The Galois Representations Associated to a Drinfeld Module in Special Characteristic, II: Openness

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Abstract

Let φ be a Drinfeld A-module in special characteristic $\mathfrak{p}_0$ over a finitely generated field K. For any finite set P of primes $\mathfrak{p} \neq \mathfrak{p}_0$ of A let Γ_P denote the image of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ in its representation on the product of the $\mathfrak{p}$-adic Tate modules of φ for all $\mathfrak{p} \in P$. We determine Γ_P up to commensurability.

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1 Introduction

Let $\mathbb{F}_p$ be the finite prime field with p elements. Let F be a finitely generated field of transcendence degree 1 over $\mathbb{F}_p$. Let A be the ring of elements of F which are regular outside a fixed place ∞ of F. Let K be another finitely generated field over $\mathbb{F}_p$ of arbitrary transcendence degree, and let $\varphi : A \to K\{\tau\}$ be a Drinfeld A-module of rank $r \geq 1$ over K in special characteristic $\mathfrak{p}_0$.

Let $K^{\mathrm{sep}} \subset \bar{K}$ denote a separable, respectively an algebraic closure of K. Then for any place $\mathfrak{p} \neq \mathfrak{p}_0$, ∞ of F the rational $\mathfrak{p}$-adic Tate module $V_{\mathfrak{p}}(\varphi)$ is a vector space of dimension r over the completion $F_{\mathfrak{p}}$, and it carries a natural continuous representation of $\mathrm{Gal}(K^{\mathrm{sep}}/K) = \mathrm{Aut}(\bar{K}/K)$. For any non-empty finite set P of places $\mathfrak{p} \neq \mathfrak{p}_0$, ∞ of F we set $V_P(\varphi) := \bigoplus_{\mathfrak{p}\in P} V_{\mathfrak{p}}(\varphi)$, which is a free module over $F_P := \bigoplus_{\mathfrak{p}\in P} F_{\mathfrak{p}}$ of rank r. We are interested in the combined representation

$$\rho_P : \mathrm{Gal}(K^{\mathrm{sep}}/K) \longrightarrow \mathrm{Aut}_{F_P}\big(V_P(\varphi)\big) \cong \mathrm{GL}_r(F_P)$$

and in particular in its image

$$\Gamma_P \subset \mathrm{GL}_r(F_P) = \prod_{\mathfrak{p}\in P} \mathrm{GL}_r(F_{\mathfrak{p}}).$$

Furthermore let k denote the finite field of constants of K and $\bar{k}$ its algebraic closure in K^{sep}. Then $\mathrm{Gal}(\bar{k}/k)$ is the free pro-cyclic group topologically generated by the element Frob_k which acts on $\bar{k}$ by $u \mapsto u^{|k|}$, and we have a natural short exact sequence

$$1 \longrightarrow \mathrm{Gal}(K^{\mathrm{sep}}/K\bar{k}) \longrightarrow \mathrm{Gal}(K^{\mathrm{sep}}/K) \longrightarrow \mathrm{Gal}(\bar{k}/k) \longrightarrow 1.$$

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We are equally interested in the image Γ_P^{geom} of $\mathrm{Gal}(K^{\mathrm{sep}}/K\bar{k})$. By construction this is a closed normal subgroup of Γ_P and the quotient is pro-cyclic.

The aim of this article is to characterize these groups up to commensurability. The corresponding problem for Drinfeld modules of generic characteristic was solved in [10], where we showed that Γ_P is open in the general linear group if $\mathrm{End}_{\bar{K}}(\varphi) = A$. In special characteristic one cannot expect openness in GL_r, because the image of Γ_P^{geom} under the determinant is finite; hence the subgroup $\det(\Gamma_P) \subset F_P^*$ is essentially pro-cyclic and thus cannot be open. The main job is therefore to describe $\Gamma_P^{\mathrm{geom}} \cap \mathrm{SL}_r$. Of course this is interesting only in the case $r > 1$. The following theorem achieves it in the case $\mathrm{End}_{\bar{K}}(\varphi) = A$:

Theorem 1.1 *Let $\varphi : A \to K\{\tau\}$ be a Drinfeld A-module of rank $r > 1$ over K and in special characteristic $\mathfrak{p}_0$, such that $\mathrm{End}_{\bar{K}}(\varphi) = A$. Then there exists a unique subfield $E \subset F$ with $[F/E] < \infty$ and the following properties. For every non-empty finite set P of places $\neq \mathfrak{p}_0$, ∞ of F let Q denote the set of places of E below those in P. Then there exists an inner form G_Q of GL_{r,F_P} over E_Q with derived group G_Q^{der} such that:*

- *(a) $G_Q^{\mathrm{der}}(E_Q) \cap \Gamma_P^{\mathrm{geom}}$ is open in both $G_Q^{\mathrm{der}}(E_Q)$ and Γ_P^{geom}.*
- *(b) There exists an element $f \in E^*$ such that*

$$\overline{f^{\mathbb{Z}}} \cdot \big(G_Q^{\mathrm{der}}(E_Q) \cap \Gamma_P^{\mathrm{geom}}\big)$$

  is an open subgroup of Γ_P, where $\overline{f^{\mathbb{Z}}}$ denotes the pro-cyclic subgroup of the group of scalars in $G_Q(E_Q)$ that is topologically generated by f.

A full answer must also characterize E and G_Q and explain when and why E can be smaller than F. The reason is that Drinfeld modules obtained by restricting φ to subrings of A can have more endomorphisms than φ. This phenomenon occurs only in special characteristic, where endomorphism rings can be non-commutative.

Theorem 1.2 *Let φ be as in Theorem 1.1. Then there exists a unique subfield $E \subset F$ with the following properties:*

- *(a) The intersection $B := E \cap A$ is infinite with quotient field E, and $d := [F/E]$ is finite.*
- *(b) The restriction $\psi := \varphi|B$ is a Drinfeld B-module of rank rd whose endomorphism ring $\mathrm{End}_{\bar{K}}(\psi)$ is an order in a central simple algebra over E of dimension d^2.*
- *(c) For every other infinite subring $C \subset A$ we have $\mathrm{End}_{\bar{K}}(\varphi|C) \subset \mathrm{End}_{\bar{K}}(\psi)$.*

Moreover, the field E is the same as in Theorem 1.1 and the group G_Q is the centralizer of $\mathrm{End}_{\bar{K}}(\psi) \otimes_B E_Q$ in the algebraic group $\underline{\mathrm{Aut}}_{E_Q}\big(V_Q(\psi)\big)$.

Unfortunately Theorem 1.2 does not lend itself well to explicit calculation, because there are infinitely many candidates $C \subset A$ to consider. But our method yields the following characterization of E by characteristic polynomials of Frobenius elements. Let Ad denote the adjoint representation of GL_r.

Theorem 1.3 *Let φ, E, and ψ be as in Theorems 1.1 and 1.2. Let X be an integral scheme of finite type over $\mathbb{F}_p$, whose function field K' is a finite extension of K, and over which φ has good reduction. Let Σ be any set of closed points $x \in X$ of Dirichlet density 1. Then each of the following subfields of F coincides with E:*

(a) If $p \neq 2$ or $r \neq 2$, the subfield generated by the traces of $\mathrm{Ad}(\rho_P(\mathrm{Frob}_x))$ *for all* $x \in \Sigma$.

(b) If $p = r = 2$, either the subfield generated by the traces of $\mathrm{Ad}(\rho_P(\mathrm{Frob}_x))$ *for all* $x \in \Sigma$, *or the subfield generated by their square roots.*

(c) If $\mathrm{End}_{\bar{K}}(\psi) = \mathrm{End}_{K'}(\psi)$, *the subfield generated by the traces of* $\rho_P(\mathrm{Frob}_x)$ *for all* $x \in \Sigma$.

Furthermore, these statements remain true when the traces are replaced by all coefficients of the characteristic polynomials of $\mathrm{Ad}(\rho_P(\mathrm{Frob}_x))$, *respectively of* $\rho_P(\mathrm{Frob}_x)$.

The above results are proved in Sections 2 through 5. In Section 2 we construct E_Q and G_Q^{der} by group theory and obtain a close approximation to Theorem 1.1. Two crucial ingredients, namely the fact that the image of Γ_P in $\mathrm{GL}_r(F_\mathfrak{p})$ is Zariski dense for every $\mathfrak{p} \in P$, and the general description of Zariski dense compact subgroups of $\mathrm{SL}_r(F_P)$, were provided in previous articles [11], [9] by the same author. The fact that E_Q comes from a global subfield $E \subset F$ is proved in Section 3 with the help of characteristic polynomials of Frobeniuses, which at the same time proves Theorem 1.3 (a) and (b). We also derive certain structural properties of E which imply in particular that $B := E \cap A$ is infinite. This allows us to analyze the Drinfeld B-module $\psi := \varphi|B$ in Section 4. Using representation theory, the Tate conjecture for ψ, and a subtle argument involving weights of t-motives that was also used in [11], we succeed in establishing the one remaining cornerstone, Theorem 1.2 (b). In Section 5 we combine the results of the preceding sections and prove the rest of the above theorems. We also work out an explicit example.

The whole discussion so far concerns Drinfeld A-modules with $\mathrm{End}_{\bar{K}}(\varphi) = A$. This is not really a big restriction, because for every Drinfeld A-module φ one can select a maximal commutative subring $\hat{A} \subset \mathrm{End}_{\bar{K}}(\varphi)$ and pass to the corresponding Drinfeld $\hat{A}$-module $\hat{\varphi}$, which satisfies $\mathrm{End}_{\bar{K}}(\hat{\varphi}) = \hat{A}$. Applying the above results to $\hat{\varphi}$ one can obtain generalizations for arbitrary φ which do not involve $\hat{\varphi}$. This is done in Section 6 for Theorems 1.1 and 1.2. The common feature in all these results is that to φ we associate a new Drinfeld B-module ψ for a certain ring B, as in Theorem 1.2, that governs the image of Galois and can be characterized by endomorphisms.

2 Group theoretic analysis

We keep the notations of the introduction. From here until the end of Section 5 we impose the additional assumption

$$\mathrm{End}_{\bar{K}}(\varphi) = A.$$

The first crucial property of Γ_P was proved in [11, Thm. 1.1]:

Theorem 2.1 *The image of* Γ_P *in* $\mathrm{GL}_r(F_\mathfrak{p})$ *is Zariski dense for every* $\mathfrak{p} \in P$.

Next we note:

Proposition 2.2 *The following statements are equivalent:*

(a) φ *is isomorphic over* $\bar{K}$ *to a Drinfeld module defined over a finite field.*

(b) Γ_P^{geom} *is finite.*

(c) $r = 1$.

Proof. Clearly (a) implies (b). Next, since $\Gamma_P/\Gamma_P^{\rm geom}$ is abelian, (b) implies that an open subgroup of Γ_P is abelian, which by Theorem 2.1 shows (c). Thirdly the moduli stack of Drinfeld A-modules of rank 1 and characteristic $\mathfrak{p}_0$ is finite over the residue field of $\mathfrak{p}_0$. Since that residue field is finite, every such Drinfeld module over $\bar{K}$ is isomorphic to a Drinfeld module defined over a finite field. This proves the remaining implication (c)$\Rightarrow$(a). **q.e.d.**

Proposition 2.3 *Let* $\det : {\rm GL}_r \to \mathbb{G}_m$ *denote the determinant homomorphism. Then* $\det(\Gamma_P^{\rm geom})$ *is finite, and an open subgroup of* $\det(\Gamma_P)$ *is the pro-cyclic subgroup* $\overline{f^{\mathbb{Z}}} \subset F_P^*$ *topologically generated by a non-zero element* $f \in A$ *which has a pole at* ∞ *and a zero at* $\mathfrak{p}_0$ *and no other zeroes or poles.*

Proof. By Anderson [1, § 4.2] there exists a Drinfeld A-module ψ over K of characteristic $\mathfrak{p}_0$ and of rank 1, such that $V_\mathfrak{p}(\psi) \cong \Lambda^r V_\mathfrak{p}(\varphi)$ as Galois representations for every prime $\mathfrak{p}$. Thus the groups $\det(\Gamma_P^{\rm geom})$ and $\det(\Gamma_P)$ are simply the groups $\Gamma_P^{\rm geom}$ and Γ_P for ψ instead of φ. After replacing φ by ψ we may therefore assume that $r = 1$.

Next note that the desired assertions are invariant under replacing K by a finite extension and φ by an isomorphic Drinfeld module. Thus by Proposition 2.2 we may reduce ourselves to the case that φ is defined over the finite field k. Then $\Gamma_P^{\rm geom} = 1$, and the eigenvalue of ${\rm Frob}_k$ on $V_\mathfrak{p}(\psi)$ is an element $f \in F^*$ which is independent of $\mathfrak{p}$ and possesses the other listed properties by [3, Prop. 2.1], [4, Thm. 3.2.3]. The proposition follows from this. **q.e.d.**

In particular Proposition 2.3 describes the Galois groups completely in the case $r = 1$. From here until the end of Section 5 we therefore assume

$$r > 1.$$

Let $\Gamma_P^{\rm ad}$ denote the image of Γ_P in ${\rm PGL}_r(F_P)$. Theorem 2.1 implies that its image in ${\rm PGL}_r(F_\mathfrak{p})$ is Zariski dense for every $\mathfrak{p} \in P$. Let $\Gamma_P^{\rm der}$ denote the closure of the commutator subgroup of Γ_P. The description [9, Thm. 0.2] of Zariski dense compact subgroups yields:

Theorem 2.4 *There exists a closed subring* $E_P \subset F_P$ *and a model* H_P *of* ${\rm SL}_{r,F_P}$ *over* E_P *such that*

(a) E_P *is a finite direct sum of local fields,*

(b) F_P *is a finitely generated* E_P*-module,*

(c) $\Gamma_P^{\rm ad}$ *is contained in the adjoint group* $H_P^{\rm ad}(E_P)$*, and*

(d) $\Gamma_P^{\rm der}$ *is an open subgroup of* $H_P(E_P)$*.*

Our job will be to determine E_P and H_P. In the rest of this section we first determine the precise relation of $H_P(E_P)$ with Γ_P and $\Gamma_P^{\rm geom}$ up to commensurability. Since at several points we want to replace K by a finite extension, we note:

Proposition 2.5 E_P *and* H_P *do not change on replacing* K *by a finite extension.*

Proof. Replacing K by a finite extension amounts to replacing $\Gamma_P^{\rm ad}$ by an open subgroup, say by $\Gamma_P^{\rm ad\prime}$. Without loss of generality we may assume it to be normal. Its image in ${\rm PGL}_r(F_\mathfrak{p})$ is still Zariski dense for every $\mathfrak{p} \in P$. Now the data (E_P, H_P) amounts to what is called a *minimal quasi-model of* $(F_P, {\rm PGL}_{r,F_P}, \Gamma_P^{\rm ad})$ following [9, Def. 0.1, Thm. 3.6]. By [9, Cor. 3.8] it remains a minimal quasi-model when $\Gamma_P^{\rm ad}$ is replaced by $\Gamma_P^{\rm ad\prime}$. Thus E_P and H_P do not change, as desired. **q.e.d.**

Next we need some information on inertia. Let K_v denote the completion of K with respect to any valuation v. One says that φ has *semi-stable reduction at v* if φ is isomorphic to a Drinfeld module φ' which has coefficients in the ring of integers $\mathcal{O}_{K_v}$ and whose reduction modulo the maximal ideal is a Drinfeld module φ'_v of some rank $r_v > 0$ over the residue field k_v. Every Drinfeld module acquires semi-stable reduction over some finite extension of K_v. One says that φ has *good reduction at v* if one can achieve $r_v = r$. In this case the inertia group at any place of K^{sep} above v has trivial image in Γ_P.

If φ has semi-stable but not good reduction at v, the rank discrepancy is explained by the local uniformization theorem. For this we view φ'_v as a Drinfeld module over K_v via any lift $k_v \hookrightarrow K_v$. We let $\bar{K}_v$ denote an algebraic closure of K_v and view it as an A-module via φ'_v. The local uniformization theorem of Drinfeld [2, § 7] says that there exists a locally free A-module $\Lambda_v \subset \bar{K}_v$ of rank $r - r_v$, such that φ' is the 'quotient of φ'_v by Λ_v'. It implies that for every $\mathfrak{p}$ there is a natural short exact sequence

$$0 \longrightarrow V_{\mathfrak{p}}(\varphi'_v) \longrightarrow V_{\mathfrak{p}}(\varphi) \longrightarrow \Lambda_v \otimes_A F_{\mathfrak{p}} \longrightarrow 0 \tag{2.6}$$

which is equivariant under the local Galois group $\mathrm{Gal}(K_v^{\text{sep}}/K_v)$. This group acts on Λ_v through a finite quotient, because the action is continuous and the module finitely generated over A. Note also that the action on $V_{\mathfrak{p}}(\varphi'_v)$ factors through the Galois group of k_v. We can thus deduce that an open subgroup of the inertia group acts unipotently on $V_{\mathfrak{p}}(\varphi)$.

Proposition 2.7 *$H_P(E_P)$ contains an open subgroup of Γ_P^{geom}.*

Proof. By Theorem 2.4 (d) we have $\Gamma_P^{\text{der}} \subset H_P(E_P) \cap \Gamma_P$. Thus $H_P(E_P) \cap \Gamma_P$ is a normal subgroup of Γ_P and the quotient $\Delta_P := \Gamma_P / H_P(E_P) \cap \Gamma_P$ is abelian. Let Δ_P^{geom} denote the image of Γ_P^{geom} in Δ_P. We must prove that Δ_P^{geom} is finite.

We first look at the ramification in Δ_P^{geom}. Consider any valuation v of K where φ has bad reduction. The above remarks show that some open subgroup of the inertia group acts unipotently on $V_{\mathfrak{p}}(\varphi)$ and hence on $V_P(\varphi)$. Thus its image consists of unipotent elements of $\mathrm{GL}_r(F_P)$. Being unipotent, they lie already in $\mathrm{SL}_r(F_P) = H_P(F_P)$. Now any unipotent element of $H_P(F_P)$ is defined over E_P if and only if its image in $H_P^{\text{ad}}(F_P)$ is defined over E_P. The latter property being guaranteed by Theorem 2.4 (c), we deduce that the image of some open subgroup of the inertia group at v is contained in $H_P(E_P)$. It follows that the image in Δ_P^{geom} of the inertia group at v is finite.

Now as above let k denote the constant field of K. Let $\bar{X}$ be an integral proper scheme over k with function field K. Since we may replace K by a finite extension, by de Jong [7] we may apply an alteration to $\bar{X}$ to make it smooth. Let $X \subset \bar{X}$ be an open dense scheme such that φ extends to a family of Drinfeld modules of rank r over X (compare [11, § 3]). Then the Galois representation factors through the étale fundamental group $\pi_1^{\text{ét}}(X)$. Now $\bar{X} \smallsetminus X$ possesses only finitely many points of codimension 1 in $\bar{X}$, and each of these corresponds to a unique equivalence class of valuations of K. Thus it follows that the subgroup $\Delta_P^{\text{inert}} \subset \Delta_P^{\text{geom}}$ generated by the images of the inertia groups at these valuations is finite. It suffices therefore to prove that the quotient $\bar{\Delta}_P^{\text{geom}} := \Delta_P^{\text{geom}}/\Delta_P^{\text{inert}}$ is finite. By the purity of the branch locus [15] this group is a quotient of the étale fundamental group $\pi_1^{\text{ét}}(\bar{X}_{\bar{k}})$ of $\bar{X}_{\bar{k}} := X \times_k \bar{k}$.

Next observe that $\bar{\Delta}_P^{\text{geom}}$ is the quotient of two compact subgroups of $\mathrm{GL}_r(F_P)$. Since F_P is a finite direct sum of local fields of positive characteristic p, every compact subgroup of $\mathrm{GL}_r(F_P)$ possesses an open pro-p subgroup. Thus the same follows for $\bar{\Delta}_P^{\mathrm{geom}}$. As $\bar{\Delta}_P^{\mathrm{geom}}$ is abelian, it must be the product of a finite group with a pro-p group. It suffices therefore to prove that the maximal pro-p quotient $\bar{\bar{\Delta}}_P^{\mathrm{geom}}$ of $\bar{\Delta}_P^{\mathrm{geom}}$ is finite.

Now $\bar{\bar{\Delta}}_P^{\mathrm{geom}}$ is a quotient of the maximal pro-p abelian quotient of the étale fundamental group $\pi_{1,p\text{-ab}}^{\text{ét}}(\bar{X}_{\bar{k}})$. Moreover this surjection is equivariant with respect to the action of Frob_k. Since the action of Frob_k on Δ_P^{geom} is given by conjugation within the abelian group Δ_P, the action on Δ_P^{geom} and hence on $\bar{\bar{\Delta}}_P^{\mathrm{geom}}$ is trivial. It follows that $\bar{\bar{\Delta}}_P^{\mathrm{geom}}$ is a quotient of the group of coinvariants $\pi_{1,p\text{-ab}}^{\text{ét}}(\bar{X}_{\bar{k}})_{\mathrm{Frob}_k}$. But this group is known to be finite by Katz and Lang [8, Thm. 2]; hence $\bar{\bar{\Delta}}_P^{\mathrm{geom}}$ is finite, as desired. **q.e.d.**

Proposition 2.8 *(a) $H_P(E_P) \cap \Gamma_P^{\mathrm{geom}}$ is open in both $H_P(E_P)$ and Γ_P^{geom}.*

(b) There exists an element $f \in A$ which has a pole at ∞ and a zero at $\mathfrak{p}_0$ and no other zeroes or poles, such that the following holds. Let $\overline{f^{\mathbb{Z}}}$ denote the pro-cyclic subgroup of the group of scalars F_P^* that is topologically generated by f. Then

$$\overline{f^{\mathbb{Z}}} \cdot \big(H_P(E_P) \cap \Gamma_P^{\mathrm{geom}}\big)$$

is an open subgroup of Γ_P.

Proof. Set $\Gamma_{P,H}^{\mathrm{geom}} := H_P(E_P) \cap \Gamma_P^{\mathrm{geom}}$. By Proposition 2.7 this is an open subgroup of Γ_P^{geom}. On the other hand we have $\Gamma_P^{\mathrm{der}} \subset \Gamma_P^{\mathrm{geom}}$, because the quotient $\Gamma_P/\Gamma_P^{\mathrm{geom}}$ is pro-cyclic. But Γ_P^{der} is an open subgroup of $H_P(E_P)$ by Theorem 2.4 (d); hence so is $\Gamma_{P,H}^{\mathrm{geom}}$, proving (a).

Next choose any element $\sigma \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ whose image in $\mathrm{Gal}(\bar{k}/k)$ is Frob_k. Consider its images $\gamma \in \Gamma_P$ and $\gamma^{\mathrm{ad}} \in \Gamma_P^{\mathrm{ad}}$. Recall that by Galois and flat cohomology applied to the short exact sequence $1 \to (\text{center of } H_P) \to H_P \to H_P^{\mathrm{ad}} \to 1$ the cokernel of the natural homomorphism $H_P(E_P) \to H_P^{\mathrm{ad}}(E_P)$ is an abelian group annihilated by r. Since $\gamma^{\mathrm{ad}} \in H_P^{\mathrm{ad}}(E_P)$ by Theorem 2.4 (c), we deduce that $\gamma^r = \lambda h$ for a scalar $\lambda \in F_P^*$ and an element $h \in H_P(E_P)$. As γ^{ad} lies in a compact subgroup of $H_P^{\mathrm{ad}}(E_P)$, the element h lies in a compact subgroup of $H_P(E_P)$. Thus by (a) some positive integral power h^m lies in Γ_P^{geom}. Modifying σ^{rm} by a suitable element of $\mathrm{Gal}(K^{\mathrm{sep}}/K\bar{k})$ then yields an element $\tau \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ whose image in $\mathrm{Gal}(\bar{k}/k)$ is Frob_k^{rm} and whose image in Γ_P is $\gamma^{rm}h^{-m} = \lambda^m$. This element is scalar, and calling it g we find that $\overline{g^{\mathbb{Z}}} \cdot \Gamma_{P,H}^{\mathrm{geom}}$ is an open subgroup of Γ_P.

Finally $g^r = \det(g \cdot \mathrm{id})$ topologically generates an open subgroup of $\det(\Gamma_P)$. Thus by Proposition 2.3 some open subgroup of $\overline{g^{r\mathbb{Z}}}$ has the form $\overline{f^{\mathbb{Z}}}$ for a non-zero element $f \in A$ which has a pole at ∞ and a zero at $\mathfrak{p}_0$ and no other zeroes or poles. Then $\overline{f^{\mathbb{Z}}} \cdot \Gamma_{P,H}^{\mathrm{geom}}$ is an open subgroup of Γ_P, and we are done. **q.e.d.**

3 Characteristic polynomials of Frobeniuses

This section is devoted to a first characterization of the ring E_P. In Theorem 3.4 we will show that E_P is the completion of a certain subfield $E \subset F$ that is independent of P. This subfield will be constructed using characteristic polynomials of Frobenius elements. We also use Frobeniuses to derive certain structural properties of E.

For later use we note the following fact. For any subfield $E' \subset F$ we let E'_P denote the closure of E' in F_P.

Proposition 3.1 *Consider infinite subfields E', $E'' \subset F$.*

(a) *Then* $E' \subset F$ *is a finite extension.*

(b) *If* $E''_P \subset E'_P$ *for all* P*, then* $E'' \subset E'$*.*

(c) *If* $E''_P = E'_P$ *for all* P*, then* $E'' = E'$*.*

Proof. (a) follows from the fact that F is finitely generated of transcendence degree 1 over $\mathbb{F}_p$. To prove (b) consider the finite subextension $E' \subset E'E'' \subset F$. Choose any place $\mathfrak{q}'$ of E' which does not lie below the place $\mathfrak{p}_0$ or ∞ of F. Let P be the set of places of F above $\mathfrak{q}'$. Then E'_P is simply the completion of E' at $\mathfrak{q}'$, and $(E'E'')_P$ is the direct sum of the completions of $E'E''$ at all places above $\mathfrak{q}'$. But the assumption in (b) implies that $(E'E'')_P = E'_P E''_P = E'_P$. It follows that $E'E'' = E'$ and hence $E'' \subset E'$, proving (b). Finally (b) implies (c) by symmetry. **q.e.d.**

Now consider any finite extension K' of K. Let X be any integral scheme of finite type over $\mathbb{F}_p$ with function field K' over which φ has good reduction (compare [11, § 3]). For any closed point $x \in X$ we let $\mathrm{Frob}_x \in \mathrm{Gal}(K^{\mathrm{sep}}/K')$ be any element of a decomposition group above x which acts by $u \mapsto u^{|k_x|}$ on the residue fields. Recall [4, Thm. 3.2.3 b] that for every $x \in \Sigma$ the characteristic polynomial of Frob_x on $V_{\mathfrak{p}}(\varphi)$ has coefficients in F and is independent of $\mathfrak{p}$. Thus the same holds for the characteristic polynomial of $\rho_P(\mathrm{Frob}_x)$ on the free F_P-module $V_P(\varphi)$. Let Ad denote the adjoint representation of GL_r. Then the same follows again for the characteristic polynomial of $\mathrm{Ad}(\rho_P(\mathrm{Frob}_x))$.

Consider any set Σ of closed points $x \in X$ of Dirichlet density 1. (For the concept of Dirichlet density in the case $\dim X > 1$ see [10, Appendix B].)

Definition 3.2 (a) $E^{\mathrm{trad}}(K', \Sigma)$ *is the subfield of* F *generated by the traces of* $\mathrm{Ad}(\rho_P(\mathrm{Frob}_x))$ *for all* $x \in \Sigma$*.*

(b) $E^{\mathrm{chad}}(K', \Sigma)$ *is the subfield of* F *generated by all coefficients of the characteristic polynomials of* $\mathrm{Ad}(\rho_P(\mathrm{Frob}_x))$ *for all* $x \in \Sigma$*.*

Clearly $E^{\mathrm{trad}}(K', \Sigma) \subset E^{\mathrm{chad}}(K', \Sigma)$, and these fields do not depend on P. But they bear a close relation with E_P. For any commutative $\mathbb{F}_2$-algebra B we set $B^2 := \{b^2 \mid b \in B\}$.

Proposition 3.3 (a) *If* $p \neq 2$ *or* $r \neq 2$*, then for all* K'*,* Σ*,* P *we have*

$$E^{\mathrm{trad}}(K', \Sigma)_P = E^{\mathrm{chad}}(K', \Sigma)_P = E_P.$$

(b) *If* $p = r = 2$*, then for all* K'*,* Σ*,* P *we have*

$$E_P^2 \subset E^{\mathrm{trad}}(K', \Sigma)_P \subset E^{\mathrm{chad}}(K', \Sigma)_P \subset E_P.$$

(c) *If* $p = r = 2$*, for every* P *there exist* K' *and* Σ *such that*

$$E^{\mathrm{trad}}(K', \Sigma)_P = E^{\mathrm{chad}}(K', \Sigma)_P = E_P^2.$$

Proof. The adjoint representation Ad of GL_r is an extension of the adjoint representation $\overline{\mathrm{Ad}}$ of PGL_r with a trivial representation of dimension 1. Thus the fields do not change if Ad is replaced by $\overline{\mathrm{Ad}}$. Now since H_P^{ad} is a model of PGL_{r,F_P} over E_P, its adjoint representation is a model over E_P of the representation $\overline{\mathrm{Ad}}$. As $\Gamma_P^{\mathrm{ad}} \subset H_P^{\mathrm{ad}}(E_P)$ by Theorem 2.4 (c), it follows that all the coefficients generating $E^{\mathrm{chad}}(K', \Sigma)$ lie in E_P. In particular this implies that $E^{\mathrm{chad}}(K', \Sigma)_P \subset E_P$.

In the case $p = r = 2$ this can be strengthened as follows. By Proposition 2.8 there exists a finite extension K' of K whose corresponding open subgroup of Γ_P

is contained in $F_P^* \cdot H_P(E_P)$. In the case $p = r = 2$ the representation $\overline{\mathrm{Ad}}$ is, as a representation of H_P, the extension of a trivial representation of dimension 1 with the twist by Frob_2 of the standard representation of SL_2. Now the standard representation of H_P exists over E_P up to an inner twist, so the coefficients of the characteristic polynomial of any element of $H_P(E_P)$ in it lie in E_P. It follows that all the coefficients generating $E^{\mathrm{chad}}(K',\Sigma)$ lie in E_P^2. In particular we have $E^{\mathrm{chad}}(K',\Sigma)_P \subset E_P^2$ in this case. This shows that (b) implies (c).

To prove the remaining inclusions in (a) and (b) note first that by Proposition 2.5 we may replace K by K'. Thus without loss of generality we may assume that $K' = K$. Let $\mathcal{O}_P^{\mathrm{trad}} \subset F_P$ denote the closure of the subring that is generated by the traces of all elements of Γ_P^{ad} on the adjoint representation of H_P^{ad}. Let E_P^{trad} denote the total ring of quotients of $\mathcal{O}_P^{\mathrm{trad}}$. Then [9, Prop. 3.10] implies that $E_P^{\mathrm{trad}} = E_P$ in the case (a) and $E_P^2 \subset E_P^{\mathrm{trad}} \subset E_P$ in the case (b). On the other hand the elements $\rho_P(\mathrm{Frob}_x)$ for $x \in \Sigma$ form a dense subset of Γ_P by the Čebotarev density theorem [10, Thm. B.9], because Σ has Dirichlet density 1. Thus by approximation we find that $E^{\mathrm{trad}}(K',\Sigma)_P$ contains the trace of every element of Γ_P^{ad}. It follows that $E_P^{\mathrm{trad}} \subset E^{\mathrm{trad}}(K',\Sigma)_P$, which together with the other stated inclusions proves (a) and (b). **q.e.d.**

Theorem-Definition 3.4 *There exists a unique subfield $E \subset F$ such that:*

(a) F is a finite extension of E.

(b) E_P is the closure of E in F_P for every P.

(c) If $p \neq 2$ or $r \neq 2$, then for all K', Σ we have

$$E^{\mathrm{trad}}(K',\Sigma) = E^{\mathrm{chad}}(K',\Sigma) = E.$$

(d) If $p = r = 2$, then for all K', Σ we have

$$E^2 \subset E^{\mathrm{trad}}(K',\Sigma) \subset E^{\mathrm{chad}}(K',\Sigma) \subset E,$$

and there exist K' and Σ such that

$$E^{\mathrm{trad}}(K',\Sigma) = E^{\mathrm{chad}}(K',\Sigma) = E^2.$$

Proof. Let $\mathcal{C}$ denote the collection of all subfields $E^{\mathrm{trad}}(K',\Sigma)$ and $E^{\mathrm{chad}}(K',\Sigma)$ for all K' and Σ. Consider any $E' \in \mathcal{C}$. If E' were finite, Proposition 3.3 would imply that E_P is finite, contradicting Theorem 2.4 (b). Thus E' is infinite. The same follows for any other $E'' \in \mathcal{C}$.

Thus if $p \neq 2$ or $r \neq 2$, by Propositions 3.1 (c) and 3.3 (a) we can deduce that $E'' = E'$. Calling this field E, properties (a) and (b) follow from Propositions 3.1 (a) and 3.3 (a). This proves the theorem in the case (c).

If $p = r = 2$, we begin with a field $E' \in \mathcal{C}$ such that $E'_P = E_P^2$, which exists by Proposition 3.3 (c). Then for any other $E'' \in \mathcal{C}$ Proposition 3.3 (b) implies that $(E''_P)^2 \subset E_P^2 = E'_P \subset E''_P$. Using Proposition 3.1 (b) we deduce that $(E'')^2 \subset E' \subset E''$. Now since $E' \subset E_P^2 \cap F \subset F_P^2 \cap F = F^2$, we have $E' = E^2$ for a subfield $E \subset F$. By construction the closure of E^2 in F_P is E_P^2, so the closure of E is E_P. On the other hand the resulting inclusions $(E'')^2 \subset E^2 \subset E''$ are equivalent to $E^2 \subset E'' \subset E$, which proves the theorem in the case (d). **q.e.d.**

Proposition 3.5 *Let $\mathfrak{q}_0$ denote the place of E below the place $\mathfrak{p}_0$ of F. Then $\mathfrak{p}_0$ is the unique place of F above $\mathfrak{q}_0$.*

Proof. Consider any closed point $x \in X$ and let α_i for $1 \leq i \leq r$ denote the eigenvalues of $\rho_P(\mathrm{Frob}_x)$. Then the eigenvalues of $\mathrm{Ad}(\rho_P(\mathrm{Frob}_x))$ are the ratios α_i/α_j. Recall [3, Prop. 2.1], [4, Thm. 3.2.3 c, d] that the α_i are algebraic over F, with valuation zero at all places not above $\mathfrak{p}_0$ or ∞, and with some fixed valuation at all places above ∞. Thus the ratios α_i/α_j are units at all places not above $\mathfrak{p}_0$. It follows that the coefficients of the characteristic polynomial of $\mathrm{Ad}(\rho_P(\mathrm{Frob}_x))$ are regular outside $\mathfrak{p}_0$. Now as x varies, these coefficients generate the field E or E^2, which by Theorem 3.4 has transcendence degree 1 over $\mathbb{F}_p$. Thus for some x, some coefficient is transcendent. Being transcendent, it must have a pole at at least one place $\mathfrak{q}$ of E. It then has a pole at every place $\mathfrak{p}$ of F above $\mathfrak{q}$. By the above remarks this implies $\mathfrak{p} = \mathfrak{p}_0$ and thus $\mathfrak{q} = \mathfrak{q}_0$. In particular we deduce that $\mathfrak{p}_0$ is the unique place of F above $\mathfrak{q}_0$, as desired. **q.e.d.**

Proposition 3.6 *Let $\bar{\infty}$ denote the place of E below the place ∞ of F. Then ∞ is the unique place of F above $\bar{\infty}$.*

Proof. (Following a suggestion of Francis Gardeyn.) Recall that $r > 1$ by assumption. Thus from Proposition 2.2 we know that φ is not isomorphic over $\bar{K}$ to a Drinfeld module defined over a finite field. On the other hand recall that the moduli stack of Drinfeld A-modules of rank r is affine. Thus any compactification $\bar{X}$ of X possesses a point $\bar{x} \in \bar{X} \smallsetminus X$ at which φ does not have potential good reduction. After replacing K' by a finite extension we may suppose that φ has semi-stable reduction at $\bar{x}$, that is, that φ is isomorphic to a Drinfeld module φ' which has coefficients in the local ring $\mathcal{O}_{\bar{X},\bar{x}}$ and whose reduction modulo the maximal ideal is a Drinfeld module $\varphi'_{\bar{x}}$ of some rank $r_{\bar{x}} > 0$ over the residue field $k_{\bar{x}}$.

We may also specialize $\bar{x}$ to a closed point of $\bar{X}$. Then the action of $\mathrm{Frob}_{\bar{x}} \in \mathrm{Gal}(K^{\mathrm{sep}}/K')$ on $V_{\mathfrak{p}}(\varphi)$ is described by applying the exact sequence 2.6 to any valuation of K' centered on $\bar{x}$. By [4, Thm. 3.2.3 b] its characteristic polynomial on $V_{\mathfrak{p}}(\varphi'_{\bar{x}})$ has coefficients in F and is independent of $\mathfrak{p}$. The same holds for the characteristic polynomial on $\Lambda_{\bar{x}} \otimes_A F_{\mathfrak{p}}$, because the action comes from an action on $\Lambda_{\bar{x}}$. Together this implies that the characteristic polynomial of $\rho_P(\mathrm{Frob}_{\bar{x}})$ has coefficients in F and is independent of $\mathfrak{p}$. Again the same follows for the characteristic polynomial of $\mathrm{Ad}(\rho_P(\mathrm{Frob}_{\bar{x}}))$.

Lemma 3.7 *The coefficients of the characteristic polynomial of* $\mathrm{Ad}(\rho_P(\mathrm{Frob}_{\bar{x}}))$ *lie in E.*

Proof. Let E' be the subfield of F generated by E and the coefficients in question. Then we must prove that the inclusion $E \subset E'$ is an equality. By Proposition 3.1 (c) it suffices to show that $E_P = E'_P$ for all P. Now as φ has good reduction at almost all places of K, the element $\rho_P(\mathrm{Frob}_{\bar{x}})$ can be approximated by the images of Frobeniuses at places of good reduction. Thus the coefficients of the characteristic polynomial of $\mathrm{Ad}(\rho_P(\mathrm{Frob}_{\bar{x}}))$ can be approximated in F_P by elements of E. It follows that these coefficients lie in E_P; hence $E'_P = E_P$, as desired. **q.e.d.**

Lemma 3.8 *The characteristic polynomial of* $\mathrm{Ad}(\rho_P(\mathrm{Frob}_{\bar{x}}))$ *possesses a coefficient b which has a pole at ∞ and at most one other pole at $\mathfrak{p}_0$.*

Proof. Let α_i for $1 \leq i \leq r_{\bar{x}}$ denote the eigenvalues of $\rho_P(\mathrm{Frob}_{\bar{x}})$. By [3, Prop. 2.1], [4, Thm. 3.2.3 c, d] they are algebraic over F, with valuation zero at all places not above $\mathfrak{p}_0$ or ∞, and with some fixed negative valuation at all places above ∞. Let ζ_j for $r_{\bar{x}} + 1 \leq j \leq r$ denote the eigenvalues of $\mathrm{Frob}_{\bar{x}}$ on $\Lambda_{\bar{x}}$, which are roots of unity. Then the eigenvalues of $\mathrm{Ad}(\rho_P(\mathrm{Frob}_{\bar{x}}))$ are all possible ratios of the α_i and ζ_j. Among these only the ratios α_i/ζ_j have a pole above ∞, and there are precisely

$n := r_{\bar{x}}(r - r_{\bar{x}})$ of them. Let b denote the n^{th} elementary symmetric polynomial in the eigenvalues of $\mathrm{Ad}(\rho_P(\mathrm{Frob}_{\bar{x}}))$. This is one of the coefficients in question; in particular it is an element of F. By construction the product of the α_i/ζ_j is the unique summand of b which has the largest pole above ∞. Thus b has a non-trivial pole at ∞. On the other hand, all the α_i and ζ_j are units at all places not above $\mathfrak{p}_0$ or ∞. Thus b can have at most one other pole at $\mathfrak{p}_0$, as desired. **q.e.d.**

To finish the proof of Proposition 3.6 let b be as in Lemma 3.8. By Lemma 3.7 it is an element of E. Since b has a pole at the place ∞ of F, it has a pole at the corresponding place $\bar{\infty}$ of E. Suppose now that F possesses another place $\mathfrak{p} \neq \infty$ above $\bar{\infty}$. Then b has a pole at $\mathfrak{p}$, which by Lemma 3.8 is possible only for $\mathfrak{p} = \mathfrak{p}_0$. But then we have $\mathfrak{q}_0 = \bar{\infty}$ and thus $\mathfrak{p}_0 = \infty$ by Proposition 3.5, a contradiction. Therefore ∞ is the unique place of F above $\bar{\infty}$, as desired. **q.e.d.**

Proposition 3.9 *Let f be any element of F which has a pole at ∞ and a zero at $\mathfrak{p}_0$ and no other zeroes or poles. Then some positive integral power of f lies in E.*

Proof. Since $\mathfrak{p}_0 \neq \infty$, Proposition 3.5 or 3.6 shows in particular that $\mathfrak{q}_0 \neq \bar{\infty}$. Let $d_{\mathfrak{q}_0}$ and $d_{\bar{\infty}}$ denote the degrees of the corresponding residue fields over $\mathbb{F}_p$. Then $D := d_{\bar{\infty}} \cdot (\bar{\mathfrak{q}}_0) - d_{\mathfrak{q}_0} \cdot (\bar{\infty})$ is a divisor of degree 0 on E. Since E is a function field with finite residue field, some positive integral multiple of D is a principal divisor. Thus there exists a function $g \in E^*$ which possesses a pole at $\bar{\infty}$ and a zero at $\mathfrak{q}_0$ and no other zeroes or poles.

Viewing g now as a function in F, Propositions 3.5 and 3.6 imply that g possesses a pole at ∞ and a zero at $\mathfrak{p}_0$ and no other zeroes or poles. Some positive integral power of f has the same pole at ∞ as some positive integral power of g. The ratio thus has no zero or pole outside $\mathfrak{p}_0$. The product formula implies that the ratio then has no zero or pole anywhere, so it lies in the constant field and is therefore a root of unity. After enlarging the exponents we find that some positive integral power of f is equal to some positive integral power of g. It is therefore an element of E, as desired. **q.e.d.**

Proposition 3.10 *There exists an element $f \in E^*$ such that*

$$\overline{f^{\mathbb{Z}}} \cdot \left(H_P(E_P) \cap \Gamma_P^{\mathrm{geom}}\right)$$

is an open subgroup of Γ_P, where $\overline{f^{\mathbb{Z}}}$ denotes the pro-cyclic subgroup of the group of scalars in $\mathrm{GL}_r(F_P)$ that is topologically generated by f.

Proof. Let $f \in F^*$ be as in Proposition 2.8 (b). Then by Proposition 3.9 some positive integral power f^n lies in E. Since the statement of 2.8 (b) is preserved under replacing f by f^n, the assertion follows. **q.e.d.**

4 Restriction of scalars

In this section we analyze the subfield $E \subset F$ by restricting the Drinfeld module φ to subrings of A. Set $d := [F/E]$. The first observation is:

Proposition 4.1 *The ring $B := E \cap A$ is infinite with quotient field E.*

Proof. Recall that A is the ring of elements of F which are regular outside ∞. Thus Proposition 3.6 implies that $E \cap A$ is the ring of elements of E which are regular outside $\bar{\infty}$. It is a standard fact that its quotient field is E. **q.e.d.**

Let $\psi : B \to K\{\tau\}$ denote the restriction of φ. This is a Drinfeld B-module of rank rd. Consider any place $\mathfrak{q} \neq \mathfrak{q}_0, \tilde{\infty}$ of E, and let P be the set of places of F above $\mathfrak{q}$. Then $E_\mathfrak{q}$ can be identified with the closure of E in F_P, which by Theorem 3.4 coincides with the ring E_P from the preceding sections. Moreover there is a natural $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-equivariant isomorphism $V_P(\varphi) \cong V_\mathfrak{q}(\psi)$. In particular the image of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ on $V_\mathfrak{q}(\psi)$ is equal to Γ_P. By the Tate conjecture [12], [13], [14] for the Drinfeld module ψ we have a natural isomorphism

$$\mathrm{End}_{K'}(\psi) \otimes_B E_\mathfrak{q} \xrightarrow{\sim} \mathrm{End}_{E_\mathfrak{q},\mathrm{Gal}(K^{\mathrm{sep}}/K')}\big(V_\mathfrak{q}(\psi)\big) \tag{4.2}$$

for every finite extension $K' \subset K^{\mathrm{sep}}$ of K. We exploit this as follows:

Proposition 4.3 *View $V_\mathfrak{q}(\psi)$ as an algebraic representation of H_P over $E_\mathfrak{q}$. Then*

$$\mathrm{End}_{\bar{K}}(\psi) \otimes_B E_\mathfrak{q} \xrightarrow{\sim} \mathrm{End}_{E_\mathfrak{q},H_P}\big(V_\mathfrak{q}(\psi)\big).$$

Proof. We show that both sides coincide with those in 4.2 for every sufficiently large K'. For the left hand side see Section 6. For the right hand side by Proposition 3.10 we can achieve that the image of $\mathrm{Gal}(K^{\mathrm{sep}}/K')$ is contained in $E_P^*{\cdot}H_P(E_\mathfrak{q})$. On the other hand this image contains an open subgroup of $H_P(E_\mathfrak{q})$ by Theorem 2.4 (d). Since equivariance is not affected by scalars, and every open subgroup of $H_P(E_\mathfrak{q})$ is Zariski dense in H_P, the right hand sides are equal, as desired. **q.e.d.**

Next recall that H_P is a model of SL_{r,F_P} over $E_\mathfrak{q}$. Choose an algebraic closure $\bar{E}_\mathfrak{q}$ of $E_\mathfrak{q}$ and an isomorphism $H_P \times_{E_\mathfrak{q}} \bar{E}_\mathfrak{q} \cong \mathrm{SL}_{r,\bar{E}_\mathfrak{q}}$. Via this isomorphism $\bar{V}_\mathfrak{q} := V_\mathfrak{q}(\psi) \otimes_{E_\mathfrak{q}} \bar{E}_\mathfrak{q}$ becomes a representation of $\mathrm{SL}_{r,\bar{E}_\mathfrak{q}}$. Let $\bar{W}_\mathfrak{q} := \bar{E}_\mathfrak{q}^{\oplus r}$ denote the standard representation of $\mathrm{SL}_{r,\bar{E}_\mathfrak{q}}$ and $\bar{W}_\mathfrak{q}^*$ its dual. Note that $\bar{W}_\mathfrak{q}^* \cong \bar{W}_\mathfrak{q}$ if and only if $r = 2$.

Proposition 4.4 *$\bar{V}_\mathfrak{q}$ is isomorphic to a direct sum of copies of $\bar{W}_\mathfrak{q}$ and $\bar{W}_\mathfrak{q}^*$.*

Proof. Fix any $\mathfrak{p} \in P$ and any minimal non-trivial H_P-invariant $E_\mathfrak{q}$-subspace $U \subset V_\mathfrak{p}(\varphi)$. Then U is an irreducible representation of the reductive group H_P, so by representation theory $U \otimes_{E_\mathfrak{q}} \bar{E}_\mathfrak{q}$ is a direct sum of irreducible representations of $H_P \times_{E_\mathfrak{q}} \bar{E}_\mathfrak{q}$ whose equivalence classes are conjugate under outer automorphisms. Now recall that we have an isomorphism $H_P \times_{E_\mathfrak{q}} F_\mathfrak{p} \cong \mathrm{SL}_{r,F_\mathfrak{p}}$ making $V_\mathfrak{p}(\varphi)$ the standard representation of $\mathrm{SL}_{r,F_\mathfrak{p}}$. Since the natural homomorphism $U \otimes_{E_\mathfrak{q}} F_\mathfrak{p} \to V_\mathfrak{p}(\varphi)$ is non-zero, it follows that the constituents of $U \otimes_{E_\mathfrak{q}} \bar{E}_\mathfrak{q}$ are conjugate to the standard representation under outer automorphisms. Thus they must be among $\bar{W}_\mathfrak{q}$ and $\bar{W}_\mathfrak{q}^*$.

On the other hand the irreducibility of $V_\mathfrak{p}(\varphi)$ over $F_\mathfrak{p}$ implies that $V_\mathfrak{p}(\varphi)$ is the sum of the subspaces λU for all $\lambda \in F_\mathfrak{p}^*$. It is thus the direct sum of some of them. It follows that $V_\mathfrak{p}(\varphi) \otimes_{E_\mathfrak{q}} \bar{E}_\mathfrak{q}$ is isomorphic to a direct sum of copies of $\bar{W}_\mathfrak{q}$ and $\bar{W}_\mathfrak{q}^*$. Since $V_\mathfrak{q}(\psi)$ is the direct sum of the spaces $V_\mathfrak{p}(\varphi)$ for all $\mathfrak{p} \in P$, the proposition follows. **q.e.d.**

Proposition 4.5 *Let $\tilde{E}$ denote the center of $\mathrm{End}^\circ_{\bar{K}}(\psi) := \mathrm{End}_{\bar{K}}(\psi) \otimes_B E$.*

- *(a) If $\bar{V}_\mathfrak{q}$ is isotypic, then $\tilde{E} = E$.*
- *(b) If $\bar{V}_\mathfrak{q}$ is not isotypic, then $\tilde{E}$ is a separable quadratic extension of E.*

Proof. Suppose that $\bar{V}_\mathfrak{q} \cong \bar{W}_\mathfrak{q}^{\oplus n} \oplus (\bar{W}_\mathfrak{q}^*)^{\oplus n^*}$, with $n^* = 0$ if $r = 2$. Then

$$\begin{aligned}
\mathrm{End}^\circ_{\bar{K}}(\psi) \otimes_E \bar{E}_\mathfrak{q} &\cong \mathrm{End}_{\bar{K}}(\psi) \otimes_B \bar{E}_\mathfrak{q} \\
&\overset{4.3}{\cong} \mathrm{End}_{E_\mathfrak{q},H_P}\big(V_\mathfrak{q}(\psi)\big) \otimes_{E_\mathfrak{q}} \bar{E}_\mathfrak{q} \\
&\cong \mathrm{End}_{\bar{E}_\mathfrak{q},\mathrm{SL}_{r,\bar{E}_\mathfrak{q}}}\big(\bar{V}_\mathfrak{q}\big) \\
&\cong \mathrm{Mat}_{n\times n}(\bar{E}_\mathfrak{q}) \oplus \mathrm{Mat}_{n^*\times n^*}(\bar{E}_\mathfrak{q}).
\end{aligned}$$

Its center is therefore

$$\tilde{E} \otimes_E \bar{E}_{\mathfrak{q}} \cong \begin{cases} \bar{E}_{\mathfrak{q}} & \text{if } \bar{V}_{\mathfrak{q}} \text{ is isotypic, and} \\ \bar{E}_{\mathfrak{q}} \oplus \bar{E}_{\mathfrak{q}} & \text{if } \bar{V}_{\mathfrak{q}} \text{ is not isotypic.} \end{cases}$$

The proposition follows from this. **q.e.d.**

Proposition 4.6 *The case (b) in Proposition 4.5 does not occur.*

Proof. Suppose that $\bar{V}_{\mathfrak{q}}$ is not isotypic and $\tilde{E}$ is a separable quadratic extension of E. This can happen only for $r \geq 3$. Recall that $\mathrm{End}^\circ_{\bar{K}}(\psi)$ is a central division algebra over $\tilde{E}$, say of dimension n^2. Then

$$\mathrm{End}^\circ_{\bar{K}}(\psi) \otimes_E \bar{E}_{\mathfrak{q}} \;\cong\; \mathrm{Mat}_{n\times n}(\tilde{E} \otimes_E \bar{E}_{\mathfrak{q}}) \;\cong\; \mathrm{Mat}_{n\times n}(\bar{E}_{\mathfrak{q}})^{\oplus 2},$$

so the proof of Proposition 4.5 shows that $\bar{V}_{\mathfrak{q}} \cong \bar{W}_{\mathfrak{q}}^{\oplus n} \oplus (\bar{W}_{\mathfrak{q}}^*)^{\oplus n}$. From φ we will construct two new Drinfeld modules with Tate modules essentially isomorphic to $\bar{W}_{\mathfrak{q}}$ and $\bar{W}_{\mathfrak{q}}^*$. Using weights of t-motives we will then show that the resulting duality between them forces $r \leq 2$, yielding a contradiction.

Lemma 4.7 *There exist finite extensions $B \subset \tilde{A}$ and $K \subset K' \subset K^{\mathrm{sep}}$, Drinfeld $\tilde{A}$-modules $\tilde{\varphi}$, $\tilde{\varphi}' : \tilde{A} \to K'\{\tau\}$ of rank r, a place $\tilde{\mathfrak{p}}$ of $\tilde{F} := \mathrm{Quot}(\tilde{A})$ above $\mathfrak{q}$, and an extension of $E_{\mathfrak{q}} \hookrightarrow \bar{E}_{\mathfrak{q}}$ to an embedding $j : \tilde{F}_{\tilde{\mathfrak{p}}} \hookrightarrow \bar{E}_{\mathfrak{q}}$, such that*

$$V_{\tilde{\mathfrak{p}}}(\tilde{\varphi}) \otimes_{\tilde{F}_{\tilde{\mathfrak{p}}},j} \bar{E}_{\mathfrak{q}} \;\cong\; \bar{W}_{\mathfrak{q}} \qquad \text{and} \qquad V_{\tilde{\mathfrak{p}}}(\tilde{\varphi}') \otimes_{\tilde{F}_{\tilde{\mathfrak{p}}},j} \bar{E}_{\mathfrak{q}} \;\cong\; \bar{W}_{\mathfrak{q}}^*$$

as representations of $\mathrm{Gal}(K^{\mathrm{sep}}/K')$ over $\bar{E}_{\mathfrak{q}}$, up to twists by scalar characters with values in $E_{\mathfrak{q}}$.

Proof. Let S be a finite set of places of $\tilde{E}$ containing all those where $\mathrm{End}^\circ_{\bar{K}}(\psi)$ does not split. After enlarging S we may suppose that S is invariant under the non-trivial automorphism $\sigma \in \mathrm{Gal}(\tilde{E}/E)$. Choose any separable field extension $\tilde{F}$ of $\tilde{E}$ of degree n which possesses exactly one place above every place in S. Then the two embeddings id, $\sigma : \tilde{E} \xrightarrow{\sim} \tilde{E} \hookrightarrow \mathrm{End}^\circ_{\bar{K}}(\psi)$ can be extended to embeddings i, $i' : \tilde{F} \hookrightarrow \mathrm{End}^\circ_{\bar{K}}(\psi)$. Set

$$\hat{A} := i^{-1}\big(\mathrm{End}_{\bar{K}}(\psi)\big) \cap i'^{-1}\big(\mathrm{End}_{\bar{K}}(\psi)\big).$$

By construction this ring contains B. It is therefore infinite and its quotient field is $\tilde{F}$. Recall that $\mathrm{End}_{\bar{K}}(\psi)$ is a subring of $\bar{K}\{\tau\}$. Composing its tautological embedding with i, i' therefore yields two homomorphisms $\hat{\varphi}$, $\hat{\varphi}' : \hat{A} \to \bar{K}\{\tau\}$. These are Drinfeld $\hat{A}$-modules extending ψ, except that the ring $\hat{A}$ is not necessarily a maximal order in $\tilde{F}$. Let $\tilde{A}$ denote the integral closure of $\hat{A}$ in $\tilde{F}$, and choose Drinfeld $\tilde{A}$-modules $\tilde{\varphi}$, $\tilde{\varphi}' : \tilde{A} \to \bar{K}\{\tau\}$ whose restrictions to $\hat{A}$ are isogenous to $\hat{\varphi}$, $\hat{\varphi}'$, as in Section 6. Let $\tilde{P}$ be the set of places of $\tilde{F}$ above $\mathfrak{q}$. Then $V_{\tilde{P}}(\tilde{\varphi}) \cong V_{\tilde{P}}(\hat{\varphi}) = V_{\mathfrak{q}}(\psi)$, where the $\tilde{F}_{\tilde{P}}$-module structure is deduced from

$$\tilde{F}_{\tilde{P}} \;\cong\; \tilde{F} \otimes_E E_{\mathfrak{q}} \xhookrightarrow{i\otimes\mathrm{id}} \mathrm{End}^\circ_{\bar{K}}(\psi) \otimes_E E_{\mathfrak{q}}.$$

Thus $V_{\tilde{P}}(\tilde{\varphi}) \otimes_{E_{\mathfrak{q}}} \bar{E}_{\mathfrak{q}} \cong \bar{V}_{\mathfrak{q}}$ with the $\tilde{F}_{\tilde{P}} \otimes_{E_{\mathfrak{q}}} \bar{E}_{\mathfrak{q}}$-module structure deduced from

$$\tilde{F}_{\tilde{P}} \otimes_{E_{\mathfrak{q}}} \bar{E}_{\mathfrak{q}} \;\cong\; \tilde{F} \otimes_E \bar{E}_{\mathfrak{q}} \xhookrightarrow{i\otimes\mathrm{id}} \mathrm{End}^\circ_{\bar{K}}(\psi) \otimes_E \bar{E}_{\mathfrak{q}} \;\cong\; \mathrm{Mat}_{n\times n}(\bar{E}_{\mathfrak{q}})^{\oplus 2}.$$

Since $\tilde{F}$ is separable of degree $2n$ over E, the left hand side is isomorphic to a direct sum of $2n$ copies of $\bar{E}_{\mathfrak{q}}$, and its image in the matrix algebra is a maximal commutative subalgebra. Choose any place $\tilde{\mathfrak{p}} \in \tilde{P}$ and extend $E_{\mathfrak{q}} \subset \bar{E}_{\mathfrak{q}}$ to an embedding

$j : \tilde{F}_{\tilde{\mathfrak{p}}} \hookrightarrow \bar{E}_{\mathfrak{q}}$. These choices amount to the selection of a simple summand of $\tilde{F}_{\tilde{P}} \otimes_{E_{\mathfrak{q}}} \bar{E}_{\mathfrak{q}}$. This summand lands in one of the simple summands of $\mathrm{Mat}_{n\times n}(\bar{E}_{\mathfrak{q}})^{\oplus 2}$, say in that corresponding to $\bar{W}_{\mathfrak{q}}$. It follows that

$$V_{\tilde{\mathfrak{p}}}(\tilde{\varphi}) \otimes_{\tilde{F}_{\tilde{\mathfrak{p}}},j} \bar{E}_{\mathfrak{q}} \;\cong\; \bar{W}_{\mathfrak{q}}.$$

In particular $\tilde{\varphi}$ has rank $r = \dim \bar{W}_{\mathfrak{q}}$. The same arguments apply to $\tilde{\varphi}'$ in place of $\tilde{\varphi}$. Since i must be replaced by i' and $i'|\tilde{E} = \sigma$ interchanges the two simple summands of $\mathrm{Mat}_{n\times n}(\bar{E}_{\mathfrak{q}})^{\oplus 2}$, we deduce that

$$V_{\tilde{\mathfrak{p}}}(\tilde{\varphi}') \otimes_{\tilde{F}_{\tilde{\mathfrak{p}}},j} \bar{E}_{\mathfrak{q}} \;\cong\; \bar{W}_{\mathfrak{q}}^{*}.$$

Now take any sufficiently large finite extension $K \subset K' \subset K^{\mathrm{sep}}$ over which $\tilde{\varphi}$, $\tilde{\varphi}'$ are defined and such that the image of $\mathrm{Gal}(K^{\mathrm{sep}}/K')$ is contained in $E_{\mathfrak{q}}^{*} \cdot H_P(E_{\mathfrak{q}})$ by Proposition 3.10. Then the above isomorphisms are equivariant under $\mathrm{Gal}(K^{\mathrm{sep}}/K')$ up to twists by scalar characters with values in $E_{\mathfrak{q}}$, as desired. **q.e.d.**

In particular we deduce:

Lemma 4.8 *(a) The Zariski closure of the image of* $\mathrm{Gal}(K^{\mathrm{sep}}/K')$ *in the group* $\mathrm{Aut}_{\tilde{F}_{\tilde{\mathfrak{p}}}}\big(V_{\tilde{\mathfrak{p}}}(\tilde{\varphi})\big) \cong \mathrm{GL}_r(\tilde{F}_{\tilde{\mathfrak{p}}})$ *contains* $\mathrm{SL}_{r,\tilde{F}_{\tilde{\mathfrak{p}}}}$*, and*

(b) $V_{\tilde{\mathfrak{p}}}(\tilde{\varphi}') \cong V_{\tilde{\mathfrak{p}}}(\tilde{\varphi})^{*} \otimes \chi$ *for some scalar character* χ *of* $\mathrm{Gal}(K^{\mathrm{sep}}/K')$.

Proof. (a) follows from Lemma 4.7 and the corresponding property of $\bar{W}_{\mathfrak{q}}$. The analogue of (b) over an algebraic closure of $\tilde{F}_{\tilde{\mathfrak{p}}}$ also follows from Lemma 4.7. Since the twisting character χ takes values in $E_{\mathfrak{q}} \subset \tilde{F}_{\tilde{\mathfrak{p}}}$, the isomorphism already exists over $\tilde{F}_{\tilde{\mathfrak{p}}}$, as desired. **q.e.d.**

Lemma 4.9 *For every field extension L of $\tilde{F}_{\tilde{\mathfrak{p}}}$ there exists up to scalar multiples exactly one* $\mathrm{Gal}(K^{\mathrm{sep}}/K')$*-equivariant endomorphism of* $V_{\tilde{\mathfrak{p}}}(\tilde{\varphi})^{*} \otimes_{\tilde{F}_{\tilde{\mathfrak{p}}}} V_{\tilde{\mathfrak{p}}}(\tilde{\varphi}')^{*} \otimes_{\tilde{F}_{\tilde{\mathfrak{p}}}} L$ *of rank* 1.

Proof. Note that this statement is not affected by scalar twists. For any field L let $W := L^{\oplus r}$ denote the standard representation of $H := \mathrm{SL}_{r,L}$. Then in view of Lemma 4.8 we must prove that up to scalar multiples there exists exactly one H-equivariant endomorphism of $W^{*} \otimes_L W$ of rank 1. The image of any such endomorphism is an H-invariant subspace of dimension 1. As H is connected semisimple, it must act trivially on this subspace. Thus the desired assertion is equivalent to

$$\dim_L \mathrm{Hom}_H\big(W^{*} \otimes_L W, L\big) = \dim_L \mathrm{Hom}_H\big(L, W^{*} \otimes_L W\big) = 1.$$

But these equalities follow at once from the absolute irreducibility of W. **q.e.d.**

The rest of the proof proceeds as in [11, Lem. 7.1], using the properties of A-motives collected in [11, § 5]. Let $M_{\tilde{\varphi}}$, $M_{\tilde{\varphi}'}$ be the $\tilde{A}$-motives over K corresponding to the Drinfeld modules $\tilde{\varphi}$, $\tilde{\varphi}'$ by [11, Prop. 5.7], and set $M := M_{\tilde{\varphi}} \otimes M_{\tilde{\varphi}'}$. Then [11, Prop. 5.8, 5.5] shows that

$$V_{\tilde{\mathfrak{p}}}(\tilde{\varphi})^{*} \otimes_{\tilde{F}_{\tilde{\mathfrak{p}}}} V_{\tilde{\mathfrak{p}}}(\tilde{\varphi}')^{*} \;\cong\; V_{\tilde{\mathfrak{p}}}(M_{\tilde{\varphi}}) \otimes_{\tilde{F}_{\tilde{\mathfrak{p}}}} V_{\tilde{\mathfrak{p}}}(M_{\tilde{\varphi}'}) \;\cong\; V_{\tilde{\mathfrak{p}}}(M)$$

as representations of $\mathrm{Gal}(K^{\mathrm{sep}}/K')$ over $\tilde{F}_{\tilde{\mathfrak{p}}}$. Thus Lemma 4.9 implies that for every field extension L of $\tilde{F}_{\tilde{\mathfrak{p}}}$ there exists up to scalar multiples exactly one $\mathrm{Gal}(K^{\mathrm{sep}}/K')$-equivariant endomorphism of $V_{\tilde{\mathfrak{p}}}(M) \otimes_{\tilde{F}_{\tilde{\mathfrak{p}}}} L$ of rank 1. Applying [11, Prop. 5.6] to $M' = M$ we deduce that this endomorphism comes from an endomorphism h of the $\tilde{A}$-motive M. Let $N \subset M$ denote its image. Then $V_{\tilde{\mathfrak{p}}}(N)$ is the image of the endomorphism $V_{\tilde{\mathfrak{p}}}(h)$ of $V_{\tilde{\mathfrak{p}}}(M)$ of rank 1; hence N is an $\tilde{A}$-motive of rank 1. On the other hand $M_{\tilde{\varphi}}$, $M_{\tilde{\varphi}'}$ are pure $\tilde{A}$-motives of weight $\frac{1}{r}$ by [11, Prop. 5.7]; hence M and N are pure $\tilde{A}$-motives of weight $\frac{2}{r}$. Thus [11, Prop. 5.3] implies that $\frac{2}{r} \in \mathbb{Z}$. Since that is impossible for $r \geq 3$, this finishes the proof of Proposition 4.6. **q.e.d.**

Since F is a maximal commutative subalgebra of $\mathrm{End}^\circ_{\bar{K}}(\psi)$, Propositions 4.5 and 4.6 together imply:

Proposition 4.10 $\mathrm{End}^\circ_{\bar{K}}(\psi)$ *is a central simple algebra over E of dimension d^2.*

5 Proof of the main results

We will now combine the results of the preceding sections to prove the theorems in the introduction. Let P be any non-empty finite set of places $\neq \mathfrak{p}_0, \infty$ of F. Let Q be the set of places of E below those in P, and $\tilde{P}$ the set of places of F above those in Q. Since E_P, $E_{\tilde{P}}$ are the closures of E in F_P, $F_{\tilde{P}}$ by Theorem 3.4, both of them can be identified with $E_Q := \bigoplus_{\mathfrak{q}\in Q} E_{\mathfrak{q}}$. Note that the inclusion $P \subset \tilde{P}$ yields natural surjections $F_{\tilde{P}} \twoheadrightarrow F_P$ and $V_Q(\psi) \cong V_{\tilde{P}}(\varphi) \twoheadrightarrow V_P(\varphi)$.

Let G_Q be the centralizer of $\mathrm{End}_{\bar{K}}(\psi) \otimes_B E_Q$ in the algebraic group $\underline{\mathrm{Aut}}_{E_Q}\big(V_Q(\psi)\big) \cong \mathrm{GL}_{dr,E_Q}$. Since $\mathrm{End}_{\bar{K}}(\psi) \otimes_B E_Q$ is a form over E_Q of the algebra of $d \times d$-matrices and $V_Q(\psi)$ is a free E_Q-module of rank rd, the algebraic group G_Q is an inner form of GL_{r,E_Q}. Moreover G_Q still acts faithfully on the quotient $V_P(\varphi)$, so we can identify it with a subgroup of the algebraic group $\underline{\mathrm{Aut}}_{E_Q}\big(V_P(\varphi)\big)$. Let G_Q^{der} denote the derived group of G_Q.

Proof of Theorem 1.1. The assertions for P follow from those for $\tilde{P}$ by projection. Thus after replacing P by $\tilde{P}$ we may assume that $V_P(\varphi) = V_Q(\psi)$. Let $K' \subset K^{\mathrm{sep}}$ be any finite extension of K such that $\mathrm{End}_{\bar{K}}(\psi) = \mathrm{End}_{K'}(\psi)$. Then the image of $\mathrm{Gal}(K^{\mathrm{sep}}/K')$ is an open subgroup of Γ_P which is contained in $G_Q(E_Q)$. Now Theorem 2.4 implies that every open subgroup of Γ_P contains a Zariski dense subgroup of H_P. Thus $H_P \subset G_Q$, and since these are forms of SL_{r,E_Q} and GL_{r,E_Q} respectively, we must have $H_P = G_Q^{\mathrm{der}}$. Now the assertions 1.1 (a) and (b) are simply restatements of Propositions 2.8 (a) and 3.10.

It remains to show that the subfield $E \subset F$ is uniquely characterized by the properties 1.1 (a) and (b). Let $E' \subset F$ be any other field with these properties. Let E'_P denote the closure of E' in F_P. Recall from Proposition 2.5 that any open subgroup of Γ_P yields the same ring E_P. Thus by the uniqueness [9, Thm. 0.2] of the ring E_P associated to any open subgroup of Γ_P we have $E'_P = E_P$. As this holds for all P, Proposition 3.1 (c) implies that $E' = E$, as desired. **q.e.d.**

Proof of Theorem 1.2. Properties (a) and (b) follow from Propositions 4.1 and 4.10, and the description of G_Q was part of the construction above.

To prove (c) consider any infinite subring $C \subset A$. Let E' denote the center of $\mathrm{End}^\circ_{\bar{K}}(\varphi|C)$. Set $B' := E' \cap A$ and consider the Drinfeld B'-module $\psi' := \varphi|B'$. Then $\mathrm{End}_{\bar{K}}(\varphi|C)$ commutes with $\varphi_{b'}$ for all $b' \in B'$; hence $\mathrm{End}_{\bar{K}}(\varphi|C) = \mathrm{End}_{\bar{K}}(\psi')$. Now $\mathrm{End}^\circ_{\bar{K}}(\psi')$ is a central division algebra over E' of dimension $(d')^2$, where $d' := [F/E']$. Let Q' be the set of places of E' below those in P; then $E'_{Q'}$ is the closure of E' in F_P. Let $G'_{Q'}$ be the centralizer of $\mathrm{End}^\circ_{\bar{K}}(\psi') \otimes_{E'} E'_{Q'}$ in the algebraic group $\underline{\mathrm{Aut}}_{E'_{Q'}}\big(V_{Q'}(\psi')\big) \cong \mathrm{GL}_{rd',E'_{Q'}}$. As with G_Q we find that $G'_{Q'}$ is an inner form of GL_r over $E'_{Q'}$ that acts faithfully on $V_P(\varphi)$, such that $G'_{Q'}(E'_{Q'})$ contains an open subgroup of Γ_P. Recall from Proposition 2.5 that passing from Γ_P to any open subgroup does not change the ring E_P. Thus the uniqueness [9, Thm. 3.6] of the minimal quasi-model of $(F_P, \mathrm{PGL}_{r,F_P}, \Gamma_P^{\mathrm{ad}})$ implies that $E_P \subset E'_P$. As this holds for all P, Proposition 3.1 (b) then shows that $E \subset E'$. This implies that $B \subset B'$ and therefore $\mathrm{End}_{\bar{K}}(\varphi|C) = \mathrm{End}_{\bar{K}}(\psi') \subset \mathrm{End}_{\bar{K}}(\psi)$, proving 1.2 (c).

This shows that the field E constructed above has all the desired properties. For the uniqueness note first that $C = B$ is one possible choice in 1.2 (c). Thus this property implies that $\mathrm{End}_{\bar{K}}(\psi)$ is the union of the rings $\mathrm{End}_{\bar{K}}(\varphi|C)$ for all $C \subset A$, which determines $\mathrm{End}_{\bar{K}}(\psi)$ uniquely. This in turn determines E by 1.2 (b), as desired. **q.e.d.**

Proof of Theorem 1.3. Assertions (a) and (b) in both versions are restatements of Theorem 3.4. It remains to prove (c). Let K' and Σ be as in Theorem 1.3. Let $E^{\mathrm{tr}}(K',\Sigma) \subset E^{\mathrm{ch}}(K',\Sigma)$ be the subfields of F generated by the traces, respectively by all coefficients of the characteristic polynomials, of $\rho_P(\mathrm{Frob}_x)$ for all $x \in \Sigma$. As in Section 3 we let $(\_)_P$ denote the closure in F_P.

Lemma 5.1 *Under the conditions in 1.3 (c) we have*

$$E^{\mathrm{tr}}(K',\Sigma)_P = E^{\mathrm{ch}}(K',\Sigma)_P = E_P = E_Q.$$

Proof. Let $\Gamma'_P \subset \Gamma_P$ be the open subgroup corresponding to K'. For 1.3 (c) we assume that $\mathrm{End}_{\bar{K}}(\psi) = \mathrm{End}_{K'}(\psi)$, which by the construction of G_Q implies that $\Gamma'_P \subset G_Q(E_Q)$. Now as G_Q is an inner form of GL_{r,E_Q}, all coefficients of the characteristic polynomial in the standard representation correspond to algebraic morphisms $G_Q \to \mathbb{A}^1_{E_Q}$ defined over E_Q. It follows that the coefficients of the characteristic polynomials of all $\rho_P(\mathrm{Frob}_x)$ lie in E_Q. Therefore $E^{\mathrm{ch}}(K',\Sigma)_P \subset E_Q$.

On the other hand the Frobeniuses $\rho_P(\mathrm{Frob}_x)$ for $x \in \Sigma$ form a dense subset of Γ'_P, because Σ has Dirichlet density 1. Thus $E^{\mathrm{tr}}(K',\Sigma)_P$ is the total ring of quotients of the closure of the subring of F_P generated by the traces of all elements of Γ'_P. By [9, Thm. 2.14] this implies that Γ'_P is contained in a model of GL_r over the subring $E^{\mathrm{tr}}(K',\Sigma)_P$. In particular $(\Gamma'_P)^{\mathrm{ad}}$ is contained in a model of PGL_r, which by the uniqueness [9, Thm. 3.6] of the minimal quasi-model of $(F_P, \mathrm{PGL}_{r,F_P}, \Gamma_P^{\mathrm{ad}})$ implies that $E_P \subset E^{\mathrm{tr}}(K',\Sigma)_P$. **q.e.d.**

From Lemma 5.1 and Proposition 3.1 (c) we deduce that $E^{\mathrm{tr}}(K',\Sigma) = E^{\mathrm{ch}}(K',\Sigma) = E$, proving 1.3 (c). **q.e.d.**

We finish this section with an explicit example. It turns out that the description of E by characteristic polynomials of Frobeniuses in the adjoint representation is the most practical one, because it does not involve passage to an a priori unknown finite extension K'.

Example 5.2 *Let $F := \mathbb{F}_p(t)$ and $A := \mathbb{F}_p[t]$ and $K := \mathbb{F}_{p^2}(x)$ with t and x transcendent over $\mathbb{F}_p$. Consider the Drinfeld module $\varphi : A \to K\{\tau\}$ of rank 3 with $\varphi_t = x\tau + \tau^3$. Then:*

(a) $\mathrm{End}_{\bar{K}}(\varphi) = A$.

(b) $E = \mathbb{F}_p(t^2)$ *and* $B = \mathbb{F}_p[t^2]$.

(c) $\mathrm{End}_{\bar{K}}(\varphi|B)$ *is the non-commutative polynomial ring* $\mathbb{F}_{p^2}\{t\}$ *with* $t\alpha = \alpha^p t$ *for all* $\alpha \in \mathbb{F}_{p^2}$.

Proof. If (a) fails, choose a maximal commutative subring $\hat{A} \subset \mathrm{End}_{\bar{K}}(\varphi)$ and let $\hat{\varphi} : \hat{A} \hookrightarrow \bar{K}\{\tau\}$ be its tautological embedding. Let $d > 1$ be the rank of $\hat{A}$ over A and r' the rank of $\hat{\varphi}$. Then dr' is the rank of φ, which is 3; hence $r' = 1$. Thus

Proposition 2.2 implies that $\hat{\varphi}$ is isomorphic over $\bar{K}$ to a Drinfeld module defined over a finite field. By restriction the same follows for φ, so there exists $y \in \bar{K}^*$ such that $y^{-1}\varphi_t y = y^{p-1}x\tau + y^{p^3-1}\tau^3$ has coefficients in $\bar{\mathbb{F}}_p$. But this implies that $x^{p^2+p+1} = (y^{p-1}x)^{p^2+p+1}/y^{p^3-1}$ and hence x lies in $\bar{\mathbb{F}}_p$, contrary to the assumption. This proves (a).

Next consider any element $u \in \mathbb{F}_{p^2}$. Then φ has good reduction at the place $x = u$ of K. We calculate

$$\varphi_{t^2} \;=\; (x\tau+\tau^3)^2 \;=\; x^{p+1}\tau^2 + (x+x^{p^3})\tau^4 + \tau^6 \;\equiv\; v\tau^2 + w\tau^4 + \tau^6 \mod (x-u),$$

where $v := u^{p+1} \in \mathbb{F}_p$ and $w := u + u^{p^3} = u + u^p \in \mathbb{F}_p$. Since the residue field at u is $\mathbb{F}_{p^2}$, the associated Frobenius acts like τ^2 and its characteristic polynomial is $vX + wX^2 + X^3 - t^2$. If $\lambda_1, \lambda_2, \lambda_3$ denote its roots in an extension of F, we find that

$$\sum_{i,j} \frac{\lambda_i}{\lambda_j} \;=\; (\lambda_1+\lambda_2+\lambda_3)\cdot\frac{\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3}{\lambda_1\lambda_2\lambda_3} \;=\; \frac{-vw}{t^2}.$$

By Theorem 3.4 this is an element of E. Any choice of $u \neq 0$ with $w = u + u^p \neq 0$ therefore implies that $t^2 \in E$.

In particular $C := \mathbb{F}_p[t^2]$ is contained in $B := E \cap A$. Since φ_{t^2} contains only even powers of τ, the ring $\mathrm{End}_{\bar{K}}(\varphi|C)$ contains both $\mathbb{F}_{p^2}$ and φ_t and hence the non-commutative polynomial ring $\mathbb{F}_{p^2}\{\varphi_t\} \cong \mathbb{F}_{p^2}\{t\}$ described in (c). By Theorem 1.2 (c) it follows that $\mathbb{F}_{p^2}\{\varphi_t\} \subset \mathrm{End}_{\bar{K}}(\varphi|B)$. Thus $\mathbb{F}_{p^2}\{\varphi_t\}$ commutes with the subring B, which means that B is contained in the center of $\mathbb{F}_{p^2}\{\varphi_t\}$. But this center is $\mathbb{F}[\varphi_{t^2}] \cong C$; hence $B \subset C$ and therefore $B = C$. This implies (b).

Finally note that A is a maximal commutative subalgebra of $\mathrm{End}_{\bar{K}}(\varphi|B)$ by (a), and of rank 2 over B. Thus $\mathrm{End}_{\bar{K}}(\varphi|B)$ is a B-order in a central quaternion algebra over E. But it already contains $\mathbb{F}_{p^2}\{\varphi_t\}$, which is a maximal order. Thus the two orders are equal, proving (c). **q.e.d.**

6 Drinfeld modules with non-scalar endomorphisms

In this section we discuss the consequences of the preceding results for a Drinfeld module $\varphi : A \to K\{\tau\}$ in special characteristic with an arbitrary endomorphism ring $\mathrm{End}_{\bar{K}}(\varphi)$. We begin by reviewing some basic properties of endomorphism rings.

By $K\{\tau\}$ we denote the non-commutative polynomial ring in one variable over K, where τ satisfies the commutation relation $\tau u = u^p\tau$ for all $u \in K$. A ring homomorphism $\varphi : A \to K\{\tau\}$, $a \mapsto \varphi_a$ is a Drinfeld module if and only if its image does not lie in $K \subset K\{\tau\}$. For any overfield L of K the endomorphism ring $\mathrm{End}_L(\varphi)$ is the set of elements of $L\{\tau\}$ which commute with φ_a for all $a \in A$. The map φ then defines an embedding $A \hookrightarrow \mathrm{End}_L(\varphi)$ which makes $\mathrm{End}_L(\varphi)$ a finitely generated torsion free A-module. Moreover $\mathrm{End}_L^\circ(\varphi) := \mathrm{End}_L(\varphi) \otimes_A F$ is a division algebra of finite dimension over F (cf. [2, § 2]) and all endomorphisms over L are defined already over a finite separable extension of K (cf. [5, Prop. 4.7.4, Rem. 4.7.5]). In particular we have $\mathrm{End}_{\bar{K}}(\varphi) = \mathrm{End}_{K^{\mathrm{sep}}}(\varphi) = \mathrm{End}_{K'}(\varphi)$ for some separable finite extension K' of K.

Now consider any infinite commutative subring $\hat{A} \subset \mathrm{End}_{\bar{K}}(\varphi)$ and let $\hat{\varphi} : \hat{A} \to \bar{K}\{\tau\}$ denote its tautological embedding. This is a Drinfeld $\hat{A}$-module, except that $\hat{A}$ is not necessarily a maximal order in its quotient field. But that is only a small problem, because most results about Drinfeld modules carry over directly to this more general case, as in Hayes [6]. One can also modify $\hat{\varphi}$ by a suitable isogeny, as follows. Let $\tilde{A}$ denote the integral closure of $\hat{A}$ in its quotient field. Then by [6, Prop. 3.2] there exists a Drinfeld module $\tilde{\varphi} : \tilde{A} \to \bar{K}\{\tau\}$ such that $\tilde{\varphi}|\hat{A}$ is isogenous to $\hat{\varphi}$, that is, there exists a non-zero $h \in \bar{K}\{\tau\}$ such that $h\,\hat{\varphi}_{\hat{a}} = \tilde{\varphi}_{\hat{a}}\,h$ for all $\hat{a} \in \hat{A}$. Let $\tilde{F}$ denote the common quotient field of $\hat{A}$ and $\tilde{A}$. Then after tensoring with $\tilde{F}$ the isogeny h induces an isomorphism $\mathrm{End}^\circ_{\bar{K}}(\hat{\varphi}) \cong \mathrm{End}^\circ_{\bar{K}}(\tilde{\varphi})$.

Moreover, let $\tilde{P}$ be the set of places of $\tilde{F}$ above those in P. Then $V_{\tilde{P}}(\tilde{\varphi}) \cong V_{\tilde{P}}(\hat{\varphi}) = V_P(\varphi)$, where the $\tilde{F}_{\tilde{P}}$-module structure on the latter is induced by

$$\tilde{F}_{\tilde{P}} \;\cong\; \tilde{A} \otimes_A F_P \;\hookrightarrow\; \mathrm{End}_{\bar{K}}(\varphi) \otimes_A F_P \;\hookrightarrow\; \mathrm{End}_{F_P}\big(V_P(\varphi)\big).$$

All this is equivariant under $\mathrm{Gal}(K^{\mathrm{sep}}/K')$ for any sufficiently large K'; hence the image of $\mathrm{Gal}(K^{\mathrm{sep}}/K')$ on $V_P(\varphi)$ coincides with that on $V_{\tilde{P}}(\tilde{\varphi})$.

Using this we can extend Theorems 1.1 and 1.2 as follows:

Theorem 6.1 *Let $\varphi : A \to K\{\tau\}$ be a Drinfeld A-module in special characteristic $\mathfrak{p}_0$, which is not isomorphic over $\bar{K}$ to a Drinfeld module defined over a finite field. Let Z denote the center of $\mathrm{End}^\circ_{\bar{K}}(\varphi)$. Write $[Z/F] = d$ and $\dim_Z \mathrm{End}^\circ_{\bar{K}}(\varphi) = e^2$. Then*

$$r' := \mathrm{rank}(\varphi)/de > 1.$$

Moreover there exists a unique subfield $E \subset Z$ with $[Z/E] < \infty$ and the following properties. For every non-empty finite set P of places $\neq \mathfrak{p}_0, \infty$ of F let $\tilde{P}$ denote the set of places of Z above those in P, and Q the set of places of E below those in $\tilde{P}$. Then $E_Q \subset Z_{\tilde{P}} \cong Z \otimes_F F_P$ acts naturally on $V_P(\varphi)$ and there exists an inner form G_Q of $\mathrm{GL}_{r'}$ over E_Q acting on $V_P(\varphi)$ such that:

(a) $G_Q^{\mathrm{der}}(E_Q) \cap \Gamma_P^{\mathrm{geom}}$ is open in both $G_Q^{\mathrm{der}}(E_Q)$ and Γ_P^{geom}.

(b) There exists an element $f \in E^*$ such that

$$\overline{f^{\mathbb{Z}}} \cdot \big(G_Q^{\mathrm{der}}(E_Q) \cap \Gamma_P^{\mathrm{geom}}\big)$$

is an open subgroup of Γ_P, where $\overline{f^{\mathbb{Z}}}$ denotes the pro-cyclic subgroup of the group of scalars in $G_Q(E_Q)$ that is topologically generated by f.

Proof. We apply the above reduction to the case that $\hat{A}$ is any maximal commutative subring of $\mathrm{End}_{\bar{K}}(\varphi)$. The definition of endomorphisms then implies that

$$\mathrm{End}^\circ_{\bar{K}}(\tilde{\varphi}) \;\cong\; \mathrm{End}^\circ_{\bar{K}}(\hat{\varphi}) \;\cong\; \mathrm{Cent}_{\mathrm{End}^\circ_{\bar{K}}(\varphi)}(\hat{A}\,) \;=\; \tilde{F}$$

and thus $\mathrm{End}_{\bar{K}}(\tilde{\varphi}) = \tilde{A}$. Note also that $[\tilde{F}/F] = de$, so the rank of $\tilde{\varphi}$ is $r' := \mathrm{rank}(\varphi)/de$. If $\tilde{\varphi}$ were isomorphic over $\bar{K}$ to a Drinfeld module defined over a finite field, then so would $\hat{\varphi}$ and hence φ. Thus Proposition 2.2 shows that $r' > 1$. In particular we can apply the earlier results to the Drinfeld module $\tilde{\varphi}$.

Let $E \subset \tilde{F}$ be the subfield associated to $\tilde{\varphi}$ by Theorem 1.1. Set $\tilde{B} := E \cap \tilde{A}$ and $\tilde{\psi} := \tilde{\varphi}|\tilde{B}$. Then applying Theorem 1.2 (b) and (c) to $\tilde{\varphi}$ with $A \subset \tilde{A}$ in place of $C \subset A$ we deduce that $\mathrm{End}^\circ_{\bar{K}}(\varphi) \cong \mathrm{End}^\circ_{\bar{K}}(\tilde{\varphi}|A) \subset \mathrm{End}^\circ_{\bar{K}}(\tilde{\psi})$ and that the center of the latter is E. Thus E commutes with $\mathrm{End}^\circ_{\bar{K}}(\varphi)$, which shows that $E \subset Z$. The other stated properties of E follow directly from Theorem 1.1.

Only the uniqueness of E is not yet guaranteed, because the construction depends on the choice of $\hat{A}$. But any subfield E with the stated properties also has the properties in Theorem 1.1 for the Drinfeld $\tilde{A}$-module $\tilde{\varphi}$. It is therefore unique by Theorem 1.1, as desired. **q.e.d.**

Theorem 6.2 *Let φ be as in Theorem 6.1. Then there exists a unique subfield E of the center Z of $\mathrm{End}^\circ_{\bar{K}}(\varphi)$ with the following properties:*

(a) The intersection $B := E \cap \mathrm{End}_{\bar{K}}(\varphi)$ is infinite with quotient field E, and $[Z/E]$ is finite.

(b) The tautological embedding $\psi : B \to \bar{K}\{\tau\}$ is a Drinfeld B-module (except that B is not necessarily a maximal order in E) whose endomorphism ring $\mathrm{End}_{\bar{K}}(\psi)$ is an order in a central simple algebra over E.

(c) For any other infinite commutative subring $C \subset \mathrm{End}_{\bar{K}}(\varphi)$ let $\chi : C \to \bar{K}\{\tau\}$ denote the tautological embedding. Then $\mathrm{End}_{\bar{K}}(\chi) \subset \mathrm{End}_{\bar{K}}(\psi)$.

Moreover, the field E is the same as in Theorem 6.1 and the group G_Q is the centralizer of $\mathrm{End}_{\bar{K}}(\psi) \otimes_B E_Q$ in the algebraic group $\underline{\mathrm{Aut}}_{E_Q}\big(V_Q(\psi)\big)$.

Proof. Let $\hat{A}$, $\hat{\varphi}$, $\tilde{A}$, $\tilde{\varphi}$, $\tilde{F}$, E be as above. Then G_Q has the given description and Theorem 1.2 implies:

($\tilde{\text{a}}$) The intersection $\tilde{B} := E \cap \tilde{A}$ is infinite with quotient field E, and $d := [F/E]$ is finite.

($\tilde{\text{b}}$) The restriction $\tilde{\psi} := \tilde{\varphi}|\tilde{B}$ is a Drinfeld $\tilde{B}$-module whose endomorphism ring $\mathrm{End}_{\bar{K}}(\tilde{\psi})$ is an order in a central simple algebra over E.

($\tilde{\text{c}}$) For every other infinite subring $\tilde{C} \subset \tilde{A}$ we have $\mathrm{End}_{\bar{K}}(\tilde{\varphi}|\tilde{C}) \subset \mathrm{End}_{\bar{K}}(\tilde{\psi})$.

Set $B := E \cap \mathrm{End}_{\bar{K}}(\varphi) = E \cap \hat{A}$. Since $\hat{A} \subset \tilde{A}$ has finite index, so does $B \subset \tilde{B}$; hence ($\tilde{\text{a}}$) implies (a). Next $\psi := \hat{\varphi}|B$ is a Drinfeld module isogenous to $\tilde{\psi}|B$, except that B is not necessarily a maximal order in E. Since any isogeny induces an isomorphism of endomorphism rings up to finite index, we find that ($\tilde{\text{b}}$) implies (b). Similarly ($\tilde{\text{c}}$) implies that for every infinite subring $C \subset \hat{A}$ we have $\mathrm{End}^\circ_{\bar{K}}(\hat{\varphi}|C) \subset \mathrm{End}^\circ_{\bar{K}}(\psi)$. In particular $\mathrm{End}_{\bar{K}}(\hat{\varphi}|C) \subset \mathrm{End}^\circ_{\bar{K}}(\hat{\varphi}|C)$ commutes with the center B of $\mathrm{End}_{\bar{K}}(\psi) \subset \mathrm{End}^\circ_{\bar{K}}(\psi)$, hence:

($\hat{\text{c}}$) For every infinite subring $C \subset \hat{A}$ we have $\mathrm{End}_{\bar{K}}(\hat{\varphi}|C) \subset \mathrm{End}_{\bar{K}}(\psi)$.

This is already a part of the remaining property (c), but only for subrings of $\hat{A}$. However, the field E is independent of the choice of $\hat{A}$ by Theorem 6.1. Thus for any infinite commutative subring $C \subset \mathrm{End}_{\bar{K}}(\varphi)$ we can simply choose $\hat{A}$ to be a maximal commutative subring of $\mathrm{End}_{\bar{K}}(\varphi)$ containing C; hence ($\hat{\text{c}}$) implies (c) in general.

We have thus shown that the subfield E from Theorem 6.1 has all the stated properties. For the uniqueness note that $C = B$ is one possible choice in (c). Thus (c) implies that $\mathrm{End}_{\bar{K}}(\psi)$ is the union of the rings $\mathrm{End}_{\bar{K}}(\chi)$ for all C, which determines $\mathrm{End}_{\bar{K}}(\psi)$ uniquely. This in turn determines E by (b), as desired. **q.e.d.**

To interpret the above theorem further let us say that a Drinfeld A-module φ and a Drinfeld C-module χ are *brothers* if and only if φ_a and χ_c commute for all $a \in A$ and $c \in C$. Then ψ from 6.2 (b) is a brother of φ, and 6.2 (c) says that $\mathrm{End}_{\bar{K}}(\chi) \subset \mathrm{End}_{\bar{K}}(\psi)$ for all other brothers of φ. Thus ψ is a brother of φ with a unique maximal endomorphism ring. Since $\mathrm{End}_{\bar{K}}(\psi)$ can be larger than $\mathrm{End}_{\bar{K}}(\varphi)$, one can ask whether one obtains yet more endomorphisms from brothers of ψ. The following strengthening of property 6.2 (c) shows that this is not the case. In other words applying Theorem 6.2 to ψ in place of φ simply yields ψ again.

Proposition 6.3 *In the situation of Theorem 6.2 we also have:*

(c$^+$) For any infinite commutative subring $C \subset \mathrm{End}_{\bar K}(\psi)$ let $\chi : C \to \bar K\{\tau\}$ denote the tautological embedding. Then $\mathrm{End}_{\bar K}(\chi) \subset \mathrm{End}_{\bar K}(\psi)$.

Proof. Theorem 6.2 (b) implies that the center of $\mathrm{End}_{\bar K}(\psi)$ is B. Thus applying Theorem 6.2 to ψ in place of φ (or to $\varphi|B_1$ for any integrally closed infinite subring $B_1 \subset B$) yields an infinite subring $B' \subset B$ which among other properties satisfies:

(c$'$) For any infinite commutative subring $C \subset \mathrm{End}_{\bar K}(\psi)$ let $\chi : C \to \bar K\{\tau\}$ denote the tautological embedding. Then $\mathrm{End}_{\bar K}(\chi) \subset \mathrm{End}_{\bar K}(\psi|B')$.

Since $\mathrm{End}_{\bar K}(\psi|B') \subset \mathrm{End}_{\bar K}(\psi)$ by 6.2 (c), this proves (c$^+$). **q.e.d.**

We finish with a criterion for when $E = F$:

Proposition 6.4 *In the situation of Theorem 6.1 we have $E = F$ if and only if:*

(a) the center of $\mathrm{End}_{\bar K}(\varphi)$ is A, and

(b) for any infinite commutative subring $C \subset A$ we have $\mathrm{End}_{\bar K}(\varphi|C) \subset \mathrm{End}_{\bar K}(\varphi)$.

Proof. If $E = F$, these properties follow directly from Theorem 6.2. Conversely assume (a) and (b). Then (a) implies $E \subset F$. We can therefore apply (b) with $C = B$ to deduce that $\mathrm{End}_{\bar K}(\psi) \subset \mathrm{End}_{\bar K}(\varphi)$. But the reverse inclusion follows from Theorem 6.2 (c) with $C = A$, so we have equality. Taking centers we deduce from (a) and 6.2 (b) that $B = A$ and thus $E = F$, as desired. **q.e.d.**

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