Automorphism groups of polynomial rings

Bachelor Thesis

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Contents

1 Introduction

Polynomial rings in n variables over a field K appear everywhere in mathematics. Since these rings are so fundamental we believe it is very natural to study the structure of their automorphism groups. This would be the way an algebraist would think about this problem, whereas an algebraic geometer would regard this automorphism group as the automorphism group of affine n-space, a geometric object. There are many problems surrounding this group which have not been solved for general n. An example is the linearization problem, which asks if every element of finite order in the group is conjugate to an affine automorphism. A counterexample to this conjecture is given in [1] by Asanuma over a field of positive characteristic, but it is still an open problem for fields of characteristic 0. A more famous problem on polynomial automorphisms is the Jacobian conjecture which states that a K-algebra homomorphism whose Jacobian determinant is a nonzero constant is an automorphism. Again the conjecture is wrong for fields of positive characteristic but open for fields of characteristic 0. The kind of problem we will be focusing on in this thesis is the problem of giving elementary generators for the group.

The aim of this thesis is to study the cases $n = 1$ and $n = 2$ and give a summary of the progress in the case $n \geq 3$. The case $n = 2$ will be the main part of this paper. In this case we will show that the group can be generated by two kinds of automorphisms which have a simple description.

I would like to thank my advisor, Professor Richard Pink for his support, and encouragement to pursue this subject.

2 First definitions and results

We will now introduce some notation which will be used throughout the thesis: Fix $R := R'[X_1, ..., X_n]$ the polynomial ring in n variables over a commutative ring R'. The automorphism group of R we will denote by $\mathrm{Aut}_{R'}(R)$. We notice that an element $\phi \in \mathrm{Aut}_{R'}(R)$ is uniquely determined by the tuple $(\phi(X_1), ..., \phi(X_n))$. Therefore we will often write $(f_1, .., f_n) \in R^n$ for the unique automorphism that sends X_i to f_i for all i. We write X for the set $\{X_1, .., X_n\}$.

Definition 2.1. We call $\phi \in \mathrm{Aut}_{R'}(R)$ *affine* if for all $i = 1, ..., n : \phi(X_i)$ is a linear polynomial.

Definition 2.2. Let $i \in \{1, ..., n\}$. We call $\phi \in \mathrm{Aut}_{R'}(R)$ an $X \setminus \{X_i\}$*-based shear* if for all $j \in \{1, ..., n\} \setminus \{i\} : \phi(X_j) = X_j$ and $\phi(X_i) = X_i + f$ for some fixed polynomial f in the variables $X \setminus \{X_i\}$

An easy computation shows that the inverse of a shear is again a shear where the polynomial f in Definition 2.2 is replaced by $-f$. More generally we have the following definition:

Definition 2.3. Let $i \in \{1, ..., n\}$. We call $\phi \in \mathrm{Aut}_{R'}(R)$ an $X \setminus \{X_i\}$*-based de Jonquières automorphism* if $\phi(R'[X \setminus \{X_i\}]) = R'[X \setminus \{X_i\}]$.

Example 2.4. An $X \setminus \{X_i\}$-based shear is an $X \setminus \{X_i\}$-based de Jonqières automorphism.

Definition 2.5. We call $\phi \in \mathrm{Aut}_{R'}(R)$ *tame* if it can be written as a composition of affine and shear automorphisms. The group of all tame automorphisms will be denoted as $\mathrm{T}_n(R')$.

Definition 2.6. We call $\phi \in \mathrm{Aut}_{R'}(R)$ *triangular* if $\phi(X_i) = X_i + f_i(X_{i+1}, ..., X_n)$ for f_i in $K[X_{i+1}, ..., X_n]$ and $i = 1, .., n$.

Remark 2.7. We notice that all triangular automorphisms are tame.

As mentioned before, the main part of this thesis will be about the case $n = 2$ and R' a field. Our aim will be to prove the following theorem:

Theorem 2.8. *(Jung, Van der Kulk) All automorphisms of $K[X_1, X_2]$ are tame. Furthermore $\mathrm{Aut}_K(K[X_1, X_2]) = A *_C B$ the amalgamated product of A and B over C, where A is the group of affine automorphisms of $K[X_1, X_2]$ and B the group of X_2-based de Jonquières automorphism and $C = A \cap B$.*

This theorem was first proven by Jung [2] in 1942 in the case where $K = \mathbb{C}$ and without the amalgamated product decomposition. Van der Kulk [3] then proved the general case over an arbitrary field with the product decomposition in 1953. In later years many different kind of proofs have been proposed, some (algebro)-geometric in nature such as in [4] and others purely algebraic like the proof by Makar-Limanov [5]. Our interest in this paper lies in this purely algebraic approach. Dicks [6] simplified the proof of Makar-Limanov and we will be closely following this proof of Dicks as it is explained in [7]. But first we will study the case $n = 1$:

Theorem 2.9. *Every $\phi \in \mathrm{Aut}_K(K[X])$ is affine.*

Proof. Since ϕ is surjective there exists $f \in K[X]$ such that

$$\phi(f) = X, \tag{1}$$

If we write $f(X) = \sum_{i=0}^{n} a_i X^i$ with $n \in \mathbb{Z}^{\geq 0}$ and a_i in K for all i and a_n nonzero, the equation (1) becomes $\sum_{i=0}^{n} a_i \phi(X)^i = X$ by the homomorphism property and ϕ being the identity function on K. This tells us that the degree of $\phi(X)$ cannot be greater than 1 or else

$$1 = \deg(X) = \deg(\sum_{i=0}^{n} a_i \phi(X)^i) = \deg(\phi(X)^n) = n \deg(\phi(X)) \neq 1, \tag{2}$$

which is a contradiction. So we have $\deg(\phi(X)) \leq 1$ but $\phi(X)$ cannot be in K, because then $X = \phi(f)$ would be in K. Therefore $\deg(\phi(X)) = 1$. □

This theorem also tells us that we can embed $\mathrm{Aut}_K(K[X])$ into a subgroup of $GL_2(K)$: Since every $\phi \in \mathrm{Aut}_K(K[X])$ is affine there exist unique $a \in K^{\times}$ and $b \in K$ such that $\phi(X) = aX + b$. This implies that

$$\phi \mapsto \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \tag{3}$$

defines a well-defined map, and an easy computation shows that this is a group homomorphism. So this means that in the case $n = 1$ we can think of the automorphism group as a finite dimensional object over K. This makes the case $n = 1$ quite easy to study compared to the cases where $n \geq 2$. In the cases where $n \geq 2$ the shear automorphisms allow us to embed $K[X_1]$, which is a infinite-dimensional vector space over K, into the automorphism group $\mathrm{Aut}_K(K[X_1, ..., X_n])$. A way we can do this is by sending an $f \in K[X_1]$ to the $X \setminus \{X_2\}$-based shear automorphism defined by $f(X_1)$.

3 Basic notions on weighted degrees

In this section we will prove Theorem 2.8. For an automorphism $(g_1, g_2) \in \mathrm{Aut}_K(K[X_1, X_2])$, let (f_1, f_2) be its inverse. The basic idea of the proof is to introduce a weighted degree on $K[X_1, X_2]$ using the $(1, 0)$-degree d_1 and $(0, 1)$-degree d_2 of f_1 (see Definition 3.1). With this weighted degree we can prove that d_1 divides d_2 or the other way around. This is a key step that also appears in all other proofs of Theorem 2.8. Fix $(n_1, n_2) \in \mathbb{Z}^2$ with $(n_1, n_2) \neq (0, 0)$.

Definition 3.1. We define a function $d : K[X_1, X_2] \to \mathbb{Z} \cup \{-\infty\}$ by setting

$$d(f) := \sup\{n_1 i + n_2 j \mid a_{ij} \neq 0\}$$

where $f = \sum_{i,j \geq 0} a_{ij} X_1^i X_2^j$. It is called the (n_1, n_2)-*degree* of f.

The $(1,1)$-degree is the usual total degree and instead of $d(f)$ we denote this by $\deg(f)$. We call the $(1,0)$-degree the X_1-degree. Similarly we call the $(0,1)$-degree the X_2-degree.

Definition 3.2. For (a,b) and (a',b') in $\mathbb{Z}^2$ we say $(a,b) \leq (a',b')$ if $an_1 + bn_2 < a'n_1 + b'n_2$ or $an_1 + bn_2 = a'n_1 + b'n_2$ with $a'n_2 \geq an_2$ and $bn_1 \geq b'n_1$.

Lemma 3.3. $(\mathbb{Z}^2, \leq)$ *is a totally ordered abelian group*

Proof. We first show that $\leq$ defines a total order on $\mathbb{Z}^2$:

Totality: Let (a,b) and (a',b') be elements in $\mathbb{Z}^2$ such that $(a,b) \not\leq (a',b')$. By Definition 3.2 this is equivalent to saying that $n_1a+n_2b > n_1a'+n_2b'$ or $n_1a+n_2b = n_1a'+n_2b'$ with $a'n_2 < an_2$ or $bn_1 < b'n_1$. If $n_1a + n_2b > n_1a' + n_2b'$ we are finished since then by Definition 3.2 we have $(a',b') \leq (a,b)$. Therefore we assume that $n_1a+n_2b = n_1a'+n_2b'$ with $a'n_2 < an_2$ or $bn_1 < b'n_1$. Suppose that $a'n_2 < an_2$. This implies that n_2 is nonzero. We multiply the equation $n_1a + n_2b = n_1a' + n_2b'$ by n_2 and get the equation $n_1n_2a + n_2^2 b = n_1n_2a' + n_2^2 b'$. Let us assume $n_1 \leq 0$. From $a'n_2 < an_2$ we get that $n_1n_2a \leq n_1n_2a'$ which implies that $n_2^2 b \geq n_2^2 b'$. Since $n_2^2 > 0$ we get that $b \geq b'$. If instead we assume $n_1 > 0$ we would get $b \leq b'$. But in both cases we get $bn_1 \leq b'n_1$. By Definition 3.2 this means that $(a',b') \leq (a,b)$.

Transitivity: Let $(a,b), (a',b')$ and (a'',b'') be elements in $\mathbb{Z}$ such that $(a,b) \leq (a',b')$ and $(a',b') \leq (a'',b'')$. If $n_1a + n_2b < n_1a' + n_2b'$ or $n_1a' + n_2b' < n_1a'' + n_2b''$, the inequality $n_1a + n_2b < n_1a'' + n_2b''$ follows via the transitivity of $\leq$ on $\mathbb{Z}$. This implies $(a,b) \leq (a'',b'')$ by Definition 3.2. So assume that $n_1a+n_2b = n_1a'+n_2b' = n_1a''+n_2b''$. By Definition 3.2 we know have the inequalities $a'n_2 \geq an_2$, $bn_1 \geq b'n_1$, $a''n_2 \geq a'n_2$, $b'n_1 \geq b''n_1$. The transitivity of $\leq$ on $\mathbb{Z}$ implies that $a''n_2 \geq an_2$ and $bn_1 \geq b''n_1$. The case where $bn_1 < b'n_1$ works analogously. By Definition 3.2 this implies $(a,b) \leq (a'',b'')$.

Antisymmetry: Let (a,b) and (a',b') be elements in $\mathbb{Z}^2$ such that $(a,b) \leq (a',b')$ and $(a',b') \leq (a,b)$. This implies $an_1 + bn_2 = a'n_1 + b'n_2$ and $a'n_2 = an_2$ and $bn_1 = b'n_1$ by the antisymmetry of the usual total order on $\mathbb{Z}$. Since $(n_1, n_2) \neq (0,0)$ we know $n_1 \neq 0$ or $n_2 \neq 0$. Let us assume $n_1 \neq 0$. From $bn_1 = b'n_1$, we get $b = b'$. But this implies $a = a'$ via the equation $an_1 + bn_2 = a'n_1 + b'n_2$. The other case works analogously.

Compatibility: The only thing left to show is the compatibility of the total order with the additive structure of $\mathbb{Z}^2$. This means for all $(a,b), (a',b'), (c,d)$ in $\mathbb{Z}^2$ such that $(a,b) \leq (a',b')$ it follows that $(a,b) + (c,d) \leq (a',b') + (c,d)$. This immediately follows from Definition 3.2 and the fact that $(\mathbb{Z}, \leq)$ is a totally ordered abelian group. $\square$

Definition 3.4. We define a function $D : K[X_1, X_2] \to \mathbb{Z}^2 \cup \{-\infty\}$ by setting

$$D(f) := \sup\{(i,j) \mid a_{ij} \neq 0\}$$

where $f = \sum_{i,j\geq 0} a_{ij} X_1^i X_2^j$ and the total order on $\mathbb{Z}^2$ as in Definition 3.1.2.

Remark 3.5. In all the equations and inequalities involving elements of $G \cup \{-\infty\}$ we use the convention that the symbol $-\infty$ satisfies the following: $\forall g \in G : -\infty < g$ and $g + -\infty = -\infty$.

Definition 3.6. Let $(G, \leq, +)$ be a totally ordered abelian group. Let R be a commutative ring. We call a function $\delta : R \to G \cup \{-\infty\}$ a *degree function* if it satisfies the following properties:
For all $x, y \in R$:
(i) $\delta(xy) = \delta(x) + \delta(y)$
(ii) $\delta(x + y) \leq \max\{\delta(x), \delta(y)\}$
(iii) $\delta(x) = -\infty \Leftrightarrow x = 0$

Remark 3.7. Let $\delta : R \to G \cup \{-\infty\}$ be a degree function. Since $\delta(1) = \delta(1 \cdot 1) = 2\delta(1)$ we get $\delta(1) = 0$. Now we want to compute $\delta(-1)$. We have $0 = \delta(1) = \delta((-1) \cdot (-1)) = 2\delta(-1)$ which implies $\delta(-1) = 0$ since a totally ordered abelian group has no torsion elements.

Lemma 3.8. *For a degree function $\delta : R \to G \cup \{-\infty\}$ we have the following property: For all $x, y \in R$: if $\delta(x) \neq \delta(y)$ it follows that $\delta(x + y) = \max\{\delta(x), \delta(y)\}$.*

Proof. Assume $\delta(x) > \delta(y)$. We leave the case where $y = 0$ to the reader. Therefore we assume that y is nonzero. we know that

$$\delta(x) = \delta((x + y) + (-y)) \leq \max\{\delta(x + y), \delta(-y)\}.$$

If $\delta(x + y)$ were strictly smaller than $\max\{\delta(x), \delta(y)\} = \delta(x)$ it would imply $\delta(x) \leq \delta(-y) = \delta(-1) + \delta(y) = \delta(y)$ which is a contradiction. Therefore $\delta(x+y) = \max\{\delta(x), \delta(y)\}$. □

Lemma 3.9. *Let R be an integral domain and δ a degree-function on R. We can extend δ to $\tilde{\delta} : \mathrm{Quot}(R) \to G \cup \{-\infty\}$ by setting $\tilde{\delta}(\frac{a}{b}) = \delta(a) - \delta(b)$ for an $\frac{a}{b} \in \mathrm{Quot}(R)$. Then $\tilde{\delta}$ is well-defined and is again a degree-function on $\mathrm{Quot}(R)$.*

Proof. We leave the proof to the reader. □

Definition 3.10. Let G, G' be totally ordered abelian groups. We call a function $\phi : G \cup \{-\infty\} \to G' \cup \{-\infty\}$ a *homomorphism* of totally ordered abelian groups if it satisfies the following properties:
For all $g, h \in G \cup \{-\infty\}$:
(i) $g \leq h \Rightarrow \phi(g) \leq \phi(h)$
(ii) $\phi(g) = -\infty \Leftrightarrow g = -\infty$
(iii) $\phi(g + h) = \phi(g) + \phi(h)$

Lemma 3.11. *Let $\phi : G \cup \{-\infty\} \to G' \cup \{-\infty\}$ be a homomorphism of totally ordered abelian groups. Let S be a finite subset of $G \cup \{-\infty\}$. Then we have the following equation:*

$$\sup\{\phi(S)\} = \phi(\sup\{S\}) \tag{4}$$

Proof. This immediately follows by Definition 3.10 property (i) and if S is empty property (ii). □

Lemma 3.12. *Let $\delta : R \to G \cup \{-\infty\}$ be a degree function. Let $\phi : G \cup \{-\infty\} \to G' \cup \{-\infty\}$ be a homomorphism of totally ordered abelian groups. Then the composition $\phi \circ \delta$ is again a degree-function on R.*

Proof. We leave the proof to the reader. □

Lemma 3.13. *Both d and D are degree functions on $K[X_1, X_2]$.*

Proof. We first define the function $\phi : \mathbb{Z}^2 \cup \{-\infty\} \to \mathbb{Z} \cup \{-\infty\}$ by setting $\phi((a, b)) = n_1 a + n_2 b$ and $\phi(-\infty) = -\infty$ and notice this is a homomorphism of totally ordered abelian groups. Then by Lemma 3.11 it follows that $\phi \circ D = d$. Then Lemma 3.12 tells us that it is enough to prove that D is a degree function on $K[X_1, X_2]$. We leave the proof that D satisfies property (ii) and (iii) in Definition 3.6 to the reader and only prove property (i): Let f, g be elements in $K[X_1, X_2]$. The case where $fg = 0$ is clear since then f or g will be zero, so we assume that f and g are both nonzero. We write f and g in the following form:

$$f = \sum_{i,j \geq 0} a_{ij} X_1^i X_2^j, \; g = \sum_{i,j \geq 0} b_{ij} X_1^i X_2^j. \tag{5}$$

We write $D(f) = (i_0, j_0)$ and $D(g) = (i'_0, j'_0)$. The product fg can be written in the following form:

$$\sum_{\substack{i'',j'' \\ \geq 0}} \Big(\sum_{\substack{(i,j)+(i',j') \\ =(i'',j'')}} a_{ij} b_{i'j'} \Big) X_1^{i''} X_2^{j''} \tag{6}$$

If we have a (i'', j'') such that the inner sum is nonzero then we know that a term $a_{ij} b_{i'j'}$ which is nonzero must appear in the inner sum. For this term to be nonzero both a_{ij} and $b_{i'j'}$ have to be nonzero. This implies $(i, j) \leq D(f)$ and $(i', j') \leq D(g)$. And from these two inequalities we get that $(i'', j'') = (i, j) + (i', j') \leq D(f) + D(g)$. This gives us the inequality $D(fg) \leq D(f) + D(g)$.

Claim 3.14. *Let $(i, j), (i', j')$ be elements in $(\mathbb{Z}^{\geq 0})^2$ such that $(i, j) + (i', j') = D(f) + D(g)$ and the term $a_{ij} b_{i'j'}$ is nonzero. Then $(i, j) = (i_0, j_0)$ and $(i', j') = (i'_0, j'_0)$.*

Proof. Consider $(i, j, i', j') \in (\mathbb{Z}^{\geq 0})^4$ which differs from (i_0, j_0, i'_0, j'_0) and which satisfies the equation $(i, j) + (i', j') = D(f) + D(g)$. Let us assume $(i, j) \neq (i_0, j_0)$ or else we work with $(i', j') \neq (i'_0, j'_0)$. Then we must have $(i, j) < (i_0, j_0)$ or the other way around. Suppose $(i, j) < (i_0, j_0)$. This together with the equation $(i, j) + (i', j') = (i_0, j_0) + (i'_0, j'_0)$ implies $(i', j') > (i'_0, j'_0)$. This means that $b_{i'j'}$ is zero so the term $a_{ij} b_{i'j'}$ is also zero. □

This shows us that the inner sum for $(i'', j'') = D(f) + D(g)$ consist only of the term $a_{i_0 j_0} b_{i'_0 j'_0}$ which is nonzero. This implies the inequality $D(fg) \geq D(f) + D(g)$. Since we have both inequalities the equality $D(fg) = D(f) + D(g)$ follows. $\square$

Lemma 3.9 tells us that we can extend d and D to $K(X_1, X_2)$. We will still denote these extensions by the same letters d and D. The equation $\phi \circ D = d$ in the beginning of the proof of Lemma 3.13 tells us that D is a kind of refinement of d. The next lemma tells us this equation still holds for the extensions of d and D to $K(X_1, X_2)$

Lemma 3.15. *The equation $\phi \circ D = d$ holds on $K(X_1, X_2)$.*

Proof. We know the equation holds for polynomials. Let $q = \frac{f}{g}$ be a rational function where $f, g \in K[X_1, X_2]$ and g nonzero. We now have $\phi(D(q)) = \phi(D(f) - D(g)) = \phi(D(f)) - \phi(D(g)) = d(f) - d(g) = d(\frac{f}{g})$. $\square$

Definition 3.16. We define the function $|\cdot| : K[X_1, X_2] \to K[X_1, X_2]$ by setting

$$|f| = \sum_{\substack{i,j \geq 0 \\ n_1 i + n_2 j = d(f)}} a_{ij} X_1^i X_2^j. \tag{7}$$

It is called the leading term of f.

Definition 3.17. Let R, R' be a commutative rings. We call a function $\psi : R \to R'$ multiplicative if for all $x, y \in R : \psi(xy) = \psi(x)\psi(y)$.

Lemma 3.18. *If R and R' are integral domains and ψ a multiplicative function from R to R' such that $\psi(x) \neq 0$ for all $x \neq 0$, we can extend ψ to $\tilde{\psi} : \mathrm{Quot}(R) \to \mathrm{Quot}(R')$ by setting $\tilde{\psi}(\frac{a}{b}) := \frac{\psi(a)}{\psi(b)}$ where $\frac{a}{b} \in \mathrm{Quot}(R)$. The extension $\tilde{\psi}$ is well defined and is again a multiplicative function.*

Proof. We leave the proof to the reader. $\square$

Lemma 3.19. *The leading term function is multiplicative.*

Proof. Let f, g be elements in $K[X_1, X_2]$. We write f and g in the following form:

$$f = \sum_{i,j \geq 0} a_{ij} X_1^i X_2^j, \ g = \sum_{i,j \geq 0} b_{ij} X_1^i X_2^j. \tag{8}$$

Both $|fg|$ and $|f||g|$ are polynomials where the monomials $X_1^{i''} X_2^{j''}$ with nonzero coefficients satisfy $n_1 i'' + n_2 j'' = d(fg) = d(f) + d(g)$. Therefore to show the equality $|fg| = |f||g|$ it is enough to show that for any $(i'', j'') \in (\mathbb{Z}^{\geq 0})^2$ such that $n_1 i'' + n_2 j'' = d(f) + d(g)$, the corresponding coefficients of the monomial $X_1^{i''} X_2^{j''}$ are the same. In $|fg|$ this corresponding coefficient is the sum over all $a_{ij} b_{i'j'}$ such that $(i, j) + (i', j') = (i'', j'')$. In $|f||g|$ this coefficient is the sum over all $a_{ij} b_{i'j'}$ such that $n_1 i + n_2 j = d(f)$ and $n_1 i' + n_2 j' = d(g)$ and $(i, j) + (i', j') = (i'', j'')$. These sums are the same since for any $(i, j, i', j') \in (\mathbb{Z}^{\geq 0})^4$ such that $a_{ij} b_{i'j'}$ is nonzero and $(i, j) + (i', j') = (i'', j'')$, we know that $n_1 i + n_2 j \leq d(f)$ and $n_1 i' + n_2 j' \leq d(g)$. Since $(n_1 i + n_2 j) + (n_1 i' + n_2 j') = n_1 i'' + n_2 j'' = d(f) + d(g)$ we can conclude that $n_1 i + n_2 j = d(f)$ and $n_1 i' + n_2 j' = d(g)$. $\square$

Using Lemma 3.18 we extend the leading term function to $K(X_1, X_2)$, again denoted by $|\cdot|$.

Definition 3.20. We call an element f in $K(X_1, X_2)$ (n_1, n_2)-homogeneous if $|f| = f$.

Example 3.21. The notion of a $(1, 1)$-homogeneous polynomial coincides with the usual notion of a homogeneous polynomial.

Lemma 3.22. *Let f_1, f_2 be in $K(X_1, X_2)$ such that $d(f_1) = d(f_2) = c \in \mathbb{Z}$. Then we have*

$$d(f_1 + f_2) < c \Leftrightarrow |f_1| + |f_2| = 0. \tag{9}$$

Proof. In the case where f_1 and f_2 are polynomials the lemma is clear since $d(f_1+f_2) < c$ if and only if the highest terms of f_1 and f_2 cancel each other out, in other words $d(f_1 + f_2) < c$ if and only if $|f_1| + |f_2| = 0$. For the general case we write $f_1 = \frac{p_1}{q_1}$ and $f_2 = \frac{p_2}{q_2}$ with p_1, q_1, p_2, q_2 polynomials where q_1 and q_2 are nonzero. The inequality $d(f_1+f_2) < c$ is equivalent to $d(p_1q_2+p_2q_1) < c+d(q_1)+d(q_2) = d(p_1q_2) = d(p_2q_1)$. This is equivalent to $|p_1q_2| + |p_2q_1| = 0$ by applying the lemma in the case of polynomials. This equation is then equivalent to $|f_1| + |f_2| = 0$, using Lemma 3.19. □

Remark 3.23. Lemma 3.22 can be easily extended to a sum of more than 2 polynomials of the same (n_1, n_2)-degree.

Lemma 3.24. *Let f_1, f_2 be in $K(X_1, X_2)$ such that $d(f_1) > d(f_2)$. Then we have*

$$|f_1 + f_2| = |f_1|. \tag{10}$$

Proof. We notice $d(f_1 + f_2) = d(f_1)$ due to Lemma 3.8. Since $d((f_1 + f_2) + (-f_1)) = d(f_2) < d(f_1)$, Lemma 3.22 tells us that $|f_1 + f_2| + |-f_1| = 0$. This implies (10). □

Lemma 3.25. *Let f_1,f_2 be in $K(X_1, X_2)$ such that $|f_1|$ and $|f_2|$ are algebraically independent over K. Then we have*

$$|K[f_1, f_2]| \subseteq K[|f_1|, |f_2|]. \tag{11}$$

Proof. A nonzero element g in $|K[f_1, f_2]|$ is of the form $|p(f_1, f_2)|$ with p a nonzero polynomial in two variables. Let us write $p(Y_1, Y_2)$ in the following form:

$$p(Y_1, Y_2) = \sum_{i,j\geq 0} a_{ij} Y_1^i Y_2^j. \tag{12}$$

We now isolate the monomials of highest degree. If $(d(f_1), d(f_2)) = (0, 0)$ we set $p_0 := p$ and $\mu := 0$. Notice that the cases $d(f_1) = -\infty$ or $d(f_2) = -\infty$ do not occur since then $|f_1|$ and $|f_2|$ would satisfy the equation $|f_1||f_2| = 0$ which would contradict the assumption that $|f_1|$ and $|f_2|$ are algebraically independent over K. If $(d(f_1), d(f_2)) \neq (0, 0)$ we define p_0 to be equal to the leading term of p with respect to the $(d(f_1), d(f_2))$-degree function. We also define μ to be the $(d(f_1), d(f_2))$-degree of p.

Claim 3.26. $d(p_0(f_1, f_2)) = \mu$.

Proof. The element $p_0(f_1, f_2)$ is a sum of monomial expressions in f_1 and f_2 where each of those expressions has (n_1, n_2)-degree μ. If $d(p_0(f_1, f_2))$ were strictly smaller than μ, Lemma 3.22 with Remark 3.23 would imply $p_0(|f_1|, |f_2|) = 0$. But this is not possible since $|f_1|$ and $|f_2|$ are algebraically independent over K. □

By Lemma 3.24 we know that $|p(f_1, f_2)| = |p_0(f_1, f_2)+(\text{terms of lower degree})| = |p_0(f_1, f_2)|$. By Lemma 3.22 with Remark 3.23 and Claim 3.26 we know that

$$d(p_0(f_1, f_2) - p_0(|f_1|, |f_2|)) < \mu = d(p_0(f_1, f_2)) = d(p_0(|f_1|, |f_2|)).$$

This tells us that

$$|p_0(f_1, f_2)| = |p_0(|f_1|, |f_2|) + p_0(f_1, f_2) - p_0(|f_1|, |f_2|)| = |p_0(|f_1|, |f_2|)|$$

by using Lemma 3.24. Since $p_0(|f_1|, |f_2|)$ is already homogeneous (it is the sum of homogeneous elements of the same degree), we have $|p(f_1, f_2)| = |p_0(|f_1|, |f_2|)| = p_0(|f_1|, |f_2|)$ which lies in $K[|f_1|, |f_2|]$. □

4 Key lemmas

In this section we will prove some rather technical lemmas which play an important role in the proof of Theorem 2.0.2.

Lemma 4.1. *If f_1 and f_2 are non-zero (n_1, n_2)-homogeneous elements of $K(X_1, X_2)$ that are algebraically dependent over K, then there exists $\lambda \in K^\times$ such that*

$$f_1^{d(f_2)} = \lambda f_2^{d(f_1)}. \tag{13}$$

Proof. Since f_1 and f_2 are algebraically dependent over K, we have $p(f_1, f_2) = 0$ for some nonzero polynomial p. Since the case where $(d(f_1), d(f_2)) = (0, 0)$ is clear we assume $(d(f_1), d(f_2)) \neq (0, 0)$. We can assume p to be $(d(f_1), d(f_2))$-homogeneous by possibly replacing p by its $(d(f_1), d(f_2))$-leading term. Let $(d(f_1), d(f_2)) = (sd_1, sd_2)$ with $s = gcd(d(f_1), d(f_2))$ then p is also (d_1, d_2)-homogeneous. We write p in the following form:

$$p = \sum_{i,j\geq 0} a_{ij} Y_1^i Y_2^j. \tag{14}$$

Since p is (d_1, d_2)-homogeneous we know that if a_{ij} is nonzero, then $id_1 + jd_2$ is a constant independent of i and j. We fix an i_0 and j_0 such that $a_{i_0 j_0}$ is nonzero. For any other pair (i', j') such that $a_{i'j'}$ is nonzero we have the following equation:

$$i_0 d_1 + j_0 d_2 = i' d_1 + j' d_2. \tag{15}$$

Using the fact that d_1 and d_2 are coprime this equation implies that

$$(i', j') = (i_0 - kd_2, j_0 + kd_1) \tag{16}$$

for some k in $\mathbb{Z}$. If we divide out $f_1^{i_0} f_2^{j_0}$ from the equation $p(f_1, f_2) = 0$ and possibly multiply by an appropriate power of $f_1^{-d_2} f_2^{d_1}$ we get a nonzero polynomial equation for $f_1^{-d_2} f_2^{d_1}$ with coefficients in K. Since K is relatively algebraically closed in $K(X_1, X_2)$ we get that $f_1^{-d_2} f_2^{d_1}$ lies in $K^\times$. This implies the lemma. □

Definition 4.2. We call an element p in $K(X_1, X_2)$ a *proper power* if $p = \lambda q^a$ for some $q \in K(X_1, X_2)$ and $\lambda \in K^\times$ and $a \in \mathbb{Z}^{>1}$.

Lemma 4.3. *For every non-constant p in $K(X_1, X_2)$ we have $p = \lambda q^a$ for some $q \in K(X_1, X_2)$ and $\lambda \in K^\times$ and $a \in \mathbb{Z}^{\geq 1}$, where q is not a* proper power.

Proof. After fixing a system of representatives of irreducible polynomials $\{p_i \mid i \in I\}$ under the relation of associatedness, we uniquely decompose p into a product of integer powers of the p_i's. Let l denote the positive greatest common divisor of the exponents in this decomposition. If we divide each exponent by l we get a q. This q is not a proper power since its exponents in the decomposition are coprime. □

Lemma 4.4. *Let f and q be elements in $K(X_1, X_2)$ with q not a proper power. If we have an equation of the form $f^l = \lambda q^m$ with $(l, m) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ and $\lambda \in K^\times$, we can conclude $f = \tilde{\lambda} q^b$ for some $b \in \mathbb{Z}$ and $\tilde{\lambda} \in K^\times$.*

Proof. After fixing a system of representatives of irreducible polynomials $\{p_i \mid i \in I\}$ under the relation of associatedness, we look at the unique decompositions of f and q into products of integer powers of the p_i's. So we write

$$f = up_{i_1}^{e_1} p_{i_2}^{e_2} \cdot .... \cdot p_{i_n}^{e_n},\ q = vp_{i_1}^{e'_1} p_{i_2}^{e'_2} \cdot .... \cdot p_{i_n}^{e'_n},$$

with u and v nonzero elements in K and the i_j's pairwise distinct for $j = 1, .., n$. The fact that q is not a proper power implies that $\gcd(e'_1, ..., e'_n) \sim 1$. Using the equation $f^l = \lambda q^m$ we get the equations $le_i = me'_i$ with $i = 1, ..., n$. The coprimeness of $e'_1, ..., e'_n$ tells us that we can write 1 as a $\mathbb{Z}$-linear combination of $e'_1, ..., e'_n$. This together with the equations imply that $l \mid m$. Therefore we get the equations $le_i = lbe'_i$ for some b in $\mathbb{Z}$ and for all $i = 1, ..., n$. Since l is nonzero we get the equations $e_i = be'_i$ for all $i = 1, ...n$. These equations imply the lemma. □

Lemma 4.5. *Let $p_1, p_2 \in K(X_1, X_2)$ with $d(p_1) \neq 0$. Then there exist (n_1, n_2)-homogeneous $q_1, q_2 \in K(X_1, X_2)$ with q_1 not a proper power such that the following holds:*

$$|K[p_1^{\pm 1}, p_2]| \subseteq K[q_1^{\pm 1}, q_2] \text{ and } |p_1| = \lambda q_1^a, \tag{17}$$

for some $\lambda \in K^\times$ and $a \in \mathbb{Z}^{>0}$.

Proof. We first prove that to show (17) it is enough to prove

$$|K[p_1, p_2]| \subseteq K[q_1^{\pm 1}, q_2]\ . \tag{18}$$

Let w be an element of $K[p_1^{\pm 1}, p_2]$. Let $b \in \mathbb{Z}^{\geq 0}$ such that wp_1^b lies in $K[p_1, p_2]$. Then $|w| = |wp_1^b||p_1^{-b}|$ lies in $K[q_1^{\pm 1}, q_2]$ since $|wp_1^b|$ lies in $K[q_1^{\pm 1}, q_2]$ and $|p_1^{-b}| = |p_1|^{-b} = \lambda^{-b} q_1^{-ab}$ also lies in $K[q_1^{\pm 1}, q_2]$. Therefore (17) holds.
Now we prove the existence of q_1 and q_2 such that (17) holds. Lemma 4.3 gives us a q_1 not a proper power such that $|p_1| = \lambda q_1^a$ for some $\lambda \in K^\times$ and $a \in \mathbb{Z}^{\geq 1}$. We assume that p_2 is not an element of $K[p_1^{\pm 1}]$ since if it were, we could set $q_2 = 0$.

Case 1: $\forall h \in K[p_1^{\pm 1}] : |p_2 - h| \in K[|p_1|^{\pm 1}]$.

Claim 4.6. *There exists a sequence $(\mu_k p_1^{b_k})_{k\geq 1}$ in $K[p_1^{\pm 1}]$ with $\mu_k \in K^\times$ and $b_k \in \mathbb{Z}$ for all k in $\mathbb{Z}^{\geq 1}$, such that the sequence*

$$p_{2,k} := p_2 - \sum_{i=1}^{k} \mu_i p_1^{b_i}$$

satisfies $d(p_{2,k+1}) < d(p_{2,k})$ for all $k \in \mathbb{Z}^{\geq 0}$. Moreover if $d(p_1)$ is negative, only finitely many b_k's are negative.

Proof. We construct μ_k and b_k inductively. Suppose that $\mu_1 p_1^{b_1}$ upto $\mu_k p_1^{b_k}$ for a $k \in \mathbb{Z}^{\geq 0}$ are already constructed such that $d(p_{2,l+1}) < d(p_{2,l})$ for all $l = 0, .., k-1$. We know that $|p_{2,k}|$ lies in $K[|p_1|^{\pm 1}]$. Since $|p_{2,k}|$ is also homogeneous and nonzero we get that $|p_{2,k}|$ is of the form $\alpha |p_1|^\beta$ with α in $K^\times$ and β in $\mathbb{Z}$. We set $\mu_{k+1} := \alpha$ and $b_{k+1} := \beta$. By Lemma 3.22 it follows that

$$d(p_{2,k+1}) = d(p_{2,k} - \mu_{k+1} p_1^{b_{k+1}}) < d(p_{2,k}). \tag{19}$$

We notice that $b_{k+1} d(p_1) = d(p_{2,k})$ for all $k \in \mathbb{Z}^{\geq 1}$. For large enough k we know that $d(p_{2,k})$ is negative. If $d(p_1)$ is also negative it follows that for large enough k, the exponent b_{k+1} must be positive. □

We can assume that $d(p_1) < 0$ after possibly replacing p_1 and q_1 by p_1^{-1} and q_1^{-1}. Let f be a nonzero element in $K[p_1, p_2]$. We write f as $r(p_1, p_2)$ where r is an element in $K[Y_1, Y_2]$. Denote by t_2 the Y_2-degree of r. We now use the sequence $p_{2,k}$ constructed in Claim 4.6. Since there are only finitely many b_k's which are negative, $l := \inf\{b_k \mid k \in \mathbb{Z}^{\geq 1}\}$ exists and is finite. We now choose a k large enough such that $d(p_{2,k}) < d(f) + t_2 |l| d(p_1)$ and $d(p_{2,k})$ is negative. We notice that

$$r(p_1, p_2) = r(p_1, p_{2,k} + \sum_{i=1}^{k} \mu_i p_1^{b_i}) = \sum_{i \in \mathbb{Z}, j \geq 0} a_{ij} p_1^i p_{2,k}^j$$

with a_{ij} nonzero for only finitely many $(i, j) \in \mathbb{Z} \times \mathbb{Z}^{\geq 0}$, where we get the last expression by expanding out $r(p_1, p_{2,k} + \sum_{i=1}^{k} \mu_i p_1^{b_i})$. If there are b_i's in $b_1, ..., b_n$ which are negative we know that the smallest i, such that there exists a j such that a_{ij} is nonzero, is bounded from below by $t_2 l$. This means that multiplying f with $p_1^{t_2|l|}$ ensures that the resulting element $\tilde{f}$ lies in $K[p_1, p_{2,k}]$. Since $d(p_{2,k}) < d(\tilde{f})$ and $d(p_{2,k})$ is negative, monomial expressions in $\tilde{f}$ which contain $p_{2,k}$ have (n_1, n_2)-degree strictly less than $d(\tilde{f})$. Then $|\tilde{f}|$ is an element in $|K[p_1]|$ by Lemma 3.24. But $|K[p_1]|$ is contained in $K[|p_1|]$. This shows that $|f|$ is an element of $K[|p_1|^{\pm 1}]$. Therefore $q_2 = 0$ does the job.

Case 2 $\exists h \in K[p_1^{\pm 1}] : |p_1|$ and $|p_2 - h|$ are algebraically independent over K.
In this case the lemma follows via Lemma 3.25 with $q_2 = |p_2 - h|$ and the following equality:

$$K[p_1^{\pm 1}, p_2 - h] = K[p_1^{\pm 1}, p_2]. \tag{20}$$

For the remaining case we construct q_2 via induction on a. More precisely we assume the lemma is true for all $a' < a$ with $a' \in \mathbb{Z}^{\geq 1}$.

Case 3: $\exists h \in K[p_1^{\pm 1}] : |p_2 - h| \notin K[|p_1|^{\pm 1}]$ and $|p_1|$ and $|p_2 - h|$ are algebraically dependent over K.
Due to Lemma 4.1 we get that $|p_1|^{d(p_2-h)}$ is a scalar multiple of $|p_2 - h|^{d(p_1)}$. Then we notice that $|p_2 - h|^{d(p_1)}$ is a scalar multiple of a power of q_1. Therefore we can use Lemma 4.4 to show that $|p_2 - h| = \mu q_1^b$, for some $\mu \in K^\times$ and $b \in \mathbb{Z}$. Since $|p_2 - h|$ does not lie in $K[|p_1|^{\pm 1}]$ we know that a cannot divide b. This also shows that the base case $a = 1$ has already been taken care of in the cases we did before. Then we can write $b = ra + \tilde{a}$ with $r, \tilde{a} \in \mathbb{Z}$ and $0 < \tilde{a} < a$. This implies that $|(p_2 - h)p_1^{-r}| = \tilde{\mu} q_1^{\tilde{a}}$ where $\tilde{\mu} = \mu\lambda^{-r}$. This means for $p_1' = (p_2 - h)p_1^{-r}$ and $p_2' = p_1$ the induction hypothesis implies that a (n_1, n_2)-homogeneous element q_2' exists such that $|K[(p_2 - h)p_1^{-r}, p_1]| \subseteq K[q_1^{\pm 1}, q_2']$. The claim follows by setting $q_2 := q_2'$ and with these inclusions:

$$K[q_1^{\pm 1}, q_2'] \supseteq |K[(p_2 - h)p_1^{-r}, p_1^{\pm 1}]| \supseteq |K[p_1, p_2]|, \tag{21}$$

where the first inclusion follows by the same argument as in the beginning of the proof. □

We use this lemma to say something about automorphisms of the polynomial ring in Lemma 4.9.

Lemma 4.7. *Let f be an element of $K[X_1, X_2]$. Let $d_i \geq 0$ be the X_i-degree of f ($i = 1, 2$). Let d_1 and d_2 be both nonzero. If X_i^a appears in the (d_2, d_1)-leading term of f with a nonzero coefficient for some $i \in \{1, 2\}$ and some $a \in \mathbb{Z}^{\geq 1}$, it follows that $a = d_i$ and both terms $X_1^{d_1}$ and $X_2^{d_2}$ appear in the (d_2, d_1)-leading term of f with nonzero coefficients. Moreover $D(f) = (d_1, 0)$ where D denotes the (d_2, d_1)-bidegree.*

Proof. In the following proof d will always mean the (d_2, d_1)-degree function and $|\cdot|$ will always mean the (d_2, d_1)-leading term. WLOG we assume X_1^a appears in $|f|$ with a nonzero coefficient. Since d_1 is the X_1-degree of f it follows that a monomial $X_1^{d_1} X_2^l$ appears in f with a nonzero coefficient for some $l \in \mathbb{Z}^{\geq 0}$. Since X_1^a appears in $|f|$ and $a \leq d_1$ we get the following inequalities:

$$d_1 d_2 \geq a d_2 = d(X_1^a) \geq d(X_1^{d_1} X_2^l) = d_1 d_2 + d_1 l \geq d_1 d_2$$

It follows that all these inequalities are equalities, which shows that $a = d_1$. Since X_1^a appears in $|f|$ with a nonzero coefficient we also know that $d(f) = d(X_1^a) = d_1 d_2$. Since d_2 is the X_2-degree of f we get that a monomial $X_1^{l'} X_2^{d_2}$ appears in f with a nonzero coefficient for some $l' \in \mathbb{Z}^{\geq 0}$. We then get the following inequalities:

$$d_1 d_2 = d(f) \geq d(X_1^{l'} X_2^{d_2}) = l' d_2 + d_1 d_2 \geq d_1 d_2$$

Again it follows that all these inequalities are equalities, which shows that $l' = 0$. Therefore $X_2^{d_2}$ appears in $|f|$. Now we only need to show that $D(f) = (d_1, 0)$. We know that $d(f) = d(X_1^{d_1}) = d_1 d_2$. We now make the following claim:

Claim 4.8. *For all $(l,k) \in (\mathbb{Z}^{\geq 0})^2$ such that $X_1^l X_2^k$ appears in f with a nonzero coefficient, we have $(d_1, 0) \geq (l,k)$.*

Proof. Since $d(f) = d_1 d_2$ we now that $d(X_1^l X_2^k) = d_2 l + d_1 k \leq d_1 d_2$. By Definition 3.2 we only need to check that when $d_2 l + d_1 k = d_1 d_2$ we have $d_1 d_1 \geq l d_1$ and $k d_2 \geq 0 d_2$ which is clearly the case since $l \leq d_1$ and $0 d_2 = 0$. □

Since $X_1^{d_1}$ appears in f the claim implies that $D(f_1) = (d_1, 0)$. □

Lemma 4.9. *Let $(f_1, f_2) \in \mathrm{Aut}_K(K[X_1, X_2])$. Let $d_i \geq 0$ be the X_i-degree of f_1 ($i = 1, 2$). Then d_1 divides d_2 or the other way around. Moreover we have $d(f_1) = d_1 d_2$ where d is the (d_2, d_1)-degree function.*

Proof. In the following proof D will always mean the (d_2, d_1)-bidegree function.

Case 1: $d_1 d_2 = 0$.
In this case $d_1 = 0$ or $d_2 = 0$ which implies d_1 divides d_2 or the other way around. For the second statement of the lemma we assume $d_1 \neq 0$. The case $d_2 \neq 0$ works analogously. Then $d_2 = 0$ which implies that f_1 lies in $K[X_1]$. Since $d(X_1) = 0$ and $f_1 \neq 0$ we have $d(f_1) = 0$.

Case 2: $d_1 d_2 \neq 0$.
This implies that $d(f_1) \neq 0$. Now by Lemma 4.5 there exists (d_2, d_1)-homogeneous q_1, q_2 with q_1 not a proper power such that the following holds:

$$|K[f_1, f_2]| \subseteq K[q_1^{\pm 1}, q_2] \text{ and } |f_1| = \lambda q_1^a, \tag{22}$$

for some $\lambda \in K^\times$ and $a \in \mathbb{Z}^{>0}$. Since (f_1, f_2) is an automorphism we know that X_1 and X_2 are elements of $K[f_1, f_2]$. Moreover they are both (d_2, d_1)-homogeneous, so by (22) they both lie in $K[q_1^{\pm 1}, q_2]$. This means that $D(K[q_1^{\pm 1}, q_2])$ contains both $(1, 0)$ and $(0, 1)$. Therefore $(\mathbb{Z}^{\geq 0})^2$ is contained in $D(K[q_1^{\pm 1}, q_2])$. We now distinguish two subcases:

Case 2a: $D(q_1)$ and $D(q_2)$ are $\mathbb{Z}$-independent:
We have

$$(\mathbb{Z}^{\geq 0})^2 \subseteq D(K[q_1^{\pm 1}, q_2] \setminus \{0\}) = \mathbb{Z} D(q_1) + \mathbb{Z}^{\geq 0} D(q_2). \tag{23}$$

The equality in (23) is due to the fact that for any nonzero element in $K[q_1^{\pm 1}, q_2]$ the monomial expressions in q_1 and q_2 have distinct bidegrees. This is due to the $\mathbb{Z}$-independence of $D(q_1)$ and $D(q_2)$. The right-hand side of (23) can be viewed as a half-plane in $\mathbb{Z}^2$. We notice that $D(q_1)$ lies in the first quadrant (by (22)). But since this half plane has the line defined by $D(q_1)$ as boundary and contains the first quadrant, $D(q_1)$ must lie on one of the coordinate axes. Furthermore we know that $(D(q_1), D(q_2))$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^2$. Therefore the transformation matrix defined by this basis must have determinant ± 1. This is only possible if $D(q_1) = (1, 0)$ or $(0, 1)$. Now we know that $D(q_1) = a^{-1} D(f_1)$, in other words we know that $D(f_1) = (a, 0)$ or $(0, a)$. Lemma 4.7 then implies that $D(f_1) = (d_1, 0)$ and both terms $X_1^{d_1}$ and $X_2^{d_2}$ appear in $|f_1|$ with nonzero coefficients. Therefore $a = d_1$ and by using (22) we see that $d_2 = X_2$-degree of $|f_1| = X_2$-degree of $q_1^{d_1} = (X_2\text{-degree of } q_1)d_1$. This shows that d_1 divides d_2. Since $D(f_1) = (d_1, 0)$ we get $d(f_1) = d_1 d_2$.

Case 2b: $D(q_1)$ and $D(q_2)$ are $\mathbb{Z}$-dependent:
This means that $D(q_1)$ and $D(q_2)$ generate a cyclic subgroup of $\mathbb{Z}^2$. We can choose a generator (i_1, i_2) of this cyclic subgroup in the following fashion:

$$D(q_r) = j_r(i_1, i_2),\ k_1 j_1 + k_2 j_2 = 1,\ j_r, i_r, k_r \in \mathbb{Z}\ (r = 1, 2). \tag{24}$$

So we have $D(q_1^{j_2}) = j_1 j_2 (i_1, i_2) = D(q_2^{j_1})$. Then there exists a μ in $K^\times$ such that $D(q_1^{j_2} - \mu q_2^{j_1}) < j_1 j_2 (i_1, i_2)$. This can be shown by considering the case where q_1 and q_2 are polynomials, then it easily extends to rational functions. We now claim that $q_1^{j_2} - \mu q_2^{j_1}$ is nonzero. If this were not the case it would follow by Lemma 4.4 that q_2 would lie in $K[q_1^{\pm 1}]$. This would imply that $D(K[q_1^{\pm 1}, q_2] \setminus \{0\}) = \mathbb{Z}D(q_1)$. But $(1, 0)$ and $(0, 1)$ lie in $D(K[q_1^{\pm 1}, q_2] \setminus \{0\})$ therefore $D(q_1)$ would generate all of $\mathbb{Z}^2$ which is not possible. We set $(i'_1, i'_2) := D(q_1^{j_2} - \mu q_2^{j_1})$. Since q_1 and q_2 are (d_2, d_1)-homogeneous and $q_1^{j_2} - \mu q_2^{j_1}$ is nonzero we have

$$i'_1 d_2 + i'_2 d_1 = d(q_1^{j_2} - \mu q_2^{j_1}) = j_1 j_2 (i_1 d_2 + i_2 d_1). \tag{25}$$

by Lemma 3.22. If $(i_1, i_2), (i'_1, i'_2)$ were $\mathbb{Z}$-dependent, we would have $(i'_1, i'_2) = j_1 j_2 (i_1, i_2)$ using the last equality in (25) . But this is not the case so we now know that $(i_1, i_2), (i'_1, i'_2)$ are $\mathbb{Z}$-independent. We define $q'_1 := q_1^{k_1} q_2^{k_2}$ and $q'_2 := q_1^{j_2} q_2^{-j_1}$. We now claim the following chain of equalities:

$$\begin{aligned}(\mathbb{Z}^{\geq 0})^2 &\subseteq D(K[q_1^{\pm 1}, q_2^{\pm 1}] \setminus \{0\}) = D(K[q_1'^{\pm 1}, q_2'^{\pm 1}] \setminus \{0\}) \\ &= D(K[q'_1, q'_2] \setminus \{0\}) + \mathbb{Z}(i_1, i_2) = D(K[q'_1, q'_2 - \mu] \setminus \{0\}) + \mathbb{Z}(i_1, i_2) \\ &= \mathbb{Z}^{\geq 0}(i'_1, i'_2) + \mathbb{Z}(i_1, i_2) = \mathbb{Z}(j_1 a)^{-1} D(f_1) + \mathbb{Z}^{\geq 0}(i'_1, i'_2).\end{aligned}$$

The first equality follows from $K[q_1^{\pm 1}, q_2^{\pm 1}] = K[q_1'^{\pm 1}, q_2'^{\pm 1}]$. For the second equality we notice that any element w' in $K[q_1'^{\pm 1}, q_2'^{\pm 1}] \setminus \{0\}$ can always be written as $w q_1'^{l_1} q_2'^{l_2}$ with $w \in K[q'_1, q'_2] \setminus \{0\}$ and some $(l_1, l_2) \in \mathbb{Z}^2$. Since $D(q'_1) = (i_1, i_2)$ and $D(q'_2) = (0, 0)$ we get that

$$D(w') = D(w) + l_1 D(q'_1) + l_2 D(q'_2) \in D(K[q'_1, q'_2] \setminus \{0\}) + \mathbb{Z}(i_1, i_2).$$

The inclusion in the other direction is clear. The third equality follows from

$$K[q'_1, q'_2] \setminus \{0\} = K[q'_1, q'_2 - \mu] \setminus \{0\}.$$

For the fourth equality we notice that

$$D(q'_2 - \mu) = D((q_1^{j_2} - \mu q_2^{j_1}) q_2^{-j_1}) = (i'_1, i'_2) - j_1 j_2 (i_1, i_2) \in \mathbb{Z}^{\geq 0}(i'_1, i'_2) + \mathbb{Z}(i_1, i_2).$$

Since (i_1, i_2) and (i'_1, i'_2) are $\mathbb{Z}$-independent it follows that $D(q'_1)$ and $D(q'_2 - \mu)$ are $\mathbb{Z}$-independent. Now we can use the same argument as in Case 2a to show that

$$D(K[q'_1, q'_2 - \mu]) = \mathbb{Z}^{\geq 0}(i_1, i_2) + \mathbb{Z}^{\geq 0}((i'_1, i'_2) - j_1 j_2 (i_1, i_2)).$$

This explains the fourth equality. Just as in Case 2a we can use the halfplane argument to show that $(j_1 a)^{-1} D(f_1) = (1, 0)$ or $(0, 1)$. Then Lemma 4.7 implies $D(f_1) = (d_1, 0)$. Therefore $(j_1 a)^{-1} D(f_1) = (1, 0)$. Since $(i_1, i_2) = (j_1 a)^{-1} D(f_1) = (1, 0)$ we get that $D(q_r) = (j_r, 0)$ for $r = 1, 2$. This implies $d(q_r) = j_r d_2$ for $r = 1, 2$. Since q_1 and q_2 are (d_2, d_1)-homogeneous we have the following inclusion:

$$d(K[q_1^{\pm 1}, q_2]) \subseteq \mathbb{Z} d(q_1) + \mathbb{Z}^{\geq 0} d(q_2). \tag{26}$$

The right-hand side of this inclusion is contained in $\mathbb{Z} d_2$, and since $K[q_1^{\pm 1}, q_2]$ contains the element X_2 we know that $d(X_2) = d_1$ lies in $\mathbb{Z} d_2$. This shows that d_2 divides d_1. Since $D(f_1) = (d_1, 0)$ we get $d(f) = d_1 d_2$. □

Remark 4.10. If $(f_1, f_2) \in \mathrm{Aut}_K(K[X_1, X_2])$ then $(f_2, f_1) \in \mathrm{Aut}_K(K[X_1, X_2])$ by precomposing (f_1, f_2) with (X_2, X_1). This means Lemma 4.9 also holds for f_2 and its X_1-degree d_1' and X_2-degree d_2'.

5 The proof of Theorem 2.8

In this section we prove a theorem from which we can deduce the tameness of $\mathrm{Aut}_K(K[X_1, X_2])$. The theorem shows more or less that if an automorphism (g_1, g_2) is not affine, we can lower the total degree of g_1 by precomposing an X_2-based shear to (g_1, g_2). By an easy induction it follows that every element in $\mathrm{Aut}_K(K[X_1, X_2])$ is a composition of affine automorphisms and shears, therefore tame. We remind the reader that with deg we mean the $(1, 1)$-degree, in other words the total degree.

Lemma 5.1. *Let f, g_1 and g_2 be nonzero elements in $K[X_1, X_2]$ such that f is not a constant. Let $d_i \geq 0$ be the X_i-degree of f $(i = 1, 2)$. We assume $d(f) = d_1 d_2$ where d denotes the (d_2, d_1)-degree function. If $\deg(g_1^{d_1}) \neq \deg(g_2^{d_2})$ then*

$$\deg(f(g_1, g_2)) = \max\{\deg(g_1^{d_1}),\ \deg(g_2^{d_2})\}.$$

Proof. We assume $\deg(g_1^{d_1}) > \deg(g_2^{d_2})$. The other case works analogously. This implies $d_1 \neq 0$.

Claim 5.2. *For terms of the form $X_1^i X_2^j$ with nonzero coefficient in f and $i < d_1$ we have $\deg(g_1^i g_2^j) < \deg(g_1^{d_1})$*

Proof. Let $X_1^i X_2^j$ be a term in f with a nonzero coefficient and $i < d_1$. Since $d(f) = d_1 d_2$ we know that $d_2 i + d_1 j \leq d_1 d_2$. We can rewrite this inequality as $j \leq d_2 - \frac{d_2}{d_1} i$. This gives us the following inequality:

$$\begin{aligned}\deg(g_1^i g_2^j) &= i \deg(g_1) + j \deg(g_2) \leq i \deg(g_1) + (d_2 - \frac{d_2}{d_1} i) \deg(g_2) \\ &= d_2 \deg(g_2) + i(\deg(g_1) - \frac{d_2}{d_1} \deg(g_2)).\end{aligned}$$

Since $i < d_1$ and $\deg(g_1) - \frac{d_2}{d_1} \deg(g_2) > 0$ we get $\deg(g_1^i g_2^j) < \deg(g_1^{d_1})$. □

Since $d_1 \neq 0$ and $d(f) = d_1 d_2$ the term $X_1^{d_1}$ appears in f with a nonzero coefficient. Therefore using Claim 5.1 we see that $\deg(f(g_1, g_2)) = \deg(g_1^{d_1})$. $\square$

Theorem 5.3. *Let* (g_1, g_2) *be an automorphism of* $K[X_1, X_2]$ *such that* $\deg(g_1) \geq \deg(g_2)$. *Then either* (g_1, g_2) *is affine or there exists a unique* $\mu \in K^\times$ *and* $d \in \mathbb{Z}^{>0}$ *such that* $\deg(g_1 - \mu g_2^d) < \deg(g_1)$.

Proof. If $\deg(g_1) = 1$, then $\deg(g_2) = 1$ and (g_1, g_2) is affine. So we can assume $\deg(g_1) > 1$. Let (f_1, f_2) be the inverse automorphism, this is also not affine. Therefore f_1 or f_2 is not linear. Let i be in $\{1, 2\}$ such that f_i is not linear. Let $d_r = X_r$-degree of f_i $(r = 1, 2)$. We view $K[X_1, X_2]$ with the (d_2, d_1)-degree function d. We know that $f_i(g_1, g_2) = X_i$ and by Lemma 4.9 with Remark 4.10 that $d(f_i) = d_1 d_2$.

Claim 5.4. $\deg(g_1^{d_1}) = \deg(g_2^{d_2})$.

Proof. We prove this via contradiction. We assume $\deg(g_1^{d_1}) \neq \deg(g_2^{d_2})$. Then we know by Lemma 5.1 that

$$\max\{\deg(g_1^{d_1}), \deg(g_2^{d_2})\} = \deg(f_i(g_1, g_2)) = \deg(X_i) = 1.$$

Since $\deg(g_1) > 1$ we know that $d_1 = 0$. This would then imply $\deg(g_2^{d_2}) = 1$. This tells us that $d_2 = 1$. But this contradicts the assumption that f_i is not linear. $\square$

Claim 5.4 tells us that d_1 and d_2 are nonzero. Since $\deg(g_1) \geq \deg(g_2)$ we know that $d_1 \leq d_2$, so using Lemma 4.9 we know that d_1 divides d_2. This means $dd_1 = d_2$ for some d in $\mathbb{Z}^{>0}$. This implies $\deg(g_1) = \deg(g_2^d)$. Let us denote by $\tilde{g}_1$ and $\tilde{g}_2$ the $(1, 1)$-leading terms of g_1 and g_2. Since $f_1(g_1, g_2) = X_1$ it follows that $\tilde{g}_1$ and $\tilde{g}_2$ are algebraically dependent over K. Then we can use Lemma 4.1 to deduce that $\tilde{g}_1^{\deg(g_2)} = \mu' \tilde{g}_2^{\deg(g_1)}$ for some μ' in $K^\times$. This implies $\tilde{g}_1 = \mu \tilde{g}_2^{d}$ for some $\mu \in K^\times$ by considering the prime factorizations of $\tilde{g}_1$ and $\tilde{g}_2$. This implies that $\deg(g_1 - \mu g_2^d) < \deg(g_1)$. $\square$

Theorem 5.5. *Every element in* $Aut_K(K[X_1, X_2])$ *is tame.*

Proof. Let (g_1, g_2) be an element of $\mathrm{Aut}_K(K[X_1, X_2])$. We do induction on $\deg(g_1) + \deg(g_2)$. By possibly precomposing with the automorphism (X_2, X_1) we can assume $\deg(g_1) \geq \deg(g_2)$. The base case is $\deg(g_1) + \deg(g_2) = 2$. In this case the theorem is true since affine automorphisms are tame. Let us assume $\deg(g_1) + \deg(g_2) > 2$. Then by using Theorem 5.3 we know that $\deg(g_1 - \mu g_2^d) < \deg(g_1)$ for some μ in $K^\times$ and d in $\mathbb{Z}^{>0}$. This tells us that by precomposing with the X_2-based shear $(X_1 - \mu X_2^d, X_2)$ we get an automorphism ϕ for which we can use our induction hypothesis since $\deg(\phi(X_1)) + \deg(\phi(X_2)) < \deg(g_1) + \deg(g_2)$. Therefore ϕ is tame. This tells us that (g_1, g_2) is tame. $\square$

Remark 5.6. We notice that Theorem 5.3 actually gives us an algorithm to compute a factorization of an automorphism into affine and shear automorphisms. We denote by $\bar{g}$ the $(1, 1)$-leading term of a g in $K[X_1, X_2]$. The algorithm is given on the next page. We notice that this algorithm not only computes a factorization into affine and shear automorphisms of a given automorphism (g_1, g_2), but also checks for any given pair $(f, g) \in K[X_1, X_2]$ with total degrees greater than or equal to 1, if this pair defines an automorphism of $K[X_1, X_2]$. To illustrate this algorithm we apply it to the following concrete example, where $\mathrm{char}(K) \neq 2$:

$$(g_1, g_2) = (X_1^4 - 2(X_1^3 - X_1^2 X_2 - X_1^2 + X_1 X_2) + X_2^2 + X_1 + X_2, X_1^2 - X_1 + X_2).$$

The algorithm will display the following:

$$(X_1 + X_2^2, X_2), (X_1 + X_2, X_2), (X_2, X_1), (X_1 + \frac{1}{4}X_2^2, X_2), (-X_1 + X_2, 2X_1).$$

This tells us that our example defines an automorphism. To get the factorization we have to read the output in a backward fashion:

$$(g_1, g_2) = (-X_1 + X_2, 2X_1) \circ (X_1 + \frac{1}{4}X_2^2, X_2) \circ (X_2, X_1) \circ (X_1 + X_2, X_2) \circ (X_1 + X_2^2, X_2)$$

Algorithm 1 $TameFact(g_1, g_2)$

H

if (g_1, g_2) is affine **then**
    if $det(g_1, g_2) \neq 0$ **then**
        print (g_1, g_2).
        return
    else
        print "not an automorphism"
        return
    end if
end if

if $deg(g_1) < deg(g_2)$ **then**
    print "(X_2, X_1),"
    $TameFact(g_2, g_1)$
    return
end if

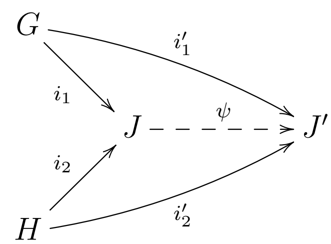
if $deg(g_2) \mid deg(g_1)$ **then**
    $d \leftarrow \frac{deg(g_1)}{deg(g_2)}$
    if $\frac{\bar{g_2}^d}{\bar{g_1}} \in K$ **then**
        $\mu \leftarrow \frac{\bar{g_2}^d}{\bar{g_1}}$
        print "$(X_1 + \mu X_2^d, X_2)$,"
        $TameFact(g_1 - \mu g_2^d, g_2)$
        return
    else
        print "not an automorphism"
        return
    end if
else
    print "not an automorphism"
    return
end if

6 Introduction to amalgamated products

Theorem 2.0.2 not only says that every automorphism is tame, but also gives us a decomposition of the group as an amalgamated product of two subgroups. In this section we will introduce the basic definitions concerning such products.

We first introduce a special case of the amalgamated product, called the free product of two groups:

Definition 6.1. Given two groups G and H a free product of G and H is a group J together with homomorphisms $i_1 : G \to J$, $i_2 : H \to J$ satisfying the following universal property: For all groups J' with homomorphisms $i_1' : G \to J'$, $i_2' : H \to J'$ there exists a unique homomorphism $\psi : J \to J'$ such that $\psi \circ i_1 = i_1'$ and $\psi \circ i_2 = i_2'$.



If such a J exists, it is unique up to unique isomorphism by the universal property. We then denote J by $G * H$.

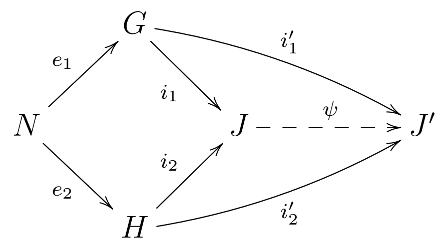
Lemma 6.2. *The free product $G * H$ exists for any groups G and H.*

Proof. (Sketch of construction)
A word in G and H is a formal expression of the form $s_1 s_2 \cdots s_n$, where each s_i is either an element of G or an element of H. Such a word may be reduced using the following operations:
(i) Remove an instance of the identity element (of either G or H)
(ii) Replace a pair of the form $g_1 g_2$ by its product in G, or a pair $h_1 h_2$ by its product in H.
It follows that every reduced word is an alternating product of elements of G and elements of H, e.g. $g_1 h_1 g_2 h_2 \cdots g_k h_k$. The group J whose elements are the reduced words in G and H, under the operation of concatenation followed by reduction satisfies the universal property of the free product together with the obvious embeddings from G and H into J. For a more detailed proof we refer the reader to [8]. □

The name free product is slightly misleading since in the category theoretical sense it is the coproduct in the category of groups and not the product. Now we introduce the definition of an amalgamated product.

Definition 6.3. Given groups G, H and N with homomorphisms $e_1 : N \to G$ and $e_2 : N \to H$, an *amalgamated product of G and H over N* is a group J together with homomorphisms $i_1 : G \to J$ and $i_2 : H \to J$ with $i_1 \circ e_1 = i_2 \circ e_2$ such that the following universal property is satisfied: For all groups J' with homomorphisms $i_1' : G \to J'$,

$i'_2 : H \to J'$ with $i'_1 \circ e_1 = i'_2 \circ e_2$ there exists a unique homomorphism $\psi : J \to J'$ such that $\psi \circ i_1 = i'_1$ and $\psi \circ i_2 = i'_2$.



As in the case of a free product, if such a J exists it is unique up to unique isomorphism by the universal property. We then denote J by $G *_N H$.

Lemma 6.4. *For any given groups G, H and N with homomorphisms $e_1 : N \to G$ and $e_2 : N \to H$, $G *_N H$ exists.*

Proof. (Sketch of construction)
$G *_N H$ can be realised as a factor group of $G * H$. We define L to be the smallest normal subgroup of $G * H$ which contains all expressions of the form $e_1(n)(e_2(n))^{-1}$ with n in N. It can be shown that $(G*H)/L$ with the obvious homomorphisms satisfies the universal property of the amalgamated product. □

Example 6.5. $G *_N H$ with N the trivial group is just the free product $G * H$.

Lemma 6.6. *Let G be a group and G_1 and G_2 be subgroups of G which together generate G. For any element $g \in G \setminus (G_1 \cup G_2)$ we have*

$$g = x_{j_1} \cdot ... \cdot x_{j_l}$$

with $l \in \mathbb{Z}^{>1}$ and $(j_1, ..., j_l) \in \{1, 2\}^l$ such that for all $i = 1, ..., j - 1$ we have $j_i \neq j_{i+1}$ and for all $i = 1, ..., j$ we have $x_{j_i} \in G_{j_i} \setminus (G_1 \cap G_2)$.

Proof. We write g as a product of elements in G_1 and G_2. In this product we simplify by replacing neighbouring factors by their product if they lie in the same subgroup. We stop this simplification algorithm when we achieve the product as claimed in the lemma or there is only one factor left in the product. Since every simplification reduces the number of factors the algorithm will terminate. But the case where only one factor is left does not happen for g since by assumption g lies in $G \setminus (G_1 \cup G_2)$. □

Lemma 6.7. *Let G_1, G_2 and N be groups with homomorphisms $e_1 : N \to G_1$ and $e_2 : N \to G_2$. For any element $s \in G_1 *_N G_2 \setminus (i_1(G_1) \cup i_2(G_2))$ we have*

$$s = i_{j_1}(x_{j_1}) \cdot ... \cdot i_{j_l}(x_{j_l})$$

with $(j_1, ..., j_l) \in \{1, 2\}^l$ such that for all $i = 1, ..., j - 1$ we have $j_i \neq j_{i+1}$ and for all $i = 1, ..., j$ we have $x_{j_i} \in G_{j_i} \setminus e_{j_i}(N)$.

Proof. The universal property of the amalgamated product implies that $G_1 *_N G_2$ is generated by $i_1(G_1)$ and $i_2(G_2)$. And we also have $i_1(e_1(N)) = i_2(e_2(N)) \subseteq i_1(G_1) \cap i_2(G_2)$. Therefore we can apply Lemma 6.6 to prove this lemma. □

Remark 6.8. The amalgamated product is used in topology to describe the fundamental group of a topological space. Let $X = U \cup V$ be a path-connected topological space with U and V open path-connected subsets of X such that their intersection is path-connected. Then the following holds:

$$\pi_1(X) = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$$

where the homomorphisms from $\pi_1(U \cap V)$ into $\pi_1(U)$ and $\pi_1(V)$ are the ones induced by the inclusions from $U \cap V$ into U and V. This is known as the Seifert-Van Kampen theorem.

7 Group actions on trees

The idea of the proof of the product decomposition is to study the group action of $\mathrm{Aut}_K(K[X_1, X_2])$ on a tree. To do this we will introduce basic definitions concerning group actions on graphs and we will also define the graph on which $\mathrm{Aut}_K(K[X_1, X_2])$ will act. The fact that the graph really is a tree is also shown in this section.

We recall some basic definitions regarding graphs:

Definition 7.1. A *graph* Γ is a pair (V, E) where V is a set and $E \subseteq \{$subsets of V consisting of 2 elements$\}$. The set V is called the set of vertices and E the set of edges.

Definition 7.2. A *vertex* u in a graph $\Gamma = (V, E)$ is called a neighbour of a vertex v if $\{v, u\} \in E$

Definition 7.3. A (positive) *weight function* on a graph $\Gamma = (V, E)$ is a function $w : V \to \mathbb{R}^{>0}$.

Definition 7.4. A *path* in a graph $\Gamma = (V, E)$ is an ordered sequence of vertices $(u_0, ..., u_k)$ for some $k \in \mathbb{Z}^{\geq 0}$ such that $\{u_i, u_{i+1}\} \in E$ for all $i = 0, ..., k-1$. We call k the length of the path.

Definition 7.5. A graph $\Gamma = (V, E)$ is called *connected* if for any vertices v_0 and v_1 there exists a path in Γ which starts in v_0 and ends in v_1.

Definition 7.6. A *circuit* in a graph $\Gamma = (V, E)$ is a path $(u_0, ..., u_k)$ of length strictly greater than 1 with $u_0 = u_k$ and for any $i \neq j$ such that $u_i = u_j$ it follows that $\{i, j\} = \{0, k\}$.

Definition 7.7. A *tree* $T = (V, E)$ is a graph which is connected and contains no circuits.

The weight function is only needed in this section to prove that a certain graph is a tree using the following lemma:

Lemma 7.8. *Let w be a weight function on the graph $\Gamma = (V, E)$ such that neighbouring vertices have different weights, and there exists a positive constant C such that for any $x \neq y$ in $w(V)$ we have $|x - y| \geq C$. If there exists a vertex $v_0 \in V$ such that all neighbours of v_0 have higher weight and every vertex $u \neq v_0$ has a unique neighbour of lower weight, we can conclude that Γ is a tree.*

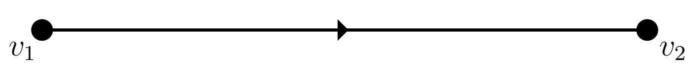
Proof. Γ **is connected:** It suffices to show that for any $u \neq v_0$ there exists a path from u to v_0: We define a sequence of vertices beginning with u in an iterative fashion. If the vertex we are at is not v_0, the next vertex in the sequence is the unique neighbour of lower weight. If we arrive at v_0 the sequence terminates. The assumption on the absolute difference between two weights tells us that a weight cannot be lowered infinitely many times without becoming negative. This means that the sequence must terminate. Therefore we get a path from u to v_0.
Γ **has no circuits:** If it had a circuit we could choose the vertex $\tilde{u}$ of highest weight in the circuit. This cannot be the vertex v_0. Therefore $\tilde{u}$ has a unique neighbour of lower weight, but since we are in a circuit there are at least two neighbours of lower weight which gives us a contradiction. □

Definition 7.9. An orientation on a graph $\Gamma = (V, E)$ is a pair of functions $t : E \to V$ and $o : E \to V$ which satisfy the following properties: $\forall e \in E$:
(i) $o(e) \in e$ and $t(e) \in e$
(ii) $o(e) \neq t(e)$
A graph together with an orientation is called an oriented graph.

Definition 7.10. Let G be a group and $\Gamma = (V, E, o, t)$ an oriented graph. A group action of G on Γ is a pair of group actions of G on the sets V and E with the following property:
$\forall g \in G : \forall e \in E : o(ge) = go(e)$ and $t(ge) = gt(e)$

Remark 7.11. We notice that for a group action of G on an oriented graph $\Gamma = (V, E, o, t)$ the group action on the set V already determines the group action on E. This is due to the fact that $ge = g\{v_1, v_2\} = \{gv_1, gv_2\}$ for g in G and $e = \{v_1, v_2\}$ in E.

Let $\Gamma = (V, E, o, t)$ be the oriented graph whose vertices are the K-subspaces of $K[X_1, X_2]$ and the orientation and edges are induced by the inclusions. The group $G := \mathrm{Aut}_K(X_1, X_2)$ acts on Γ by sending a K-subspace of $K[X_1, X_2]$ to its image via an automorphism. This action clearly respects the orientation of Γ. Now we look at the subgraph T_0 consisting of the two vertices $v_1 = K + KX_2$ and $v_2 = K + KX_1 + KX_2$ and the edge connecting them:



Let T be the subgraph generated from T_0 by G. This means that $T = (V', E', o, t)$ is defined in the following way (the orientation is just the one induced from Γ):

$$V' := Gv_1 \cup Gv_2, \; E' := \{\{u_1, u_2\} \in E \mid \exists g \in G : g\{v_1, v_2\} = \{u_1, u_2\}\}. \tag{27}$$

We also define a weight function on T:

Lemma 7.12. *The function $w : V' \to \mathbb{R}^{>0}$ defined by the following formula:*

$$w(K + Kg_2) = \deg(g_2), \; w(K + Kg_1 + Kg_2) = -\frac{1}{2} + \max\{\deg(g_1), \deg(g_2)\}, \tag{28}$$

for $(g_1, g_2) \in \mathrm{Aut}_K(K[X_1, X_2])$ is well defined.

Proof. A vertex u in the graph T is the image of $K+KX_2$ or $K+KX_1+KX_2$ under an automorphism (g_1, g_2). Therefore the vertex u is of the form $K+Kg_2$ or $K+Kg_1+Kg_2$. In the case where $u = K + Kg_2$ if $u = K + Kg_2'$ for a possibly different automorphism (g_1', g_2') it follows that $g_2' = \lambda g_2 + \mu$ for some λ in $K^\times$ and μ in K. This tells us that $\deg(g_2') = \deg(g_2)$. In the case where $u = K + Kg_1 + Kg_2$ let $u = K + Kg_1' + Kg_2'$ for a possibly different automorphism (g_1', g_2'). This means we can express g_1' and g_2' in terms of g_1 and g_2:

$$g_1' = \lambda_1 + \mu_1 g_1 + \epsilon_1 g_2,\ \ g_2' = \lambda_2 + \mu_2 g_1 + \epsilon_2 g_2, \tag{29}$$

for some $\lambda_i, \mu_i, \epsilon_i \in K$ for $i = 1, 2$. This tells us that the transformation matrix M, which transforms coordinates in the basis $(1, g_1', g_2')$ into coordinates in the basis $(1, g_1, g_2)$, is of the following form:

$$M = \begin{pmatrix} 1 & \lambda_1 & \lambda_1 \\ 0 & \mu_1 & \mu_2 \\ 0 & \epsilon_1 & \epsilon_2 \end{pmatrix}. \tag{30}$$

Since M is invertible this implies

$$0 \neq \det(M) = \det(\begin{pmatrix} \mu_1 & \mu_2 \\ \epsilon_1 & \epsilon_2 \end{pmatrix})$$

Therefore (μ_1, ϵ_1) and (μ_2, ϵ_2) are linearly independent. This implies that it is not possible that $\mu_1 = \mu_2 = 0$ or $\epsilon_1 = \epsilon_2 = 0$. This means in the case where $\deg(g_1) \neq \deg(g_2)$ we have $\max\{\deg(g_1), \deg(g_2)\} = \max\{\deg(g_1'), \deg(g_2')\}$. In the case where $\deg(g_1) = \deg(g_2)$ we have the same equality since it is not possible that in both the terms $\mu_1 g_1 + \epsilon_1 g_2$ and $\mu_2 g_1 + \epsilon_2 g_2$ the $(1,1)$-leading terms of g_1 and g_2 cancel each other out. Again this is due to the linear independence of (μ_1, ϵ_1) and (μ_2, ϵ_2). ☐

Lemma 7.13. *Let (g_1, g_2) be in $\mathrm{Aut}_K(K[X_1, X_2])$. Then the neighbours of $K + Kg_2$ in the graph T are the vertices*

$$K + K(g_1 + p(g_2)) + Kg_2$$

for each $p \in K[X_2]$.

Proof. Let x be a neighbour of $K + Kg_2$. Since G acts on the graph T we know that applying $(g_1, g_2)^{-1}$ to the edge $\{K+Kg_2, x\}$ gives us again an edge in the graph T. This edge will consist of $K + KX_2$ and a neighbour x'. We know that this edge comes from the edge $\{K+KX_2, K+KX_1+KX_2\}$ via an automorphism (g_1', g_2'). Since $K+KX_2$ gets sent to itself we know that g_2' is linear and lies in $K[X_2]$. For (g_1', g_2') to be surjective, g_1' must have X_1-degree 1. Using Lemma 4.9 this implies g_1' lies in $KX_1 + q$ where q lies in $K[X_2]$. This means x' is of the form $K + Kg_1' + Kg_2' = K + K(X_1 + p) + KX_2$ for some $p \in K[X_2]$. Conversely any $p \in K[X_2]$ defines a neighbour of $K + KX_2$ via the formula $K + K(X_1 + p) + KX_2$. The claim follows by applying (g_1, g_2) to the edge $\{K + KX_2, x'\}$. ☐

Lemma 7.14. *Let* (g_1, g_2) *be in* $\mathrm{Aut}_K(K[X_1, X_2])$. *Then the neighbours of* $K + Kg_1 + Kg_2$ *are the vertices*

$$K + K(\lambda_1 g_1 + \lambda_2 g_2)$$

where $(\lambda_1, \lambda_2) \in K^2 \setminus \{(0, 0)\}$.

Proof. Now let x be a neighbour of $K + Kg_1 + Kg_2$. Like in the previous lemma we transform the edge $\{x, K + Kg_1 + Kg_2\}$ via $(g_1, g_2)^{-1}$. We then get the edge $\{x', K + KX_1 + KX_2\}$. Since this is an edge in the graph T it comes from the edge $\{K+KX_2, K+KX_1+KX_2\}$ via some automorphism. But this automorphism stabilizes the vertex $K + KX_1 + KX_2$, so it is affine. This tells us that x' is of the form $K + K(\lambda_1 X_1 + \lambda_2 X_2)$ with λ_1, λ_2 in K not both zero. Conversely any vertex of that form is a neighbour of $K + KX_1 + KX_2$. The claim follows by applying (g_1, g_2) to the edge $\{x', K + KX_1 + KX_2\}$. □

Lemma 7.15. *In the graph* T *every vertex* u *aside from* $K + KX_1 + KX_2$ *has a unique neighbour of lower weight. And every neighbour of* $K + KX_1 + KX_2$ *has higher weight than* $K + KX_1 + KX_2$.

Proof. **Case** $u \in Gv_1$:
This means that u is of the form $K + Kg_2$ for an automorphism (g_1, g_2). Using Theorem 5.5 we can assume that $\deg(g_1) \leq \deg(g_2)$ by possibly precomposing (g_1, g_2) with X_2-based shears. Then Lemma 7.13 gives us a description of all neighbours of the vertex u. We see that the vertices $K + K(g_1 + p(g_2)) + Kg_2$ where $\deg(p) > 1$ have higher weight than the vertex $K + Kg_2$. If $\deg(p) \leq 1$ we get the vertex $K + K(g_1 + p(g_2)) + Kg_2 = K + Kg_1 + Kg_2$. This vertex has lower weight than $K + Kg_2$ and is therefore the unique neighbour of $K + Kg_2$ with lower weight than $K + Kg_2$.
Case $u \in Gv_2$:
In this case u is of the form $K + Kg_1 + Kg_2$ for an automorphism (g_1, g_2) (not affine since $u \neq K + KX_1 + KX_2$). We can assume $\deg(g_1) \neq \deg(g_2)$ by using Theorem 5.5 to correct the automorphism if $\deg(g_1)$ were equal to $\deg(g_2)$. Since the correction will be via the automorphism $(X_1 - \mu X_2, X_2)$ for some $\mu \in K^\times$, the vertex $K + Kg_1 + Kg_2$ stays the same. Since Lemma 7.14 gives us a description of the neighbours of u we see that depending on the degrees of g_1 and g_2 one of the vertices $K + Kg_1$ or $K + Kg_2$ will be the unique neighbour of $K + Kg_1 + Kg_2$ of lower weight. Lemma 7.14 also tells us that the neighbours of $K + KX_1 + KX_2$ are of the form $K + K(\lambda_1 X_1 + \lambda_2 X_2)$ with λ_1, λ_2 in K not both zero. Therefore all neighbours of $K + KX_1 + KX_2$ have higher weight than $K + KX_1 + KX_2$. □

Lemma 7.16. *The graph* T *is a tree*

Proof. Let w be the weight function defined in Lemma 7.12. It follows from the definition of w that neighbouring vertices in T have different weights. It is also clear from the definition that the absolute difference between two different weights is bounded by $\frac{1}{2}$ from below. Due to Lemma 7.15 we know that the weight function w satisfies the remaining hypotheses of Lemma 7.8, which finishes the proof. □

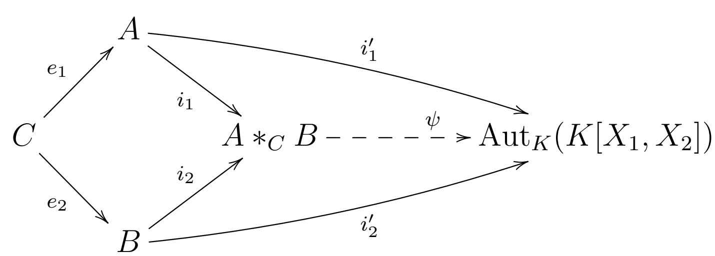
8 Proof of the decomposition

In this section we use the fact that T is a tree to prove the decomposition of $\mathrm{Aut}_K(K[X_1, X_2])$ as an amalgamated product.

We define A to be the group of affine automorphisms of $K[X_1, X_2]$ and B the group of X_2-based de Jonquiéres automorphisms and $C = A \cap B$. A more explicit description of B and C are

$$B = \{(\alpha X_1 + f(X_2), \beta X_2 + \gamma) \in \mathrm{Aut}_K(K[X_1, X_2]) \mid \alpha, \beta \in K^\times, \gamma \in K, f \in K[X_2]\},$$
$$C = \{(\alpha X_1 + \beta X_2 + \gamma, \epsilon X_2 + \theta) \in \mathrm{Aut}_K(K[X_1, X_2]) \mid \alpha, \beta, \gamma, \epsilon, \theta \in K, \alpha\epsilon \neq 0\}.$$

With the inclusions of A and B into $\mathrm{Aut}_K K[X_1, X_2])$ we get a natural group homomorphism ψ from $A *_C B$ into $\mathrm{Aut}_K(K[X_1, X_2])$ via the universal property of the amalgamated product.

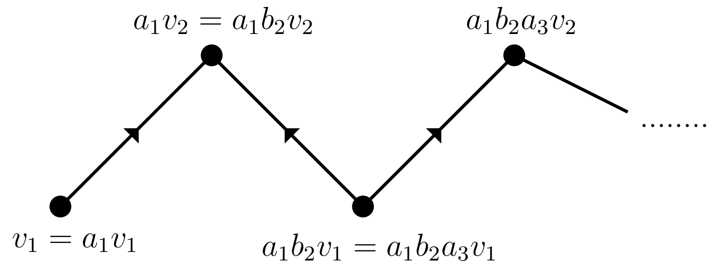


Theorem 8.1. *ψ is an isomorphism*

Proof. **Surjectivity**: To prove this it is enough to show that A and B together generate all of $\mathrm{Aut}_K(K[X_1, X_2])$: We first analyse the neighbours of v_1 and v_2 in the graph T. We notice that the stabilizer of v_1 is B and the stabilizer of v_2 is A. A neighbour x of v_1 comes from the edge $\{v_1, v_2\}$ via an automorphism g. This means that g fixes v_1 and therefore lies in B. So x is of the form gv_2 with g in B. The same argument shows that any neighbour of v_2 is of the form $g'v_1$ with g' in A. Let G' be the subgroup of $\mathrm{Aut}_K(K[X_1, X_2])$ generated by A and B. Let T' be the subgraph of Γ generated by T_0 via the subgroup G'. We notice that T' is a subgraph of T. An arbitrary vertex of T' is of the form hv_1 or hv_2 for an h in G'. A neighbour of hv_1 in T will be of the form hgv_2 with g in B since by applying h^{-1} to the edge consisting of hv_1 and its neighbour we get an edge in T connecting v_1 and a neighbour. The same argument shows that a neighbour of hv_2 in T is of the form $hg'v_1$ with g' in A. Since hg and hg' lie in G', the graph T' is closed under passage to neighbours in T. Since T is connected we get $T' = T$. This tells us that for any g in G we have $gv_1 = g'v_1$ for some g' in G'. This means $g'^{-1}g$ lies in B which implies that g lies in G'.

Injectivity: Assume $\ker(\psi) \neq \{1\}$. Then there exists an element z in $\ker(\psi)$ which is not the identity. We realise $A *_C B$ as $(A * B)/L$ where L is the smallest normal subgroup generated by elements of the form $i_1(c)i_2(c)^{-1}$ with $c \in C$. The map ψ then sends an element $g_1 \cdot \ldots \cdot g_n L$ to $g_1 \cdot \ldots \cdot g_n$ in $\mathrm{Aut}_K(K[X_1, X_2])$. This shows that z cannot be of the form gL with g in A or B or else $\psi(gL) = g = id$ which is a contradiction since we assumed z not to be the identity. Lemma 6.7 then tells us that z can be written as

$z = a_1 b_2 \cdot \ldots \cdot a_{k-1} b_k L$ with a_i in $A \setminus C$ and b_i in $B \setminus C$. It is not necessary that the word starts with an element in $A \setminus C$ and end in an element $B \setminus C$ but all the other cases can be handled similarly. The element z defines a path in T as seen in the following diagram:



This path ends at $\psi(z)v_1$. Since $\psi(z)$ is the identity the path ends at vertex v_1. In the path there is no backtracking or else somewhere in the path an edge is followed by the same edge which would imply that for some l we have $a_l \in C$ or $b_l \in C$ which is not possible. Therefore this path contains a circuit which gives us a contradiction since T is a tree. □

Remark 8.2. We notice that in the injectivity part of the proof of Theorem 8.1 we also show the following property of $\mathrm{Aut_K}(K[X_1, X_2])$: A nonempty alternating product of elements in $A \setminus C$ and $B \setminus C$ is not equal to the identity.

9 Case $n \geq 3$

In this section we illustrate some results on the case $n \geq 3$. A natural question to ask is if the automorphisms of polynomial rings over fields in more than two variables are tame. This turns out to be wrong. The first counterexample is the Nagata automorphism in a polynomial ring in 3 variables. As the name suggests the counterexample is due to a mathematician named Masayoshi Nagata who conjectured that this automorphism would not be tame. This conjecture was proven in 2003 by I.P. Shestakov and U.U. Umirbaev [10] for fields of characteristic 0. In this section the field K has characteristic 0.

Definition 9.1. The automorphism

$$(X - 2Y(Y^2 + XZ) - Z(Y^2 + XZ)^2, Y + Z(Y^2 + XZ), Z)$$

of $K[X, Y, Z]$ is called the Nagata automorphism.

Remark 9.2. The Nagata automorphism fixes $Y^2 + XZ$. Therefore its inverse is

$$(X + 2Y(Y^2 + XZ) - Z(Y^2 + XZ)^2, Y - Z(Y^2 + XZ), Z).$$

Theorem 9.3. *(Shestakov-Umirbaev) If K has characteristic zero the Nagata automorphism is not tame.*

One of the reasons that Nagata thought this automorphism wasn't tame was due to the following lemma.

Lemma 9.4. *Since the Nagata automorphism ϕ fixes Z we can view it as an element of* $\mathrm{Aut}_{K[Z]}((K[Z])[X,Y])$. *Then ϕ is not in $T_2(K[Z])$.*

Proof. Consider the natural extension of ϕ to $(K(Z))[X,Y]$ and call it ψ. We denote by $J(K(Z))$ resp. $J(K[Z])$ the set of Y-based de Jonquières automorphisms of $K(Z)[X,Y]$ resp. $K[Z][X,Y]$. We also denote by $A(K(Z))$ resp. $A(K[Z])$ the set of affine automorphisms of $K(Z)[X,Y]$ resp. $K[Z][X,Y]$. We can factorize ψ in the following form:

$$\psi = \sigma_1\sigma_2\sigma_1^{-1},\ \sigma_1 = (X + Y^2/Z, Y),\ \sigma_2 = (X, Z^2X + Y).$$

We notice that σ_1 is in $J(K(Z)) \setminus A(K(Z))$ and σ_2 is in $A(K(Z)) \setminus J(K(Z))$. Let us assume that ϕ lies in $\mathrm{T}_2(K[Z])$. This implies that we can write ψ^{-1} in the following form:

$$\psi^{-1} = \lambda_1\tau_1\lambda_2...\tau_n\lambda_{n+1},\ \tau_i \in J(K[Z]), \lambda_i \in A(K[Z]),\ i = 1, ..., n+1.$$

Furthermore we can also choose τ_i such that they do not lie in $J(K[Z]) \cap A(K[Z])$ for $i = 1, ..., n$. The same for λ_i for $i = 1, ..., n+1$ except for λ_1 or λ_{n+1} there is the possibility that at least one of them is the identity. Now we get the following equation:

$$\lambda_1\tau_1\lambda_2...\tau_n\lambda_{n+1}\sigma_1\sigma_2\sigma_1^{-1} = \mathrm{id}$$

Remark 8.2 tells us that λ_{n+1} is the identity. And since $\tau_n\sigma_1$ is in $J(K(Z))$ Remark 8.2 tells us that $\tau_n\sigma_1$ also lies in $A(K(Z))$. We know τ_n is of the following form:

$$\tau_n = (a_1X + f(Y), a_2Y + \gamma),\ a_1, a_2 \in K^\times\ \gamma \in K\ f \in (K[Z])[Y].$$

This means $\tau_n\sigma_1(X) = a_1X + f(Y) + \frac{1}{Z}(a_2Y + \gamma)^2$. Since $f(Y)$ cannot kill the term $\frac{1}{Z}a_2^2Y^2$ we get a contradiction with the fact that $\tau_n\sigma_1$ is affine. □

Martha K. Smith showed in 1989 [10] that the Nagata automorphism becomes tame after extending it to a polynomial ring in four variables by sending the new variable to itself. This leads us to the notion of stable tameness.

Definition 9.5. An element ϕ in $\mathrm{Aut}_K(K[X_1, .., X_n])$ is called *stably tame* if there exists an m in $\mathbb{Z}^{\geq 0}$ such that the natural extension of ϕ to an automorphism of $K[X_1, ..., X_n, X_{n+1}, ..., X_m]$ by leaving $X_{n+1}, ..., X_m$ fixed, is tame.

Now we will go into the details of Martha K. Smith's proof. For this we first introduce some definitions regarding derivations on polynomial rings. Let us assume $R = K[X_1, ..., X_n]$.

Definition 9.6. A *K-derivation* on R is a map $D : R \to R$ such that following properties hold:
For all $f, g \in R$:
(i) $D(f + g) = D(f) + D(g)$
(ii) $D(fg) = fD(g) + D(f)g$
(iii) $D(K) = \{0\}$
The set of all K-derivations on R is denoted by $\mathrm{Der_K}(R)$.

Definition 9.7. Let D be a K-derivation. We define the set

$$\mathrm{Nil}(D) := \{f \in R \mid \exists n \in \mathbb{Z}^{\geq 1} : D^n(f) = 0\},$$

where D^n is D composed with itself n times.

Lemma 9.8. *Let D be a K-derivation. Then* $\mathrm{Nil}(D)$ *is a K-subalgebra of R.*

Proof. Due to $D(K) = 0$ we know that 1 and 0 lie in $\mathrm{Nil}(D)$. Since we have property (i) in Definition 9.6 we know that $\mathrm{Nil}(D)$ is closed under addition. To show that $\mathrm{Nil}(D)$ is closed under multiplication we use the following formula:

$$D^n(fg) = \sum_{i=0}^{n} \binom{n}{k} D^i(f) D^{n-i}(g), \;\; f, g \in R, \;\; n \in \mathbb{Z}^{\geq 0}.$$

This can be shown via induction using property (ii) in Definition 9.6. If we take f and g in $\mathrm{Nil}(D)$ and choosing $n \in \mathbb{Z}^{\geq 0}$ big enough, the formula tells us that $D^n(fg) = 0$. Therefore fg lies in $\mathrm{Nil}(D)$. The property that $\mathrm{Nil}(D)$ is also a K-subspace of R follows from the fact that $D(K) = \{0\}$ and that $\mathrm{Nil}(D)$ is closed under multiplication. □

Definition 9.9. A K-derivation D is called *locally nilpotent* if $\mathrm{Nil}(D) = R$. The set of all locally nilpotent K-derivations is denoted by $\mathrm{LND_K}(R)$.

Definition 9.10. A K-derivation D is called triangular if $D(X_i) \in K[X_{i+1}, ..., X_n]$ for all $i = 1, ..., n$.

Lemma 9.11. *Triangular K-derivations are locally nilpotent.*

Proof. Let D be a triangular K-derivation. Due to Lemma 9.8 it suffices to show that the X_i lie in $\mathrm{Nil}(D)$ for all $i = 1, ..., n$. We notice that $D(X_n)$ lies in K so therefore also in $\mathrm{Nil}(D)$. This implies that X_n lies in $\mathrm{Nil}(D)$. Since $D(X_{n-1})$ lies in $K[X_n]$ which is contained in $\mathrm{Nil}(D)$, it follows that X_{n-1} lies in $\mathrm{Nil}(D)$. If we repeat this argument recursively we get that the X_i lie in $\mathrm{Nil}(D)$ for all $i = 1, ..., n$. □

Remark 9.12. A K-derivation D is uniquely determined by $(D(X_1), ..., D(X_n))$. The unique K-derivation D such that $(D(X_1), ..., D(X_n)) = (f_1, ..., f_n)$ for $(f_1, ..., f_n)$ in R^n, can be written in the following form:

$$D = \sum_{i=1}^{n} f_i \frac{\partial}{\partial X_i}.$$

Lemma 9.13. *The map* $\exp : LND_K(R) \to \mathrm{Aut}_K(R)$ *defined by the formula*

$$\exp(D) = \sum_{i=0}^{\infty} \frac{1}{i!} D^i,$$

is well defined and satisfies the following property: For D and E in $LND_K(R)$ such that $D \circ E = E \circ D$, it follows that $\exp(D + E) = \exp(D) \circ \exp(E)$.

Proof. We leave the proof to the reader to look up in [12] page 26. □

Example 9.14. Let us consider the triangular K-derivation $-2Y\frac{\partial}{\partial X}+Z\frac{\partial}{\partial Y}$ on $K[X,Y,Z]$. We notice that (Y^2+XZ) lies in the kernel of this K-derivation. This implies that $(Y^2+XZ)(-2Y\frac{\partial}{\partial X}+Z\frac{\partial}{\partial Y})$ is a locally nilpotent K-derivation. An easy computation shows that $\exp((Y^2+XZ)(-2Y\frac{\partial}{\partial X}+Z\frac{\partial}{\partial Y}))$ is equal to the Nagata automorphism.

Lemma 9.15. *Let D be a locally nilpotent K-derivation on R. Let w be in* $\ker(D)$. *We extend D to $R[T]$ by setting $D(T):=0$. We define $\tau\in\operatorname{Aut}_K(R[T])$ by setting $\tau(X_i):=X_i$ for $i=1,..,n$ and $\tau(T):=T+w$. Then we have the following equation:*

$$\exp(wD)=\tau^{-1}\circ\exp(-TD)\circ\tau\circ\exp(TD).$$

Proof. We will show the the following equivalent equation:

$$\exp(TD)\circ\tau\circ\exp(wD)\circ\exp(-TD)=\tau. \tag{31}$$

Due to Lemma 9.13 and the fact that $-TD$ and wD commute with each other, we know that $\exp(wD)\circ\exp(-TD)=\exp((w-T)D)$. Now we compute

$$(\tau\circ\exp((w-T)D))(X_i)=\tau(\sum_{j=0}^{\infty}\frac{1}{j!}(w-T)^jD^j(X_i))=\exp(-TD)(X_i),$$

for $i=1,...,n$. And since $D(T)=0$ we have

$$(\tau\circ\exp((w-T)D))(T)=\tau(T)=w+T,$$

by composing these two equations with $\exp(TD)$ we get (31). □

Theorem 9.16. *Let D be a triangular K-derivation on R. Let w be in* $\ker(D)$. *We extend D to $R[T]$ by setting $D(T):=0$. Then $exp(wD)$ is a tame automorphism of $R[T]$.*

Proof. Since TD is a triangular K-derivation of $R[T]$ it follows that $\exp(TD)$ is a triangular automorphism(See Definition 2.6). This implies that $\exp(TD)$ is tame. Since τ in Lemma 9.15 is also tame we get that $\exp(wD)$ is tame. □

Due to this theorem and Example 9.14 we now see that the Nagata automorphism is stably tame.

In general a lot is still unknown about the case $n=3$, such as if there exist automorphisms which are not stably tame. It is still being studied on how to weaken the notion of tameness in a natural way such that it generates the full automorphism group. One such notion is that of tamizability.

Definition 9.17. An element ϕ in $\operatorname{Aut}_K(K[X_1,..,X_n])$ is called *tamizable* if there exists a Ψ in $\operatorname{Aut}_K(K[X_1,..,X_n])$ such that $\Psi\circ\phi\circ\Psi^{-1}$ is tame.

It is not known if all automorphisms are tamizable in the case $n=3$. Further conjectures on $\operatorname{Aut}_K(K[X_1,..,X_n])$ are described in [13].

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