Estimation of state process via Kalman filter

notation: $Z_t \mid Y_1^s \sim f_{t|s}(z_t \mid Y_1^s)\, dz_t$

$t > s$: prediction density

$t = s$: filter density

$t < s$: smoothing density

example: maximum a-posteriori (MAP) estimator

$$\hat{Z}_t = \arg\max_{z_t} f_{t|n}(z_t \mid Y_1^n)$$

recursive computation of $f_{t|s}(\cdot \mid \cdot)$:

prediction density:

$$f_{t|t-1}(z_t \mid Y_1^{t-1}) = \int p(z_t, z_{t-1} \mid Y_1^{t-1})\, dz_{t-1}$$

$$= \int p(z_t \mid z_{t-1}, \cancel{Y_1^{t-1}}) \cdot \underbrace{p(z_{t-1} \mid Y_1^{t-1})}_{f_{t-1|t-1}(z_{t-1} \mid Y_1^{t-1})}\, dz_{t-1}$$

$$= \int p(z_t \mid z_{t-1}) \cdot f_{t-1|t-1}(z_{t-1} \mid Y_1^{t-1})\, dz_{t-1}$$

filter density at time $t-1$

filter density

$$f_{t|t}(x_t \mid Y_1^t) = p(z_t \mid Y_t, Y_1^{t-1})$$

$$= \frac{p(Y_t \mid z_t, \cancel{Y_1^{t-1}}) \cdot p(z_t \mid Y_1^{t-1})}{p(Y_t \mid Y_1^{t-1})}$$

$$= \frac{p(Y_t \mid z_t) \cdot f_{t|t-1}(z_t \mid Y_1^{t-1})}{\int p(Y_t \mid z_t) \cdot f_{t|t-1}(z_t \mid Y_1^{t-1}) \, dz_t}$$

prediction density at time t

recursive scheme:

$$\begin{tikzcd}
\text{prediction density} & & f_{1|0} \arrow[d, "Y_1"] & f_{2|1} \arrow[d, "Y_2"] & {} & \cdots \\
\text{filter density} & f_{0|0} \arrow[ur] & f_{1|1} \arrow[ur] & f_{2|2} \arrow[ur] & &
\end{tikzcd}$$

smoother density

$$f_{t|n}(z_t \mid Y_1^n) = \int p(z_t \mid z_{t+1}, Y_1^n) \cdot f_{t+1|n}(z_{t+1} \mid Y_1^n) \, dz_{t+1}$$

smoother density at time $t+1$

can show that:

$$p(z_t \mid z_{t+1}, Y_1^n) = \frac{p(z_{t+1} \mid z_t) \cdot f_{t|t}(z_t \mid Y_1^t)}{f_{t+1|t}(z_{t+1} \mid Y_1^t)}$$

recursive computation of smoother densities:

$$f_{n|n} \longrightarrow f_{n-1|n} \longrightarrow f_{n-2|n} \longrightarrow \dots$$

filter dens. — $f_{n-1|n-1}$ $\qquad$ $f_{n-2|n-2}$

pred. dens. — $f_{n|n-1}$ $\qquad$ $f_{n-1|n-2}$

note: we need to do integration

if z_t is high-dimensional $\rightsquigarrow$ not easy!

linear state space model

linear transformations and Gaussian errors

$$\rightsquigarrow \quad f_{t|s} \sim \mathcal{N}(m_{t|s}, R_{t|s}) \quad \begin{matrix} s \le t \\ s > t \end{matrix}$$

Kalman filter $\qquad$ filter and prediction

$$m_{t|t-1} = G_t \, m_{t-1|t-1} \qquad \text{(clear)}$$

$$R_{t|t-1} = \Sigma_t + G_t \, R_{t-1|t-1} \, G_t^T \qquad \text{(clear)}$$

$$R_{t|t} = \left(H_t \, \Omega_t^{-1} H_t + R_{t|t-1}^{-1}\right)^{-1}$$

$$m_{t|t} = m_{t|t-1} + R_{t|t} \, H_t^T \, \Omega_t^{-1} \left(Y_t - H_t \, m_{t|t-1}\right)$$

smoother

$$m_{t|n} = m_{t|t} + S_t \left(m_{t+1|n} - m_{t+1|t} \right)$$

$$R_{t|n} = R_{t|t} + S_t \left(R_{t+1|t} - R_{t+1|n} \right) S_t^T$$

$$S_t = R_{t|t} \, G_{t+1}^T \, R_{t+1|t}^{-1}$$

$\rightsquigarrow$ everything can be computed recursively and in an efficient way