Abelian varieties, l-adic representations, and l-independence

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Let A be an abelian variety of dimension g over a global field K. Let $\bar{K}$ denote a separable closure of K. If ℓ is a rational prime distinct from the characteristic of K, the Galois group $\mathrm{Gal}(\bar{K}/K)$ acts on the group $A[\ell^n] \cong (\mathbb{Z}/\ell^n\mathbb{Z})^{2g}$ of ℓ^n-torsion points of $A(\bar{K})$. Therefore, it acts continuously on the vector space

$$V_\ell := (\varprojlim A[\ell^n]) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \cong \mathbb{Q}_\ell^{2g}.$$

This system of representations is *strictly compatible* in the sense of Serre [13]. Let ρ_ℓ denote the homomorphism $\mathrm{Gal}(\bar{K}/K) \to \mathrm{Aut}(V_\ell)$ and G_ℓ the Zariski closure of $\rho_\ell(\mathrm{Gal}(\bar{K}/K))$ in $\mathrm{GL}_{2g,\mathbb{Q}_\ell}$. Let G_ℓ° be the identity component of G_ℓ and V_ℓ the representation of G_ℓ° on V_ℓ by ρ_ℓ^{alg}.

This paper is motivated by the following conjecture:

Conjecture. *There exists a connected reductive group G over $\mathbb{Q}$, and a faithful representation ρ of G on a $\mathbb{Q}$-vector space V, such that for all $\ell \gg 0$,*

$$(G_\ell^\circ, V_\ell, \rho_\ell^{\mathrm{alg}}) \cong (G, V, \rho) \times_{\mathbb{Q}} \mathbb{Q}_\ell.$$

In the case that K is a number field, this conjecture goes back almost thirty years. The Lie algebra analogue has been formulated by J. Tate [18], D. Mumford [10], and J.-P. Serre [13]. A more precise conjecture involves comparison with the singular homology group $H_1(A(\mathbb{C}), \mathbb{Q})$ for a fixed embedding $K \subset \mathbb{C}$. If G_∞ denotes the associated Hodge group (cf. §4), the "Mumford-Tate" conjecture states that the comparison isomorphism induces an isomorphism $G_\ell^\circ \cong G_\infty \times \mathbb{Q}_\ell$ for every ℓ. Serre's conjecture [14] C.3.3, which is phrased in the language of algebraic groups, is even more precise.

In the function field case the (present) lack of natural comparison isomorphisms raises delicate questions. For instance, there does not seem to be a natural choice of isomorphism. Moreover, the abstract Tannakian point of view alone does not furnish full justification for the conjecture. Nevertheless, in §5 we show how it follows from other, well-known conjectures.

Almost all the existing unconditional evidence for the conjecture concerns the number field case. For $g = 1$, it is due to Serre [12]. He extended the method to $g \in \{2, 6\} \cup (1 + 2\mathbb{Z})$ under the hypothesis $\mathrm{End}_{\bar{K}}(A) = \mathbb{Z}$ in [16], and this result has undergone some improvement in work of W. Chi [2], [3]. In a different direction, Y. Zarhin [22] proved the conjecture for abelian varieties which admit a place of reduction "of K3 type." Serre proved [15] that the absolute reductive rank of G_ℓ is independent of ℓ, following Zarhin's proof in the function field case [21]. P. Deligne ([6] I Prop. 6.2) proved "one half" of the Mumford-Tate conjecture for abelian varieties over number fields, namely the inclusion $G_\ell^\circ \subset G_\infty \times \mathbb{Q}_\ell$.

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* Supported by N.S.A. Grant No. MDA 904-92-H-3026

Our results fall considerably short of a proof of the general conjecture, but they provide supporting evidence of several different kinds. Our main theorems are as follows:

Theorem 3.3–3.4. *Assume that the center of* $\mathrm{End}_{\bar{K}}(A)$ *is equal to* $\mathbb{Z}$. *Suppose the "splitting field" associated with* A *(cf. §1) is* $\mathbb{Q}$ or *that* $g = \dim(A)$ *is divisible neither by* 3^{15} *nor by* $2^4 m^5$ *for any integer* $m \geq 2$. *Then there exists a connected reductive group* G *over* $\mathbb{Q}$, *and a faithful representation* ρ *of* G *on a vector space* V, *such that*

$$(G_\ell^\circ, V_\ell, \rho_\ell^{\mathrm{alg}}) \cong (G, V, \rho) \times_{\mathbb{Q}} \mathbb{Q}_\ell$$

for all ℓ *belonging to a set of primes of Dirichlet density one. If* K *is a function field, such an isomorphism exists for all* $\ell \gg 0$.

Theorem 4.1. *Suppose that* K *is a function field. Let* σ_ℓ *denote the action of* $\mathrm{End}_{\bar{K}}(A)$ *on* V_ℓ. *There exists a complex vector space* V *with a representation* σ *of* $\mathrm{End}_{\bar{K}}(A)$ *and a faithful representation* ρ *of a connected complex reductive group* G, *such that*

$$(G_\ell^\circ, V_\ell, \rho_\ell^{\mathrm{alg}}, \sigma_\ell) \times_{\mathbb{Q}_\ell} \mathbb{C} \cong (G, V, \rho, \sigma)$$

for all ℓ *and all embeddings* $\mathbb{Q}_\ell \hookrightarrow \mathbb{C}$.

Theorem 4.3. *Suppose that* K *is a number field, and let* G_∞ *denote the Hodge group of* A *for a fixed embedding* $K \subset \mathbb{C}$. *If* $\mathrm{rank}(G_\ell^\circ) = \mathrm{rank}(G_\infty)$ *for some* ℓ, *then* $G_\ell^\circ = G_\infty \times \mathbb{Q}_\ell$ *for every* ℓ. *In particular, if the Mumford-Tate conjecture holds for one prime, then it holds for every prime.*

We would like to thank W. Barker for allowing us to present his unpublished result (Theorem 3.2) below. We would also like to acknowledge useful discussions with C.-L. Chai and J. Milne.

§1. Pure compatible systems of ℓ-adic representations: Generalities

Let K be a global field, *i.e.*, a number field or the field of rational functions on an algebraic curve over a finite field. Let $\bar{K}$ denote a separable closure of K. We recall Serre's definition [13] of a *strictly compatible system of* ℓ*-adic representations* of $\mathrm{Gal}(\bar{K}/K)$: Let S be a finite set of non-archimedean primes of K. The compatible system consists of a continuous representation ρ_ℓ of $\mathrm{Gal}(\bar{K}/K)$ on a finite dimensional $\mathbb{Q}_\ell$-vector space V_ℓ, for all $\ell \nmid \mathrm{char}(K)$. One assumes that ρ_ℓ is unramified at every non-archimedean place $v \notin S$ whose residue characteristic is not ℓ. For all such ℓ, v, the characteristic polynomial of the image $\rho_\ell(\mathrm{Frob}_v)$ of Frobenius is well-defined, and the compatibility condition states that its coefficients lie in $\mathbb{Q}$ and depend only on v. Clearly, this condition implies that the dimension n of V_ℓ is independent of ℓ. We assume that the system is *pure of weight* $w \in \mathbb{Z}$, *i.e.* that the eigenvalues of $\rho_\ell(\mathrm{Frob}_v)$ have absolute value $q_v^{w/2}$ for every complex embedding, where q_v is the number of elements in the residue field of v. Throughout this paper, *compatible system* always means strictly compatible system in the sense of Serre.

For example, when X is a smooth proper variety over K, then by Deligne [5] 3.3.9, the ℓ-adic cohomology $V_\ell := H^w(X \times_K \bar{K}, \mathbb{Q}_\ell)$ is a pure compatible system of weight w, with S the set of primes of bad reduction for X.

The *algebraic monodromy group* of ρ_ℓ, denoted G_ℓ, is defined as the Zariski closure of $\rho_\ell(\mathrm{Gal}(\bar{K}/K))$ in the algebraic group $\mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell) \cong \mathrm{GL}_{n,\mathbb{Q}_\ell}$. Replacing ρ_ℓ by its semisimplification does not affect characteristic polynomials. Thus none of our basic assumptions changes and no information is lost, except that G_ℓ is replaced by its reductive part. We assume throughout that ρ_ℓ is semisimple, so G_ℓ is reductive. In fact, as we will see, the theorem of Zarhin-Faltings (Th. 3.1 below) implies that ρ_ℓ is semisimple in the case of the system of Tate modules of an abelian variety over a global field.

We are interested mainly in the connected component of the identity G_ℓ°. By Serre ([15] p. 17, [17] 2.2.3, cf. also [9] 6.14), we know:

Proposition 1.1. *(i) If G_ℓ is connected for some ℓ, then it is so for every ℓ.*
(ii) The open subgroup $\rho_\ell^{-1}(G_\ell^\circ(\mathbb{Q}_\ell))$ is independent of ℓ.
(iii) The groups G_ℓ/G_ℓ° for different ℓ are canonically isomorphic.

Of course G_ℓ° does not change if K is replaced by a finite extension. Thus, Prop. 1.1 allows us to reduce to the case that every G_ℓ is connected, whenever desired.

Let us recall some definitions from [9]. First consider a connected reductive subgroup G of GL_n over a field F of characteristic zero. A regular semisimple element $g \in G(F)$ lies in a unique maximal torus $T_g \subset G$. We say that g is Γ*-regular* if it is regular semisimple, if every automorphism of $T_g \times_F \bar{F}$ which fixes g and preserves the formal character of $T_g \subset \mathrm{GL}_n$ is trivial, and if the only $\mathrm{GL}_n(\bar{F})$-conjugate of T_g that contains g is T_g itself. (The equivalence of this definition with that in [9] 4.5 can be proved easily, using [9] 4.4–7.) A non-archimedean place $v \notin S$ of K is called *good* if the image of Frobenius $\rho_\ell(\mathrm{Frob}_v)$ belongs to G_ℓ° and is Γ-regular with respect to this group. By [9] 4.5, 6.14 this condition does not depend on the choice of ℓ. Moreover, by [9] 7.2, the set of good primes has positive Dirichlet density.

For every good v there exists a *characteristic torus* $T_v \subset \mathrm{GL}_{n,\mathbb{Q}}$, an element $t_v \in T_v(\mathbb{Q})$, and a family of isomorphisms $\phi_\ell : \mathrm{GL}_{n,\mathbb{Q}_\ell} \to GL(V_\ell)$ coming from a choice of basis on each V_ℓ, such that for every ℓ,

$$\phi_\ell(t_v) = \rho_\ell(\mathrm{Frob}_v),$$

and $\phi_\ell(T_v \times_{\mathbb{Q}} \mathbb{Q}_\ell)$ is a maximal torus of G_ℓ° ([9] 4.7). (Note that the characteristic torus usually but not always coincides with Serre's Frobenius torus [15].) The splitting field of T_v is equal to the splitting field of the characteristic polynomial of $\rho_\ell(\mathrm{Frob}_v)$. The intersection of these fields, for all good v, is called the *splitting field* of $\{\rho_\ell\}$ and denoted E. In particular, it is a finite Galois extension of $\mathbb{Q}$. The splitting field does not change when K is replaced by a finite extension (this follows, e.g., from [9] 8.4). Since our compatible system is pure, the eigenvalues of Frobenius lie in CM fields; thus E is either totally real or CM.

The term “splitting field” is justified by the fact that whenever there exists a $G/\mathbb{Q}$ such that $G_\ell^\circ = G \times_{\mathbb{Q}} \mathbb{Q}_\ell$ for all $\ell \gg 0$, then E is the splitting field of the quasi-split inner form of G. Note that the existence of such a G requires all but finitely many G_ℓ° to be unramified, *i.e.*, quasi-split over $\mathbb{Q}_\ell$ and split over an unramified extension. In [9] 8.9 we proved something weaker, namely:

Proposition 1.2. *For all ℓ belonging to a set of primes of Dirichlet density one, G_ℓ° is unramified over $\mathbb{Q}_\ell$ and split over $E \otimes \mathbb{Q}_\ell$.*

Suppose that for every ℓ we are given a Galois invariant $\mathbb{Z}_\ell$-lattice $\Lambda_\ell \subset V_\ell$. This is the case, in particular, for a compatible system of representations arising as the ℓ-adic cohomology of a smooth proper variety. Let $\mathcal{G}_\ell^\circ$ be the Zariski closure of G_ℓ° in the algebraic group $\mathrm{Aut}_{\mathbb{Z}_\ell}(\Lambda_\ell) \cong \mathrm{GL}_{n,\mathbb{Z}_\ell}$, endowed with the unique structure of reduced closed subscheme. This is a subgroup scheme that is flat over $\mathbb{Z}_\ell$. If $\mathcal{G}_\ell^\circ$ is smooth with reductive fibres, then G_ℓ must be unramified. We know somewhat less, namely

Proposition 1.3. *For all $\ell \gg 0$, $\mathcal{G}_\ell^\circ$ is smooth and of constant reductive rank over $\mathbb{Z}_\ell$.*

Proof. By Prop. 1.1 we may assume that all G_ℓ are connected. We abbreviate $\mathcal{G}_\ell := \mathcal{G}_\ell^\circ$. Fix a good place v of K. If $\lambda_i \in \bar{\mathbb{Q}}$ are the pairwise distinct eigenvalues of Frob_v, we may assume that ℓ does not divide the discriminant $\prod_{i\neq j}(\lambda_i - \lambda_j)^2$. In other words, no two distinct eigenvalues can become congruent modulo a prime above ℓ. By the definition of “good” we have $t_\ell := \rho_\ell(\mathrm{Frob}_v) \in T_\ell(\mathbb{Q}_\ell)$ for a unique maximal torus $T_\ell \subset G_\ell$. Our assumption on the eigenvalues implies that T_ℓ splits over some unramified extension $F/\mathbb{Q}_\ell$. The eigenspace decomposition of $V_\ell \otimes_{\mathbb{Q}_\ell} F$ under T_ℓ coincides with that under t_ℓ, and by our assumption on ℓ this is also the same as the decomposition under the prime-to-ℓ part of t_ℓ. But the latter induces a direct sum decomposition on the lattice $\mathcal{O}_F^n \cong \Lambda_\ell \otimes_{\mathbb{Z}_\ell} \mathcal{O}_F$. It follows that $T_{\ell,F} \cong \mathbb{G}_{m,F}^r$ extends to a closed subgroup scheme $\mathcal{T}_{\mathcal{O}_F} \cong \mathbb{G}_{m,\mathcal{O}_F}^r \hookrightarrow \mathrm{GL}_{n,\mathcal{O}_F}$. Of course, $\mathcal{T}_{\mathcal{O}_F}$ is contained in $\mathcal{G}_{\ell,\mathcal{O}_F}$.

Let L_α be a root space with respect to $T_{\ell,F}$ in the Lie algebra of $G_{\ell,F}$. Then $L_\alpha \cap \mathfrak{gl}_{n,\mathcal{O}_F}$ is a free $\mathcal{O}_F$-module of rank 1. If $\ell \geq n$, we may use exp and log to go back and forth between nilpotent subalgebras and unipotent subgroups. Thus $L_\alpha \cap \mathfrak{gl}_{n,\mathcal{O}_F}$ is the Lie algebra of a subgroup scheme $\mathcal{U}_\alpha \subset \mathrm{GL}_{n,\mathcal{O}_F}$ that is isomorphic to the additive group. Again, this subgroup scheme must be contained in $\mathcal{G}_{\ell,\mathcal{O}_F}$, and $\mathcal{T}_{\mathcal{O}_F}$ acts on it through the character α. Consider the product morphism

$$\mathcal{T}_{\mathcal{O}_F} \times_{\mathcal{O}_F} \prod_\alpha \mathcal{U}_\alpha \longrightarrow \mathcal{G}_{\ell,\mathcal{O}_F}.$$

The induced map of relative tangent spaces at the identity section is equivariant under $\mathcal{T}_{\mathcal{O}_F}$, hence it must be injective. Since the identity section on both sides splits off the horizontal tangent space, our morphism induces an injection of the full tangent space. Thus it is a closed embedding in a Zariski neighborhood of the identity section. On the other hand it is a local isomorphism in the generic fibre. It follows that, near the identity section, the Zariski closure of $G_{\ell,F}$ in $\mathrm{GL}_{n,\mathcal{O}_F}$, with the unique structure of reduced closed subscheme, is contained in the image of the product morphism. As $\mathcal{O}_F$ is unramified over $\mathbb{Z}_\ell$, the definition of $\mathcal{G}_\ell$ as Zariski closure with unique reduced structure commutes with base change. Thus our morphism must be a local isomorphism near the identity section. Since the left hand side is smooth over $\mathcal{O}_F$, the proposition follows. □

§2. The function field case

In this section we restrict to the case that K is a function field. Let $\mathbb{F}_q \subset K$ be the field of constants and X a smooth geometrically connected algebraic curve over $\mathbb{F}_q$ with function field K. We remove from X the finite set where the ρ_ℓ may be ramified, and fix a geometric point $\bar{x}$ of X. Then each ρ_ℓ comes from a representation of the étale fundamental group $\pi_1(X, \bar{x})$ which we denote again by ρ_ℓ. Every V_ℓ is the stalk at $\bar{x}$ of a lisse ℓ-adic sheaf $\mathcal{F}_\ell$ on X, which is pointwise pure of weight w.

There is a short exact sequence of étale fundamental groups

$$0 \to \pi_1(\bar{X}, \bar{x}) \to \pi_1(X, \bar{x}) \xrightarrow{\pi} \widehat{\mathbb{Z}} \to 0,$$

where $\bar{X}$ obtained from X by extension of scalars to $\overline{\mathbb{F}}_q$. The term on the left hand side is called the geometric fundamental group of X. The Zariski closure, G_ℓ^{geom}, of $\rho_\ell(\pi_1(\bar{X}, \bar{x}))$ is called the *geometric monodromy group*. It is a normal subgroup of G_ℓ. Since $\mathcal{F}_\ell$ is pure, it follows from [5] 1.3.9 and 3.4.1 (iii) that G_ℓ^{geom} is semisimple. In fact, $(G_\ell^{\mathrm{geom}})^\circ$ is the derived group of G_ℓ°.

Proposition 2.1. *The dimension of the space of invariants $V_\ell^{G_\ell^{\mathrm{geom}}}$ is independent of ℓ.*

Proof. The cohomology with compact support $H_c^i(\bar{X}, \mathcal{F}_\ell)$ vanishes in degrees $i > 2$. For $i = 2$ it is canonically isomorphic to $V_\ell^{G_\ell^{\mathrm{geom}}}(-1)$, where (-1) denotes Tate twist; it is therefore pure of weight $w + 2$. In degrees $i < 2$ it is mixed of weight $\leq w + i$, by [5] 3.3.1; in particular it has weights $< w + 2$. It follows that the dimension in question can be described as the sum of the multiplicities of all Frobenius eigenvalues of weight $w+2$ in the virtual representation $\sum(-1)^i H_c^i(\bar{X}, \mathcal{F}_\ell)$. By the Lefschetz trace formula ([4] Rapport, 3.1) this number depends only on the zeta function of $(X, \mathcal{F}_\ell)$, which is, by the compatibility assumption, independent of ℓ. □

Of course the proposition can be applied to any compatible system of representations that is obtained from $\{V_\ell\}$ by linear algebra. That is, consider any algebraic representation of $\mathrm{GL}_{n,\mathbb{Q}}$ on a space W. On $W \otimes \mathbb{Q}_\ell$ we obtain a representation of G_ℓ^{geom}, unique up to isomorphism, such that $\dim(W \otimes \mathbb{Q}_\ell)^{G_\ell^{\mathrm{geom}}}$ is independent of ℓ. In the terminology of [8] all the pairs $(G_\ell^{\mathrm{geom}}, V_\ell)$ have the same *dimension data.*

Proposition 2.2. *(i) If G_ℓ^{geom} is connected for some ℓ, then it is so for every ℓ.*
(ii) The open subgroup $\rho_\ell^{-1}((G_\ell^{\mathrm{geom}})^\circ(\mathbb{Q}_\ell)) \cap \pi_1(\bar{X}, \bar{x})$ is independent of ℓ.
(iii) The groups $G_\ell^{\mathrm{geom}}/(G_\ell^{\mathrm{geom}})^\circ$ for different ℓ are canonically isomorphic.

Proof. First we show that (i), if universally true, implies the rest. The open subgroup $\rho_\ell^{-1}((G_\ell^{\mathrm{geom}})^\circ(\mathbb{Q}_\ell)) \cap \pi_1(\bar{X}, \bar{x})$ belongs to a finite extension of $K\bar{\mathbb{F}}_q$. We can write this extension as $L\bar{\mathbb{F}}_q$ for a finite extension L/K. We can apply the same constructions to the representations $\rho_{\ell'}|_{\mathrm{Gal}(\bar{L}/L)}$. By construction the geometric monodromy group of this representation is equal to $(G_\ell^{\mathrm{geom}})^\circ$ for $\ell' = \ell$, so by (i) it is connected for any ℓ'. This implies that every $\rho_\ell^{-1}((G_\ell^{\mathrm{geom}})^\circ(\mathbb{Q}_\ell)) \cap \pi_1(\bar{X}, \bar{x})$ is contained in every other, so all are equal. Part (iii) follows immediately from (ii).

To prove (i) we consider the dimension functions

$$W \mapsto \dim\Big((W \otimes \mathbb{Q}_\ell)^{\mathrm{Gal}(\bar{K}/L\bar{\mathbb{F}}_q)}\Big)$$

as L ranges over all finite extensions of K. Of course invariant dimensions cannot decrease when L increases. The lemma below implies that G_ℓ^{geom} is connected if and only if this function stays the same for every finite extension L. By Prop. 2.1 this condition is independent of ℓ, so the corollary follows. □

Lemma 2.3. *Let G be a reductive algebraic subgroup of GL_n over a field. Suppose that for every representation of GL_n on a finite dimensional vector space W we have $\dim(W^{G^\circ}) = \dim(W^G)$. Then G is connected.*

Proof. The diagram of algebraic groups

$$\begin{array}{ccccc} \mathrm{GL}_n & \hookleftarrow & G & \twoheadrightarrow & G/G^\circ \\ \| & & \| & & \| \\ \mathrm{Spec}\, S & \hookleftarrow & \mathrm{Spec}\, A & \twoheadrightarrow & \mathrm{Spec}\, B \end{array}$$

corresponds to the equivariant diagram of G-representations $S \twoheadrightarrow A \hookleftarrow B$. There exists a finite dimensional GL_n-invariant subspace $W \subset S$ whose image in A contains B. By complete reducibility under G the representation $W|_G$ contains a direct summand isomorphic to B. Now by assumption

$$0 = \dim(W^{G^\circ}) - \dim(W^G) \geq \dim(B^{G^\circ}) - \dim(B^G) = [G : G^\circ] - 1,$$

so $[G : G^\circ] = 1$, as desired. □

We fix embeddings $\mathbb{Q}_\ell \subset \mathbb{C}$ which we use without further mention. We write $\rho_\ell^{\mathrm{geom}}$ (resp. ρ_ℓ^{alg}) for the tautological representation of $(G_\ell^{\mathrm{geom}})^\circ$ (resp. G_ℓ°) on V_ℓ. Note that if one $\rho_\ell^{\mathrm{geom}}$ is absolutely irreducible, then so are all others; this follows at once by applying Prop. 2.1 to the representation on $\mathrm{End}(V_\ell)$.

Theorem 2.4. *The connected component of the identity $(G_\ell^{\mathrm{geom}})^\circ \times_{\mathbb{Q}_\ell} \mathbb{C}$ is, up to isomorphism, independent of ℓ. If the ρ_ℓ are absolutely irreducible, then the complexified triple $((G_\ell^{\mathrm{geom}})^\circ, V_\ell, \rho_\ell^{\mathrm{alg}}) \times_{\mathbb{Q}_\ell} \mathbb{C}$ is, up to isomorphism, independent of ℓ.*

Proof. By Prop. 2.2 we are reduced to the case that the G_ℓ^{geom} are already connected. Then the theorem follows from [8] Th. 1 and Th. 2. □

After choosing a basis for V_ℓ the second part of this theorem says that $(G_\ell^{\mathrm{geom}})^\circ \times_{\mathbb{Q}_\ell} \mathbb{C}$ is independent of ℓ *up to conjugation in* $\mathrm{GL}_{n,\mathbb{C}}$. For compatible systems coming from cohomology one expects this to be true without irreducibility assumptions. But this cannot be proved by invariant theory alone, for it is known ([8] Th. 3) that for $n \gg 0$ there exist non-conjugate, though isomorphic, connected semisimple subgroups of $\mathrm{GL}_{n,\mathbb{C}}$ with the same dimension data. In the case of abelian varieties this gap can be closed: see §4.

The results 2.1–4 extend directly to pure compatible systems of λ-adic representations. More precisely, let F be a number field and denote its completion at a finite prime λ by F_λ. A system of continuous representations ρ_λ of $\mathrm{Gal}(\bar{K}/K)$ on F_λ-vector spaces V_λ, for $\lambda \nmid \mathrm{char}(K)$, is called compatible if ρ_λ is unramified at every place $v \notin S$ whose residue characteristic is different from that of λ, and if the characteristic polynomial of $\rho_\lambda(\mathrm{Frob}_v)$ has coefficients in F and is independent of λ. The geometric monodromy groups are then linear algebraic groups over F_λ, and the independence of ℓ holds for the extension of scalars under any embedding $F_\lambda \subset \mathbb{C}$.

In the next section we shall need information about the $\mathbb{Q}_\ell$-rational structure of G_ℓ° that holds without Dirichlet density restriction. For this purpose we take up the method of [9]. Let T be the characteristic torus associated to a fixed good place of K; after choosing a basis of each V_ℓ we may assume that $T \times \mathbb{Q}_\ell$ is a maximal torus of $G_\ell^\circ \subset \mathrm{GL}_{n,\mathbb{Q}_\ell}$ (cf. the definition of "good" in §1). Let Γ denote the group of all automorphisms of $T \times \bar{\mathbb{Q}}$ that fix the formal character of its tautological representation. For every ℓ, let W_ℓ denote the absolute Weyl group of G_ℓ° with respect to $T \times \mathbb{Q}_\ell$, and N_ℓ its normalizer in Γ. Consider any unramified maximal torus $T'_\ell \subset G_\ell^\circ$ and conjugate it into $T \times \mathbb{Q}_\ell$ under $G_\ell^\circ(\bar{\mathbb{Q}}_\ell)$. The action of Frob_ℓ on the character group of T'_ℓ corresponds to an element $\sigma(T'_\ell) \in N_\ell$, whose W_ℓ-conjugacy class depends only on T'_ℓ.

Now assume that G_ℓ° is unramified. Then the W_ℓ-conjugacy classes thus obtained form a full coset $\sigma(G_\ell^\circ)W_\ell$ ([9] 3.10). Moreover, G_ℓ° is determined up to $\mathrm{GL}_n(\mathbb{Q}_\ell)$-conjugacy by its $\mathrm{GL}_n(\mathbb{C})$-conjugacy class and this coset modulo N_ℓ. Fix ℓ and finitely many maximal tori $T_{\ell,i} \subset G_\ell^\circ$ such that the $\sigma(T_{\ell,i})$ meet all the conjugacy classes in $\sigma(G_\ell^\circ)W_\ell$. By [9] 8.2 we can find characteristic tori $T_i \subset \mathrm{GL}_{n,\mathbb{Q}}$ associated to good places of K, such that every $T_i \times \mathbb{Q}_\ell$ is conjugate to $T_{\ell,i}$ under $\mathrm{GL}_n(\mathbb{Q}_\ell)$. The set of all primes ℓ' with the same Frobenius conjugacy class as ℓ in the splitting field of every T_i has positive Dirichlet density. By Prop. 1.2 the same holds for the subset, denoted $\mathbb{L}$, of all those for which, in addition, $G_{\ell'}^\circ$ is unramified. By construction every $T_i \times \mathbb{Q}_{\ell'}$ is $\mathrm{GL}_n(\mathbb{Q}_{\ell'})$-conjugate to a maximal torus of $G_{\ell'}^\circ$. For $\ell' \in \mathbb{L}$ this implies that $\sigma(T_{\ell,i})$ and $\sigma(T_{\ell',i})$ are conjugate under Γ. To express this state of affairs in a concise form, denote by $[S]$, for any subset $S \subset \Gamma$, the set of all elements of Γ that are conjugate to an element of S. We have proved:

Lemma 2.5. *For every $\ell' \in \mathbb{L}$, we have $[\sigma(G_\ell^\circ)W_\ell] \subset [\sigma(G_{\ell'}^\circ)W_{\ell'}]$.*

Proposition 2.6. *For any ℓ that splits in E, if G_ℓ° is unramified, then it is split.*

Proof. Let $\ell' \in \mathbb{L}$ as above; by Prop. 1.2 we may even suppose that $G_{\ell'}^\circ$ splits over $E \otimes \mathbb{Q}_{\ell'}$. Since ℓ splits in E, so does ℓ', and $G_{\ell'}^\circ$ is split. In other words $\sigma(G_{\ell'}^\circ) \in W_{\ell'}$. By the first part of Th. 2.4 (this is the only place in the proof where we use the hypothesis that K is a function field), the pairs $(T \times \mathbb{C}, W_\ell)$ and $(T \times \mathbb{C}, W_{\ell'})$ are isomorphic. (Caution:

we do not know that W_ℓ and $W_{\ell'}$ are conjugate under Γ; cf. [8] Th. 3.) For any set S of automorphisms of T let $[[S]]$ denote the set of all characteristic polynomials of $\gamma \in S$ acting on the character group of T. So far we know

$$[[\sigma(G_\ell^\circ)W_\ell]] \subset [[\sigma(G_{\ell'}^\circ)W_{\ell'}]] = [[W_{\ell'}]] = [[W_\ell]].$$

We have therefore reduced the proposition to the following lemma.

Lemma 2.7. *Let $T \subset G$ be a maximal torus in a connected reductive group over $\mathbb{C}$. Let W denote the associated Weyl group. Let σ be an automorphism of finite order of T that preserves the root system of G. If $[[\sigma W]] \subset [[W]]$, then $\sigma \in W$.*

Proof. This is done as in [9] §2. Let Φ_r denote the r^{th} cyclotomic polynomial. We define a kind of lexicographic order on the set of all polynomials which are products of powers of the Φ_r. For two such polynomials we write $f \succ g$ if, for some r, the multiplicity of Φ_r in f is greater than that in g, but for all $s > r$ the multiplicity of Φ_s is the same. Clearly $f \succeq g \Leftrightarrow f \succ g$ or $f = g$ defines a total order. It suffices to prove $\max[[\sigma W]] \succ \max[[W]]$ whenever $\sigma \notin W$. Now $\max[[\sigma W]]$ is preserved under isogenies and is multiplicative for decompositions of (G, T, σ). Thus it suffices to prove the inequality in the case that (G, T, σ) cannot be decomposed further, up to isogeny. For these the inequality follows from [9] 2.4 and Tables 1 and 2. □

§3. Abelian varieties over global fields

In this section we consider an abelian variety A over an arbitrary global field K. Letting T_ℓ denote its Tate module, we have a pure compatible system of Galois representations on $V_\ell := T_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. We keep the notations of §1.

The following is a celebrated theorem of Faltings ([7] Satz 3, Satz 4, and the Bemerkung at the end of the article) in the number field case, and of Zarhin [19], [20] in the function field case.

Theorem 3.1. *(i) The action of $\mathrm{Gal}(\bar{K}/K)$ on V_ℓ is semisimple for every ℓ.*
(ii) The map $\mathrm{End}_K(A) \otimes \mathbb{Z}_\ell \to \mathrm{End}_{\mathrm{Gal}(\bar{K}/K)}(T_\ell)$ is an isomorphism for every ℓ.
(iii) For every $\ell \gg 0$, the subalgebra of $\mathrm{End}_{\mathbb{Z}_\ell}(T_\ell)$ generated by $\rho_\ell(\mathrm{Gal}(\bar{K}/K))$ is the full commutant of $\mathrm{End}_K(A) \otimes \mathbb{Z}_\ell$.

Parts (i) and (ii) imply that G_ℓ is reductive and that the canonical map $\mathrm{End}_K(A) \otimes \mathbb{Q}_\ell \to \mathrm{End}_{G_\ell}(V_\ell)$ is an isomorphism. It was observed by W. Barker (unpublished) that part (iii) implies the following strengthening of Prop. 1.2.

Theorem 3.2. *G_ℓ° is unramified for all $\ell \gg 0$.*

Proof. First we use Prop. 1.1 to replace K by a finite extension so that every G_ℓ is connected. Then, as in §1 we define $\mathcal{G}_\ell = \mathcal{G}_\ell^\circ$ as the Zariski closure of G_ℓ in the algebraic group $\mathrm{Aut}_{\mathbb{Z}_\ell}(T_\ell)$. By Prop. 1.3 this is smooth over $\mathbb{Z}_\ell$ for all $\ell \gg 0$. We shall prove that its closed fibre is reductive; this then implies the theorem.

Excluding at most finitely many primes, we may assume that $\mathrm{End}_K(A) \otimes \mathbb{Z}_\ell$ is isomorphic to a direct sum of full matrix algebras over unramified extensions of $\mathbb{Z}_\ell$. As a representation of any summand, T_ℓ must be isomorphic to a direct sum of copies of the standard representation. It follows that the commutant of $\mathrm{End}_K(A) \otimes \mathbb{Z}_\ell$ is also isomorphic to a direct sum of matrix algebras over unramified extensions of $\mathbb{Z}_\ell$, that it maps surjectively to the commutant in $\mathrm{End}(T_\ell/\ell T_\ell)$, and that the latter is semisimple. By Th. 3.1 (iii) this commutant is generated by the image of $\mathrm{Gal}(\bar{K}/K)$, for all $\ell \gg 0$. It follows that the Galois representation on $T_\ell/\ell T_\ell$ is semisimple and that, dually, its commutant is equal to the image of $\mathrm{End}_K(A)$. This last fact implies that every $\mathrm{Gal}(\bar{K}/K)$-invariant subspace of $T_\ell/\ell T_\ell$ is invariant under $\mathcal{G}_\ell$ in the algebraic sense. Moreover, every irreducible $\mathrm{Gal}(\bar{K}/K)$-subspace of $T_\ell/\ell T_\ell$ is *a fortiori* irreducible under $\mathcal{G}_\ell$. Thus the representation of the closed fibre of $\mathcal{G}_\ell$ is semisimple. By the definition of $\mathcal{G}_\ell$ it is also faithful, so the closed fibre of $\mathcal{G}_\ell$ is reductive, as desired. □

As in §1 we denote the splitting field of $\{\rho_\ell\}$ by E. Let $g := \dim(A)$. For the next two theorems we assume (1) and either (2) or (3):

(1) The center of $\mathrm{End}_{\bar{K}}(A)$ is equal to $\mathbb{Z}$,
(2) $E = \mathbb{Q}$,
(3) The dimension g is divisible neither by 3^{15} nor by $2^4 m^5$ for any integer $m \geq 2$.

Theorem 3.3. *Under the above assumptions (1) and (2), or (1) and (3), there exists a connected reductive group G over $\mathbb{Q}$, and a faithful representation ρ of G on a vector space V, such that*

$$(G_\ell^\circ, V_\ell, \rho_\ell^{\mathrm{alg}}) \cong (G, V, \rho) \times_{\mathbb{Q}} \mathbb{Q}_\ell$$

for all ℓ belonging to a set of primes of Dirichlet density one.

Proof. The hypotheses remain valid when K is replaced by a finite extension, so using Prop. 1.1 we may assume that every G_ℓ is already connected. By (1), $\mathrm{End}_K(A) \otimes \mathbb{Q}$ is a central simple algebra over $\mathbb{Q}$, say of dimension d^2. Excluding a finite number of primes, we may assume that $\mathrm{End}_K(A) \otimes \mathbb{Q}_\ell$ is isomorphic to the algebra of $d \times d$ matrices over $\mathbb{Q}_\ell$. From Th. 3.1 it then follows that ρ_ℓ is a direct sum of d copies of an absolutely irreducible Galois representation ρ'_ℓ on a $\mathbb{Q}_\ell$-vector space V'_ℓ of dimension $2g/d$. Clearly the ρ'_ℓ are again compatible. Using assumption (2) or (3), [9] 9.10 implies that

$$(G_\ell, V'_\ell, {\rho'_\ell}^{\mathrm{alg}}) \cong (G, V', \rho') \times_{\mathbb{Q}} \mathbb{Q}_\ell$$

for suitable (G, V', ρ') over $\mathbb{Q}$, and a set of primes of Dirichlet density one. The theorem follows. □

Theorem 3.4. *Suppose that K is a function field. Then, under the above assumptions (1) and (2), or (1) and (3), there exists a connected reductive group G over $\mathbb{Q}$, and a faithful representation ρ of G on a vector space V, such that*

$$(G_\ell^\circ, (G_\ell^{\mathrm{geom}})^\circ, V_\ell, \rho_\ell) \cong (G, G^{\mathrm{der}}, V, \rho) \times_{\mathbb{Q}} \mathbb{Q}_\ell$$

for all $\ell \gg 0$.

Proof. Applying Prop. 1.1 and Prop. 2.2, we may reduce the theorem to the case that that both G_ℓ and G_ℓ^{geom} are connected. As in the preceding proof, after excluding a finite number of primes we are reduced to studying the quadruples $(G_\ell, G_\ell^{\text{geom}}, V'_\ell, {\rho'_\ell}^{\text{alg}})$. Since G_ℓ^{geom} is the derived group of G_ℓ, absolute irreducibility implies that $G_\ell \subset G_\ell^{\text{geom}} \cdot \{\text{scalars}\}$. But the representation is pure of weight -1, so the determinant of G_ℓ cannot be trivial. It follows that $G_\ell = G_\ell^{\text{geom}} \cdot \{\text{scalars}\}$. With (G, V', ρ') as in the preceding proof it follows that

$$(G_\ell, G_\ell^{\text{geom}}, V'_\ell, {\rho'_\ell}^{\text{alg}}) \cong (G, G^{\text{der}}, V', \rho') \times_{\mathbb{Q}} \mathbb{Q}_\ell$$

for a set of primes of Dirichlet density one. Without loss of generality we may assume that G is quasi-split. Then, by Prop. 1.2, these isomorphisms already imply that G splits over E. Note also that, by the second part of Th. 2.4, the above isomorphism exists over $\mathbb{C}$ for every ℓ.

By Th. 3.2, after excluding a finite number of primes we may assume that G_ℓ is unramified. In the case $E = \mathbb{Q}$, this together with Prop. 2.6 implies that G_ℓ is split. But G splits over $E = \mathbb{Q}$, so the isomorphism over $\mathbb{C}$ can be realized over $\mathbb{Q}_\ell$, as desired. In the case of (3) we use Lemma 2.5 directly. With the notations of §2, we may assume that $T \subset G \subset \mathrm{GL}_{n,\mathbb{Q}}$. Let W be the absolute Weyl group of G with respect to T, and N its normalizer in Γ. The $\mathbb{Q}$-rational structure of G is determined by a homomorphism $\varphi : \mathrm{Gal}(E/\mathbb{Q}) \to N$. The desired isomorphism exists if and only if

$$\sigma(G_\ell^\circ)W_\ell = \gamma(\varphi(\mathrm{Frob}_\ell)W)\gamma^{-1}$$

for some $\gamma \in \Gamma$. Since we know this already on a set of Dirichlet density one, by Lemma 2.5 we find that

$$[\sigma(G_\ell^\circ)W_\ell] \subset [\sigma(G_{\ell'}^\circ)W_{\ell'}] = [\varphi(\mathrm{Frob}_{\ell'})W] = [\varphi(\mathrm{Frob}_\ell)W]$$

for suitable ℓ'. The rest is an elementary calculation of finite groups. If $W_1 \subset W$ is the largest subgroup that is normal in Γ, it suffices to prove the above conjugacy modulo W_1. The possible forms of the factor groups $W/W_1 \subset \Gamma/W_1$ are deduced from the proof of [9] 9.7, so the theorem follows from the lemma below. □

Lemma 3.5. *Let S_n denote the symmetric group on n letters, and consider one of the following cases:*
(i) $H = \{\pm 1\}^j \times \{1\}^{k-j} \subset G = \{\pm 1\}^k \rtimes S_k$ for $0 \le j \le k$, or
(ii) $H = S_{k_1} \times \ldots \times S_{k_r} \subset G = S_k$ for a partition $k = k_1 + \ldots + k_r$.
Let $N \subset G$ be the normalizer of H, and $n_1, n_2 \in N$. Suppose that every element of the coset $n_1 H$ is conjugate in G to an element of $n_2 H$. Then $n_1 H = n(n_2 H)n^{-1}$ for some $n \in N$.

Proof. For case (i) we first classify the conjugacy classes in $G = \{\pm 1\}^k \rtimes S_k$. Consider an element $g \in G$ and a cycle of length i of its image in S_k. The restriction of g^i to the stabilizer $\{\pm 1\}^i \rtimes S_i$ of the i letters in the cycle takes the form $(\epsilon, \ldots, \epsilon) \in \{\pm 1\}^i$ for a certain $\epsilon = \pm 1$. We call this ϵ the *sign* of the cycle. One easily checks that two elements of G are conjugate if and only if they have the same number of cycles of every length and sign.

The normalizer of $H = \{\pm 1\}^j \times \{1\}^{k-j}$ in G is equal to

$$N = \{\pm 1\}^k \rtimes (S_j \times S_{k-j}),$$

so

$$N/H \cong S_j \times (\{\pm 1\}^{k-j} \rtimes S_{k-j}).$$

For $\nu \in \{1, 2\}$, let $a_{i,\nu}$ (resp. $a^{\epsilon}_{i,\nu}$) denote the number of cycles of n_ν of length i in the S_j-factor (resp. of sign ϵ and length i in the $\{\pm 1\}^{k-j} \rtimes S_{k-j}$-factor). The total number of cycles of length i of the image of n_ν in S_k is $a_{i,\nu} + a^1_{i,\nu} + a^{-1}_{i,\nu}$. This is independent of ν, so it suffices to prove $a^\epsilon_{i,1} = a^\epsilon_{i,2}$ for all i and ϵ.

We claim that $a^\epsilon_{i,\nu}$ is the minimum number of cycles of length i and sign ϵ for any element in the coset $n_\nu H$. Indeed, a cycle of length i in S_j can lift to a cycle of sign $-\epsilon$, so it does not contribute to the minimum. The contribution of the other cycles is by definition $a^\epsilon_{i,\nu}$. Since any element of $n_1 H$ can be conjugated into $n_2 H$ under G, we find

$$a^\epsilon_{i,1} \geq a^\epsilon_{i,2}.$$

On the other hand, the $a^\epsilon_{i,\nu}$ determine the cycle decomposition of an element of S_{k-j}, so

$$\sum_{i,\epsilon} i a^\epsilon_{i,\nu} = k - j.$$

The resulting inequality

$$k - j = \sum_{i,\epsilon} i a^\epsilon_{i,1} \geq \sum_{i,\epsilon} i a^\epsilon_{i,2} = k - j$$

must be an equality, so $a^\epsilon_{i,1} = a^\epsilon_{i,2}$ for all i and ϵ. This implies that n_1 and n_2 are conjugate in N/H and therefore that $n_1 H$ and $n_2 H$ are conjugate in N.

For case (ii), we define a_i to be the number of factors of H of type S_i. Then

$$N/H \cong S_{a_1} \times S_{a_2} \times \cdots,$$

where the constituent S_{a_i} permutes the a_i factors of H of type S_i. Let $b_{ij\nu}$ denote the number of cycles of length j of n_ν inside S_{a_i}. It suffices to prove that $b_{ij1} = b_{ij2}$ for all i and j, since this implies that the images of n_1 and n_2 in N/H are conjugate, and the result follows.

For any $m, \ell \geq 1$ and any $\sigma \in S_k$, let $c_{m,\ell}(\sigma)$ be the number of letters occurring in cycles of length ℓ, 2ℓ ..., or $m\ell$. Set

$$c_{m\ell\nu} = \min\{c_{m\ell}(\sigma) \mid \sigma \in n_\nu H\}.$$

We claim that

$$c_{m\ell\nu} = \sum_{\{(i,j):\ \ell|j \text{ and } ij\le m\ell\}} ijb_{ij\nu}.$$

Indeed, to minimize $c_{m\ell}(\sigma)$, we can minimize independently the contributions from each cycle of length j in the factor S_{a_i}. In the group S_k, such a cycle interchanges j disjoint sets of i letters, and each of these sets is permuted arbitrarily by an element of H. Clearly any element of the coset acts through cycles of lengths divisible by j and no larger than ij. On the other hand, the coset contains an element that acts through precisely i cycles of length j and another than acts through a single ij-cycle. Thus the contribution to $c_{m\ell\nu}$ is zero unless $\ell|j$ and $ij \le m\ell$. In that case, all the cycles lengths are contained in $\{\ell, 2\ell, \ldots, m\ell\}$, so the contribution is ij, as claimed.

It follows from the definition that $c_{m\ell}(\sigma)$ depends only on the S_k-conjugacy class of σ. Since any element of n_1H can be conjugated into n_2H, we conclude

$$c_{m\ell 1} \ge c_{m\ell 2}.$$

We now use induction on m to prove that $b_{m\ell 1} = b_{m\ell 2}$ for all ℓ. If the claim holds for all values smaller than a given m, then the only non-zero terms in the sum

$$c_{m\ell 1} - c_{m\ell 2} = \sum_{\{(i,j):\ \ell|j \text{ and } ij\le m\ell\}} ij(b_{ij1} - b_{ij2})$$

are those for which $i = m$ and $j = \ell$. Thus,

$$0 \le c_{m\ell 1} - c_{m\ell 2} = m\ell(b_{m\ell 1} - b_{m\ell 2}).$$

On the other hand,

$$\sum_j jb_{ij\nu} = a_i$$

for any i, ν, by definition of $b_{ij\nu}$. It follows that

$$0 \le \sum_\ell \ell(b_{m\ell 1} - b_{m\ell 2}) = a_m - a_m = 0.$$

Hence $b_{m\ell 1} = b_{m\ell 2}$ for all ℓ, and the lemma follows. □

With a suitable generalization of the results of [8] and [9] it should be possible to eliminate the hypothesis (1) in Th. 3.3 and Th. 3.4 (while modifying the divisibility hypothesis in (3) appropriately). Then one should be able to prove that for all global fields (resp. function fields) and most (resp. all sufficiently large) primes ℓ, the $\mathrm{End}_{\bar{K}}(A)$-linear action of G_ℓ° on V_ℓ comes from a fixed $\mathbb{Q}$-group G endowed with a representation defined over $\mathrm{End}_{\bar{K}}(A)$. The next section contains some results in this direction.

§4. Abelian varieties with arbitrary endomorphism ring

The results of this section make essential use of the fact that, after suitable extension of scalars, the system of Tate modules of an abelian variety over a global field can be decomposed into a sum of systems of absolutely irreducible representations. Notations remain the same as in §3. Without loss of generality we assume that all G_ℓ and, if K is a function field, all G_ℓ^{geom} are connected. Then in particular $\text{End}_K(A) = \text{End}_{\bar{K}}(A)$.

By the Wedderburn theorem, the semisimple algebra $\text{End}_K(A) \otimes \mathbb{Q}$ splits over some number field F. Choose an isomorphism with a direct sum of matrix algebras

$$\text{End}_K(A) \otimes F \cong \prod_{i=1}^{r} M_{n_i}(F).$$

For every finite prime $\lambda \nmid \text{char}(K)$ of F this induces a decomposition

$$V_\ell \otimes_{\mathbb{Q}_\ell} F_\lambda \cong \bigoplus_{i=1}^{r} W_{i,\lambda} \otimes_F F^{n_i},$$

where F^{n_i} denotes the standard representation of $M_{n_i}(F)$ and $W_{i,\lambda}$ is a representation of $\text{Gal}(\bar{K}/K)$, unique up to isomorphism. By Th. 3.1 the $W_{i,\lambda}$ are absolutely irreducible and pairwise inequivalent. For fixed i and varying λ, they form a pure compatible system of λ-adic representations (see, e.g., [11] §8). In particular, $d_i := \dim W_{i,\lambda}$ is independent of λ. The algebraic monodromy group G_λ (resp. the geometric monodromy group G_λ^{geom}, if K is a function field) of $\bigoplus_i W_{i,\lambda}$ is obtained from G_ℓ (resp. G_ℓ^{geom}) by extension of scalars to F_λ. Let us abbreviate

$$\text{GL}_{\underline{d}} := \prod_{i=1}^{r} \text{GL}_{d_i}.$$

Choosing a basis for each $W_{i,\lambda}$, our G_λ (resp. G_λ^{geom}) become subgroups of $\text{GL}_{\underline{d},F_\lambda}$, unique up to conjugation. Our problem is then to study whether they come — up to conjugation in $\text{GL}_{\underline{d}}$ — from a fixed subgroup over F.

Taking the determinant in each factor, G_λ maps onto a subtorus T_λ of the product of multiplicative groups $\mathbb{G}_{m,F_\lambda}^r$. Every character of $\mathbb{G}_{m,F}^r$ gives rise to a compatible system of representations, so whether it is trivial on T_λ is independent of λ. Thus each $T_\lambda = T \times_F F_\lambda$ for some subtorus $T \subset \mathbb{G}_{m,F}^r$. The center of $\text{GL}_{\underline{d},F_\lambda}$ maps onto $\mathbb{G}_{m,F_\lambda}^r$, and the pre-image of T_λ is the identity component of the center of G_λ. In other words, these identity components come from a fixed torus in the center of $\text{GL}_{\underline{d},F}$. Thus, for questions of independence of ℓ it suffices to deal with G_λ^{der}.

The following result is an analogue of Th. 2.4. Let σ_ℓ denote the action of $\text{End}_{\bar{K}}(A)$ on V_ℓ.

Theorem 4.1. *If K is a function field, the data $(G_\ell^\circ, (G_\ell^{\text{geom}})^\circ, V_\ell, \rho_\ell^{\text{alg}}, \sigma_\ell) \times_{\mathbb{Q}_\ell} \mathbb{C}$ is, up to isomorphism, independent of ℓ.*

Proof. By the above remarks it suffices to prove that the subgroups

$$G_\lambda^{\mathrm{geom}} \times_{F_\lambda} \mathbb{C} = G_\lambda^{\mathrm{der}} \times_{F_\lambda} \mathbb{C} \subset \mathrm{GL}_{\underline{d},\mathbb{C}}$$

are all conjugate. Every representation of $\mathrm{GL}_{\underline{d},F}$, on a finite dimensional vector space U, can be obtained from the standard representations by means of linear algebra. Thus it gives rise to a compatible system of λ-adic representations to which we can apply Proposition 2.1. It follows that the dimension of invariants of $G_\lambda^{\mathrm{geom}}$ in $U \otimes_F F_\lambda$ is independent of λ. The desired assertion is now a consequence of the next theorem, which is an easy extension of [8] Th. 2.

Theorem 4.2. *Let G be a connected semisimple algebraic subgroup of $\mathrm{GL}_{\underline{d},\mathbb{C}}$, such that each standard representation $G \to \mathrm{GL}_{d_i,\mathbb{C}}$ is irreducible. Then the data assigning $\dim(U^G)$ to every representation of $\mathrm{GL}_{\underline{d},\mathbb{C}}$ on a finite dimensional vector space U determines G up to conjugation in $\mathrm{GL}_{\underline{d},\mathbb{C}}$.*

Proof. Let $T \subset G$ be a maximal torus, Φ the associated root system, and ρ_i the formal character of the representation $T \to \mathrm{GL}_{d_i,\mathbb{C}}$. If $X^*(T)$ is the character group of T, we can view ρ_i as an element of the group ring $\mathbb{Z}[X]$ of the vector space $X = X^*(T) \otimes \mathbb{Q}$. It suffices to prove that the dimension function $U \mapsto \dim(U^G)$ determines the tuple $(X, \rho_1, \ldots, \rho_r, \Phi)$ up to isomorphism. Let Γ be the (finite) group of all automorphisms of X that fix each ρ_i. Consider the subspace generated by the element

$$F := \sum_{\gamma \in \Gamma} \gamma \left(\prod_{\alpha \in \Phi} (1 - \alpha) \right) \in \mathbb{Q}[X].$$

We claim that the dimension function determines the isomorphism class of the tuple $(X, \rho_1, \ldots, \rho_r, \mathbb{Q} \cdot F)$.

To see this, fix a maximal compact subgroup K of G and a maximal torus S of K. Let dk and ds denote Haar measure with integral 1 on K and S respectively. As T is any maximal torus of G, we may choose it to be the complexification of S. The image $\rho(G) \subset \mathrm{GL}_{\underline{d}}(\mathbb{C})$ is compact and therefore lies in a subgroup $U_{\underline{d}} := \prod_i U(d_i)$. Choosing maximal tori $U(1)^{d_i} \subset U(d_i)$ such that $\rho_i(S) \subset U(1)^{d_i}$, we obtain the diagram

$$\begin{array}{ccc} K & \stackrel{\rho}{\longrightarrow} & U_{\underline{d}} \\ p_K \downarrow & & \downarrow p_U \\ K^\natural & \stackrel{\rho^\natural}{\longrightarrow} & U_{\underline{d}}^\natural \\ \pi_S \uparrow & & \uparrow \pi_U \\ S & \stackrel{\rho_S}{\longrightarrow} & U(1)^{\sum_i d_i} \end{array}$$

where the superscript $\natural$ denotes set of conjugacy classes. For any representation $\sigma : \mathrm{GL}_{\underline{d},\mathbb{C}} \to \mathrm{GL}(U)$, we have

$$\dim(U^G) = \dim(U^K) = \int_K \mathrm{tr}(\sigma(\rho(k))dk = \int_{U_{\underline{d}}} \mathrm{tr}(\sigma)\rho_* dk.$$

By the Peter-Weyl theorem, the dimension function determines the measure

$$p_{U*}\rho_* dk = \rho^\natural_* p_{K*} dk$$

on $U^\natural_{\underline{d}}$. As π_U is finite, we can pull back measures and set

$$Y := \operatorname{supp}(\pi_U^* \rho^\natural_* p_{K*} dk) = \pi_U^{-1} \operatorname{supp}(\rho^\natural_* p_{K*} dk) = \pi_U^{-1}\rho(K^\natural) = \bigcup_{\sigma \in S_{\underline{d}}} \rho_S(S)^\sigma.$$

Here we have abbreviated $S_{\underline{d}} := \prod_i S_{d_i}$. As ρ is faithful, we may identify S with a single irreducible component $\rho_S(S)$ of the real-algebraic variety Y, and let S° denote the complement of the intersection of S with the other components of Y. By the Weyl integration formula,

$$\pi_U^* \rho^\natural_* p_{K*} dk|_{S^\circ} = \frac{1}{|W|} \sum_{\sigma \in \mathrm{Stab}_{S_{\underline{d}}} S} \sigma_* \left(\prod_{\alpha \in \Phi} (1 - \alpha(s)) ds \right).$$

A regular function on a torus is determined by its restriction to an open set, so $\mathbb{Q}F$ is determined as a one-dimensional subspace of $\mathbb{Q}[X^*(S) \otimes \mathbb{Q}] = \mathbb{Q}[X^*(T) \otimes \mathbb{Q}]$. We can read off $\rho_i \in \mathbb{Z}[X^*(T)]$ from the formal character of the restriction of the projection $U_{\underline{d}} \to U(d_i)$ to S. Thus the tuple $(X, \rho_1, \ldots, \rho_r, \mathbb{Q}F)$ is determined up to isomorphism.

Let $X = \bigoplus X_j$ be the isotypic decomposition under Γ. Since Γ contains the Weyl group of Φ, this induces a decomposition of the root system. Moreover, since X decomposes into pairwise inequivalent representations under the Weyl group, each X_j is already Γ-irreducible. As the formal character of an irreducible representation, every ρ_i is the tensor product of unique $\rho_{i,j} \in \mathbb{Z}[X_j]$. This, in turn, implies that Γ is a product of groups acting only on X_j, and hence that F is a tensor product as well. The theorem therefore reduces to the Γ-irreducible case.

In this case every non-trivial ρ_i must be faithful, (up to center). Thus, by [8] Th. 4, any single non-trivial ρ_i determines Φ completely, unless it is a tensor power of a basic “ambiguous” representation in an explicit, short list. On the other hand, the abstract isomorphism class of Φ is determined in any case, by [8] Th. 1. Since the list of basic “ambiguous” representations is such that the abstract isomorphism class of Φ determines Φ up to Γ-conjugacy, the theorem is proved. □

Another consequence of [8] Th. 4 concerns the Mumford-Tate conjecture. Suppose that K is a number field, given with an embedding $K \subset \mathbb{C}$. The singular homomology group $V := H_1(A(\mathbb{C}), \mathbb{Q})$ carries a natural Hodge structure of weight -1: $V \otimes \mathbb{C} = V^{-1,0} \oplus V^{0,-1}$. Let $z \in \mathbb{C}^\times$ act on $V^{-1,0}$ through multiplication by z, on $V^{0,-1}$ through multiplication by $\bar{z}$. The *Hodge group* is by definition the smallest algebraic subgroup $G_\infty \subset \mathrm{Aut}(V)$, defined over $\mathbb{Q}$, such that this action factors through $G_\infty(\mathbb{C})$.

Via the comparison isomorphisms $V_\ell \cong V \otimes \mathbb{Q}_\ell$ one can compare G_∞ and G_ℓ. The *Mumford-Tate conjecture* asserts that $G_\ell^\circ = G_\infty \times \mathbb{Q}_\ell$ for every ℓ. One half of this, namely the inclusion $G_\ell^\circ \subset G_\infty \times \mathbb{Q}_\ell$, has been proved by [6] I Prop. 6.2.

Theorem 4.3. *Suppose that K is a number field, and let G_∞ denote the Hodge group of A for a fixed embedding $K \subset \mathbb{C}$. If $\operatorname{rank}(G_\ell^\circ) = \operatorname{rank}(G_\infty)$ for some ℓ, then $G_\ell^\circ = G_\infty \times \mathbb{Q}_\ell$ for every ℓ. In particular, if the Mumford-Tate conjecture holds for one prime, then it holds for every prime.*

Proof. By Serre [15] p. 6 (cf. also [9] Prop. 6.12) the rank of G_ℓ° is independent of ℓ. Therefore we have $\operatorname{rank}(G_\ell^\circ) = \operatorname{rank}(G_\infty)$ for every ℓ. The comparison isomorphisms are equivariant with respect to the action of $\operatorname{End}_{\bar{K}}(A)$ on V_ℓ and on V. Thus our decomposition into compatible systems of absolutely irreducible representations is induced by a similar decomposition

$$V \otimes F \cong \bigoplus_{i=1}^{r} W_i \otimes_F F^{n_i}.$$

After a choice of bases $G_\infty \times F$ is a subgroup of $\mathrm{GL}_{\underline{d},F}$, and by Deligne [6] I Prop. 6.2 we have $G_\lambda^\circ \subset G_\infty \times F_\lambda$ for every λ. By assumption these groups have equal rank. Thus the desired assertion is a consequence of the following lemma.

Lemma 4.4. *Let $H \subset G$ be connected reductive algebraic subgroups of $\mathrm{GL}_{\underline{d},\mathbb{C}}$, such that each standard representation $H \to \mathrm{GL}_{d_i,\mathbb{C}}$ is irreducible. If $\operatorname{rank}(H) = \operatorname{rank}(G)$, then $H = G$.*

Proof. Let Z_H denote the center of H. By irreducibility it is equal to the intersection of H with the scalars $\mathbb{G}_{m,\mathbb{C}}^r \subset \mathrm{GL}_{\underline{d},\mathbb{C}}$. Fix a maximal torus T of H. As Z_H is contained in T, $Z_H = T \cap \mathbb{G}_{m,\mathbb{C}}^r$. The same reasoning applies to G. Since T is also a maximal torus of G, we find $Z_H = Z_G$.

Let $\Psi \subset \Phi \subset X^*(T)$ denote the roots of H, respectively of G. Then the cocharacter group of Z_H (resp. Z_G) is equal to $X^*(T)/\mathbb{Z}\Psi$ (resp. $X^*(T)/\mathbb{Z}\Phi$). The equality of centers therefore implies $\mathbb{Z}\Psi = \mathbb{Z}\Phi$. On the other hand, by [1] Ch. VI no. 1.7, Prop. 23, $\Psi = \Phi \cap \mathbb{Z}\Psi$. It follows that $\Psi = \Phi$, hence that $H = G$, as claimed. ∎

§5. Independence-of-ℓ conjectures in the function field case

In this section we suppose that our compatible system of representations comes from the cohomology of a smooth proper variety X over K, *i.e.*, that $V_\ell = H^w(X \times_K \bar{K}, \mathbb{Q}_\ell)$ for every $\ell \nmid \operatorname{char}(K)$. If K is a number field, it is conjectured that $G_\ell^\circ = G_\infty \times \mathbb{Q}_\ell$ via the comparison isomorphism, where G_∞ is the Hodge group of $H^w(X(\mathbb{C}), \mathbb{Q})$ with respect to any given complex embedding of K. In fact, this is a consequence of the general Hodge and Tate conjectures. In the function field case one lacks natural comparison isomorphisms for the different V_ℓ, so that it is far from obvious what ℓ-independence properties should be expected. We shall present two implications of other well-known conjectures. From now on K is a function field.

Conjecture 5.1. *There exists a connected reductive group G over $\mathbb{Q}$, and a faithful representation ρ of G on a $\mathbb{Q}$-vector space V, such that for all finite primes $\ell \neq \mathrm{char}(K)$, $(G_\ell^\circ, V_\ell, \rho_\ell^{\mathrm{alg}})$ is an inner twist of $(G, V, \rho) \times_\mathbb{Q} \mathbb{Q}_\ell$.*

Theorem 5.2. *Conjecture 5.1 is implied by the general Tate conjecture and the semisimplicity conjecture.*

Proof. The well-known semisimplicity conjecture states that, in the given situation, the representation ρ_ℓ is semisimple. This is equivalent to saying that G_ℓ is reductive. By means of Prop. 1.1 we reduce to the case that every G_ℓ is connected.

The Tate conjecture states that the space of Galois-invariants in every $V_\ell^{\otimes m} \otimes (V_\ell^\vee)^{\otimes n} \otimes \mathbb{Q}_\ell(p)$ is generated by algebraic cycle classes. Let $\mathcal{C}$ denote the Tannakian category generated by the motive $h^w(X)$ and the Tate motive. (For this and the following see [6]). The fibre functor $h^w(X) \mapsto V_\ell$ yields a $\otimes$-isomorphism between $\mathcal{C} \otimes \mathbb{Q}_\ell$ and the category of representations of G_ℓ, and furthermore there exist isomorphisms $V_\ell \otimes_{\mathbb{Q}_\ell} \mathbb{C} \cong V_{\ell'} \otimes_{\mathbb{Q}_{\ell'}} \mathbb{C}$ which respect the classes of algebraic cycles and which identify $G_\ell \times_{\mathbb{Q}_\ell} \mathbb{C}$ with $G_{\ell'} \times_{\mathbb{Q}_{\ell'}} \mathbb{C}$ [*loc. cit.* II Th. 2.11, Th. 3.2]. These isomorphisms are unique up to composition with a point of G_ℓ.

By an argument of Serre ([15] pp. 13–14, or [3] Th. 3.7) we may choose a non-archimedean place $v \notin S$ of K such that the Zariski closure of the subgroup generated by $\rho_\ell(\mathrm{Frob}_v)$ is a maximal torus of G_ℓ, for every ℓ. Let T_ℓ denote this maximal torus. If k_v is the residue field of v and X_v the reduction of X, then T_ℓ is the algebraic monodromy group of the representation of $\mathrm{Gal}(\bar{k}_v/k_v)$ on $V_\ell = H^w(X_v \times_{k_v} \bar{k}_v, \mathbb{Q}_\ell)$. Now we apply the above discussion to the Tannakian category $\mathcal{C}_v$ generated by the Tate motive and the motive $h^w(X_v)$ (here we use the Tate conjecture a second time!). The fact that our fibre functors factor through the $\otimes$-morphism $\mathcal{C} \to \mathcal{C}_v$ gives us additional information. Namely, the requirement that the morphisms in $\mathcal{C}_v$ be preserved implies that the isomorphisms for the V_ℓ are unique up to composition with a point of T_ℓ, and that the $T_\ell \times_{\mathbb{Q}_\ell} \mathbb{C}$ are identified with each other. As these groups are abelian, their identification, being unique up to conjugation, is independent of any choice.

Lemma 5.3. *(i) The elements $\rho_\ell(\mathrm{Frob}_v) \in T_\ell(\mathbb{Q}_\ell)$ correspond under this identification.*
(ii) There exist a torus T over $\mathbb{Q}$ and a set of isomorphisms $T_\ell \cong T \times \mathbb{Q}_\ell$ that are compatible with this identification.

Proof. Let A denote the endomorphism ring of $h^w(X_v)$ in $\mathcal{C}_v$. Since $A \otimes \mathbb{Q}_\ell$ is the commutant of $\rho_\ell(\mathrm{Frob}_v)$ in $\mathrm{End}(V_\ell)$, it is semisimple and $\rho_\ell(\mathrm{Frob}_v)$ is an element of its center. The elements of A can be represented by algebraic cycles, hence the system of representations is A-linearly compatible, i.e. for any $a \in A$ the characteristic polynomial of $a \cdot \rho_\ell(\mathrm{Frob}_v)$ is independent of ℓ ([11] §8). This implies that all $\rho_\ell(\mathrm{Frob}_v)$ come from a fixed element t_v of the center of A. This proves part (i) of the lemma. Part (ii) is clear if we define T as the Zariski closure in the $\mathbb{Q}$-algebraic group $\mathrm{Aut}(A)$ of the subgroup generated by t_v. □

Now Th. 5.2 is proved by exploiting the "common maximal torus" T. The identifications over $\mathbb{C}$ imply that the root system of G_ℓ, as a subset of the character group $X^*(T)$, is independent of ℓ. Let Φ denote this root system, W its Weyl group, and ρ_T the formal character of ρ_ℓ^{alg}. The form of T over $\mathbb{Q}$ corresponds to a homomorphism

$$\varphi : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}(X^*(T), \Phi, \rho_T).$$

There is a quasi-split connected reductive group G over $\mathbb{Q}$ with maximal torus T, root system Φ, and a representation ρ on a vector space V with formal character ρ_T. In fact, such G is unique up to isomorphism, and determined by the composite map

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\varphi} \mathrm{Aut}(X^*(T), \Phi, \rho_T) \longrightarrow \mathrm{Aut}(X^*(T), \Phi, \rho_T)/W.$$

The class of $(G_\ell^\circ, V_\ell, \rho_\ell^{\mathrm{alg}})$ up to inner twist corresponds to its restriction to $\mathrm{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$. The theorem is proved. $\square$

Conjecture 5.4. *G_ℓ° is unramified for all $\ell \gg 0$.*

The weaker assertion for a set of primes of density one is known: Prop. 1.2. In Th. 3.2 the full conjecture is proved for abelian varieties. The general conjecture is also justified by the fact that the proof of Th. 3.2 works in general provided that an analogue of Th. 3.1 (iii), a conjecture of the Shafarevich type, is available. Together, conjectures 5.1 and 5.4 imply the conjecture in the introduction in the function field case.

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