$. Hence, L^0-boundedness of $I_X\left(\beta\left(\mathbb{F}^2\right)\right)$ implies L^0-boundedness of $I_X\left(\beta\left(\mathbb{F}^1\right)\right)$. This shows that if X is not a weak semimartingale with respect to the filtration generated by X, then it is not a weak semimartingale with respect to any other filtration. Therefore it is natural to introduce the following definition.

Definition 1.6 *Let $(X_t)_{t\in[0,T]}$ be a stochastic process. We call X a weak semimartingale if it is a weak semimartingale with respect to $\mathbb{F}^X$. We call X a semimartingale if it is a semimartingale with respect to $\bar{\mathbb{F}}^X$.*

Example 1.7 It is easy to see that the deterministic process

$$X_t = \begin{cases} 0 & \text{for } t \in [0,1] \\ 1 & \text{for } t \in (1,2] \end{cases} ,$$

is a weak semimartingale. But it is not a semimartingale because it is not a.s. right-continuous.

However, the following proposition shows that every a.s right-continuous $\mathbb{F}$-weak semimartingale is also an $\bar{\mathbb{F}}$-semimartingale.

Proposition 1.8 *Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be a filtration. Then every stochastically right-continuous $\mathbb{F}$-weak semimartingale is also an $\bar{\mathbb{F}}$-weak semimartingale. In particular, if X is a.s. right-continuous, it is an $\bar{\mathbb{F}}$-semimartingale.*

Proof. Define $\mathbb{F}^0 = \left(\mathcal{F}_t^0\right)_{t\in[0,T]}$ as follows: Let $\mathcal{F}_T^0$ be the completion of $\mathcal{F}_T$, $\mathcal{N}$ the null sets of $\mathcal{F}_T^0$ and set

$$\mathcal{F}_t^0 = \sigma\left(\mathcal{F}_t \cup \mathcal{N}\right),\ t \in [0,T]\,.$$

Let $t \in [0,T]$ and $g \in L^0\left(\mathcal{F}_t^0\right)$ such that $|g| \le 1$ almost surely. We set

$$A = \{g > \mathrm{E}\left[g|\mathcal{F}_t\right]\} \quad \text{and} \quad B = \{g < \mathrm{E}\left[g|\mathcal{F}_t\right]\}\ .$$

Since

$$\mathcal{F}_t^0 = \{G \subset \Omega\ :\ \exists F \in \mathcal{F}_t \text{ such that } G \triangle F \in \mathcal{N}\}\ ,$$

there exist $\tilde{A}, \tilde{B} \in \mathcal{F}_t$ with $A\triangle\tilde{A}, B\triangle\tilde{B} \in \mathcal{N}$. The equalities

$$\int_A g - \mathrm{E}\left[g|\mathcal{F}_t\right] dP = \int_{\tilde{A}} g - \mathrm{E}\left[g|\mathcal{F}_t\right] dP = 0$$

and

$$\int_B g - \mathrm{E}\left[g|\mathcal{F}_t\right] dP = \int_{\tilde{B}} g - \mathrm{E}\left[g|\mathcal{F}_t\right] dP = 0$$

imply $P[A] = P[B] = 0$. Hence,

$$g = \mathrm{E}\left[g|\mathcal{F}_t\right] \quad \text{almost surely.} \tag{1.3.4}$$

Let $(X_t)_{t\in[0,T]}$ be an $\mathbb{F}$-weak semimartingale. It follows from (1.3.4) that for every $\vartheta \in \beta\left(\mathbb{F}^0\right)$ there exists a $\tilde{\vartheta} \in \beta(\mathbb{F})$ with $I_X(\tilde{\vartheta}) = I_X(\vartheta)$ almost surely. Therefore,

$$I_X\left(\beta\left(\mathbb{F}\right)\right) = I_X\left(\beta\left(\mathbb{F}^0\right)\right) \quad \text{in } L^0\,.$$

This shows that X is also an $\mathbb{F}^0$-weak semimartingale.
Now let

$$\gamma = \sum_{j=0}^{n-1} g_j 1_{(t_j, t_{j+1}]} \in \beta\left(\bar{\mathbb{F}}\right) .$$

For all $t \in [0, T]$,

$$\bar{\mathcal{F}}_t = \bigcap_{s>t} \mathcal{F}^0_{s \wedge T} .$$

Therefore,

$$\gamma^\varepsilon = \sum_{j=0}^{n-1} g_j 1_{(t_j+\varepsilon, t_{j+1}]} \quad \text{is in} \quad \beta\left(\mathbb{F}^0\right) \tag{1.3.5}$$

for all ε with $0 < \varepsilon < \min_j \left(t_{j+1} - t_j\right)$. If $(X_t)_{t\in[0,T]}$ is stochastically right-continuous, then

$$\lim_{\varepsilon \searrow 0} I_X\left(\gamma^\varepsilon\right) = I_X\left(\gamma\right) \quad \text{in probability.}$$

This, together with (1.3.5) and the fact that $I_X\left(\beta\left(\mathbb{F}^0\right)\right)$ is bounded in L^0, implies that $I_X\left(\beta\left(\bar{\mathbb{F}}\right)\right)$ is also bounded in L^0, and therefore X is an $\bar{\mathbb{F}}$-weak semimartingale. □

It follows from Lemma 1.4 d) that for $H \in \left(0, \frac{1}{2}\right)$, B^H has infinite quadratic variation. The next proposition shows that this implies that B^H cannot be a weak semimartingale if $H \in \left(0, \frac{1}{2}\right)$.

Proposition 1.9 *Let $(X_t)_{t\in[0,T]}$ be an a.s. càdlàg process and denote by τ the set of all finite partitions*

$$0 = t_0 < t_1 < \cdots < t_n = T \ , \ n \in \mathbb{N} \, ,$$

of $[0, T]$. *If*

$$\left\{ \sum_{j=0}^{n-1} \left(X_{t_{j+1}} - X_{t_j}\right)^2 \ : \ (t_0, t_1, \ldots, t_n) \in \tau \right\}$$

is unbounded in L^0, then X is not a weak semimartingale.

Proof. To simplify calculations we define $Y_t = X_t - X_0$, $t \in [0, T]$. Then $(Y_t)_{t\in[0,T]}$ is an $\mathbb{F}^X$-adapted, a.s. càdlàg process with $Y_0 = 0$. It is clear that $I_Y = I_X$ and

$$\sum_{j=0}^{n-1} \left(Y_{t_{j+1}} - Y_{t_j}\right)^2 = \sum_{j=0}^{n-1} \left(X_{t_{j+1}} - X_{t_j}\right)^2$$

for all partitions

$$(t_0, t_1, \dots, t_n) \in \tau\,.$$

To prove the lemma we must show that $I_Y\left(\beta\left(\mathbb{F}^X\right)\right)$ is unbounded in L^0. The key ingredient in our derivation of this from the L^0-unboundedness of

$$\left\{\sum_{j=0}^{n-1} \left(Y_{t_{j+1}} - Y_{t_j}\right)^2 \,:\, (t_0, t_1, \dots, t_n) \in \tau\right\}$$

is the equality

$$\sum_{j=0}^{n-1} \left(Y_{t_{j+1}} - Y_{t_j}\right)^2 = Y_T^2 - 2\sum_{j=1}^{n-1} Y_{t_j}\left(Y_{t_{j+1}} - Y_{t_j}\right)\,, \tag{1.3.6}$$

which holds for all partitions

$$(t_0, t_1, \dots, t_n) \in \tau\,.$$

That

$$\left\{\sum_{j=0}^{n-1} \left(Y_{t_{j+1}} - Y_{t_j}\right)^2 \,:\, (t_0, t_1, \dots, t_n) \in \tau\right\}$$

is unbounded in L^0 means that

$$c := \lim_{L\to\infty} \sup_{\tau} P\left[\sum_{j=0}^{n-1} \left(Y_{t_{j+1}} - Y_{t_j}\right)^2 > L\right] > 0\,. \tag{1.3.7}$$

We will deduce from this that

$$\lim_{L\to\infty} \sup_{\vartheta\in\beta\left(\mathbb{F}^X\right)} P\left[|I_X(\vartheta)| > L\right] \geq \frac{c}{4}\,, \tag{1.3.8}$$

which implies L^0-unboundedness of $I_Y\left(\beta\left(\mathbb{F}^X\right)\right)$. To do this we choose $L > 0$. Since Y is a.s. càdlàg, $\sup_{t\in[0,T]}|Y_t| < \infty$ almost surely. Therefore there exists an $N > 0$ such that

$$P\left[\sup_{t\in[0,T]}|Y_t| > N\right] < \frac{c}{4}. \tag{1.3.9}$$

(1.3.7) implies that there exists a partition

$$(t_0, t_1, \ldots, t_n) \in \tau$$

with

$$P\left[\sum_{j=0}^{n-1}\left(Y_{t_{j+1}} - Y_{t_j}\right)^2 > 2LN + N^2\right] > \frac{c}{2}. \tag{1.3.10}$$

It follows from (1.3.9) and (1.3.10) that

$$P\left[\left\{\sup_{t\in[0,T]}|Y_t| > N\right\} \cup \left\{\sum_{j=0}^{n-1}\left(Y_{t_{j+1}} - Y_{t_j}\right)^2 \le 2LN + N^2\right\}\right]$$

$$\le P\left[\sup_{t\in[0,T]}|Y_t| > N\right] + P\left[\sum_{j=0}^{n-1}\left(Y_{t_{j+1}} - Y_{t_j}\right)^2 \le 2LN + N^2\right] < 1 - \frac{c}{4}.$$

Hence,

$$P\left[\left\{\sup_{t\in[0,T]}|Y_t| \le N\right\} \cap \left\{\sum_{j=0}^{n-1}\left(Y_{t_{j+1}} - Y_{t_j}\right)^2 > 2LN + N^2\right\}\right] > \frac{c}{4}. \tag{1.3.11}$$

It is clear that

$$\vartheta = \sum_{j=1}^{n-1} -1_{\left\{\left|Y_{t_j}\right| \le N\right\}} \frac{Y_{t_j}}{N} 1_{(t_j, t_{j+1}]}$$

is in $\beta\left(\mathbb{F}^X\right)$ and it can be seen from (1.3.6) that on the event

$$\left\{\sup_{t\in[0,T]}|Y_t| \le N\right\} \cap \left\{\sum_{j=0}^{n-1}\left(Y_{t_{j+1}} - Y_{t_j}\right)^2 > 2LN + N^2\right\},$$

we have

$$\begin{aligned} I_Y(\vartheta) &= \frac{1}{2N}\left(\sum_{j=0}^{n-1}\left(Y_{t_{j+1}} - Y_{t_j}\right)^2 - Y_T^2\right) \\ &> \frac{1}{2N}\left(2LN + N^2 - N^2\right) = L\,. \end{aligned}$$

Together with (1.3.11), this implies that

$$P\left[I_Y(\vartheta) > L\right] > \frac{c}{4}.$$

Since L was chosen arbitrarily, this shows (1.3.8), and the proposition is proved. □

Corollary 1.10 $\left(B_t^H\right)_{t\in[0,T]}$ *is not a weak semimartingale if* $H \in \left(0, \frac{1}{2}\right)$.

Proof. It follows from Lemma 1.4 d) that

$$\sum_{j=0}^{n-1}\left(B^H_{\frac{(j+1)}{n}T} - B^H_{\frac{j}{n}T}\right)^2 \stackrel{(n\to\infty)}{\longrightarrow} \infty \text{ in probability.}$$

This implies that

$$\left\{\sum_{j=0}^{n-1}\left(B^H_{\frac{(j+1)}{n}T} - B^H_{\frac{j}{n}T}\right)^2 \;:\; n \in \mathbb{N}\right\}$$

is unbounded in L^0. Since B^H is continuous, the corollary follows from Proposition 1.9. □

For $H \in \left(\frac{1}{2}, 1\right)$, a direct proof of the fact that $\left(B_t^H\right)_{t\in[0,T]}$ is not a weak semimartingale seems to be difficult. But Proposition 1.8 permits us to use already existing results on classical semimartingales.

Proposition 1.11 *Let* $(X_t)_{t\in[0,T]}$ *be an a.s. right-continuous process such that*

$$P\left[(X_t)_{t\in[0,T]} \text{ is of finite variation}\right] < 1 \tag{1.3.12}$$

and, for all $\varepsilon > 0$*, there exists a partition*

$$0 = t_0 < t_1 < \cdots < t_n = T\,,\; n \in \mathbb{N}\,,$$

with

$$\max_{0 \le j \le n-1} \left(t_{j+1} - t_j\right) < \varepsilon \tag{1.3.13}$$

and

$$P\left[\sum_{j=0}^{n-1} \left(X_{t_{j+1}} - X_{t_j}\right)^2 > \varepsilon\right] < \varepsilon\,. \tag{1.3.14}$$

Then X is not a weak semimartingale.

Proof. Suppose X is a weak semimartingale. By Proposition 1.8, X is also an $\bar{\mathbb{F}}^X$-semimartingale. Hence, X is of the form

$$X_t = X_0 + M_t + A_t\,,$$

where X_0 is an $\bar{\mathcal{F}}_0$-measurable random variable, $M_0 = A_0 = 0$, M is an a.s. right-continuous local martingale with respect to $\bar{\mathbb{F}}$ and A an a.s. right-continuous, $\bar{\mathbb{F}}$-adapted finite variation process. It follows from (1.3.13), (1.3.14) and Theorem II.22 of Protter (1990) that

$$[X, X]_t = X_0 \text{ a.s.}\,, \; t \in [0, T]\,.$$

Hence,

$$[M, M]_t = 0 \text{ a.s.}\,, \; t \in [0, T]\,.$$

Therefore, Theorem II.27 of Protter (1990) implies $M_t = 0$ a.s., $t \in [0, T]$. Hence, X is a finite variation process. This contradicts (1.3.12). Therefore X cannot be a weak semimartingale. □

Corollary 1.12 $\left(B_t^H\right)_{t \in [0,T]}$ *is not a weak semimartingale if* $H \in \left(\frac{1}{2}, 1\right)$.

Proof. It follows from Lemma 1.4 d) that

$$\sum_{j=0}^{n-1} \left| B^H_{\frac{(j+1)}{n}T} - B^H_{\frac{j}{n}T} \right| \stackrel{(n \to \infty)}{\longrightarrow} \infty \text{ in probability.}$$

Therefore, there exists a sequence $(n_k)_{k=0}^{\infty}$ of natural numbers such that

$$\sum_{j=0}^{n_k-1} \left| B^H_{\frac{(j+1)}{n_k}T} - B^H_{\frac{j}{n_k}T} \right| \stackrel{(k \to \infty)}{\longrightarrow} \infty \text{ almost surely.}$$

Hence,

$$P\left[\left(B_t^H\right)_{t\in[0,T]} \text{ is of finite variation}\right] = 0\,.$$

On the other hand, Lemma 1.4 c) shows that

$$\sum_{j=0}^{n-1}\left(B^H_{\frac{(j+1)}{n}T} - B^H_{T\frac{j}{n}T}\right)^2 \overset{(n\to\infty)}{\longrightarrow} 0 \text{ in } L^1\,.$$

Hence, $\left(B_t^H\right)_{t\in[0,T]}$ satisfies the assumptions of Proposition 1.11. Therefore it is not a weak semimartingale. □

1.4 The market

Throughout this thesis we will consider a market that consists of a money market account and a stock that pays no dividends. All economic activity takes place in a time interval $[0, T]$ for some $T \in (0, \infty)$. Borrowing and short-selling are allowed, the borrowing rate is equal to the lending rate, and it is possible to buy and sell any fraction of stock shares. Moreover, there exist no transaction costs and stock shares can be bought and sold at the same price. We assume that money in the money market account evolves according to a stochastic process $\left(\tilde{S}_t^0\right)_{t\in[0,T]}$ and the stock price follows a stochastic process $\left(\tilde{S}_t\right)_{t\in[0,T]}$. Since we want to use $\tilde{S}^0$ as a numéraire, we require it to be positive. By S we denote the discounted stock price $\tilde{S}/\tilde{S}^0$. To make clear how derivative prices depend on the explicit modelling of $(\tilde{S}^0, \tilde{S})$, we will analyse the price of a European call option on the stock. Such an option is specified by its maturity T and the strike price K. It has a random pay-off at time T which is given by

$$\left(\tilde{S}_T - K\right)^+\,.$$

The first continuous-time stochastic model for a financial asset appeared in the thesis of Bachelier (1900). He proposed modelling the price of a stock as follows:

$$\tilde{S}_t = \tilde{S}_0 + \mu t + \sigma B_t\,,$$

where $\tilde{S}_0$, μ and σ are constants and B is a Brownian motion. The drawbacks of this model are that $\tilde{S}_t$ can become negative and the relative returns are lower for higher stock prices.

Samuelson (1965) introduced the more realistic model

$$\tilde{S}_t = \tilde{S}_0 \exp\left(\left\{\mu - \frac{\sigma^2}{2}\right\} t + \sigma B_t\right), \tag{1.4.1}$$

where $\tilde{S}_0$, μ and σ are constants and B is a Brownian motion. Black and Scholes (1973) noticed that if $\tilde{S}$ is as in (1.4.1) and there is a constant r such that $\tilde{S}^0_t = \exp(rt)$, then the pay-off of a European call option on $\tilde{S}$ can be replicated by continuous trading in $\tilde{S}^0$ and $\tilde{S}$, and they derived an explicit formula for the price of such an option. However, the Samuelson model also has deficiencies and up to now there have been many efforts to build better models. Cutland et al. (1995) discuss the empirical evidence that suggests that long-range dependence should be accounted for when modelling stock price movements and present a fractional version of the Samuelson model.

For constants $\tilde{S}_0 > 0$, ν, $\sigma > 0$ and r, we call

$$\tilde{S}^0_t = 1, \quad \tilde{S}_t = \tilde{S}_0 + \nu t + \sigma B^H_t \, , \; t \in [0, T] \, , \tag{1.4.2}$$

the fractional Bachelier model and

$$\tilde{S}^0_t = \exp(rt), \quad \tilde{S}_t = \tilde{S}_0 \exp\left(\{r + \nu\} \, t + \sigma B^H_t\right) \, , \; t \in [0, T] \, , \tag{1.4.3}$$

the fractional Samuelson model or, alternatively, the fractional Black-Scholes model.

Chapter 2

Arbitrage in fractional Brownian motion models

2.1 Introduction

In Section 1.3 we showed that for $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$, $\left(B_t^H\right)_{t\in[0,T]}$ is not a weak semimartingale. In particular, it is not a $\bar{\mathbb{F}}^{B^H}$-semimartingale, neither is $S = \tilde{S}/\tilde{S}^0$ in the models (1.4.2) and (1.4.3). Therefore, it follows immediately from Theorem 7.2 of Delbaen and Schachermayer (1994) that (1.4.2) and (1.4.3) admit a “free lunch with vanishing risk” consisting of simple predictable integrands adapted to $\bar{\mathbb{F}}^{B^H}$. Rogers (1997), Shiryaev (1998) and Salopek (1998) even give arbitrage strategies for fractional Brownian motion models.

Rogers (1997) constructs arbitrage for the fractional Bachelier model (1.4.2). His strategy consists of a combination of buy and hold strategies and works for all Hurst parameters $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$. However, as self-similarity of $\tilde{S}$ is essential for its construction, Rogers' arbitrage only exists in the case $\nu = 0$, i.e. $\tilde{S}_t = \tilde{S}_0 + \sigma B_t^H$. Moreover, Rogers models $\tilde{S}_t$ for $t \in (-\infty, 0]$ and to generate a profit on the time interval $[-1, 0)$, his arbitrage needs to know the whole history of $\tilde{S}$ from time $-\infty$ until the present.

In Shiryaev (1998) only the case $H \in \left(\frac{1}{2}, 1\right)$ is treated. An integral with respect to B^H is defined and it is indicated how it can be shown that for regular enough functions F, the modified Itô formula

$$dF(t, B_t^H) = \partial_1 F(t, B_t^H)dt + \partial_2 F(t, B_t^H)dB_t^H \tag{2.1.1}$$

holds. Using this for the fractional Bachelier model (1.4.2) with $H \in \left(\frac{1}{2}, 1\right)$, one can choose a $c > 0$ and set

$$\vartheta_t^0 = -c\left(\nu t + \sigma B_t^H\right)^2 - 2c\tilde{S}_0\left(\nu t + \sigma B_t^H\right), \quad \vartheta_t^1 = 2c\left(\nu t + \sigma B_t^H\right)$$

to obtain

$$\vartheta_t^0 \tilde{S}_t^0 + \vartheta_t^1 \tilde{S}_t = \vartheta_0^0 \tilde{S}_0^0 + \vartheta_0^1 \tilde{S}_0 + \int_0^t \vartheta_u^1 d\tilde{S}_u = c\left(\nu t + \sigma B_t^H\right)^2 .$$

Hence, if continuous adjustment of the portfolio is allowed, $(\vartheta^0, \vartheta^1)$ is a self-financing arbitrage strategy for the fractional Bachelier model.

For the fractional Samuelson model (1.4.3) with $H \in \left(\frac{1}{2}, 1\right)$, one can set for all $c > 0$,

$$\vartheta_t^0 = c\tilde{S}_0\left(1 - \exp\left(2\nu t + 2\sigma B_t^H\right)\right), \quad \vartheta_t^1 = 2c\left(\exp\left(\nu t + \sigma B_t^H\right) - 1\right) .$$

It follows from (2.1.1) that

$$\begin{aligned}\vartheta_t^0 \tilde{S}_t^0 + \vartheta_t^1 \tilde{S}_t &= \vartheta_0^0 \tilde{S}_0^0 + \vartheta_0^1 \tilde{S}_0 + \int_0^t \vartheta_u^0 d\tilde{S}_u^0 + \int_0^t \vartheta_u^1 d\tilde{S}_u \\ &= c\tilde{S}_0 \exp(rt)\left(\exp\left(\nu t + \sigma B_t^H\right) - 1\right)^2 ,\end{aligned}$$

which shows that $(\vartheta^0, \vartheta^1)$ is a self-financing arbitrage strategy for the fractional Samuelson model.

More generally, it is shown in Salopek (1998) that if a stochastic process $(X_t)_{t\geq 0}$ is almost surely continuous and of bounded p-variation for some $p < 2$ (this is the case for the processes $\tilde{S}^0$ and $\tilde{S}$ in (1.4.2) and (1.4.3) when $H \in \left(\frac{1}{2}, 1\right)$), then for a real function f on $\mathbb{R}$ that is locally Lipschitz, $t \geq 0$ and a sequence of partitions $0 = t_0^n < t_1^n < \cdots < t_{J(n)}^n = t$, $n \in \mathbb{N}$ with

$$\lim_{n\to\infty} \max_j \left|t_{j+1}^n - t_j^n\right| = 0 ,$$

the finite sums

$$\sum_{j=0}^{J(n)-1} f\left(X_{t_j^n}\right)\left(X_{t_{j+1}^n} - X_{t_j^n}\right)$$

almost surely converge to a limit $\int_0^t f(X_u)dX_u$ and

$$\int_0^t f(X_u)dX_u \overset{a.s.}{=} F(X_t) - F(X_0)\,,$$

where $F(x) = \int_0^x f(u)du,\ x \in \mathbb{R}$. This is used in Salopek (1998) to construct a self-financing arbitrage strategy for two financial assets X and Y that are both almost surely continuous, of bounded p-variation for some $p < 2$ and such that $X_t \neq Y_t$ almost surely for all t.

In this chapter we construct arbitrage strategies for a class of fractional Brownian motion models that contains (1.4.2) and (1.4.3) for all $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$, and we show how arbitrage can be excluded from these models by putting restrictions on the class of trading strategies.

In Section 2 we define the notions of 'free lunch with vanishing risk', 'arbitrage' and 'strong arbitrage'. Then we introduce different classes of trading strategies. In Section 3 we construct arbitrage strategies. As in the case of Rogers (1997) our arbitrage strategies consist of combinations of buy and hold strategies. Therefore we need no integration theory for fractional Brownian motion. Moreover, to generate a profit on the time interval $[0, T]$, our strategies need only know the history of $S = \tilde{S}/\tilde{S}^0$ on $[0, T]$. However, to perform these strategies it must be allowed to buy and sell within arbitrarily small time intervals. In Section 4 we show that arbitrage can be ruled out from models of the form (1.4.2) and (1.4.3) by introducing a minimal amount of time $h > 0$ that must lie between two consecutive transactions.

2.2 The trading strategies

In this section the time interval is an arbitrary closed interval $[a, b]$. Money can be invested in a money market account where money grows according to a positive stochastic process $\left(\tilde{S}_t^0\right)_{t\in[a,b]}$ and a stock whose price follows a stochastic process $\left(\tilde{S}_t\right)_{t\in[a,b]}$. A trading strategy is a pair $\vartheta = \left(\vartheta^0, \vartheta^1\right)$ of stochastic processes $\left(\vartheta_t^0\right)_{t\in[a,b]}$ and $\left(\vartheta_t^1\right)_{t\in[a,b]}$. $\vartheta_t^0\tilde{S}_t^0$ describes the money in the money market account at time t and ϑ_t^1 the number of stock shares held at time t. Hence, the evolution of the portfolio value of a strategy ϑ is given by

$$\tilde{V}_t^\vartheta = \vartheta_t^0\tilde{S}_t^0 + \vartheta_t^1\tilde{S}_t\,,\ t \in [a, b]\,.$$

We set

$$V_t^\vartheta = \frac{\tilde{V}_t^\vartheta}{\tilde{S}_t^0} = \vartheta_t^0 + \vartheta_t^1 S_t \,,\, t \in [a,b]\,.$$

Definition 2.1 *Let ξ be a $[0,\infty]$-valued random variable with $P[\xi > 0] > 0$.*

a) A sequence of trading strategies $\{\vartheta(n)\}_{n=1}^\infty$ is a ξ-FLVR (ξ-free lunch with vanishing risk) if

$$\lim_{n\to\infty} \left(V_b^{\vartheta(n)} - V_a^{\vartheta(n)}\right) = \xi \quad \textit{in probability}$$

and

$$\lim_{n\to\infty} \left\| \left(V_b^{\vartheta(n)} - V_a^{\vartheta(n)}\right)^- \right\|_\infty = 0\,.$$

$\{\vartheta(n)\}_{n=1}^\infty$ is a FLVR if it is a ξ'-FLVR for some $[0,\infty]$-valued random variable ξ' with $P[\xi' > 0] > 0$.

b) A trading strategy ϑ is a ξ-arbitrage if

$$V_b^\vartheta - V_a^\vartheta = \xi \quad \textit{almost surely.}$$

ϑ is an arbitrage if it is a ξ'-arbitrage for some $[0,\infty]$-valued random variable ξ' with $P[\xi' > 0] > 0$.

c) A trading strategy ϑ is a strong arbitrage if there exists a constant $c > 0$ such that

$$V_b^\vartheta - V_a^\vartheta \ge c \quad \textit{almost surely.}$$

It is clear that we must put certain restrictions on a trading strategy to give it an economic meaning. First of all, trading strategies should only be based on available information. To describe the evolution of information we introduce a family of σ-algebras $\mathbb{F} = (\mathcal{F}_t)_{t\in[a,b]}$. We assume that at any time $t \in [a,b]$, $\tilde{S}_t^0$ and $\tilde{S}_t$ can be observed and no information is lost over time. In other words, $\mathbb{F}$ is a filtration and

$$\mathcal{F}_t^{\tilde{S}^0,\tilde{S}} := \sigma\left(\left(\tilde{S}_u^0\right)_{u\in[0,t]}, \left(\tilde{S}_u\right)_{u\in[0,t]}\right) \subset \mathcal{F}_t \quad \text{for all } t \in [a,b]\,.$$

Note that

$$\mathcal{F}_t^S := \sigma\left((S_u)_{u\in[0,t]}\right) \subset \mathcal{F}_t^{\tilde{S}^0,\tilde{S}} \quad \text{for all } t \in [a,b]\,.$$

Furthermore, we require $\tilde{S}^0$ and $\tilde{S}$ to be progressively measurable with respect to $\mathbb{F}$. This is in particular the case when $\tilde{S}^0$ and $\tilde{S}$ are right-continuous, and it ensures that for all $\mathbb{F}$-stopping times τ, the stopped processes $\left(\tilde{S}^0_{\tau\wedge t}\right)_{t\in a,b}$ and $\left(\tilde{S}_{\tau\wedge t}\right)_{t\in a,b}$ are also progressively measurable with respect to $\mathbb{F}$. To construct arbitrage in fractional Brownian models of the form (1.4.2) or (1.4.3) it is enough to consider combinations of buy and hold strategies. We start our discussion of different classes of combinations of buy and hold strategies by recalling the definition of the class $\mathbf{S}(\mathbb{F})$ of simple predictable integrands and introducing the class $\mathbf{aS}(\mathbb{F})$ of almost simple predictable integrands.

Definition 2.2

a) $\mathbf{S}(\mathbb{F}) := \{g_0 1_{\{a\}} + \sum_{j=1}^{n-1} g_j 1_{(\tau_j,\tau_{j+1}]} \,:\, n \geq 2,\ a = \tau_1 \leq \cdots \leq \tau_n = b;$ *all* τ_j*'s are* $\mathbb{F}$*-stopping times;* g_0 *is a real,* $\mathcal{F}_a$*-measurable random variable; and the other* g_j*'s are real,* $\mathcal{F}_{\tau_j}$*-measurable random variables*$\}$

b) $\mathbf{aS}(\mathbb{F}) := \{g_0 1_{\{a\}} + \sum_{j=1}^{\infty} g_j 1_{(\tau_j,\tau_{j+1}]} \,:\, a = \tau_1 \leq \tau_2 \leq \cdots \leq b;$ *all* τ_j*'s are* $\mathbb{F}$*-stopping times;* g_0 *is a real,* $\mathcal{F}_a-$ *measurable random variable; the other* g_j*'s are real,* $\mathcal{F}_{\tau_j}$*-measurable random variables;* $P\left[\exists\, j \text{ such that } \tau_j = b\right] = 1\}$

c) *For* $\vartheta^1 = g_0 1_{\{a\}} + \sum_{j=1}^{\infty} g_j 1_{(\tau_j,\tau_{j+1}]} \in \mathbf{aS}(\mathbb{F})$ *we define* $\left(\vartheta^1 \cdot \tilde{S}\right)_t := \sum_{j=1}^{\infty} g_j (\tilde{S}_{\tau_{j+1}\wedge t} - \tilde{S}_{\tau_j\wedge t}),\ t \in [a,b]\,.$
(Note that this is almost surely a sum of finitely many terms, and the process $\left(\left(\vartheta^1 \cdot \tilde{S}\right)_t\right)_{t\in[a,b]}$ *is progressively measurable because* $\left(\tilde{S}_t\right)_{t\in[a,b]}$ *is.)*

Remark 2.3 For $\vartheta^1 = g_0 1_{\{a\}} + \sum_{j=1}^{\infty} g_j 1_{(\tau_j,\tau_{j+1}]} \in \mathbf{aS}(\mathbb{F})$ we can define the sets $A_n = \{\tau_n < b\} \cap \{\tau_{n+1} = b\}$, $n \in \mathbb{N}$. Then $P\left[\bigcup_{n=1}^{\infty} A_n\right] = 1$, the function $N : \Omega \to \mathbb{N}$ defined by

$$N(\omega) := \begin{cases} n, & \omega \in A_n \\ 0, & \omega \notin \bigcup_{n=1}^{\infty} A_n \end{cases}$$

is $\mathcal{F}_b$-measurable and

$$\vartheta^1 = g_0 1_{\{a\}} + \sum_{j=1}^{\infty} g_j 1_{(\tau_j,\tau_{j+1}]} = g_0 1_{\{a\}} + \sum_{j=1}^{N} g_j 1_{(\tau_j,\tau_{j+1}]} \text{ almost surely.}$$

If an investor buys and sells stock shares according to ϑ^1, he will almost surely carry out only finitely many transactions. But he does not know from the beginning how many. Note that if we take an arbitrary $\mathcal{F}_b$-measurable function $N : \Omega \to \mathbb{N}$, an increasing sequence of $\mathbb{F}$-stopping times $a = \tau_1 \le \tau_2 \le \cdots \le b$, a real, $\mathcal{F}_a$-measurable function g_0 and real, $\mathcal{F}_{\tau_j}$-measurable functions g_j, $j \in \mathbb{N}$,then

$$g_0 1_{\{a\}} + \sum_{j=1}^{N} g_j 1_{(\tau_j, \tau_{j+1}]} = g_0 1_{\{a\}} + \sum_{j=1}^{\infty} 1_{\{j \le N\}} g_j 1_{(\tau_j, \tau_{j+1}]}$$

need not be in $\mathbf{aS}(\mathbb{F})$.

Definition 2.4

$$\Theta^{\mathbf{S}}(\mathbb{F}) := \left\{ \vartheta \ : \ \vartheta^0, \vartheta^1 \in \mathbf{S}(\mathbb{F}) \right\} \quad , \quad \Theta^{\mathbf{aS}}(\mathbb{F}) := \left\{ \vartheta \ : \ \vartheta^0, \vartheta^1 \in \mathbf{aS}(\mathbb{F}) \right\} .$$

Definition 2.5 *Let $\vartheta = \left(\vartheta^0, \vartheta^1\right) \in \Theta^{\mathbf{aS}}(\mathbb{F})$. There exist $\mathbb{F}$-stopping times*

$$a = \tau_1 \le \tau_2 \le \cdots \le b$$

such that ϑ^0 and ϑ^1 can be written in the form

$$\vartheta^0 = f_0 1_{\{a\}} + \sum_{j=1}^{\infty} f_j 1_{(\tau_j, \tau_{j+1}]}, \quad \vartheta^1 = g_0 1_{\{a\}} + \sum_{j=1}^{\infty} g_j 1_{(\tau_j, \tau_{j+1}]} . \tag{2.2.1}$$

We set $\tau_0 = a - 1$ and call ϑ self-financing for $(\tilde{S}^0, \tilde{S})$ if for all $j \ge 1$, $k = 1, \ldots, j$ and $l \ge 0$,

$$1_{\{\tau_{j-k} < \tau_{j-k+1} = \tau_{j+l} < \tau_{j+l+1}\}} \left\{ \left(f_{j+l} - f_{j-k} \right) \tilde{S}^0_{\tau_j} + \left(g_{j+l} - g_{j-k} \right) \tilde{S}_{\tau_j} \right\} \overset{\text{a.s.}}{=} 0 . \tag{2.2.2}$$

(Note that the property (2.2.2) is independent of the representation (2.2.1) of ϑ.)

$$\Theta^{\mathbf{S}}_{\mathrm{sf}}(\mathbb{F}) := \left\{ \vartheta \in \Theta^{\mathbf{S}}(\mathbb{F}) \ : \ \vartheta \text{ is self-financing} \right\} .$$

$$\Theta^{\mathbf{aS}}_{\mathrm{sf}}(\mathbb{F}) := \left\{ \vartheta \in \Theta^{\mathbf{aS}}(\mathbb{F}) \ : \ \vartheta \text{ is self-financing} \right\} .$$

Proposition 2.6 *Let $\vartheta = \left(\vartheta^0, \vartheta^1\right) \in \Theta^{\mathbf{aS}}(\mathbb{F})$. Then the following are equivalent:*

- *(i)* ϑ *is self-financing for* $(\tilde{S}^0, \tilde{S})$
- *(ii)* $\tilde{V}^\vartheta_t \overset{\text{a.s.}}{=} \tilde{V}^\vartheta_a + \left(\vartheta^0 \cdot \tilde{S}^0\right)_t + \left(\vartheta^1 \cdot \tilde{S}\right)_t$ *for all* $t \in [a, b]$
- *(iii)* ϑ *is self-financing for* $(1, S)$
- *(iv)* $V^\vartheta_t \overset{\text{a.s.}}{=} V^\vartheta_a + \left(\vartheta^1 \cdot S\right)_t$ *for all* $t \in [a, b]$

Proof. Let $a = \tau_1 \le \tau_2 \le \cdots \le b$ be an increasing sequence of $\mathbb{F}$-stopping times such that

$$\vartheta^0 = f_0 1_{\{a\}} + \sum_{j=1}^{\infty} f_j 1_{(\tau_j, \tau_{j+1}]}, \quad \vartheta^1 = g_0 1_{\{a\}} + \sum_{j=1}^{\infty} g_j 1_{(\tau_j, \tau_{j+1}]} .$$

(i) $\Rightarrow$ (ii): For $t = a$, (ii) is trivially satisfied. So let us assume $t \in (a, b]$. For almost all $\omega \in \Omega$, there exists a $j \in \mathbb{N}$, such that $t \in (\tau_j, \tau_{j+1}]$, and

$$\begin{aligned} &\tilde{V}_a^{\vartheta} + \left(\vartheta^0 \cdot \tilde{S}^0\right)_t + \left(\vartheta^1 \cdot \tilde{S}\right)_t \\ &= f_0 \tilde{S}^0_{\tau_1} + g_0 \tilde{S}_{\tau_1} + \sum_{i=1}^{j-1} f_i \left(\tilde{S}^0_{\tau_{i+1}} - \tilde{S}^0_{\tau_i}\right) + f_j \left(\tilde{S}^0_t - \tilde{S}^0_{\tau_j}\right) \\ &\quad + \sum_{i=1}^{j-1} g_i \left(\tilde{S}_{\tau_{i+1}} - \tilde{S}_{\tau_i}\right) + g_j \left(\tilde{S}_t - \tilde{S}_{\tau_j}\right) \\ &= \sum_{i=1}^{j} \tilde{S}^0_{\tau_i} \left(f_{i-1} - f_i\right) + \sum_{i=1}^{j} \tilde{S}_{\tau_i} \left(g_{i-1} - g_i\right) + f_j \tilde{S}^0_t + g_j \tilde{S}_t = \vartheta^0_t \tilde{S}^0_t + \vartheta^1_t \tilde{S}_t , \end{aligned}$$

where the last inequality follows from (i) and the fact that $f_j = \vartheta^0_t$, $g_j = \vartheta^1_t$.
(ii) $\Rightarrow$ (i): Let $j \ge 1$, $k = 1, \ldots, j$ and $l \ge 0$. On

$$\left\{\tau_{j-k} < \tau_{j-k+1} = \tau_{j+l} < \tau_{j+l+1}\right\}$$

we have

$$\begin{aligned} &\left(f_{j+l} - f_{j-k}\right) \tilde{S}^0_{\tau_j} + \left(g_{j+l} - g_{j-k}\right) \tilde{S}_{\tau_j} \\ &= \left(f_{j+l} \tilde{S}^0_{\tau_{j+l+1}} + g_{j+l} \tilde{S}_{\tau_{j+l+1}}\right) - \left(f_{j-k} \tilde{S}^0_{\tau_j} + g_{j-k} \tilde{S}_{\tau_j}\right) \\ &\quad - f_{j+l} \left(\tilde{S}^0_{\tau_{j+l+1}} - \tilde{S}^0_{\tau_j}\right) - g_{j+l} \left(\tilde{S}_{\tau_{j+l+1}} - \tilde{S}_{\tau_j}\right) \\ &= \left(\vartheta^0_{\tau_{j+l+1}} \tilde{S}^0_{\tau_{j+l+1}} + \vartheta^1_{\tau_{j+l+1}} \tilde{S}_{\tau_{j+l+1}}\right) - \left(\vartheta^0_{\tau_j} \tilde{S}^0_{\tau_j} + \vartheta^1_{\tau_j} \tilde{S}_{\tau_j}\right) \\ &\quad - \left[\vartheta^0_a \tilde{S}^0_a + \vartheta^1_a \tilde{S}_a + \sum_{i=1}^{j+l} f_i \left(\tilde{S}^0_{\tau_{i+1}} - \tilde{S}^0_{\tau_i}\right) + \sum_{i=1}^{j+l} g_i \left(\tilde{S}_{\tau_{i+1}} - \tilde{S}_{\tau_i}\right)\right] \\ &\quad + \left[\vartheta^0_a \tilde{S}^0_a + \vartheta^1_a \tilde{S}_a + \sum_{i=1}^{j-1} f_i \left(\tilde{S}^0_{\tau_{i+1}} - \tilde{S}^0_{\tau_i}\right) + \sum_{i=1}^{j-1} g_i \left(\tilde{S}_{\tau_{i+1}} - \tilde{S}_{\tau_i}\right)\right] \overset{\text{a.s.}}{=} 0 , \end{aligned}$$

where the last inequality follows from (ii).
The equivalence of (i) and (iii) is trivial, and the equivalence of (iii) and (iv) can be shown in the same way as the equivalence of (i) and (ii). □

Remark 2.7 It follows from Proposition 2.6 that for all $\vartheta \in \Theta^{\mathbf{aS}}_{\mathrm{sf}}(\mathbb{F})$,

$$\vartheta^0_t \overset{\text{a.s.}}{=} V^{\vartheta}_a + \left(\vartheta^1 \cdot S\right)_t - \vartheta^1_t S_t \, , \; t \in [a,b] \, . \tag{2.2.3}$$

This shows that if we identify indistinguishable processes, the map

$$\vartheta = \left(\vartheta^0, \vartheta^1\right) \mapsto \left(V^{\vartheta}_a, \vartheta^1\right)$$

is a bijection from $\Theta^{\mathbf{aS}}_{\mathrm{sf}}(\mathbb{F})$ to $L^0(\mathcal{F}_a) \times \mathbf{aS}(\mathbb{F})$. In particular, there exists for all $\left(\xi, \vartheta^1\right) \in L^0(\mathcal{F}_a) \times \mathbf{aS}(\mathbb{F})$, a unique $\vartheta^0 \in \mathbf{aS}(\mathbb{F})$ such that $\vartheta = \left(\vartheta^0, \vartheta^1\right)$ is in $\Theta^{\mathbf{aS}}_{\mathrm{sf}}(\mathbb{F})$ and $V^{\vartheta}_a = \xi$.

In $\Theta^{\mathbf{aS}}_{\mathrm{sf}}(\mathbb{F}^S)$ there exist so called doubling strategies which can create arbitrage even in the standard Samuelson model, where

$$S_t = S_0 \exp\left(\nu t + \sigma B_t\right) \, , \; t \in [0,T] \, ,$$

for constants $S_0 > 0$, ν, σ and a Brownian motion B. It was noticed by Harrison and Pliska (1981) that they can be ruled out by putting an admissibility condition on the trading strategies. We use the admissibility condition of Delbaen and Schachermayer (1994). It is more liberal than the one of Harrison and Pliska (1981) but restrictive enough to exclude arbitrage in the Samuelson model.

Definition 2.8 *Let $c \geq 0$. We call $\vartheta \in \Theta^{\mathbf{aS}}_{\mathrm{sf}}(\mathbb{F})$, c-admissible if*

$$\inf_{t\in[a,b]} \left(V^{\vartheta}_t - V^{\vartheta}_a\right) = \inf_{t\in[a,b]} \left(\vartheta^1 \cdot S\right)_t \geq -c \quad \textit{almost surely.}$$

We call ϑ admissible if it is c-admissible for some $c \geq 0$.

$$\Theta^{\mathbf{S}}_{\mathrm{sf,adm}}(\mathbb{F}) := \left\{\vartheta \in \Theta^{\mathbf{S}}_{\mathrm{sf}}(\mathbb{F}) \; : \; \vartheta \textit{ is admissible}\right\}.$$

$$\Theta^{\mathbf{aS}}_{\mathrm{sf,adm}}(\mathbb{F}) := \left\{\vartheta \in \Theta^{\mathbf{aS}}_{\mathrm{sf}}(\mathbb{F}) \; : \; \vartheta \textit{ is admissible}\right\}.$$

2.3 Construction of arbitrage

Theorem 2.9 *Let B^H be a fractional Brownian motion. Let $T \in (0, \infty)$, $\nu \in C^1[0, T]$ and $\sigma > 0$. Then in all four cases:*

$$\begin{array}{lll} (i) & H \in (\frac{1}{2}, 1), & S_t = \nu(t) + \sigma B_t^H,\ t \in [0, T] \\ (ii) & H \in (\frac{1}{2}, 1), & S_t = \exp\left(\nu(t) + \sigma B_t^H\right),\ t \in [0, T] \\ (iii) & H \in (0, \frac{1}{2}), & S_t = \nu(t) + \sigma B_t^H,\ t \in [0, T] \\ (iv) & H \in (0, \frac{1}{2}), & S_t = \exp\left(\nu(t) + \sigma B_t^H\right),\ t \in [0, T] \end{array}$$

there exists for every constant $c > 0$ and all $n \in \mathbb{N}$, a $\vartheta^1(n) \in \mathbf{S}(\mathbb{F}^S)$ such that

$$\begin{array}{ll} a) & P\left[\left(\vartheta^1(n) \cdot S\right)_T = c\right] > 1 - \frac{1}{n} \quad \textit{and} \\ b) & \inf_{t\in[0,T]} \left(\vartheta^1(n) \cdot S\right)_t \geq -\frac{1}{n}\,. \end{array}$$

In particular, the strategies $\vartheta(n) = \left(\vartheta^0(n), \vartheta^1(n)\right) \in \Theta^{\mathbf{S}}_{\mathrm{sf,adm}}(\mathbb{F}^S)$, $n \in \mathbb{N}$, where $\vartheta^0(n)$ is given by

$$\vartheta_t^0(n) = \left(\vartheta^1(n) \cdot S\right)_t - \vartheta_t^1(n) S_t\,,\ t \in [0, T]\,,\ n \in \mathbb{N}\,,$$

form a c-FLVR. In the cases (iii) and (iv), $\vartheta^1(n)$ can be chosen such that also

$$c) \quad \left|\vartheta^1(n)\right| \leq \frac{1}{n}\,.$$

Theorem 2.10 *In all four cases (i)-(iv) of Theorem 2.9 there exists for every constant $c > 0$, a $\frac{1}{c}$-admissible c-arbitrage $\vartheta \in \Theta^{\mathbf{aS}}_{\mathrm{sf,adm}}(\mathbb{F}^S)$. In the cases (iii) and (iv), ϑ can be chosen such that $\left|\vartheta^1\right| \leq \frac{1}{c}$.*

In order to prove Theorems 2.9 and 2.10 we need the following two lemmas.

Lemma 2.11 *Let $(Z_t)_{t\in[a,b]}$ be a continuous stochastic process. If*

$$P\left[Z_b = Z_a\right] = 0\,, \tag{2.3.1}$$

and for all $\varepsilon > 0$ there exist deterministic times $a = t_0 < \cdots < t_n = b$ such that

$$P\left[\max_{t\in[a,b]} \sum_{j=0}^{n-1} \left(Z_{t_{j+1}\wedge t} - Z_{t_j\wedge t}\right)^2 \geq \varepsilon\right] < \varepsilon, \tag{2.3.2}$$

then there exists for all $M > 0$ a $\gamma \in \mathbf{S}(\mathbb{F}^Z)$ such that

$$\begin{array}{ll} a) & P[(\gamma \cdot Z)_b < M] < \frac{1}{M} \quad \textit{and} \\ b) & \inf_{t\in[a,b]} (\gamma \cdot Z)_t \geq -\frac{1}{M}. \end{array}$$

Proof. Let $M > 0$. It follows from (2.3.1) and (2.3.2) that there exist an $\varepsilon > 0$ such that

$$P\left[(Z_b - Z_a)^2 < \varepsilon\right] < \frac{1}{2M} \tag{2.3.3}$$

and a partition $a = t_0 < \cdots < t_n = b$, such that

$$P\left[\max_{t\in[a,b]} \sum_{j=0}^{n-1} \left(Z_{t_{j+1}\wedge t} - Z_{t_j\wedge t}\right)^2 \geq \frac{\varepsilon}{M^2+1}\right] < \frac{1}{2M}\,. \tag{2.3.4}$$

Since Z is continuous,

$$\xi = \inf\left\{t \in [a,b] \,:\, \sum_{j=0}^{n-1} \left(Z_{t_{j+1}\wedge t} - Z_{t_j\wedge t}\right)^2 \geq \frac{\varepsilon}{M^2+1}\right\} \tag{2.3.5}$$

$$(\text{we set } \inf\emptyset = b)$$

is an $\mathbb{F}^Z$-stopping time (see e.g. Problem 1.2.7 of Karatzas and Shreve (1988)) and (2.3.4) implies

$$P\left[\xi < b\right] < \frac{1}{2M}\,. \tag{2.3.6}$$

Furthermore,

$$\gamma = \frac{2}{\varepsilon}\left(M + \frac{1}{M}\right)\sum_{j=1}^{n-1}\left(Z_{t_j} - Z_a\right) 1_{(t_j,t_{j+1}]}1_{[0,\xi]} \tag{2.3.7}$$

is in $\mathbf{S}(\mathbb{F}^Z)$ and a calculation shows that for all $t \in [a,b]$,

$$(\gamma\cdot Z)_t = \frac{M+\frac{1}{M}}{\varepsilon}\left[\left(Z_{t\wedge\xi} - Z_a\right)^2 - \sum_{j=0}^{n-1}\left(Z_{t_{j+1}\wedge t\wedge\xi} - Z_{t_j\wedge t\wedge\xi}\right)^2\right]. \tag{2.3.8}$$

This together with (2.3.5) implies b). From (2.3.8), (2.3.6) and (2.3.3) it follows that

$$\begin{aligned}
&P[(\gamma\cdot Z)_b < M]\\
= \;& P\left[\frac{M+\frac{1}{M}}{\varepsilon}\left\{\left(Z_\xi - Z_a\right)^2 - \sum_{j=0}^{n-1}\left(Z_{t_{j+1}\wedge\xi} - Z_{t_j\wedge\xi}\right)^2\right\} < M\right]\\
\leq \;& P\left[\left(Z_\xi - Z_a\right)^2 < \varepsilon\right] \leq P\left[\xi < b\right] + P\left[(Z_b - Z_a)^2 < \varepsilon\right] < \frac{1}{M}.
\end{aligned}$$

This shows a), and the lemma is proved. □

Lemma 2.12 *Let* $(Z_t)_{t\in[a,b]}$ *be a continuous stochastic process. If for all* $L > 0$ *there exist deterministic times* $a = t_0 < \cdots < t_n = b$*, such that*

$$P\left[\sum_{j=0}^{n-1}\left(Z_{t_{j+1}} - Z_{t_j}\right)^2 < L\right] < \frac{1}{L}, \tag{2.3.9}$$

then there exists for all $M > 0$ *a* $\gamma \in \mathbf{S}(\mathbb{F}^Z)$ *such that*

a) $P[(\gamma \cdot Z)_b < M] < \frac{1}{M}$,
b) $\inf_{t\in[a,b]}(\gamma \cdot Z)_b \geq -\frac{1}{M}$ *and*
c) $|\gamma| \leq \frac{1}{M}$.

Proof. Let $M > 0$. Since Z is continuous,

$$\xi_N = \inf\{t \in [a,b] : |Z_t - Z_a| \geq N\} \quad \text{(we set } \inf\emptyset = b) \tag{2.3.10}$$

is for all $N > 0$ an $\mathbb{F}^Z$-stopping time and $\{\xi_N < b\} \to \emptyset$, as $N \to \infty$. Therefore there exists an $N \geq 2$, such that

$$P\left[\xi_N < b\right] < \frac{1}{2M}. \tag{2.3.11}$$

By assumption (2.3.9) there exists a partition $a = t_0 < \cdots < t_n = b$, such that

$$P\left[\sum_{j=0}^{n-1}\left(Z_{t_{j+1}} - Z_{t_j}\right)^2 < N^2(M^2+1)\right] < \frac{1}{2M}. \tag{2.3.12}$$

It is easy to see that

$$\gamma = -\frac{2}{MN^2}\sum_{j=1}^{n-1}\left(Z_{t_j} - Z_a\right)1_{(t_j,t_{j+1}]}1_{[0,\xi_N]}$$

is in $\mathbf{S}(\mathbb{F}^Z)$ and satisfies c). As in the proof of Lemma 2.11 a calculation shows that for all $t \in [a,b]$,

$$(\gamma \cdot Z)_t = \frac{1}{MN^2}\left[\sum_{j=0}^{n-1}\left(Z_{t_{j+1}\wedge t\wedge\xi_N} - Z_{t_j\wedge t\wedge\xi_N}\right)^2 - \left(Z_{t\wedge\xi_N} - Z_a\right)^2\right]. \tag{2.3.13}$$

This together with (2.3.10) implies b). From (2.3.13), (2.3.11) and (2.3.12) follows that

$$
\begin{aligned}
& P\left[(\gamma \cdot Z)_b < M\right] \\
= \;& P\left[\frac{1}{MN^2}\left\{\sum_{j=0}^{n-1}\left(Z_{t_{j+1}\wedge\xi_N} - Z_{t_j\wedge\xi_N}\right)^2 - \left(Z_{\xi_N} - S_a\right)^2\right\} < M\right] \\
\leq \;& P\left[\sum_{j=0}^{n-1}\left(Z_{t_{j+1}\wedge\xi_N} - Z_{t_j\wedge\xi_N}\right)^2 < M^2N^2 + N^2\right] \\
\leq \;& P\left[\xi_N < b\right] + P\left[\sum_{j=0}^{n-1}\left(Z_{t_{j+1}} - Z_{t_j}\right)^2 < N^2(M^2+1)\right] < \frac{1}{M}\,.
\end{aligned}
$$

This shows a) and the lemma is proved. □

Remark 2.13 The conclusions of Lemmas 2.11 and 2.12 remain true if (2.3.2) or (2.3.9) are satisfied for general stopping times $a = \tau_0 \leq \cdots \leq \tau_n = b$ instead of deterministic times $a = t_0 < \cdots < t_n = b$. However, for the proof of Theorems 2.9 and 2.10 the versions with deterministic times are sufficient.

Proof of Theorem 2.9 By self-similarity of B^H it is enough to prove Theorem 2.9 for $T = 1$.

(i) $H \in (\frac{1}{2}, 1)$, $\quad S_t = \nu(t) + \sigma B_t^H,\ t \in [0,1]$:
It is clear that $(S_t)_{t\in[0,1]}$ satisfies (2.3.1). It follows from Lemma 1.4 a) and the fact that ν is Lipschitz that

$$
\max_{t\in[0,1]} \sum_{j=0}^{n-1}\left(S_{\frac{j+1}{n}\wedge t} - S_{\frac{j}{n}\wedge t}\right)^2 \stackrel{(n\to\infty)}{\longrightarrow} 0 \text{ almost surely.} \tag{2.3.14}
$$

This shows that $(S_t)_{t\in[0,1]}$ satisfies (2.3.2). Thus, it follows from Lemma 2.11 that for all $n \in \mathbb{N}$, there exists a $\gamma(n) \in \mathbf{S}(\mathbb{F}^S)$ such that

a) $\quad P[(\gamma(n)\cdot S)_1 < c] < \frac{1}{n}$ and
b) $\quad \inf_{t\in[0,1]}(\gamma(n)\cdot S)_t \geq -\frac{1}{n}$.

For every $n \in \mathbb{N}$,

$$
\xi_n = \inf\left\{t \,:\, (\gamma(n)\cdot S)_t = c\right\} \quad (\text{we set } \inf\emptyset = 1)
$$

is an $\mathbb{F}^Z$-stopping time and for $\vartheta^1(n) = \gamma(n) \cdot 1_{[0,\xi_n]} \in \mathbf{S}(\mathbb{F}^Z)$ we have

a) $P[(\vartheta^1(n) \cdot S)_1 = c] > 1 - \frac{1}{n}$ and
b) $\inf_{t\in[0,1]} (\vartheta^1(n) \cdot S)_t \geq -\frac{1}{n}$.

(ii) $H \in (\frac{1}{2}, 1), \quad S_t = \exp(\nu(t) + \sigma B_t^H), t \in [0,1]$:
It is clear that $(S_t)_{t\in[0,1]}$ satisfies (2.3.1). That $(S_t)_{t\in[0,1]}$ satisfies (2.3.2) follows from

$$|S_t - S_u| \leq \left(\max_{v\in[0,1]} S_v\right) |\ln S_t - \ln S_u|, \; u,t \in [0,1],$$

and (2.3.14). Now the assertion can be deduced from Lemma 2.11 as before.

(iii) $H \in \left(0, \frac{1}{2}\right)$, $S_t = \nu(t) + \sigma B_t^H$, $t \in [0,1]$:
To show that $(S_t)_{t\in[0,1]}$ satisfies (2.3.9) we choose an $L > 0$. It follows from Lemma 1.4 c) that

$$\frac{1}{n}\sum_{j=0}^{n-1}\left|B^H_{\frac{j+1}{n}} - B^H_{\frac{j}{n}}\right| \overset{(n\to\infty)}{\longrightarrow} 0 \text{ in } L^1.$$

Hence,

$$\sum_{j=0}^{n-1} 2\left|\left(\nu\left(\frac{j+1}{n}\right) - \nu\left(\frac{j}{n}\right)\right)\left(\sigma B^H_{\frac{j+1}{n}} - \sigma B^H_{\frac{j}{n}}\right)\right|$$

$$\leq 2\left\|\nu'\right\|_\infty \frac{1}{n}\sigma\sum_{j=0}^{n-1}\left|B^H_{\frac{j+1}{n}} - B^H_{\frac{j}{n}}\right| \overset{(n\to\infty)}{\longrightarrow} 0 \text{ in } L^1.$$

In particular, there exists an $n_1 \in \mathbb{N}$, such that for all $n \geq n_1$,

$$P\left[\sum_{j=0}^{n-1}\left|2\left(\nu\left(\frac{j+1}{n}\right) - \nu\left(\frac{j}{n}\right)\right)\left(\sigma B^H_{\frac{j+1}{n}} - \sigma B^H_{\frac{j}{n}}\right)\right| > L\right] < \frac{1}{2L}.$$

On the other hand, Lemma 1.4 d) implies that there exists an $n_2 \in \mathbb{N}$, such that for all $n \geq n_2$,

$$P\left[\sum_{j=0}^{n-1}\left(\sigma B^H_{\frac{j+1}{n}} - \sigma B^H_{\frac{j}{n}}\right)^2 < 2L\right] < \frac{1}{2L}.$$

Hence, for all $n \geq \max(n_1, n_2)$,

$$P\left[\sum_{j=0}^{n-1}\left(S_{\frac{j+1}{n}} - S_{\frac{j}{n}}\right)^2 < L\right] \leq P\left[\sum_{j=0}^{n-1}\left(\sigma B^H_{\frac{j+1}{n}} - \sigma B^H_{\frac{j}{n}}\right)^2\right.$$
$$\left. +2\left(\nu\left(\frac{j+1}{n}\right) - \nu\left(\frac{j}{n}\right)\right)\left(\sigma B^H_{\frac{j+1}{n}} - \sigma B^H_{\frac{j}{n}}\right) < L\right]$$
$$\leq P\left[\sum_{j=0}^{n-1}\left(\sigma B^H_{\frac{j+1}{n}} - \sigma B^H_{\frac{j}{n}}\right)^2 < 2L\right]$$
$$+P\left[\sum_{j=0}^{n-1} 2\left|\left(\nu\left(\frac{j+1}{n}\right) - \nu\left(\frac{j}{n}\right)\right)\left(\sigma B^H_{\frac{j+1}{n}} - \sigma B^H_{\frac{j}{n}}\right)\right| > L\right] < \frac{1}{L}\,.$$

This shows that $(S_t)_{t\in[0,1]}$ satisfies (2.3.9). By Lemma 2.12 there exists for all $n \in \mathbb{N}$, a $\gamma(n) \in \mathbf{S}(\mathbb{F}^Z)$ such that

a) $P[(\gamma(n)\cdot S)_1 < c] < \frac{1}{n}$
b) $\inf_{t\in[0,1]}(\gamma(n)\cdot S)_t \geq -\frac{1}{n}$
c) $|\gamma(n)| \leq \frac{1}{n}$.

Having shown this, we can construct $\vartheta^1(n)$ as in (i). By c) we get $\left|\vartheta^1(n)\right| \leq \frac{1}{n}$.

(iv) $H \in (0, \frac{1}{2})$, $\quad S_t = \exp\left(\nu(t) + \sigma B^H_t\right), t \in [0,1]$:
Since $(S_t)_{t\in[0,1]}$ is positive and continuous, $\min_{v\in[0,1]} S_v > 0$. Therefore, there exists an $\varepsilon > 0$ such that

$$P\left[\min_{v\in[0,1]} S_v \leq \varepsilon\right] < \frac{1}{2L}\,.$$

It follows from what we have shown in the proof of (iii) that there exists a partition $0 = t_0 < \cdots < t_n = 1$, such that

$$P\left[\sum_{j=0}^{n-1}\left(\ln S_{t_{j+1}} - \ln S_{t_j}\right)^2 < \frac{1}{\varepsilon^2}L\right] < \frac{1}{2L}\,.$$

Since for all j,

$$\left|S_{t_{j+1}} - S_{t_j}\right| \geq \left(\min_{v\in[0,1]} S_v\right)\left|\ln S_{t_{j+1}} - \ln S_{t_j}\right|,$$

we obtain

$$P\left[\sum_{j=0}^{n-1}\left(S_{t_{j+1}}-S_{t_j}\right)^2<L\right]$$

$$\le P\left[\min_{v\in[0,1]} S_v \le \varepsilon\right]+P\left[\sum_{j=0}^{n-1}\left(\ln S_{t_{j+1}}-\ln S_{t_j}\right)^2<\frac{1}{\varepsilon^2}L\right]<\frac{1}{L}.$$

This shows that $(S_t)_{t\in[0,1]}$ satisfies (2.3.9). Thus, $\vartheta^1(n)$ can be constructed as in (iii). Again $\left|\vartheta^1(n)\right| \le \frac{1}{n}$. This completes the proof of the theorem. □

Proof of Theorem 2.10 Since B^H is self-similar, it is enough to prove the theorem for $T=1$. We split (0, 1] into the subintervals

$$I_n=(a_n=1-2^{1-n}, b_n=1-2^{-n}],\ n\in\mathbb{N}.$$

By S^n we denote the restriction of S to I_n and by $\mathbb{F}^{S^n}=\left(\mathcal{F}_t^{S^n}\right)_{t\in I_n}$ the filtration generated by S^n. Note that $\mathcal{F}_t^{S^n}\subset\mathcal{F}_t^S$ for all $n\in\mathbb{N}$ and $t\in I_n$.

Since B^H has stationary increments, it follows from Theorem 2.9 that there exists for all $n\in\mathbb{N}$, a $\gamma(n)\in\mathbf{S}(\mathbb{F}^{S^n})$ such that

a) $P[(\gamma(n)\cdot S^n)_{b_n}<c+\frac{1}{c}]<\frac{1}{n}$
b) $\inf_{t\in I_n}(\gamma(n)\cdot S^n)_t\ge-\frac{1}{2^n c}$.

For

$$\gamma=\sum_{n=1}^{\infty}\gamma(n)1_{I_n},$$

$$\xi=\inf\left\{t\in[0,1]\,:\,(\gamma\cdot S)_t=c\right\}\quad(\text{we set }\inf\emptyset=1)$$

is an $\mathbb{F}^S$-stopping time. a) and b) imply $P[\xi<1]=1$. Therefore, $\vartheta^1=\gamma\cdot 1_{[0,\xi]}$ belongs to $\mathbf{aS}(\mathbb{F}^S)$ and

$$\left(\vartheta^0,\vartheta^1\right)$$

with

$$\vartheta_t^0=\left(\vartheta^1\cdot S\right)_t-\vartheta_t^1 S_t\,,\ t\in[0,1],$$

is a $\frac{1}{c}$-admissible c-arbitrage in $\Theta^{\mathbf{aS}}_{\mathrm{sf,adm}}(\mathbb{F}^S)$. In the cases (iii) and (iv), all $\gamma(n)$'s can be chosen such that $|\gamma(n)|\le\frac{1}{c}$. Then $\left|\vartheta^1\right|\le\frac{1}{c}$ too, and the theorem is proved. □

Remarks 2.14

1. In a market model $\left(\left(\tilde{S}^0_t\right)_{t\in[0,T]}, \left(\tilde{S}_t\right)_{t\in[0,T]}\right)$ with strong arbitrage it is possible to super-replicate a European call option with time-T pay-off $\tilde{C}_T = \left(\tilde{S}_T - K\right)^+$, $K > 0$, without initial endowment in the following way: At time 0 one borrows money from the money market account to buy one stock share. Then one applies a strong arbitrage strategy to generate the amount of money needed to pay back ones debts without selling the stock share. At time T one owns a stock share and has no debts. This hedges the option. The following example shows that a European call option can have a positive super-replication price if the model $\left(\left(\tilde{S}^0_t\right)_{t\in[0,T]}, \left(\tilde{S}_t\right)_{t\in[0,T]}\right)$ only admits arbitrage:
Let $(\Omega, \mathcal{A}, P)$ be a probability space with a Brownian motion B and an independent fractional Brownian motion B^H, $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Furthermore, let ξ be a random variable on $(\Omega, \mathcal{A}, P)$ that is independent of B and B^H and such that $P[\xi = 0] = P[\xi = 1] = \frac{1}{2}$. Let r, ν and $\sigma > 0$, be constants. The model $\left(\left(\tilde{S}^0_t\right)_{t\in[0,T]}, \left(\tilde{S}_t\right)_{t\in[0,1]}\right)$ with

$$\tilde{S}^0_t = \exp(rt) \quad \text{and} \quad \tilde{S}_t = \exp\left\{(r+\nu)t + \sigma\left((1-\xi)B_t + \xi B^H_t\right)\right\}, \quad t \in [0,1],$$

has arbitrage but no strong arbitrage in $\Theta^{\mathbf{aS}}_{\mathrm{sf,adm}}(\mathbb{F}^{\tilde{S}})$. Is is clear that the super-replication of $\tilde{C}_1$ with a strategy from $\Theta^{\mathbf{aS}}_{\mathrm{sf,adm}}(\mathbb{F}^{\tilde{S}})$ costs at least the Black-Scholes price.
2. As we mentioned in the introduction, it is shown in Salopek (1998) that a stochastic process Z which is almost surely continuous and of bounded p-variation for some $p < 2$, can be integrated path-wise with respect to itself, and

$$\int_0^t 2\,(Z_u - Z_0)\,dZ_u = (Z_t - Z_0)^2 \quad \text{for all} \quad t \in [0,T]. \tag{2.3.15}$$

The process (2.3.7), which is the building block for our arbitrage strategy in the cases (i) and (ii) of Theorem 2.9, is a multiple of a discrete version of the integrand in (2.3.15).
3. It is clear that Theorem 2.9 cannot only be applied to models

$$\left(\left(\tilde{S}^0_t\right)_{t\in[0,T]}, \left(\tilde{S}_t\right)_{t\in[0,T]}\right)$$

with

$$S_t = \nu(t) + \sigma B_t^H \quad \text{or} \quad S_t = \exp\left(\nu(t) + \sigma B_t^H\right) ,$$

but to all models $\left(\left(\tilde{S}_t^0\right)_{t\in[0,T]}, \left(\tilde{S}\right)_{t\in[0,T]}\right)$ such that $(S_t)_{t\in[0,T]}$ satisfies conditions (2.3.1) and (2.3.2) of Lemma 2.11 or condition (2.3.9) of Lemma 2.12. In particular, condition (2.3.2) is fulfilled by all processes with vanishing quadratic variation, and all processes with infinite quadratic variation satisfy condition (2.3.9). For different generalizations of Lemma 1.4 see e.g. Shao (1996), Takashima (1989) or Kôno and Maejima (1991). Shao (1996) contains results on p-variation of Gaussian processes with stationary increments. Takashima (1989) gives sample path properties of ergodic self-similar processes, and in Kôno and Maejima (1991), results on Hölder continuity of sample paths of some self-similar stable processes can be found.

2.4 Exclusion of arbitrage

The arbitrage strategies that we constructed in Section 3 act on ever smaller time intervals. They can be excluded by introducing a minimal amount of time $h > 0$ that must lie between two consecutive transactions.

Definition 2.15 *Let $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be a filtration and $h > 0$.*

$$\mathbf{S}^h(\mathbb{F}) := \left\{ g_0 1_{\{0\}} + \sum_{j=1}^{n-1} g_j 1_{(\tau_j, \tau_{j+1}]} \in \mathbf{S}(\mathbb{F}) \;:\; \tau_{j+1} \geq \tau_j + h \,, \; \forall j \right\} .$$

$$\Theta_{\mathrm{sf}}^h(\mathbb{F}) := \left\{ \vartheta \in \Theta_{\mathrm{sf}}^{\mathbf{S}} \;:\; \vartheta^0, \vartheta^1 \in \mathbf{S}^h(\mathbb{F}) \right\} . \tag{2.4.1}$$

In the following we will show that none of the models (i)-(iv) of Theorem 2.9 has an arbitrage in $\bigcup_{h>0} \Theta_{\mathrm{sf}}^h(\mathbb{F}^S)$.

Lemma 2.16 *Let $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ and $(B_t)_{t\geq 0}$ a one-sided Brownian motion. Let $(Z_t)_{t\geq 0}$ be a continuous version of $\left(\int_0^t (t-s)^{H-\frac{1}{2}} dB_s\right)_{t\geq 0}$. Then, for all $c \geq 0$ and all h and T such that $0 < h \leq T$,*

$$P\left[\inf_{t\in[h,T]} Z_t \geq c\right] = P\left[\sup_{t\in[h,T]} Z_t \leq -c\right] > 0 .$$

Proof. Let $c \geq 0$ and $0 < h \leq T$.

$$P\left[\inf_{t\in[h,T]} Z_t \geq c\right] = P\left[\sup_{t\in[h,T]} Z_t \leq -c\right]$$

follows from the fact that $(-Z_t)_{t\geq 0}$ has the same distribution as $(Z_t)_{t\geq 0}$. Theorem 2.9.25 of Karatzas and Shreve (1988) shows that for all $n \in \mathbb{N}$, there exists a measurable set $\Omega_n \subset \Omega$ with $P[\Omega_n] = 1$ such that for all $\omega \in \Omega_n$ and all $t \in [0, n]$,

$$\lim_{s\to t} \frac{B_t - B_s}{\sqrt{|t-s|}\log\left(\frac{1}{|t-s|}\right)} = 0\,. \tag{2.4.2}$$

For $\tilde{\Omega} = \bigcap_{n=1}^{\infty}$, $P[\tilde{\Omega}] = 1$, and (2.4.2) holds for all $\omega \in \tilde{\Omega}$ and $t \geq 0$. Hence, $(B_t)_{t\geq 0}$ induces Wiener measure Q_W on $\left(\hat{\Omega}, \mathcal{B}\right)$, where

$$\hat{\Omega} = \left\{\omega \in C[0,\infty) \,:\, \omega(0) = 0\,,\ \lim_{s\to t} \frac{\omega(t) - \omega(s)}{\sqrt{|t-s|}\log\left(\frac{1}{|t-s|}\right)} = 0\,,\ \forall t \geq 0\right\}$$

and $\mathcal{B}$ is the σ-algebra of subsets of $\hat{\Omega}$ generated by the cylinder sets. Note that for all $\omega \in \hat{\Omega}$,

$$\int_0^t (t-s)^{H-\frac{1}{2}} d\omega(s)$$

can for all $t \geq 0$, be defined as an improper Riemann-Stieltjes integral which is continuous in t. Hence,

$$P\left[\inf_{t\in[h,T]} Z_t \geq c\right] = Q_W\left[\inf_{t\in[h,T]} \int_0^t (t-s)^{H-\frac{1}{2}} d\omega(s) \geq c\right].$$

Let us first assume $H \in (\frac{1}{2}, 1)$. In this case we set

$$m = \frac{H+\frac{1}{2}}{h^{H+\frac{1}{2}}}\left[c + T^{H-\frac{1}{2}}\right], \quad \omega_m(t) = \omega(t) - mt\,,\ t \in [0,T]$$

and

$$A_m = \left\{\omega \in \hat{\Omega} \,:\, \sup_{t\in[0,T]} |\omega_m(t)| \leq 1\right\}.$$

By Girsanov's Theorem there exists a probability measure Q_m that is equivalent to Q_W such that $(\omega_m(t))_{t\in[0,T]}$ is a Brownian motion under Q_m. It is well known that $Q_m[A_m] > 0$. Equivalence of Q_W and Q_m implies that also

$$Q_W[A_m] > 0\,. \tag{2.4.3}$$

For all $\omega \in \hat{\Omega}$ and $t \geq 0$,

$$\int_0^t (t-s)^{H-\frac{1}{2}} d\omega(s) = \int_0^t \omega(s)(H-\frac{1}{2})(t-s)^{H-\frac{3}{2}} ds\,,$$

$$= (H-\frac{1}{2})\int_0^t \omega_m(s)(t-s)^{H-\frac{3}{2}} ds + (H-\frac{1}{2})m\int_0^t s(t-s)^{H-\frac{3}{2}} ds$$

$$= (H-\frac{1}{2})\int_0^t \omega_m(s)(t-s)^{H-\frac{3}{2}} ds + m\frac{t^{H+\frac{1}{2}}}{H+\frac{1}{2}}$$

For $\omega \in A_m$, we obtain for all $t \in [h, T]$ the following estimates:

$$(H-\frac{1}{2})\int_0^t \omega_m(s)(t-s)^{H-\frac{3}{2}} ds \geq -(H-\frac{1}{2})\int_0^t (t-s)^{H-\frac{3}{2}} ds$$

$$= -t^{H-\frac{1}{2}} \geq -T^{H-\frac{1}{2}}$$

and, by our choice of m,

$$m\frac{t^{H+\frac{1}{2}}}{H+\frac{1}{2}} = \left(\frac{t}{h}\right)^{H+\frac{1}{2}} \left(c + T^{H-\frac{1}{2}}\right) \geq c + T^{H-\frac{1}{2}}\,.$$

Hence,

$$\int_0^t (t-s)^{H-\frac{1}{2}} d\omega(s) ds \geq -T^{H-\frac{1}{2}} + c + T^{H-\frac{1}{2}} = c\,.$$

It follows that

$$A_m \subset \left\{ \inf_{t\in[h,T]} \int_0^t (t-s)^{H-\frac{1}{2}} d\omega(s) \geq c \right\}\,.$$

This and (2.4.3) prove the lemma for $H \in (\frac{1}{2}, 1)$.
For $H \in (0, \frac{1}{2})$, the proof is slightly more delicate. It follows from

$$Q_W\left[\sup_{t\in[0,T]} |\omega(t)| \leq \frac{1}{2}\right] > 0$$

and Lemma 1.4 a) that there exist constants $\varepsilon \in (0, h)$ and $\delta > 0$ such that

$$Q_W\left[A(\frac{1}{2}, \varepsilon, \delta)\right] > 0\,,$$

where

$$A(\frac{1}{2}, \varepsilon, \delta) = \left\{ \omega \in \hat{\Omega} \,:\, \sup_{t \in [0,T]} |\omega(t)| \leq \frac{1}{2} \text{ and } \sup_{\substack{t,s \in [0,T]; \\ 0<t-s<\varepsilon}} \frac{|\omega(t) - \omega(s)|}{(t-s)^{\frac{1}{2}-\frac{H}{2}}} \leq \delta \right\}$$

We set

$$m = \frac{H + \frac{1}{2}}{h^{H+\frac{1}{2}}} \left[c + \varepsilon^{H-\frac{1}{2}} + (\frac{1}{2} - H) \frac{2\delta}{H} \varepsilon^{\frac{H}{2}} + \frac{1}{2} h^{H-\frac{1}{2}} \right],$$

$$\omega_m(t) = \omega(t) - mt \,,\ t \in [0, T]$$

and Q_m as before. Furthermore, we define

$$A_m(\frac{1}{2}, \varepsilon, \delta) = \left\{ \omega \in \hat{\Omega} \,:\, \omega_m \in A(\frac{1}{2}, \varepsilon, \delta) \right\}.$$

Since $(\omega_m(t))_{t \in [0,T]}$ is a Brownian motion under Q_m,

$$Q_m \left[A_m(\frac{1}{2}, \varepsilon, \delta) \right] = Q_W \left[A(\frac{1}{2}, \varepsilon, \delta) \right] > 0 \,.$$

Hence, also

$$Q_W \left[A_m(\frac{1}{2}, \varepsilon, \delta) \right] > 0 \,. \tag{2.4.4}$$

For $\omega \in \hat{\Omega}$ and $t \geq h$, we can write

$$\begin{aligned}
&\int_0^t (t-s)^{H-\frac{1}{2}} d\omega(s) = \int_0^t (t-s)^{H-\frac{1}{2}} d\,[\omega(s) - \omega(t)] \\
=\ & (\frac{1}{2} - H) \int_0^t [\omega(t) - \omega(s)]\,(t-s)^{H-\frac{3}{2}} ds + t^{H-\frac{1}{2}} \omega(t) \\
=\ & (\frac{1}{2} - H) \int_0^t [\omega_m(t) - \omega_m(s)]\,(t-s)^{H-\frac{3}{2}} ds \\
+\ & (\frac{1}{2} - H) m \int_0^t (t-s)^{H-\frac{1}{2}} ds + t^{H-\frac{1}{2}} \omega_m(t) + m t^{H+\frac{1}{2}} \\
=\ & (\frac{1}{2} - H) \int_0^{t-\varepsilon} [\omega_m(t) - \omega_m(s)]\,(t-s)^{H-\frac{3}{2}} ds
\end{aligned}$$

$$+(\frac{1}{2} - H) \int_{t-\varepsilon}^t [\omega_m(t) - \omega_m(s)]\,(t-s)^{H-\frac{3}{2}} ds + t^{H-\frac{1}{2}} \omega_m(t) + m \frac{t^{H+\frac{1}{2}}}{H + \frac{1}{2}} \,.$$

If $\omega \in A_m(\frac{1}{2}, \varepsilon, \delta)$ and $t \in [h, T]$, we can estimate the four preceding terms as follows:

$$\begin{aligned}
& (\frac{1}{2} - H) \int_0^{t-\varepsilon} [\omega_m(t) - \omega_m(s)]\,(t-s)^{H-\frac{3}{2}} ds \\
\geq\ & -(\frac{1}{2} - H) \int_0^{t-\varepsilon} (t-s)^{H-\frac{3}{2}} ds = -\varepsilon^{H-\frac{1}{2}} + t^{H-\frac{1}{2}} \geq -\varepsilon^{H-\frac{1}{2}}\,, \\
& (\frac{1}{2} - H) \int_{t-\varepsilon}^{t} [\omega_m(t) - \omega_m(s)]\,(t-s)^{H-\frac{3}{2}} ds \\
\geq\ & -(\frac{1}{2} - H) \int_{t-\varepsilon}^{t} \delta (t-s)^{\frac{H}{2}-1} ds = -(\frac{1}{2} - H)\frac{2\delta}{H}\varepsilon^{\frac{H}{2}}\,, \\
& t^{H-\frac{1}{2}} \omega_m(t) \geq -\frac{1}{2} h^{H-\frac{1}{2}}
\end{aligned}$$

and

$$\begin{aligned}
m\frac{t^{H+\frac{1}{2}}}{H+\frac{1}{2}} &= \left(\frac{t}{h}\right)^{H+\frac{1}{2}} \left[c + \varepsilon^{H-\frac{1}{2}} + (\frac{1}{2} - H)\frac{2\delta}{H}\varepsilon^{\frac{H}{2}} + \frac{1}{2}h^{H-\frac{1}{2}}\right] \\
&\geq c + \varepsilon^{H-\frac{1}{2}} + (\frac{1}{2} - H)\frac{2\delta}{H}\varepsilon^{\frac{H}{2}} + \frac{1}{2}h^{H-\frac{1}{2}}\,.
\end{aligned}$$

Hence,

$$\int_0^t (t-s)^{H-\frac{1}{2}} d\omega(s) \geq$$

$$-\varepsilon^{H-\frac{1}{2}} - (\frac{1}{2} - H)\frac{2\delta}{H}\varepsilon^{\frac{H}{2}} - \frac{1}{2}h^{H-\frac{1}{2}} + c + \varepsilon^{H-\frac{1}{2}} + (\frac{1}{2} - H)\frac{2\delta}{H}\varepsilon^{\frac{H}{2}} + \frac{1}{2}h^{H-\frac{1}{2}} = c\,.$$

This and (2.4.4) prove the lemma for $H \in (0, \frac{1}{2})$. □

Theorem 2.17 *Let B^H be a fractional Brownian motion with $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. Let $T \in (0, \infty)$, $\sigma > 0$ and $\nu : [0, T] \to \mathbb{R}$ be a measurable function such that* $\sup_{t\in[0,T]} |\nu(t)| < \infty$. *Consider the two cases*

(i) $\quad S_t = \nu(t) + \sigma B_t^H,\ t \in [0, T]$

(ii) $\quad S_t = \exp\left(\nu(t) + \sigma B_t^H\right),\ t \in [0, T]$

If

$$\vartheta^1 = g_0 1_{\{0\}} + \sum_{j=1}^{n-1} g_j 1_{(\tau_j, \tau_{j+1}]} \in \bigcup_{h>0} \mathbf{S}^h(\mathbb{F}^S)$$

and there exists a $j \in \{1, \ldots, n-1\}$ *with* $P\left[g_j \neq 0\right] > 0$,
then in case (i),

$$P\left[\left(\vartheta^1 \cdot S\right)_T \le -c\right] > 0 \quad \textit{for all } c \ge 0\,,$$

and in case (ii),

$$P\left[\left(\vartheta^1 \cdot S\right)_T < 0\right] > 0\,.$$

Proof. For notational simplicity we give the proof for $S_t = B_t^H$ and $S_t = \exp\left(B_t^H\right)$. The generalizations to the cases (i) and (ii) are obvious. To prove the theorem for $S_t = B_t^H$ we fix an $h > 0$, and take a

$$\vartheta^1 = g_0 1_{\{0\}} + \sum_{j=1}^{n-1} g_j 1_{(\tau_j, \tau_{j+1}]} \in \mathbf{S}^h(\mathbb{F}^{B^H})\,,$$

such that there exists a $j \in \{1, \ldots, n-1\}$ with $P\left[g_j \neq 0\right] > 0$. If

$$k = \max\left\{j \in \{1, \ldots, n-1\} \,:\, P\left[g_j \neq 0\right] > 0\right\}\,,$$

then

$$\left(\vartheta^1 \cdot B^H\right)_T = \sum_{j=1}^{k} g_j \left(B^H_{\tau_{j+1}} - B^H_{\tau_j}\right) \text{ almost surely.}$$

Let $c \ge 0$. It is clear that

$$P\left[\sum_{j=1}^{k} g_j \left(B^H_{\tau_{j+1}} - B^H_{\tau_j}\right) \le -c\right] \tag{2.4.5}$$

$$\ge P\left[\sum_{j=1}^{k-1} g_j \left(B^H_{\tau_{j+1}} - B^H_{\tau_j}\right) + \sup_{t \in [h,T]} g_k \left(B^H_{\tau_k + t} - B^H_{\tau_k}\right) \le -c\right]\,.$$

Let

$$\hat{\Omega} = \left\{\omega \in C(\mathbb{R}) \,:\, \omega(0) = 0\,;\ \lim_{s \to t} \frac{\omega(t) - \omega(s)}{\sqrt{|t-s|}\log\left(\frac{1}{|t-s|}\right)} = 0\,,\ \forall t \ge \mathbb{R}\right\}\,,$$

$\mathcal{B}$ the σ-algebra of subsets of $\hat{\Omega}$ that is generated by the cylinder sets and P the Wiener measure on $\left(\hat{\Omega}, \mathcal{B}\right)$. Without loss of generality we can assume that

$\left(B_t^H\right)_{t\geq 0}$ is defined on $\left(\hat{\Omega}, \mathcal{B}, P\right)$ by the improper Riemann-Stieltjes integrals

$$B_t^H(\omega) = \int_{-\infty}^{t} \left[(t-s)^{H-\frac{1}{2}} - 1_{\{s\leq 0\}}(-s)^{H-\frac{1}{2}}\right] d\omega(s)\,,\ t \geq 0\,. \qquad (2.4.6)$$

We define the filtration $\mathbb{F}^{\hat{\Omega}} = \left(\mathcal{F}_t^{\hat{\Omega}}\right)_{t\in[0,T]}$ by

$$\mathcal{F}_t^{\hat{\Omega}} = \sigma\left\{\left\{\omega \in \hat{\Omega}\ :\ \omega(s) \leq a\right\}\ :\ -\infty < s \leq t\,,\ a \in \mathbb{R}\right\}\,.$$

It is clear that $\mathbb{F}^{\hat{\Omega}}$ is bigger than the filtration $\mathbb{F}^{B^H} = \left(\mathcal{F}_t^{B^H}\right)_{t\in[0,T]}$, which is given by

$$\mathcal{F}_t^{B^H} = \sigma\left\{B_s^H\ :\ 0 \leq s \leq t\right\}\,.$$

Therefore the $\mathbb{F}^{B^H}$-stopping times $\tau_1, \ldots \tau_k$, are also $\mathbb{F}^{\hat{\Omega}}$-stopping times. In the following we split each function $\omega \in \hat{\Omega}$ at the time point $\tau_k(\omega)$. We set

$$\pi_1\omega(s) = \omega(s)1_{(-\infty,\tau_k(\omega)]}(s)\,,\ s \in \mathbb{R}\,,$$

$$\pi_2\omega(s) = \omega(\tau_k(\omega)+s) - \omega(\tau_k(\omega))\,,\ s \geq 0\,,$$

and let

$$\Omega_1 = \left\{\pi_1(\omega) \in \mathbb{R}^{\mathbb{R}}\ :\ \omega \in \hat{\Omega}\right\}\,,$$

$\mathcal{B}_1$ the σ-algebra of subsets of Ω_1 that is generated by the cylinder sets,

$$\Omega_2 = \left\{\pi_2(\omega) \in C[0,\infty)\ :\ \omega \in \hat{\Omega}\right\}$$

and $\mathcal{B}_2$ the σ-algebra of subsets of Ω_2 that is generated by the cylinder sets. It can easily be checked that the mapping

$$\pi_1 : \left(\hat{\Omega}, \mathcal{B}\right) \to (\Omega_1, \mathcal{B}_1)$$

is $\mathcal{F}_{\tau_k}^{\hat{\Omega}}$-measurable. On the other hand, it follows from Theorem I.32 of Protter (1990) that $(\pi_2\omega(s))_{s\geq 0}$ is a Brownian motion under P which is independent of $\mathcal{F}_{\tau_k}^{\hat{\Omega}}$. It can be seen from (2.4.6) that for all $\omega \in \hat{\Omega}$ and $t \in [h, T]$,

$$\left(\sum_{j=1}^{k-1} g_j\left(B_{\tau_{j+1}}^H - B_{\tau_j}^H\right) + g_k\left(B_{\tau_k+t}^H - B_{\tau_k}^H\right)\right)(\omega) = U_t(\pi_1\omega, \pi_2\omega)$$

where for $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$ and $t \in [h, T]$,

$$U_t(\omega_1, \omega_2) = U^0(\omega_1) + g_k(\omega_1) \left(U_t^1(\omega_1) + U_t^2(\omega_2) \right) ,$$

and

$$U^0(\omega_1) = \left(\sum_{j=1}^{k-1} g_j \left(B^H_{\tau_{j+1}} - B^H_{\tau_j} \right) \right) (\omega_1) ,$$

$$U_t^1(\omega_1) = \int_{-\infty}^{\tau_k(\omega_1)} \left[(\tau_k(\omega_1) + t - s)^{H-\frac{1}{2}} - (\tau_k(\omega_1) - s)^{H-\frac{1}{2}} \right] d\omega_1(s) ,$$

$$U_t^2(\omega_2) = \int_0^t (t-s)^{H-\frac{1}{2}} d\omega_2(s) .$$

Since $(U_t)_{t \in [h,T]}$ is a continuous stochastic process on $(\Omega_1 \times \Omega_2, \mathcal{B}_1 \otimes \mathcal{B}_1)$, the set

$$A = \left\{ (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 \, : \, \sup_{t \in [h,T]} U_t(\omega_1, \omega_2) \leq -c \right\}$$

is $\mathcal{B}_1 \otimes \mathcal{B}_2$-measurable. It follows from Proposition A.2.5 of Lamberton and Lapeyre (1996) that for almost every $\omega \in \hat{\Omega}$,

$$\mathrm{E} \left[1_A \left(\pi_1, \pi_2 \right) \mid \mathcal{F}^{\hat{\Omega}}_{\tau_k} \right] (\omega) = \phi \left(\pi_1 \omega \right) ,$$

where

$$\phi : \Omega_1 \to \mathbb{R}$$

is defined by

$$\phi \left(\omega_1 \right) = \mathrm{E} \left[1_A \left(\omega_1, \pi_2 \right) \right] , \omega_1 \in \Omega_1 .$$

Since $U_t^1(\omega_1)$ is for all $\omega_1 \in \Omega_1$ continuous in t, $\sup_{t \in [h,T]} U_t^1(\omega_1)$ is for all $\omega_1 \in \Omega_1$ finite. Therefore and since $(\pi_2 \omega(t))_{t \geq 0}$ is a Brownian motion under P, it follows from Lemma 2.16 that for all $\omega_1 \in \Omega_1$ with $g_k(\omega_1) \neq 0$,

$$\phi(\omega_1) = P \left[\sup_{t \in [h,T]} U_t(\omega_1, \pi_2) \leq -c \right]$$

$$\geq P \left[U^0(\omega_1) + \sup_{t \in [h,T]} g_k(\omega_1) U_t^1(\omega_1) + \sup_{t \in [h,T]} g_k(\omega_1) U_t^2(\pi_2) \leq -c \right] > 0 .$$

Since

$$P \left[g_k \circ \pi_1 \neq 0 \right] > 0 ,$$

we have

$$P\left[\sum_{j=1}^{k-1} g_j\left(B^H_{\tau_{j+1}} - B^H_{\tau_j}\right) + \sup_{t\in[h,T]} g_k\left(B^H_{\tau_k+t} - B^H_{\tau_k}\right) \le -c\right]$$

$$= \mathrm{E}\left[1_A(\pi_1,\pi_2)\right] = \mathrm{E}\left[\mathrm{E}\left[1_A(\pi_1,\pi_2) \mid \mathcal{F}^{\hat{\Omega}}_{\tau_k}\right]\right] = \mathrm{E}\left[\phi\circ\pi_1\right] > 0\,.$$

This and (2.4.5) prove the theorem in the case $S_t = B^H_t$.
If $S_t = \exp\left(B^H_t\right)$, let us assume there exists an $h > 0$ and a

$$\vartheta^1 = g_0 1_{\{0\}} + \sum_{j=1}^{n-1} g_j 1_{(\tau_j,\tau_{j+1}]} \in \mathbf{S}^h(\mathbb{F}^{B^H})$$

such that $\left(\vartheta^1\cdot S\right)_T \ge 0$ almost surely and there exists a $j \in \{1,\dots,n-1\}$ with $P\left[g_j \neq 0\right] > 0$. If

$$k = \min\left\{ l\ :\ P\left[g_l \neq 0\right] > 0 \text{ and } \sum_{j=1}^{l} g_j\left(\mathrm{e}^{B^H_{\tau_{j+1}}} - \mathrm{e}^{B^H_{\tau_j}}\right) \ge 0 \text{ a.s.}\right\}\,,$$

then either

$$g_1 = \cdots = g_{k-1} = 0 \text{ almost surely}$$

or

$$P\left[\sum_{j=1}^{k-1} g_j\left(\mathrm{e}^{B^H_{\tau_{j+1}}} - \mathrm{e}^{B^H_{\tau_j}}\right) < 0\right] > 0\,.$$

In both cases, $P[C] > 0$ for

$$C = \left\{\sum_{j=1}^{k-1} g_j\left(\mathrm{e}^{B^H_{\tau_{j+1}}} - \mathrm{e}^{B^H_{\tau_j}}\right) \le 0\,,\ g_k \neq 0\right\}\,.$$

With the same method that we used in the first part of the proof one can deduce from Lemma 2.16 that for almost all $\omega \in C$,

$$P\left[\sum_{j=1}^{k-1} g_j\left(\mathrm{e}^{B^H_{\tau_{j+1}}} - \mathrm{e}^{B^H_{\tau_j}}\right) + \sup_{t\in[h,T]} g_k\left(\mathrm{e}^{B^H_{\tau_k+t}} - \mathrm{e}^{B^H_{\tau_k}}\right) < 0 \,\middle|\, \mathcal{F}^{\hat{\Omega}}_{\tau_k}\right](\omega) > 0\,.$$

Hence,

$$P\left[\sum_{j=1}^{k} g_j \left(e^{B^H_{\tau_{j+1}}} - e^{B^H_{\tau_j}}\right) < 0\right]$$

$$\geq P\left[\sum_{j=1}^{k-1} g_j \left(e^{B^H_{\tau_{j+1}}} - e^{B^H_{\tau_j}}\right) + \sup_{t\in[h,T]} g_k \left(e^{B^H_{\tau_k+t}} - e^{B^H_{\tau_k}}\right) < 0\right]$$

$$= \mathrm{E}\left[P\left[\sum_{j=1}^{k-1} g_j \left(e^{B^H_{\tau_{j+1}}} - e^{B^H_{\tau_j}}\right) + \sup_{t\in[h,T]} g_k \left(e^{B^H_{\tau_k+t}} - e^{B^H_{\tau_k}}\right) < 0 \,\middle|\, \mathcal{F}^{\hat{\Omega}}_{\tau_k}\right]\right]$$

$$\geq \mathrm{E}\left[1_C P\left[\sum_{j=1}^{k-1} g_j \left(e^{B^H_{\tau_{j+1}}} - e^{B^H_{\tau_j}}\right) \right.\right.$$
$$\left.\left. + \sup_{t\in[h,T]} g_k \left(e^{B^H_{\tau_k+t}} - e^{B^H_{\tau_k}}\right) < 0 \,\middle|\, \mathcal{F}^{\hat{\Omega}}_{\tau_k}\right]\right] > 0\,.$$

This contradicts our assumption and the theorem is proved. □

It follows from Theorem 2.17 that in both cases

$$\begin{array}{ll} \text{(i)} & S_t = \nu(t) + \sigma B^H_t,\ t \in [0,T]\,, \quad \text{and} \\ \text{(ii)} & S_t = \exp\left(\nu(t) + \sigma B^H_t\right),\ t \in [0,T]\,, \end{array}$$

the model $\left(\left(\tilde{S}^0_t\right)_{t\in[0,T]}, \left(\tilde{S}_t\right)_{t\in[0,T]}\right)$ has no arbitrage in $\bigcup_{h>0}\Theta^h_{\rm sf}(\mathbb{F}^S)$. Moreover, in case (i) there exist no non-trivial admissible strategies in $\bigcup_{h>0}\Theta^h_{\rm sf}(\mathbb{F}^S)$. An inspection of the proof of Theorem 2.17 shows that in case (ii), a $\vartheta \in \bigcup_{h>0}\Theta^h_{\rm sf}(\mathbb{F}^S)$ can only be admissible if ϑ^1 is almost surely non-negative.

Clearly, the class $\Theta^{\mathbf{S}}_{\rm sf}(\mathbb{F}^S)$ is bigger than $\bigcup_{h>0}\Theta^h_{\rm sf}(\mathbb{F}^S)$. It is an open problem whether or not models of the form (i) and (ii) have arbitrage in $\Theta^{\mathbf{S}}_{\rm sf}(\mathbb{F}^S)$ or $\Theta^{\mathbf{S}}_{\rm sf,adm}(\mathbb{F}^S)$.

It follows from similar arguments to the ones in the proof of Theorem 2.17 that in both cases (i) and (ii) the cheapest way to super-replicate a European call option with a strategy $\vartheta \in \bigcup_{h>0}\Theta^h_{\rm sf}(\mathbb{F}^S)$ is to buy the stock. In particular, in both cases (i) and (ii) of Theorem 2.17 the model $\left(\left(\tilde{S}^0\right)_{t\in[0,T]}, \left(\tilde{S}_t\right)_{t\in[0,T]}\right)$ is incomplete when trading strategies are restricted to $\bigcup_{h>0}\Theta^h_{\rm sf}(\mathbb{F}^S)$.

Chapter 3

Regularized fractional Brownian motion and option pricing

3.1 Introduction

For simplicity we will from now on consider market models

$$\left(\left(\tilde{S}^0_t\right)_{t\in[0,T]}, \left(\tilde{S}_t\right)_{t\in[0,T]}\right)$$

with $\tilde{S}^0_t = \mathrm{e}^{rt}$, $t \in [0, T]$, for some $r > 0$. In this case, $\mathbb{F}^{\tilde{S}} = \mathbb{F}^S$, and the model is specified if the evolution of the discounted stock price S is given.

A way to make the fractional Brownian motion models

$$S_t = S_0 + \nu t + \sigma B^H_t \ , \ t \in [0, T] \, , \tag{3.1.1}$$

$$S_t = S_0 \exp\left(\nu t + \sigma B^H_t\right) \ , \ t \in [0, T] \, , \tag{3.1.2}$$

arbitrage-free without restricting the trading strategies is indicated in the last section of Rogers (1997). Rogers (1997) regularizes fractional Brownian motion by changing the convolution kernel φ_H (1.2.4) in the Mandelbrot-Van Ness representation (1.2.3) of fractional Brownian motion. He gives a class of functions φ such that the stochastic process

$$R^\varphi_t = \int_{-\infty}^{t} [\varphi(t-s) - \varphi(-s)] \, dW_s \, , t \geq 0 \, , \tag{3.1.3}$$

is a Gaussian semimartingale with the same long-range dependence as fractional Brownian motion and proposes to use such a process for modelling a discounted stock price. However, the semimartingale property of the process (3.1.3) is not enough to ensure that the model

$$S_t = S_0 \exp\left(\nu t + \sigma \frac{R_t^{\varphi}}{\|R_1^{\varphi}\|_2}\right),\ t \in [0,T], \tag{3.1.4}$$

where $S_0 > 0$, $\nu, \sigma > 0$ are constants, is arbitrage-free.

Definition 3.1 *Let $(C[0,T],\mathcal{B})$ be the space of continuous functions with the σ-algebra generated by the cylinder sets. If $(Y_t)_{t\in[0,T]}$ is an a.s. continuous stochastic process, we denote by Q_Y the measure induced by Y on $(C[0,T],\mathcal{B})$. We call two a.s. continuous stochastic processes $(Y_t)_{t\in[0,T]}$ and $(Z_t)_{t\in[0,T]}$ equivalent if Q_Y and Q_Z are equivalent.*

The main result of this chapter is that for a larger class of functions φ than the one in Rogers (1997), the process $\left(R_t^{\varphi}\right)_{t\in[0,T]}$, given by (3.1.3), is not only a semimartingale but also equivalent to Brownian motion. This implies that the model (3.1.4) has a unique equivalent martingale measure. Hence, it is arbitrage-free and complete.

In Section 2 we construct for each $H \in (0,1)$, a class of processes whose finite-dimensional distributions are close to those of B^H and which have a unique equivalent martingale measure. In Section 3 we use these processes to build regularized fractional Samuelson models. Since these models have a unique equivalent martingale measure, option prices can be obtained by calculating conditional expectations. We discuss the pricing of a European call option in such a framework.

3.2 Regularizing fractional Brownian motion

3.2.1 General idea

In this subsection we give some heuristic arguments that indicate why for $H \in (0,\frac{1}{2}) \cup (\frac{1}{2},1)$, the behaviour of the function φ_H (1.2.4) near zero is responsible for the existence of arbitrage in the models (3.1.1), (3.1.2) and how φ_H can be regularized to yield a process that can be used to build an arbitrage-free stock price model with long-range dependence.

The arbitrage strategies in Section 2.3 consist of combinations of buy and hold strategies that act on ever smaller time intervals. For $H \in (0,\frac{1}{2})$, they exploit the fact that B^H has infinite quadratic variation. For $H \in (\frac{1}{2}, 1)$ they use that B^H is a non-constant process with vanishing quadratic variation. To exclude these arbitrage strategies we vary the local path behaviour of fractional Brownian motion in such a way that we obtain a process with non-zero, finite quadratic variation.

To sketch how this can be achieved we first show that the quadratic variation of B^H is related to the rate of convergence of $\mathrm{E}\left[\left(B_t^H\right)^2\right]$ to 0, as $t \searrow 0$. Since fractional Brownian motion has stationary increments, we have for all $t \geq 0$ and $s \geq 0$,

$$\mathrm{E}\left[\left(B_{s+t}^H - B_s^H\right)^2\right] = \mathrm{E}\left[\left(B_t^H\right)^2\right] = t^{2H}\,.$$

For $H \in (\frac{1}{2}, 1)$, we get for every partition

$$0 = t_0 < \cdots < t_n = T\,,$$

of $[0, T]$, the estimate

$$\mathrm{E}\left[\sum_{j=1}^{n}\left(B_{t_j}^H - B_{t_{j-1}}^H\right)^2\right] = \sum_{j=1}^{n}\left(t_j - t_{j-1}\right)^{2H} \leq \max_j \left(t_j - t_{j-1}\right)^{2H-1} T\,.$$

This shows that $\mathrm{E}\left[\sum_{j=1}^{n}\left(B_{t_j}^H - B_{t_{j-1}}^H\right)^2\right]$ converges to zero as the grid size of the partition goes to zero. Hence, B^H has vanishing quadratic variation for $H \in (\frac{1}{2}, 1)$. On the other hand, if $H \in (0, \frac{1}{2})$, then

$$\mathrm{E}\left[\sum_{j=1}^{n}\left(B_{j\frac{T}{n}}^H - B_{(j-1)\frac{T}{n}}^H\right)^2\right] = n\left(\frac{T}{n}\right)^{2H} \to \infty,\ \text{for } n \to \infty\,.$$

This indicates that B^H has infinite quadratic variation, for $H \in (0, \frac{1}{2})$. We have shown this rigorously in the proof of Lemma 1.4.

To see which part of the function φ_H (1.2.4) accounts for the behaviour of $\mathrm{E}\left[\left(B_t^H\right)^2\right]$ for small $t > 0$, we fix a small $\delta > 0$, and write

$$t^{2H} = \mathrm{E}\left[\left(B_t^H\right)^2\right] = c_H^2 \int_{-\infty}^{t} \left[\varphi_H(t-s) - \varphi_H(-s)\right]^2 ds$$

$$= c_H^2 \int_{-\infty}^{-\delta} \left[\varphi_H(t-s) - \varphi_H(-s)\right]^2 ds + c_H^2 \int_{-\delta}^{t} \left[\varphi_H(t-s) - \varphi_H(-s)\right]^2 ds$$

If $H = \frac{1}{2}$, then

$$\int_{-\infty}^{-\delta} [\varphi_H(t-s) - \varphi_H(-s)]^2 \, ds = 0 \, .$$

If $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$, then

$$\int_{-\infty}^{-\delta} [\varphi_H(t-s) - \varphi_H(-s)]^2 \, ds = \int_{\delta}^{\infty} \left[(t+x)^{H-\frac{1}{2}} - x^{H-\frac{1}{2}}\right]^2 dx$$

$$\leq \int_{\delta}^{\infty} \left[t\left(H - \frac{1}{2}\right) x^{H-\frac{3}{2}}\right]^2 dx = t^2 \frac{\left(H-\frac{1}{2}\right)^2}{2(1-H)} \delta^{2(H-1)} \, . \tag{3.2.1}$$

This shows that for all $H \in (0, 1)$, for small $t > 0$, the essential contribution to $\mathrm{E}\left[\left(B_t^H\right)^2\right]$ comes from the term

$$c_H^2 \int_{-\delta}^{t} [\varphi_H(t-s) - \varphi_H(-s)]^2 \, ds \, .$$

Hence, the behaviour of the function φ_H near zero determines the rate of convergence of $\mathrm{E}\left[\left(B_t^H\right)^2\right]$ to 0, as $t \searrow 0$. To change B^H into a process with similar distribution but non-zero, finite quadratic variation, we vary φ_H in a neighbourhood of zero so that the resulting function φ satisfies

$$\int_{-\delta}^{t} [\varphi(t-s) - \varphi(-s)]^2 \, ds \approx t \, , \quad \text{as} \quad t \searrow 0 \, ,$$

where we write for two functions f and g, $f(t) \approx g(t)$, as $t \to t_0$, if there exists a constant $c \in (0, \infty)$ such that $\lim_{t \to t_0} \frac{f(t)}{g(t)} = c$. To give a concrete example for the sort of functions we have in mind we set for $H \in (0, 1)$, $a \in \mathbb{R}$ and $b > 0$,

$$\varphi_H^{a,b}(x) := \begin{cases} a + \frac{\varphi_H(b)-a}{b} x & x \in [0, b] \\ \varphi_H(x) & x \in (-\infty, 0) \cup (b, \infty) \end{cases} .$$

As φ_H, the functions $\varphi_H^{a,b}$ satisfy

$$\int_{\mathbb{R}} \left[\varphi_H^{a,b}(t-s) - \varphi_H^{a,b}(-s)\right]^2 ds < \infty \, , \quad \text{for all } t \in \mathbb{R} \, .$$

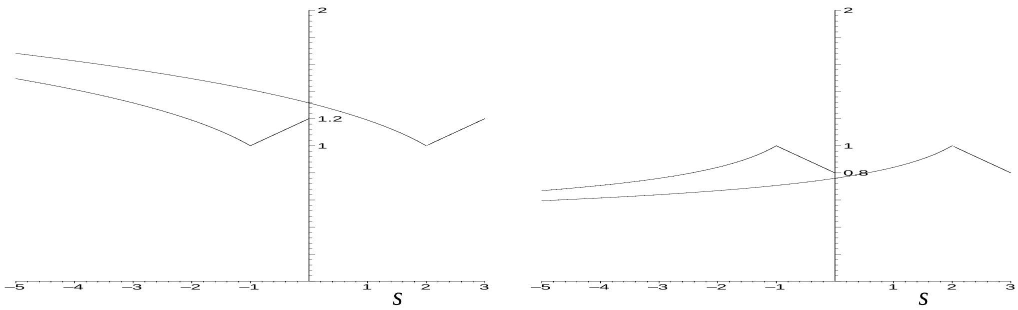

Figure 3.1: *Left: The functions* $\varphi_H^{a,b}(t-s)$ *and* $\varphi_H^{a,b}(-s)$ *for* $H=\frac{3}{4}$, $a=1.2$, $b=1$ *and* $t=3$. *Right: The functions* $\varphi_H^{a,b}(t-s)$ *and* $\varphi_H^{a,b}(-s)$ *for* $H=\frac{1}{4}$, $a=0.8$, $b=1$ *and* $t=3$.

Therefore, they can be used to define the integrals

$$R_t^{\varphi_H^{a,b}} = \int_{\mathbb{R}} \left[\varphi_H^{a,b}(t-s) - \varphi_H^{a,b}(-s)\right] dW_s, \quad t \in \mathbb{R}, \tag{3.2.2}$$

in the L^2-sense.
It is clear that $\left(R_t^{\varphi_H^{a,b}}\right)_{t\in\mathbb{R}}$ is a centred Gaussian process with stationary increments. The same calculation as in (3.2.1) yields

$$\int_{-\infty}^{-b} \left[\varphi_H^{a,b}(t-s) - \varphi_H^{a,b}(-s)\right]^2 dx \quad = O(t^2)\,, \text{ for } t \searrow 0\,.$$

Similarly, it can be checked that

$$\int_{-b}^{0} \left[\varphi_H^{a,b}(t-s) - \varphi_H^{a,b}(-s)\right]^2 dx \quad = O(t^2)\,, \text{ for } t \searrow 0\,.$$

and

$$\int_{0}^{t} \left[\varphi_H^{a,b}(t-s) - \varphi_H^{a,b}(-s)\right]^2 dx \quad \begin{cases} \approx t & \text{if } a \neq 0 \\ \approx t^3 & \text{if } a = 0 \end{cases}, \text{ for } t \searrow 0\,.$$

Hence,

$$\int_{-\infty}^{t} \left[\varphi_H^{a,b}(t-s) - \varphi_H^{a,b}(-s)\right]^2 dx \quad \begin{cases} \approx t & \text{if } a \neq 0 \\ = O(t^2) & \text{if } a = 0 \end{cases}, \text{ for } t \searrow 0\,.$$

It will follow from Proposition 3.2 and Corollary 3.8 that $\left(\frac{1}{a} R_t^{\varphi_H^{a,b}}\right)_{t\in[0,T]}$ is a finite variation process if $a=0$ and equivalent to Brownian motion if $a \neq 0$.

On the other hand, for small b, $\left(\frac{R_t^{\varphi_H^{a,b}}}{\left\|R_1^{\varphi_H^{a,b}}\right\|_2}\right)_{t\in[0,T]}$ is similar to $\left(B_t^H\right)_{t\in[0,T]}$ in the following sense: It can be checked that for all $t \in [0, T \vee 1]$,

$$\left|\int_{-\infty}^{t} [\varphi_H(t-s) - \varphi_H(-s)]^2\, ds - \int_{-\infty}^{t} \left[\varphi_H^{a,b}(t-s) - \varphi_H^{a,b}(-s)\right]^2 ds\right|$$

$$\leq \left|\int_{-b}^{0} [\varphi_H(t-s) - \varphi_H(-s)]^2\, ds - \int_{-b}^{0} \left[\varphi_H^{a,b}(t-s) - \varphi_H^{a,b}(-s)\right]^2 ds\right|$$

$$+ \left|\int_{t-b}^{t} [\varphi_H(t-s) - \varphi_H(-s)]^2\, ds - \int_{t-b}^{t} \left[\varphi_H^{a,b}(t-s) - \varphi_H^{a,b}(-s)\right]^2 ds\right|$$

$$\leq \int_{-b}^{0} [\varphi_H(t-s) - \varphi_H(-s)]^2\, ds \vee \int_{-b}^{0} \left[\varphi_H^{a,b}(t-s) - \varphi_H^{a,b}(-s)\right]^2 ds$$

$$+ \int_{t-b}^{t} [\varphi_H(t-s) - \varphi_H(-s)]^2\, ds \vee \int_{t-b}^{t} \left[\varphi_H^{a,b}(t-s) - \varphi_H^{a,b}(-s)\right]^2 ds$$

$$\leq \begin{cases} 2\left(((T\vee 1)+b)^{H-\frac{1}{2}} + |a|\right)^2 b & \text{if } H \in \left(\frac{1}{2}, 1\right) \\ 2\int_0^b \left(x^{H-\frac{1}{2}} + |a|\right)^2 dx & \text{if } H \in (0, \frac{1}{2}] \end{cases} .$$

This shows that

$$\lim_{b\searrow 0} \sup_{t\in[0,T\vee 1]} \left|\operatorname{Var}\left(\frac{B_t^H}{c_H}\right) - \operatorname{Var}\left(R_t^{\varphi_H^{a,b}}\right)\right| = 0 \tag{3.2.3}$$

and in particular,

$$\lim_{b\searrow 0} \left|\frac{1}{c_H^2} - \operatorname{Var}\left(R_1^{\varphi_H^{a,b}}\right)\right| = 0\,. \tag{3.2.4}$$

It follows from (3.2.4) and (3.2.3) that for given $H \in (0,1)$, $a \in \mathbb{R}$ and $\varepsilon > 0$ there exists a $b > 0$ such that

$$\left|c_H^2 - \frac{1}{\left\|R_1^{\varphi_H^{a,b}}\right\|_2^2}\right| \leq \frac{\varepsilon}{3} \frac{c_H^2}{\operatorname{Var}\left(B_T^H\right)} \tag{3.2.5}$$

and, for all $t \in [0, T]$,

$$\frac{1}{\left\| R_1^{\varphi_H^{a,b}} \right\|_2^2} \left| \mathrm{Var}\left(\frac{B_t^H}{c_H} \right) - \mathrm{Var}\left(R_t^{\varphi_H^{a,b}} \right) \right| \leq \frac{\varepsilon}{3}\,. \tag{3.2.6}$$

From (3.2.5) and (3.2.6) follows that for all $t \in [0, T]$,

$$\begin{aligned}
&\left| \mathrm{Var}\left(B_t^H \right) - \mathrm{Var}\left(\frac{R_t^{\varphi_H^{a,b}}}{\left\| R_1^{\varphi_H^{a,b}} \right\|_2} \right) \right| \\
\leq\ & \left| \mathrm{Var}\left(B_t^H \right) - \frac{1}{\left\| R_1^{\varphi_H^{a,b}} \right\|_2^2} \mathrm{Var}\left(\frac{B_t^H}{c_H} \right) \right| \\
+\ & \left| \frac{1}{\left\| R_1^{\varphi_H^{a,b}} \right\|_2^2} \mathrm{Var}\left(\frac{B_t^H}{c_H} \right) - \mathrm{Var}\left(\frac{R_t^{\varphi_H^{a,b}}}{\left\| R_1^{\varphi_H^{a,b}} \right\|_2} \right) \right| \\
=\ & \left| \mathrm{Var}\left(\frac{B_t^H}{c_H} \right) \right| \left| c_H^2 - \frac{1}{\left\| R_1^{\varphi_H^{a,b}} \right\|_2^2} \right| \\
+\ & \frac{1}{\left\| R_1^{\varphi_H^{a,b}} \right\|_2^2} \left| \mathrm{Var}\left(\frac{B_t^H}{c_H} \right) - \mathrm{Var}\left(R_t^{\varphi_H^{a,b}} \right) \right| \leq \frac{2\varepsilon}{3}\,.
\end{aligned}$$

By stationarity of the increments of B^H and $R^{\varphi_H^{a,b}}$, this implies for all $t, s \in$

$[0, T]$ with $s \leq t$,

$$\begin{aligned}
&\left|\mathrm{Cov}\left(B_t^H, B_s^H\right) - \mathrm{Cov}\left(\frac{R_t^{\varphi_H^{a,b}}}{\left\|R_1^{\varphi_H^{a,b}}\right\|_2}, \frac{R_s^{\varphi_H^{a,b}}}{\left\|R_1^{\varphi_H^{a,b}}\right\|_2}\right)\right| \\
= \quad & \frac{1}{2}\Bigg| \mathrm{Var}\left(B_t^H\right) + \mathrm{Var}\left(B_s^H\right) - \mathrm{Var}\left(B_{t-s}^H\right) \\
& -\mathrm{Var}\left(\frac{R_t^{\varphi_H^{a,b}}}{\left\|R_1^{\varphi_H^{a,b}}\right\|_2}\right) - \mathrm{Var}\left(\frac{R_s^{\varphi_H^{a,b}}}{\left\|R_1^{\varphi_H^{a,b}}\right\|_2}\right) + \mathrm{Var}\left(\frac{R_{t-s}^{\varphi_H^{a,b}}}{\left\|R_1^{\varphi_H^{a,b}}\right\|_2}\right)\Bigg| \\
\leq \quad & \frac{1}{2}\left|\mathrm{Var}\left(B_t^H\right) - \mathrm{Var}\left(\frac{R_t^{\varphi_H^{a,b}}}{\left\|R_1^{\varphi_H^{a,b}}\right\|_2}\right)\right| \\
& +\frac{1}{2}\left|\mathrm{Var}\left(B_s^H\right) - \mathrm{Var}\left(\frac{R_s^{\varphi_H^{a,b}}}{\left\|R_1^{\varphi_H^{a,b}}\right\|_2}\right)\right| \\
& +\frac{1}{2}\left|\mathrm{Var}\left(B_{t-s}^H\right) - \mathrm{Var}\left(\frac{R_{t-s}^{\varphi_H^{a,b}}}{\left\|R_1^{\varphi_H^{a,b}}\right\|_2}\right)\right| \leq \epsilon \,.
\end{aligned}$$

3.2.2 R^φ and its semimartingale decomposition

The largest class of functions φ of the form

(R1) $\quad \varphi : \mathbb{R} \to \mathbb{R}$ is measurable and $\varphi(x) = 0 \quad$ for $\quad x < 0\,,$

that can be used to define the integrals

$$R_t^\varphi = \int_{\mathbb{R}} [\varphi(t-s) - \varphi(-s)]\, dW_s, \quad t \in \mathbb{R}, \tag{3.2.7}$$

in the L^2-sense, is the class of functions that besides (R1) also satisfy

(R2) $\quad \int_{\mathbb{R}} [\varphi(t-s) - \varphi(-s)]^2\, ds < \infty, \quad$ for all $\quad t \in \mathbb{R}\,.$

If φ satisfies (R1) and (R2), it can easily be seen that the process $\left(R_t^{\varphi}\right)_{t\in\mathbb{R}}$ defined in (3.2.7) is a centred Gaussian process with stationary increments. But in contrast to fractional Brownian motion it is in general not self-similar. If φ is of the form

$$\text{(R3)} \qquad \varphi(x) = \begin{cases} \varphi(0) + \int_0^x \psi(y)dy & x \geq 0 \\ 0 & x < 0 \end{cases} ,$$

for some $\psi \in L^2(\mathbb{R}_+)$, then it also satisfies (R1) and (R2). Hence, $\left(R_t^{\varphi}\right)_{t\in\mathbb{R}}$ is well-defined.

Proposition 3.2 *If φ satisfies* (R3)*, then for all $t \geq 0$:*

$$R_t^{\varphi} = \varphi(0)W_t + \int_0^t \int_{-\infty}^s \psi(s-u)dW_u ds \,. \tag{3.2.8}$$

Proof.

$$\begin{aligned} R_t^{\varphi} &= \int_{-\infty}^t [\varphi(t-u) - \varphi(-u)]\, dW_u \\ &= \int_{-\infty}^0 [\varphi(t-u) - \varphi(-u)]\, dW_u + \int_0^t \varphi(t-u)dW_u \\ &= \int_{-\infty}^0 \int_0^t \psi(s-u)ds dW_u + \int_0^t \left[\int_u^t \psi(s-u)ds + \varphi(0)\right] dW_u \end{aligned}$$

By the stochastic version of Fubini's theorem (see e.g. Theorem IV.46 of Protter (1990)), we can change the order of integration. Hence, the above equals

$$\begin{aligned} &\int_0^t \int_{-\infty}^0 \psi(s-u)dW_u ds + \int_0^t \int_0^s \psi(s-u)dW_u ds + \varphi(0)W_t \\ = &\int_0^t \int_{-\infty}^s \psi(s-u)dW_u ds + \varphi(0)W_t \,, \end{aligned}$$

i.e. (3.2.8) holds. □

Corollary 3.3 *If φ satisfies* (R3)*, then $\left(R_t^{\varphi}\right)_{t\geq 0}$ is a continuous semimartingale with respect to the smallest filtration $\bar{\mathbb{F}}^W = \left(\bar{\mathcal{F}}_t^W\right)_{t\geq 0}$ that satisfies the usual assumptions and contains the filtration $\mathbb{F}^W = \left(\mathcal{F}_t^W\right)_{t\geq 0}$, where for all $t \geq 0$, $\mathcal{F}_t^W = \sigma\left\{W_s \,:\, -\infty < s \leq t\right\}$.*
The canonical semimartingale decomposition of R^{φ} with respect to $\bar{\mathbb{F}}^W$ is given by (3.2.8).

Proof. The corollary follows immediately from Proposition 3.2. □

Corollary 3.4 *If φ satisfies* (R3)*, then* $\left(R^\varphi_t\right)_{t\geq 0}$ *is a continuous semimartingale with respect to the smallest filtration* $\bar{\mathbb{F}}^{R^\varphi} = \left(\bar{\mathcal{F}}^{R^\varphi}_t\right)_{t\geq 0}$ *that satisfies the usual assumptions and contains the filtration* $\mathbb{F}^{R^\varphi} = \left(\mathcal{F}^{R^\varphi}_t\right)_{t\geq 0}$*, where for all* $t \geq 0$, $\mathcal{F}^{R^\varphi}_t = \sigma\left\{R^\varphi_s \,:\, 0 \leq s \leq t\right\}$.

Proof. It follows from (3.2.7) that $\mathcal{F}^{R^\varphi}_t \subset \mathcal{F}^W_t$, for all $t \geq 0$. Hence, $\bar{\mathcal{F}}^{R^\varphi}_t \subset \bar{\mathcal{F}}^W_t$, for all $t \geq 0$, and the corollary follows from Corollary 3.3 by Stricker's Theorem (see Protter (1990), p. 45). □

Remark 3.5 Let φ be of the form (R3). The author guesses that in general, $(W_t)_{t\geq 0}$ is not adapted to the filtration $\bar{\mathbb{F}}^{R^\varphi}$. In this case, (3.2.8) is not the $\bar{\mathbb{F}}^{R^\varphi}$-semimartingale decomposition of R^φ and it is not obvious how to find it.

3.2.3 Equivalence of $\left(\frac{1}{\varphi(0)} R^\varphi\right)_{t\in[0,T]}$ to Brownian motion

Let φ be a function that satisfies (R3). It can be seen from (3.2.8) that $\left(R^\varphi_t\right)_{t\geq 0}$ is a finite variation process if and only if $\varphi(0) = 0$. In this subsection we show that $\left(\frac{1}{\varphi(0)} R^\varphi_t\right)_{t\in[0,T]}$ is equivalent to Brownian motion if $\varphi(0) \neq 0$. The key to this result is the following theorem.

Theorem 3.6 *Let $(W_t)_{t\in\mathbb{R}}$ be a two-sided Brownian motion. Let $T \in (0,\infty)$ and $k \in L^2(G)$, where*

$$G = \{(s,u) \in [0,T] \times (-\infty, T] \,:\, s \geq u\} \,.$$

Then

$$\exp\left\{\int_0^t \int_{-\infty}^s k(s,u)dW_u dW_s - \frac{1}{2}\int_0^t \left(\int_{-\infty}^s k(s,u)dW_u\right)^2 ds\right\}\,, \quad t \in [0,T],$$

is a martingale.

Proof. We show that the Novikov condition is satisfied. By Corollary 3.5.14 of Karatzas and Shreve (1988) it is enough to show that there exists a partition $0 = t_0 < \cdots < t_l = T$ such that

$$\mathrm{E}\left[\exp\left\{\frac{1}{2}\int_{t_{n-1}}^{t_n} \left(\int_{-\infty}^s k(s,u)dW_u\right)^2 ds\right\}\right] < \infty \quad \text{for all } n = 1,\ldots,l\,. \tag{3.2.9}$$

Since $k \in L^2(G)$, there exists a partition $0 = t_0 < \cdots < t_l = T$, such that

$$\int_{t_{n-1}}^{t_n} \int_{-\infty}^{s} (k(s,u))^2 \, du ds \le \frac{1}{9} \quad \text{for all } n = 1, \ldots, l\,. \tag{3.2.10}$$

To show that (3.2.9) holds for this partition we fix n and set

$$\tilde{k}(s,u) = k(s,u) 1_{(t_{n-1},t_n]}(s)\,,\ (s,u) \in G\,.$$

(3.2.10) implies

$$\left\| \tilde{k} \right\|_2 \le \frac{1}{3}\,. \tag{3.2.11}$$

For every $m \in \mathbb{N}$, there exists a partition $t_{n-1} = s_1^m < \cdots < s_{J(m)}^m = t_n$, and a $k^m \in L^2(G)$, of the form

$$k^m(s,u) = \sum_{j=1}^{J(m)} k_j^m(u) 1_{(s_{j-1}^m, s_j^m]}(s)$$

with

$$\left\| \tilde{k} - k^m \right\|_2 \le \frac{1}{m}\,. \tag{3.2.12}$$

For all $m \in \mathbb{N}$, let X^m be the centred, $J(m)$-dimensional Gaussian vector with j-th component

$$\sqrt{s_j^m - s_{j-1}^m} \int_{-\infty}^{s_{j-1}^m} k_j^m(u) dW_u, \quad j = 1, \ldots, J(m).$$

There exists an orthogonal $J(m) \times J(m)$-matrix A^m such that $Y^m = A^m X^m$ is a centred Gaussian vector with independent components. Since A^m is orthogonal, we have

$$\sum_{j=1}^{J(m)} \left(Y_j^m\right)^2 = \left(Y^m\right)^T Y^m = \left(X^m\right)^T \left(A^m\right)^T A^m X^m$$

$$= \left(X^m\right)^T X^m = \sum_{j=1}^{J(m)} \left(X_j^m\right)^2\,. \tag{3.2.13}$$

Together with (3.2.11) and (3.2.12) this implies for all $m \ge 3$,

$$\sum_{j=1}^{J(m)} \mathrm{Var}\left(Y_j^m\right) = \sum_{j=1}^{J(m)} E\left(X_j^m\right)^2$$

$$= \sum_{j=1}^{J(m)} \left(s_j^m - s_{j-1}^m\right) \int_{-\infty}^{s_{j-1}^m} \left(k_j^m(u)\right)^2 du$$

$$= \|k^m\|_2^2 \le \left(\left\|k^m - \tilde{k}\right\|_2 + \left\|\tilde{k}\right\|_2\right)^2 < \frac{1}{2}\,. \tag{3.2.14}$$

Furthermore, it follows from (3.2.13) and the independence of the Y_j^m, $j = 1, \ldots, J(m)$, that

$$\begin{aligned}
& \mathrm{E}\left[\exp\left\{\frac{1}{2}\int_0^t \left(\int_{-\infty}^s k^m(s,u)dW_u\right)^2 ds\right\}\right] \\
= \; & \mathrm{E}\left[\exp\left\{\frac{1}{2}\sum_{j=1}^{J(m)} \left(X_j^m\right)^2\right\}\right] = \mathrm{E}\left[\exp\left\{\frac{1}{2}\sum_{j=1}^{J(m)} \left(Y_j^m\right)^2\right\}\right] \\
= \; & \prod_{j=1}^{J(m)} \mathrm{E}\left[\exp\left\{\frac{1}{2}\left(Y_j^m\right)^2\right\}\right] = \prod_{j=1}^{J(m)} \left(1 - \mathrm{Var}\left(Y_j^m\right)\right)^{-\frac{1}{2}} .
\end{aligned}$$

It can easily be shown by induction on $J(m)$ that

$$\prod_{j=1}^{J(m)} \left(1 - \mathrm{Var}\left(Y_j^m\right)\right) \ge 1 - \sum_{j=1}^{J(m)} \mathrm{Var}\left(Y_j^m\right) .$$

Therefore, it follows from (3.2.14) that for all $m \ge 3$,

$$\prod_{j=1}^{J(m)} \left(1 - \mathrm{Var}\left(Y_j^m\right)\right) > \frac{1}{2}\,.$$

Hence, for all $m \ge 3$,

$$\begin{aligned}
& \mathrm{E}\left[\exp\left\{\frac{1}{2}\int_0^t \left(\int_{-\infty}^s k^m(s,u)dW_u\right)^2 ds\right\}\right] \\
& \le \left(1 - \sum_{j=1}^{J(m)} \mathrm{Var}\left(Y_j^m\right)\right)^{-\frac{1}{2}} < \sqrt{2}\,.
\end{aligned} \tag{3.2.15}$$

Since

$$\mathrm{E}\left[\int_0^T \left(\int_{-\infty}^s \tilde{k}(s,u)dW_u - \int_{-\infty}^s k^m(s,u)dW_u\right)^2 ds\right]$$

$$= \int_0^T \mathrm{E}\left[\left\{\int_{-\infty}^s \left(\tilde{k}(s,u) - k^m(s,u)\right) dW_u\right\}^2\right] ds$$

$$= \int_0^T \int_{-\infty}^s \left(\tilde{k}(s,u) - k^m(s,u)\right)^2 du ds = \left\|\tilde{k} - k^m\right\|_2^2 \overset{(m\to\infty)}{\longrightarrow} 0\,,$$

there exists a subsequence $\{k^{m_i}\}_{i=1}^{\infty}$ such that for almost every $\omega \in \Omega$,

$$\int_0^T \left\{\left(\int_{-\infty}^s \tilde{k}(s,u) dW_u\right)(\omega) - \left(\int_{-\infty}^s k^{m_i}(s,u) dW_u\right)(\omega)\right\}^2 ds \overset{(i\to\infty)}{\longrightarrow} 0\,,$$

i.e. for almost every $\omega \in \Omega$,

$$\left\|\left(\int_{-\infty}^s \tilde{k}(s,u) dW_u\right)(\omega) - \left(\int_{-\infty}^s k^{m_i}(s,u) dW_u\right)(\omega)\right\|_{L^2[0,T]}^2 \overset{(i\to\infty)}{\longrightarrow} 0\,.$$

This implies that for almost every $\omega \in \Omega$,

$$\left\|\left(\int_{-\infty}^s k^{m_i}(s,u) dW_u\right)(\omega)\right\|_{L^2[0,T]}^2$$

$$\overset{(i\to\infty)}{\longrightarrow} \left\|\left(\int_{-\infty}^s \tilde{k}(s,u) dW_u\right)(\omega)\right\|_{L^2[0,T]}^2\,.$$

Hence,

$$\int_0^T \left(\int_{-\infty}^s k^{m_i}(s,u) dW_u\right)^2 ds \overset{(i\to\infty)}{\longrightarrow} \int_0^T \left(\int_{-\infty}^s \tilde{k}(s,u) dW_u\right)^2 ds$$

almost surely. By Fatou's lemma and (3.2.15) we obtain

$$\mathrm{E}\left[\exp\left\{\frac{1}{2}\int_{t_{n-1}}^{t_n} \left(\int_{-\infty}^s k(s,u) dW_u\right)^2 ds\right\}\right]$$

$$= \mathrm{E}\left[\exp\left\{\frac{1}{2}\int_0^T \left(\int_{-\infty}^s \tilde{k}(s,u) dW_u\right)^2 ds\right\}\right]$$

$$\leq \liminf_i \mathrm{E}\left[\exp\left\{\frac{1}{2}\int_0^T \left(\int_{-\infty}^s k^{m_i}(s,u) dW_u\right)^2 ds\right\}\right] \leq \sqrt{2}\,.$$

This completes the proof of the Theorem. □

Remark 3.7 Theorem 3.6 is only a slight generalization of Theorem 2 of Hitsuda (1968). But our proof is simpler and does not need results from the theory of Volterra integral equations.

Corollary 3.8 *Let φ be a function with $\varphi(0) \neq 0$ that satisfies* (R3). *Let $\bar{\mathbb{F}}^W = \left(\bar{\mathcal{F}}_t^W\right)_{t\in[0,T]}$ be the smallest filtration that satisfies the usual assumptions and contains the filtration $\mathbb{F}^W = \left(\mathcal{F}_t^W\right)_{t\in[0,T]}$, where $\mathcal{F}_t^W = \sigma\left(W_s \,:\, -\infty < s \leq t\right)$. Then*

$$\left(\frac{1}{\varphi(0)} R_t^{\varphi}\right)_{t\in[0,T]}$$

is a $\bar{\mathbb{F}}^W$-Brownian motion with respect to the probability measure

$$Q = \exp\left\{-\int_0^T \int_{-\infty}^s \frac{\psi(s-u)}{\varphi(0)} dW_u dW_s - \frac{1}{2}\int_0^T \left(\int_{-\infty}^s \frac{\psi(s-u)}{\varphi(0)} dW_u\right)^2 ds\right\} \cdot P\,.$$

Proof. It follows from Proposition 3.2 that

$$\frac{1}{\varphi(0)} R_t^{\varphi} = W_t + \int_0^t \int_{-\infty}^s \frac{\psi(s-u)}{\varphi(0)} dW_u ds\,.$$

Theorem 3.6 implies that

$$\exp\left\{-\int_0^t \int_{-\infty}^s \frac{\psi(s-u)}{\varphi(0)} dW_u dW_s - \frac{1}{2}\int_0^t \left(\int_{-\infty}^s \frac{\psi(s-u)}{\varphi(0)} dW_u\right)^2 ds\right\}\,, \quad t \in [0,T]\,,$$

is a martingale. Therefore, the corollary follows from Girsanov's (1960) theorem. □

Remark 3.9 It can easily be checked that the functions $\varphi_H^{a,b}$ from Subsection 3.2.1 satisfy (R3). For $a \neq 0$ we call the stochastic process

$$\left(\frac{R_t^{\varphi_H^{a,b}}}{\left\|R_1^{\varphi_H^{a,b}}\right\|}\right)_{t\geq 0}$$

a regularized fractional Brownian motion because for small $b > 0$ its finite-dimensional distributions are similar to those of B^H and at the same time, for every $T \in (0, \infty)$, $\left(\frac{1}{a} R_t^{\varphi_H^{a,b}}\right)_{t\in[0,T]}$ is equivalent to Brownian motion. However, note that for $H \in \left(\frac{1}{2}, 1\right)$, the paths of $\frac{R^{\varphi_H^{a,b}}}{\left\| R_1^{\varphi_H^{a,b}} \right\|}$ are less regular than those of B^H so far as the degree of local Hölder continuity is concerned.

3.3 Option pricing with regularized fractional Brownian motion

As in Section 1.4, we consider a frictionless market that consists of a money market account and a stock. One unit of money in the money market account grows like $\left(e^{rt}\right)_{t\in[0,T]}$ for a constant interest rate r. The discounted stock price follows a stochastic process $(S_t)_{t\in[0,T]}$. The information obtained by observing the stock is given by the filtration $\mathbb{F}^S = \left(\mathcal{F}_t^S\right)_{t\in[0,T]}$, where $\mathcal{F}_t^S = \sigma\left(S_u : 0 \le u \le t\right)$, $t \in [0, T]$. By $\bar{\mathbb{F}} = \left(\bar{\mathcal{F}}_t^S\right)_{t\in[0,T]}$ we denote the smallest filtration that contains $\mathbb{F}^S$ and fulfils the usual assumptions. We are interested in the price of a European call option on the stock with strike price K and maturity T. In discounted terms this option pays at time T a random amount of

$$(S_T - e^{-rT} K)^+ .$$

If S is not a Markov process, it might be useful to know the history of S when pricing the option. Therefore we examine the discounted price C_{t_0} of the European call option at some time $t_0 \in (0, T)$. To avoid trivial arbitrage opportunities, C_{t_0} has to lie in the interval

$$\left(\left(S_{t_0} - e^{-rT} K\right)^+ , S_{t_0}\right) .$$

In the Samuelson model

$$S_t = S_0 \exp\left(\nu t + \sigma B_t\right) , \; t \in [0, T] , \tag{3.3.1}$$

where $S_0 > 0$, ν and $\sigma > 0$ are constants and B is a Brownian motion, Black and Scholes (1973) gave an explicit formula for C_{t_0}. For given r, K and T the Black-Scholes price only depends on t_0, S_{t_0} and the volatility σ but not on the whole trajectory $(S_t)_{t\in[0,t_0]}$ and not on the parameter ν. For given r, K, T and fixed t_0, S_{t_0}, the discounted Black-Scholes price $\mathrm{dBS}(t_0, S_{t_0}, .)$ of a European call option is a continuous, strictly increasing function of σ which maps the interval $(0, \infty)$ bijectively to the interval

$$\left(\left(S_{t_0} - \mathrm{e}^{-rT} K\right)^+, S_{t_0}\right) .$$

Alternatively, let us assume that empirical data suggests that the discounted price of a particular stock should be modelled as a fractional Samuelson process

$$S_t = S_0 \exp\left(\nu t + \sigma B_t^H\right) , \; t \in [0, T] , \tag{3.3.2}$$

where $S_0 > 0$, ν and $\sigma > 0$ are constants and B^H is a fractional Brownian motion with $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$. We have shown in Section 2.4 that this model is arbitrage-free if one confines the trading strategies to $\bigcup_{h>0} \Theta_{\mathrm{sf}}^h(\mathbb{F}^S)$. On the other hand, we have shown that the model admits a FLVR consisting of integrands in $\Theta_{\mathrm{sf,adm}}^{\mathbf{S}}(\mathbb{F}^S)$ and strong arbitrage in $\Theta_{\mathrm{sf,adm}}^{\mathbf{aS}}(\mathbb{F}^S)$. However, we can use the processes $R^{\varphi_H^{a,b}}$, given in (3.2.2), to regularize the fractional Samuelson model (3.3.2).

3.3.1 Naive option pricing in regularized fractional Samuelson models

It is clear from what we have shown in Subsection 3.2.1 that for given $H \in (0, 1)$, there exists for every $\varepsilon > 0$ a continuous function

$$b : (0, \infty) \to (0, \infty) ,$$

such that for all $a > 0$,

$$\left| c_H - \frac{1}{\left\| R_1^{\varphi_H^{a,b(a)}} \right\|_2} \right| \leq \frac{c_H}{2} \tag{3.3.3}$$

and for all $t, s \in [0, T]$,

$$\left| \mathrm{Cov}\left(B_t^H, B_s^H\right) - \mathrm{Cov}\left(\frac{R_t^{\varphi_H^{a,b(a)}}}{\left\| R_1^{\varphi_H^{a,b(a)}} \right\|_2} , \frac{R_s^{\varphi_H^{a,b(a)}}}{\left\| R_1^{\varphi_H^{a,b(a)}} \right\|_2} \right) \right| \leq \varepsilon . \tag{3.3.4}$$

On the other hand, Corollary 3.8 shows that for all $a > 0$,

$$\left(\frac{1}{a} R_t^{\varphi_H^{a,b(a)}}\right)_{t\in[0,T]}$$

is equivalent to Brownian motion. Therefore there exists for each $a > 0$ a unique probability measure Q^a on $(\Omega, \bar{\mathcal{F}}_T^S)$ which is equivalent to P such that the regularized fractional Samuelson process

$$S_t = S_0 \exp\left(\nu t + \sigma \frac{R_t^{\varphi_H^{a,b(a)}}}{\left\| R_1^{\varphi_H^{a,b(a)}} \right\|_2}\right), \quad t \in [0,T], \tag{3.3.5}$$

is a martingale on $\left(\Omega, \mathbb{F}^S, Q^a\right)$. According to current practice in mathematical finance, in such a framework discounted option prices are calculated by taking the conditional expectation of the options discounted pay-off under the equivalent martingale measure. In the model (3.3.5) with $a > 0$, this leads to the following discounted price for our European call option at time t_0:

$$\begin{aligned} C_{t_0}(a) &= E_{Q^a}\left[\left(S_0 \exp\left(\nu T + \sigma \frac{R_T^{\varphi_H^{a,b(a)}}}{\left\| R_1^{\varphi_H^{a,b(a)}} \right\|_2}\right) - \mathrm{e}^{-rT} K \right)^+ \middle| \bar{\mathcal{F}}_{t_0} \right] \\ &= \mathrm{dBS}\left(t_0, S_{t_0}, \frac{\sigma a}{\left\| R_1^{\varphi_H^{a,b(a)}} \right\|_2}\right) \end{aligned} \tag{3.3.6}$$

Hence, $C_{t_0}(a)$ given in (3.3.6), only depends on S_{t_0} and not on the whole past $(S_u)_{u\in[0,t_0]}$ even though $R^{\varphi_H^{a,b(a)}}$ is not a Markov process under P. Furthermore, it follows from the continuity of the function b and from (3.3.3) that the mapping

$$a \mapsto \frac{a}{\left\| R_1^{\varphi_H^{a,b(a)}} \right\|_2}$$

is a continuous surjection from $(0,\infty)$ to $(0,\infty)$. This shows that although for every $a > 0$, the model (3.3.5) is close to the model (3.3.2) in the sense of (3.3.4), the discounted option prices $C_{t_0}(a)$ in (3.3.6) fill the whole interval

$$\left(\left(S_{t_0} - \mathrm{e}^{-rT} K\right)^+, S_{t_0}\right),$$

as a is running through $(0, \infty)$.

Remark 3.10 Note that for the special case $H = \frac{1}{2}$, the model (3.3.2) is the Samuelson model (3.3.1) and the models (3.3.5) are rather perturbations than regularizations of (3.3.1). Whereas in the Samuelson model (3.3.1), calculating the discounted option price by taking the conditional expectation under the equivalent martingale measure Q, leads to the discounted Black-Scholes price

$$C_{t_0} = E_Q\left[\left(S_0 \exp\left(\nu T + \sigma B_T\right) - \mathrm{e}^{-rT} K\right)^+ \middle| \bar{\mathcal{F}}_{t_0}\right] = \mathrm{dBS}\left(t_0, S_{t_0}, \sigma\right) ,$$

in the models (3.3.5), this method yields the prices $C_{t_0}(a)$ given by (3.3.6).

A similar result was obtained by Brigo and Mercurio (2000). For a given finite time-grid $\Gamma \subset [0, T]$ they constructed a class of processes $\left(Y_t^a\right)_{t\in[0,T]}$, $a \in (0, \infty)$, such that each process Y^a has the same finite-dimensional distribution on Γ as the geometric Brownian motion (3.3.1), the same one-dimensional marginal distributions as (3.3.1) for all $t \in [0, T]$, and a unique equivalent martingale measure. As in our case the quadratic variation of the processes Y^a in Brigo and Mercurio (2000) can be very different from that of (3.3.1), and for every constant

$$c \in \left(\left(S_0 - e^{-rT} K\right)^+, S_0\right)$$

there exists an $a \in (0, \infty)$ such that the time zero price of a European call option with maturity T and strike price K on a stock modelled with Y^a equals c. In contrast to our processes $R^{\varphi_{\frac{1}{2}}^{a,b(a)}}$, $a > 0$, the processes Y^a, $a > 0$, in Brigo and Mercurio (2000) have exactly the same distribution as (3.3.1) on the finite-time grid Γ. On the other hand, the log-processes $\left(\log Y_t^a\right)_{t\in[0,T]}$, $a > 0$, do not have stationary increments whereas our log-processes $\left(R_t^{\varphi_{\frac{1}{2}}^{a,b(a)}}\right)_{t\in[0,T]}$, $a > 0$, do.

3.3.2 Discussion

In order to understand why (3.3.6) can lead to totally different option prices in models that are close to each other in the sense of (3.3.4), we take a closer look at the mechanism of option pricing by calculating the conditional expectation under the equivalent martingale measure.

Let us assume that the present time $t_0 \in (0, T)$ is equal to $\frac{N_0}{N}T$, where $N_0 < N$ are two natural numbers, and after observing the discounted stock price s_t at times

$$0, h, 2h, \ldots, N_0 h = t_0 ,$$

where $h = \frac{T}{N}$, we have come to the conclusion that

$$\ln\left(\frac{s_{jh}}{s_{(j-1)h}}\right), \ j = 1, \ldots, N_0 , \tag{3.3.7}$$

could be the realisation of a stationary random sequence

$$\left(jh\nu + X_j\right)_{j=1}^{N_0} ,$$

where ν is a constant and $\left(X_j\right)_{j=1}^{N_0}$ is a centred Gaussian vector that has up to some statistical tolerance $\varepsilon > 0$ the same covariance structure as

$$\left(\sigma B_{jh}^H - \sigma B_{(j-1)h}^H\right)_{j=1}^{N_0} ,$$

where $\sigma > 0$ is a constant and B^H a fractional Brownian motion with Hurst parameter $H \in (0, 1)$.

If we think that there will be no considerable change in market conditions after time t_0, it is natural to model the discounted stock price on the time-grid

$$0, h, \ldots, Nh = T ,$$

as

$$S_t = S_0 \exp\left(\nu t + \sigma B_t^H\right), \ t = 0, h, \ldots, Nh = T . \tag{3.3.8}$$

It can easily be checked that the model (3.3.8) is arbitrage-free if we allow all discrete-time predictable processes as trading strategies. But at the same time it is not possible to replicate the discounted option pay-off

$$\left(S_T - \mathrm{e}^{-rT} K\right)^+$$

and the cheapest way to super-replicate it at time t_0 is to buy a stock share.

In reality transactions are not restricted to a pre-specified time-grid. The assumption that $\ln\left(\frac{S_t}{S_0}\right)$ has stationary increments suggests to extend the model (3.3.8) to

$$S_t = S_0 \exp\left(\nu t + \sigma B_t^H\right), \ t \in [0, T] . \tag{3.3.9}$$

Since the model is based on observations on a time-grid with mesh-width h, in a first step we only allow trading strategies from the class $\Theta_{\text{sf}}^h(\mathbb{F}^S)$, which is given in (2.4.1). We have shown in Theorem 2.17 that this makes the model (3.3.9) arbitrage-free. But then again, as we have seen at the end of Section 2.4, the cheapest way to super-replicate the option is to buy the stock. Instead of super-replicating the option, one could choose an incomplete market criterion and try to hedge the option with a strategy from $\Theta_{\text{sf}}^h(\mathbb{F}^S)$ in an optimal way according to the chosen criterion.

However, for $H = \frac{1}{2}$, the process (3.3.9) is the Samuelson process (3.3.1). There exists an equivalent probability measure $Q \sim P$ under which the process (3.3.1) is a martingale. Therefore a reasonable option price can be obtained by enlarging the class of trading strategies as follows:

$$\begin{aligned}\Theta_{\text{sf,adm}}\left(\bar{\mathbb{F}}^S\right) = \{\vartheta \, : \, & \vartheta^0 \text{ and } \vartheta^1 \text{ are } \bar{\mathbb{F}}^S\text{-predictable;} \\ & \textstyle\int_0^T \left|\vartheta_u^0\right| du < \infty \text{ and } \int_0^T \left(\vartheta_u^1\right)^2 du < \infty \text{ a.s.;} \\ & V_t^\vartheta = V_0^\vartheta + \textstyle\int_0^t \vartheta_u^1 dS_u \text{ for all } t \in [0,T]; \\ & \text{and there exists a constant } c \geq 0 \text{ such that} \\ & \inf_{t\in[0,T]} \textstyle\int_0^t \vartheta_t^1 dS_u \geq -c \text{ a.s.}\}\,.\end{aligned} \tag{3.3.10}$$

It can be shown (see one of the many textbooks on mathematical finance) that in a market model $\left(\left(\tilde{S}_t^0\right)_{t\in[0,T]}, \left(\tilde{S}_t\right)_{t\in[0,T]}\right)$ where $\left(\tilde{S}_t^0\right)_{t\in[0,T]}$ is a continuous, finite variation process and $S = \tilde{S}/\tilde{S}^0$ is as in (3.3.1), there exists no arbitrage in the class $\Theta_{\text{sf,adm}}\left(\bar{\mathbb{F}}^S\right)$, and there is a unique $\hat{\vartheta} = \left(\hat{\vartheta}^0, \hat{\vartheta}^1\right) \in \Theta_{\text{sf,adm}}\left(\bar{\mathbb{F}}^S\right)$ such that

$$\left(\int_0^t \hat{\vartheta}_u^1 dS_u\right)_{t\in[0,T]}$$

is a square-integrable martingale under Q with

$$V_{t_0}^{\hat{\vartheta}} + \int_{t_0}^T \hat{\vartheta}_u^1 dS_u = V_T^{\hat{\vartheta}} = \left(S_T - \mathrm{e}^{-rT}K\right)^+ .$$

$\hat{\vartheta}$ is an optimal hedging strategy in the sense that $V_{t_0}^{\hat{\vartheta}} \leq V_{t_0}^{\vartheta}$ a.s., for all $\vartheta \in \Theta_{\text{sf,adm}}(\bar{\mathbb{F}}^S)$ that satisfy

$$V_{t_0}^{\vartheta} + \int_{t_0}^T \vartheta_u^1 dS_u = V_T^{\vartheta} \geq \left(S_T - \mathrm{e}^{-rT}K\right)^+ .$$

Hence, $V_{t_0}^{\hat{\vartheta}}$ is the minimal discounted amount of money needed at time t_0 to produce a perfect replication of the option pay-off in the model (3.3.1) with a trading strategy from $\Theta_{\mathrm{sf,adm}}\left(\bar{\mathbb{F}}^S\right)$. Therefore

$$C_{t_0} = V_{t_0}^{\hat{\vartheta}} = \mathrm{E}_Q\left[\left(S_T - \mathrm{e}^{-rT}K\right)^+ |\bar{\mathcal{F}}_{t_0}\right] = \mathrm{dBS}(t_0, S_{t_0}, \sigma)\,. \qquad (3.3.11)$$

Since the Black-Scholes hedging strategy $\hat{\vartheta}$ is almost surely of unbounded variation, it is not possible to perform it in practice. It remains to be checked whether the discounted Black-Scholes price (3.3.11) can be justified by an approximation of $\hat{\vartheta}$ with a strategy from $\Theta_{\mathrm{sf}}^h(\mathbb{F}^S)$ for fixed $h > 0$.

If the discounted stock is modelled with a fractional Samuelson process (3.3.9) with a $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$, an extension of the trading strategies beyond $\Theta_{\mathrm{sf}}^h(\mathbb{F}^S)$ is problematic. We have shown in Section 2.3 that the models (3.1.1) and (3.1.2) admit FLVR consisting of strategies in $\Theta_{\mathrm{sf,adm}}^{\mathbf{S}}(\mathbb{F}^S)$ and strong arbitrage in $\Theta_{\mathrm{sf,adm}}^{\mathbf{aS}}(\mathbb{F}^S)$. However, we can regularize the process (3.3.9) by replacing fractional Brownian motion B^H with a regularized fractional Brownian motion $R^{\varphi_H^{a,b(a)}}$ for some $a > 0$, where $b(a)$ is chosen so that (3.3.3) and (3.3.4) are satisfied for some statistical tolerance $\varepsilon > 0$. As the model (3.3.9), the model

$$S_t = S_0 \exp\left(\nu t + \sigma \frac{R_t^{\varphi_H^{a,b(a)}}}{\left\| R_1^{\varphi_H^{a,b(a)}} \right\|_2}\right)\,, \; t \in [0, T]\,, \qquad (3.3.12)$$

is for all $a > 0$, consistent with our observation of the past stock prices (3.3.7). It is clear that for a fixed trading strategy $\vartheta \in \Theta_{\mathrm{sf}}^h(\mathbb{F}^S)$ the discounted gain process $\int \vartheta dS$ is probabilistically similar in the model (3.3.9) and all models (3.3.12), $a > 0$. On the other hand, if strategies from $\Theta_{\mathrm{sf,adm}}\left(\bar{\mathbb{F}}^S\right)$ are considered, the space of discounted trading outcomes

$$\left\{\int_0^T \vartheta_u^1 dS_u \;:\; \vartheta \in \Theta_{\mathrm{sf,adm}}\left(\bar{\mathbb{F}}^S\right)\right\}$$

varies considerably in models of the form (3.3.12) with different parameters $a > 0$. For instance, since for each $a > 0$, the model (3.3.12) is a Samuelson model under the equivalent martingale measure Q^a, there exists a

$\vartheta(a) \in \Theta_{\mathrm{sf,adm}}\left(\bar{\mathbb{F}}^S\right)$ such that

$$\left(S_T - K\mathrm{e}^{-rT}\right)^+ = \mathrm{E}_{Q^a}\left[\left(S_T - K\mathrm{e}^{-rT}\right)^+ \mid \bar{\mathcal{F}}_{t_0}\right] + \int_{t_0}^T \vartheta_t^1(a)dS_t \quad (3.3.13)$$

almost surely,

and

$$\mathrm{E}_{Q^a}\left[\left(S_T - K\mathrm{e}^{-rT}\right)^+ |\bar{\mathcal{F}}_{t_0}\right]$$

$$= \mathrm{dBS}\left(t_0, S_{t_0}, \frac{\sigma a}{\left\| R_1^{\varphi_H^{a,b(a)}} \right\|_2}\right) \to \begin{cases} \left(S_{t_0} - \mathrm{e}^{-rT}K\right)^+ & \text{for } a \to 0 \\ S_{t_0} & \text{for } a \to \infty \end{cases},$$

Hence, if strategies from $\Theta_{\mathrm{sf,adm}}\left(\bar{\mathbb{F}}^S\right)$ are permitted, it is possible to replicate the option perfectly. But the replication costs increase with the quantity

$$\frac{\sigma a}{\left\| R_1^{\varphi_H^{a,b(a)}} \right\|_2}.$$

Whereas the parameter a does not have a big influence on the probabilistic properties of the process $R^{\varphi_H^{a,b(a)}}$, it can be seen from Proposition 3.2 that the quadratic variation of $R^{\varphi_H^{a,b(a)}}$ over a fixed time interval is proportional to

$$\frac{a^2}{\left\| R_1^{\varphi_H^{a,b(a)}} \right\|_2^2}.$$

This shows that in the models (3.3.12) option prices obtained by calculating the conditional expectation under the equivalent martingale measure heavily depend on the local path behaviour of the stochastic process that models the stock price, whereas the finite-dimensional distributions of the process do not seem to have an essential influence.

It is not clear whether for given $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, there exists an $a > 0$ such that the strategy $\vartheta(a) \in \Theta_{\mathrm{sf,adm}}\left(\bar{\mathbb{F}}^S\right)$ that replicates the option in (3.3.13) can be interpreted as an idealisation of strategies from $\Theta_{\mathrm{sf}}^h(\mathbb{F}^S)$ and therefore for this a,

$$\mathrm{dBS}\left(t_0, S_{t_0}, \frac{\sigma a}{\left\| R_1^{\varphi_H^{a,b(a)}} \right\|_2}\right)$$

is the “right” option price in this situation.

Chapter 4

Mixed fractional Brownian motion

4.1 Introduction

By mixed fractional Brownian motion we mean a linear combination of different fractional Brownian motions. In this chapter we examine whether a mixed fractional Brownian motion is a semimartingale when it is of the special form

$$M_t^{H,\alpha} := B_t + \alpha B_t^H \, , \; t \in [0, T] \, ,$$

where B is a Brownian motion, B^H an independent fractional Brownian motion, $\alpha \in \mathbb{R} \setminus \{0\}$ and $T \in (0, \infty)$.

It follows from self-similarity of fractional Brownian motion that the process

$$\left(B_t + \alpha B_t^H \right)_{t \in [0,T]}$$

has the same distribution as

$$\left(T^{\frac{1}{2}} B_{\frac{t}{T}} + \alpha T^H B^H_{\frac{t}{T}} \right)_{t \in [0,T]} = T^{\frac{1}{2}} \left(B_{\frac{t}{T}} + \alpha T^{H - \frac{1}{2}} B^H_{\frac{t}{T}} \right)_{t \in [0,T]} \, .$$

This shows that there is no loss of generality in assuming $T = 1$.

Remark 4.1 Let $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$ and define the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,1]}$ by

$$\mathcal{F}_t = \sigma \left((B_s)_{0 \le s \le t} \, , \left(B_s^H \right)_{0 \le s \le t} \right) , \; t \in [0, 1] \, .$$

Since B is an $\mathbb{F}$-Brownian motion and therefore also an $\mathbb{F}$-weak semimartingale and B^H is not an $\mathbb{F}$-weak semimartingale, $M^{H,\alpha} = B + \alpha B^H$ cannot be an $\mathbb{F}$-weak semimartingale. This does not imply that $M^{H,\alpha}$ is not a weak semimartingale, that is, not a weak semimartingale with respect to its own filtration $\mathbb{F}^{M^{H,\alpha}}$.

The problem of determining whether $M^{H,\alpha}$ is a semimartingale is easiest when $H \in \left\{\frac{1}{2}, 1\right\}$. It is clear that

$$\frac{1}{\sqrt{1+\alpha^2}} M^{\frac{1}{2},\alpha}$$

is a Brownian motion. In particular, it is an $\bar{\mathbb{F}}^{M^{\frac{1}{2},\alpha}}$-semimartingale. Hence, $M^{\frac{1}{2},\alpha}$ is a semimartingale. $M^{1,\alpha}$ can be represented as

$$M_t^{1,\alpha} = B_t + \alpha t \xi,\ t \in [0,1]\,,$$

where B is a Brownian motion and ξ an independent standard normal random variable. This shows that $M^{1,\alpha}$ is a semimartingale with respect to $\bar{\mathbb{F}} = \left(\bar{\mathcal{F}}_t\right)_{t\in[0,1]}$, where

$$\mathcal{F}_t = \sigma\left(\xi, (B_s)_{0\le s\le t}\right)\ , t \in [0,1]\,.$$

With the help of Girsanov's (1960) theorem we can show even more. Unlike $M^{\frac{1}{2},\alpha}$, $M^{1,\alpha}$ is not a multiple of a Brownian motion under the measure P. But it is a Brownian motion under an equivalent measure Q. It can be deduced from Fubini's theorem that

$$\mathrm{E}\left[\exp\left(-\alpha\xi B_1 - \frac{1}{2}(\alpha\xi)^2\right)\right] = 1\,.$$

Therefore,

$$Q = \exp\left(-\alpha\xi B_1 - \frac{1}{2}(\alpha\xi)^2\right)\cdot P$$

is a probability measure that is equivalent to P, and it follows from Girsanov's (1960) theorem that $M^{1,\alpha}$ is a Brownian motion under Q. Hence, $M^{1,\alpha}$ is equivalent to Brownian motion in the sense of Definition 3.1.

It can be seen from Definition 1.5 that the weak semimartingale property is invariant under a change of the probability measure within the same equivalence class. The same is true for the semimartingale property. Hence, all processes that are equivalent to Brownian motion are semimartingales.

We express the main results of this chapter in the following theorem.

Theorem 4.2 $\left(M^{H,\alpha}\right)_{t\in[0,1]}$ *is not a weak semimartingale if* $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4}]$*, it is equivalent to* $\sqrt{1+\alpha^2}$ *times Brownian motion if* $H = \frac{1}{2}$ *and equivalent to Brownian motion if* $H \in (\frac{3}{4}, 1]$.

For $H \in \left\{\frac{1}{2}, 1\right\}$, we have already proved Theorem 4.2. For $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$, we have shown in Section 1.3 that B^H is not a weak semimartingale. For $H \in \left(0, \frac{1}{2}\right)$, the proof was based on the fact that the quadratic variation of B^H is infinite. The same argument can be used to show that $M^{H,\alpha}$ is not a weak semimartingale for $H \in \left(0, \frac{1}{2}\right)$, because, as we will show in Section 2, in this case $M^{H,\alpha}$ has also infinite quadratic variation. For $H \in \left(\frac{1}{2}, 1\right)$, B^H is not a weak semimartingale because it is a stochastic process with vanishing quadratic variation and paths of infinte variation. This reasoning cannot be applied to treat $M^{H,\alpha}$ for $H \in \left(\frac{1}{2}, 1\right)$, because then, $M^{H,\alpha}$ has the same quadratic variation as Brownian motion. In this case we need more refined methods to see whether $M^{H,\alpha}$ is a semimartingale. Surprisingly, $M^{H,\alpha}$ is not a weak semimartingale if $H \in (\frac{1}{2}, \frac{3}{4}]$ and it is equivalent to Brownian motion if $H \in (\frac{3}{4}, 1]$. In Section 3 we prove Theorem 4.2 for $H \in (\frac{1}{2}, \frac{3}{4}]$. The proof depends on a theorem of Stricker (1984) on Gaussian processes. In Section 4 we prove Theorem 4.2 for $H \in (\frac{3}{4}, 1]$. In this case we use the concept of relative entropy and the fact that two Gaussian measures are either equivalent or singular. In Section 5 we discuss the price of a European call option on a stock that is modelled as an exponential mixed fractional Brownian motion with drift. In Section 6 we discuss general results of Shepp (1966) and Hitsuda (1968) on represetations of Gaussian processes that are equivalent to Brownian motion. In Section 7 we solve a linear integral equation to obtain the Radon-Nikodym derivative of $M^{H,\alpha}$ with respect to Wiener measure for certain values of H and α. In Section 8 we solve a quadratic integral equation to obtain the canonical semimartingale decomposition of $M^{H,\alpha}$ for certain values of H and α.

4.2 Proof of Theorem 4.2 for $H \in \left(0, \frac{1}{2}\right)$

From now on we use the following notation. For a stochastic process $(X_t)_{t\in[0,1]}$ and $n \in \mathbb{N}$, we set for $j = 1, \ldots n$, $\Delta_j^n X = X_{\frac{j}{n}} - X_{\frac{j-1}{n}}$.

Like B^H, $M^{H,\alpha}$ cannot be a weak semimartingale for $H \in \left(0, \frac{1}{2}\right)$, because it has infinite quadratic variation. To show this we write for $n \in \mathbb{N}$,

$$\sum_{j=1}^{n}\left(\Delta_j^n M^{H,\alpha}\right)^2 = \sum_{j=1}^{n}\left(\Delta_j^n B\right)^2 + 2\alpha\sum_{j=1}^{n}\Delta_j^n B\,\Delta_j^n B^H + \alpha^2\sum_{j=1}^{n}\left(\Delta_j^n B^H\right)^2 .$$

It is known that

$$\sum_{j=1}^{n}\left(\Delta_j^n B\right)^2 \overset{(n\to\infty)}{\longrightarrow} 1 \quad \text{in } L^2$$

(see e.g. Theorem I.28 of Protter (1990)). From

$$\mathrm{E}\left[\left(\sum_{j=1}^{n}\Delta_j^n B\,\Delta_j^n B^H\right)^2\right] = \sum_{j,k=1}^{n}\mathrm{E}\left[\Delta_j^n B\,\Delta_j^n B^H\,\Delta_k^n B\,\Delta_k^n B^H\right]$$

$$= \sum_{j=1}^{n}\mathrm{E}\left[\left(\Delta_j^n B\right)^2\right]\mathrm{E}\left[\left(\Delta_j^n B^H\right)^2\right] = n\frac{1}{n}\left(\frac{1}{n}\right)^{2H}$$

it follows that

$$\sum_{j=1}^{n}\Delta_j^n B\,\Delta_j^n B^H \overset{(n\to\infty)}{\longrightarrow} 0 \quad \text{in } L^2 .$$

On the other hand, it follows from Lemma 1.4 d) that

$$\sum_{j=1}^{n}\left(\Delta_j^n B^H\right)^2 \overset{(n\to\infty)}{\longrightarrow} \infty \quad \text{in probability.}$$

Hence,

$$\sum_{j=1}^{n}\left(\Delta_j^n M^{H,\alpha}\right)^2 \overset{(n\to\infty)}{\longrightarrow} \infty \quad \text{in probability} .$$

In particular,

$$\left\{\sum_{j=1}^{n}\left(\Delta_j^n M^{H,\alpha}\right)^2 \;:\; n\in\mathbb{N}\right\}$$

is unbounded in L^0 and $M^{H,\alpha}$ is not a weak semimartingale by Proposition 1.9.

□

4.3 Proof of Theorem 4.2 for $H \in (\frac{1}{2}, \frac{3}{4}]$

For $H \in (\frac{1}{2}, \frac{3}{4}]$, the key in the proof of Theorem 4.2 is Lemma 4.4 below. It is based on Theorem 1 of Stricker (1984). Before we can formulate Lemma 4.4, we must specify our notion of a quasimartingale. We call a stochastic process $(X_t)_{t\in[0,1]}$ a quasimartingale if it is a quasimartingale with respect to $\mathbb{F}^X = \left(\mathcal{F}_t^X\right)_{t\in[0,1]}$, where $\mathcal{F}_t^X = \sigma\left((X_s)_{0\le s\le t}\right)$, $t \in [0,1]$.

Definition 4.3 *A stochastic process $(X_t)_{t\in[0,1]}$ is a quasimartingale if*

$$X_t \in L^1 \quad \textit{for all } t \in [0,1], \quad \textit{and}$$

$$\sup_{\tau} \sum_{j=0}^{n-1} \left\| \mathrm{E}\left[X_{t_{j+1}} - X_{t_j} | \mathcal{F}_{t_j}^X\right]\right\|_1 < \infty ,$$

where τ is the set of all finite partitions

$$0 = t_0 < t_1 < \cdots < t_n = 1 , \; n \in \mathbb{N} , \quad \textit{of} \quad [0,1] .$$

Lemma 4.4 *If $M^{H,\alpha}$ is not a quasimartingale, it is not a weak semimartingale.*

Proof. Let us assume that $M^{H,\alpha}$ is a weak semimartingale. Then Theorem 1 of Stricker (1984) implies that $I_{M^{H,\alpha}}(\beta(\mathbb{F}^{M^{H,\alpha}}))$ is bounded in L^2 (for the definition of $I_{M^{H,\alpha}}(\beta(\mathbb{F}^{M^{H,\alpha}}))$ see Section 1.3). Therefore it is also bounded in L^1. For any finite partition

$$0 = t_0 < t_1, \cdots < t_n = 1 , \; n \in \mathbb{N} ,$$

$$\sum_{j=0}^{n-1} \mathrm{sgn}\left(\mathrm{E}\left[M_{t_{j+1}}^{H,\alpha} - M_{t_j}^{H,\alpha} | \mathcal{F}_{t_j}\right]\right) 1_{(t_j, t_{j+1}]} \in \beta(\mathbb{F}^{M^{H,\alpha}}) ,$$

and

$$\left\| I_{M^{H,\alpha}}\left(\sum_{j=0}^{n-1} \mathrm{sgn}\left(\mathrm{E}\left[M_{t_{j+1}}^{H,\alpha} - M_{t_j}^{H,\alpha} | \mathcal{F}_{t_j}^{M^{H,\alpha}}\right]\right) 1_{(t_j, t_{j+1}]}\right)\right\|_1$$

$$\ge \mathrm{E}\left[I_{M^{H,\alpha}}\left(\sum_{j=0}^{n-1} \mathrm{sgn}\left(\mathrm{E}\left[M_{t_{j+1}}^{H,\alpha} - M_{t_j}^{H,\alpha} | \mathcal{F}_{t_j}^{M^{H,\alpha}}\right]\right) 1_{(t_j, t_{j+1}]}\right)\right]$$

$$= \sum_{j=0}^{n-1} \left\| \mathrm{E}\left[M^{H,\alpha}_{t_{j+1}} - M^{H,\alpha}_{t_j} | \mathcal{F}^{M^{H,\alpha}}_{t_j} \right] \right\|_1 .$$

It follows that $M^{H,\alpha}$ is a quasimartingale. Hence, if $M^{H,\alpha}$ is not a quasimartingale, it cannot be a weak semimartingale. □

It remains to prove that $M^{H,\alpha}$ is not a quasimartingale if $H \in (\frac{1}{2}, \frac{3}{4}]$. We do this in the next two lemmas.

Lemma 4.5 *For $H \in (\frac{1}{2}, \frac{3}{4})$, $M^{H,\alpha}$ is not a quasimartingale.*

Proof. Since conditional expectation is a contraction with respect to the L^1-norm, we have for all $n \in \mathbb{N}$ and all $j = 1, \ldots, n-1$,

$$\left\| \mathrm{E}\left[\Delta^n_{j+1} M^{H,\alpha} | \mathcal{F}^{M^{H,\alpha}}_{\frac{j}{n}} \right] \right\|_1 \geq \left\| \mathrm{E}\left[\Delta^n_{j+1} M^{H,\alpha} | \Delta^n_j M^{H,\alpha} \right] \right\|_1 . \tag{4.3.1}$$

Moreover,

$$\left\| \mathrm{E}\left[\Delta^n_{j+1} M^{H,\alpha} | \Delta^n_j M^{H,\alpha} \right] \right\|_1 = \sqrt{\frac{2}{\pi}} \left\| \mathrm{E}\left[\Delta^n_{j+1} M^{H,\alpha} | \Delta^n_j M^{H,\alpha} \right] \right\|_2 \tag{4.3.2}$$

because $\mathrm{E}\left[\Delta^n_{j+1} M^{H,\alpha} | \Delta^n_j M^{H,\alpha} \right]$ is a centred Gaussian random variable. Using (4.3.1) and (4.3.2) we obtain

$$\sum_{j=0}^{n-1} \left\| \mathrm{E}\left[\Delta^n_{j+1} M^{H,\alpha} | \mathcal{F}^{M^{H,\alpha}}_{\frac{j}{n}} \right] \right\|_1 \geq \sqrt{\frac{2}{\pi}} \sum_{j=1}^{n-1} \left\| \mathrm{E}\left[\Delta^n_{j+1} M^{H,\alpha} | \Delta^n_j M^{H,\alpha} \right] \right\|_2$$

$$= \sqrt{\frac{2}{\pi}} \sum_{j=1}^{n-1} \left\| \frac{\mathrm{Cov}\left(\Delta^n_{j+1} M^{H,\alpha}, \Delta^n_j M^{H,\alpha} \right)}{\mathrm{Cov}\left(\Delta^n_j M^{H,\alpha}, \Delta^n_j M^{H,\alpha} \right)} \Delta^n_j M^{H,\alpha} \right\|_2$$

$$= \sqrt{\frac{2}{\pi}} \sum_{j=1}^{n-1} \frac{\mathrm{Cov}\left(\Delta^n_{j+1} M^{H,\alpha}, \Delta^n_j M^{H,\alpha} \right)}{\sqrt{\mathrm{Cov}\left(\Delta^n_j M^{H,\alpha}, \Delta^n_j M^{H,\alpha} \right)}}$$

$$= \sqrt{\frac{2}{\pi}} \sum_{j=1}^{n-1} \frac{\alpha^2 n^{-2H} \left(\frac{2^{2H}}{2} - 1 \right)}{\sqrt{\frac{1}{n} + \alpha^2 n^{-2H}}} \geq \sqrt{\frac{2}{\pi}} \alpha^2 \left(\frac{2^{2H}}{2} - 1 \right) \sum_{j=1}^{n-1} \frac{n^{-2H}}{\sqrt{\frac{1}{n} + \alpha^2 \frac{1}{n}}}$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{2^{2H}}{2} - 1 \right) \frac{\alpha^2}{\sqrt{1+\alpha^2}} \sum_{j=1}^{n-1} n^{\frac{1}{2}-2H}$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{2^{2H}}{2} - 1 \right) \frac{\alpha^2}{\sqrt{1+\alpha^2}} (n-1) n^{\frac{1}{2}-2H} \to \infty\,, \text{ as } n \to \infty\,.$$

This proves the lemma. □

Lemma 4.6 *$M^{\frac{3}{4},\alpha}$ is not a quasimartingale.*

Proof. For $H = \frac{3}{4}$, the estimate (4.3.1) is not good enough. Now we need that, for all $n \in \mathbb{N}$ and all $j = 1, \ldots, n-1$,

$$\left\| \mathrm{E}\left[\Delta_{j+1}^n M^{\frac{3}{4},\alpha} | \mathcal{F}_{\frac{j}{n}}^{M^{\frac{3}{4},\alpha}} \right] \right\|_1$$

$$\geq \left\| \mathrm{E}\left[\Delta_{j+1}^n M^{\frac{3}{4},\alpha} | \Delta_j^n M^{\frac{3}{4},\alpha}, \ldots, \Delta_1^n M^{\frac{3}{4},\alpha} \right] \right\|_1 \,,$$

which follows, like (4.3.1) from the fact that conditional expectation is a contraction with respect to the L^1-norm. Since

$$\mathrm{E}\left[\Delta_{j+1}^n M^{\frac{3}{4},\alpha} | \Delta_j^n M^{\frac{3}{4},\alpha}, \ldots, \Delta_1^n M^{\frac{3}{4},\alpha} \right]$$

is centred Gaussian,

$$\left\| \mathrm{E}\left[\Delta_{j+1}^n M^{\frac{3}{4},\alpha} | \Delta_j^n M^{\frac{3}{4},\alpha}, \ldots, \Delta_1^n M^{\frac{3}{4},\alpha} \right] \right\|_1$$

$$= \sqrt{\frac{2}{\pi}} \left\| \mathrm{E}\left[\Delta_{j+1}^n M^{\frac{3}{4},\alpha} | \Delta_j^n M^{\frac{3}{4},\alpha}, \ldots, \Delta_1^n M^{\frac{3}{4},\alpha} \right] \right\|_2 \,.$$

Hence,

$$\sum_{j=0}^{n-1} \left\| \mathrm{E}\left[\Delta_{j+1}^n M^{\frac{3}{4},\alpha} | \mathcal{F}_{\frac{j}{n}}^{M^{\frac{3}{4},\alpha}} \right] \right\|_1$$

$$\geq \sqrt{\frac{2}{\pi}} \sum_{j=1}^{n-1} \left\| \mathrm{E}\left[\Delta_{j+1}^n M^{\frac{3}{4},\alpha} | \Delta_j^n M^{\frac{3}{4},\alpha}, \ldots, \Delta_1^n M^{\frac{3}{4},\alpha} \right] \right\|_2 \,,$$

and the lemma is proved if we can show that

$$\sum_{j=1}^{n-1} \left\| \mathrm{E}\left[\Delta_{j+1}^n M^{\frac{3}{4},\alpha} | \Delta_j^n M^{\frac{3}{4},\alpha}, \ldots, \Delta_1^n M^{\frac{3}{4},\alpha} \right] \right\|_2 \stackrel{(n\to\infty)}{\longrightarrow} \infty\,. \tag{4.3.3}$$

For $n \in \mathbb{N}$ and $j \in \{1, \ldots, n-1\}$,

$$\left(\Delta_{j+1}^n M^{\frac{3}{4},\alpha}, \Delta_j^n M^{\frac{3}{4},\alpha}, \ldots, \Delta_1^n M^{\frac{3}{4},\alpha}\right)$$

is a Gaussian vector. Therefore,

$$\mathrm{E}\left[\Delta_{j+1}^n M^{\frac{3}{4},\alpha} | \Delta_j^n M^{\frac{3}{4},\alpha}, \ldots, \Delta_1^n M^{\frac{3}{4},\alpha}\right] = \sum_{k=1}^{j} b_k \Delta_k^n M^{\frac{3}{4},\alpha}, \tag{4.3.4}$$

where the vector $b = (b_1, \ldots, b_j)^T$ solves the system of linear equations

$$m = Ab, \tag{4.3.5}$$

in which m is a j-vector whose k-th component m_k is

$$\mathrm{Cov}\left(\Delta_{j+1}^n M^{\frac{3}{4},\alpha}, \Delta_k^n M^{\frac{3}{4},\alpha}\right)$$

and A is the covariance matrix of the Gaussian vector

$$\left(\Delta_1^n M^{\frac{3}{4},\alpha}, \ldots, \Delta_j^n M^{\frac{3}{4},\alpha}\right).$$

Note that A is symmetric and, since the random variables

$$\Delta_1^n M^{\frac{3}{4},\alpha}, \ldots, \Delta_j^n M^{\frac{3}{4},\alpha}$$

are linearly independent, also positive definite. It follows from (4.3.4) and (4.3.5) that

$$\begin{aligned} &\left\| \mathrm{E}\left[\Delta_{j+1}^n M^{\frac{3}{4},\alpha} | \Delta_j^n M^{\frac{3}{4},\alpha}, \ldots, \Delta_1^n M^{\frac{3}{4},\alpha}\right] \right\|_2^2 \\ &= b^T A b = m^T A^{-1} m \geq \|m\|_2^2 \lambda^{-1}, \end{aligned} \tag{4.3.6}$$

where λ is the largest eigenvalue of the matrix A. Since

$$A = \frac{1}{n}\mathrm{id} + \alpha^2 C,$$

where C is the covariance matrix of the increments of fractional Brownian motion

$$\left(\Delta_1^n B^{\frac{3}{4}}, \ldots, \Delta_j^n B^{\frac{3}{4}}\right),$$

we have

$$\lambda = \frac{1}{n} + \alpha^2 \mu,$$

where μ is the largest eigenvalue of C. As

$$C_{kl} = n^{-\frac{3}{2}} \frac{1}{2} \left((|k-l|+1)^{\frac{3}{2}} - 2\,|k-l|^{\frac{3}{2}} + ||k-l|-1|^{\frac{3}{2}} \right) ,$$

$k, l = 1, \ldots, j$, it follows from the Gershgorin Circle Theorem (see e.g. Golub and Van Loan (1989)) and the special form of C that

$$\mu \le \max_{k=1,\ldots,j} \sum_{l=1}^{j} |C_{kl}| \le 2 \sum_{l=1}^{j} |C_{1l}|$$

$$= 2n^{-\frac{3}{2}} \frac{1}{2} \sum_{l=0}^{j-1} \left((l+1)^{\frac{3}{2}} - 2l^{\frac{3}{2}} + |l-1|^{\frac{3}{2}} \right) = n^{-\frac{3}{2}} \left(1 + j^{\frac{3}{2}} - (j-1)^{\frac{3}{2}} \right) .$$

Furthermore,

$$n^{-\frac{3}{2}} \left(1 + j^{\frac{3}{2}} - (j-1)^{\frac{3}{2}} \right) \le \frac{1}{n} + n^{-\frac{3}{2}} \frac{\partial}{\partial j} j^{\frac{3}{2}}$$

$$= \frac{1}{n} + n^{-\frac{3}{2}} \frac{3}{2} j^{\frac{1}{2}} \le \frac{1}{n} + n^{-\frac{3}{2}} \frac{3}{2} n^{\frac{1}{2}} \le 3 \frac{1}{n} .$$

Hence,

$$\lambda \le \frac{1}{n} + \alpha^2 3 \frac{1}{n} = \left(1 + 3\alpha^2 \right) \frac{1}{n} .$$

and

$$\lambda^{-1} \ge \frac{n}{1+3\alpha^2} . \tag{4.3.7}$$

On the other hand,

$$\begin{aligned}
\|m\|_2^2 &= \sum_{k=1}^{j} \left(\mathrm{Cov} \left(\Delta_{j+1}^n M^{\frac{3}{4},\alpha}, \Delta_k^n M^{\frac{3}{4},\alpha} \right) \right)^2 \\
&= \alpha^4 \sum_{k=1}^{j} \left(\mathrm{Cov} \left(\Delta_{j+1}^n B^{\frac{3}{4}}, \Delta_k^n B^{\frac{3}{4}} \right) \right)^2 \\
&= \alpha^4 \frac{1}{4} n^{-3} \sum_{k=1}^{j} \left((k+1)^{\frac{3}{2}} - 2k^{\frac{3}{2}} + (k-1)^{\frac{3}{2}} \right)^2 .
\end{aligned}$$

Since the function $x \mapsto x^{\frac{3}{2}}$ is analytic on $\left\{ x \in \mathbb{C} \,:\, \mathrm{Re} x > 0 \right\}$,

$$(k+1)^{\frac{3}{2}} - 2k^{\frac{3}{2}} + (k-1)^{\frac{3}{2}} = \sum_{m=1}^{\infty} \frac{1}{m!} \frac{\partial^m}{\partial k^m} k^{\frac{3}{2}} + (-1)^m \frac{1}{m!} \frac{\partial^m}{\partial k^m} k^{\frac{3}{2}}$$

$$\geq \frac{\partial^2}{\partial k^2} k^{\frac{3}{2}} = \frac{3}{4} k^{-\frac{1}{2}}, \; k = 2, \ldots j \, .$$

That

$$(k+1)^{\frac{3}{2}} - 2k^{\frac{3}{2}} + (k-1)^{\frac{3}{2}} \geq \frac{3}{4} k^{-\frac{1}{2}}$$

also holds for $k = 1$, can be checked directly. It follows that

$$\|m\|_2^2 \geq \alpha^4 \frac{1}{4} n^{-3} \frac{9}{16} \sum_{k=1}^{j} \frac{1}{k} \geq \alpha^4 \frac{9}{64} n^{-3} \int_1^j \frac{1}{x} dx = \alpha^4 \frac{9}{64} n^{-3} \log j \, . \quad (4.3.8)$$

Putting (4.3.6), (4.3.7) and (4.3.8) together, we obtain

$$\sum_{j=1}^{n-1} \left\| \mathrm{E} \left[\Delta_{j+1}^n M^{\frac{3}{4},\alpha} | \Delta_j^n M^{\frac{3}{4},\alpha}, \ldots, \Delta_1^n M^{\frac{3}{4},\alpha} \right] \right\|_2$$

$$\geq \frac{3}{8} \frac{\alpha^2}{\sqrt{1+3\alpha^2}} \frac{1}{n} \sum_{j=1}^{n-1} \sqrt{\log j} \;\to\; \infty \, , \text{ as } n \to \infty \, .$$

Hence, (4.3.3) holds and the lemma is proved. □

4.4 Proof of Theorem 4.2 for $H \in (\frac{3}{4}, 1]$

To show that for $H \in (\frac{3}{4}, 1]$, $M^{H,\alpha}$ is equivalent to Brownian motion we use the concept of relative entropy. The following definition and all results on relative entropy that we need in this section can be found in chapter 6 of Hida and Hitsuda (1976).

Definition 4.7 *Let Q_1 and Q_2 be probability measures on a measurable space $(\Omega, \mathcal{E})$ and denote by $\mathcal{P}$ all finite partitions*

$$\Omega = \bigcup_{j=1}^{n} E_j, \quad \textit{where} \quad E_j \in \mathcal{E}\, , \quad E_j \cap E_k = \emptyset \textit{ if } j \neq k \, ,$$

of Ω. The entropy $H\,(Q_1|Q_2)$ of Q_1 relative to Q_2 is given by

$$H\,(Q_1|Q_2) := \sup_{\mathcal{P}} \sum_{j=1}^{n} \log \left(\frac{Q_1[E_j]}{Q_2[E_j]} \right) Q_1[E_j] \, ,$$

where we assume $\frac{0}{0} = 0 \log 0 = 0$.

For all $n \in \mathbb{N}$, we define $Y_n : C[0,1] \to \mathbb{R}^n$ by

$$Y_n(\omega) = \left(\omega\left(\frac{1}{n}\right) - \omega(0), \ldots, \omega(1) - \omega\left(\frac{n-1}{n}\right)\right)^T ,$$

and set $\mathcal{B}_n = \sigma(Y_n)$. Note that $\bigvee_{n=1}^{\infty} \mathcal{B}_n$ equals the σ-algebra $\mathcal{B}$ generated by the cylinder sets. We denote by $Q_{M^{H,\alpha}}$ the measure induced by $M^{H,\alpha}$ on $(C[0,1], \mathcal{B})$ and by Q_W Wiener measure on $(C[0,1], \mathcal{B})$. Further, we let for all $n \in \mathbb{N}$, $Q^n_{M^{H,\alpha}}$ and Q^n_W be the restrictions of $Q_{M^{H,\alpha}}$ and Q_W, respectively, to $\mathcal{B}_n$.

To show that $M^{H,\alpha}$ is equivalent to Brownian motion, we make use of the following lemma.

Lemma 4.8 *If*

$$\sup_n H\left(Q^n_{M^{H,\alpha}} | Q^n_W\right) < \infty, \tag{4.4.1}$$

then $Q_{M^{H,\alpha}}$ and Q_W are equivalent.

Proof. From (4.4.1) it follows by Lemma 6.3 of Hida and Hitsuda (1976) that $Q_{M^{H,\alpha}}$ is absolutely continuous with respect to Q_W. But two Gaussian measures on $(C[0,1], \mathcal{B})$ can only be equivalent or singular (see e.g. Theorem 6.1 of Hida and Hitsuda). Therefore $Q_{M^{\alpha,H}}$ and Q_W must be equivalent. □

In the following lemma we show that (4.4.1) holds.

Lemma 4.9

$$\sup_n H\left(Q^n_{M^{H,\alpha}} | Q^n_W\right) < \infty$$

Proof. For all $n \in \mathbb{N}$, Y_n is a centred Gaussian vector under both measures $Q^n_{M^{H,\alpha}}$ and Q^n_W. The covariance matrices of Y_n under $Q^n_{M^{H,\alpha}}$ and Q^n_W are

$$\mathrm{E}_{Q^n_{M^{H,\alpha}}}\left[Y_n Y_n^T\right] = \frac{1}{n}\mathrm{id} + \alpha^2 C_n ,$$

where C_n is the covariance matrix of the increments of fractional Brownian motion

$$\left(\Delta^n_1 B^H, \ldots, \Delta^n_n B^H\right)$$

and

$$\mathrm{E}_{Q^n_W}\left[Y_n Y_n^T\right] = \frac{1}{n}\mathrm{id} .$$

Since C_n is symmetric, there exists an orthogonal $n \times n$-matrix U_n such that $U_n C_n U_n^T$ is a diagonal matrix $D_n = \mathrm{diag}\left(\lambda^n_1, \ldots \lambda^n_n\right)$. $X_n = \sqrt{n} U_n Y_n$ is still a centred Gaussian vector under both measures $Q^n_{M^{H,\alpha}}$ and Q^n_W. The covariance matrices of X_n under these two measures are

$$\mathrm{E}_{Q^n_{M^{H,\alpha}}}\left[X_n X_n^T\right] = \mathrm{id} + n\alpha^2 D_n$$

and

$$\mathrm{E}_{Q^n_W}\left[X_n X_n^T\right] = \mathrm{id}\,.$$

Through X_n, $Q^n_{M^{H,\alpha}}$ and Q^n_W induce measures $R^n_{M^{H,\alpha}}$ and R^n_W on $\mathbb{R}^n$. It can easily be seen from Definition 4.7 that

$$H\left(Q^n_{M^{H,\alpha}}|Q^n_W\right) = H\left(R^n_{M^{H,\alpha}}|R^n_W\right)\,.$$

Since both measures $R^n_{M^{H,\alpha}}$ and R^n_W are non-degenerate Gaussian measures on $\mathbb{R}^n$, they are equivalent. We denote by ξ_n the Radon-Nikodym derivative of $R^n_{M^{H,\alpha}}$ with respect to R^n_W. Lemma 6.1 of Hida and Hitsuda (1976) and a calculation show that

$$H\left(R^n_{M^{H,\alpha}}|R^n_W\right) = \mathrm{E}_{R^n_{M^{H,\alpha}}}\left[\log \xi_n\right] = \frac{1}{2}\sum_{j=1}^{n}\left(n\alpha^2\lambda_j^n - \log\left(1 + n\alpha^2\lambda_j^n\right)\right)\,.$$

For all $x \geq 0$, we have

$$x - \log(1+x) = \int_0^x \frac{u}{1+u}du \leq \int_0^x u du = \frac{1}{2}x^2\,.$$

Therefore,

$$H\left(R^n_{M^{H,\alpha}}|R^n_W\right) \leq \frac{1}{4}n^2\alpha^4\sum_{j=1}^{n}\left(\lambda_j^n\right)^2\,.$$

Hence, the lemma is proved if we can show that

$$\sup_n n^2\sum_{j=1}^{n}\left(\lambda_j^n\right)^2 < \infty\,, \tag{4.4.2}$$

where $\lambda_1^n, \ldots, \lambda_n^n$ are the eigenvalues of the covariance matrix of the increments of fractional Brownian motion

$$\left(\Delta_1^n B^H, \ldots, \Delta_n^n B^H\right)\,.$$

Since orthogonal transformation leaves the Hilbert-Schmidt norm of a matrix invariant,

$$\sum_{j=1}^{n}\left(\lambda_j^n\right)^2 = \sum_{j,k=1}^{n}\mathrm{Cov}\left(\Delta_j^n B^H, \Delta_k^n B^H\right)^2\,.$$

As fractional Brownian motion has stationary increments,

$$\sum_{j,k=1}^{n} \mathrm{Cov}\left(\Delta_j^n B^H, \Delta_k^n B^H\right)^2 \leq 2n \sum_{k=1}^{n} \mathrm{Cov}\left(\Delta_k^n B^H, \Delta_1^n B^H\right)^2$$

$$= 2nn^{-4H}\left(1+\left(\frac{2^{2H}}{2}-1\right)^2\right) + 2n\sum_{k=3}^{n} \mathrm{Cov}\left(\Delta_k^n B^H, \Delta_1^n B^H\right)^2 .$$

Since, for $H \in (\frac{3}{4}, 1]$,

$$n^2 2nn^{-4H}\left(1+\left(\frac{2^{2H}}{2}-1\right)^2\right) \to 0\,, \quad \text{as } n \to \infty\,,$$

it is enough to show

$$\sup_n n^3 \sum_{k=3}^{n} \mathrm{Cov}\left(\Delta_k^n B^H, \Delta_1^n B^H\right)^2 < \infty \tag{4.4.3}$$

to prove (4.4.2). For all $k \geq 3$, we have

$$\begin{aligned}
\mathrm{Cov}\left(\Delta_k^n B^H, \Delta_1^n B^H\right) &= n^{-2H}\frac{1}{2}\left(k^{2H} - 2(k-1)^{2H} + (k-2)^{2H}\right)\\
\leq\ & n^{-2H}\frac{1}{2}\left(\frac{\partial}{\partial k}k^{2H} - \frac{\partial}{\partial k}(k-2)^{2H}\right)\\
=\ & Hn^{-2H}\left(k^{2H-1} - (k-2)^{2H-1}\right) \leq Hn^{-2H} 2\frac{\partial}{\partial k}(k-2)^{2H-1}\\
=\ & 2H(2H-1)\,n^{-2H}(k-2)^{2H-2} .
\end{aligned}$$

Using this, we obtain

$$\begin{aligned}
n^3 \sum_{k=3}^{n} \mathrm{Cov}\left(\Delta_k^n B^H, \Delta_1^n B^H\right)^2 &\leq 4H^2(2H-1)^2 n^{3-4H} \sum_{k=1}^{n-2} k^{4H-4}\\
&\leq 4H^2(2H-1)^2 n^{3-4H}\int_0^{n-2} x^{4H-4}dx\\
&= \frac{4H^2(2H-1)^2}{4H-3} n^{3-4H}(n-2)^{4H-3}\\
&\leq \frac{4H^2(2H-1)^2}{4H-3} .
\end{aligned}$$

Hence, (4.4.3) holds, and the lemma is proved. □

Remark 4.10 In this section we have shown that for $H \in (\frac{3}{4}, 1]$, $Q_{M^{H,\alpha}}$ and Q_W are equivalent. But our method of proof has not given us the Radon-Nikodym derivative nor have we found the semimartingale decomposition of $M^{H,\alpha}$. These problems will be addressed in Sections 6, 7 and 8.

4.5 Option pricing with mixed fractional Brownian motion

In this section we examine the price C_0 of a European call option with strike price K and maturity $T = 1$. If money in the money market account grows like $\exp(rt)$, $t \in [0, 1]$, for a constant r, the option's discounted pay-off is given by

$$\left(S_1 - e^{-r} K\right)^+ .$$

Let us assume that empirical data suggests that the discounted price of the stock S should be modelled as

$$S_t = S_0 \exp\left(\nu t + \sigma B_t^H\right), \ t \in [0, 1], \tag{4.5.1}$$

for constants $S_0 > 0$, $\nu, \sigma > 0$, and a fractional Brownian motion B^H. We have shown in Section 2.3 that for $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$, such a model admits arbitrage. However, if $H \in \left(\frac{3}{4}, 1\right)$, we can exclude all arbitrage strategies by regularizing fractional Brownian motion in the following way:

If $(B_t)_{t \in [0,1]}$ is a Brownian motion independent of B^H, Theorem 4.2 implies that for all $\varepsilon > 0$,

$$\left(\varepsilon B_t + B_t^H\right)_{t \in [0,1]} \quad \text{is equivalent to} \quad (\varepsilon B_t)_{t \in [0,1]} .$$

We observe that

$$\operatorname{Cov}\left(\varepsilon B_t + B_t^H, \varepsilon B_s + B_s^H\right) = \varepsilon^2 (t \wedge s) + \operatorname{Cov}\left(B_t^H, B_s^H\right), \ t, s \in [0, 1] .$$

Hence, $\left(\varepsilon B_t + B_t^H\right)_{t \in [0,1]}$ is a continuous centred Gaussian process that has up to ε^2 the same covariance function as $\left(B_t^H\right)_{t \in [0,1]}$. This shows that if the model (4.5.1) fits empirical data, then so does

$$S_t = S_0 \exp\left\{\nu t + \sigma \left(\varepsilon B_t + B_t^H\right)\right\}, \ t \in [0, 1], \tag{4.5.2}$$

for $\varepsilon > 0$ small enough. But in contrast to (4.5.1), (4.5.2) has, like the Samuelson model, a unique equivalent martingale measure Q^{ε}. This implies that the model (4.5.2) is arbitrage-free and also complete if we allow all strategies of $\Theta_{\text{sf,adm}}(\bar{\mathbb{F}}^S)$, given in (3.3.10). According to current practice in mathematical finance, in such a framework options are priced by taking the expected value under the equivalent martingale measure of the option's discounted pay-off. In the model (4.5.2) this leads to the following option price:

$$C_0(\varepsilon) = \mathrm{E}_{Q^{\varepsilon}}\left[\left(S_0 \exp\left\{\nu + \sigma\left(\varepsilon B_1 + B_1^H\right)\right\} - \mathrm{e}^{-r}K\right)^+\right]$$

$$= \mathrm{BS}(0, S_0, \varepsilon\sigma)\,, \qquad (4.5.3)$$

where BS is the Black-Scholes price. Since the function $\mathrm{BS}(0, S_0, .)$ is continuous, strictly increasing and bijective from the interval $(0, \infty)$ to the interval $((S_0 - \mathrm{e}^{-r}K)^+, S_0)$, $C_0(\varepsilon)$ in (4.5.3) is close to $(S_0 - \mathrm{e}^{-r}K)^+$ when $\varepsilon > 0$ is small. The deeper reason why $C_0(\varepsilon)$ is so low in this situation, is that (4.5.3) gives the initial capital necessary to replicate the pay-off of the call option with a trading strategy from $\Theta_{\text{sf,adm}}(\bar{\mathbb{F}}^S)$, and this strategy seems to exploit small movements of the stochastic process (4.5.2) over very short time intervals.

In reality a seller of the option can only carry out finitely many transactions to hedge the option. Moreover, he cannot buy and sell within nanoseconds. Therefore he will demand a higher price than $\mathrm{BS}(0, S_0, \varepsilon\sigma) \simeq \left(S_0 - \mathrm{e}^{-r}K\right)^+$.

To find a reasonable option price, one should introduce a waiting time $h > 0$ and restrict trading strategies to the class $\Theta_{\text{sf}}^h(\mathbb{F}^S)$ (2.4.1) of self-financing strategies that can buy and sell at $\mathbb{F}^S$-stopping times but after each transaction there must be a waiting period of minimal length h before the next. For small $\varepsilon > 0$, the discounted gains process of such a strategy is similar in both models (4.5.1) and (4.5.2), as should be the case. Moreover (4.5.1) has no arbitrage in $\Theta_{\text{sf}}^h\left(\mathbb{F}^S\right)$. Hence, if we confine the strategies to the class $\Theta_{\text{sf}}^h\left(\mathbb{F}^S\right)$, we can return to the model (4.5.1) to value the option. Since (4.5.1) with the strategies $\Theta_{\text{sf}}^h\left(\mathbb{F}^S\right)$ is an incomplete model, one has to decide in which sense the pay-off of the option should be approximated and then search for an optimal strategy. It is not clear whether the regularization (4.5.2) is of any use in such a procedure.

4.6 Representations of Gaussian processes that are equivalent to Brownian motion

In this section we discuss general results of Shepp (1966) and Hitsuda (1968) on representations of Gaussian processes that are equivalent to Brownian motion.

Let $0 < T \leq \infty$. For $0 < T < \infty$, we set $I_T = [0, T]$, and for $T = \infty$, $I_\infty = [0, \infty)$. Let $C(I_T)$ be the space of real-valued, continuous functions on I_T. The coordinates process $(X_t)_{t \in I_T}$ on $C(I_T)$ is given by

$$X_t(\omega) = \omega(t) \, , \; \omega \in C(I_T) \, , \; t \in I_T \, .$$

It generates the σ-algebra

$$\mathcal{B}_T := \sigma \left\{ X_t^{-1}(B) \, : \, t \in I_T, B \text{ an open subset of } \mathbb{R} \right\} \, .$$

By Q_W we denote Wiener measure on $(C(I_T), \mathcal{B}_T)$. Every almost surely continuous process $(Y_t)_{t \in I_T}$ on a probability space $(\Omega, \mathcal{A}, P)$ has a distribution Q_Y on $(C(I_T), \mathcal{B}_T)$. It is given by

$$Q_Y [B] = P [Y \in B] \, , \; B \in \mathcal{B}_T \, .$$

4.6.1 The representations of Shepp and Hitsuda

Let $(Y_t)_{t \in I_T}$ be a Gaussian process on a probability space $(\Omega, \mathcal{A}, P)$, that is, for all $n \in \mathbb{N}$ and $\{t_1, t_2, \ldots, t_n\} \subset I_T$, the distribution of $\left(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n}\right)$ with respect to P is n-dimensional Gaussian. We set

$$M_t^Y = \mathrm{E}_P [Y_t] \, , \; t \in I_T \, ,$$

and

$$\Gamma_{ts}^Y = \mathrm{E}_P \left[(Y_t - M_t^Y)(Y_s - M_s^Y) \right] \, , \; t, s \in I_T \, .$$

We call $(Y_t)_{t \in I_T}$ centred if $M_t^Y = 0$, $t \in I_T$. It follows from Theorem 1 of Shepp (1966) that an a.s. continuous Gaussian process $(Y_t)_{t \in I_T}$ is equivalent to Brownian motion if and only if

$$M_t^Y = \int_0^t m^Y(u) du \, , \; t \in I_T \, ,$$

for a $m^Y \in L^2(I_T)$ and $\left(Y_t - M_t^Y\right)_{t \in I_T}$ is equivalent to Brownian motion. Therefore, we will henceforth only treat a.s. continuous, centred Gaussian processes.

Before we start discussing representations of a.s. continuous, centred Gaussian processes $(Y_t)_{t\in I_T}$ that are equivalent to Brownian motion, we collect some properties of integral operators induced by L^2-kernels. The proofs of all these facts can be found in Smithies (1958) or Dunford and Schwartz (1963).

An L^2-kernel is a $k \in L^2(I_T^2)$. It induces a Hilbert-Schmitd operator

$$k^{\mathrm{op}} : L^2(I_T) \to L^2(I_T) ,$$

given by

$$k^{\mathrm{op}} f(t) = \int_0^T k(t,s) f(s) ds , \ t \in I_T , \ f \in L^2(I_T) .$$

The spectrum $\sigma(k^{\mathrm{op}})$ consists of at most countably many points. Every non-zero value in $\sigma(k^{\mathrm{op}})$ is an eigenvalue of finite multiplicity. If $\{\lambda_j\}_{j=1}^N$, $N \in \mathbb{N} \cup \{\infty\}$, is the family of non-zero eigenvalues of k^{op}, repeated according to their multiplicity, then

$$\sum_{j=1}^N |\lambda_j|^2 < \infty .$$

The Carleman-Fredholm determinant $\delta_k : \mathbb{C} \to \mathbb{C}$ is defined by the convergent product

$$\delta_k(\lambda) = \prod_{j=1}^N (1 - \lambda\lambda_j) \mathrm{e}^{\lambda\lambda_j} , \ \lambda \in \mathbb{C} .$$

k^{op} is said to be of trace class if $\sum_{j=1}^N |\lambda_j| < \infty$. If k^{op} is of trace class, its trace is defined to be

$$\mathrm{tr}(k^{\mathrm{op}}) = \sum_{j=1}^N \lambda_j .$$

If $k, l \in L^2(I_T^2)$, then $k * l$ given by

$$k * l(t,s) = \int_0^T k(t,u) l(u,s) du , \ t, s \in I_T ,$$

is again in $L^2(I_T^2)$, and

$$(k * l)^{\mathrm{op}} = k^{\mathrm{op}} l^{\mathrm{op}} .$$

Moreover, $k^{\mathrm{op}} l^{\mathrm{op}}$ is always of trace class and

$$\mathrm{tr}(k^{\mathrm{op}} l^{\mathrm{op}}) = \int_0^T \int_0^T k(t,s) l(s,t) ds dt \,.$$

If $k \in L^2(I_T^2)$ and $1 \notin \sigma(k^{\mathrm{op}})$, then there exists a unique kernel $n_k \in L^2(I_T^2)$ such that

$$\left(\mathrm{id} - k^{\mathrm{op}}\right)^{-1} = \mathrm{id} - n_k^{\mathrm{op}} \,.$$

We call n_k the negative resolvent kernel of k because $-n_k$ is usually called the resolvent kernel of k for the value 1. If k is continuous, then so is n_k. If k is real-valued and symmetric, k^{op} is self-adjoint. Therefore, all eigenvalues λ_j are real, the corresponding eigenfunctions e_j can be chosen orthonormal, and k can be represented as

$$k(t,s) = \sum_{j=1}^{N} \lambda_j e_j(t) e_j(s) \,,$$

where the series converges in $L^2(I_T^2)$. It follows that

$$n_k(t,s) = \sum_{j=1}^{N} \frac{-\lambda_j}{1-\lambda_j} e_j(t) e_j(s) \,.$$

In particular, n_k is again real-valued and symmetric.

The proof of the following theorem can be found in Shepp (1966).

Theorem 4.11 (Shepp)
a) *A* $\Gamma : I_T^2 \to \mathbb{R}$ *is the covariance function of an a.s. continuous, centred Gaussian process equivalent to Brownian motion if and only if it is of the form*

$$\Gamma_{ts} = t \wedge s - \int_0^t \int_0^s k(u,v) dv du \,, \ t,s \in I_T \,,$$

where k *is in* $L^2(I_T^2)$*, real-valued, symmetric and* $\sigma(k^{\mathrm{op}}) \subset (-\infty, 1)$.
b) *Let* $(Y_t)_{t \in I_T}$ *be an a.s. continuous, centred Gaussian process that is equivalent to Brownian motion. Then*

$$\frac{dQ_Y}{dQ_W}(X) = c \exp\left(\int_0^T \int_0^s n_k(s,u) dX_u dX_s\right) \,, \tag{4.6.1}$$

where n_k *is the negative resolvent kernel of the* L^2*-kernel* k *that satisfies*

$$\Gamma_{ts}^Y = t \wedge s - \int_0^t \int_0^s k(u,v) dv du \,, \ t,s \in I_T \,,$$

and

$$c = \frac{1}{\sqrt{\delta_k(1) \exp\left\{\mathrm{tr}(-n_k^{\mathrm{op}} k^{\mathrm{op}})\right\}}} .$$

Remarks 4.12
1. We call (4.6.1) the Shepp-representation of Q_Y.
2. n_k is the unique L^2-kernel that solves the equation

$$n_k(t,s) + k(t,s) = \int_0^T n_k(t,u)k(u,s)du \; , \; t,s \in I_T \; .$$

It is also the unique L^2-kernel that solves the equation

$$n_k(t,s) + k(t,s) = \int_0^T k(t,u)n_k(u,s)du \; , \; t,s \in I_T \; .$$

3. Let $\Gamma : I_T^2 \to \mathbb{R}$ be of the form

$$\Gamma_{ts} = t \wedge s - \int_0^t \int_0^s k(u,v)dvdu \; , \; t,s \in I_T \; , \tag{4.6.2}$$

for a real-valued, symmetric L^2-kernel k. Γ is the covariance function of a centred Gaussian process if and only if it is positive semi-definite, that is,

$$\sum_{j,k=1}^{n} z_j \Gamma_{t_j t_k} \bar{z}_k \geq 0 \, ,$$

for all $n \in \mathbb{N}$, $\{t_1, \ldots, t_n\} \subset I_T$ and $z \in \mathbb{C}^n$. But (4.6.2) can be written as

$$\Gamma_{ts} = \left(1_{[0,t]}, 1_{[0,s]}\right) - \left(1_{[0,t]}, k^{\mathrm{op}} 1_{[0,s]}\right) \; , \; t,s \in I_T \; ,$$

where $(f,g) = \int_0^T f(t)\bar{g}(t)dt$, $f,g \in L^2(I_T)$. This and the fact that the functions of the form

$$\sum_{j=1}^{n} z_j 1_{[0,t_j]} \, , \quad n \in \mathbb{N} \, , \; \{t_1, \ldots, t_n\} \subset I_T \; , \; z \in \mathbb{C}^n \; ,$$

are dense in $L^2(I_T)$, imply that (4.6.2) is the covariance function of a centred Gaussian process if and only if $\sigma(k^{\mathrm{op}}) \subset (-\infty, 1]$.

Corollary 4.13
a) *Let* $(Y_t)_{t\in I_T}$ *be an a.s. continuous, centred Gaussian process equivalent to Brownian motion on a probability space* $(\Omega, \mathcal{A}, P)$ *and* k *the real-valued, symmetric* L^2*-kernel that satisfies*

$$\Gamma^Y_{ts} = t \wedge s - \int_0^t \int_0^s k(u, v)dvdu \; , \; t, s \in I_T \; .$$

Then the following two are equivalent:

(i) There exists a probability space $(\Omega', \mathcal{A}', P')$ *with a Brownian motion* $(B_t)_{t\in I_T}$ *and an independent, a.s. continuous, centred Gaussian process* $(Z_t)_{t\in I_T}$ *such that*

$$(Y_t)_{t\in I_T} = (B_t + Z_t)_{t\in I_T} \quad \text{in distribution.}$$

(ii) $\sigma(k^{\text{op}}) \subset (-\infty, 0]$.

b) *Let* $(B_t)_{t\in I_T}$ *be a Brownian motion and* $(Z_t)_{t\in I_T}$ *an independent, a.s. continuous, centred Gaussian process. Then the following two are equivalent:*

(i) $(B_t + Z_t)_{t\in I_T}$ *is equivalent to Brownian motion.*

(ii) There exists a real-valued, symmetric L^2*-kernel* k *such that*

$$\Gamma^Z_{ts} = -\int_0^t \int_0^s k(u, v)dvdu \; , \; t, s \in I_T \; .$$

Proof.
a) Let $(B_t)_{t\in I_T}$ be a Brownian motion and $(Z_t)_{t\in I_T}$ an independent, a.s. continuous, centred Gaussian process on a probability space $(\Omega', \mathcal{A}', P')$ such that

$$(Y_t)_{t\in I_T} = (B_t + Z_t)_{t\in I_T} \quad \text{in distribution}\,.$$

Then

$$\Gamma^Z_{ts} = \Gamma^Y_{ts} - t \wedge s = -\int_0^t \int_0^s k(u, v)dvdu = -\left(1_{[0,t]}, k^{\text{op}} 1_{[0,s]}\right) \; , \; t, s \in I_T \; ,$$

where $(f, g) = \int_0^T f(t)\bar{g}(t)dt, \; f, g \in L^2(I_T)$. Since Γ^Z_{ts} is a covariance function and therefore positive semi-definite and the functions of the form

$$\sum_{j=1}^n z_j 1_{[0,t_j]} \; , \; n \in \mathbb{N} \; , \; \{t_1, \ldots, t_n\} \subset I_T \; , \; z \in \mathbb{C}^n \; ,$$

are dense in $L^2(I_T)$, it follows that $\sigma(k^{\text{op}}) \subset (-\infty, 0]$.

On the other hand, if $\sigma(k^{\text{op}}) \subset (-\infty, 0]$, then

$$-\int_0^t \int_0^s k(u,v)dvdu\ ,\ t,s \in I_T\ ,$$

is positive semi-definite. It follows that there exists a probability space $\left(\Omega', \mathcal{A}', P'\right)$ with a Brownian motion $(B_t)_{t\in I_T}$ and an independent centred Gaussian process $(Z_t)_{t\in I_T}$ such that $\Gamma^Z_{ts} = -\int_0^t \int_0^s k(u,v)dvdu\ ,\ t,s \in I_T$. Hence,

$$\Gamma^{B+Z}_{ts} = t\wedge s - \int_0^t \int_0^s k(u,v)dvdu = \Gamma^Y_{ts}\ ,\ t,s \in I_T\ ,$$

and therefore,

$$(B_t + Z_t)_{t\in I_T} = (Y_t)_{t\in I_T} \quad \text{in distribution.}$$

Since $(B_t)_{t\in 0,T}$ and $(Y_t)_{t\in 0,T}$ are a.s. continuous, $(Z_t)_{t\in 0,T}$ can also be chosen to be a.s. continuous.

b) If $(B_t + Z_t)_{t\in I_T}$ is equivalent to Brownian motion, then it follows from Theorem 4.11 a) that there exists a real-valued, symmetric L^2-kernel k such that

$$t\wedge s + \Gamma^Z_{ts} = \Gamma^{B+Z}_{ts} = t\wedge s - \int_0^t \int_0^s k(u,v)dvdu\ ,\ t,s \in I_T\ .$$

Hence,

$$\Gamma^Z_{ts} = -\int_0^t \int_0^s k(u,v)dvdu\ ,\ t,s \in I_T\ .$$

On the other hand, if there exists a real-valued, symmetric L^2-kernel k such that

$$\Gamma^Z_{ts} = -\int_0^t \int_0^s k(u,v)dudv\ ,\ t,s \in I_T\ ,$$

then $\sigma(k^{\text{op}}) \subset (-\infty, 0]$ because Γ^Z is positive semi-definite. Since

$$\Gamma^{B+Z}_{ts} = t\wedge s - \int_0^t \int_0^s k(u,v)dudv\ ,\ t,s \in I_T\ ,$$

it follows from Theorem 4.11 a) that $(B_t + Z_t)_{t\in I_T}$ is equivalent to Brownian motion. □

Remark 4.14 Let $0 < T < \infty$, $(B_t)_{0\le t\le T}$ a Brownian motion and $\left(B_t^H\right)_{t\in[0,T]}$ an independent fractional Brownian motion with Hurst parameter $H \in (0,1]$. Since

$$\Gamma_{ts}^{B^H} = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

$$= H(2H-1)\int_0^t \int_0^s |u-v|^{2H-2}\,dv du\ ,\ t,s \in [0,T]\,,$$

it follows from Corollary 4.13 b) that for every $\alpha \in \mathbb{R}\setminus\{0\}$, the mixed fractional Brownian motion

$$M_t^{H,\alpha} = B_t + \alpha B_t^H\ ,\ t \in [0,T]\,,$$

is equivalent to Brownian motion if and only if $H \in (\frac{3}{4}, 1]$. This assertion is the part of Theorem 4.2 which we proved in Section 4.4. Note that for the mixed fractional Brownian motion $M^{H,\alpha}$, condition (ii) of Corollary 4.13 b) is equivalent to condition (4.4.2).

For $H \in (\frac{3}{4}, 1]$, let λ_H be the largest eigenvalue of $H(2H-1)\,f_{2H-2}^{\mathrm{op}}$, where

$$f_p(t,s) = |t-s|^p\ ,\ t,s \in [0,T]\,,\ p \in (-\frac{1}{2}, \infty)\,.$$

Since $H(2H-1)\,f_{2H-2}^{\mathrm{op}}$ is positive semi-definite, λ_H is equal to the operator norm $\left\|H(2H-1)\,f_{2H-2}^{\mathrm{op}}\right\| > 0$ of $H(2H-1)\,f_{2H-2}^{\mathrm{op}}$. Hence, if $c \in (0, \frac{1}{\lambda_H})$, then

$$t\wedge s - \frac{c}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)\ ,\ t,s \in [0,T]\,,$$

is the covariance function of an a.s. continuous, centred Gaussian process equivalent to Brownian motion which cannot have the same distribution as the sum of a Brownian motion and an independent Gaussian process.

$$t\wedge s - \frac{1}{2\lambda_H}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)\ ,\ t,s \in [0,T]\,,$$

is the covariance function of a centred Gaussian process that is neither equivalent to Brownian motion nor equal in distribution to the sum of a Brownian motion and an independent Gaussian process.

A kernel $h \in L^2(I_T^2)$ is called a Volterra kernel if $h(t,s) = 0$ for all $s > t$. In this case h^{op} is quasi-nilpotent, that is, the spectral radius

$$\sup\left\{|\lambda|\ :\ \lambda \in \sigma(h^{\mathrm{op}})\right\} = \liminf_{n\to\infty} \left\|\left(h^{\mathrm{op}}\right)^n\right\|^{\frac{1}{n}}$$

is zero. Hence, the negative resolvent kernel n_h of h always exists, and it can be shown that it is also a Volterra kernel. Moreover, if h is a real-valued Volterra kernel, then so is n_h.

The proof of the following theorem can be found in Hitsuda (1968).

Theorem 4.15 (Hitsuda)
a) *Let $(Y_t)_{t\in I_T}$ be an a.s. continuous, centred Gaussian process on a probability space $(\Omega, \mathcal{A}, P)$ that is equivalent to Brownian motion. Then there exists a unique real-valued Volterra kernel $h \in L^2(I_T^2)$ such that*

$$W_t = Y_t - \int_0^t \int_0^s h(s,u) Y_u ds \, , \; t \in I_T \, , \tag{4.6.3}$$

is a Brownian motion on $(\Omega, \mathcal{A}, P)$. Furthermore,

$$Y_t = W_t - \int_0^t \int_0^s n_h(s,u) dW_u ds \, , \; t \in I_T \, , \tag{4.6.4}$$

where $n_h \in L^2(I_T^2)$ is the negative resolvent kernel of h, and the representation (4.6.4) is unique in the following sense: If $(B_t)_{t\in I_T}$ is a Brownian motion on $(\Omega, \mathcal{A}, P)$ and $l \in L^2(I_T^2)$ a real-valued Volterra kernel such that

$$Y_t = B_t - \int_0^t \int_0^s l(s,u) dB_u ds \, , \; t \in I_T \, ,$$

then $B = W$ and $l = n_h$.
b) *Let $(B_t)_{t\in I_T}$ be a Brownian motion on a probability space $(\Omega, \mathcal{A}, P)$ and $l \in L^2(I_T^2)$ a real-valued Volterra kernel. Then*

$$\mathrm{E}_P \left[\exp\left(\int_0^T \int_0^s l(s,u) dB_u dB_s - \frac{1}{2} \int_0^T \left(\int_0^s l(s,u) dB_u \right)^2 ds \right) \right] = 1 \, , \tag{4.6.5}$$

and, by Girsanov's (1960) theorem,

$$B_t - \int_0^t \int_0^s l(s,u) dB_u ds \, , \; t \in I_T \, ,$$

is a Brownian motion on $\left(\Omega, \mathcal{A}, \tilde{P}\right)$, where

$$\tilde{P} = \exp\left(\int_0^T \int_0^s l(s,u) dB_u dB_s - \frac{1}{2} \int_0^T \left(\int_0^s l(s,u) dB_u \right)^2 ds \right) \cdot P \, .$$

Remarks 4.16
1. It follows from (4.6.3) and (4.6.4) that

$$\mathcal{F}_t^W = \sigma\left\{W_s \,:\, 0 \le s \le t\right\} = \sigma\left\{Y_s \,:\, 0 \le s \le t\right\} = \mathcal{F}_t^Y \;,\; t \in I_T \;.$$

Therefore, (4.6.4) is the canonical semimartingale decomposition of Y on

$$\left(\Omega, \mathcal{A}, \left(\mathcal{F}_t^Y\right)_{t \in I_T}, P\right) .$$

We call it the Hitsuda representation of Y.
2. n_h is the unique Volterra kernel that solves the equation

$$n_h(t,s) + h(t,s) = \int_s^t n_h(t,u)h(u,s)du \;,\; t,s \in I_T \;,\; s \le t \;.$$

It is also the unique Volterra kernel that solves the equation

$$n_h(t,s) + h(t,s) = \int_s^t h(t,u)n_h(u,s)du \;,\; t,s \in I_T \;,\; s \le t \;.$$

4.6.2 The Girsanov-Hitsuda representation and relations between different representations

Let $(Y_t)_{t \in I_T}$ be an a.s. continuous, centred Gaussian process that is equivalent to Brownian motion and k, n_k, h, n_h the kernels from Subsection 4.6.1.

Proposition 4.17

$$\frac{dQ_Y}{dQ_W}(X) = \tag{4.6.6}$$

$$\exp\left(\int_0^T \int_0^s h(s,u)dX_u dX_s - \frac{1}{2}\int_0^T \left(\int_0^s h(s,u)dX_u\right)^2 ds\right) .$$

Proof. It follows from Theorem 4.15 a) that

$$\tilde{W} = X_t - \int_0^t \int_0^s h(s,u)dX_u ds \;,\; t \in I_T \;,$$

is a Brownian motion on $(C(I_T), \mathcal{B}_T, Q_Y)$. On the other hand, Theorem 4.15 b) implies that $\tilde{W}$ is also a Brownian motion on $\left(C(I_T), \mathcal{B}_T, \tilde{Q}\right)$, where

$$\tilde{Q} = \exp\left(\int_0^T \int_0^s h(s,u)dX_u dX_s - \frac{1}{2}\int_0^T \left(\int_0^s h(s,u)dX_u\right)^2 ds\right) \cdot Q_W \;.$$

Since $\mathcal{F}_T^{\tilde{W}} = \mathcal{F}_T^X = \mathcal{B}_T$, it follows that $Q_Y = \tilde{Q}$ on $(C(I_T), \mathcal{B}_T)$. This proves the proposition. □

Remark 4.18 Since the second part of Theorem 4.15 b) follows from (4.6.5) by Girsanov's (1960) theorem, we call (4.6.6) the Girsanov-Hitsuda representation of Q_Y.

Corollary 4.19

$$n_k(t,s) = h(t,s) - \int_t^T h(v,t)h(v,s)dv\,,\ t,s \in I_T\,,\ s \le t\,. \tag{4.6.7}$$

$$\log c = -\frac{1}{2}\int_0^T \int_0^s h^2(s,u)duds\,. \tag{4.6.8}$$

Proof. By comparing (4.6.1) with (4.6.6) we get

$$\begin{aligned}
\log c \quad & + \int_0^T \int_0^s n_k(s,u)dX_u dX_s \\
= \quad & \int_0^T \int_0^s h(s,u)dX_u dX_s - \frac{1}{2}\int_0^T \left(\int_0^s h(s,u)dX_u\right)^2 ds \\
= \quad & \int_0^T \int_0^s h(s,u)dX_u dX_s \\
& -\frac{1}{2}\int_0^T \left[2\int_0^s \int_0^v h(s,u)dX_u h(s,v)dX_v + \int_0^s h^2(s,v)dv\right] ds \\
= \quad & \int_0^T \int_0^s h(s,u)dX_u dX_s - \int_0^T \int_0^s \int_s^T h(v,s)h(v,u)dv dX_u dX_s \\
& -\frac{1}{2}\int_0^T \int_0^s h^2(s,v)dvds\,.
\end{aligned}$$

Now (4.6.8) follows by taking expectation with respect to Q_W. The fact that for every real-valued Volterra kernel $l \in L^2(I_T^2)$,

$$\mathrm{E}_{Q_W}\left[\left(\int_0^T \int_0^s l(s,u)dX_u dX_s\right)^2\right] = \int_0^T \int_0^s l^2(s,u)duds\,,$$

entails that the linear mapping

$$l \mapsto \int_0^T \int_0^s l(s,u)dX_u dX_s$$

is an injection from the real-valued Volterra kernels to $L^2(C(I_T), \mathcal{B}_T, Q_W)$. Thogether with (4.6.8), this implies (4.6.7). □

Proposition 4.20

$$k(t,s) = n_h(t,s) - \int_0^s n_h(t,v)n_h(s,v)dv \ , \ t,s \in I_T \ , \ s \le t \ . \tag{4.6.9}$$

Proof. Recall that $n_h(t,s) = 0$ for $s > t$. We define $\tilde{n}_h$ by

$$\tilde{n}_h(t,s) = \begin{cases} n_h(t,s) & \text{for} \quad s \le t \\ n_h(s,t) & \text{for} \quad s > t \end{cases} \ .$$

For all $t, s \in I_T$ with $s \le t$, we have

$$\begin{aligned}
&\int_0^t \int_0^s k(u,v)dvdu = s - \Gamma_{ts}^Y \\
= \quad & s - \mathrm{E}_P\Bigg[\bigg(W_t - \int_0^t \int_0^u n_h(u,v)dW_v du\bigg) \\
& \qquad \bigg(W_s - \int_0^s \int_0^u n_h(u,v)dW_v du\bigg)\Bigg] \\
= \quad & s - \mathrm{E}_P\Bigg[\bigg(W_t - \int_0^t \int_v^t n_h(u,v)du dW_v\bigg) \\
& \qquad \bigg(W_s - \int_0^s \int_v^s n_h(u,v)du dW_v\bigg)\Bigg] \\
= \quad & \int_0^s \int_v^t n_h(u,v)dudv + \int_0^s \int_v^s n_h(u,v)dudv \\
& - \int_0^s \left(\int_v^t n_h(u,v)du\right)\left(\int_v^s n_h(u,v)du\right)dv \\
= \quad & \int_0^t \int_0^s \tilde{n}_h(u,v)dvdu - \int_0^t \int_0^s \int_0^{u\wedge v} n_h(u,x)n_h(v,x)dxdvdu \ .
\end{aligned}$$

Hence,

$$k(t,s) = \tilde{n}_h(t,s) - \int_0^{t\wedge s} n_h(t,v)n_h(s,v)dv \quad \text{for all } t,s \in I_T \ ,$$

which implies (4.6.9). □

Remark 4.21 Since k and n_h are the negative resolvent kernels of n_k and h, respectively, it should in principle be possible to derive (4.6.9) from (4.6.7) directly and vice versa.

4.7 The Shepp-represention of mixed fractional Brownian motion

It follows from Theorem 4.2 and Theorem 4.11 b) that for all $H \in (\frac{3}{4}, 1]$ and $\alpha \in \mathbb{R} \setminus \{0\}$ there exists a real-valued, symmetric $q \in L^2([0,1]^2)$ and a $c > 0$ such that

$$\frac{dQ_{M^{H,\alpha}}}{dQ_W}(X) = c \exp\left(\int_0^1 \int_0^t q(t,s) dX_s dX_t\right) .$$

As we have seen in Remark 4.12.2, q is the unique L^2-kernel that solves the equation

$$q(t,s) + \alpha^2 H(2H-1) \int_0^1 q(t,x)\, |x-s|^{2H-2}\, dx \tag{4.7.1}$$

$$= \alpha^2 H(2H-1)\, |t-s|^{2H-2} \;,\; (t,s) \in [0,1]^2 ,$$

and

$$c = \left(\mathrm{E}\left[\exp\left(\int_0^1 \int_0^t q(t,s) dX_s dX_t\right)\right]\right)^{-1} .$$

For $H = 1$, equation (4.7.1) reduces to

$$q(t,s) + \alpha^2 \int_0^1 q(t,x) dx = \alpha^2 \;,\; (t,s) \in [0,1]^2 ,$$

which is solved by

$$q(t,s) = \frac{\alpha^2}{1+\alpha^2} .$$

It follows that

$$\frac{dQ_{M^{1,\alpha}}}{dQ_W}(X) = \frac{1}{\sqrt{1+\alpha^2}} \exp\left(\frac{\alpha^2}{1+\alpha^2} \frac{1}{2} X_1^2\right) . \tag{4.7.2}$$

To treat the case $H \in (\frac{3}{4}, 1)$ we introduce the following notation. We let

$$p = 2H - 2 \in \left(-\frac{1}{2}, 0\right), \quad \mu = \alpha^2 H(2H-1) \geq 0 ,$$

and we define the bounded linear operator $A : L^2[0,1]^2 \to L^2[0,1]^2$ by

$$Al(t,s) = \int_0^1 l(t,x) dx \;,\; l \in L^2[0,1]^2 .$$

Theorem 4.22 *Let*

$$\mu \geq 0 \quad \textit{and} \quad \rho = \min\left\{\frac{1}{2}, \frac{1+\mu}{1+3\mu+4\mu^2}\right\} .$$

Then for all $p \in (-\rho, \rho)$*, the unique* $q_p \in L^2[0,1]^2$ *that solves*

$$q_p(t,s) + \mu \int_0^1 q_p(t,x)\,|x-s|^p\,dx = \mu\,|t-s|^p \ , \ (t,s) \in [0,1]^2 \,, \quad (4.7.3)$$

is given by

$$q_p(t,s) = \mu\,|t-s|^p - \mu \sum_{n=0}^{\infty} \tilde{q}_n(t,s)\,p^n \,, \quad (4.7.4)$$

where

$$\tilde{q}_0(t,s) = \frac{\mu}{1+\mu} \quad (4.7.5)$$

and for $n \geq 1$*,*

$$\tilde{q}_n(t,s) = \mu \left(\mathrm{id} - \frac{\mu}{1+\mu} A\right) \quad (4.7.6)$$

$$\left(\int_0^1 \frac{1}{1+\mu} \frac{\ln^n|x-s|}{n!} dx + \sum_{j=1}^{n} \int_0^1 \frac{\ln^j|t-x|}{j!} \frac{\ln^{(n-j)}|x-s|}{(n-j)!} dx \right.$$

$$\left. - \sum_{j=1}^{n-1} \int_0^1 \tilde{q}_j(t,x) \frac{\ln^{(n-j)}|x-s|}{(n-j)!} dx \right) .$$

In particular, all $\tilde{q}_n$ *are continuous, bounded, symmetric and for all* $p \in [0, \rho)$*,*

$$\sum_{n=0}^{\infty} \|\tilde{q}_n\|_\infty\, p^n < \infty \,,$$

(It follows that for all $p \in (-\rho, \rho)$*,*

$$\sum_{n=0}^{\infty} \tilde{q}_n(t,s)\,p^n$$

converges uniformly in $(t,s) \in [0,1]^2$*, to a continuous, bounded and symmetric function* $\tilde{q}_p$*.)*

Proof. It follows inductively from (4.7.6) that all $\tilde{q}_n$ are continuous and bounded on $[0, 1]^2$. Next we show that for all $p \in [0, \rho)$,

$$\sum_{n=0}^{\infty} \|\tilde{q}_n\|_\infty \, p^n < \infty \,.$$

For $n \geq 1$,

$$\frac{1}{1+\mu} \frac{\ln^n |x-s|}{n!} + \sum_{j=1}^{n} \frac{\ln^j |t-x|}{j!} \frac{\ln^{(n-j)} |x-s|}{(n-j)!}$$

has for all $t, s, x \in [0, 1]$ with $x \neq s$ and $x \neq t$, the same sign and its absolute value is everywhere smaller or equal to the absolute value of

$$\sum_{j=0}^{n} \frac{\ln^j |t-x|}{j!} \frac{\ln^{(n-j)} |x-s|}{(n-j)!} = \frac{1}{n!} \left(\ln |t-x| + \ln |x-s|\right)^n \,.$$

Moreover,

$$|\ln |t-x| + \ln |x-s||^n \leq 2^{n-1} \left|\ln^n |t-x| + \ln^n |x-s|\right| \,.$$

This implies for all $(t, s) \in [0, 1]^2$,

$$\begin{aligned}
&\Bigg|\left(\mathrm{id} - \frac{\mu}{1+\mu} A\right) \\
&\int_0^1 \left(\frac{1}{1+\mu} \frac{\ln^n |x-s|}{n!} + \sum_{j=1}^{n} \frac{\ln^j |t-x|}{j!} \frac{\ln^{(n-j)} |x-s|}{(n-j)!}\right) dx\Bigg| \\
\leq\ & \int_0^1 \left|\frac{1}{1+\mu} \frac{\ln^n |x-s|}{n!} + \sum_{j=1}^{n} \frac{\ln^j |t-x|}{j!} \frac{\ln^{(n-j)} |x-s|}{(n-j)!}\right| dx \\
&\vee \int_0^1 \int_0^1 \left|\frac{1}{1+\mu} \frac{\ln^n |x-s|}{n!} + \sum_{j=1}^{n} \frac{\ln^j |t-x|}{j!} \frac{\ln^{(n-j)} |x-s|}{(n-j)!}\right| dxds \\
\leq\ & \frac{2^{n-1}}{n!} \int_0^1 \left|\ln^n |t-x| + \ln^n |x-s|\right| dx \\
&\vee \frac{2^{n-1}}{n!} \int_0^1 \int_0^1 \left|\ln^n |t-x| + \ln^n |x-s|\right| dxds \leq 2^{n+1} \,,
\end{aligned}$$

where the last inequality follows from the fact that

$$\int_0^1 \left|\ln^k |t-x|\right| dx \le 2k! \text{ for all } t \in [0,1] \text{ and all } k \in \mathbb{N}\,.$$

Furthermore,

$$\left|\left(\mathrm{id} - \frac{\mu}{1+\mu}A\right)\left(\sum_{j=1}^{n-1}\int_0^1 \tilde{q}_j(t,x)\frac{\ln^{(n-j)}|x-s|}{(n-j)!}dx\right)\right|$$

$$\le \left(1+\frac{\mu}{1+\mu}\right)2\sum_{j=1}^{n-1}\|\tilde{q}_j\|_\infty = \frac{2+4\mu}{1+\mu}\sum_{j=1}^{n-1}\|\tilde{q}_j\|_\infty\,.$$

Hence, for $\tilde{q}_n$ as in (4.7.6), we obtain for all $n \ge 1$,

$$\|\tilde{q}_n\|_\infty \le 2\mu 2^n + \frac{2\mu+4\mu^2}{1+\mu}\sum_{j=1}^{n-1}\|\tilde{q}_j\|_\infty\,.$$

Therefore, we can estimate the $\|\tilde{q}_n\|_\infty$ as follows:

$$\|\tilde{q}_n\|_\infty \le a_n\,,\ n \ge 1\,,$$

where the sequence $(a_n)_{n=1}^\infty$ is recursively defined by

$$a_1 = 4\mu \tag{4.7.7}$$

and

$$a_n = 2\mu 2^n + \left(\frac{2\mu+4\mu^2}{1+\mu}\right)\sum_{j=1}^{n-1}a_j\,,\ n \ge 2\,. \tag{4.7.8}$$

The radius of convergence of $\sum_{n=0}^\infty \|\tilde{q}_n\|_\infty p^n$ is at least as big as the radius of convergence of $\sum_{n=1}^\infty a_n z^n$. It follows from (4.7.7) and (4.7.8) that for sufficiently small z,

$$\begin{aligned}\sum_{n=1}^\infty a_n z^n &= 2\mu\sum_{n=1}^\infty (2z)^n + \frac{2\mu+4\mu^2}{1+\mu}\sum_{n=1}^\infty a_n z^n \sum_{n=1}^\infty z^n \\ &= 2\mu\sum_{n=1}^\infty (2z)^n + \frac{2\mu+4\mu^2}{1+\mu}\sum_{n=1}^\infty a_n z^n \frac{z}{1-z}\,.\end{aligned}$$

Hence,

$$\frac{1-\frac{1+3\mu+4\mu^2}{1+\mu}z}{1-z}\sum_{n=1}^{\infty}a_n z^n = 2\mu\sum_{n=1}^{\infty}(2z)^n\,,$$

and

$$\sum_{n=1}^{\infty}a_n z^n = \frac{1-z}{1-\frac{1+3\mu+4\mu^2}{1+\mu}z}2\mu\sum_{n=1}^{\infty}(2z)^n\,.$$

Since the right-hand side of the above equation is analytic in $z \in \mathbb{C}$ on an open disc with center 0 and radius

$$\rho = \min\left\{\frac{1}{2}\,,\ \frac{1+\mu}{1+3\mu+4\mu^2}\right\}\,,$$

ρ is the radius of convergence of $\sum_{n=1}^{\infty} a_n z^n$. This shows that for all $p \in [0,\rho)$,

$$\sum_{j=0}^{\infty}\|\tilde{q}_n\|_\infty\, p^n < \infty\,.$$

To show that

$$(4.7.4)\qquad q_p(t,s) = \mu\,|t-s|^p - \mu\sum_{n=0}^{\infty}\tilde{q}_n(t,s)p^n$$

solves equation (4.7.3), we note that for all $(t,s) \in [0,1]^2$ with $t \neq s$, we can write

$$|t-s|^p = \sum_{n=0}^{\infty}\frac{\ln^n|t-s|}{n!}p^n\,. \qquad (4.7.9)$$

Plugging (4.7.4) and (4.7.9) into (4.7.3) yields

$$\mu\sum_{n=0}^{\infty}\tilde{q}_n(t,s)p^n + \mu^2\int_0^1\sum_{n=0}^{\infty}\tilde{q}_n(t,x)p^n\sum_{n=0}^{\infty}\frac{\ln^n|x-s|}{n!}p^n dx \qquad (4.7.10)$$

$$= \mu^2\int_0^1\sum_{n=0}^{\infty}\frac{\ln^n|t-x|}{n!}p^n\sum_{n=0}^{\infty}\frac{\ln^n|x-s|}{n!}p^n dx\,,\ (t,s)\in[0,1]^2\,.$$

Since for all $p \in [0,\rho)$ and all $(t,x) \in [0,1]^2$,

$$\sum_{n=0}^{\infty}|\tilde{q}_n(t,x)|\,p^n \le \sum_{n=0}^{\infty}\|\tilde{q}_n\|_\infty\,p^n < \infty\,,$$

and for all $p \geq 0$ and all $(x,s) \in [0,1]^2$ with $x \neq s$,

$$\sum_{n=0}^{\infty} \left| \frac{\ln^n |x-s|}{n!} \right| p^n = \sum_{n=0}^{\infty} \frac{\ln^n |x-s|}{n!} (-p)^n = |x-s|^{-p} \ ,$$

we obtain for all $p \in (-\rho, \rho)$ and all $t, x, s \in [0,1]$ with $x \neq s$,

$$\sum_{n=0}^{\infty} \tilde{q}_n(t,x) p^n \sum_{n=0}^{\infty} \frac{\ln^n |x-s|}{n!} p^n$$

$$= \sum_{n=0}^{\infty} p^n \sum_{j=0}^{n} \tilde{q}_j(t,x) \frac{\ln^{(n-j)} |x-s|}{(n-j)!}$$

and

$$\begin{aligned} & \sum_{n=0}^{\infty} \left| p^n \sum_{j=0}^{n} \tilde{q}_j(t,x) \frac{\ln^{(n-j)} |x-s|}{(n-j)!} \right| \\ \leq \quad & \sum_{n=0}^{\infty} |p|^n \sum_{j=0}^{n} \left| \tilde{q}_j(t,x) \right| \left| \frac{\ln^{(n-j)} |x-s|}{(n-j)!} \right| \\ = \quad & \sum_{n=0}^{\infty} |\tilde{q}_n(t,x)| \, |p|^n \sum_{n=0}^{\infty} \left| \frac{\ln^n |x-s|}{n!} \right| |p|^n \\ \leq \quad & \left(\sum_{n=0}^{\infty} \|\tilde{q}_n\|_\infty |p|^n \right) |x-s|^{-|p|} \ . \end{aligned}$$

Hence, it follows from Lebesgue's Dominated Convergence Theorem that for all $(t,s) \in [0,1]^2$,

$$\int_0^1 \sum_{n=0}^{\infty} \tilde{q}_n(t,x) p^n \sum_{n=0}^{\infty} \frac{\ln^n |x-s|}{n!} p^n dx$$

$$= \int_0^1 \sum_{n=0}^{\infty} p^n \sum_{j=0}^{n} \tilde{q}_j(t,x) \frac{\ln^{(n-j)} |x-s|}{(n-j)!} dx$$

$$= \sum_{n=0}^{\infty} p^n \sum_{j=0}^{n} \int_0^1 \tilde{q}_j(t,x) \frac{\ln^{(n-j)} |x-s|}{(n-j)!} dx \ .$$

Similarly, it can be shown that for all $(t,s) \in [0,1]^2$,

$$\int_0^1 \sum_{n=0}^{\infty} \frac{\ln^n |t-x|}{n!} p^n \sum_{n=0}^{\infty} \frac{\ln^n |x-s|}{n!} p^n dx$$

$$= \sum_{n=0}^{\infty} p^n \sum_{j=0}^{n} \int_0^1 \frac{\ln^j |t-x|}{j!} \frac{\ln^{(n-j)} |x-s|}{(n-j)!} dx \,.$$

Hence, (4.7.10) becomes

$$\sum_{n=0}^{\infty} p^n \left[\mu \tilde{q}_n(t,s) + \mu^2 \left(\sum_{j=0}^{n} \int_0^1 \tilde{q}_j(t,x) \frac{\ln^{(n-j)} |x-s|}{(n-j)!} dx \right) \right] \tag{4.7.11}$$

$$= \sum_{n=0}^{\infty} p^n \mu^2 \left(\sum_{j=0}^{n} \int_0^1 \frac{\ln^j |t-x|}{j!} \frac{\ln^{(n-j)} |x-s|}{(n-j)!} dx \right) \,,\ (t,s) \in [0,1]^2 \,.$$

which is satisfied if, for every $n \in \mathbb{N}_0$, the following linear equation is fulfilled:

$$\tilde{q}_n(t,s) + \mu \sum_{j=0}^{n} \int_0^1 \tilde{q}_j(t,x) \frac{\ln^{(n-j)} |x-s|}{(n-j)!} dx \tag{4.7.12}$$

$$= \mu \sum_{j=0}^{n} \int_0^1 \frac{\ln^j |t-x|}{j!} \frac{\ln^{(n-j)} |x-s|}{(n-j)!} dx \,,\ (t,s) \in [0,1]^2 \,.$$

For $n = 0$, (4.7.12) reduces to

$$\tilde{q}_0(t,s) + \mu \int_0^1 \tilde{q}_0(t,x) dx = \mu \,,\ (t,s) \in [0,1]^2 \,,$$

which has the solution

$$\text{(4.7.5)} \quad \tilde{q}_0(t,s) = \frac{\mu}{1+\mu} \,,\ (t,s) \in [0,1]^2 \,.$$

For $n \geq 1$, we plug (4.7.5) into (4.7.12) and obtain

$$(\mathrm{id} + \mu A)\, \tilde{q}_n(t,s) \tag{4.7.13}$$

$$= \mu \left[\left(\int_0^1 \frac{1}{1+\mu} \frac{\ln^n |x-s|}{n!} dx + \sum_{j=1}^{n} \int_0^1 \frac{\ln^j |t-x|}{j!} \frac{\ln^{(n-j)} |x-s|}{(n-j)!} dx \right) \right.$$

$$- \left(\sum_{j=1}^{n-1} \int_0^1 \tilde{q}_j(t,x) \frac{\ln^{(n-j)} |x-s|}{(n-j)!} dx \right) \Bigg] \ , \ (t,s) \in [0,1]^2 \ .$$

That this equation is solved for $\tilde{q}_n$ given by (4.7.6) follows from the fact that

$$\left(\mathrm{id} - \frac{\mu}{1+\mu} A \right) = (\mathrm{id} + \mu A)^{-1} \ .$$

It remains to be shown that all $\tilde{q}_n$ are symmetric. Reasoning as above, one can show that for all $p \in (-\rho, \rho)$,

$$g_p(t,s) + \mu \int_0^1 |t-x|^p \, g_p(x,s) dx = \mu \, |t-s|^p \ , \ (t,s) \in [0,1]^2 \quad (4.7.14)$$

has the unique solution

$$g_p(t,s) = \mu \, |t-s|^p - \mu \sum_{n=0}^{\infty} \tilde{g}_n(t,s) p^n \ , \ (t,s) \in [0,1]^2 \ ,$$

where

$$\tilde{g}_0(t,s) = \frac{\mu}{1+\mu}$$

and for $n \geq 1$,

$$\tilde{g}_n(t,s) = \mu \left(\mathrm{id} - \frac{\mu}{1+\mu} \tilde{A} \right) \quad (4.7.15)$$

$$\Bigg[\left(\int_0^1 \frac{\ln^n |t-x|}{n!} \frac{1}{1+\mu} dx + \sum_{j=1}^{n} \int_0^1 \frac{\ln^{(n-j)} |t-x|}{(n-j)!} \frac{\ln^j |x-s|}{j!} dx \right)$$

$$- \left(\sum_{j=1}^{n-1} \int_0^1 \frac{\ln^{(n-j)} |t-x|}{(n-j)!} \tilde{g}_j(x,s) dx \right) \Bigg] \ ,$$

where

$$\tilde{A} l(t,s) = \int_0^1 l(x,s) dx \quad \text{for} \quad l \in L^2[0,1]^2 \ .$$

A comparison of (4.7.15) with (4.7.6) shows that for all $n \in \mathbb{N}_0$,

$$\tilde{g}_n(t,s) = \tilde{q}_n(s,t) \ , \ (t,s) \in [0,1]^2 \ .$$

But since for all $p \in (-\rho, \rho)$, q_p also solves (4.7.14), we have $q_p = g_p$ for all $p \in (\rho, \rho)$. Hence, $\tilde{q}_n = \tilde{g}_n$ for all $n \in \mathbb{N}_0$, i.e. all $\tilde{q}_n$ are symmetric. □

Remarks 4.23
1. Let $f_p(t,s) = |t-s|^p$, $t, s \in [0,t]$, $p \in \mathbb{R}$. For all $(\mu, p) \in \mathbb{R}^2$, such that

$$\mu \geq 0 \quad \text{and} \quad p \in (-\rho, \rho) \ , \tag{4.7.16}$$

where

$$\rho = \min\left\{\frac{1}{2}\ ,\ \frac{1+\mu}{1+3\mu+4\mu^2}\right\} \ ,$$

the L^2-kernel q_p from Theorem 4.22 solves

$$\mathrm{id} - q_p^{\mathrm{op}} = \left(\mathrm{id} + \mu f_p^{\mathrm{op}}\right)^{-1} \ .$$

Alternatively,

$$\left(\mathrm{id} + \mu f_p^{\mathrm{op}}\right)^{-1}$$

can be expressed as the Neumann series

$$\sum_{n=0}^{\infty} (-\mu f_p^{\mathrm{op}})^n \tag{4.7.17}$$

whenever

$$\left\|\mu f_p^{\mathrm{op}}\right\| < 1 \ . \tag{4.7.18}$$

The operator norm of f_p^{op} can be estimated as follows:

$$\| f_p^{\mathrm{op}} \| \leq \left(\int_0^1 \int_0^1 |t-s|^{2p} \, ds dt \right)^{\frac{1}{2}} = [(2p+1)(p+1)]^{-\frac{1}{2}} \ .$$

Hence, (4.7.18) holds if

$$\mu^2 < (2p+1)(p+1) \ . \tag{4.7.19}$$

It can be seen in Figure 4.1 that for $(\mu, p) \in \mathbb{R}_+ \times (-\frac{1}{2}, 0]$, condition (4.7.19) is more restrictive than condition (4.7.16).
2. For $H = 1$, we can use the formula

$$\text{(4.7.2)} \qquad \frac{dQ_{M^{1,\alpha}}}{dQ_W}(X) = \frac{1}{\sqrt{1+\alpha^2}} \exp\left(\frac{\alpha^2}{1+\alpha^2} \frac{1}{2} X_1^2 \right)$$

to compute for all $d > 0$,

$$P\left[M_t^{1,\alpha} < d \ , \ t \in [0,1]\right] = \Phi\left(\frac{d}{\sqrt{1+\alpha^2}}\right) - \mathrm{e}^{2d^2\alpha^2}\, \Phi\left(-d\frac{1+2\alpha^2}{\sqrt{1+\alpha^2}}\right) \ , \tag{4.7.20}$$

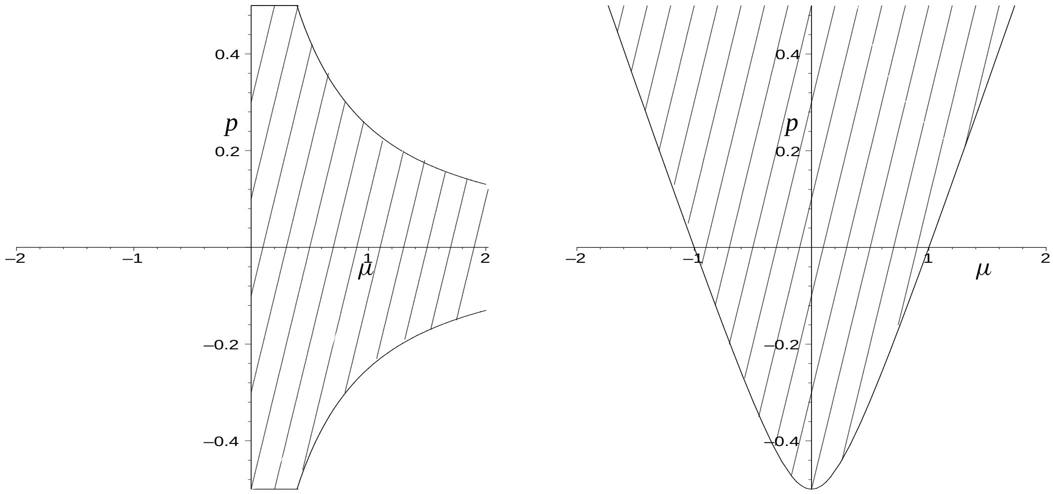

Figure 4.1: *Left: All* $(\mu, p) \in \mathbb{R} \times [-\frac{1}{2}, \infty)$ *that satisfy (4.7.16). Right: All* $(\mu, p) \in \mathbb{R} \times [-\frac{1}{2}, \infty)$ *that satisfy (4.7.19).*

where Φ is the standard normal distribution function.
To prove (4.7.20) we let

$$D = \{X \;:\; X_t < d\,,\; t \in [0,1]\}$$

and note that

$$P\left[M_t^{1,\alpha} < d\,,\; t \in [0,1]\right] = Q_{M^{1,\alpha}}[D] = \mathrm{E}_{Q_W}\left[1_D(X)\frac{dQ_{M^{1,\alpha}}}{dQ_W}(X)\right]$$

$$= \frac{1}{\sqrt{1+\alpha^2}}\mathrm{E}_{Q_W}\left[1_D(X)\exp\left(\frac{\alpha^2}{1+\alpha^2}\frac{1}{2}X_1^2\right)\right].$$

The reflection principle gives that for all Borel sets $B \subset \mathbb{R}$,

$$Q_W[D \cap \{X_1 \in B\}] = \int_{\mathbb{R}} 1_B(x) f(x) dx\,,$$

where

$$f(x) = 1_{\{x \le d\}}[\varphi(x) - \varphi(2d - x)] \quad \text{and} \quad \varphi(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{x^2}{2}}\,.$$

It follows that

$$\frac{1}{\sqrt{1+\alpha^2}}\mathrm{E}_{Q_W}\left[1_D(X)\exp\left(\frac{\alpha^2}{1+\alpha^2}\frac{1}{2}X_1^2\right)\right]$$

$$= \frac{1}{\sqrt{1+\alpha^2}} \int_{-\infty}^{d} \exp\left(\frac{\alpha^2}{1+\alpha^2}\frac{1}{2}x^2\right) f(x)dx \, .$$

Now (4.7.20) follows from a simple calculation.
For $H \in (\frac{3}{4}, 1)$, we are not able to deduce an explicit formula for

$$P\left[M_t^{H,\alpha} < d \, , \; t \in [0,1]\right]$$

because

$$\frac{dQ_{M^{H,\alpha}}}{dQ_W}(X)$$

depends not only on X_1 but on the whole path of X.

4.8 The Hitsuda-representation of mixed fractional Brownian motion

It follows from Theorem 4.2 and Theorem 4.15 a) that for all $H \in (\frac{3}{4}, 1]$ and $\alpha \in \mathbb{R} \setminus \{0\}$ there exists a real-valued Volterra kernel $r \in L^2([0,1]^2)$ and a Brownian motion $(W_t)_{0 \le t \le 1}$ such that

$$M_t^{H,\alpha} = W_t + \int_0^t \int_0^s r(s,u)dW_u ds \, , \; t \in [0,1] \, .$$

It follows from Proposition 4.20 that r is the unique Volterra kernel that solves the equation

$$\alpha^2 H(2H-1)(t-s)^{2H-2} = r(t,s) + \int_0^s r(t,x)r(s,x)dx \, , \; (t,s) \in \triangle \, , \tag{4.8.1}$$

where we set $\triangle = \left\{(t,s) \in [0,1]^2 \; : \; s < t\right\}$. For $H = 1$, (4.8.1) reduces to

$$\alpha^2 = r(t,s) + \int_0^s r(t,x)r(s,x)dx \, , \; (t,s) \in \triangle \, ,$$

and is easy to solve. Its solution is given by

$$r(t,s) = \frac{\alpha^2}{1+\alpha^2 s} \, , \; (t,s) \in \triangle \, .$$

Hence, for all $t \in [0,1]$,

$$M_t^{1,\alpha} = W_t + \int_0^t \int_0^s \frac{\alpha^2}{1+\alpha^2 u} dW_u ds = W_t + \int_0^t \int_u^t \frac{\alpha^2}{1+\alpha^2 u} ds dW_u$$

$$= \int_0^t \left(1 + (t-u)\frac{\alpha^2}{1+\alpha^2 u}\right) dW_u = \left(1+\alpha^2 t\right)\int_0^t \frac{dW_u}{1+\alpha^2 u}\,.$$

As in Section 4.7, it is much harder to solve (4.8.1) for $H \in (\frac{3}{4}, 1)$. We set

$$p = 2H - 2 \in \left(-\frac{1}{2}, 0\right) \quad \text{and} \quad \mu = \alpha^2 H(2H-1) \geq 0\,.$$

Further, for $\mu \geq 0$, we define the bounded linear operator

$$A_\mu : L^2(\triangle) \to L^2(\triangle)$$

by

$$A_\mu l(t,s) = \frac{\mu}{1+\mu s}\int_0^s [l(t,x) + l(s,x)]\,dx$$
$$-\frac{2\mu^2}{(1+\mu s)^2}\int_0^s \int_0^x l(x,y)dydx\,,\ l \in L^2(\triangle)\,.$$

Theorem 4.24 *Let*

$$\mu \geq 0 \quad \textit{and} \quad \rho = \frac{1}{5.08 + 26.88\mu + 13.952\mu^2}\,. \tag{4.8.2}$$

Then for all $p \in (-\rho, \rho)$, the equation

$$\mu(t-s)^p = r_p(t,s) + \int_0^s r_p(t,x) r_p(s,x)dx\,,\ (t,s) \in \triangle\,, \tag{4.8.3}$$

is solved by the $L^2(\triangle)$-function

$$r_p(t,s) = \mu(t-s)^p - \mu\sum_{n=0}^{\infty}\tilde{r}_n(t,s)p^n\,,\ (t,s) \in \triangle\,, \tag{4.8.4}$$

where

$$\tilde{r}_0(t,s) = \frac{\mu s}{1+\mu s}\,, \tag{4.8.5}$$

and for $n \geq 1$,

$$\tilde{r}_n(t,s) = A_\mu \frac{\ln^n(t-s)}{n!} + \mu\left(\mathrm{id} - A_\mu\right) \tag{4.8.6}$$

$$\left[\sum_{j=1}^{n-1}\int_0^s \left(\frac{\ln^j(t-x)}{j!} - \tilde{r}_j(t,x)\right)\left(\frac{\ln^{(n-j)}(s-x)}{(n-j)!} - \tilde{r}_{n-j}(s,x)\right)dx\right]\,.$$

In particular, all $\tilde{r}_n$ are continuous and bounded on $\triangle$ and for all $p \in [0, \rho)$,

$$\sum_{n=0}^{\infty} \|\tilde{r}_n\|_\infty \, p^n < \infty \, .$$

(It follows that

$$\sum_{n=0}^{\infty} \tilde{r}_n(t,s) p^n$$

converges for all $p \in (-\rho, \rho)$ uniformly in $(t, s) \in \triangle$ to a bounded and continuous function $\tilde{r}_p$.)

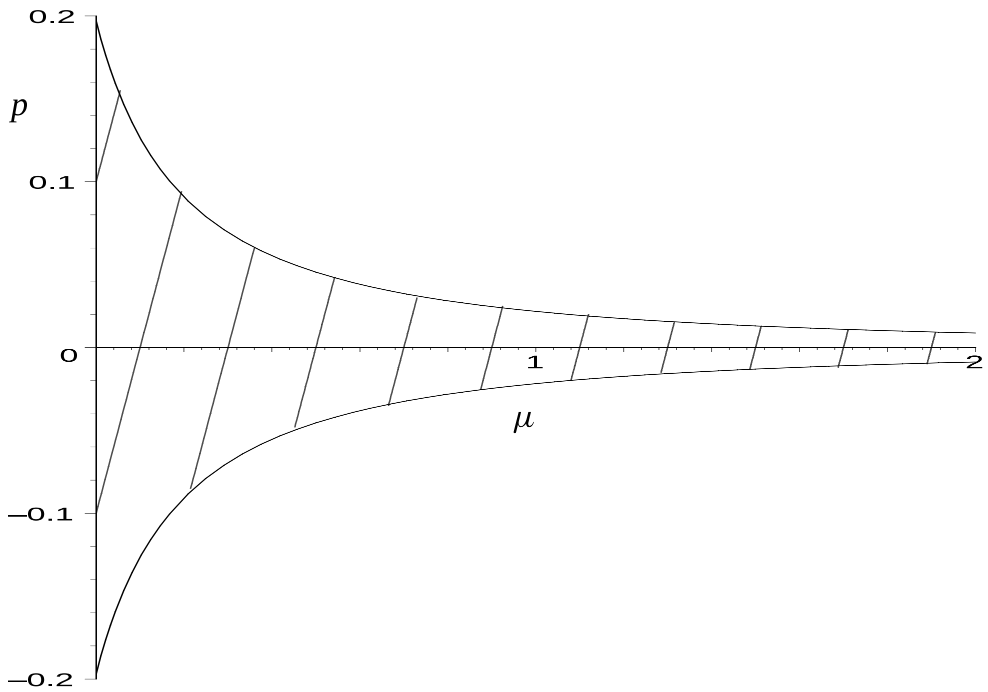

Figure 4.2: *All $(\mu, p) \in \mathbb{R}_+ \times [-\frac{1}{2}, \infty)$ such that $p \in (-\rho, \rho)$, where ρ is given by (4.8.2).*

Proof. It follows inductively from (4.8.6) that all $\tilde{r}_n$ are continuous and bounded on $\triangle$. In order to prove that for all $p \in [0, \rho)$,

$$\sum_{n=0}^{\infty} \|\tilde{r}_n\|_\infty \, p^n < \infty \, ,$$

we construct recursively for all $n \geq 1$, a polynomial

$$a_n(s) = \sum_{j=0}^{n-1} a_{n,j} s^j \tag{4.8.7}$$

with non-negative coefficients $a_{n,0}, \ldots, a_{n,n-1}$ such that for all $(t, s) \in \triangle$,

$$|\tilde{r}_n(t, s)| \leq a_n(s) \,.$$

Let $n \geq 1$. Since

$$\int_0^s \left[\frac{\ln^n(t-x)}{n!} + \frac{\ln^n(s-x)}{n!} \right] dx$$

and

$$\int_0^s \int_0^x \frac{\ln^n(x-y)}{n!} dy dx$$

have for all $(t, s) \in \triangle$ the same sign, we have for all $(t, s) \in \triangle$,

$$\left| A_\mu \frac{\ln^n(t-s)}{n!} \right| \tag{4.8.8}$$

$$\leq \frac{\mu}{1+\mu s} \left| \int_0^s \left[\frac{\ln^n(t-x)}{n!} + \frac{\ln^n(s-x)}{n!} \right] dx \right|$$
$$\vee \frac{2\mu^2}{(1+\mu s)^2} \left| \int_0^s \int_0^x \frac{\ln^n(x-y)}{n!} dy dx \right| \leq 2\mu \,,$$

where the second inequality follows from

$$\begin{aligned} & \frac{\mu}{1+\mu s} \left| \int_0^s \left[\frac{\ln^n(t-x)}{n!} + \frac{\ln^n(s-x)}{n!} \right] dx \right| \\ = \; & \frac{\mu}{1+\mu s} \left| \int_{t-s}^t \frac{\ln^n(x)}{n!} dx + \int_0^s \frac{\ln^n(x)}{n!} dx \right| \\ \leq \; & \mu 2 \left| \int_0^t \frac{\ln^n(x)}{n!} dx \right| \leq 2\mu \,, \end{aligned}$$

and

$$\frac{2\mu^2}{(1+\mu s)^2} \left| \int_0^s \int_0^x \frac{\ln^n(x-y)}{n!} dy dx \right| \leq \frac{2\mu^2}{(1+\mu s)^2} \left| \int_0^s dx \right|$$

$$= \frac{2\mu^2 s}{(1+\mu s)^2} \leq \max_{s\geq 0} \frac{2\mu^2 s}{(1+\mu s)^2} = \frac{\mu}{2}\,.$$

In particular,

$$|\tilde{r}_1(t,s)| \leq 2\mu \quad \text{for all} \quad (t,s) \in \triangle\,.$$

Hence, we can set

$$a_1 = 2\mu\,. \tag{4.8.9}$$

Let us now suppose that for $n \geq 2$, there exist polynomials $a_1, \ldots, a_{n-1}$ of the form (4.8.7) such that for all $(t,s) \in \triangle$,

$$|\tilde{r}_m(t,s)| \leq a_m(s) = \sum_{j=0}^{m-1} a_{m,j} s^j\ ,\ m = 1, \ldots, n-1\,.$$

To obtain an estimate on $|\tilde{r}_n(t,s)|$, we write

$$\sum_{j=1}^{n-1} \int_0^s \left(\frac{\ln^j(t-x)}{j!} - \tilde{r}_j(t,x) \right) \left(\frac{\ln^{(n-j)}(s-x)}{(n-j)!} - \tilde{r}_{(n-j)}(s,x) \right) dx$$

as

$$\sum_{j=1}^{n-1} \int_0^s \frac{\ln^j(t-x)}{j!} \frac{\ln^{(n-j)}(s-x)}{(n-j)!} dx \tag{4.8.10}$$

$$-\sum_{j=1}^{n-1} \int_0^s \left(\tilde{r}_j(t,x) \frac{\ln^{(n-j)}(s-x)}{(n-j)!} + \tilde{r}_j(s,x) \frac{\ln^{(n-j)}(t-x)}{(n-j)!} \right) dx$$

$$+\sum_{j=1}^{n-1} \int_0^s \tilde{r}_j(t,x) \tilde{r}_{(n-j)}(s,x) dx\,.$$

and estimate the three terms separately. For all $(t,s) \in \triangle$,

$$\begin{aligned} & \left| \sum_{j=1}^{n-1} \int_0^s \frac{\ln^j(t-x)}{j!} \frac{\ln^{(n-j)}(s-x)}{(n-j)!} dx \right| \\ \leq\ & \left| \sum_{j=0}^{n} \int_0^s \frac{\ln^j(t-x)}{j!} \frac{\ln^{(n-j)}(s-x)}{(n-j)!} dx \right| \\ =\ & \frac{1}{n!} \left| \int_0^s [\ln(t-x) + \ln(s-x)]^n \, dx \right| \\ \leq\ & \frac{1}{n!} \int_0^s |2\ln(s-x)|^n \, dx \leq 2^n\,. \end{aligned}$$

Furthermore,

$$\left|\sum_{j=1}^{n-1}\int_0^s \left(\tilde{r}_j(t,x)\frac{\ln^{(n-j)}(s-x)}{(n-j)!}+\tilde{r}_j(s,x)\frac{\ln^{(n-j)}(t-x)}{(n-j)!}\right)dx\right|$$

$$\leq \sum_{j=1}^{n-1}\int_0^s a_j(x)\left|\frac{\ln^{(n-j)}(t-x)}{(n-j)!}+\frac{\ln^{(n-j)}(s-x)}{(n-j)!}\right|dx$$

$$\leq 2\sum_{j=1}^{n-1}\int_0^s a_j(x)\left|\frac{\ln^{(n-j)}(s-x)}{(n-j)!}\right|dx \leq 2\sum_{j=1}^{n-1}a_j(s)\,,$$

where the last inequality follows from the fact that for all $m, l \in \mathbb{N}_0$,

$$\int_0^s x^m\left|\frac{\ln^l(s-x)}{l!}\right|dx \leq s^m\int_0^s\left|\frac{\ln^l(s-x)}{l!}\right|dx \leq s^m\,.$$

The last term of (4.8.10) can be estimated as follows:

$$\left|\sum_{j=1}^{n-1}\int_0^s \tilde{r}_j(t,x)\tilde{r}_{(n-j)}(s,x)dx\right|$$

$$\leq \sum_{j=1}^{n-1}\int_0^s a_j(x)a_{n-j}(x)dx\,.$$

Combining the three preceding estimates we obtain

$$\left|\sum_{j=1}^{n-1}\int_0^s\left(\frac{\ln^j(t-x)}{j!}-\tilde{r}_j(t,x)\right)\left(\frac{\ln^{(n-j)}(s-x)}{(n-j)!}-\tilde{r}_{(n-j)}(s,x)\right)dx\right|$$

$$\leq 2^n+2\sum_{j=1}^{n-1}a_j(s)+\sum_{j=1}^{n-1}\int_0^s a_j(x)a_{n-j}(x)dx,\ ,(t,s)\in\triangle\,. \qquad (4.8.11)$$

Next observe that for $l \in L^2(\triangle)$, such that there exists an $m \in \mathbb{N}_0$ with

$$l(t,s) \leq s^m \quad \text{for all} \quad (t,s)\in\triangle\,,$$

we have

$$\left|\left(\mathrm{id}-A_\mu\right)l(t,s)\right|$$

$$\leq s^m + \frac{\mu}{1+\mu s}\int_0^s 2x^m dx + \frac{2\mu^2}{(1+\mu s)^2}\int_0^s\int_0^x y^m dydx$$
$$\leq s^m + \frac{\mu}{1+\mu s}2\frac{s^{m+1}}{m+1} + \frac{2\mu^2}{(1+\mu s)^2}\frac{s^{m+2}}{(m+1)(m+2)}$$
$$\leq s^m + 2s^m + s^m = 4s^m\,.$$

This together with (4.8.6), (4.8.8) and (4.8.11) implies

$$|\tilde{r}_n(t,s)| \leq 2\mu + 4\mu\left(2^n + 2\sum_{j=1}^{n-1} a_j(s) + \sum_{j=1}^{n-1}\int_0^s a_j(x)a_{(n-j)}(x)dx\right)\,.$$

Therefore, for $n \geq 2$ and $s \in [0,1]$, we can set

$$a_n(s) = \mu\left(2 + 2^{n+2} + 8\sum_{j=1}^{n-1} a_j(s) + 4\sum_{j=1}^{n-1}\int_0^s a_j(x)a_{(n-j)}(x)dx\right)\,. \tag{4.8.12}$$

Since

$$|\tilde{r}_n(t,s)| \leq a_n(1) \quad \text{for all } n \geq 1 \text{ and all } (t,s) \in \triangle\,,$$

the radius of convergence of

$$\sum_{n=0}^{\infty} \|\tilde{r}\|_\infty\, p^n$$

is at least as big as the radius of convergence of

$$\sum_{n=1}^{\infty} a_n(1)z^n\,.$$

To obtain an estimate on the radius of convergence of

$$\sum_{n=1}^{\infty} a_n(1)z^n\,,$$

we set for $s \in [0,1]$ and $z \in \mathbb{C}$ sufficiently small

$$a(z,s) = \sum_{n=1}^{\infty} a_n(s)z^n\,.$$

It follows from (4.8.9) and (4.8.12) that

$$a(z,s) = 2\mu\left[\sum_{n=1}^{\infty} z^n + 2\sum_{n=2}^{\infty}(2z)^n + 4a(z,s)\sum_{n=1}^{\infty} z^n + 2\int_0^s a^2(z,x)dx\right],$$

which for small enough z can be written as

$$4\mu\int_0^s a^2(z,x)dx + \frac{(8\mu+1)z-1}{1-z}a(z,s) = 2\mu z\frac{8z^2-6z-1}{(1-z)(1-2z)}. \tag{4.8.13}$$

The unique solution of (4.8.13) is given by

$$a(z,s) = \frac{2\mu z(8\mu z+z-1)(8z^2-6z-1)}{8\mu^2 z(1-z)(8z^2-6z-1)s + (8\mu z+z-1)^2(1-2z)}.$$

In particular,

$$a(z,1) = \frac{2\mu z(8\mu z+z-1)(8z^2-6z-1)}{8\mu^2 z(1-z)(8z^2-6z-1) + (8\mu z+z-1)^2(1-2z)}.$$

This shows that the radius of convergence of

$$\sum_{n=1}^{\infty} a_n(1)z^n = a(z,1)$$

is the radius r of the largest open disc $D_r(0) \subset \mathbb{C}$ with center 0 on which the denominator

$$\begin{aligned}
1 - (4+16\mu+8\mu^2)z + (5+48\mu+24\mu^2)z^2 - (2+32\mu+16\mu^2)z^3 - 64\mu^2 z^4 \\
= 8\mu^2 z(1-z)(8z^2-6z-1) + (8\mu z+z-1)^2(1-2z)
\end{aligned} \tag{4.8.14}$$

of the rational function $a(z,1)$ does not vanish. For $|z| \le \frac{1}{5}$, one can estimate the terms of expression (4.8.14) that depend on z as follows:

$$\begin{aligned}
& \Big| -(4+16\mu+8\mu^2)z + (5+48\mu+24\mu^2)z^2 \\
& -(2+32\mu+16\mu^2)z^3 - 64\mu^2 z^4 \Big| \\
\le\ & |z|\left[4+16\mu+8\mu^2 + \frac{1}{5}\left(5+48\mu+24\mu^2\right)\right. \\
& \left. +\frac{1}{5^2}\left(2+32\mu+16\mu^2\right) + \frac{1}{5^3}64\mu^2\right] \\
=\ & |z|\left(5.08 + 26.88\mu + 13.952\mu^2\right).
\end{aligned}$$

This shows that for

$$|z| < \rho = \frac{1}{5.08 + 26.88\mu + 13.952\mu^2}$$

the denominator (4.8.14) does not vanish. It follows that

$$\sum_{n=0}^{\infty} \|\tilde{r}_n\|_\infty \, p^n < \infty \, ,$$

for all $p \in (-\rho, \rho)$. To show that

$$(4.8.4) \quad r_p(t,s) = \mu(t-s)^p - \mu \sum_{n=0}^{\infty} \tilde{r}_n(t,s) p^n \, , \; (t,s) \in \triangle \, ,$$

is a solution of (4.8.3), we write it as

$$r_p(t,s) = \mu \sum_{n=0}^{\infty} \left(\frac{\ln^n(t-s)}{n!} - \tilde{r}_n(t,s) \right) p^n$$

and plug it into (4.8.3). This yields the equation

$$\mu \sum_{n=0}^{\infty} \tilde{r}_n(t,s) p^n = \tag{4.8.15}$$

$$\mu^2 \int_0^s \sum_{n=0}^{\infty} \left(\frac{\ln^n(t-x)}{n!} - \tilde{r}_n(t,x) \right) p^n \sum_{n=0}^{\infty} \left(\frac{\ln^n(s-x)}{n!} - \tilde{r}_n(s,x) \right) p^n dx \, ,$$

$$(t,s) \in \triangle \, .$$

As in the proof of Theorem 4.22, one can show that for all $(t,s) \in \triangle$,

$$\int_0^s \sum_{n=0}^{\infty} \left(\frac{\ln^n(t-x)}{n!} - \tilde{r}_n(t,x) \right) p^n \sum_{n=0}^{\infty} \left(\frac{\ln^n(s-x)}{n!} - \tilde{r}_n(s,x) \right) p^n dx =$$

$$\sum_{n=0}^{\infty} p^n \sum_{j=0}^{n} \int_0^s \left(\frac{\ln^j(t-x)}{j!} - \tilde{r}_j(t,x) \right) \left(\frac{\ln^{(n-j)}(s-x)}{(n-j)!} - \tilde{r}_{(n-j)}(s,x) \right) dx \, .$$

Hence, (4.8.15) becomes

$$\mu \sum_{n=0}^{\infty} \tilde{r}_n(t,s) p^n = \tag{4.8.16}$$

$$\mu^2 \sum_{n=0}^{\infty} p^n \sum_{j=0}^{n} \int_0^s \left(\frac{\ln^j(t-x)}{j!} - \tilde{r}_j(t,x) \right) \left(\frac{\ln^{(n-j)}(s-x)}{(n-j)!} - \tilde{r}_{(n-j)}(s,x) \right) dx \,,$$

$$(t,s) \in \triangle \,,$$

which is satisfied if, for all $n \geq 0$, the equation

$$\tilde{r}_n(t,s) = \tag{4.8.17}$$

$$\mu \sum_{j=0}^{n} \int_0^s \left(\frac{\ln^j(t-x)}{j!} - \tilde{r}_j(t,x) \right) \left(\frac{\ln^{(n-j)}(s-x)}{(n-j)!} - \tilde{r}_{(n-j)}(s,x) \right) dx \,,$$

$$(t,s) \in \triangle \,,$$

holds. For $n = 0$, (4.8.17) reduces to

$$\tilde{r}_0(t,s) = \mu \int_0^s (1 - \tilde{r}_0(t,x))\,(1 - \tilde{r}_0(s,x))\, dx \,,\ (t,s) \in \triangle \,,$$

and is solved by

$$\text{(4.8.5)} \quad \tilde{r}_0(t,s) = \frac{\mu s}{1 + \mu s} \,, (t,s) \in \triangle \,.$$

For $n \geq 1$, we plug (4.8.5) into (4.8.17). Then (4.8.17) becomes

$$(\mathrm{id} + B_\mu)\tilde{r}_n(t,s) = B_\mu \frac{\ln^n(t-s)}{n!} \tag{4.8.18}$$

$$+\mu \sum_{j=1}^{(n-1)} \int_0^s \left(\frac{\ln^j(t-x)}{j!} - \tilde{r}_j(t,x) \right) \left(\frac{\ln^{(n-j)}(s-x)}{(n-j)!} - \tilde{r}_{(n-j)}(s,x) \right) dx \,,$$

$$(t,s) \in \triangle \,,$$

where the bounded linear operator

$$B_\mu : L^2(\triangle) \to L^2(\triangle)$$

is given by

$$B_\mu l(t,s) = \int_0^s \frac{\mu}{1 + \mu x} \left[l(t,x) + l(s,x) \right] dx \,,\ l \in L^2(\triangle) \,.$$

It can be checked by performing integration by parts several times that

$$B_\mu A_\mu = B_\mu - A_\mu \quad \text{on } L^2(\triangle) \,.$$

This implies

$$(\mathrm{id} + B_\mu) A_\mu = B_\mu \quad \text{and} \quad (\mathrm{id} + B_\mu)(\mathrm{id} - A_\mu) = \mathrm{id}$$

and hence shows that the $\tilde{r}_n$ given in (4.8.6) solve (4.8.18). □

Bibliography

Bachelier, L. (1900). Théorie de la spéculation. *Ann. Sci. Ecole Norm. Sup.* 17, 21-86.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *J. Polit. Econom.* 81, 637-659.

Brigo, D. and Mercurio, F. (2000). Option pricing impact of alternative continuous-time dynamics for discretely-observed stock prices. *Finance and Stochastics* 4, 147-159.

Cutland, N.J., Kopp P.E. and Willinger, W. (1995). Stock price returns and the Joseph effect: a fractional version of the Black-Scholes model. *Progress in Probability* 36, 327-351.

Delbaen, F. and Schachermayer, W. (1994). A general version of the fundamental theorem of asset pricing. *Math. Ann.* 300, 463-520.

Dellacherie, C. and Meyer, P.A. (1980). *Probabilités et Potentiel. Chap. V - VIII.* Hermann.

Doléans-Dade, C. and Meyer, P.A. (1970). Intégrales stochastiques par rapport aux martingales locales. *Séminaire Proba. IV. Lecture Notes in Mathematics* 124, 77-107.

Dunford, N. and Schwartz, J.T. (1963). *Linear Operators Part II.* Intersience Publishers, New York.

Girsanov, I.V. (1960). On transforming a certain class of stochastic processes by absolutely continuous substitution of measures. *Theory of Probability and its Applications* 5, 285-301.

Golub, G.H. and Van Loan, C.F. (1989). *Matrix Computations.* The Johns Hopkins University Press.

Hida, T. and Hitsuda, M. (1976). *Gaussian Processes.* American Mathematical Society: Translations of Mathematical Monographs.

Hitsuda, M. (1968). Representation of Gaussian processes equivalent to Wiener process. *Osaka Journal of Mathematics* 5, 299-312.

Itô, K. (1944). Stochastic integral. *Proc. Imp. Acad. Tokyo* 20, 519-524.

Karatzas, I. and Shreve, S.E. (1988). *Brownian Motion and Stochastic Calculus.* Springer: Graduate Texts in Mathematics.

Kolmogorov, A.N. (1940). Wienersche Spiralen und einige andere interessante Kurven im Hilbertschen Raum. *C.R. (Doklady) Acad. Sci. URSS (N.S.)* 26, 115-118.

Kôno, N. and Maejima, M. (1991). Hölder continuity of sample paths of some self-similar stable processes. *Tokyo Journal of Mathematics* Vol. 14, No. 1, 93-100.

Lamberton, D. and Lapeyre, B. (1996). *Stochastic Calculus Applied to Finance.* Chapman & Hall.

Lin, S.J. (1995). Stochastic analysis of fractional Brownian motions. *Stochastics and Stochastics Reports* 55, 121-140.

Lipster, R.Sh. and Shiryaev, A.N. (1989). *Theory of Martingales.* Kluwer Acad. Publ., Dordrecht.

Mandelbrot, B.B. and Van Ness, J.W. (1968). Fractional Brownian motions, fractional noises and applications. *SIAM Review* 10, 422-437.

Merton, R.C. (1973). Theory of rational option pricing. *Bell J. Econom. Management Sci.* 4, 141-183.

Protter, P. (1990). *Stochastic Integration and Differential Equations.* Berlin: Springer-Verlag.

Rogers, L.C.G. (1997). Arbitrage with fractional Brownian motion. *Mathematical Finance* 7, 95-105.

Salopek, D.M. (1998). Tolerance to arbitrage. *Stochastic Processes and their Applications* 76, 217-230.

Samuelson, P.A. (1965). Rational theory of warrant pricing. *Industrial Management Review* Vol. 6, No. 2, 13-31.

Shao, Q.M. (1996). p-variation of Gaussian processes with stationary increments. *Studia Sci. Math. Hungar.* 31, 237-247.

Shepp, L.A. (1966). Radon-Nikodym derivatives of Gaussian measures. *Ann. Math. Statist.* 37, 321-354.

Shiryaev, A.N. (1984). *Probability.* Springer-Verlag, New York.

Shiryaev, A.N. (1998). On arbitrage and replication for fractal models. *Research Report no. 20, MaPhySto, Department of Mathematical Sciences, University of Aarhus, Denmark.*

Smithies, F. (1958). *integral equations.* Cambridge Univ. Press, London and New York.

Stricker, C. (1984). Quelques remarques sur les semimartingales Gaussiennes et le problème de l'innovation. *Lecture Notes in Control and Information Sciences* 61, 260-276.

Takashima, K. (1989). Sample path properties of ergodic self-similar processes. *Osaka J. Math.* 26, 159-189.

Wheeden, R.L. and Zygmund, A. (1977). *Measure and Integral.* Marcel Dekker, Inc.