ARCH(1) – Modell

ARCH: **A**uto**R**egressive **C**onditional **H**eteroscedasticity

$$X_t = \sigma_t \varepsilon_t \qquad (t \in \mathbb{Z})$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 \qquad (\alpha_0, \alpha_1 > 0)$$

ε_t i.i.d. , $E[\varepsilon_t] = 0$

$\qquad Var(\varepsilon_t) = E[\varepsilon_t^2] = 1$

ε_t unabhängig von $\{X_s ; s < t\}$
independent of

Eigenschaften: Falls X_t respektive X_t^2 schwach
If and are weakly

stationär (und Momente existieren, d.h. Bedingungen an
stationary and moments exist, i.e. assumptions on
α_1):

(1) $E[X_t \mid X_{t-1}] = 0 \implies E[X_t] = 0$

(2) $Var(X_t \mid X_{t-1}) = \sigma_t^2$

(3) $Cor(X_0, X_k) = 0$ für $h \neq 0$

(4) $Corr(X_0^2, X_k^2) = \alpha_1^k$ für $h \neq 0$

Beweis:

(1) $\mathbb{E}[X_t \mid X_{t-1}] = \sigma_t \underbrace{\mathbb{E}[\varepsilon_t]}_{=0} = 0$

(2) $\operatorname{Var}(X_t \mid X_{t-1}) = \sigma_t^2 \underbrace{\operatorname{Var}(\varepsilon_t)}_{=1} = \sigma_t^2$

(3) Sei $k \geq 1$.
consider

$$\operatorname{Cor}(X_0, X_k) \overset{(1)}{=} \mathbb{E}[X_0 X_k]$$

$$= \mathbb{E}\Big[\underbrace{\mathbb{E}[X_0 X_k \mid X_{k-1}, \dots, X_0]}_{\substack{= X_0 \mathbb{E}[X_k \mid X_{k-1}, \dots, X_0] \\ = X_0 \mathbb{E}[X_k \mid X_{k-1}] \overset{(1)}{=} 0}}\Big] = 0$$

(4) Schreibe
Write

$$X_t^2 = \sigma_t^2 \varepsilon_t^2 = \sigma_t^2 + \sigma_t^2(\varepsilon_t^2 - 1)$$
$$= \alpha_0 + \alpha_1 X_{t-1}^2 + \sigma_t^2(\varepsilon_t^2 - 1)$$

Sei $k = 1$.
consider

$$\operatorname{Cor}(X_0^2, X_1^2) = \mathbb{E}\Big[\underbrace{\operatorname{Cov}(X_0^2, X_1^2 \mid X_0)}_{=0}\Big] +$$

$$\operatorname{Cov}\Big(\underbrace{\mathbb{E}[X_0^2 \mid X_0]}_{= X_0^2}, \underbrace{\mathbb{E}[X_1^2 \mid X_0]}_{= \alpha_0 + \alpha_1 X_0^2}\Big)$$

$$= \alpha_1 \operatorname{Var}(X_0^2)$$

$$\Longrightarrow \operatorname{Corr}(X_0^2, X_1^2) = \alpha_1$$

Mit Induktion: für $k \geq 2$

with induction

$$\operatorname{Cov}(X_0^2, X_k^2) = \mathbb{E}\big[\underbrace{\operatorname{Cov}(X_0^2, X_k^2 \mid X_{k-1}, \dots, X_0)}_{=0}\big] +$$

$$\operatorname{Cov}\big(\underbrace{\mathbb{E}[X_0^2 \mid X_{k-1}, \dots, X_0]}_{=X_0^2}, \underbrace{\mathbb{E}[X_k^2 \mid X_{k-1}, \dots, X_0]}_{=\alpha_0 + \alpha_1 X_{k-1}^2}\big)$$

$$= \alpha_1 \operatorname{Cov}(X_0^2, X_{k-1}^2) \overset{\substack{\text{induction} \\ \text{Induktion}}}{=} \alpha_1^k \operatorname{Var}(X_0)$$

$$\Longrightarrow \operatorname{Corr}(X_0^2, X_k^2) = \alpha_1^k$$