On ℓ-independence of Algebraic Monodromy Groups in Compatible Systems of Representations

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Consider a profinite group $\mathcal{G}$, and a collection of continuous representations $\rho_\ell: \mathcal{G} \to GL_n(\mathbf{Q}_\ell)$, indexed by a set $\mathbf{L}$ of rational primes ℓ. Suppose that $\mathcal{G}$ is endowed with a dense subset of "Frobenius" elements $\{F_\alpha | \alpha \in \mathbf{A}\}$. The system $\{\rho_\ell\}$ is called a compatible system of ℓ-adic representations if, for every $\alpha \in \mathbf{A}$, the characteristic polynomial of $\rho_\ell(F_\alpha)$ has coefficients in the field of rational numbers and does not depend on ℓ. In 6.5, we will give a more precise and less restrictive definition which allows us to throw out some bad pairs (ℓ, α) in order to accomodate ramification. Our notion slightly generalizes Serre's original definition [14]; to recover Serre's definition, we take $\mathcal{G}$ to be the Galois group of a number field K and F_α to be Frobenius representatives for primes of K.

Let G_ℓ be the Zariski closure of $\rho_\ell(\mathcal{G})$ in $GL_{n,\mathbf{Q}_\ell}$. This is the *algebraic monodromy group* at ℓ. Our question is the following: *How does G_ℓ vary with ℓ?* One hopes for some kind of "ℓ-independence." At best, there can exist a *global* algebraic group $G \subset GL_{n,\mathbf{Q}}$ such that every G_ℓ is conjugate to $G \times_{\mathbf{Q}} \mathbf{Q}_\ell$. Unfortunately, this does not always happen in the abstract setting in which we work, so we must settle for weaker ℓ-independence results.

We first recall what is already known in this direction. The compatibility condition bears only on the semisimple part of the elements $\rho_\ell(F_\alpha)$, so we lose no information by assuming all ρ_ℓ to be semisimple representations. (Alternatively, we could take arbitrary representations and define G_ℓ as the quotient of the Zariski closure of $\rho_\ell(\mathcal{G})$ by its unipotent radical.) Thus $\{G_\ell\}$ is a family of reductive groups. Serre has proved

Proposition **(6.12)** *The formal character of the representation $G_\ell^\circ \hookrightarrow GL_{n,\mathbf{Q}_\ell}$ is independent of ℓ.*

Proposition **(6.14)** *The group of connected components G_ℓ/G_ℓ° is independent of ℓ.*

These results are stated ([15] p.6, p.17 and [17] 2.2.3) for representations associated to abelian varieties, but the proofs work generally. We use (6.14) and passage to an open subgroup of $\mathcal{G}$ of finite index to reduce to the case that G_ℓ is connected for all ℓ. This hypothesis remains in force for the remainder of the introduction.

Our ℓ-independence results depend essentially on the Cebotarev density theorem and may therefore fail on a set of primes of Dirichlet density 0. So we always assume that the index set $\mathbf{L}$ of $\{\rho_\ell\}$ is of Dirichlet density 1. Each α defines a characteristic polynomial over $\mathbf{Q}$ and therefore a splitting field. The intersection of these fields, for all sufficiently regular F_α, defines the *splitting field of* $\{\rho_\ell\}$. The precise definition is given in 8.1, but we want to emphasize that this is one of the points where the connectedness of G_ℓ plays an essential role. We prove

Proposition **(8.9)** *For all ℓ belonging to a subset $\mathbf{L}' \subset \mathbf{L}$ of Dirichlet density 1, G_ℓ is unramified (in particular quasi-split) and split over $E\mathbf{Q}_\ell$.*

Our main result is:

Theorem **(Part of 9.1)** *For all ℓ belonging to a subset $\mathbf{L}' \subset \mathbf{L}$ of Dirichlet density 1, the Weyl group of G_ℓ depends only on the conjugacy class of the Frobenius at ℓ in $Gal(E/\mathbf{Q})$.*

This implies in particular that, for $\ell \in \mathbf{L}'$, the dimension of G_ℓ depends only on the conjugacy class of $Frob(\ell)$. If the representations ρ_ℓ are all absolutely irreducible, using results from [11], we can prove a stronger result:

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Theorem **(Part of 9.4)** *For all ℓ belonging to a subset $\mathbf{L}' \subset \mathbf{L}$ of Dirichlet density 1, the root datum and representation of G_ℓ depend only on the conjugacy class of $Frob(\ell)$ in $Gal(E/\mathbf{Q})$.*

Under certain dimension restrictions we can even prove that the root datum is independent of ℓ (see 9.6 and 9.7). Under these restrictions, or when the splitting field of $\{\rho_\ell\}$ is $\mathbf{Q}$, the groups G_ℓ (for $\ell \in \mathbf{L}'$) do in fact come from a common *global* group $G \subset GL_{n,\mathbf{Q}}$ (see 9.10).

We conclude with examples to show that we have pushed our axiomatic set-up 6.5 about as far as it will go. To obtain the stronger results which are believed to hold for cohomology representations, additional information will be needed.

The proof of Theorem 9.1 is quite technical, but the strategy is fairly simple. Suppose we have a compatible system $\{\rho_\ell\}$ of 2-dimensional representations. By Prop. 6.12, the rank of G_ℓ is independent of ℓ, so we suppose for definiteness that it is 2. The only connected, rank-2 reductive subgroups of $GL_{2,\mathbf{Q}_\ell}$ are the full group and the maximal tori $T_\ell \subset GL_{2,\mathbf{Q}_\ell}$. They are distinguished in that T_ℓ contains at most one unramified torus of rank 2 while GL_2 contains two up to conjugation: the split torus, and a non-split unramified torus which is the restriction of scalars of the multiplicative group from the unramified quadratic extension of $\mathbf{Q}_\ell$.

Each Frobenius element F_α has an associated (quadratic) characteristic polynomial with coefficients in $\mathbf{Q}$. We write d_α for the discriminant, and $E_\alpha = \mathbf{Q}(\sqrt{d_\alpha})$ for the splitting field. The set $\{F_\alpha \mid d_\alpha \neq 0\}$ is dense in $\mathcal{G}$, so we may assume that $d_\alpha \neq 0$ for all $\alpha \in \mathbf{A}$. Let Ω denote the compositum of all the E_α. We claim that either $[\Omega : \mathbf{Q}] = \infty$, in which case $G_\ell = GL_2$ for a set of primes of Dirichlet density 1; or $[\Omega : \mathbf{Q}] \leq 2$, in which case Ω is the splitting field of $\{\rho_\ell\}$, and G_ℓ is obtained from a torus over $\mathbf{Q}$. If $\Omega = \mathbf{Q}$, this is the rank-2 split torus, and if $[\Omega : \mathbf{Q}] = 2$, it is the restriction of the multiplicative group of Ω to $\mathbf{Q}$.

Suppose first that $[\Omega : \mathbf{Q}] = \infty$. Then we can take $\alpha_1, \ldots, \alpha_N$ such that the fields E_{α_i} are linearly disjoint. If for some ℓ, G_ℓ is a torus T_ℓ, then the $\rho_\ell(F_{\alpha_i})$ all lie in $T_\ell(\mathbf{Q}_\ell)$, so they all have the same splitting field, namely the splitting field of T_ℓ. Thus $\mathbf{Q}_\ell E_{\alpha_1} = \cdots = \mathbf{Q}_\ell E_{\alpha_2}$, or in terms of Legendre symbols, there exists X such that

$$X = \left(\frac{d_{\alpha_1}}{\ell}\right) = \left(\frac{d_{\alpha_2}}{\ell}\right) = \cdots = \left(\frac{d_{\alpha_N}}{\ell}\right).$$

The number of possible ℓ such that $X = 0$ is finite. By the Cebotarev density theorem, the density of primes ℓ such that $X = 1$ is 2^{-N}, and likewise for the set of primes ℓ such that $X = -1$. Letting $N \to \infty$, the density of $\{\ell | G_\ell = T_\ell\}$ is zero.

If Ω is a number field of degree > 2, we can find α and β such that E_α and E_β are distinct quadratic fields. Whenever $\left(\frac{d_\alpha}{\ell}\right) \neq \left(\frac{d_\beta}{\ell}\right)$, G_ℓ is GL_2. Let $\ell_1, \ell_2, \ldots$ denote an infinite sequence of such primes. If ℓ is one such, the Frobenius images $\rho_\ell(F_\alpha)$ are dense in some open subset of $GL_2(\mathbf{Q}_\ell)$. Now, $GL_{2,\mathbf{Q}_\ell}$ contains a split torus and an unramified non-split torus. Therefore, there exists α such that $\left(\frac{d_\alpha}{\ell}\right) = 1$ and α' such that $\left(\frac{d_{\alpha'}}{\ell}\right) = -1$. It is not difficult to see that more generally, the image of the map

$$\alpha \mapsto \left(\left(\frac{d_\alpha}{\ell_1}\right), \ldots, \left(\frac{d_\alpha}{\ell_M}\right)\right)$$

is $\{\pm 1\}^M$. As M can be made as large as we please and all d_α belong to a fixed number field Ω, this is absurd. If $[\Omega : \mathbf{Q}] \leq 2$, all the E_α are the same, so G_ℓ cannot be GL_2 for any ℓ. We conclude that all G_ℓ are tori, and they split in $\mathbf{Q}_\ell \Omega$. Thus Ω is the splitting field of $\{\rho_\ell\}$.

The analysis is, of course, much more involved for more complicated groups than GL_2, but the philosophy is the same. "Usually", $\rho_\ell(F_\alpha)$ lies in a unique maximal torus of G_ℓ, and this torus is determined up to $GL_n(\mathbf{Q}_\ell)$-conjugation by the characteristic polynomial of $\rho_\ell(F_\alpha)$. (Cf. 4.7. This situation is very similar to that in [15] Th. p.17.) Hence a compatible system of ℓ-adic representations gives simultaneous information about maximal tori of all G_ℓ (see 7.5). This suggests a question of some intrinsic interest: Consider a connected reductive group $G \subset GL_{n,K}$, where K is a non-archimedean local field. To what extent is G, and its given representation, determined by the $GL_n(K)$-conjugacy classes of all maximal tori of G? We do not treat this problem in full generality, but it underlies the material in §§2, 3 and 8.

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Part I. Preparation: Reductive Groups

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§1. Root Data.

We recall some notions defined in [18]. Let Λ be a free $\mathbf{Z}$-module of finite rank, $\Lambda^\vee = \mathrm{Hom}(\Lambda, \mathbf{Z})$ its dual, and $\langle\, ,\, \rangle$ the pairing between them. For $\alpha \in \Lambda$ and $\alpha^\vee \in \Lambda^\vee$, we consider the endomorphism of Λ defined by $s_{\alpha,\alpha^\vee}(\lambda) = \lambda - \langle \alpha^\vee, \lambda\rangle\alpha$. It is a reflection if and only if $\langle\alpha^\vee, \alpha\rangle = 2$. Its dual is given by $(s_{\alpha,\alpha^\vee})^\vee(\lambda^\vee) = s_{\alpha^\vee,\alpha}(\lambda^\vee) = \lambda^\vee - \langle\lambda^\vee, \alpha\rangle\alpha^\vee$.

(1.1) Consider finite subsets $\Phi \subset \Lambda$ and $\Phi^\vee \subset \Lambda^\vee$ endowed with a bijection $\Phi \ni \alpha \mapsto \alpha^\vee \in \Phi^\vee$. Assuming that for all α we have

(1.1.1) $\langle\alpha^\vee, \alpha\rangle = 2$,
(1.1.2) $s_{\alpha,\alpha^\vee}(\Phi) \subset \Phi$, and
(1.1.3) $s_{\alpha^\vee,\alpha}(\Phi^\vee) \subset \Phi^\vee$,

the bijection $\alpha \mapsto \alpha^\vee$ is determined by the remaining data. The quadruple $\Psi = (\Lambda, \Phi, \Lambda^\vee, \Phi^\vee)$ is called a *root datum* (cf. [18] 1.1).

(1.2) Axiom (1.1.1) implies $0 \notin \Phi$. Thus Φ is a root system in the vector space $\mathbf{Q}\Phi \subset \Lambda_{\mathbf{Q}}$, in the sense of [1] ch.VI, 1.1, Déf.1, and $\Phi^\vee$ is the dual root system to Φ. We write $\mathrm{Aut}(\Psi)$ for the group of $w \in GL(\Lambda)$ such that $w(\Phi) = \Phi$ and $w^\vee(\Phi^\vee) = \Phi^\vee$. We call Ψ *reduced* if Φ is reduced. The $s_{\alpha,\alpha^\vee}$ generate a finite normal subgroup $W(\Psi) \subset \mathrm{Aut}(\Psi)$, called the *Weyl group* of Ψ ([18] 1.3). We write $\mathrm{Out}(\Psi) := \mathrm{Aut}(\Psi)/W(\Psi)$.

(1.3) Consider a root datum $(\Lambda, \Phi, \Lambda^\vee, \Phi^\vee)$. A sub-root datum is any root datum of the form $(\Lambda, \Phi', \Lambda^\vee, \Phi'^\vee)$, where $\Phi' \subset \Phi$ and $\Phi'^\vee \subset \Phi^\vee$. It is not required that Φ' be a closed subset of Φ in the sense of [1] VI 1.7 Déf.4, or that it be of the same rank.

(1.4) An *isogeny* from a root datum $(\Lambda', \Phi', \Lambda'^\vee, \Phi'^\vee)$ to a root datum $(\Lambda, \Phi, \Lambda^\vee, \Phi^\vee)$ is an injective homomorphism $f : \Lambda' \to \Lambda$ with finite cokernel, that induces a bijection $\Phi' \to \Phi$, and whose transpose induces a bijection $\Phi^\vee \to \Phi'^\vee$ ([18] 1.7).

(1.5) Suppose that instead of Λ we are given a finite dimensional $\mathbf{Q}$-vector space V, with dual $V^\vee$. Suppose that $\Phi \subset V$, $\Phi^\vee \subset V^\vee$ satisfy the conditions 1.1. Consider the axiom axiom:

(1.5.1) For all $\alpha \in \Phi$ and $\beta^\vee \in \Phi^\vee$ we have $\langle\beta^\vee, \alpha\rangle \in \mathbf{Z}$.

It is equivalent to the existence of a lattice $\Lambda \subset V$ that contains Φ and whose dual contains $\Phi^\vee$. Hence the functor

$$\Psi = (\Lambda, \Phi, \Lambda^\vee, \Phi^\vee) \mapsto \Psi_{\mathbf{Q}} = (\Lambda_{\mathbf{Q}}, \Phi, \Lambda^\vee_{\mathbf{Q}}, \Phi^\vee)$$

identifies the category of all root data in which all isogenies have been inverted with the category of all quadruples $(V, \Phi, V^\vee, \Phi^\vee)$ as above, satisfying 1.1 and 1.5.1. We call an element of the latter category a *root datum up to isogeny.* We write $W(\Psi_{\mathbf{Q}}) = W(\Psi)$.

(1.6) Direct sums of root data, or of root data up to isogeny are defined in the obvious way. We use additive notation: $\Psi = \Psi' + \Psi''$. Let us call a root datum up to isogeny irreducible if it is not isomorphic to the direct sum of two non-trivial root data up to isogeny. It is easy to prove the following facts:

Lemma **(1.7)** *Every root datum up to isogeny possesses a unique decomposition into irreducible pieces. The irreducible root data up to isogeny are the following:*
(1.7.1) Semisimple case: Φ *is a simple root system in the vector space* V *(generated by* Φ*), and* $\Phi^\vee$ *(the dual root system) is uniquely determined by* Φ*.*
(1.7.2) Toral case: $\dim(V) = 1$ *and* $\Phi = \Phi^\vee = \emptyset$*.*

Variant **(1.8)** Consider pairs $(\Psi_{\mathbf{Q}}, \sigma W(\Psi_{\mathbf{Q}}))$, where $\Psi_{\mathbf{Q}}$ is a root system up to isogeny, and $\sigma \in \mathrm{Aut}(\Psi_{\mathbf{Q}})$ an element of finite order. With direct sums defined in the obvious way, we have:

Lemma **(1.9)** *Every such pair possesses a unique decomposition into irreducible ones. The irreducible ones are the following:*
(1.9.1) Semisimple case: Let Ω *be a simple root system,* n *a positive integer, and* $\tau \in Out(\Omega)$*. Let* Φ *be the direct sum of* n *copies of* Ω*,* V *the vector space generated by* Φ*, and* $\Phi^\vee$ *its dual root system. Let* σ *be an automorphism of* Φ *that cyclically permutes the* n *simple factors of* Φ*. Then* σ^n *maps every simple factor to itself, and its image under the homomorphism* $Aut(\Omega)^n \to Out(\Omega)^n$ *is the element* $(\tau_1, \ldots, \tau_n)$*, where the* τ_i *all belong to a single conjugacy class* $[\tau] \subset Out(\Omega)$*. The isomorphy class of* $(\Psi_{\mathbf{Q}}, \sigma W(\Psi_{\mathbf{Q}}))$ *is uniquely determined by* Ω*,* n*, and* $[\tau]$*.*
(1.9.2) Toral case: Let n *be an integer,* V *a faithful irreducible representation over* $\mathbf{Q}$ *of a cyclic group of order* n*, and* σ *the image of a generator of this group. Let* $\Phi = \Phi^\vee = \emptyset$*. The isomorphy class of* $(\Psi_{\mathbf{Q}}, \sigma W(\Psi_{\mathbf{Q}})) = (\Psi_{\mathbf{Q}}, \sigma)$ *is uniquely determined by* n*.*
Proof. We say a root datum up to isogeny is *isotypic* if it is a direct sum of isomorphic irreducible pieces. Every Ψ has a unique decomposition into isotypic pieces $\Psi_1 + \cdots + \Psi_k$. Every automorphism of Ψ preserves this decomposition, and irreducible pieces can only be mapped to irreducible pieces in the same isotypic component, so $\mathrm{Aut}(\Psi) = \mathrm{Aut}(\Psi_1) \times \cdots \times \mathrm{Aut}(\Psi_k)$. This reduces the problem to the case Ψ is isotypic: $\Psi = m\Psi_0$, Ψ_0 irreducible. Assume Ψ is semisimple. Every $\sigma \in \mathrm{Aut}(\Psi) = \mathrm{Aut}(\Psi_0)^m \rtimes S_m$ induces a permutation of factors. If the permutation respects a partition of $\{1, \ldots, m\}$, σ respects the corresponding decomposition of Ψ. This reduces the problem to the case that σ acts transitively on $\{1, \ldots, m\}$. Let τ be the permutation $(1\,2 \cdots n)$ viewed as an element of $\mathrm{Aut}(\Psi)$, and let $v = (v_1, \ldots, v_m)$, $w = (w_1, \ldots, w_m)$ be elements of $\mathrm{Aut}(\Psi_0)^m$. Then

$$v(\tau w)v^{-1} = \tau v^\tau w v^{-1} = \tau(v_2 w_1 v_1^{-1}, v_3 w_2 v_2^{-1}, \ldots, v_1 w_m v_m^{-1}).$$

In other words, in the coset $\tau\mathrm{Aut}(\Psi_0)^m$, the $\mathrm{Aut}(\Psi_0)^m$-conjugacy classes are indexed by elements of $\mathrm{Aut}(\Psi_0)$; the conjugacy class of τw is $w_1 w_2 \cdots w_m$. Therefore, up to conjugacy, $\tau w W(m\Psi_0)$ is determined by the image of $w_1 \ldots w_m$ in $\mathrm{Aut}(\Psi_0)/W(\Psi_0)$, or equivalently, the image of any coordinate of $(\tau w)^m$ in $\mathrm{Aut}(\Psi_0)/W(\Psi_0)$. This second formulation shows that different elements of $\mathrm{Aut}(\Psi_0)/W(\Psi_0)$ give different $\mathrm{Aut}(\Psi)$-conjugacy classes of $\sigma = \tau w$. This gives the classification 1.9.1.

A toral root datum up to isogeny is just a $\mathbf{Q}$-vector space. If $\Psi_{\mathbf{Q}}$ is toral, the pair $(\Psi_{\mathbf{Q}}, \sigma W(\Psi_{\mathbf{Q}}))$ consists of a $\mathbf{Q}$-vector space and an endomorphism of finite order. Then 1.9.2 just claims that a faithful irreducible $\mathbf{Q}$-representation of a finite cyclic group is determined up to isomorphism by the order of the group. This is an easy consequence of the character theory of cyclic groups. □

§2. Characteristic Polynomials of Root Datum Automorphisms.

Consider a pair $(\Psi_{\mathbf{Q}}, \sigma W(\Psi_{\mathbf{Q}}))$ as in 1.8, with $\Psi_{\mathbf{Q}} = (V, \Phi, V^\vee, \Phi^\vee)$. Denote by F_τ the characteristic polynomial of an automorphism τ of V. Let $\mathcal{F}$ be the set of F_τ for all $\tau \in \sigma W(\Psi_{\mathbf{Q}})$. The aim of this section is to prove the following theorem.

Theorem **(2.1)** $\mathcal{F}$ *determines the triple* $(V, W(\Psi_{\mathbf{Q}}), \sigma W(\Psi_{\mathbf{Q}}))$ *uniquely up to isomorphism.*

(2.2) By [1] ch.VI, 1.5, Th. 2 (iv), the set of reflections given by roots in Φ is determined by the Weyl group. Therefore the triple $(V, W(\Psi_{\mathbf{Q}}), \sigma W(\Psi_{\mathbf{Q}}))$ determines the roots up to rational multiples. Clearly, irreducible factors of $(\Psi_{\mathbf{Q}}, \sigma W(\Psi_{\mathbf{Q}}))$ of type B_n, C_n, BC_n cannot distinguished. The multiplicity of every irreducible direct factor which is not of type B, C, or BC is determined, while for each n, only the sum of the multiplicities of B_n, C_n, and BC_n factors is determined.

(2.3) Before proceeding to the proof of 2.1, we introduce some notation. We write $\Theta = (\Psi_{\mathbf{Q}}, \sigma W(\Psi_{\mathbf{Q}}))$, and $\mathcal{F} = \mathcal{F}(\Theta)$. The irreducible pairs in 1.9.1 are written as follows. Let Ω, n, τ be as in 1.9.1. The group of outer automorphisms of a simple root system is trivial, cyclic of order 2, or dihedral of order 6. Therefore, τ is determined up to conjugacy by its order in $\mathrm{Out}(\Omega)$, which is an integer $1 \leq k \leq 3$. We write (Ω, n, k) for the irreducible pair defined by 1.9.1. If Θ has an irreducible factor of the form $(C_m, n, 1)$ or $(BC_m, n, 1)$, we may replace this factor by the corresponding $(B_m, n, 1)$. Having done this, we may assume that Θ has no irreducible factors of type BC_m or C_m. This assumption will remain in force for the rest of §2. We want to show that it implies that Θ is uniquely determined by $\mathcal{F}(\Theta)$.

Lemma **(2.4)** $\mathcal{F}((\Omega, n, k)) = \{F(X^n) \mid F \in \mathcal{F}((\Omega, 1, k))\}$.
Proof. The automorphism group of $n\Omega$ is a semi-direct product $\mathrm{Aut}(\Omega)^n \rtimes S_n$. For every element $\sigma \in \mathrm{Aut}(n\Omega)$ that permutes the n simple factors transitively, we have $\sigma^n \in \mathrm{Aut}(\Omega)^n$, and each component lies in the same conjugacy class. Moreover, for each element $x \in \tau W(\Omega)$, we have

$$((1\,2\,\cdots\,n)(1, 1, \ldots, 1, x))^n = (x, x, \ldots, x),$$

so the n^{th} power map maps onto the set of $\mathrm{Aut}(\Omega)$-conjugacy classes of $(\Omega, 1, k)$. The assertion follows. □

(2.5) Every $\tau \in \sigma W(\Psi_{\mathbf{Q}})$ is an automorphism of finite order of the $\mathbf{Q}$-vector space V. Therefore, every element $F \in \mathcal{F}(\Theta)$ is a product of cyclotomic polynomials. If Φ_n denotes the n^{th} cyclotomic polynomial, we have a unique factorization $F = \prod_{n \geq 1} \Phi_n^{a_n}$, where the a_n are non-negative integers. Since $\dim(V) = \deg(F)$, we have an *a priori* bound on the possible n for which a_n can be non-zero. Mapping

$$F \mapsto \sum a_i\,[i] \in H := \bigoplus_{i=1}^{N} \mathbf{R} \cdot [i] \cong \mathbf{R}^N$$

for some fixed, sufficiently large, N, we can consider $\mathcal{F}(\Theta)$ as a subset of H. If Θ is a direct sum of Θ_i, we clearly have $\mathcal{F}(\Theta) = \sum \mathcal{F}(\Theta_i) = \{\sum h_i \mid h_i \in \mathcal{F}(\Theta_i)\}$ in H. Viewing $\mathcal{F}(\Theta)$ in this way, we can use convex geometry to extract information.

Lemma **(2.6)** *For a subset $A \subset H$ denote by $\hat{A}$ the convex closure of A. Consider non-empty subsets $A, B \subset H$.*
(2.6.1) $\widehat{A + B} = \hat{A} + \hat{B}$.
(2.6.2) If A, B are compact and convex, then B is determined by A and $A + B$.
Proof. The first assertion is trivial. For the second, let B' be the set of all $h \in H$ such that $A + h \subset A + B$. Clearly it contains B. Assume that there exists an element $h \in B' \setminus B$. Since B is compact and convex, it is contained in a closed half-space that does not contain h. In other words, there exists a linear form ℓ on H such that $\ell(b) > \ell(h)$ for all $b \in B$. Since A is compact, there exists an element $a_\circ \in A$ such that $\ell(a) \geq \ell(a_\circ)$ for all $a \in A$. Now by assumption $a_\circ + h \in A + B$, but by construction $\ell(a + b) > \ell(a_\circ + h)$ for all $a \in A$ and $b \in B$. This is a contradiction. □

(2.7) Write $\hat{\mathcal{F}}(\Theta)$ for the convex closure of $\mathcal{F}(\Theta)$ in H. We shall prove that $\hat{\mathcal{F}}(\Theta)$ already determines Θ. For this, the preceding lemma allows us to use induction on the rank of Θ. Indeed, assume that a non-trivial direct factor Θ_1 of Θ is already known. Letting Θ_2 be the complement, we find that $\hat{\mathcal{F}}(\Theta_2)$ is determined by $\hat{\mathcal{F}}(\Theta_1)$ and $\hat{\mathcal{F}}(\Theta)$. Thus it suffices to prove: If Θ is non-trivial, then $\hat{\mathcal{F}}(\Theta)$ determines at least one non-trivial direct factor of Θ.

(2.8) Let $F_\circ(\Theta)$ be the greatest common divisor of all characteristic polynomials in $\mathcal{F}(\Theta)$. If its image in H is $\sum a_i^\circ\,[i]$, then each $a_i^\circ = \min\{a_i\}$, the minimum taken over all $h = \sum a_i\,[i] \in \hat{\mathcal{F}}(\Theta)$. Thus $F_\circ(\Theta)$ is determined by $\hat{\mathcal{F}}(\Theta)$.

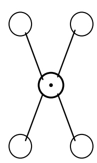
Lemma **(2.9)** *Let Θ_1 be the toral part of Θ. Then $\{F_\circ(\Theta)\} = \mathcal{F}(\Theta_1)$. In particular, the toral part of Θ is determined by $\hat{\mathcal{F}}(\Theta)$.*
Proof. Clearly $F_\circ(\Theta)$ is multiplicative for direct sums, so we may assume that Θ is irreducible. If it is toral, then $\mathcal{F}(\Theta)$ consists of precisely one element which will be $F_\circ(\Theta)$. If Θ is irreducible but not toral, we have to prove that $F_\circ(\Theta) = 1$. By 2.4, the case $\Theta = (\Omega, n, k)$ reduces to the case $\Theta = (\Omega, 1, k)$. First

assume that $\Omega = A_m$. Let $\epsilon = 1$ if $k = 1$, and $= -1$ if $k = 2$. Then $\mathcal{F}(\Theta)$ contains both $(1 - \epsilon X)^m$ and $(1 - (\epsilon X)^{m+1})/(1 - \epsilon X)$, which are relatively prime, as desired. For arbitrary $\Theta = (\Omega, 1, k)$, we use lemma 2.10 below to obtain another pair Θ' for which the assertion is already proved, and such that $F_\circ(\Theta)$ divides $F_\circ(\Theta') = 1$. □

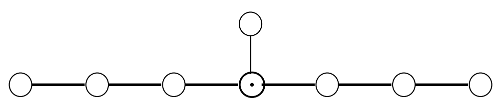
Lemma **(2.10)** *Let Ω be an irreducible semisimple root datum, and $\sigma \in Aut(\Omega)$. There exists a root subdatum $\Omega' \subset \Omega$ which is composed of simple root data of type A and has no toral part, such that the coset $\sigma W(\Omega)$ possesses a representative that stabilizes Ω'.*
Proof. Although by 2.3, we do not need to treat the cases C_n, BC_n, this lemma holds for all Ω. The short roots in B_n, BC_n, C_3, G_2 form root subsystems consisting entirely of factors of type A. Taking the corresponding coroots, we obtain root subdata of the desired sort. The root data of types C_n $(n \geq 4)$ and F_4 contain root data of type D_n. When Ω is of type D, we have inclusions $2nA_1 \subset D_{2n}$ and $A_3 + 2nA_1 \subset D_{2n+3}$ corresponding to the standard embeddings $SO(4)^n \hookrightarrow SO(4n)$ and $SO(6) \times SO(4)^n \hookrightarrow SO(4n+6)$, respectively. Clearly we can lift every outer automorphism of order ≤ 2 to an automorphism of the embedded root datum. It suffices, then, to treat the exceptional cases of type E and the D_4 triality case.

We recall ([1] VI 4.3) the "extended Dynkin diagram" of a simple root system Φ, obtained from the Cartan matrix of a set of roots Σ consisting of a basis of Φ together with the most negative root in Φ. Clearly, Σ satisfies condition (C1) of [4]; that is, $\alpha, \beta \in \Sigma$ implies $\alpha - \beta \notin \Sigma$. Therefore, by the remarks following [4], Table 6, any proper subset of Σ forms the basis of some root subsystem Φ' of Φ. The Dynkin diagram of Φ' is obtained from the extended Dynkin diagram of Φ by deleting the missing vertices, together with their incident edges. The outer automorphisms of Φ extend to automorphisms of the extended Dynkin diagram. If we delete a vertex which is fixed by this action, then $rank(\Phi') = rank(\Phi)$, and every outer automorphism of Φ can be represented by an automorphism that stabilizes Φ'. The extended diagrams for D_4, E_6, E_7, and E_8 are as follows ([1] VI 4.3 Th. 4):

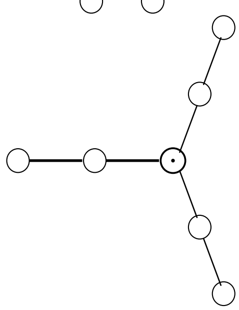
$\tilde{D}_4$:



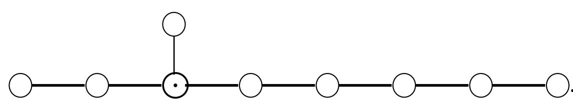
$\tilde{E}_7$:



$\tilde{E}_6$:



$\tilde{E}_8$:



In each case, the marked vertex is fixed by every automorphism of the completed Dynkin diagram, and its complement is a union of ordinary Dynkin diagrams of type A. □

(2.11) Fix $1 \leq m \leq N$. Consider all elements $a = \sum a_i\,[i] \in \hat{\mathcal{F}}(\Theta)$ for which a_N is as large as possible. Of these, take those elements for which a_{N-1} is as large as possible. Continue like this through a_m. The remaining indices $(i < m)$ are unconstrained. We obtain in this way a closed convex subset of $\hat{\mathcal{F}}(\Theta)$ which we denote $\hat{X}_m(\Theta)$. It is clearly additive in Θ:

$$\hat{X}_m(\Theta_1 + \Theta_2) = \hat{X}_m(\Theta_1) + \hat{X}_m(\Theta_2).$$

For any subset $X \subset H$, we define $m(X)$ as the largest index m such that for some $a = \sum a_i\,[i] \in X$ we have $a_m \neq 0$. Evidently $m(\Theta) = m(\mathcal{F}(\Theta))$ depends only on $\hat{\mathcal{F}}(\Theta)$, and $\hat{X}_m(\Theta) = \hat{\mathcal{F}}(\Theta)$ whenever $m > m(\Theta)$.

Lemma **(2.12)** *For Θ irreducible and $m \leq m(\Theta)$, $\hat{X}_m(\Theta)$ contains precisely one element.*
Proof. See 2.18 below.

Reduction **(2.13)** By 2.9 and the argument in 2.7 we may assume that Θ contains no toral part. Let Θ_1 consist of all irreducible factors Θ' of Θ with $m(\Theta') = m(\Theta)$, and Θ_2 of all remaining factors. Then we have $\hat{X}_{m(\Theta)}(\Theta) = \hat{X}_{m(\Theta)}(\Theta_1) + \hat{\mathcal{F}}(\Theta_2)$, and by the preceding lemma $\hat{X}_{m(\Theta)}(\Theta_1)$ contains precisely one element. By 2.9 this element is determined by $\hat{X}_{m(\Theta)}(\Theta)$. Thus $\hat{\mathcal{F}}(\Theta_2)$ is determined. By induction, Θ_2 is determined. By the argument in 2.7, $\hat{\mathcal{F}}(\Theta_1)$ is determined. We are thus reduced to the case where all irreducible factors Θ' of Θ have the same $m(\Theta')$.

(2.14) Fix an integer $1 \leq k \leq N$. In contrast to 2.11 we now consider all elements $a = \sum a_i\,[i] \in \hat{\mathcal{F}}(\Theta)$ for which a_N is as *small* as possible. Of these, we take those for which a_{N-1} is as small as possible, and so on through a_k. From here on we take those for which a_{k-1} is as *large* as possible, then those for which a_{k-2} is as large as possible, and so on through a_1. In this way we construct an extremal point of $\hat{\mathcal{F}}(\Theta)$, hence an element of $\mathcal{F}(\Theta)$. Let us call this element $x_k(\Theta)$. Clearly, it is additive in Θ:

$$x_k(\Theta_1 + \Theta_2) = x_k(\Theta_1) + x_k(\Theta_2).$$

(2.15) Now consider a semisimple root datum Θ such that for all irreducible factors Θ' of Θ, $m(\Theta')$ is a fixed integer m. By 2.9, $x_m(\Theta)$ is a linear combination of $[1]$, …, $[m-1]$. Let ℓ be the smallest integer such that $[m-\ell]$ occurs in $x_m(\Theta)$. Let Θ_1, …, Θ_k be the pairwise inequivalent irreducible pairs with $m(\Theta_i) = m$ and such that $[m-\ell]$ is the largest term that occurs in $x_m(\Theta_i)$. If n_i is the multiplicity with which Θ_i occurs in Θ, we write $\Theta = \Theta_0 + \sum_{i=1}^{k} n_i\Theta_i$ where Θ_0 contains all remaining factors. By construction, we have $x_m(\Theta_0) = x_{m-\ell}(\Theta_0)$, whence

$$x_m(\Theta) - x_{m-\ell}(\Theta) = \sum_{i=1}^{k} n_i \left(x_m(\Theta_i) - x_{m-\ell}(\Theta_i)\right).$$

Lemma **(2.16)** *If $\Theta_i \cong (A_1, \ell, 1)$, then $x_m(\Theta_i) - x_{m-\ell}(\Theta_i) = 0$. For all other Θ_i (keeping m and ℓ fixed), the vectors $x_m(\Theta_i) - x_{m-\ell}(\Theta_i)$ are linearly independent.*
Proof. See 2.18 below.

(2.17) By lemma 2.16, all coefficients n_i, with at most one exception, are determined. Thus it remains to treat the case where $\Theta = \Theta_0 + n(A_1, \ell, 1)$. But here, too, n is determined since the coefficient of $[m-\ell]$ in $x_m(\Theta)$ is n times the coefficient in $x_m((A_1, \ell, 1))$. This finishes the proof of 2.1, modulo lemmas 2.12 and 2.16.

(2.18) Lemma 2.12 and 2.16 are proved by explicit calculation. In the classical cases, this is very elementary. The necessary information about the automorphism groups of the exceptional root systems is not so easy to obtain. Fortunately, the set of all possible eigenvalues of elements of the Weyl group, together with their maximal multiplicities, is determined by the exponents, which are tabulated. (For a proof, see 2.19 below.) In particular, for $\Theta = (\Omega, 1, 1)$, $m(\Theta)$ is equal to the Coxeter number $h(\Omega)$ (which is 1+the largest exponent), and ℓ is the difference of the two largest exponents of Ω. By easy considerations the precise values of $x_{m+1}(\Theta)$, $x_m(\Theta)$ and $x_{m-\ell}(\Theta)$ can be completely determined from 2.19 in the cases $(E_8, 1, 1)$, $(F_4, 1, 1)$, $(G_2, 1, 1)$. The case $(D_4, 1, 3)$ can be reduced completely to the F_4-case. Since the non-trivial outer automorphism of E_6 can be represented by the scalar -1, the values of m and ℓ are easily calculated for the twisted type $(E_6, 1, 2)$. For this type as well as for $(E_6, 1, 1)$ and $(E_7, 1, 1)$, the information in 2.19 does not suffice for computing the $x_i(\Theta)$. In these cases we resorted to the Atlas of Finite Groups ([3]).

For convenience we have included two tables (Tables 1 and 2) showing, for every simple type $\Theta = (\Omega, 1, k)$, the values $m = m(\Theta)$, the unique integer ℓ such that the highest non-zero term in $x_m(\Theta)$ is $[m-\ell]$, the lexicographically largest element of $\hat{\mathcal{F}}(\Theta)$, which is just $x_{m+1}(\Theta)$ and is the unique element of $\hat{X}_m(\Theta)$, the second largest element, $x_m(\Theta)$, and the lexicographically largest element with the smallest multiplicity of $[m-\ell]$, $x_{m-\ell}(\Theta)$. The information about arbitrary types (Ω, n, k) follows from this by 2.4. In Table 1 (classical cases) we use $\{n\}$ to denote the polynomial $X^n - 1$. For simplicity we have included the case $(A_3, 1, 2) \cong (D_3, 1, 2)$ in the D_n-series. In Table 2 (exceptional cases), the expression $[n]$ denotes, as in 2.5, the n^{th} cyclotomic polynomial. For the E_n-cases, we have included a Table 3 showing the conjugacy classes of the entries of Table 2 in Atlas notation, with the innovation that for a class x in G of order n, the lift to $2.G$ of order $2n$ is written $\tilde{x}$.

Proposition **(2.19)** *Let Ω be a simple root system, and $k_1, \dots, k_r$ the exponents of $W(\Omega)$. Then the least common multiple of the characteristic polynomials of all $w \in W(\Omega)$ (acting on $\mathbf{Q}\Omega$) is*

$$(1 - X^{k_1+1}) \cdots (1 - X^{k_r+1}).$$

Proof. Abbreviate $P(X) = (1 - X^{k_1+1}) \cdots (1 - X^{k_r+1})$, and let χ be the character of an irreducible representation of $W(\Omega)$. By [2] Prop. 11.1.1,

$$P(X) \sum_{w \in W} \frac{\chi(w)}{det(1 - wX)}$$

is a polynomial in X, and in the special case $\chi = 1$ it is a non-zero constant. Taking linear combinations, we obtain a polynomial for every class function χ. Applying this to the characteristic function of the conjugacy class of a fixed element $w \in W(\Omega)$, it follows that $det(1 - wX)$ divides $P(X)$. This gives one direction of the asserted equality. For the other, take $\chi = 1$. We have just seen that all terms in the sum

$$\sum_{w \in W} \frac{P(X)}{det(1 - wX)}$$

are polynomials. The fact that this sum is a non-zero constant shows that these terms have no non-trivial common divisor. This is the other direction. □

§3. Maximal Tori of Reductive Groups.

We collect some (mostly) known results on the classification of quasi-split connected reductive groups, and of their maximal tori. We begin with generalities which hold for any perfect field K.

(3.1) Let G be a connected reductive group over K, and T a maximal torus. There is a canonical way to associate a reduced root datum $\Psi = (\Lambda, \Phi, \Lambda^\vee, \Phi^\vee)$, where Λ is the character group of T and Φ the set of non-zero weights of T in the adjoint representation on $Lie(G)$ ([18] 2.2). The Galois action induces a continuous homomorphism $\mathrm{Gal}(K^{sep}/K) \to \mathrm{Aut}(\Psi)$ which we denote by φ_T.

(3.2) Any two different maximal tori of G are conjugate over the separable closure K^{sep}. This conjugation induces an isomorphism between the root data associated to the two tori which is uniquely determined up to conjugation by the Weyl group. Let $\mathrm{Out}(\Psi) = \mathrm{Aut}(\Psi)/W(\Psi)$ denote the group of so–called outer automorphisms and π the projection $\mathrm{Aut}(\Psi) \to \mathrm{Out}(\Psi)$. Given two maximal tori the associated groups $\mathrm{Out}(\Psi)$ are canonically isomorphic, and the composite homomorphism $\bar\varphi = \pi \circ \varphi_T : \mathrm{Gal}(K^{sep}/K) \to \mathrm{Out}(\Psi)$ is independent of T.

(3.3) Conversely, fix a root datum Ψ and such a homomorphism $\bar\varphi$. Then there exists a quasi-split connected reductive group over K, unique up to isomorphism, with these invariants ([6] Satz 3.1.1, also [18] Prop. 2.13).

(3.4) A central isogeny between two connected reductive groups induces an isogeny of the corresponding root data, and conversely ([18] 2.5 and 2.8). Thus, by 3.3, a quasi-split connected reductive group *up to central isogeny* determines and is determined by a root datum up to isogeny $\Psi_{\mathbf{Q}}$ and a continuous homomorphism $\mathrm{Gal}(K^{sep}/K) \to \mathrm{Out}(\Psi_{\mathbf{Q}})$.

(3.5) Let us call φ_T the *type* of a maximal torus T; it is uniquely determined up to conjugation by $W(\Psi)$. We know that it must be a lift of $\bar\varphi$ to a continuous homomorphism $\mathrm{Gal}(K^{sep}/K) \to \mathrm{Aut}(\Psi_{\mathbf{Q}})$, but in general not every such lift occurs for a maximal torus. We have, however:

Theorem **(3.6)** *If G is quasi-split, then every lift of $\bar\varphi$ occurs for some maximal torus T of G.*
Proof. Fix a lift $\varphi : \mathrm{Gal}(K^{sep}/K) \to \mathrm{Aut}(\Psi_{\mathbf{Q}})$ of $\bar\varphi$. Choose an abstract torus T over K and an isomorphism from its character group to Λ, such that the Galois action is φ. This data determines a unique $G(K^{sep})$-conjugacy class of embeddings $T_{K^{sep}} \hookrightarrow G_{K^{sep}}$, and the assumption $\pi \circ \varphi = \bar\varphi$ means that this conjugacy class is defined over K. By [10] Cor. 2.2 this conjugacy class contains a K-rational element, *i.e.* an isomorphism from T to a maximal torus T' of G. By definition, $\varphi_{T'} = \varphi$, as desired. □

(3.7) In the rest of this section, K will be a non-archimedean local field. Letting K^{nr} denote its maximal unramified extension, we have a short exact sequence

$$1 \to \mathrm{Gal}(K^{sep}/K^{nr}) \to \mathrm{Gal}(K^{sep}/K) \to \mathrm{Gal}(K^{nr}/K) \to 1.$$

We denote by $Frob \subset \mathrm{Gal}(K^{sep}/K)$ the inverse image of arithmetic Frobenius in $\mathrm{Gal}(K^{nr}/K)$; this is a subset.

(3.8) A connected reductive group over K is usually called *unramified* if it is quasi-split over K and split over K^{nr}. In particular, a torus is unramified if it splits over K^{nr}. If a connected reductive group G over K possesses an unramified maximal torus, then it splits over K^{nr}, but it need not be quasi-split. If G splits over K^{nr}, the homomorphism $\bar{\varphi}$ of 3.2 is uniquely determined by the element (sic) $\bar{\varphi}(Frob) \in \mathrm{Out}(\Psi)$. By 3.3–4, unramified groups up to isogeny are classified by pairs $(\Psi_{\mathbf{Q}}, \sigma W(\Psi_{\mathbf{Q}}))$, where $\Psi_{\mathbf{Q}}$ is a root datum up to isogeny, and $\sigma \in \mathrm{Aut}(\Psi_{\mathbf{Q}})$ is of finite order.

(3.9) Consider a connected reductive group G. We define

$$\mathcal{F} = \bigcup_{T} \phi_T(Frob) \qquad \text{and} \qquad \mathcal{F}^{nr} = \bigcup_{T\ unramified} \phi_T(Frob)$$

where the union is extended over all maximal tori, respectively all unramified maximal tori of G. Clearly we have

$$\mathcal{F}^{nr} \subset \mathcal{F} \subset \pi^{-1}(\bar{\varphi}(Frob)) \subset \mathrm{Aut}(\Psi).$$

Theorem 3.6 implies:

Corollary **(3.10)** *If G is unramified, then $\mathcal{F}^{nr} = \mathcal{F} = \pi^{-1}(\bar{\varphi}(Frob))$.*

(3.11) For an element $\alpha \in \mathrm{Aut}(\Lambda)$ let $\mu(\alpha)$ be the multiplicity of the eigenvalue $+1$ on $\Lambda_{\mathbf{Q}}$. For a subset $X \subset \mathrm{Aut}(\Lambda)$ let $\mu(X) = \max_{\alpha \in X} \mu(\alpha)$. We shall prove the following converse to 3.10.

Theorem **(3.12)** *If $\mu(\mathcal{F}^{nr}) = \mu(\mathcal{F})$, then G is unramified.*

(3.13) **First reduction.** Every G has some maximal K-rational torus T, so $\mathcal{F}$ and hence $\mathcal{F}^{nr}$ is non-empty. Therefore G possesses an unramified maximal torus. Fix a maximal K-split torus $A \subset G$, and let M be its centralizer in G. This is a connected reductive (Levi-) subgroup. As G splits over K^{nr}, every K^{nr}-split torus extends to a maximal torus which is K^{nr}-split; in particular, M is K^{nr}-split. If M is a torus, then G is quasi-split, as desired. Otherwise, the semisimple part M^{ss} of M is non-trivial and anisotropic. By lemma 3.14 below there exists a maximal torus S of M^{ss} and an element $\sigma \in Frob$ which has the eigenvalue 1 at least once on the character group of S. Now the multiplicity of this eigenvalue of σ on the character group of the center of M is just the dimension of A. Thus, $\phi_{Z(M)S}(\sigma) \in \mathcal{F}$ has eigenvalue 1 with multiplicity $m > \dim(A)$. By assumption there exists an unramified maximal torus T for which $\phi_T(Frob)$ has eigenvalue 1 with multiplicity $\geq m$. But this means that the maximal split subtorus of T has dimension greater than $\dim(A)$, which is impossible since all maximal split tori are conjugate. To finish the proof of the theorem, we have to prove:

Lemma **(3.14)** *Let G be a non-trivial anisotropic connected semisimple group over K that splits over K^{nr}. There exists a maximal torus $T \subset G$ and an element $\sigma \in Frob$ which has the eigenvalue 1 at least once on the character group of T.*

Proof. Without loss of generality we may assume that G is simply connected and almost K-simple. Then, by a theorem of Kneser ([9] Satz 3) it must be isomorphic to the (algebraic) group of all elements of reduced norm 1 in a division algebra D over K. Replacing G by the full multiplicative group of D, we have to construct a maximal torus and a $\sigma \in Frob$ which has the eigenvalue 1 at least *twice*.

Let L be the center of D, and $\dim_L(D) = n^2$. Since G splits over K^{nr}, L/K is unramified. As G is non-abelian, $n \geq 2$. By the theory of central simple algebras over local fields, every field extension M/L of degree n injects into D, so every $M^\times$ is a maximal torus of $D^\times$ (in fact, every maximal torus has this form). The character group of this torus has $Hom_K(M, K^{sep})$ as a canonical base, and the Galois action comes from the permutation representation on this set. The multiplicity of the eigenvalue 1 for an element of the Galois group is just the number of cycles in this permutation representation. Thus the assertion is just the following:

Lemma **(3.15)** *Let L/K be an unramified extension, and $n \geq 2$. There exists a field extension M/L of degree n, and an element $\sigma \in Frob$ that does not act transitively on $Hom_K(M, K^{sep})$.*

Proof. Take any field extension N/K that is totally ramified of degree n. Then $M = N \otimes L$ is a field, and the natural map

$$Hom_K(M, K^{sep}) \to Hom_K(N, K^{sep}) \times Hom_K(L, K^{sep})$$

is a bijection. Moreover, there exists a unique element in $\mathrm{Gal}(M/N)$ that restricts to Frobenius in $\mathrm{Gal}(L/K)$. If $\sigma \in \mathrm{Gal}(K^{sep}/N)$ represents this element, it fixes at least one element of $Hom_K(N, K^{sep})$. Since $n \geq 2$, it cannot act transitively. □

§4. The Space of Characteristic Polynomials of a Representation.

Fix a field K of characteristic zero. In this section the field of definition of every algebraic variety or group is K, unless otherwise specified. Denote by $ch : GL_n \to \mathbf{G}_m \times \mathbf{A}^{n-1}$ the morphism associating to a matrix the coefficients of its characteristic polynomial. We want to study images under ch of reductive subgroups $G \subset GL_{n,K}$. We begin with tori.

(4.1) Let $\mathbf{G}_m^n$ be a split maximal torus in GL_n. The Weyl group of GL_n with respect to $\mathbf{G}_m^n$ is the symmetric group S_n, acting by permutation of factors. The restriction of ch to $\mathbf{G}_m^n$ identifies $\mathbf{G}_m \times \mathbf{A}^{n-1}$ with the scheme-theoretic quotient $\mathbf{G}_m^n/S_n$.

(4.2) Let T_0 be a subtorus of $\mathbf{G}_m^n$. Let ρ_0 denote the inclusion map, which we view as a representation of T_0. Write $\Gamma = \mathrm{Aut}(T_0, \rho_0)$ for the subgroup of $\mathrm{Aut}(T_0)$ which preserves ρ_0. If χ_i denotes the composition of ρ_0 with the i^{th} projection map, every element of Γ permutes the n-tuple $(\chi_1, \ldots, \chi_n)$. Thus, the canonical monomorphism

$$Norm_{S_n}(T_0)/Cent_{S_n}(T_0) \to \mathrm{Aut}(T_0, \rho_0)$$

is an isomorphism.

(4.3) Now let $G \subset GL_n$ be a connected reductive subgroup. Then there exists a subtorus $T_0 \subset \mathbf{G}_m^n$ such that every maximal torus of G is, over an algebraic closure of K, conjugate to T_0. The semisimple part of any point of G can, over an algebraic closure, be conjugated into T_0, hence we have $ch(T_0) = ch(G)$ pointwise. Since $ch|_{\mathbf{G}_m^n}$ is the finite morphism onto its quotient by S_n, we find that $ch(G) = ch(T_0)$ is Zariski-closed and that T_0 is, up to conjugation by S_n, uniquely determined by $ch(G)$. In other words, the pair (T_0, ρ_0) is determined by $ch(G)$, up to isomorphism (Cf. [11] §1.) The irreducibility of T_0 implies that of $ch(T_0)$. As every split torus T_0/K is obtained from a split torus over $\mathbf{Q}$ by extension of scalars, $ch(G) = ch(T_0)$ is defined over $\mathbf{Q}$.

(4.4) Let T_0 be a subtorus of $\mathbf{G}_m^n$. For every $\sigma \in S_n \setminus Cent_{S_n}(T_0)$, we define a proper subgroup $H_\sigma \subset T_0$. If $\sigma(T_0) = T_0$, we let $H_\sigma = \{t \in T_0 \mid \sigma(t) = t\}$; otherwise, $H_\sigma = T_0 \cap \sigma(T_0)$. We let Y be the union of $ch(H_\sigma)$ for all these σ; this is a Zariski-closed proper subset of $ch(T_0)$. Since up to conjugation T_0 is determined by $ch(T_0)$, Y depends only on $ch(T_0)$. Observe that, for any $t \in T_0$, $ch(t) \notin Y$ implies (but is in general not equivalent to the fact) that the eigenspace decomposition of t coincides with that of T_0. If T_0 is associated to a connected reductive subgroup $G \subset GL_n$ as above, Y is a Zariski-closed proper subset of $ch(G) = ch(T_0)$, depending only on $ch(G)$.

Definition **(4.5)** *Let $G \subset GL_n$ be a connected reductive subgroup, and $Y \subset ch(G)$ as above. A point $g \in G$ is* Γ-regular *if and only if $ch(g) \notin Y$.*

Proposition **(4.6)** *Every Γ-regular point of G is regular semisimple in G.*

Proof. Without loss of generality we may suppose that K is algebraically closed, and that $g \in G(K)$. Let g_{ss} denote its semisimple part, and T a maximal torus of G that contains g_{ss}. If $ch(g) \notin Y$, the definition of Y implies that $\sigma(g_{ss}) \neq g_{ss}$ for every $1 \neq [\sigma] \in Norm_{S_n}(T_L)/Cent_{S_n}(T_L) \cong \Gamma$. In particular $w(g_{ss}) \neq g_{ss}$ for every non-trivial w in the Weyl group of G with respect to T; *i.e.*, g_{ss} is regular semisimple in G. Thus, $g_{ss} = g$, as desired. □

Proposition **(4.7)** *Let G and Y be as above.*
(4.7.1) For every $x \in (ch(G) \setminus Y)(K)$, there exists a torus $T \subset GL_n$, and an element $t \in T(K)$, such that $ch(T) = ch(G)$ and $ch(t) = x$. The pair (t, T) is unique up to conjugation by $GL_n(K)$.
(4.7.2) Let $g \in G(K)$ be Γ-regular. Then g lies in a unique maximal torus T of G, and the $GL_n(K)$-conjugacy class of the pair (g, T) is uniquely determined by $ch(g)$ and $ch(G)$.
Proof. To prove (4.7.2), note first that as g is regular semisimple, the connected centralizer $T = Z_G(g)^\circ$ is the only maximal torus of G containing g. By (4.3), $ch(T) = ch(G)$. The uniqueness statement is now a consequence of (4.7.1).

For (4.7.1), first observe that there exists a unique semisimple conjugacy class of $t \in GL_n(K)$ with $ch(t) = x$. Fixing a representative, it suffices to show that there exists a unique torus $T \subset GL_n$ containing t with $ch(T) = ch(G)$.

Fix an arbitrary maximal torus $S \subset GL_n$ containing t, and a splitting field $L \supset K$ of S. Since $ch(t) \in ch(G)$, there exists an irreducible component T_L of $ch^{-1}(ch(G_L)) \cap S_L$ that contains t. Since $S_L \cong \mathbf{G}_m^n$ is split, T_L is a torus, and every other irreducible component is of the form $\sigma(T_L)$ for some $\sigma \in S_n \setminus Norm_{S_n}(T_L)$. Since $ch(t) \notin Y$, the definition of Y implies that none of these other irreducible components contains t. Thus T_L is unique. This shows that there exists a unique irreducible component T of $ch^{-1}(ch(G)) \cap S$ that contains t, and T is a torus, defined over K. The equality $ch(T) = ch(G)$, proves the existence claim.

For the uniqueness let S be as above, and consider any T with the desired properties. By the first part of the proof it suffices to show that T is contained in S. For this recall that the eigenspace decomposition of t coincides with that of T. In other words T is contained in the center of $Cent_{GL_n}(t)$. Since S is a maximal torus of the latter group, it automatically contains T, as desired. $\square$

(4.8) So far we have only studied connected reductive groups. If G is no longer connected, the map ch still tells us something about its connected components. The following results are due to Serre ([16]):

Proposition **(4.9)** *Let $G \subset GL_n$ be a reductive subgroup, G° its identity component, and $g \in G(K)$. Then $ch(gG^\circ)$ is Zariski-closed.*

This fact is important because information that is a priori given for the Zariski-closure applies to $ch(gG^\circ)$ itself. For instance:

Lemma **(4.10)** *Let $G \subset GL_n$ be a reductive subgroup, G° its identity component, and $g \in G(K)$. Then $ch(gG^\circ) = ch(G^\circ)$ if and only if $g \in G^\circ$.*
Proof. The "if" direction is obvious. Conversely we have $ch(id) \in ch(G^\circ) = ch(gG^\circ)$, hence $ch(id) = ch(gh)$ for some $h \in G^\circ$. This means that $u := gh$ is unipotent. Since every unipotent element of G is contained in the identity component, it follows that $g = uh^{-1} \in G^\circ$, as desired. $\square$

Actually, in the remainder of this article we could get by with the following corollary.

Proposition **(4.11)** *Let $G \subset GL_n$ be a reductive subgroup, G° its identity component, and $g \in G(K)$. Then the Zariski-closure of $ch(gG^\circ)$ is equal to $ch(G^\circ)$ if and only if $g \in G^\circ$.*

In [16] this is proved independently of 4.9. Here is anotherproof:
Proof. The "if" direction is trivial. Without loss of generality we may assume that K is algebraically closed. By (4.3), $\dim(ch(G^\circ)) = rank(G^\circ)$. We show first that $\dim(ch(gG^\circ)) = rank(G^\circ)$ implies that g induces an inner automorphism on G°. For this let L be an algebraically closed overfield of K of sufficiently large transcendence degree, and $\eta : Spec(L) \to gG^\circ$ a generic point in the algebro-geometric sense. Then $ch(gG^\circ) = \mathrm{Zar}(ch(\eta))$, where Zar denotes Zariski closure. Let d be a positive integer such that η^d is a point of G°. Observe that there exists a finite morphism $[d]$ from $\mathbf{G}_m \times \mathbf{A}^{n-1}$ to itself such that $ch(x^d) = [d](ch(x))$ for every $x \in GL_n$. This implies that $\dim(\mathrm{Zar}(ch(\eta^d))) = \dim(\mathrm{Zar}(ch(\eta))) = rank(G^\circ)$. Let $T_L \subset G_L^\circ$ be the smallest closed subgroup containing the semisimple part of η^d. Since K is algebraically closed, there exists a subgroup $T' \subset G^\circ$, defined over K, such that T'_L is conjugate to T_L under $G^\circ(L)$. Since T_L is contained in a maximal torus, so is T', and the inequalities $rank(G^\circ) = \dim(\mathrm{Zar}(ch(\eta^d))) \leq \dim(T') = \dim(T_L) \leq rank(G^\circ)$ imply that T_L is itself a maximal torus. But by definition, η centralizes T_L, so it (and hence g) induces an inner automorphism on G°.

If g induces an inner automorphism on G°, we may replace it by an element of gG° that centralizes G°. Fixing a maximal torus $T \subset G^\circ$, any semisimple element of gG° can be conjugated into gT. As the image under ch of the group generated by g and G° is the same as the image of the group generated by g and T, we may replace G by the latter group, which is abelian. Up to conjugation, G now lies inside $\mathbf{G}_m^n$. As

$$ch^{-1}(ch(gT)) \cap \mathbf{G}_m^n = \bigcup_{\sigma \in S_n} \sigma(gT),$$

$ch(gT) = ch(T)$ if and only if $gT = \sigma(T)$ for some $\sigma \in S_n$. Since $\sigma(T)$ contains the identity, it follows that $g^{-1} \in T$, whence $g \in T$, as desired. $\square$

§5. Absolutely Irreducible Representations.

Let K be an algebraically closed field of characteristic zero. Consider a connected reductive group G over K together with a faithful irreducible representation ρ. In theorem 4 of [11] we determined the extent to which (G, ρ) is determined by the formal character of this representation, in the case that G is semisimple. We recall this result, allowing G to be reductive.

(5.1) Let (G_i, ρ_i) be such pairs, $i = 1$, 2. We write $(G_1, \rho_1) \sim (G_2, \rho_2)$ and call these pairs *similar* if and only if for some maximal tori $T_i \subset G_i$ the pairs $(T_i, \rho_i|_{T_i})$ are isomorphic.

(5.2) Since ρ is a faithful representation, we may identify G with its image. Since ρ is irreducible, the center Z of G is just the intersection with the scalars. Thus either G is semisimple, or Z is equal to the group of scalars. In either case, $G^{der} = \ker(det(\rho))^\circ$. For any maximal torus T of G, we have $Z \subset T$ and $G^{der} \cap T = \ker(det(\rho|_T))^\circ$ and $T = \ker(det(\rho|_T))^\circ \cdot Z^\circ$. This shows that $(G_1, \rho_1) \sim (G_2, \rho_2)$ if and only if $\dim(Z(G_1)) = \dim(Z(G_2))$ and $(G_1^{der}, \rho_1|_{G_1^{der}}) \sim (G_2^{der}, \rho_2|_{G_2^{der}})$.

(5.3) For semisimple G, there exist the following basic similarity relations (see [11] Theorem 4). (The symbols C_n, D_n, etc. denote simple semisimple groups of the indicated type. The isomorphism class in the isogeny class is determined by the representation which is assumed to be faithful.)
(5.3.1) For all integers $3 \le n > i > 0$ there exist unique representations V_i, W_i of the simple semisimple groups of type C_n, respectively D_n, such that $(C_n, V_i) \sim (D_n, W_i)$. Here D_3 is taken to mean A_3. In the standard coordinates of the root systems, V_i and W_i have the highest weight $(i, i-1, \ldots, 1, 0, \ldots, 0)$.
(5.3.2) There exists a 4096-dimensional representation U of the simple semisimple group of type F_4 such that $(F_4, U) \sim (C_4, V_3) \sim (D_4, W_3)$.
(5.3.3) There exist 27-dimensional representations V, W, such that $(A_2, V) \sim (G_2, W)$.
(5.3.4) Fix $m \ge 2$, and consider a partition $m = m_1 + \ldots + m_k$. Let G be a semisimple group with root system $B_{m_1} \oplus \ldots \oplus B_{m_k}$, and ρ the (exterior) tensor product of the spin representations of all factors (Here B_1 is taken to mean A_1, for which the spin representation is the standard representation.) All pairs (G, ρ) thus obtained, with the same m, are similar.

(5.4) We call the similarity classes in 5.3 *basic ambiguous* classes. We call these and the similarity classes that contain just one isomorphy class with simple Lie algebra *basic* classes.

(5.5) Suppose that $(G_1, \rho_1) \sim (G_2, \rho_2)$ and $(G_1', \rho_1') \sim (G_2', \rho_2')$. Let $G_i \cdot G_i'$ denote the image of $G_i \times G_i'$ in the (exterior) tensor product of the representations ρ_i, ρ_i'. Then we have $(G_1 \cdot G_1', \rho_1 \otimes \rho_1') \sim (G_2 \cdot G_2', \rho_2 \otimes \rho_2')$. Thus the basic similarity relations generate many others.

Theorem **(5.5)** *Every similarity relation is induced by 5.5 from basic similarity relations.*
Proof. For semisimple G, this is theorem 4 of [11]. The general case follows from 5.2. $\square$

Proposition **(5.7)** *Let ρ be an irreducible representation of a connected reductive group G. Let T be a maximal torus of G.*
(5.7.1) There is, up to permutation, a unique factorization

$$(G, \rho) = (G_1 \cdots G_k, \rho_1 \otimes \cdots \otimes \rho_k)$$

where the similarity class of each (G_i, ρ_i) is basic, such that the basic similarity class 5.3.4 occurs at most once.

(5.7.2) If $(G,\rho) \sim (G',\rho')$, then for a suitable numbering the factors in the decompositions of both pairs are again pairwise similar.
(5.7.3) The corresponding decomposition of T is invariant under $Aut(T,\rho|_T)$.
Proof. The decomposition is almost the decomposition of (G,ρ) that comes from the decomposition of G as an almost direct sum of simple groups (or one-dimensional tori, if the center has positive dimension). The difference is only that all factors which are isomorphic to a spin representation of a simple group of type B_m are lumped together. Thus the first two assertions are clear.

To prove the third we may assume without loss of generality that G is semisimple. We argue as in [11], writing Γ for $\mathrm{Aut}(T,\rho|_T)$. Let Φ be the root system of G with respect to T. Let $\Phi^\circ \subset \Phi$ be the set of all roots that are short in the simple factors to which they belong. By the proposition in [11] §4, Φ° is determined by $(T,\rho|_T)$. In particular, it is invariant under Γ. By lemma 2 of [11] §2, Φ° is again a root system. The decomposition of Φ° into simple factors is in general *finer* than the decomposition into simple factors of Φ. Recall, however, that the short roots in any simple reduced root system not of type B form again a simple root system. Thus the factors of T which come from factors of G that are not of type B are uniquely determined by Φ°. This part of the decomposition is therefore invariant under Γ, and we are reduced to the case where Φ° is a direct sum of root systems of type $B_1 = A_1$. We reduce to the case that Γ permutes the simple A_1-factors transitively, *i.e.* that it acts irreducibly on $X^*(T)_{\mathbf{Q}}$. In this case the proof of theorem 4 in [11] shows that either we have no ambiguous case at all and Γ respects the decomposition of Φ into simple factors, or that we have the spin representation in *every* factor of G. In both cases the desired assertion holds. □

Lemma **(5.8)** *Except for the representations 5.3.3 (dimension 27) all basic ambiguous representations have even dimension greater than 2.*
Proof. Clearly there is no ambiguity if the dimension is 2, but for every even integer greater than 2 the case 5.3.1 with $i = 1$ yields an ambiguous representation of the given dimension, namely the similarity $SO(2n) \sim Sp(2n)$. The case 5.3.2 being obvious, it remains to prove that the dimensions of the representations in 5.3.1 and 5.3.4 are even. This is clear for the spin-representation in 5.3.4, and in 5.3.1 it can easily be calculated from Weyl's dimension formula. □

Part II. Algebraic Monodromy Groups

§6. Representations of F-groups.

Definition **(6.1)** *Consider a topological group $\mathcal{G}$ and a collection of elements F_α, indexed by $\alpha \in A$, called "Frobenius elements." If the F_α are* dense *in $\mathcal{G}$, then we call $(\mathcal{G}, A, \{F_\alpha\})$ an F*-group*.*

Example **(6.2)** Let K be a global field. If p is a finite place of K, let K_p denote the completion of K at K_p. There is a short exact sequence

$$0 \to I \to \mathrm{Gal}(K_p^{sep}/K_p) \to \widehat{\mathbf{Z}} \to 0;$$

we call the pre-image of 1, the *Frobenius coset.* We say an element $x \in \mathrm{Gal}(K^{sep}/K)$ is a *Frobenius element with respect to* p if there exists an embedding $K^{sep} \hookrightarrow K_p^{sep}$, such that x is the restriction of an element of the Frobenius coset. Let A be the set of elements $\alpha \in \mathrm{Gal}(K^{sep}/K)$ which are Frobenius elements with respect to some p. By the Cebotarev density theorem, $(\mathrm{Gal}(K^{sep}/K, A, \{F_\alpha\})$ is an F-group. This is the motivating example.

Example **(6.3)** Example 6.2 has a natural generalization as follows: Let X be a connected normal scheme of finite type over $\mathrm{Spec}(\mathbf{Z})$ and of dimension ≥ 1. Let K be the function field of X and $\mathcal{G} = \mathrm{Gal}(K^{sep}/K)$. For every closed point x let K_x be the function field of the henselization of X in x, and $K_{\bar{x}}$ that of the strict henselization. We have canonical embeddings $K \hookrightarrow K_x \hookrightarrow K_{\bar{x}}$. Every extension $K^{sep} \hookrightarrow K_x^{sep} = K_{\bar{x}}^{sep}$ of this embedding induces a homomorphism

$$j : \ \mathrm{Gal}(K_x^{sep}/K_x) \to \mathrm{Gal}(K^{sep}/K).$$

Let A be the set of all triples $\alpha = (x, j, F)$ where F lies in the j-image of any representative of Frobenius in $\mathrm{Gal}(K_{\bar{x}}/K_x)$. By density of Frobenius conjugacy classes, we have an F-group.

Example **(6.4)** Choose a set $\mathbf{L}$ of rational primes, and a reductive subgroup $G_\ell \subset GL_{n,\mathbf{Q}_\ell}$ for each $\ell \in \mathbf{L}$. Let K_ℓ be an open compact subgroup of $G_\ell(\mathbf{Q}_\ell)$, and $\mathcal{G} = \prod_\ell K_\ell$. Let $\{F_\alpha\}$ be the set of elements $F_\alpha = (\ell \mapsto k_\ell) \in \mathcal{G}$ for which there exists a polynomial $P_\alpha(x) \in \mathbf{Q}[x]$ such that the characteristic polynomial of all but at most finitely many k_ℓ equals $P_\alpha(x)$. Whether $\mathcal{G}$ is an F-group (*i.e.* whether the F_α are or are not dense) depends on the collection of K_ℓ.

Definition **(6.5)** *A* compatible system of ℓ-adic representations *(or* representation *for short) of an F-group $(\mathcal{G}, A, \{F_\alpha\})$ is a collection of continuous representations $\rho_\ell : \ \mathcal{G} \to GL_n(\mathbf{Q}_\ell)$, indexed by a set $\mathbf{L}$ of rational primes, such that there is a subset $\mathcal{X} \subset A \times \mathbf{L}$ satisfying the following conditions:*

(6.5.1) For every $\alpha \in A$, $(\alpha, \ell) \in \mathcal{X}$ for all but at most finitely many $\ell \in \mathbf{L}$.
(6.5.2) For any primes $\ell_1, \ldots, \ell_m \in \mathbf{L}$, the set $\{F_\alpha | (\alpha, \ell_i) \in \mathcal{X}$ for all $i = 1, \ldots, m\}$ is dense in $\mathcal{G}$.
(6.5.3) For all $(\alpha, \ell) \in \mathcal{X}$, the characteristic polynomial of $\rho_\ell(F_\alpha)$ has coefficients in $\mathbf{Q}$ and depends only on α.
(Compare Serre's definition [12] 1.3.)

Example **(6.6)** It is well-known that ℓ-adic cohomology provides such representations, in particular in the cases 6.2–3. Here the set $\mathcal{X}$ should consist of all (α, ℓ) such that α is a Frobenius element with residue characteristic $p \neq \ell$ and ρ_ℓ is unramified at the corresponding point. In Example 6.4, let $\mathcal{X}$ be the set of all (α, ℓ) such that, with $F_\alpha = (\ell \mapsto k_\ell)$, the characteristic polynomial of k_ℓ equals $P_\alpha(x)$. Taking ρ_ℓ to be the projection map, the conditions 6.5.1 and 6.5.3 hold automatically. We get a representation of an F-group if and only if 6.5.2 holds.

Definition **(6.7)** *We call $\{\rho_\ell\}$* everywhere semisimple *(resp.* everywhere absolutely irreducible*) if each representation ρ_ℓ is semisimple (resp. absolutely irreducible).*

(6.8) Given a representation $\{\rho_\ell\}$ of an F-group $\mathcal{G}$, we denote by G_ℓ the Zariski closure of $\rho_\ell(\mathcal{G})$, for every $\ell \in \mathbf{L}$. In symbols $G_\ell = \mathrm{Zar}(\rho_\ell(\mathcal{G}))$. This is an algebraic subgroup of $GL_{n,\mathbf{Q}_\ell}$. We write G_ℓ° for the connected component of the identity. Clearly we have:

Lemma **(6.9)** *If $\{\rho_\ell\}$ is everywhere semi-simple, then every G_ℓ is reductive. If $\{\rho_\ell\}$ is everywhere absolutely irreducible, then every G_ℓ is reductive, and its natural representation is absolutely irreducible.*

(6.10) Since the condition 6.5.3 bears only on the semisimple part of $\rho_\ell(F_\alpha)$, nothing can be said about the unipotent radical of G_ℓ. Therefore, from now on we only consider everywhere semisimple representations.

Lemma **(6.11)** *Let $\mathcal{H} \subset \mathcal{G}$ be an open subgroup of finite index, and $\gamma \in \mathcal{G}$. Then the Zariski closure of $ch(\rho_\ell(\gamma\mathcal{H}))$ in $(\mathbf{G}_m \times \mathbf{A}^{n-1})_{\mathbf{Q}_\ell}$ is defined over $\mathbf{Q}$ and independent of ℓ.*
Proof. By 6.5.2, for all ℓ_1, $\ell_2 \in \mathbf{L}$ the set

$$\mathcal{F} = \{F_\alpha \in \gamma\mathcal{H} \mid (\alpha, \ell_1),\ (\alpha, \ell_2) \in \mathcal{X}\}$$

is dense in $\gamma\mathcal{H}$. Since the Zariski topology is coarser than the ℓ-adic topology, the Zariski closure of $ch(\rho_{\ell_i}(\gamma\mathcal{H}))$ is equal to that of $ch(\rho_{\ell_i}(\mathcal{F}))$ for $i = 1$, 2. But by 6.5.3 we have $ch(\rho_{\ell_1}(\mathcal{F})) = ch(\rho_{\ell_2}(\mathcal{F})) \subset (\mathbf{G}_m \times \mathbf{A}^{n-1})(\mathbf{Q})$. ◻

Proposition **(6.12)** *The variety $ch(G_\ell^\circ)$ is defined over $\mathbf{Q}$ and is independent of ℓ. In particular there exists a $\mathbf{Q}$-split torus T_0 and a faithful representation ρ_0 of T_0, such that for all ℓ, $T_{0,\bar{\mathbf{Q}}_\ell}$ is isomorphic to a maximal torus of $G^\circ_{\ell,\bar{\mathbf{Q}}_\ell}$ and ρ_0 equivalent to the representation induced by the natural representation $G_\ell^\circ \hookrightarrow GL_n$.*
Proof. Applying 6.11 to $\gamma = 1$ and all sufficiently small $\mathcal{H}$ yields the first assertion. The rest follows from 4.3. ◻

Remark **(6.13)** This was first observed by Serre (see [15], §3). In particular, the rank of G_ℓ does not depend on ℓ. The following result is also due to Serre ([16] and [17]).

Proposition **(6.14)** *The open subgroup of finite index $\rho_\ell^{-1}(G_\ell^\circ(\mathbf{Q}_\ell)) \subset \mathcal{G}$ is independent of ℓ. In particular, the groups G_ℓ/G_ℓ° for different ℓ are canonically isomorphic. If G_ℓ is connected for some ℓ, then it is so for all ℓ.*
Proof. Fix $\gamma \in \mathcal{G}$. For all sufficiently small open subgroups $\mathcal{H} \subset \mathcal{G}$ of finite index $\rho_\ell(\gamma)G_\ell^\circ$ is the Zariski closure of $\rho_\ell(\gamma\mathcal{H})$. Using 4.9 it follows that

$$ch(\rho_\ell(\gamma)G_\ell^\circ) == Zar(ch(\rho_\ell(\gamma\mathcal{H}))). \tag{6.14.1}$$

By 4.10 the left hand side is equal to $ch(G_\ell^\circ)$ if and only if $\rho_\ell(\gamma) \in G_\ell^\circ$. As 6.11 and 6.12 prove the ℓ-independence of Zar$(ch(\rho_\ell(\gamma\mathcal{H})))$ and $ch(G_\ell^\circ)$ respectively, the condition $\rho_\ell(\gamma) \in G_\ell^\circ$ does not depend on ℓ, which proves the first assertion. The remaining assertions follow immediately. ◻

(6.15) The preceding proposition allows us to reduce the general problem of studying G_ℓ° to the case in which G_ℓ is already connected for some (and hence for all) $\ell \in \mathbf{L}$. Indeed, let $\mathcal{H} = \rho_\ell^{-1}(G_\ell^\circ(\mathbf{Q}_\ell))$ and $A' = \{\alpha \in A \mid F_\alpha \in \mathcal{H}\}$. Then $\mathcal{H}$ together with the collection of F_α for $\alpha \in A'$ forms an F-group, and the restricted representations $\{\rho_\ell|_\mathcal{H}\}$ form a compatible system, with $\mathcal{X}' = \mathcal{X} \cap (A' \times \mathbf{L})$. Clearly Zar$(\rho_\ell(\mathcal{H})) = G_\ell^\circ$, as desired.

In the rest of this article we consider only the connected case.

§7. Frobenius Elements and Maximal Tori.

From now on, we fix a *profinite* F-group $\mathcal{G}$, and an everywhere semisimple compatible system of ℓ-adic representations $\{\rho_\ell\}$ of common dimension n, indexed by a set $\mathbf{L}$ of primes of Dirichlet density 1. We use the notation and terminology of §6. By 6.9, the G_ℓ are reductive, and we assume (using 6.15) that they are connected. Our ultimate aim is to study how G_ℓ varies with ℓ. In this section, we transform the data given by $\{\rho_\ell\}$ into information about maximal tori of the different groups G_ℓ.

Let G_ℓ^{ad} denote the adjoint group of G_ℓ.

Proposition **(7.1)** *For every finite collection $\ell_1, \ldots, \ell_k$ of pairwise distinct primes in $\mathbf{L}$, the image of $\mathcal{G}$ in $\prod_{i=1}^k G_{\ell_i}^{ad}(\mathbf{Q}_{\ell_i})$ is open.*
Proof. (Here we make essential use of the fact that our representations are are defined over $\mathbf{Q}_\ell$ rather than $\bar{\mathbf{Q}}_\ell$.) By the corollary to proposition 2 in [13], the image of $\mathcal{G}$ in $G_{\ell_i}^{ad}(\mathbf{Q}_{\ell_i})$ is an open subgroup for every i. There exists an open subgroup $\mathcal{H}_i \subset \mathcal{G}$ such that $\rho_{\ell_i}(\mathcal{H}_i)$ is a pro-ℓ_i-group. Let $\mathcal{H} = \bigcap_{i=1}^k \mathcal{H}_i$, and N_i be the image of $\mathcal{H}$ in $G_{\ell_i}^{ad}(\mathbf{Q}_{\ell_i})$. The image of $\mathcal{H}$ in $\prod_{i=1}^k N_i$ is a compact subgroup that maps surjectively onto every factor. Since the N_i are pro-ℓ_i-groups for pairwise distinct ℓ_i, it follows that $\mathcal{H}$ maps onto $\prod_{i=1}^k N_i$, which proves the assertion. ◻

Proposition **(7.2)** *For any $\ell \in \mathbf{L}$, the set of all $\gamma \in \mathcal{G}$ such that $\rho_\ell(\gamma)$ is Γ-regular (in the sense of 4.5) is open and dense in $\mathcal{G}$.*

Proof. (This is an analogue of [15] Th. p.13.) Take any $\gamma \in \mathcal{G}$ and any open subgroup of finite index $\mathcal{H} \subset \mathcal{G}$. Since G_ℓ is assumed to be connected, we have $\mathrm{Zar}(\rho_\ell(\gamma\mathcal{H})) = G_\ell$, so $\mathrm{Zar}(ch(\rho_\ell(\gamma\mathcal{H}))) = ch(G_\ell)$ is not contained in Y. In particular, $ch(\rho_\ell(\gamma\mathcal{H})) \not\subset Y(\mathbf{Q}_\ell)$, which shows that there are Γ-regular points arbitrarily near γ. The claim of openness follows from the continuity of $ch \circ \rho_\ell$, since Y is Zariski-closed. □

Proposition **(7.3)** *Let $\ell_1, \ldots, \ell_k$ be pairwise distinct primes in $\mathbf{L}$, and $T_i \subset G_{\ell_i}$ maximal tori. Let V be the set of all $\gamma \in \mathcal{G}$ such that, for every $1 \le i \le k$, $\rho_{\ell_i}(\gamma)$ is Γ-regular and conjugate to an element of $T_i(\mathbf{Q}_{\ell_i})$. Then V is open in $\mathcal{G}$, and its closure contains the identity.*

Proof. We first consider a fixed prime $\ell \in \mathbf{L}$ and a maximal torus $T \subset G_\ell$. Let W be the set of all regular semisimple elements of $G_\ell(\mathbf{Q}_\ell)$ which are conjugate to an element of $T(\mathbf{Q}_\ell)$. Since this set contains all regular elements of $T(\mathbf{Q}_\ell)$ itself, the closure of W contains the identity. We shall prove that it is also open. It suffices to show that for every regular element $t_0 \in T(\mathbf{Q}_\ell)$, some ℓ-adic neighborhood of t_0 in $G_\ell(\mathbf{Q}_\ell)$ is contained in W. Consider the morphism $G_\ell \times T \to G_\ell$, $(g,t) \mapsto gtg^{-1}$. It maps $(1,t_0)$ to t_0, and since t_0 is regular semisimple, its differential at $(1,t_0)$ is surjective. It follows (e.g. by [7] Satz 1.1.1) that the associated map $G_\ell(\mathbf{Q}_\ell) \times T(\mathbf{Q}_\ell) \to G_\ell(\mathbf{Q}_\ell)$ is surjective over a neighborhood of t_0. In particular, W contains a neighborhood of t_0, as desired. Note that, since regularity is only a condition on the image of an element in G_ℓ^{ad}, the set W is invariant under multiplication by the center of $G_\ell(\mathbf{Q}_\ell)$.

Coming back to the given situation, let W_i denote the set of regular semisimple elements of $G_{\ell_i}(\mathbf{Q}_{\ell_i})$ which are conjugate to elements of $T_i(\mathbf{Q}_{\ell_i})$. Then V is just the set of all Γ-regular points in $\bigcap_{i=1}^k \rho_{\ell_i}^{-1}(W_i)$. By 7.1 and the above remarks, this intersection is open and its closure contains the identity. By 7.2 these properties are inherited by V, as desired. □

(7.4) Fix $\alpha \in A$. By 6.5.1, excluding at most finitely many primes, $ch(\rho_\ell(F_\alpha))$ is a point of $(\mathbf{G}_m \times \mathbf{A}^{n-1})(\mathbf{Q})$ that depends only on α. The condition that $\rho_\ell(F_\alpha)$ is Γ-regular is therefore also independent of ℓ; when it holds, we call α or F_α Γ-*regular*. By 7.2, the assumptions 6.5 still hold if we replace A by the subset of all Γ-regular α. From now on we assume that every $\alpha \in A$ is Γ-regular.

By 4.7, there is a torus $T_\alpha \subset GL_{n,\mathbf{Q}}$, canonical up to conjugacy, associated to $ch(\rho_\ell(F_\alpha))$, and for any $\ell \in \mathbf{L}$ with $(\alpha,\ell) \in \mathcal{X}$, $T_\alpha \times_{\mathbf{Q}} \mathbf{Q}_\ell$ is conjugate to the unique maximal torus of G_ℓ that contains $\rho_\ell(F_\alpha)$. Thus, by 7.3, we have the following data:

(7.5) $\mathbf{L}$ is a set of rational primes of Dirichlet-density 1. For every $\ell \in \mathbf{L}$, we are given a connected reductive subgroup $G_\ell \subset GL_{n,\mathbf{Q}_\ell}$. For every α in some index set A we are given a torus $T_\alpha \subset GL_{n,\mathbf{Q}}$. Finally we are given a subset $\mathcal{X} \subset A \times \mathbf{L}$, subject to the following conditions:

(7.5.1) For every $\alpha \in A$, $(\alpha,\ell) \in \mathcal{X}$ for all but at most finitely many $\ell \in \mathbf{L}$.

(7.5.2) For all $(\alpha,\ell) \in \mathcal{X}$, $T_\alpha \times_{\mathbf{Q}} \mathbf{Q}_\ell$ is conjugate to a maximal torus of G_ℓ.

(7.5.3) For all pairwise distinct primes $\ell_1, \ldots, \ell_k \in \mathbf{L}$, and all maximal tori $T_i \subset G_{\ell_i}$, there exists an $\alpha \in A$ such that, for every i, $(\alpha,\ell_i) \in \mathcal{X}$ and $T_\alpha \times_{\mathbf{Q}} \mathbf{Q}_{\ell_i}$ is conjugate to T_i.

§8. Maximal Tori and Cebotarev Density.

In this section we use the results of §3 and the axioms 7.5 to extract information about the groups G_ℓ and their Weyl groups. The underlying heuristic principle is that the Weyl group of a semisimple group over a non-archimedean local field should be more or less determined by the set of maximal tori. The assumptions of §7 (G_ℓ is connected reductive and all F_α are Γ-regular) remain in force.

(8.1) Let T_0, ρ_0 be as in 6.12, and write $\Gamma = \mathrm{Aut}(T_0,\rho_0)$ as in 4.2. For every $\alpha \in A$ let E_α denote the splitting field of the torus T_α. Since $T_\alpha \times_{\mathbf{Q}} \bar{\mathbf{Q}}$ is conjugate to $T_0 \times_{\mathbf{Q}} \bar{\mathbf{Q}}$, the conjugacy class of T_α determines and is determined by a homomorphism

$$\varphi_\alpha : \mathrm{Gal}(E_\alpha/\mathbf{Q}) \to \Gamma,$$

unique up to conjugation by Γ. We fix φ_α in its conjugacy class. Let E be the intersection of all E_α. We call it the *splitting field* of $\{\rho_\ell\}$. Note that the connectedness of G_ℓ and the Γ-regularity of the F_α are essential; admitting bad F_α may give an intersection smaller than the true splitting field.

Proposition **(8.2)** *Fix a finite extension F/E, a finite set of primes $\ell_i \in \mathbf{L}$, and a collection of maximal tori $T_i \subset G_{\ell_i}$. Then there exists $\alpha \in A$ satisfying condition 7.5.3 such that E_α and F are linearly disjoint over E.*

Proof. Without loss of generality we may assume that F/E is Galois. Let $K_1, \ldots, K_m$ be the subfields of F corresponding to the different maximal proper normal subgroups of $\mathrm{Gal}(F/E)$. Since E_α/E is Galois, the linear disjointness condition is equivalent to $K_\mu \not\subset E_\alpha$ for every $1 \leq \mu \leq m$.

By definition of E, for every μ there exists $\alpha_\mu \in A$ with $K_\mu \not\subset E_{\alpha_\mu}$. For a fixed collection of α_μ, Cebotarev's density theorem guarantees an infinite set of $\lambda_\mu \in \mathbf{L}$ that split in E_{α_μ} but not in K_μ. We may choose these λ_μ pairwise distinct, distinct from the ℓ_i, and (by 7.5.1) such that $(\alpha_\mu, \lambda_\mu) \in \mathcal{X}$. By 7.5.2, $T_{\alpha_\mu} \times_{\mathbf{Q}} \mathbf{Q}_{\lambda_\mu}$ is conjugate to some maximal torus S_μ of G_{λ_μ}. Since λ_μ splits in E_{α_μ}, this torus is split. Applying 7.5.3 to the ℓ_i and the λ_μ together, with the maximal tori T_i and S_μ, respectively, we obtain $\alpha \in A$ such that $T_\alpha \times_{\mathbf{Q}} \mathbf{Q}_{\ell_i}$ is conjugate to T_i, and every λ_μ splits in E_α. Since λ_μ does not split in K_μ, this cannot be a subfield of E_α, as desired. $\square$

Lemma **(8.3)** *Let $A = \coprod_{\mu=1}^m A_\mu$ be a partition of A. There exists a finite subset $X \subset \mathbf{L}$, and an index $1 \leq \mu \leq m$, such that if $\mathbf{L}$ is replaced by $\mathbf{L} \setminus X$ and A by A_μ, the conditions 7.5 remain valid.*

Proof. The conditions 7.5.1–2 hold for any X and μ. Assume that for all X and all μ the condition 7.5.3 fails. By induction on μ we can find pairwise distinct $\ell_{\mu,i} \in \mathbf{L}$, for $1 \leq i \leq k_\mu$ and $1 \leq \mu \leq m$, and maximal tori $T_{\mu,i} \in G_{\ell_{\mu,i}}$, such that for every fixed μ, the assertion 7.5.3 fails for $\ell_{\mu,1}, \ldots, \ell_{\mu,k_\mu}$, $T_{\mu,1}, \ldots, T_{\mu,k_\mu}$, with A replaced by A_μ. Applying 7.5.3 to the set of all $\ell_{\mu,i}$ and $T_{\mu,i}$, we obtain an $\alpha \in A$ which must lie in some A_μ: a contradiction. $\square$

Proposition **(8.4)** *There exist a finite subset $X \subset \mathbf{L}$, a subgroup $\Delta \subset \Gamma$, a normal subgroup $\Delta_1 \subset \Delta$, and an isomorphism $\bar\varphi : Gal(E/\mathbf{Q}) \xrightarrow{\sim} \Delta/\Delta_1$, such that if $\mathbf{L}$ is replaced by $\mathbf{L} \setminus X$, and A by*

$$\left\{ \alpha \in A \;\middle|\; \begin{array}{l} \varphi_\alpha(Gal(E_\alpha/\mathbf{Q})) = \Delta, \\ \varphi_\alpha(Gal(E_\alpha/E)) = \Delta_1, \text{ and} \\ \pi \circ \varphi_\alpha = \bar\varphi \end{array} \right\},$$

the conditions 7.5 remain valid. Here π denotes the canonical projection $\Delta \to \Delta/\Delta_1$.

Proof. For the triple $(\Delta, \Delta_1, \bar\varphi)$ there are only finitely many possibilities, so the assertion follows from 8.3. $\square$

(8.5) Let Λ be the character group of T_0. For any $\ell \in \mathbf{L}$ and any maximal torus $T \subset G_\ell$, $T \times_{\mathbf{Q}_\ell} \bar{\mathbf{Q}}_\ell$ is conjugate to $T_0 \times_{\mathbf{Q}} \bar{\mathbf{Q}}_\ell$. This yields an isomorphism between the root datum of G_ℓ and a root datum of the form $\Psi_\ell = (\Lambda, \Phi_\ell, \Lambda^\vee, \Phi_\ell^\vee)$. Up to conjugation by Γ, this root datum and the isomorphism depend only on G_ℓ; we fix both of them. Since the Weyl group of G_ℓ stabilizes the given representation of T, we have $W(\Psi_\ell) \subset \Gamma$.

(8.6) In analogy to §3 we adopt the following notation: π_ℓ is the projection $Stab_\Gamma(\Psi_\ell) \to Stab_\Gamma(\Psi_\ell)/W(\Psi_\ell)$, and $\bar\varphi_\ell$ the homomorphism $\mathrm{Gal}(\bar{\mathbf{Q}}_\ell/\mathbf{Q}_\ell) \to Stab_\Gamma(\Psi_\ell)/W(\Psi_\ell)$ canonically associated to G_ℓ. $Frob_\ell$ denotes the subset of $\mathrm{Gal}(\bar{\mathbf{Q}}_\ell/\mathbf{Q}_\ell)$ of all elements that act as Frobenius on every unramified extension and i_ℓ the inclusion $\mathrm{Gal}(\bar{\mathbf{Q}}_\ell/\mathbf{Q}_\ell) \hookrightarrow \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ induced by any fixed embedding $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_\ell$. We define $\mathcal{F}_\ell$ and $\mathcal{F}_\ell^{nr}$ as in 3.9 (setting $G = G_\ell$). Clearly we have the inclusions

$$\mathcal{F}_\ell^{nr} \subset \mathcal{F}_\ell \subset \pi_\ell^{-1}(\bar\varphi_\ell(Frob_\ell)) \subset Stab_\Gamma(\Psi_\ell).$$

For any subset $X \subset \Gamma$, denote by $[X]$ the set of all elements of Γ conjugate to an element of X.

Proposition **(8.7)** *For a cofinite set of $\ell \in \mathbf{L}$, G_ℓ splits over $\mathbf{Q}_\ell^{nr}$, and moreover*

$$\emptyset \neq [\mathcal{F}_\ell^{nr}] \subset [\mathcal{F}_\ell] \subset \left[\pi^{-1}(\bar\varphi \circ i_\ell(Frob_\ell))\right].$$

Proof. We may assume that the replacement in 8.4 has been carried out. By 7.5.1–2, all the G_ℓ, with at most a finite set of exceptions, possess an unramified maximal torus (namely that coming from T_α), whence the first assertion and the claim that $\mathcal{F}_\ell^{nr} \neq \emptyset$. On the other hand, by 7.5.3, for every maximal torus $T \subset G_\ell$ there exists $\alpha \in A$ such that $(\alpha, \ell) \in \mathcal{X}$ and $T_\alpha \times_{\mathbf{Q}} \mathbf{Q}_\ell$ is conjugate to T. Then φ_T (see 3.1) and $\varphi_\alpha \circ i_\ell$ are Γ-conjugate, so

$$[\varphi_T(Frob_\ell)] = [\varphi_\alpha \circ i_\ell(Frob_\ell)] \subset \left[\pi^{-1}(\bar\varphi \circ i_\ell(Frob_\ell))\right].$$

$\square$

Proposition **(8.8)** *The set of $\ell \in \mathbf{L}$ for which $[\mathcal{F}_\ell^{nr}] = \left[\pi^{-1}(\bar\varphi \circ i_\ell(Frob_\ell))\right]$ has Dirichlet-density* 1.

Proof. We may assume that the replacement in 8.4 has been carried out. Fix an element $\delta \in \Delta$. We have to prove that the set

$$\left\{ \ell \in \mathbf{L} \;\middle|\; \begin{array}{l} \pi(\delta) \in \bar\varphi \circ i_\ell(Frob_\ell), \text{ but} \\ \delta \text{ is not conjugate to an element of } \mathcal{F}_\ell^{nr} \end{array} \right\} \tag{8.8.1}$$

has Dirichlet density 0. By 8.2 and induction over N, we can find $\alpha_1, \ldots, \alpha_N \in A$ such that the E_{α_i} are linearly disjoint over E. Consider the set of $\ell \in \mathbf{L}$ which are unramified in every E_{α_i} and such that $\pi(\delta) \in \bar\varphi \circ i_\ell(Frob_\ell)$, but for which δ is not conjugate to an element of $\varphi_{\alpha_i} \circ i_\ell(\mathrm{Frob}_\ell)$ for any i, $1 \le i \le N$. There are precisely $|\Delta_1|$ possibilities for each $\varphi_{\alpha_i} \circ i_\ell(\mathrm{Frob}_\ell)$ with the fixed $\bar\varphi \circ i_\ell(Frob_\ell)$. For each i the last condition rules out at most $|\Delta_1| - 1$ of these, so by Cebotarev's density theorem and the linear disjointness of the E_{α_i}, this set has Dirichlet density at most

$$\left(1 - \frac{1}{|\Delta_1|}\right)^N .$$

Since, for each i and almost all ℓ, $\varphi_{\alpha_i} \circ i_\ell$ is conjugate to φ_T for some unramified maximal torus $T \subset G_\ell$, this set contains all the "bad" primes 8.8.1, with at most finitely many exceptions, including the primes that ramify in some E_{α_i}. We send N to infinity, and the proposition follows. □

Proposition **(8.9)** *For ℓ in a subset $\mathbf{L}' \subset \mathbf{L}$ of Dirichlet-density* 1*, the connected reductive group G_ℓ is unramified (in particular quasi-split), and*

$$\left[\pi_\ell^{-1}(\bar\varphi_\ell(Frob_\ell))\right] = \left[\pi^{-1}(\bar\varphi \circ i_\ell(Frob_\ell))\right] . \tag{8.9.1}$$

Moreover, G_ℓ is split over $E\mathbf{Q}_\ell$.

Proof. By Propositions 8.7 and 8.8, we may assume $\emptyset \neq [\mathcal{F}_\ell^{nr}] = [\mathcal{F}_\ell]$. Therefore, $\mu(\mathcal{F}_\ell^{nr}) = \mu(\mathcal{F}_\ell)$, with $\mu(\ )$ defined as in 3.11. By 3.12, G_ℓ is unramified, which implies, by 3.10, $\mathcal{F}_\ell^{nr} = \pi_\ell^{-1}(\bar\varphi_\ell(Frob_\ell))$. Applying $[\,]$ to both sides, (8.9.1) follows. The quasi-split group G_ℓ is determined up to isogeny by Ψ_ℓ and $\bar\varphi_\ell$ (see 3.3). As $\bar\varphi_\ell$ factors through $\mathrm{Gal}(E\mathbf{Q}_\ell/\mathbf{Q}_\ell)$, G_ℓ is split over $E\mathbf{Q}_\ell$. □

Note that W. Barker observed (unpublished) that as a consequence of [5] Satz 2, G_ℓ is actually quasi-split for all $\ell \gg 0$ when the system of representations comes from an abelian variety over a number field.

(8.10) This result can be interpreted as saying that a substantial part of the information about G_ℓ (at least about its Weyl group) depends, on a set of primes of density 1, only on the image of $Frob_\ell$ in $\mathrm{Gal}(E/\mathbf{Q})$. In the next section we make this more precise.

§9. Main Results.

We recall our setting. The system $\{\rho_\ell\}$ is an everywhere semisimple representation of a profinite F-group $\mathcal{G}$, indexed by a set $\mathbf{L}$ of primes of Dirichlet-density 1 (see 6.1 and 6.5). For every $\ell \in \mathbf{L}$, the algebraic monodromy group $G_\ell = \mathrm{Zar}(\rho_\ell(\mathcal{G}))$ is reductive (see 6.8–9) and we assume it to be connected (see 6.15). As in 8.5 we denote the root datum of G_ℓ by $\Psi_\ell = (\Lambda, \Phi_\ell, \Lambda^\vee, \Phi_\ell^\vee)$. When G_ℓ possesses an unramified maximal torus, its structure up to inner twist is determined by the coset $\sigma_\ell W(\Psi_\ell) = \pi_\ell^{-1}(\bar\varphi_\ell(Frob_\ell)) \subset \mathrm{Aut}(\Psi_\ell)$ (3.8, 8.6). Finally, E is the splitting field of $\{\rho_\ell\}$ (see 8.1).

Theorem **(9.1)** *There exists a subset $\mathbf{L}' \subset \mathbf{L}$ of Dirichlet density* 1 *such that, for $\ell \in \mathbf{L}'$, G_ℓ is unramified, and the triple $(\Lambda_\mathbf{Q}, W(\Psi_\ell), \sigma_\ell W(\Psi_\ell))$ depends up to isomorphism only on the image of $Frob_\ell$ in Gal$(E/\mathbf{Q})$.*

Proof. Let $\mathbf{L}'$ be the set given by 8.9. Then we have the first assertion, plus the fact that the set of conjugacy classes in Γ (see 8.1) generated by $\sigma_\ell W(\Psi_\ell)$ depends only on the image of $Frob_\ell$ in $\mathrm{Gal}(E/\mathbf{Q})$. In particular, the set of characteristic polynomials on $\Lambda_\mathbf{Q}$ of the elements of $\sigma_\ell W(\Psi_\ell)$ depends only on this information. By 2.1, these characteristic polynomials determine the triple in question. □

Corollary **(9.2)** *Let $\mathbf{L}'$ be as in 9.1. Then, for $\ell \in \mathbf{L}'$, the dimension of G_ℓ, and the dimension of its center, depends only on the image of $Frob_\ell$ in Gal$(E/\mathbf{Q})$.*

Proof. By 2.2, the Weyl group determines the roots up to rational multiples. Thus the assertion follows from 9.1. Note that this gives a partial answer to [14] I Question 4 (ii). □

(9.3) The next theorem shows that the B/C indeterminacy does not arise in the absolutely irreducible case. By abuse of notation, we write ρ_ℓ for both the ambient representation $G_\ell \hookrightarrow GL_n$ and its formal character.

Theorem **(9.4)** *Assume that $\{\rho_\ell\}$ is everywhere absolutely irreducible. Then there exists a subset $\mathbf{L}' \subset \mathbf{L}$ of Dirichlet-density 1 such that, for $\ell \in \mathbf{L}'$, the triple $(\Psi_\ell, \rho_\ell, \sigma_\ell W(\Psi_\ell))$ depends, up to isomorphism, only on the image of $Frob_\ell$ in $Gal(E/\mathbf{Q})$.*

Proof. For simplicity fix an embedding of each $\mathbf{Q}_\ell$ into $\mathbf{C}$. Since, by 6.12, the formal character ρ_ℓ is independent of ℓ, the pairs $(G_{\ell,\mathbf{C}}, \rho_\ell)$ are similar, in the sense of 5.1, for *all* $\ell \in \mathbf{L}$. First we want to use the canonical decomposition into basic similarity classes (5.7) to reduce to the case of a power of a basic similarity class. For this, fix a basic similarity class, and let $k \geq 1$ be its multiplicity in $(G_{\ell,\mathbf{C}}, \rho_\ell)$. Suppose that the basic class is represented by (G_1, ρ_1). Then every $(G_{\ell,\mathbf{C}}, \rho_\ell)$ decomposes uniquely into $(G_{1,\ell,\mathbf{C}} \cdot G_{2,\ell,\mathbf{C}}, \rho_{1,\ell} \otimes \rho_{2,\ell})$, where $(G_{1,\ell,\mathbf{C}}, \rho_{1,\ell})$ is similar to $(G_1, \rho_1)^k$. By uniqueness this decomposition must already be defined over $\mathbf{Q}_\ell$; in particular we have a canonical decomposition $G_\ell = G_{1,\ell} \cdot G_{2,\ell}$. This induces a canonical decomposition for every maximal torus $T \subset G_\ell$. By 7.5.1–2 we also get a decomposition for T_α, at least over $\mathbf{Q}_\ell$. But by 5.7.3 every factor of T is invariant under Γ; this implies that every factor of T_α thus obtained is invariant under the normalizer of T_α in $GL_{n,\mathbf{Q}}$. It follows that the decomposition of T_α is defined over $\mathbf{Q}$. If we write it $T_\alpha = T_{\alpha,1} \cdot T_{\alpha,2}$, then it is clear from the construction that the pairs $(G_{i,\ell}, \rho_{i,\ell})$ together with the tori $T_{\alpha,i}$ also satisfy the conditions in 7.5, for every $i = 1, 2$. Therefore, as long as we proceed formally, using 7.5 as our axioms, we may assume that every $(G_{\ell,\mathbf{C}}, \rho_\ell)$ is in a fixed power of a basic similarity class. As theorem 9.1 depends only on 7.5, we find that, for $\ell \in \mathbf{L}'$, $(\Lambda_\mathbf{Q}, W(\Psi_\ell), \sigma_\ell W(\Psi_\ell))$ depends up to isomorphism only on the image of $Frob_\ell$ in $\mathrm{Gal}(E/\mathbf{Q})$. By 2.2, $(\Psi_{\ell,\mathbf{Q}}, \sigma_\ell W(\Psi_\ell))$ is determined by this data up to the relation generated by the equivalences $B_m \sim C_m$. But in a *basic* similarity class, a non-trivial equivalence of this form is not possible (see 5.3). It follows that, for $\ell \in \mathbf{L}'$, $(\Psi_{\ell,\mathbf{Q}}, \sigma_\ell W(\Psi_\ell))$ depends only on the image of $Frob_\ell$ in $\mathrm{Gal}(E/\mathbf{Q})$. Now within a basic similarity class or a power thereof, the formal character of any simple factor is determined by the root system of this simple factor. This means that $\Psi_{\ell,\mathbf{Q}}$ determines ρ_ℓ. This, in turn, determines Ψ_ℓ, and we are done. □

(9.5) If $E = \mathbf{Q}$, this result is particularly nice: see 9.10 below. Otherwise there is no guarantee that the root data Ψ_ℓ are the same for different conjugacy classes in $\mathrm{Gal}(E/\mathbf{Q})$. However, under dimension restrictions, we have more precise results:

Proposition **(9.6)** *Consider the situation of 9.4, and assume that the common dimension n of the representation $\{\rho_\ell\}$ is either odd and not divisible by 27, or equal to 2. Then (Ψ_ℓ, ρ_ℓ) is independent of ℓ, for all $\ell \in \mathbf{L}$ (without excluding a subset of Dirichlet density 0). In other words, the groups $G_\ell \times_{\mathbf{Q}_\ell} \mathbf{C} \subset GL_{n,\mathbf{C}}$ are all conjugate (any choice of embeddings $\mathbf{Q}_\ell \hookrightarrow \mathbf{C}$).*

Proof. By 5.8 the assumption implies that no basic ambiguous similarity class occurs in the decomposition of (G_ℓ, ρ_ℓ). □

Proposition **(9.7)** *Under the hypotheses of 9.4, if the dimension n of the representation $\{\rho_\ell\}$ is divisible neither by 3^{15} nor by the fifth power of an even integer strictly greater than 2, then, for all $\ell \in \mathbf{L}'$, (Ψ_ℓ, ρ_ℓ) is independent of ℓ.*

Proof. As in the proof of 9.4, we may reduce to the case of a power of a fixed basic similarity class. If this class is not ambiguous, we are done as in 9.6. Otherwise, using 5.8 the assumption implies that we have $k \leq 4$ copies of a basic similarity class of type 5.3.1 or 5.3.3, or that we have the spin-case 5.3.4, with rank $k \leq 9$. The case 5.3.2, having dimension 2^{12}, cannot occur at all.

In the cases 5.3.1 and 5.3.3, the ambiguity is between two root systems $\Phi \subset \Phi'$, where Φ' has no outer automorphisms but induces an outer automorphism of order 2 on Φ. Thus $\Gamma_1 = W(\Phi)^k$ is a normal subgroup of Γ, and $\bar{\Gamma} = \Gamma/\Gamma_1$ is canonically isomorphic to $\{\pm 1\}^k \rtimes S_k$. For any $\ell \in \mathbf{L}$, the Weyl group $W(\Psi_\ell)$ contains Γ_1, and its image in $\bar{\Gamma}$ is of the form $\{\pm 1\}^j \times \{1\}^{k-j}$ for some $0 \leq j \leq k$.

In the case 5.3.4 we have $\Gamma = \{\pm 1\}^k \rtimes S_k$ with $1 \leq k \leq 9$, and $W(\Psi_\ell)$ must contain the normal subgroup $\Gamma_1 = \{\pm 1\}^k$. The image of $W(\Psi_\ell)$ in $\bar{\Gamma} = \Gamma/\Gamma_1$ is of the form $\prod_{i=1}^r S_{k_i}$ for some partition $k = \sum_{i=1}^r k_i$.

In either case, let $\mathbf{L}'$ be as in 8.9, and choose a prime $\ell_0 \in \mathbf{L}'$ which splits completely in E, and which is congruent to $1 \bmod 2k!$. For any such choice, equation (8.9.1) reduces to the equality $[W(\Psi_{\ell_0})] = [\Delta_1]$. We want to prove that $W(\Psi_{\ell_0})$ is, as a subgroup of Γ, conjugate to $\Delta_1\Gamma_1$. This will follow from lemma 9.8 below applied to $G = \bar{\Gamma}$, $H = W(\Psi_{\ell_0})$, and $H' = \Delta_1\Gamma_1/\Gamma_1$. Condition 9.8.1 is already clear. For 9.8.2 observe that, since G_ℓ is split, by 3.6 every homomorphism $\mathrm{Gal}(\bar{\mathbf{Q}}_{\ell_0}/\mathbf{Q}_{\ell_0}) \to W(\Psi_{\ell_0})$ occurs for some maximal torus of G_ℓ. Since the surjection $W(\Psi_{\ell_0}) \to W(\Psi_{\ell_0})/\Gamma_1$ possesses a right inverse, every homomorphism $\mathrm{Gal}(\bar{\mathbf{Q}}_{\ell_0}/\mathbf{Q}_{\ell_0}) \to W(\Psi_{\ell_0})/\Gamma_1$ comes from some maximal torus. Now ℓ_0 has been chosen so that there exists a cyclic extension of $\mathbf{Q}_{\ell_0}$ that is totally ramified of degree $2k!$. As $\mathbf{Q}_{\ell_0}$ has an unramified extension of the same degree, there exists a surjective homomorphism $\mathrm{Gal}(\bar{\mathbf{Q}}_{\ell_0}/\mathbf{Q}_{\ell_0}) \to (\mathbf{Z}/2k!\mathbf{Z})^2$. The order of every element of $W(\Psi_{\ell_0})/\Gamma_1$ divides $2k!$, so every pair of commuting elements of this group lies in the image of a homomorphism coming from some maximal torus. By 7.5.2–3, there exists $\alpha \in A$ as in 8.4 with $(\alpha, \ell_0) \in \mathcal{X}$ and such that $T_\alpha \times_{\mathbf{Q}} \mathbf{Q}_{\ell_0}$ is conjugate to a maximal torus of G_ℓ of this given type. It follows that there exists a single element of Γ that conjugates the two given commuting elements of $W(\Psi_{\ell_0})/\Gamma_1$ to $\Delta_1\Gamma_1/\Gamma_1$. In other words, condition 9.8.2 holds.

Applying lemma 9.8 and conjugating Ψ_{ℓ_0} in Γ if necessary, we may now assume that $W(\Psi_{\ell_0}) = \Delta_1\Gamma_1$. We shall prove that $W(\Psi_\ell)$ is conjugate to $W(\Psi_{\ell_0})$ for every $\ell \in \mathbf{L}'$. In fact, writing $\sigma\Delta_1 \subset \Delta$ for the image of $Frob_\ell$ and $\sigma_\ell W(\Psi_\ell)$ for the coset determining G_ℓ, (8.9.1) reads $[\sigma\Delta_1] = [\sigma_\ell W(\Psi_\ell)]$. Since $\Gamma_1 \subset W(\Psi_\ell)$ is a normal subgroup of Γ, this equation shows that $[\sigma\Delta_1] = [\sigma\Delta_1\Gamma_1]$. This implies the equalities

$$[\sigma_\ell W(\Psi_\ell)] = [\sigma\Delta_1] = [\sigma\Delta_1\Gamma_1] = [\sigma W(\Psi_{\ell_0})]\,.$$

As in the proof of 9.4, it now follows that the *cosets* $\sigma_\ell W(\Psi_\ell)$ and $\sigma W(\Psi_{\ell_0})$ are conjugate in Γ, as desired. □

Lemma **(9.8)** *Suppose that H and G are groups such that either $H = \{\pm 1\}^j \times \{1\}^{k-j} \subset G = \{\pm 1\}^k \rtimes S_k$ for some integers $0 \leq j \leq k \leq 4$, or there exists a partition $k = \sum_{i=1}^r k_i \leq 9$ such that $H = \prod_{i=1}^r S_{k_i} \subset G = S_k$. In either case, let $H' \subset G$ be another subgroup, and assume that*
(9.8.1) Every element of H is conjugate to an element of H', and vice versa.
(9.8.2) For every pair of commuting elements h_1, $h_2 \in H$ there exists $g \in G$ such that both $gh_ig^{-1} \in H'$.
Then there exists $g \in G$ with $H' = gHg^{-1}$.
Proof. Elementary, but tedious calculation. The bounds on k cannot be improved. □

(9.9) Suppose we are given that Ψ_ℓ is independent of $\ell \in \mathbf{L}'$. Our results do no automatically imply that there exists a global group over $\mathbf{Q}$ from which all "local" groups are derived. Neither does it follow that the representations ρ_ℓ are the same, unless under some restriction like everywhere absolute irreducibility (In fact, the results of [11] imply that the ρ_ℓ may fail to correspond for different ℓ). However, we have the following special result:

Proposition **(9.10)** *Assume that $\{\rho_\ell\}$ is everywhere absolutely irreducible. Suppose further that either $E = \mathbf{Q}$, or the dimension of $\{\rho_\ell\}$ satisfies the conditions of 9.7. Then there exists a subgroup $G \subset GL_{n,\mathbf{Q}}$, defined over $\mathbf{Q}$, such that, for all ℓ in a subset $\mathbf{L}' \subset \mathbf{L}$ of Dirichlet-density 1, G_ℓ is conjugate to $G \times_{\mathbf{Q}} \mathbf{Q}_\ell$ over $\mathbf{Q}_\ell$.*
Proof. By 9.4, G_ℓ is unramified, hence quasi-split, for $\ell \in \mathbf{L}'$. It is therefore determined, up to conjugation in $GL_n(\mathbf{Q}_\ell)$, by the triple $(\Psi, \rho, \sigma_\ell W(\Psi))$ with common root data Ψ. In the case $E = \mathbf{Q}$ we have $\Delta = \Delta_1$, so all G_ℓ split. We may therefore take for G the unique $\mathbf{Q}$-split group with root datum Ψ and representation ρ. In the situation of 9.7 we have proved that, up to conjugation, $W(\Psi) = \Delta_1\Gamma_1$ for some normal subgroup $\Gamma_1 \subset \Gamma$. We can therefore define a homomorphism

$$\mathrm{Gal}(E/\mathbf{Q}) \cong \Delta/\Delta_1 \to \Delta\Gamma_1/\Delta_1\Gamma_1 \subset Stab_\Gamma(\Psi)/W(\Psi).$$

By 3.3 this homomorphism defines a unique quasi-split group G over $\mathbf{Q}$. Since G is quasi-split, and the isomorphism class of the given representation ρ is invariant under the action of the Galois group, ρ descends to a representation of G over $\mathbf{Q}$ (see [19] Th. 3.3). By construction, (G, ρ) has the desired properties. □

§10. Counter-examples.

To round things off, we want to indicate the limitations of our approach by constructing a number of examples of representations of F-groups for which the G_ℓ are not compatible in various ways. These constitute counter-examples to the most optimistic expectations one might have entertained.

(10.1) All constructions will be based on 6.4 and 6.6. If 6.5.2 holds, then the Zariski-closure of $\rho_\ell(\mathcal{G})$ is equal to G_ℓ. Thus to produce counterexamples we have only to choose the G_ℓ and K_ℓ in the desired way, and to verify 6.5.2. Let us assume that every G_ℓ is connected. Recall (4.3) that $ch(G_\ell)$ is Zariski-closed. By 6.12, it must be independent of ℓ. Assuming this, let us write Z for the corresponding reduced closed subscheme of $\mathbf{G}_m \times \mathbf{A}^{n-1}$, defined over $\mathbf{Q}$. Let us see what is needed to prove 6.5.2.

Since $\mathcal{G} = \prod_\ell K_\ell$ carries the product topology, a subset is dense if and only if its image in $\prod_{i \in T} K_\ell$ is dense, for every finite subset $T \subset \mathbf{L}$. By definition (6.4) of the set $\{F_\alpha\}$, it suffices to prove the following assertion:

(10.1.1) Given any finite subset $T \subset \mathbf{L}$, and a non-empty open subset $U_\ell \subset K_\ell$ for every $\ell \in T$, there exists $P \in Z(\mathbf{Q})$ such that
(a) $P \in ch(U_\ell)$ for all $\ell \in T$, and
(b) $P \in ch(K_\ell)$ for all but at most finitely many $\ell \in \mathbf{L} \setminus T$.

Lemma **(10.2)** *Every $ch(U_\ell)$ contains a non-empty open subset of $Z(\mathbf{Q}_\ell)$.*

Proof. Since G_ℓ is connected reductive, U_ℓ contains a non-empty open subset of some maximal torus $T_\ell \subset G_\ell$. The morphism $ch|_{T_\ell} : T_\ell \to ch(T_\ell) = ch(G_\ell)$ is generically étale, hence the associated map of $\mathbf{Q}_\ell$-points is open on an open dense subset ([7] Satz 1.1.1). This implies the assertion. □

(10.3) The above lemma implies that condition 10.1.1 (a) is satisfied whenever $P \in Z(\mathbf{Q})$ satisfies some congruence conditions at all primes $\ell \in T$. Therefore the following condition implies 10.1.1.

(10.3.1) For any $P_0 \in Z(\mathbf{Q})$ and any positive integer N that is a product of primes in $\mathbf{L}$, there exists $P \in Z(\mathbf{Q})$ such that
(a) $P \equiv P_0 \bmod N$, and
(b) $P \in ch(K_\ell)$ for all sufficiently large $\ell \in \mathbf{L}$.

Condition 10.3.1 is implied, in turn, by the following:

(10.3.2) For any $P \in Z(\mathbf{Q})$ outside some fixed Zariski-closed proper subset, $P \in ch(K_\ell)$ for all but at most finitely many $\ell \in \mathbf{L}$.

Counterexample **(10.4)** We construct an example where G_ℓ is strictly "smaller than it ought to be" for an infinite set of primes. Fix a prime $\ell_0 \equiv 1 \bmod 8$. Let $\mathbf{L}$ be the set of all rational primes different from ℓ_0, and S a subset with the property: For every positive integer N that is not divisible by ℓ_0, there are at most finitely many $\ell \in S$ which are *not* congruent to $\ell_0 \bmod N$. Any finite set has this property; it is possible for S to be infinite though its Dirichlet-density must, of course, be zero.

We take $K_\ell = SL_2(\mathbf{Z}_\ell)$ and $G_\ell = SL_2 \hookrightarrow GL_2$ for $\ell \in \mathbf{L} \setminus S$. For $\ell \in S$, however, we take G_ℓ to be the norm-1-torus of the unique unramified quadratic extension of $\mathbf{Q}_\ell$, and $K_\ell = G_\ell(\mathbf{Q}_\ell)$ which is already compact.

Let us prove 10.3.1. We are given $P_0 = X^2 + a_0 X + 1$ and a positive integer N that is prime to ℓ_0. Let b_0 be any integer such that the polynomial $X^2 + b_0 X + 1$ does not split modulo ℓ_0. Let a be any integer greater than 2 which is congruent to $a_0 \bmod N$ and to $b_0 \bmod \ell_0$. We claim that the polynomial $P = X^2 + aX + 1$ does the job. Condition 10.3.1 (a) holds by definition. There is nothing to check except at the primes in S. We need to show that for all sufficiently large $\ell \in S$, the roots of P generate an unramified quadratic extension of $\mathbf{Q}_\ell$. In terms of the Legendre symbol, this means $(\frac{a^2-4}{\ell}) = -1$. By assumption, we have $(\frac{b_0^2-4}{\ell_0}) = -1$. In particular, $a^2 - 4 \equiv b_0^2 - 4 \not\equiv 0 \bmod \ell_0$. By definition of S, $\ell \equiv \ell_0 \bmod 8(a^2-4)$ for all sufficiently large $\ell \in S$. By quadratic reciprocity,

$$\left(\frac{a^2-4}{\ell}\right) = \left(\frac{a^2-4}{\ell_0}\right) = \left(\frac{b_0^2-4}{\ell_0}\right) = -1,$$

as desired.

Counterexample **(10.5)** We construct an example where, on an infinite set of primes, G_ℓ fails to be quasi-split. Let $\mathbf{L}$ and S be as in 10.4. For $\ell \in \mathbf{L} \setminus S$ let $K_\ell = SL_2(\mathbf{Z}_\ell)$ and $G_\ell = SL_2$, but this time we take the 4-dimensional representation induced by the left regular representation of the matrix algebra. For $\ell \in S$, we let D_ℓ be the anisotropic quaternion algebra over $\mathbf{Q}_\ell$, $K_\ell \subset D_\ell^\times$ the subgroup of all elements of reduced norm 1, and $G_\ell \subset GL_4$ the Zariski-closure of K_ℓ in the representation induced by the left regular representation of D_ℓ.

The proof of 10.3.1 is essentially that of 10.4. We are dealing with polynomials of the form $(X^2+aX+1)^2$. Proceeding as in 10.4, we find such a polynomial which satisfies 10.3.1 (a) and (b), with unramified anisotropic tori in place of quaternion algebras. Our assertion now follows from the fact that over a local field any 1-dimensional anisotropic torus can be embedded into any quaternion algebra.

(10.6) Examples in the spirit of 10.4-5 can be based on groups other than SL_2. In the following examples we have to work with larger groups.

Counterexample **(10.7)** We construct an example where the splitting field is $\mathbf{Q}$ and $W(\Psi_\ell)$ is the same for all ℓ, but where Ψ_ℓ varies without any density restriction. Let $n = 2k+1$. If $n = 3$ or 5, then all Ψ_ℓ will be isogenous but the representations inequivalent; for $n \geq 7$ the Ψ_ℓ will not even be isogenous.

Let $H_1 = Sp_{2k,\mathbf{Q}}$ be a split symplectic group of rank k, embedded into $GL_{n,\mathbf{Q}}$ by the direct sum of the standard representation with the trivial representation. Let $H_2 \subset GL_{n,\mathbf{Q}}$ be a split special orthogonal group. The common subspace $Z = ch(H_1) = ch(H_2) \subset \mathbf{G}_m \times \mathbf{A}^{2k}$ consists of all monic polynomials P of degree $2k+1$ with the property $P(X) = -X^{2k+1}P(X^{-1})$. For both $i = 1, 2$, the map ch induces an isomorphism of the variety of all semisimple conjugacy classes of H_i with Z. There is a Zariski-open dense subspace $U \subset Z$ such that the centralizer in H_i of any point mapping to U is connected. Choose any partition $\mathbf{L} = \mathbf{L}_1 \cup \mathbf{L}_2$ of the set of all rational primes. For $\ell \in \mathbf{L}_i$, we put $G_\ell = H_i \times_{\mathbf{Q}} \mathbf{Q}_\ell$ and $K_\ell = G_i(\mathbf{Q}_\ell) \cap GL_n(\mathbf{Z}_\ell)$.

To prove 10.3.2 we may fix $P \in U(\mathbf{Q})$. For $i = 1, 2$ choose a semisimple element $h_i \in H_i(\bar{\mathbf{Q}})$ with $ch(h_i) = P$. The conjugacy class of h_i, being determined by P, is defined over $\mathbf{Q}$. By assumption the centralizer of h_i is connected whence, by definition ([10] §4), Kottwitz' obstruction vanishes for h_i. Since H_i is split, by [10] Th.4.7 the conjugacy class of h_i contains a $\mathbf{Q}$-rational element. We may therefore assume $h_i \in H_i(\mathbf{Q})$. Now, for all but at most finitely many $\ell \in \mathbf{L}_i$, we have $h_i \in GL_n(\mathbf{Z}_\ell)$, whence $h_i \in K_\ell$, as desired.

Counterexample **(10.8)** We construct an example where $\{\rho_\ell\}$ is everywhere absolutely irreducible, the splitting field E is larger than $\mathbf{Q}$, and the root data Ψ_ℓ depends only on, but differs with, the behavior of ℓ in E. This example shows that the dimension restrictions in 9.7 cannot be weakened.

Fix a basic ambiguous class of type 5.3.1 or 5.3.3. If the common dimension of the representation is m, we can represent the two isomorphism classes in this ambiguous class by connected split semisimple groups H, $H' \subset GL_{m,\mathbf{Q}}$. Letting Ψ, Ψ' denote the root data of H and H' respectively, we know that one of them is a proper root subdatum of the other, say Ψ a root subdatum of Ψ'. Then $\mathrm{Aut}(\Psi) = W(\Psi')$, and we can identify $W(\Psi')/W(\Psi)$ with the group $\{\pm 1\}$. For any overfield $F/\mathbf{Q}$ and any homomorphism $\chi:\ \mathrm{Gal}(\bar{F}/F) \to \{\pm 1\}$ there is, by 3.3, associated a twist H^χ of H, which is a quasi-split connected semisimple group over F with the same root datum Ψ. The isomorphism class of the given representation is invariant under this Galois action, so since H^χ is quasi-split, this representation can be realized as a representation of H^χ, defined over F (See [19] Th. 3.3).

Fix an integer $k \geq 5$, and let $n = m^k$. Let $H_1 \subset GL_{n,\mathbf{Q}}$ be the image of $(H')^k$ in the exterior tensor power of the given representation of H'. Fix a quadratic number field E, and let $\chi_E:\ \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to \{\pm 1\}$ be the associated quadratic character. Let us denote the restriction of χ_E to $\mathrm{Gal}(\bar{F}/F)$ by $\chi_{E,F}$. For any character $\chi:\ \mathrm{Gal}(\bar{F}/F) \to \{\pm 1\}$ we define $H(\chi)$ likewise as the image of $H^\chi \times H^{\chi\chi_{E,F}} \times (H' \times_{\mathbf{Q}} F)^{k-2}$. For $F = \mathbf{Q}$ and $\chi = \mathbf{1}$ the trivial character, we put $H_2 = H(\mathbf{1})$. The variety Z of all characteristic polynomials is the same for all of these groups, and we use ch indiscriminately to denote the morphisms $H_1 \to Z$, $H_2 \to Z$, and $H(\chi) \to Z$. Let $\mathbf{L}$ be the set of all odd rational primes that do not ramify in E. For $\ell \in \mathbf{L}$ which splits in E, we put $G_\ell = H_1 \times_{\mathbf{Q}} \mathbf{Q}_\ell$. For $\ell \in \mathbf{L}$ which is inert in E, we put $G_\ell = H_2 \times_{\mathbf{Q}} \mathbf{Q}_\ell$. In either case, we put $K_\ell = G_\ell(\mathbf{Q}_\ell) \cap GL_n(\mathbf{Z}_\ell)$.

(10.9) We shall prove that 10.1.1 holds in the situation of 10.8. In order to do this, we must take a closer look at the rational conjugacy classes of the groups involved.

Let T_0 and ρ_0 be as in 8.1. Then $\mathrm{Aut}(T_0, \rho_0) = W(\Psi')^k \rtimes S_k$, and $ch|_{T_0} : T_0 \to Z$ is a ramified Galois covering with this group as Galois group. Let $(\ \ )^\natural$ denote the variety of all semisimple conjugacy classes of a connected semisimple group. We have a canonical isomorphism $H_1^\natural \cong T_0/W(\Psi')^k$. Put $X = T_0/(W(\Psi)^2 \times W(\Psi')^{k-2})$. On this we have a natural action of $W(\Psi')^2/W(\Psi)^2 \cong \{\pm 1\}^2$. For any character $\chi:\ \mathrm{Gal}(\bar{F}/F) \to \{\pm 1\}$, $H(\chi)^\natural$ is canonically isomorphic to the twist of X by the homomorphism $(\chi, \chi\chi_{E,F}) :$ $\mathrm{Gal}(\bar{F}/F) \to \{\pm 1\}^2$.

For any $P \in Z(F)$, the Galois-set structure of the inverse image of P in T_0 is given by a homomorphism $\phi_P : \mathrm{Gal}(\bar{F}/F) \to \mathrm{Aut}(T_0, \rho_0)$, unique up to conjugation. If P comes from a F-rational conjugacy class of H_i or of $H(\chi)$, then ϕ_P factors through $W(\Psi')^k$. In this case we denote the components of the composite homomorphism

$$\mathrm{Gal}(\bar{F}/F) \xrightarrow{\phi_P} W(\Psi')^k \to W(\Psi')^k / W(\Psi)^k \cong \{\pm 1\}^k$$

by $\chi_{P,1}, \ldots, \chi_{P,k}$. Conversely, any such P lifts to an element of $H_1^\natural(F)$. It lifts to an element of $H(\chi)^\natural(F)$ if and only if, after permuting the $\chi_{P,i}$ if necessary, we have $\chi_{P,1} = \chi$ and $\chi_{P,2} = \chi\chi_{E,F}$.

As in 10.7, there is a Zariski-open dense subspace $U \subset Z$ such that the centralizer of any point of H_i, $H(\chi)$ mapping to U is connected. This allows us to represent rational conjugacy classes by rational elements. The validity of 10.1.1 follows from the following lemmas.

Lemma **(10.10)** *For any $\ell \in \mathbf{L}$ and any non-empty open subset $U_\ell \subset K_\ell$, there exists a character χ_ℓ : $Gal(\bar{\mathbf{Q}}_\ell/\mathbf{Q}_\ell) \to \{\pm 1\}$ and a non-empty open subset $V_\ell \subset H(\chi_\ell)(\mathbf{Q}_\ell)$ so that $ch(V_\ell) \subset ch(U_\ell)$.*

Proof. If ℓ is inert in E, the assertion holds trivially with $\chi_\ell = \mathbf{1}$ and $V_\ell = U_\ell$. So assume that ℓ splits in E. Fix any $g \in U_\ell$ so that $P := ch(g) \in U$ and that $ch(U_\ell)$ contains a neighborhood of P in $U(\mathbf{Q}_\ell)$. By hypothesis, ℓ is odd, so there are precisely 4 distinct characters $\mathrm{Gal}(\bar{\mathbf{Q}}_\ell/\mathbf{Q}_\ell) \to \{\pm 1\}$. Since $k \geq 5$, at least two of the $\chi_{P,i}$ must be equal. After conjugation by S_k we may, and do, assume that $\chi_{P,1} = \chi_{P,2}$. Put $\chi_\ell = \chi_{P,1}$. The assumption that ℓ splits in E means that $\chi_{E,\mathbf{Q}_\ell}$ is the trivial character. By the above characterization of F-rational conjugacy classes it follows that P lifts to a point in $H(\chi_\ell)^\natural(\mathbf{Q}_\ell)$. Since $P \in U$, and $H(\chi_\ell)$ is quasi-split, by the same argument as in 10.7 we can even lift P to a point $h \in H(\chi_\ell)(\mathbf{Q}_\ell)$. The desired assertion now holds if V_ℓ is any sufficiently small neighborhood of h. $\square$

Lemma **(10.11)** *Given any finite subset $T \subset \mathbf{L}$, and a non-empty open subset $U_\ell \subset K_\ell$ for every $\ell \in T$, there exists a character χ : $Gal(\bar{\mathbf{Q}}/\mathbf{Q}) \to \{\pm 1\}$ and an element $h \in H(\chi)(\mathbf{Q})$ so that $ch(h) \in U(\mathbf{Q})$, and $ch(h) \in ch(U_\ell)$ for every $\ell \in T$.*

Proof. Let χ be any global character whose restriction to the local Galois group at every $\ell \in T$ is equal to χ_ℓ given by 10.10. Then we can view V_ℓ as an open subset of $H(\chi)(\mathbf{Q}_\ell)$. To prove the assertion it remains to show that $H(\chi)$ satisfies weak approximation with respect to T. First consider two opposite $\mathbf{Q}$-rational Borel subgroups B, B' of $H(\chi)$, with unipotent radicals U, U', and let $S = B \cap B'$ be the common maximal torus. Looking at the big Bruhat cell shows that $H(\chi)$ is birational to $U \times S \times U'$, so it suffices to prove weak approximation for S with respect to T. By [8] 5.1, this is known if S splits over a cyclic extension of $\mathbf{Q}_\ell$ for every $\ell \in T$. This latter condition depends only on the restrictions of χ and $\chi\chi_E$ to $\mathrm{Gal}(\bar{\mathbf{Q}}_\ell/\mathbf{Q}_\ell)$. When ℓ splits in E, these restrictions are equal, so at worst S splits over a quadratic extension of $\mathbf{Q}_\ell$. When ℓ is inert in E, the proof of 10.10 gives $\chi_\ell = \mathbf{1}$, so S splits over the unramified quadratic extension of $\mathbf{Q}_\ell$. In either case, we are done. $\square$

Lemma **(10.12)** *For any character χ : $Gal(\bar{\mathbf{Q}}/\mathbf{Q}) \to \{\pm 1\}$ and any $h \in H(\chi)(\mathbf{Q})$ with $ch(h) \in U$, the set $\{\ell \in \mathbf{L} | ch(h) \notin ch(K_\ell)\}$ is finite.*

Proof. For all sufficiently large primes ℓ, h lies in a hyperspecial subgroup of $H(\chi)(\mathbf{Q}_\ell)$ ([20] 3.9.1). If ℓ is inert in E and χ unramified at ℓ, then the restriction of χ to $\mathrm{Gal}(\bar{\mathbf{Q}}_\ell/\mathbf{Q}_\ell)$ is either trivial or equal to $\chi_{E,\mathbf{Q}_\ell}$. In both cases $H(\chi) \times_{\mathbf{Q}} \mathbf{Q}_\ell$ is conjugate to $H_2 \times_{\mathbf{Q}} \mathbf{Q}_\ell$ in $GL_{n,\mathbf{Q}_\ell}$. Thus, for all sufficiently large primes $\ell \in \mathbf{L}$ that are inert in E, $ch(h)$ is contained in the image of a hyperspecial subgroup of $H_2(\mathbf{Q}_\ell)$. With finitely many exceptions, at most, K_ℓ is a hyperspecial subgroup of $H_2(\mathbf{Q}_\ell)$. The desired assertion, for inert primes, now follows from the fact that all hyperspecial subgroups are conjugate under $H_2^{ad}(\mathbf{Q}_\ell)$ (see [20] 2.5).

To prove the assertion for split primes observe that $ch(h)$ is the image of a rational conjugacy class in H_1. The same argument as in 10.7 shows that this conjugacy class has a rational representative, *i.e.* there exists $h_1 \in H_1(\mathbf{Q})$ with $ch(h_1) = ch(h)$. For all sufficiently large primes, this element is contained in the intersection $H_1(\mathbf{Q}_\ell) \cap GL_n(\mathbf{Z}_\ell)$. But for primes ℓ that split in E, this intersection is K_ℓ, and we are done. $\square$

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Table 1: Classical cases. For simplicity we have included the case $(A_3, 1, 2) \cong (D_3, 1, 2)$ in the D_n-series.

Θ	m	ℓ	$x_{m+1}(\Theta), \quad x_m(\Theta), \quad x_{m-\ell}(\Theta)$
$(A_n, 1, 1)$ $n \geq 1$	$n+1$	1	$x_{m+1}(\Theta) = \{n+1\} - \{1\}$ $x_m(\Theta) = \{n\}$ $x_{m-\ell}(\Theta) = \begin{cases} \{n-1\} + \{2\} - \{1\} & \text{for } n \geq 3 \\ 2\{1\} & \text{for } n = 2 \\ \{1\} & \text{for } n = 1 \end{cases}$
$(A_n, 1, 2)$ $n \geq 2$ even	$2n+2$	4	$x_{m+1}(\Theta) = \{2n+2\} - \{n+1\} - \{2\} + \{1\}$ $x_m(\Theta) = \{2n-2\} - \{n-1\} + \{2\} - \{1\}$ $x_{m-\ell}(\Theta) = \begin{cases} \{2n-6\} - \{n-3\} + \{6\} - \{3\} & \text{for } n \geq 6 \\ \{4\} & \text{for } n = 4 \\ \{2\} & \text{for } n = 2 \end{cases}$
$(A_n, 1, 2)$ $n \geq 5$ odd	$2n$	4	$x_{m+1}(\Theta) = \{2n\} - \{n\}$ $x_m(\Theta) = \{2n-4\} - \{n-2\} + \{6\} - \{3\} - \{2\} + \{1\}$ $x_{m-\ell}(\Theta) = \begin{cases} \{2n-8\} - \{n-4\} + \{10\} - \{5\} - \{2\} + \{1\} & \text{for } n \geq 9 \\ \{8\} - \{2\} + \{1\} & \text{for } n = 7 \\ \{4\} + \{2\} - \{1\} & \text{for } n = 5 \end{cases}$
$(B_n, 1, 1)$ $n \geq 2$	$2n$	2	$x_{m+1}(\Theta) = \{2n\} - \{n\}$ $x_m(\Theta) = \{2n-2\} - \{n-1\} + \{2\} - \{1\}$ $x_{m-\ell}(\Theta) = \begin{cases} \{2n-4\} - \{n-2\} + \{4\} - \{2\} & \text{for } n \geq 4 \\ \{3\} & \text{for } n = 3 \\ 2\{1\} & \text{for } n = 2 \end{cases}$
$(D_n, 1, 1)$ $n \geq 4$	$2n-2$	2	$x_{m+1}(\Theta) = \{2n-2\} - \{n-1\} + \{2\} - \{1\}$ $x_m(\Theta) = \{2n-4\} - \{n-2\} + \{4\} - \{2\}$ $x_{m-\ell}(\Theta) = \begin{cases} \{2n-6\} - \{n-3\} + \{6\} - \{3\} & \text{for } n \geq 6 \\ \{5\} & \text{for } n = 5 \\ \{3\} + \{1\} & \text{for } n = 4 \end{cases}$
$(D_n, 1, 2)$ $n \geq 3$	$2n$	2	$x_{m+1}(\Theta) = \{2n\} - \{n\}$ $x_m(\Theta) = \{2n-2\} - \{n-1\} + \{1\}$ $x_{m-\ell}(\Theta) = \{2n-4\} - \{n-2\} + 2\{2\} - 2\{1\}$

Table 2: Exceptional cases.

Θ	m	ℓ	$x_{m+1}(\Theta)$	$x_m(\Theta)$	$x_{m-\ell}(\Theta)$
$(G_2, 1, 1)$	6	3	[6]	[3]	2 [2]
$(D_4, 1, 3)$	12	6	[12]	2 [6]	2 [3]
$(F_4, 1, 1)$	12	4	[12]	[8]	2 [6]
$(E_6, 1, 1)$	12	3	[12] + [3]	[9]	[8] + [2] + [1]
$(E_6, 1, 2)$	18	6	[18]	[12] + [6]	[10] + 2 [2]
$(E_7, 1, 1)$	18	4	[18] + [2]	[14] + [2]	[12] + [6] + [2]
$(E_8, 1, 1)$	30	6	[30]	[24]	[20]

Table 3: Conjugacy classes in Atlas notation.

Θ	$\sigma W(\Psi)$	$x_{m+1}(\Theta)$	$x_m(\Theta)$	$x_{m-\ell}(\Theta)$
$(E_6, 1, 1)$	$U_4(2).2$	$12A,\ 12B$	$9A,\ 9B$	$8A$
$(E_6, 1, 2)$	$\{-1\} \times U_4(2).2$	$-9A,\ -9B$	$-12A,\ -12B$	$-5A$
$(E_7, 1, 1)$	$\{\pm 1\} \times S_6(2)$	$-9A$	$-7A$	$-12C$
$(E_8, 1, 1)$	$2.O_8^+(2).2$	$-\widetilde{15B},\ -\widetilde{15C}$	$\widetilde{12F},\ \widetilde{12G}$	$\widetilde{10B},\ \widetilde{10C}$