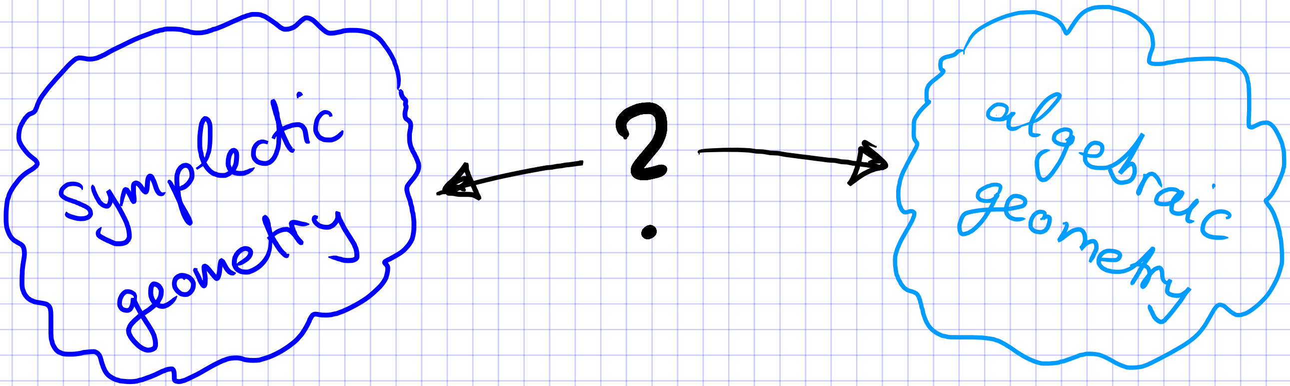
What is

HOMOLOGICAL MIRROR SYMMETRY



"Mirror Symmetry" ~1980 : observed by physicists in string theory

"Homological Mirror Symmetry" mathematical explanation for mirror symmetry phenomenon

1994 : HMS conjecture by Maxim Kontsevich

A-side		B-side
triangulated category constructed from symplectic geometry of X	Equivalence $\longleftrightarrow$	triangulated category constructed from algebraic geometry of $\tilde{X}$
"derived Fukaya category"		"derived category of coherent sheaves"

HMS conjecture

X Calabi-Yau manifold.

Then there exists a complex algebraic manifold $\tilde{X}$, and an equivalence

bounded derived category coming from the Fukaya category of X $DFuk(X)$	$\xleftrightarrow{\text{Equivalence}}$	bounded derived category of coherent sheaves on $\tilde{X}$ $DCoh(\tilde{X})$

$\tilde{X}$ is called <u>mirror-dual</u> to X

Proven cases : - Elliptic curves

suggested by Kontsevich 1994

proof by Polishchuk-Zaslow 1998

Port : "An introduction to HMS and the case of elliptic curves"

- Quartic surface (Seidel 2003)

Some proven aspects : - abelian varieties (Fukaya 2002)

- Lagrangian torus fibrations (Kontsevich, Soibelman 2000)

There are extensions to non-Calabi-Yau manifolds.

(X,ω) symplectic manifold : $\omega \in \Omega^2(X)$ non-degenerate, $d\omega = 0$

$L^n \subset X^{2n}$ Lagrangian submanifold : $\omega|_{TL} = 0$

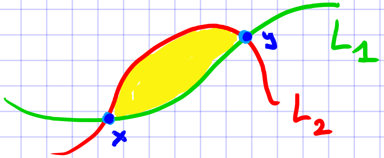
$\mathcal{L}$: a class of (decorated) Lagrangian submanifolds

Fukaya category : A_∞-category with objects : $\mathcal{L}$

Fuk(M)

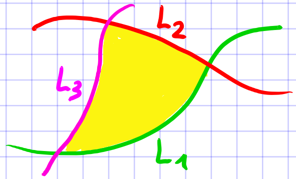
morphisms : intersections

$$\mathrm{Hom}(L_1, L_2) = \bigoplus_{x \in L_1 \cap L_2} \mathbb{C}\, x$$



chain complex with differential coming from counting holomorphic strips

composition : counting triangles



higher order composition : counting polygons

Algebra : Add cones to Fuk(X) and take cohomology.

$\rightsquigarrow$ DFuk(X)

X complex manifold

Structure sheaf: $\forall\, \mathcal{U} \subset X$ open: $\mathcal{O}_X(\mathcal{U}) := \{ f : \mathcal{U} \to \mathbb{C} \mid f \text{ holomorphic} \}$

Sheaf of $\mathcal{O}_X$-modules: $\forall\, \mathcal{U} \subset X$ open: $\mathcal{O}_X(\mathcal{U})$-module $\mathcal{F}(\mathcal{U})$

for $V \subset \mathcal{U}$:

$$\begin{tikzcd} \mathcal{O}_X(\mathcal{U}) \times \mathcal{F}(\mathcal{U}) \arrow[r] \arrow[d, "res"] \arrow[dr, phantom, "/\!/\!/"] & \mathcal{F}(\mathcal{U}) \arrow[d, "res"] \\ \mathcal{O}_X(V) \times \mathcal{F}(V) \arrow[r] & \mathcal{F}(V) \end{tikzcd}$$

$\forall\ s_i \in \mathcal{F}(\mathcal{U}_i),\ \bigcup \mathcal{U}_i = \mathcal{U},\quad s_i|_{\mathcal{U}_i \cap \mathcal{U}_j} = s_j|_{\mathcal{U}_i \cap \mathcal{U}_j}$:

$\exists!\ s \in \mathcal{F}(\mathcal{U}) : \quad s|_{\mathcal{U}_i} = s_i$.

<u>Example</u> $\quad Y \xrightarrow{p} X$ holomorphic vector bundle

$\mathcal{F}(\mathcal{U}) := \{ s : \mathcal{U} \to Y \mid \forall x \in \mathcal{U} : p(s(x)) = x,\ s \text{ holomorphic} \}$

This is an example of a <u>coherent sheaf of $\mathcal{O}_X$-modules.</u>

$\mathcal{F}$ is called <u>coherent</u> if

- $\exists$ open cover $\{\mathcal{U}_i\}$ of X, $\exists$ exact sequences

$$(\mathcal{O}_X|_{\mathcal{U}_i})^I \longrightarrow (\mathcal{O}_X|_{\mathcal{U}_i})^J \longrightarrow \mathcal{F}|_{\mathcal{U}_i} \longrightarrow 0$$

- $\forall\, \mathcal{U}$ open affine: $\mathcal{F}(\mathcal{U})$ is finitely generated $\mathcal{O}_X(\mathcal{U})$-module.

X complex manifold

Structure sheaf: $\forall\, \mathcal{U} \subset X$ open: $\mathcal{O}_X(\mathcal{U}) := \{ f : \mathcal{U} \to \mathbb{C} \mid f \text{ holomorphic} \}$

Sheaf of $\mathcal{O}_X$-modules: $\forall\, \mathcal{U} \subset X$ open: $\mathcal{O}_X(\mathcal{U})$ -module $\mathcal{F}(\mathcal{U})$

for $V \subset \mathcal{U}$:

$$\begin{array}{ccc} \mathcal{O}_X(\mathcal{U}) \times \mathcal{F}(\mathcal{U}) & \longrightarrow & \mathcal{F}(\mathcal{U}) \\ \downarrow \text{res} & /\!/\!/ & \downarrow \text{res} \\ \mathcal{O}_X(V) \times \mathcal{F}(V) & \longrightarrow & \mathcal{F}(V) \end{array}$$

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Category of coherent sheaves: $\quad \mathrm{Coh}(X)$

Algebra $\rightsquigarrow$ $D\,\mathrm{Coh}(X)$

bounded derived Fukaya category of T^2

$$DFuk(T^2)$$

Equivalence

$\longleftrightarrow$

bounded derived category of coherent sheaves on E

$$DCoh(E)$$

$$\left(T^2 = \mathbb{R}^2 / \mathbb{Z}^2 \,,\, dx \wedge dy\right)$$

$$E = \mathbb{C} / \langle 1, i \rangle$$

T^2 : 2-torus $\mathbb{R}^2/\mathbb{Z}^2$

symplectic form $\omega = dx \wedge dy$

$\mathcal{L}$: closed geodesics

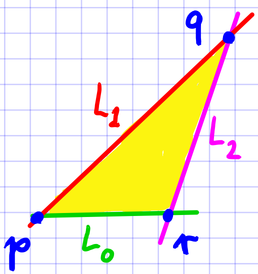
i.e. straight lines of rational slope

morphisms : $\mathrm{Hom}(L_0, L_1) := \bigoplus_{p \in L_0 \cap L_1} \mathbb{C}\, p$

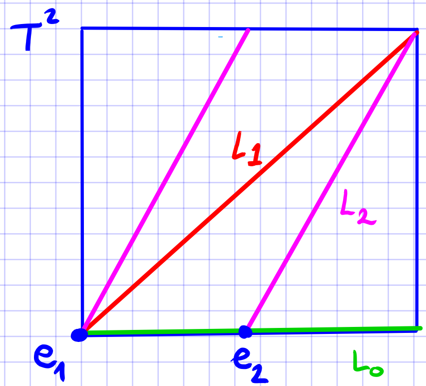
differential: 0

Composition: $\mathrm{Hom}(L_0, L_1) \otimes \mathrm{Hom}(L_1, L_2) \longrightarrow \mathrm{Hom}(L_0, L_2)$

$$\underset{L_0 \cap L_1}{\overset{\pitchfork}{p}} \otimes \underset{L_1 \cap L_2}{\overset{\pitchfork}{q}} \longmapsto \sum_{r \in L_0 \cap L_2} c(p, q; r)\, r$$



$$c(p, q; r) = \sum_{\text{triangle}} e^{-2\pi\, \text{area(triangle)}}$$



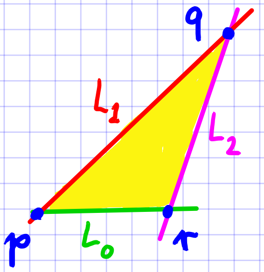
$\mathrm{Hom}(L_0, L_1) = \mathbb{C}\, e_1$

$\mathrm{Hom}(L_1, L_2) = \mathbb{C}\, e_1$

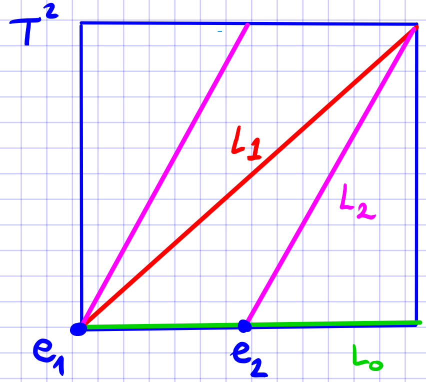
$\mathrm{Hom}(L_0, L_2) = \mathbb{C} e_1 \oplus \mathbb{C} e_2$

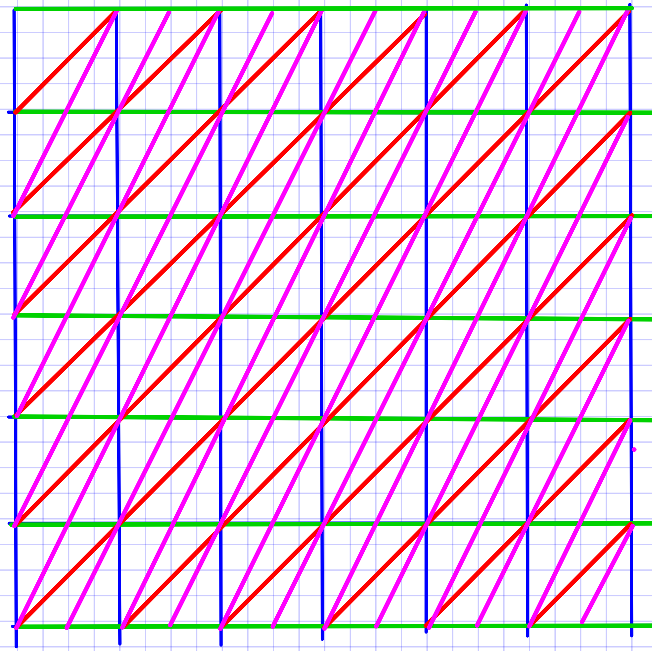
Composition: $\mathrm{Hom}(L_0,L_1)\otimes\mathrm{Hom}(L_1,L_2)\longrightarrow\mathrm{Hom}(L_0,L_2)$

$$\underset{L_0\cap L_1}{\overset{\cap}{p}}\ \otimes\ \underset{L_1\cap L_2}{\overset{\cap}{q}}\ \longmapsto\ \sum_{r\in L_0\cap L_2} C(p,q;r)\,r$$



$$C(p,q;r)=\sum_{\text{triangle}} e^{-2\pi\,\mathrm{area}(\text{triangle})}$$





$$\begin{array}{ccccc}\mathrm{Hom}(L_0,L_1) & \otimes & \mathrm{Hom}(L_1,L_2) & \longrightarrow & \mathrm{Hom}(L_0,L_2)\\ \| & & \| & & \|\\ \mathbb{C}e_1 & & \mathbb{C}e_1 & & \mathbb{C}e_1\oplus\mathbb{C}e_2\end{array}$$

$$e_1\otimes e_1\longmapsto\sum_{n\in\mathbb{Z}} e^{-2\pi n^2}e_1+\sum_{n\in\mathbb{Z}} e^{-2\pi(n+\frac12)^2}e_2$$

$$E = \mathbb{C}/\langle 1, i\rangle = \mathbb{C}^*/_{u \sim e^{-2\pi} u}$$

* Coherent Sheaves:
  - holomorphic sections of holomorphic vector bundles
  - torsion sheaves over one point

* Any holomorphic line bundle $\mathcal{L}$ is of the form

$$\mathcal{L} \cong t_x^* \mathcal{K} \otimes \mathcal{K} \otimes \dots \otimes \mathcal{K}, \quad x \in E$$

$$t_x : E \longrightarrow E$$
$$[y] \longmapsto [y+x]$$

$$\mathcal{K} = \mathbb{C}^* \times \mathbb{C} \Big/ (u,v) \sim (u e^{-2\pi}, e^{\pi} u^{-1} v)$$

* Let $\mathcal{L} = \mathcal{K}^n$, $\mathcal{L}' = \mathcal{K}^m$, $k := m-n > 0$. Then

$$\mathrm{Hom}(\mathcal{L}, \mathcal{L}') \cong \{\text{holomorphic sections of } \mathcal{K}^{m-n}\}$$

$$= \left\langle \theta[0](ki, kz), \theta\left[\tfrac{1}{k}\right](ki, kz), \dots, \theta\left[\tfrac{k-1}{k}\right](ki, kz) \right\rangle_{\mathbb{C}}$$

where

$$\theta[a](\tau, z) = \sum_{m \in \mathbb{Z}} e^{\pi i [(m+a)^2 \tau + 2(m+a)]} \quad \text{"Theta functions".}$$

* E.g. holomorphic sections of $\mathcal{K}$:

$$\theta_i(z) = \theta[0,0](i,z) = \sum_{m\in\mathbb{Z}} e^{\pi i(m^2 i + 2mz)}$$

$$= \sum_{m\in\mathbb{Z}} e^{-\pi m^2} u^m$$

So $\quad \mathrm{Hom}(\mathcal{O},\mathcal{K}) \cong \mathbb{C}\{\theta_i(z)\}$

$\mathrm{Hom}(\mathcal{K},\mathcal{K}^2) \cong \mathbb{C}\{\theta_i(z)\}$

$\mathrm{Hom}(\mathcal{O},\mathcal{K}^2) \cong \mathbb{C}\{\theta_{2i}(2z)\} \oplus \mathbb{C}\{\theta[\tfrac{1}{2}](2i,2z)\}$

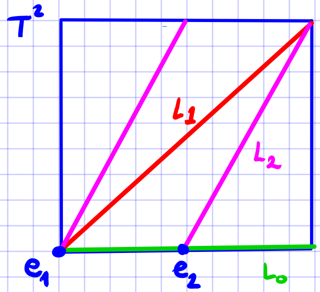
* Composition: Product of theta functions!

A-side

L_0

L_1

L_2



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$$\begin{array}{ccc}
\mathrm{Hom}(L_0,L_1)\otimes \mathrm{Hom}(L_1,L_2) & \ni & e_1 \otimes e_1 \\
\downarrow & & \downarrow \\
\mathrm{Hom}(L_0,L_2) & \ni & \displaystyle\sum_{n\in\mathbb{Z}} e^{-2\pi n^2}\, e_1 + \sum_{n\in\mathbb{Z}} e^{-2\pi (n+\frac{1}{2})^2}\, e_2
\end{array}$$

B-side

$\mathcal{L}_0 = \mathcal{O}$

$\mathcal{L}_1 = \mathcal{K}$

$\mathcal{L}_2 = \mathcal{K}^2$

$$\mathrm{Hom}(\mathcal{L}_0, \mathcal{L}_1) = \mathbb{C}\{\theta_i(z)\}$$

$$\mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2) = \mathbb{C}\{\theta_i(z)\}$$

$$\mathrm{Hom}(\mathcal{L}_0, \mathcal{L}_2) = \mathbb{C}\{\theta_{2i}(2z)\} \oplus \mathbb{C}\{\theta[\tfrac{1}{2},0](2i,2z)\}$$

$$\begin{array}{ccc}
\mathrm{Hom}(\mathcal{L}_0,\mathcal{L}_1)\otimes \mathrm{Hom}(\mathcal{L}_1,\mathcal{L}_2) & \ni & \theta_i(z) \otimes \theta_i(z) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(\mathcal{L}_0,\mathcal{L}_2) & \ni & \theta_i(z)\cdot\theta_i(z)
\end{array}$$

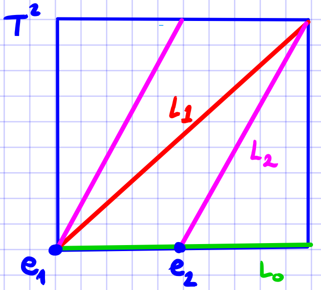
$$\theta_i(z)\cdot\theta_i(z) = \theta_{2i}(0)\,\theta_{2i}(2z) + \theta[\tfrac{1}{2},0](2i,0)\,\theta[\tfrac{1}{2},0](2i,2z)$$

A-side

L_0

L_1

L_2



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$$\mathrm{Hom}(L_0, L_2) = \mathbb{C}\, e_1 \oplus \mathbb{C}\, e_2$$

$$\mathrm{Hom}(L_0, L_1) \otimes \mathrm{Hom}(L_1, L_2) \ni e_1 \otimes e_1$$

$$\downarrow \qquad\qquad \downarrow$$

$$\mathrm{Hom}(L_0, L_2) \ni \sum_{n \in \mathbb{Z}} e^{-2\pi n^2}\, e_1 + \sum_{n \in \mathbb{Z}} e^{-2\pi (n+\frac{1}{2})^2}\, e_2$$

B-side

$$\mathcal{L}_0 = \mathcal{O}$$

$$\mathcal{L}_1 = \mathcal{K}$$

$$\mathcal{L}_2 = \mathcal{K}^2$$

$$\mathrm{Hom}(\mathcal{L}_0, \mathcal{L}_1) = \mathbb{C}\{\Theta_i(z)\}$$

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$$\mathrm{Hom}(\mathcal{L}_0, \mathcal{L}_2) = \mathbb{C}\{\Theta_{2i}(2z)\} \oplus \mathbb{C}\{\Theta[\tfrac{1}{2}, 0](2i, 2z)\}$$

$$\mathrm{Hom}(\mathcal{L}_0, \mathcal{L}_1) \otimes \mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2) \ni \Theta_i(z) \otimes \Theta_i(z)$$

$$\downarrow \qquad\qquad \downarrow$$

$$\mathrm{Hom}(\mathcal{L}_0, \mathcal{L}_2) \ni \Theta_i(z) \cdot \Theta_i(z)$$

$$\| $$

$$\Theta_{2i}(0)\, \Theta_{2i}(2z) + \Theta[\tfrac{1}{2}, 0](2i, 0)\, \Theta[\tfrac{1}{2}, 0](2i, 2z)$$

$=$

$=$

$$\Theta[a](\tau, z) = \sum_{m \in \mathbb{Z}} e^{\pi i [(m+a)^2 \tau + 2(m+a) z]}$$

A-side	B-side
* Lagrangian of slope d and y-intercept y_0	* holomorphic line bundle $t^*_{-y_0 i}\mathcal{K} \otimes \mathcal{K}^{d-1}$
* vertical Lagrangians	* torsion sheaves
* gradings on Lagrangians $\rightsquigarrow$ $\mathbb{Z}$-grading on $\mathrm{Hom}((L_0,\alpha_0),(L_1,\alpha_1))$ $\rightsquigarrow$ Shift functor $(L,\alpha) \longmapsto (L, \alpha+1)$	* Shift functor in $D\mathrm{Coh}(E)$
* local systems on Lagrangians	* higher rank holomorphic vector bundles
* B-field $(T^2, \omega_{\mathbb{C}} = B + i\omega)$ $\omega = A\,dx \wedge dy$	* $E_\tau = \mathbb{C}/\langle 1, \tau \rangle$, $\tau = B + iA$

Thank you for listening!