

Estimation of state process via Kalman filter

notation: $Z_t | Y_1^s \sim f_{t|s}(z_t | Y_1^s) dz_t$

$t > s$: prediction density

$t = s$: filter density

$t < s$: smoothing density

example: maximum a-posteriori (MAP) estimator

$$\hat{z}_t = \arg \max_{z_t} f_{t|n}(z_t | Y_1^n)$$

recursive computation of $f_{t|s}(\cdot)$:

prediction density:

$$f_{t|t-1}(z_t | Y_1^{t-1}) = \int p(z_t, z_{t-1} | Y_1^{t-1}) dz_{t-1}$$

$$= \int p(z_t | z_{t-1}, Y_1^{t-1}) \cdot \underbrace{p(z_{t-1} | Y_1^{t-1})}_{f_{t-1|t-1}(z_{t-1} | Y_1^{t-1})} dz_{t-1}$$

$$= \int p(z_t | z_{t-1}) \cdot f_{t-1|t-1}(z_{t-1} | Y_1^{t-1}) dz_{t-1}$$

↑
filter density at time $t-1$

filter density

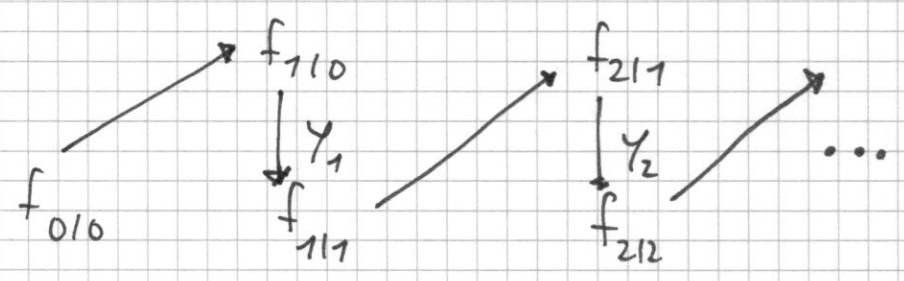
$$\begin{aligned}
 f_{t|t}(x_t | Y_1^t) &= p(z_t | Y_t, Y_1^{t-1}) \\
 &= \frac{p(Y_t | z_t, Y_1^{t-1}) \cdot p(z_t | Y_1^{t-1})}{p(Y_t | Y_1^{t-1})} \\
 &= \frac{p(Y_t | z_t) \cdot f_{t|t-1}(z_t | Y_1^{t-1})}{\int p(Y_t | z_t) \cdot f_{t|t-1}(z_t | Y_1^{t-1}) dz_t}
 \end{aligned}$$

↑ prediction density at time t

recursive scheme:

prediction density

filter density



smoother density

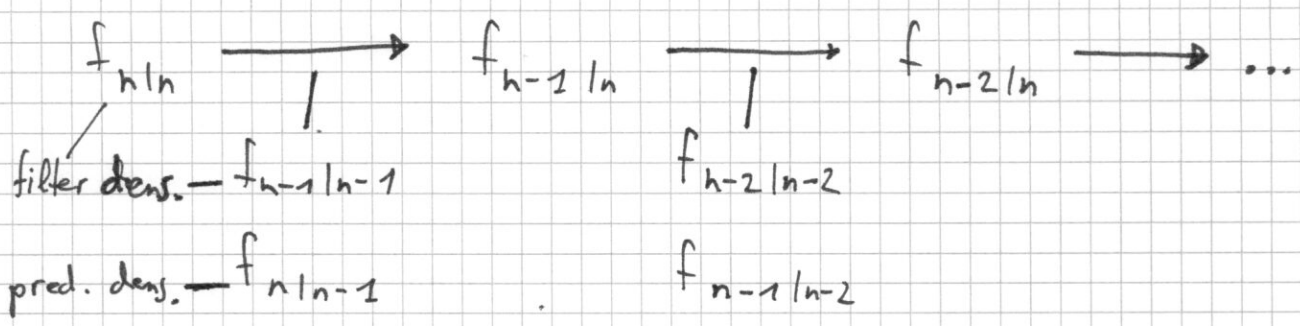
$$f_{t|n}(z_t | Y_1^n) = \int p(z_t | z_{t+1}, Y_1^n) \cdot f_{t+1|n}(z_{t+1} | Y_1^n) dz_{t+1}$$

↑ smoother density at time t+1

can show that:

$$p(z_t | z_{t+1}, Y_1^n) = \frac{p(z_{t+1} | z_t) \cdot f_{t|t}(z_t | Y_1^t)}{f_{t+1|t}(z_{t+1} | Y_1^t)}$$

recursive computation of smoother densities:



note: we need to do integration
if z_t is high-dimensional \rightsquigarrow not easy!

linear state space model

linear transformations and Gaussian errors

$\rightsquigarrow f_{t|s} \sim \mathcal{N}(m_{t|s}, R_{t|s}) \quad \begin{matrix} s \leq t \\ s > t \end{matrix}$

Kalman filter filter and prediction

$m_{t|t-1} = G_t m_{t-1|t-1}$ (clear)

$R_{t|t-1} = \Sigma_t + G_t R_{t-1|t-1} G_t^T$ (clear)

$R_{t|t} = (H_t \Omega_t^{-1} H_t^T + R_{t|t-1}^{-1})^{-1}$

$m_{t|t} = m_{t|t-1} + R_{t|t} H_t^T \Omega_t^{-1} (y_t - H_t m_{t|t-1})$

smoother

$$m_{t|h} = m_{t|t} + S_t (m_{t+1|h} - m_{t+1|t})$$

$$R_{t|h} = R_{t|t} + S_t (R_{t+2|t} - R_{t+2|h}) S_t^T$$

$$S_t = R_{t|t} G_{t+1}^T R_{t+1|t}^{-1}$$

→ everything can be computed recursively
and in an efficient way