A Common Generalization of the Conjectures of André-Oort, Manin-Mumford, and Mordell-Lang

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1 Introduction

Let S be a mixed Shimura variety over the field of complex numbers $\mathbb{C}$. By definition an irreducible component of a mixed Shimura subvariety of S, or of its image under a Hecke operator, is called a *special subvariety* of S.

Consider any irreducible closed subvariety $Z \subset S$. Since any intersection of special subvarieties is a finite union of special subvarieties, there exists a unique smallest special subvariety containing Z. We call it the *special closure of* Z and denote it by S_Z. We call the dimension of S_Z the *amplitude of* Z, and the codimension of Z in S_Z the *defect of* Z. The defect measures how far Z is away from being special; in particular Z is special if and only if its defect is zero. Moreover Z is called *Hodge generic* if S_Z is an irreducible component of S, that is, if Z is not contained in any special subvariety of codimension > 0.

For any point $s \in S$ the amplitude and the defect of $\{s\}$ coincide and are called the *amplitude of* s. The points of amplitude zero in S are precisely the special points in S. Moreover s is called *Hodge generic* if $\{s\}$ is Hodge generic.

Conjecture 1.1 *Consider a mixed Shimura variety S over $\mathbb{C}$, an integer d, and a subset $\Xi \subset S$ of points of amplitude $\leq d$. Then any irreducible component Z of the Zariski closure of Ξ has defect $\leq d$.*

Clearly this is equivalent to the following contrapositive version:

Conjecture 1.2 *Consider a mixed Shimura variety S over $\mathbb{C}$ and an irreducible closed subvariety Z. Then the intersection of Z with the union of all special subvarieties of S of dimension $< \dim S_Z - \dim Z$ is not Zariski dense in Z.*

Moreover, since these conjectures are invariant under Hecke operators, one may assume that S_Z is an irreducible component of a mixed Shimura subvariety S' of S. Replacing S by S' then leads to the following equivalent formulation:

Conjecture 1.3 *Consider a mixed Shimura variety S over $\mathbb{C}$ and a Hodge generic irreducible closed subvariety Z. Then the intersection of Z with the union of all special subvarieties of S of codimension $> \dim Z$ is not Zariski dense in Z.*

The aim of this note is to propose and explain these conjectures, and to relate them to other conjectures and known results. Conjecture 1.1 for $d = 0$ is precisely the André-Oort conjecture, which has been established in special cases or under additional assumptions by Moonen [13] [14], André [1], Edixhoven [8], [9], [10],

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Edixhoven-Yafaev [11], and Yafaev [23], [24]; and an analogue by Breuer [5]. For a recent survey see the Bourbaki talk by Noot [15]. (In [17] the author did not comment on the relation of the André-Oort conjecture with applications and with further problems, because he did not strive for completeness of that kind. But as a nice application he should have mentioned the work of Cohen and Wüstholz [6] and Cohen [7], who apply the case proved by Edixhoven-Yafaev [11] to transcendence and special values of hypergeometric functions. For a more thorough survey see Noot [15].)

In Section 3 we show that Conjecture 1.1 implies the conjecture on generalized Hecke orbits from [17, Conj. 1.6]. In Section 5 we rephrase it in two ways for subvarieties contained in a fiber of a Shimura family $A \to S$ of semiabelian varieties. The resulting conjectures about subvarieties of semiabelian varieties have been studied independently by Bombieri-Masser-Zannier [3], [4], Viada [22], Rémond-Viada [20], and Ratazzi [18], [19], who proved different special cases. Both these conjectures and the one from [17, Conj. 1.6] imply the Mordell-Lang conjecture, which is also a theorem. Furthermore, in Section 6 we apply Conjecture 1.1 to subvarieties of families of semiabelian varieties, obtaining in particular a relative version of the Manin-Mumford conjecture, motivated by a question of André.

The author hopes that these correlations with other conjectures and results are justification enough for the proposed conjectures. At least he found no trivial counterexample, though the conjectures are numerically sharp by Propositions 2.1 and 6.4. Also, there is no discussion of possible strategies of proof, or of related conjectures or generalizations involving height estimates or points of small height.

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2 Sharpness

Conjecture 1.1 is numerically sharp in the following sense.

Proposition 2.1 *Consider a mixed Shimura variety S over $\mathbb{C}$ which possesses a mixed Shimura subvariety of dimension d. Then there exists an irreducible closed subvariety $Z \subset S$ of defect d containing a Zariski dense set of points of amplitude $\leq d$.*

Proof. Fix a mixed Shimura subvariety $S' \subset S$ of dimension d. After replacing S and S' by suitable finite coverings we may assume that both are smooth. We choose a locally closed embedding $S \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$ and construct Z as an irreducible component of $S \cap L$ for a sufficiently general linear subspace $L \subset \mathbb{P}^n_{\mathbb{C}}$ of codimension d.

On the one hand we can require that L meets S' transversally at some point $s' \in Z$. It then also meets S transversally at s', so that Z is smooth at s' of codimension d in S. On the other hand we may still vary L in a nonempty open subset of its parameter space, so the possible subvarieties Z sweep out a nonempty open subset of S. As the set of Hodge generic points in S is dense for the analytic topology, we may therefore assume that Z contains a Hodge generic point. Then Z itself is Hodge generic, and hence of defect d. (This can also be achieved using Bertini's theorem.) Let Ξ denote the set of points in Z of amplitude $\leq d$, and $\overline{\Xi}$ its Zariski closure in Z. To finish the proof, we must show that $\overline{\Xi} = Z$.

For this let (P, X) be the mixed Shimura datum underlying S and (P', X') the mixed Shimura subdatum underlying S'. Let $X^+ \subset X$ and $X'^+ \subset X' \cap X^+$ be connected components and $\Gamma \subset P(\mathbb{Q})$ and $\Gamma' \subset P'(\mathbb{Q})$ arithmetic subgroups, such that $\Gamma\backslash X^+$ and $\Gamma'\backslash X'^+$ are the irreducible component of S, resp. of S', that contains Z. Let $\pi\colon X^+ \twoheadrightarrow S$ denote the projection, write $s' = \pi(x')$ for a point $x' \in X'^+$, and let $\tilde{Z}$ denote the irreducible component of $\pi^{-1}(Z)$ containing x'. Then X'^+ and $\tilde{Z}$ are complex analytic submanifolds of X^+ of complementary dimension that meet transversally in x'. Thus for any element p in a suitable open connected neighborhood of the identity $N \subset P(\mathbb{R})$, the translate pX'^+ meets $\tilde{Z}$ transversally in a point $x(p)$ near x' which varies real analytically with p. If, in addition, $p \in P(\mathbb{Q})$, then $\pi(pX'^+)$ is contained in a Hecke translate of S', and so the point $\pi(x(p)) \in \pi(pX'^+) \cap Z$ has amplitude $\leq d$. In other words, we have

$$\big\{\pi(x(p)) \bigm| p \in P(\mathbb{Q}) \cap N\big\} \;\subset\; \Xi.$$

Since $P(\mathbb{Q})$ is dense in $P(\mathbb{R})$, and $\pi(x(p))$ varies continuously with p, by taking closures we deduce that

$$\big\{\pi(x(p)) \bigm| p \in N\big\} \;\subset\; \overline{\Xi}.$$

To prove that $\overline{\Xi} = Z$ it thus suffices to show that the subset $x(N) := \{x(p) \,|\, p \in N\}$ of $\tilde{Z}$ is not contained in any proper complex analytic subvariety of $\tilde{Z}$.

This is clear if $P(\mathbb{R})$ acts transitively on X, because then pX'^+ for $p \in N$ sweeps out a whole neighborhood of x' in X^+, and so $x(p)$ sweeps out a neighborhood in $\tilde{Z}$. For the general case recall [12, Ch. IV, Prop. 1.3] that X^+ can be realized as an open subset of a flag variety $\check{X}$ associated to the complexified group $P(\mathbb{C})$. The action of $P(\mathbb{R})$ on X^+ extends to a transitive complex analytic action of $P(\mathbb{C})$ on $\check{X}$, which we can use to translate X'^+. As before, for any p in a suitable open connected neighborhood of the identity $\check{N} \subset P(\mathbb{C})$, the translate pX'^+ meets $\tilde{Z}$ transversally in a point $x(p)$ near x', which now varies complex analytically with p. In other words, the real analytic function $p \mapsto x(p)$ on N extends to a complex analytic function $\check{N} \to \tilde{Z}$. Thus any complex analytic subvariety of $\tilde{Z}$ containing $x(N)$ also contains $x(\check{N})$. On the other hand, the fact that pX'^+ for $p \in \check{N}$ sweeps out a whole neighborhood of x' in $\check{X}$ implies that $x(\check{N})$ contains a neighborhood of x' in $\tilde{Z}$. Thus any complex analytic subvariety of $\tilde{Z}$ containing $x(N)$ contains a neighborhood of x'. As $\tilde{Z}$ is irreducible, such a subvariety is equal to $\tilde{Z}$, as desired.

q.e.d.

3 Generalized Hecke orbits

A morphism between mixed Shimura varieties that is induced by a morphism between the underlying mixed Shimura data is called a *Shimura morphism.* A *generalized Hecke operator* on a mixed Shimura variety S consists of Shimura morphisms $S \xleftarrow{\varphi} S' \xrightarrow{\psi} S$ that are induced by automorphisms of the underlying mixed Shimura data. The *generalized Hecke orbit* of a point $s \in S$ is the union of the subsets $\psi(\varphi^{-1}(s)) \subset S$ for all such diagrams (compare [17, §3]).

In [17, Conj. 1.6] we formulated the following conjecture:

Conjecture 3.1 *Let S be a mixed Shimura variety over $\mathbb{C}$ and $\Lambda \subset S$ the generalized Hecke orbit of a point $s \in S$. Let $Z \subset S$ be an irreducible closed algebraic subvariety such that $Z \cap \Lambda$ is Zariski dense in Z. Then Z is a weakly special subvariety of S, that is, there exist Shimura morphisms $T' \xleftarrow{\varphi} T \xrightarrow{i} S$ and a point $t' \in T'$, such that Z is an irreducible component of $i(\varphi^{-1}(t'))$, or of its image under a Hecke operator.*

In [17, §§4–5] we showed that this implies the Mordell-Lang conjecture and the special case of the André-Oort conjecture for the generalized Hecke orbit of a special point. We now show that it follows from the much neater statement of Conjecture 1.1.

As a preparation we mention the following useful facts. It is known that the image of a special subvariety under a Shimura morphism is again a special subvariety. Dually, every irreducible component of the preimage of a special subvariety under a Shimura morphism is again a special subvariety. From this one easily deduces the following lemma:

Lemma 3.2 *Let $\varphi\colon S \to S'$ be a Shimura morphism, and Z an irreducible closed subvariety of S. Then the image under φ of the special closure of Z is the special closure of $\varphi(Z)$.*

Theorem 3.3 *Conjecture 1.1 implies Conjecture 3.1.*

Proof. Using Hecke operators we can move s to any connected component of S. Thus we may assume that the special closure of $\{s\}$ is an irreducible component of a mixed Shimura subvariety $T' \subset S$. Then for any $\lambda \in \Lambda$ the point $(\lambda, s) \in S{\times}T'$ lies in the transform of the diagonal $\operatorname{diag}(T')$ under a generalized Hecke operator on S. This transform is a finite union of special subvarieties of dimension d. In particular, it contains the special closure of $\{(\lambda, s)\}$, and so the amplitude of (λ, s) is $\leq d$. (In fact, Lemma 3.2 implies that it is equal to d, the amplitude of s.)

By applying Hecke operators we can also move Z to any connected component of S. Thus we may assume that the special closure of $Z{\times}\{s\}$ in $S{\times}T'$ is an irreducible component of a mixed Shimura subvariety $T \subset S{\times}T'$. Since by assumption $\Xi := (Z{\times}\{s\}) \cap (\Lambda{\times}\{s\})$ is Zariski dense in $Z{\times}\{s\}$, Conjecture 1.1 applied to $\Xi \subset S{\times}T'$ implies that $\dim T - \dim Z \leq d$. On the other hand, Lemma 3.2 shows that the composite morphism $T \hookrightarrow S{\times}T' \twoheadrightarrow T'$ is surjective. Being a Shimura morphism, its fiber dimension is constant. As $Z{\times}\{s\}$ is contained in a fiber, we deduce that $\dim Z \leq \dim T - \dim T' = \dim T - d$. Together we find that $\dim Z = \dim T - d$, which means that $Z{\times}\{s\}$ is an irreducible component of a fiber of $T \to T'$. Thus Z is weakly special, as desired. **q.e.d.**

4 An auxiliary result

In the next section we will need the following result:

Proposition 4.1 *For any semiabelian varieties B and C over $\mathbb{C}$, there exists an epimorphism of semiabelian varieties $\tilde{C} \twoheadrightarrow C$, such that for every epimorphism of semiabelian varieties $B \twoheadrightarrow B''$ the induced map*

$$\operatorname{Hom}(\tilde{C}, B) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \operatorname{Hom}(\tilde{C}, B'') \otimes_{\mathbb{Z}} \mathbb{Q}$$

is surjective.

When B is an abelian variety, this holds already for $\tilde{C} = C$, because B'' is an almost direct summand of B. In the general case we will construct $\tilde{C}$ as an extension of C by a torus.

Recall first that every semiabelian variety A lies in a short exact sequence $0 \to T \to A \to \bar{A} \to 0$, where $\bar{A}$ is an abelian variety and T is a torus. The extension class is an element

$$\xi_A \;\in\; \operatorname{Ext}^1(\bar{A}, T) \;\cong\; \operatorname{Hom}\big(X^*(T), \operatorname{Ext}^1(\bar{A}, \mathbb{G}_m)\big) \;\cong\; \operatorname{Hom}\big(X^*(T), \bar{A}^*\big),$$

where $X^*(T)$ denotes the character group of T, and $\bar{A}^*$ the abelian variety dual to $\bar{A}$. Moreover, the triple $(\bar{A}^*, X^*(T), \xi_A)$ is unique up to unique isomorphism and functorial in A. Let $\mathcal{S}$ denote the category whose objects are triples (B, M, φ) consisting of an abelian variety B over $\mathbb{C}$, a finitely generated free $\mathbb{Z}$-module M, and a homomorphism of abstract groups $\varphi\colon M \to B$, and where morphisms $(B, M, \varphi) \to (C, N, \psi)$ are pairs of homomorphisms $B \to C$ and $M \to N$ making the following diagram commute:

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ N \arrow[r, "\psi"] & C \end{tikzcd}$$

Then the category of semiabelian varieties over $\mathbb{C}$ is antiequivalent to $\mathcal{S}$ by the contravariant functor $A \mapsto (\bar{A}^*, X^*(T), \xi_A)$. Thus Proposition 4.1 is equivalent to the following statement about $\mathcal{S}$:

Proposition 4.2 *For any (B, M, φ) and (C, N, ψ), there exists a monomorphism $(C, N, \psi) \to (\tilde{C}, \tilde{N}, \tilde{\psi})$, such that for every monomorphism $(B', M', \varphi') \to (B, M, \varphi)$ the induced map*

$$\operatorname{Hom}_{\mathcal{S}}\big((B, M, \varphi), (\tilde{C}, \tilde{N}, \tilde{\psi})\big) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \operatorname{Hom}\big((B', M', \varphi'), (\tilde{C}, \tilde{N}, \tilde{\psi})\big) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is surjective.

To prove this we will use the following surjectivity criterion:

Proposition 4.3 *Let (B, M, φ) and (C, N, ψ) be such that $h(\varphi(M)) \subset \psi(N)$ for all homomorphisms $h\colon B \to C$. Then for any monomorphism $(B', M', \varphi') \to (B, M, \varphi)$ the induced map*

$$\operatorname{Hom}_{\mathcal{S}}\big((B, M, \varphi), (C, N, \psi)\big) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \operatorname{Hom}_{\mathcal{S}}\big((B', M', \varphi'), (C, N, \psi)\big) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is surjective.

Proof. Consider a commutative diagram

$$\begin{tikzcd} M \arrow[r, "\varphi"] & B \\ M' \arrow[u, "i"] \arrow[r, "\varphi'"] \arrow[d, "\ell'"'] & B' \arrow[u, "j"'] \arrow[d, "h'"] \\ N \arrow[r, "\psi"] & C \end{tikzcd}$$

whose upper half defines a monomorphism in $\mathcal{S}$. This means that i is injective and the kernel of j is finite. As $j(B')$ is an almost direct summand of B, there exist a homomorphism $h : B \to C$ and a positive integer r, such that $h \circ j = rh'$. On the other hand, there exist a submodule $M'' \subset M$ and a positive integer s, such that $sM \subset i(M') \oplus M'' \subset M$. Since $h(\varphi(M'')) \subset h(\varphi(M)) \subset \psi(N)$ by assumption, and M'' is a free $\mathbb{Z}$-module, we can find a homomorphism $\ell''\colon M'' \to N$, such that $\psi \circ \ell'' = h \circ \varphi|M''$. If ℓ denotes the composite homomorphism

$$M \xrightarrow{s\cdot\mathrm{id}} i(M') \oplus M'' \xrightarrow{(r\ell'\circ i^{-1}, \ell'')} N,$$

we deduce the following diagram commutes everywhere:

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[dd, bend right=60, "\ell"'] & B \arrow[dd, bend left=60, "sh"] \\ M' \arrow[u, "i"] \arrow[r, "\varphi'"] \arrow[d, "sr\ell'"'] & B' \arrow[u, "j"'] \arrow[d, "srh'"] \\ N \arrow[r, "\psi"] & C \end{tikzcd}$$

This means that $sr(h',\ell')\colon (B',M',\varphi') \to (C,N,\psi)$ is the homomorphism induced by $(sh,\ell)\colon (B,M,\varphi) \to (C,N,\psi)$, proving the desired surjectivity. **q.e.d.**

Proof of Proposition 4.2. Since $\mathrm{Hom}(B,C)$ and M are finitely generated $\mathbb{Z}$-modules and the evaluation map $\mathrm{Hom}(B,C) \times B \to C$ is bilinear,

$$\Lambda \;:=\; \sum_{h:B\to C} h(\varphi(M))$$

is a finitely generated subgroup of C. Choose a surjective homomorphism $\pi\colon F \twoheadrightarrow \Lambda$ from a finitely generated free $\mathbb{Z}$-module F, and define $(\tilde{C},\tilde{N},\tilde{\psi})$ by the commutative diagram

$$\begin{tikzcd}
N \arrow[r, "\psi"] \arrow[d, hook, "{(\mathrm{id},0)}"'] & C \arrow[d, "\mathrm{id}"] \\
N \oplus F \arrow[r, "{(\psi,\pi)}"] \arrow[d, equal] & C \arrow[d, equal] \\
\tilde{N} \arrow[r, "\tilde{\psi}"] & \tilde{C}.
\end{tikzcd}$$

Then by construction $h(\varphi(M)) \subset \tilde{\psi}(\tilde{N})$ for all homomorphisms $h\colon B \to \tilde{C}$; hence the desired surjectivity follows from Proposition 4.3. This also finishes the proof of Proposition 4.1. **q.e.d.**

5 Subvarieties of semiabelian varieties

Now we discuss what the proposed conjectures mean for subvarieties of semiabelian varieties. For any semiabelian variety A and any integer d we let $A^{[>d]}$ denote the union of all algebraic subgroups of A of codimension $> d$. We are interested in the following analog of Conjecture 1.3 that generalizes the Manin-Mumford conjecture:

Conjecture 5.1 *Consider a semiabelian variety A over $\mathbb{C}$ and an irreducible closed subvariety X of dimension d that is not contained in any proper algebraic subgroup of A. Then $X \cap A^{[>d]}$ is not Zariski dense in X.*

Next, a *subgroup of finite rank of A* is an abstract subgroup $\Gamma \subset A$ such that $\dim_{\mathbb{Q}}(\Gamma \otimes_{\mathbb{Z}} \mathbb{Q})$ is finite. The following generalizes the Mordell-Lang conjecture:

Conjecture 5.2 *Consider a semiabelian variety A over $\mathbb{C}$, a subgroup of finite rank $\Gamma \subset A$, and an irreducible closed subvariety X of dimension d that is not contained in any translate of any proper algebraic subgroup of A. Then $X \cap (A^{[>d]} + \Gamma)$ is not Zariski dense in X.*

Note that the two conjectures differ in both assumption and conclusion; hence none of them is a direct consequence of the other.

These conjectures are the outgrowth of the work of several people. In most of the known results one assumes that X is a curve (so that $d = 1$) which is not contained in any translate of any proper algebraic subgroup of A, and that X and A are defined over $\bar{\mathbb{Q}}$. One then shows that $X \cap A^{[>1]}$ is finite, i.e., one proves the conclusion of Conjecture 5.1 under the stronger condition of Conjecture 5.2. Bombieri, Masser, and Zannier [3, Thm. 2] achieve this whenever A is a torus, and in [4] they extend it to an arbitrary base field of characteristic zero. Viada [22] achieves it when $A = E^g$ for a CM elliptic curve E. Ratazzi [18], [19], building on the strategy of Rémond [21], obtains the same result when $A = B^n$ for a simple CM abelian variety B. Furthermore, Rémond and Viada [20, Thm. 1.7, Thm. 1.6] prove

Conjectures 5.1 and 5.2 themselves whenever $A = E^g$ for a CM elliptic curve E and $X \subset A$ is a curve defined over $\bar{\mathbb{Q}}$. The cited articles contain several further results in the direction of the above conjectures; in particular they contain various height estimates for points in $X \cap A^{[>d]}$ or in $X \cap (A^{[>d]} + \Gamma)$. The general case of the above conjectures is formulated with the Manin-Mumford and Mordell-Lang conjectures in mind, although any definite results are lacking.

In the rest of this section we will prove that Conjectures 5.1 and 5.2 are equivalent and that they both are consequences of Conjecture 1.3. The equivalence is proved by generalizing the method of Rémond and Viada [20, Prop. 4.2].

Theorem 5.3 *Conjecture 5.1 implies Conjecture 5.2.*

Proof. Let A, Γ, X, d be as in 5.2. Fix a maximal sequence of linearly independent elements $a_1, \ldots, a_r \in \Gamma$, and let C denote the Zariski closure of the subgroup of A^r that is generated by the point $\underline{a} := (a_1, \ldots, a_r)$. After multiplying all a_i by a suitable positive integer, if necessary, we may assume that C is connected and hence a semiabelian subvariety of A^r.

Consider the semiabelian variety $B := A \times C$ and its irreducible closed subvariety $Y := X \times \{\underline{a}\}$ of dimension d. We claim that Y is not contained in any proper algebraic subgroup of B. Indeed, since X is irreducible and not contained in any translate of any proper algebraic subgroup of A, the differences of elements of X generate A. Thus the differences of elements of Y generate $A \times \{0\}$, and so any algebraic subgroup H of B containing Y must also contain $A \times \{0\}$. On the other hand, the projection of H to the second factor contains the element $\underline{a}$, which by construction generates a Zariski dense subgroup of C. Therefore $H = A \times C$, proving the claim.

Applying Conjecture 5.1 to B, Y, d, we can now deduce that $Y \cap B^{[>d]}$ is not Zariski dense in Y. To prove that $X \cap (A^{[>d]} + \Gamma)$ is not Zariski dense in X, it therefore suffices to show the inclusion

$$\big(X \cap (A^{[>d]} + \Gamma)\big) \times \{\underline{a}\} \ \subset\ Y \cap B^{[>d]}. \tag{5.4}$$

For this let $x = g + \gamma \in X$ for elements $g \in G$ and $\gamma \in \Gamma$, where G is an algebraic subgroup of A of codimension $> d$. Choose integers $n > 0$ and $n_1, \ldots, n_r$ such that $n\gamma = n_1 a_1 + \ldots + n_r a_r$. Then we have $n\gamma = \varphi(\underline{a})$ for the homomorphism $\varphi := (n_1, \ldots, n_r)\colon A^r \to A$. Within B we therefore have

$$(nx, n\underline{a}) \ =\ (ng + n\gamma, n\underline{a}) \ =\ (ng + \varphi(\underline{a}), n\underline{a}) \ =\ (ng, 0) + (\varphi, n)(\underline{a}),$$

which shows that

$$(x, \underline{a}) \ \in\ H \ :=\ n^{-1}\big(G \times \{0\} + (\varphi, n)(C)\big).$$

Clearly $\dim H = \dim G + \dim C$, and hence $\operatorname{codim}_B H = \operatorname{codim}_A G > d$. Thus $(x, \underline{a}) \in Y \cap B^{[>d]}$, proving (5.4), as desired. **q.e.d.**

Theorem 5.5 *Conjecture 5.2 implies Conjecture 5.1.*

Proof. Let A, X, d be as in 5.1. As X is connected, the differences of all elements of X generate a connected algebraic subgroup of A. Let B denote this semiabelian subvariety; then $X \subset B + a$ for some element $a \in A$ (for instance in X). Fix such an element; then $Y := X - a$ is an irreducible closed subvariety of B of dimension d. Then B is also generated by the differences of all elements of Y, which implies that Y is not contained in any translate of any proper algebraic subgroup of B.

Let C denote the identity component of the Zariski closure of the subgroup of A that is generated by a. Choose an integer $n > 0$, such that $na \in C$. Let $\kappa\colon \tilde{C} \twoheadrightarrow C$ be the epimorphism furnished by Proposition 4.1, and fix an element $\tilde{c} \in \tilde{C}$ with $\kappa(\tilde{c}) = na$. Since $\mathrm{Hom}(\tilde{C}, B)$ is a finitely generated $\mathbb{Z}$-module, so is the subgroup

$$\Gamma_0 \;:=\; \bigl\{\varphi(\tilde{c}) \bigm| \varphi \in \mathrm{Hom}(\tilde{C}, B)\bigr\}$$

of B. Thus its division group

$$\Gamma \;:=\; \bigl\{b \in B \bigm| \exists m \in \mathbb{Z}^{>0} : mb \in \Gamma_0\bigr\}$$

is a subgroup of finite rank of B. Applying Conjecture 5.2 to B, Γ, Y, d, we can therefore deduce that $Y \cap (B^{[>d]} + \Gamma)$ is not Zariski dense in Y. To prove that $X \cap A^{[>d]}$ is not Zariski dense in X, it therefore suffices to show the inclusion

$$X \cap A^{[>d]} \;\subset\; \bigl(Y \cap (B^{[>d]} + \Gamma)\bigr) + a. \tag{5.6}$$

For this let $x \in X \cap G$ for an algebraic subgroup G of A of codimension $> d$. Then $X \subset x + B \subset G + B$, which by the assumption on X implies that $G + B = A$. Consider the projection

$$\pi\colon\; A \;=\; G{+}B \;\twoheadrightarrow\; G{+}B/G \;\cong\; B/G \cap B \;=:\, B''.$$

Applying Proposition 4.1 to the homomorphism $\pi|C\colon C \to B''$ yields a homomorphism ψ and a positive integer m making the following diagram commute. The right hand side indicates the effect on the element $\tilde{c}$:

$$\begin{tikzcd} C \arrow[r, "m\pi|C"] & B'' \\ \tilde{C} \arrow[u, two heads, "\kappa"] \arrow[r, "\psi"] & B \arrow[u, two heads, "\pi|B"'] \end{tikzcd} \qquad\qquad \begin{tikzcd} na \arrow[r, mapsto] & mn\pi(a) \\ \tilde{c} \arrow[u, mapsto] \arrow[r, mapsto] & \psi(\tilde{c}) \arrow[u, mapsto] \end{tikzcd}$$

Since $\psi(\tilde{c}) \in \Gamma_0$ by the construction of Γ_0, we find that $mn\pi(a) \in \pi(\Gamma_0)$. Back on A this means that $mna \in G + \Gamma_0$. By the construction of Γ this implies that $a \in G' + \Gamma$ with $G' := (mn)^{-1}G$. From this we deduce that

$$y \;:=\; x - a \;\in\; \bigl(G - (G' + \Gamma)\bigr) \cap Y \;\subset\; (G' + \Gamma) \cap B \;=\; (G' \cap B) + \Gamma.$$

On the other hand, the equality $G + B = A$ implies that $G' + B = A$, which shows that $G' \cap B$ is an algebraic subgroup of B with

$$\mathrm{codim}_B(G' \cap B) \;=\; \mathrm{codim}_A G' \;=\; \mathrm{codim}_A G \;>\; d.$$

Thus $y \in Y \cap (B^{[>d]} + \Gamma)$, proving (5.6), as desired. **q.e.d.**

The link between Conjecture 5.1 and those from the introduction involves a special kind of Shimura morphism $\pi : A \to S$ of mixed Shimura varieties, which is a family of semiabelian varieties, such that the group structure is given in terms of Shimura morphisms, as in [17, Rem. 2.13]. We call such a family a *Shimura family of semiabelian varieties.*

Every semiabelian variety over $\mathbb{C}$ is isomorphic to the fiber A_s over a point $s \in S$ for such a family. Moreover, using Hecke operators the point s can be moved to any connected component of S. Thus we may assume that the special closure of $\{s\}$ is an irreducible component of a mixed Shimura subvariety $S' \subset S$. After replacing $A \to S$ by $\pi^{-1}(S') \to S'$, which is again a Shimura family of semiabelian varieties, we may therefore assume that $S' = S$. Then s is Hodge generic in S.

Theorem 5.7 *Conjecture 1.3 implies Conjecture 5.1. More precisely, consider a Shimura family of semiabelian varieties $\pi\colon A \to S$, and let A_s denote the fiber above a Hodge generic point $s \in S$. Then Conjecture 1.3 for subvarieties of A that are contained in A_s is equivalent to Conjecture 5.1 for subvarieties of A_s.*

Proof. Since s is Hodge generic in S, every semiabelian subvariety of A_s extends to a Shimura family of semiabelian subvarieties over a finite covering of S. Since translation by any torsion point of A_s also extends to a Shimura morphism over a finite covering of S, we deduce that any translate of a semiabelian subvariety of A_s by a torsion point is an irreducible component of $T \cap A_s$ for some special subvariety $T \subset A$. Conversely, for any special subvariety $T \subset A$, every irreducible component of $T \cap A_s$ is a translate of a semiabelian subvariety of A_s by a torsion point. In other words, the irreducible components of all algebraic subgroups of A_s are precisely the irreducible components of $T \cap A_s$ for all special subvarieties $T \subset A$.

Moreover, if $T \cap A_s$ is non-empty, we have $s \in \pi(T)$. As $\pi(T)$ is a special subvariety of S, it must then be the irreducible component of S that contains s. The morphism from T to this irreducible component is then surjective with constant fiber dimension, which implies that the codimension in A_s of every irreducible component of $T \cap A_s$ is equal to the codimension of T in S.

Now consider any irreducible closed subvariety X of A_s of dimension d. Then by the above remarks X is contained in a proper algebraic subgroup of A_s if and only if it is contained in a special subvariety of A of codimension > 0. Thus $X \subset A_s$ satisfies the assumptions of Conjecture 5.1 if and only if $X \subset A$ satisfies the assumptions of Conjecture 1.3.

On the other hand, since X is contained in A_s, the above remarks show that its intersection with the union of all algebraic subgroups of A_s of codimension $> d$ is equal to its intersection with the union of all special subvarieties of S of codimension $> d$. Both conjectures assert that this intersection is not Zariski dense in X; hence they are equivalent in this case. **q.e.d.**

6 Subvarieties of families of semiabelian varieties

For any family of semiabelian varieties $B \to X$ we denote the fiber over a point $x \in X$ by B_x. For any integer d we set

$$B^{[>d]} := \bigcup_{x \in X} B_x^{[>d]}.$$

In other words $b \in B^{[>d]}$ if and only if b is contained in an algebraic subgroup of codimension $> d$ of its fiber. The following is a relative version of Conjecture 5.1.

Conjecture 6.1 *Consider an algebraic family of semiabelian varieties $B \to X$ over $\mathbb{C}$ and an irreducible closed subvariety $Y \subset B$ of dimension d that is not contained in any proper closed subgroup scheme of $B \to X$. Then $Y \cap B^{[>d]}$ is not Zariski dense in Y.*

Next, assume that X is irreducible, so that the relative dimension $\dim(B/X)$ of $B \to X$ is constant. Then for any $d < \dim(B/X)$, the subset $B^{[>d]}$ contains all torsion points of all fibers of $B \to X$. Thus the following is a consequence of Conjecture 6.1.

Conjecture 6.2 *Consider an algebraic family of semiabelian varieties $B \to X$ over an irreducible variety over $\mathbb{C}$, and an irreducible closed subvariety $Y \subset B$. Assume that Y is not contained in any proper closed subgroup scheme of $B \to X$, and that it contains a Zariski dense subset of torsion points. Then $\dim Y \geq \dim(B/X)$.*

Note that when X is a point, the conclusion is equivalent to $Y = B$; hence this case of Conjecture 6.2 is equivalent to the Manin-Mumford conjecture.

For another interesting special case suppose that Y is the image of a section $\beta : X \to B$. If β does not factor through a proper closed subgroup scheme, but meets torsion points on a Zariski dense subset of X, Conjecture 6.2 asserts that $\dim X \geq \dim(B/X)$. This is motivated by a result of André for elliptic pencils [2, p. 12].

Theorem 6.3 *Conjecture 1.3 implies Conjecture 6.1.*

Proof. Let $Y \subset B \to X$ be as in Conjecture 6.1. Let first X' denote the closure of the image of Y in X. Then after replacing $B \to X$ by its pullback to X', we may assume that $Y \to X$ is dominant. In particular X is then irreducible.

Next we claim that Conjecture 6.1 is invariant under pullback by unramified finite Galois coverings $X' \to X$. The essential point for this is to show that if $Y \times_X X'$ is contained in a proper closed subgroup scheme G' of $B \times_X X' \to X'$, then Y is contained in a proper closed subgroup scheme of $B \to X$. To prove this let H' be the intersection of all Galois conjugates of G'. Then by étale descent $H' = H \times_X X'$ for a proper closed subgroup scheme H of $B \to X$ that contains Y, as desired.

Now after replacing X by a suitable unramified Galois covering over which $B \to X$ acquires a sufficiently high level structure, there exists a cartesian diagram

$$\begin{tikzcd} Y \subset B \arrow[r, "\psi"] \arrow[d] & A \arrow[d] \\ X \arrow[r, "\varphi"] & S \end{tikzcd}$$

where $A \to S$ is a Shimura family of semiabelian varieties. After applying a Hecke operator on S we may assume that the special closure of $\varphi(X)$ in S is an irreducible component of a mixed Shimura subvariety S'. We can then replace $A \to S$ by its pullback to S', after which $\varphi(X)$ is Hodge generic in S.

Next let Z denote the Zariski closure of $\psi(Y)$ in A. Recall that Y is not contained in any proper closed subgroup scheme of $B \to X$. Since the special subvarieties of A that dominate S are precisely the translates of semiabelian subschemes by torsion points, it follows that Z is Hodge generic in A.

On the other hand, the irreducible components of all algebraic subgroups of a fiber A_s of $A \to S$ are precisely the irreducible components of the intersections of A_s with all special subvarieties of A (compare the proof of Theorem 5.7). Thus every algebraic subgroup of A_s of codimension $> d$ is contained in a special subvariety of A of codimension $> d$. It follows that $A^{[>d]} = \bigcup_{s \in S} A_s^{[>d]}$ is contained in the union of all special subvarieties of A of codimension $> d$.

Since $d = \dim Y \geq \dim Z$, Conjecture 1.3 now implies that $Z \cap A^{[>d]}$ is not Zariski dense in Z. Using $\psi\big(Y \cap B^{[>d]}\big) \subset Z \cap A^{[>d]}$, it follows that $Y \cap B^{[>d]}$ is not Zariski dense in Y, as desired. **q.e.d.**

Like Conjecture 1.1, Conjecture 6.1 is numerically sharp in a precise sense. Note that Conjecture 6.1 always reduces to the case that Y dominates X, which then implies that $\dim Y \geq \dim X$.

Proposition 6.4 *Consider an algebraic family of semiabelian varieties $B \to X$ over an irreducible algebraic variety over $\mathbb{C}$, which possesses a closed subgroup scheme of constant fiber codimension d over X, such that $d \geq \dim X$. Then there exists an irreducible closed subvariety $Y \subset B$ of dimension d, which dominates X and is not contained in any proper closed subgroup scheme of $B \to X$, such that $Y \cap B^{[>d-1]}$ is Zariski dense in Y.*

Proof. It suffices to construct Y over a Zariski open dense subset of X and then take its closure. Thus after shrinking X we may assume that X is affine and smooth, and that $B \to X$ is globally an extension of an abelian scheme by a torus. As $B \to X$ is quasiprojective, we may then choose a locally closed embedding $B \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$. We set $c := \dim B - d$ and construct Y as an irreducible component of $B \cap L$ for a sufficiently general linear subspace $L \subset \mathbb{P}^n_{\mathbb{C}}$ of codimension c.

Fix a closed subgroup scheme $B' \subset B$ of constant fiber codimension d over X. After shrinking X, if necessary, we may assume that $B' \to X$ is smooth. Since $\dim B' = \dim B - d = c$, we can require that L meets B' transversally at some isolated point b'. Then it also meets B transversally at b'; hence Y is smooth of dimension d at b'.

On the other hand, fix any point $x_0 \in X$. Since $\dim B_{x_0} = \dim B - \dim X \geq \dim B - d = c$, we can also require that L meets the fiber B_{x_0} transversally at some point $b_0 \in B_{x_0}$. Then L, and hence Y, meets all nearby fibers, which shows that Y dominates X.

Moreover, under the stated requirements we may still vary L in a nonempty open subset of its parameter space, so the possible intersections $Y \cap B_{x_0}$ sweep out a nonempty open subset of B_{x_0}. We may therefore also assume that b_0 is not contained in any proper algebraic subgroup of B_{x_0}. Assume this and suppose that Y is contained in a closed subgroup scheme G of B; we then claim that $G = B$. For this let U be an open dense subset of X over which G is flat. Let C be a smooth irreducible curve in X which contains x_0 and meets U, and let H be the closure of $G \times_X (U \cap C)$ in $G \times_X C$. Since C is a smooth curve, $H \to C$ is then flat. As $Y \to X$ is smooth at b_0, so is $Y \times_X C \to C$; hence $b_0 \in H \cap B_{x_0}$. By the assumption on b_0 this implies that $B_{x_0} \subset H$. By flatness, it follows that $H = B \times_X C$. Thus G is generically equal to B. Being closed, it is therefore equal to B, as desired. This shows that Y is not contained in any proper closed subgroup scheme of $B \to X$.

It remains to prove that $\Xi := Y \cap B^{[>d-1]}$ is Zariski dense in Y. We will show this without any further requirements on L. Let V denote the relative tangent bundle of $B \to X$ at the zero section and $\pi \colon V \to B$ the exponential map. We then have a natural short exact sequence

$$0 \longrightarrow \Lambda \longrightarrow V \xrightarrow{\pi} B \longrightarrow 0,$$

where Λ is a local system of finitely generated free $\mathbb{Z}$-modules on X, embedded fiberwise discretely into V. Abbreviate $\tilde{B}' := \pi^{-1}(B')$ and $\tilde{Y} := \pi^{-1}(Y)$ and select a point $v \in \pi^{-1}(b')$. Then $\tilde{B}'$ and $\tilde{Y}$ are complex analytic submanifolds of V of complementary dimension that meet transversally in v.

Let x denote the base point in X below b'. Then for any sufficiently small analytic local section t of $V \to X$ near x, the translate $\tilde{B}' + t$ is a small perturbation of $\tilde{B}'$ and meets $\tilde{Y}$ transversally in a unique point $v(t)$ near v which varies analytically with t. If, in addition, t is a (locally constant) section of $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q} \subset V$, then mt is a section of Λ for some integer $m > 0$, and so $\pi(\tilde{B}' + t)$ is contained in the subgroup scheme $m^{-1}(B') \subset B$ of fiber codimension d over X. Under these conditions we then have $\pi(v(t)) \in \Xi$.

Let $\overline{\Xi}$ denote the Zariski closure of Ξ in Y. As $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ is dense in $\Lambda \otimes_{\mathbb{R}} \mathbb{R}$ for the usual topology, and $\pi(v(t))$ varies continuously with t, by taking closures we deduce that $\pi(v(t)) \in \overline{\Xi}$ for any sufficiently small locally constant section t of $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ near x.

Moreover, let $\chi \colon \Lambda \otimes_{\mathbb{Z}} \mathbb{C} \twoheadrightarrow V$ denote the homomorphism induced by the embedding $\Lambda \hookrightarrow V$, and consider the map $u \mapsto \pi(v(\chi(u))) \in B$ for all sufficiently small locally constant sections u of $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ near x. This map is complex analytic, and its restriction to the real subspace $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ factors through the complex subvariety $\overline{\Xi} \subset B$. Therefore we also have $\pi(v(\chi(u))) \in \overline{\Xi}$ for all sufficiently small u.

But since χ is surjective, the translates $\tilde{B}' + \chi(u)$ for all such u sweep out a whole neighborhood of v in V. Thus $v(\chi(u))$ sweeps out a neighborhood of v in $\tilde{Y}$. Therefore the points $\pi(v(\chi(u))) \in \overline{\Xi}$ sweep out a neighborhood of b' in the irreducible variety Y, which proves that $\overline{\Xi} = Y$, as desired. **q.e.d.**

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