ARMA models

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Autoregressive models (recap)

$(X_t)_{t\in\mathbb{Z}}$ is autoregressive of order q (AR(p)) if

$$X_t = \sum_{j=1}^{p} \varphi_j X_{t-j} + \varepsilon_t,\ t \in \mathbb{Z},$$

$(\varepsilon_t)_{t\in\mathbb{Z}}$ a sequence of i.i.d. variables, $\mathbb{E}[\varepsilon_t] = 0$.

with backshift operator B and corresponding AR(p)-polynomial:

$$\Phi(z) = 1 - \sum_{j=1}^{p} \varphi_j z^j \ (z \in \mathbb{C})$$

rewrite the model as

$$(\Phi(B)X)_t = \varepsilon_t,\ t \in \mathbb{Z}.$$

Stationarity and causality

Assume

$$(A): \quad \Phi(z) \neq 0 \text{ for } |z| \leq 1$$

Then:

$$X_t = (\Psi(B)\varepsilon)_t, \ \Psi(z) = \frac{1}{\Phi(z)} = 1 + \sum_{j=1}^{\infty} \psi_j z^j,$$

$$\text{and thus } X_t = \sum_{j=1}^{\infty} \psi_j \varepsilon_{t-j} + \varepsilon_t$$

Conclusion:
Assume (A) and $\mathbb{E}|\varepsilon_t| < \infty$: then, $(X_t)_{t\in\mathbb{Z}}$ is stationary and causal

in fact: assume $\mathbb{E}|\varepsilon_t| < \infty$
$(X_t)_{t\in\mathbb{Z}}$ is stationary $\Leftrightarrow$ (A)

Moving average (MA(q))

$(X_t)_{t\in\mathbb{Z}}$ is a moving average of order q (MA(q)) if

$$X_t = \sum_{k=1}^{q} \theta_k \varepsilon_{t-k} + \varepsilon_t,\ t \in \mathbb{Z},$$

$(\varepsilon_t)_{t\in\mathbb{Z}}$ sequence of i.i.d. variables, $\mathbb{E}[\varepsilon_t] = 0$.

Because an MA(q) is of the form $X_t = fct.(\varepsilon_t, \varepsilon_{t-1}, \ldots, \varepsilon_{t-q})$:
$\rightsquigarrow$ it is always stationary and causal
Representation with the backshift operator:

$$X_t = (\Theta(B)\varepsilon)_t,\ t \in \mathbb{Z},$$
$$\Theta(z) = 1 + \sum_{k=1}^{q} \theta_k z^k \ (z \in \mathbb{C}).$$

Invertibility

Analogously to AR(p) models, we can invert $\Theta(\cdot)$ if its roots are outside the unit circle.

Theorem

Consider an MA(q) process and assume that $\Theta(z) \neq 0$ *for* $|z| \leq 1$ *and* $\mathbb{E}|\varepsilon_t| < \infty$. *Then,*

$$\varepsilon_t = \sum_{j=0}^{\infty} \gamma_j X_{t-j}, \ \gamma_0 = 1, \ t \in \mathbb{Z},$$

$$\Gamma(z) = \Theta^{-1}(z) = \frac{1}{\Theta(z)} = \sum_{j=0}^{\infty} \gamma_j z^j, \ \gamma_0 = 1.$$

That is, we have an AR(∞) process:

$$X_t = \sum_{j=1}^{\infty} -\gamma_j X_{t-j} + \varepsilon_t$$

Implication: can model an infinite conditional dependence with 1 or a few parameters
For example: in AR(p) model, we have that

$$\mathbb{E}[X_t | X_{t-1}, X_{t-2}, \ldots] = \mathbb{E}[X_t | X_{t-1}, \ldots, X_{t-p}] = \sum_{j=1}^{p} \phi_j X_{t-j}.$$

But with an MA(q) model,

$$\mathbb{E}[X_t | X_{t-1}, X_{t-2}, \ldots]$$

depends on the infinite past

As a concrete example: consider an MA(1)

$$X_t = \theta \varepsilon_{t-1} + \varepsilon_t.$$

Then:

$$\Theta(z) = 1 + \theta z, \quad \Gamma(z) = 1/\Theta(z) = 1 + \sum_{j=1}^{\infty} (-\theta)^j z^j.$$

For $|\theta| < 1$: $\Gamma(z)$ is well-defined for $|z| \leq 1$
$\rightsquigarrow$ for $|\theta| < 1$ and $\mathbb{E}|\varepsilon_t| < \infty$: can represent

$$X_t = \sum_{j=1}^{\infty} (-\theta)^j X_{t-j} + \varepsilon_t, \; t \in \mathbb{Z},$$

which is an AR(∞) process,
i.e., a non-Markovian process whose conditional distribution depends on an infinite past.

Autoregressive moving average of orders p and q (ARMA(p,q))

Combination of AR(p) and MA(q) provides a flexible modeling framework!

$(X_t)_{t\in\mathbb{Z}}$ is a autoregressive moving average of orders p and q (ARMA(p,q)) if

$$X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \sum_{k=1}^{q} \theta_k \varepsilon_{t-k} + \varepsilon_t, \; t \in \mathbb{Z}.$$

With the backshift operator: representation

$$(\Phi(B)X)_t = (\Theta(B)\varepsilon)_t, \; t \in \mathbb{Z},$$

$$\Phi(z) = 1 - \sum_{j=1}^{p} \phi_j z^j, \;\; \Theta(z) = 1 + \sum_{k=1}^{q} \theta_k z^k, \;\; z \in \mathbb{C}.$$

model can be over-parameterized

Example: consider the (seemingly) ARMA$(1,1)$ equation

$$X_t = 0.8X_{t-1} - 0.8\varepsilon_{t-1} + \varepsilon_t,$$
$$\text{i.e. } (\Phi(B)X)_t = (\Theta(B)\varepsilon)_t, \;\; \Phi(z) = \Theta(z) = 1 - 0.8z.$$

Note that the i.i.d. sequence $(\varepsilon)_{t\in\mathbb{Z}}$ satisfies the equation above (just use $X_t = \varepsilon_t$)
i.e., equation above is satisfied by an i.i.d. sequence (which we usually do not represent as an ARMA(1,1) process)

problem occurs because $\Phi(\cdot)$ and $\Theta(\cdot)$ have common roots (i.e. $z_0 = 1/0.8$)
$\leadsto$ can factor out terms on both sides of $(\Phi(B)X)_t = (\Theta(B)\varepsilon)_t$

problem disappears and ARMA(p, q) model is identifiable if the set of roots of $\Phi(\cdot)$ and the set of roots of $\Theta(\cdot)$ have no common element
i.e., polynomials $\Phi(\cdot)$ and $\Theta(\cdot)$ have no common factors

Stationarity and causality

with analogous arguments as before: can invert $\Phi(\cdot)$ and/or $\Theta(\cdot)$ if the corresponding roots are outside the unit circle

Theorem

Consider an ARMA(p, q) with $\Phi(z) \neq 0$ $(|z| \leq 1)$, $\Theta(z) \neq 0$ $(|z| \leq 1)$ *and assume that the roots of* $\Phi(\cdot)$ *and* $\Theta(\cdot)$ *are distinct. Then we have: MA(∞) representation*

$$X_t = \sum_{j=1}^{\infty} \psi_j \varepsilon_{t-j} + \varepsilon_t, \; t \in \mathbb{Z},$$

$$\Psi(z) = \frac{\Theta(z)}{\Phi(z)} = \sum_{j=0}^{\infty} \psi_j z^j, \; \psi_0 = 1 \; (|z| \leq 1),$$

and the AR(∞) representation

$$\varepsilon_t = \sum_{j=0}^{\infty} \gamma_j X_{t-j}, \; \gamma_0 = 1, \; t \in \mathbb{Z},$$

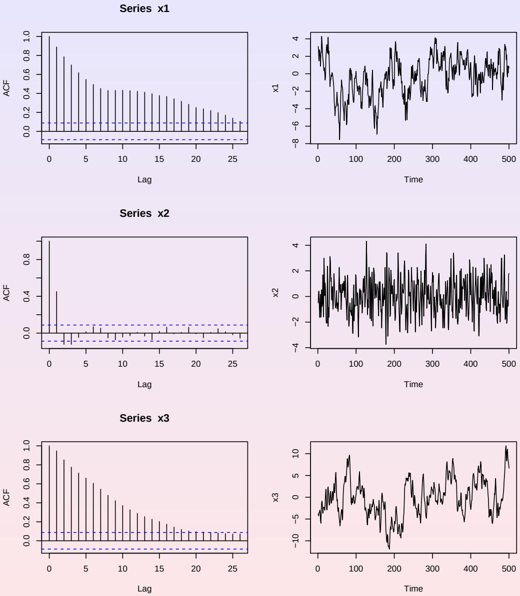
Condition $\Phi(z) \neq 0$ ($|z| \leq 1$) implies stationarity and causality of the ARMA(p, q) process since $X_t = \sum_{j=1}^{\infty} \psi_j \varepsilon_{t-j} + \varepsilon_t$ is then a function of infinitely many $\varepsilon_t, \varepsilon_{t-1}, \ldots$

Brief illustration

```
set.seed(22)
x1 ← arima.sim(n=500,model=list(ar=0.9))
acf(x1)
plot(x1)

set.seed(22)
x2 ← arima.sim(n=500,model=list(ma=0.9))
acf(x2)
plot(x2)

set.seed(22)
x3 ← arima.sim(n=500,model=list(ar=0.9,ma=0.9))
acf(x3)
plot(x3)
```


Series x1
ACF
Lag
x1
Time
Series x2
ACF
Lag
x2
Time
Series x3
ACF
Lag
x3
Time