COMPACT SUBGROUPS
OF LINEAR ALGEBRAIC GROUPS

by

Richard Pink

Fakultät für Mathematik und Informatik, Universität Mannheim
D-68131 Mannheim, Germany
e-mail: pink@math.uni-mannheim.de

Abstract:

The general problem underlying this article is to give a qualitative classification of all compact subgroups $\Gamma \subset \mathrm{GL}_n(F)$, where F is a local field and n is arbitrary. It is natural to ask whether Γ is an open compact subgroup of $H(E)$, where H is a linear algebraic group over a closed subfield $E \subset F$. We show that Γ indeed has this form, up to finite index and a finite number of abelian subquotients. When Γ is Zariski dense in a connected semisimple group, we give a precise openness result for the closure of the commutator group of Γ. In the case $\mathrm{char}(F) = 0$ the answers have long been known by results of Chevalley and Weyl. The motivation for this work comes from the positive characteristic case, where such results are needed to study Galois representations associated to function fields. We also derive openness results over a finite number of local fields.

0. Introduction

Consider a local field F, i.e. a topological field that is either complete with respect to a non-trivial discrete valuation with finite residue field, or that is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. Let n be a positive integer. The general problem underlying this article is to understand the structure of a compact subgroup $\Gamma \subset \mathrm{GL}_n(F)$ in view of properties that are shared by all open subgroups of Γ.

When $\mathrm{char}(F) = 0$, it has long been known that Γ is a real or p-adic Lie group. The main reason for this is that the logarithm and the exponential series allow us to go back and forth between $\mathrm{GL}_n(F)$ and its Lie algebra. By results of Chevalley in the p-adic case (cf. [1] ch.II §7) and Weyl in the real case (cf. [9] Ch. 4, Th. 2.1), one finds that Γ is in some sense essentially algebraic, to wit: the commutator subgroup of a suitable open subgroup of Γ is open in an algebraic group over $\mathbb{R}$ resp. $\mathbb{Z}_p$.

In positive characteristic it is not possible to translate the problem into one of Lie algebras, as in the p-adic case. What is worse, there is no subfield $E \subset F$ such that Γ is a priori an "E-adic" Lie group. Thus even the definition of an algebraic envelope, in which Γ has a chance to be open, poses substantial difficulties.

As a first step, let G be the Zariski closure of Γ in the algebraic group $\mathrm{GL}_{n,F}$. This is a linear algebraic group, which may be assumed connected after Γ is replaced by a suitable open normal subgroup. The intersection of Γ with the maximal solvable normal subgroup of G is a successive extension of at most n abelian groups and can be studied directly without much difficulty. Thus, after dividing G by its maximal solvable normal subgroup, it remains to study the hard case that G is connected adjoint. Write G as a direct product of Weil restrictions $\prod_{i=1}^m \mathcal{R}_{F_i/F} G_i$, where each G_i is an absolutely simple adjoint group over a finite extension F_i of F. Then we can view Γ as a subgroup of $\prod_{i=1}^m G_i(F_i)$. Thus we are led to the following, slightly more general question.

The Setup: For each $1 \leq i \leq m$ let G_i be an absolutely simple connected adjoint group over a local field F_i. Let $\Gamma \subset \prod_{i=1}^m G_i(F_i)$ be a compact subgroup whose image in each factor $G_i(F_i)$ is Zariski dense. The problem is to give a qualitative classification of such Γ. Note that in this formulation the F_i need not be given as extensions of one and the same local field, and the Zariski density is required only in each individual factor. We need not even assume that the F_i have the same residue characteristic. In this situation the following phenomena can force Γ to be small. First, some G_i might be defined already over a closed subfield $E_i \subset F_i$, such that the image of Γ in $G_i(F_i)$ consists of E_i-valued points. Second, there might be an isomorphism of algebraic groups $G_i \cong G_j$ over a field isomorphism $F_i \cong F_j$, for $i \neq j$, such that the image of Γ is contained in the graph of the resulting isomorphism $G_i(F_i) \cong G_j(F_j)$. Third, there are some additional pathologies involving non-standard inseparable isogenies for certain root systems in characteristics 2 and 3. Any promising concept of algebraic envelope of Γ has to take all these phenomena into account.

The use of (quite elementary) group schemes provides an elegant language for this discussion. Changing notation with regard to the beginning of this introduction, let us now consider the commutative semisimple ring $F := \bigoplus_{i=1}^m F_i$. Then the individual G_i fit together to a group scheme G over F, such that $G(F) = \prod_{i=1}^m G_i(F_i)$. It may happen that there exists a semisimple closed subring $E \subset F$ such that F is of finite type as module over E, another fiberwise absolutely simple adjoint group scheme H over E, and an isogeny $\varphi : H \times_E F \longrightarrow G$ such that $\Gamma \subset \varphi(H(E))$. In fact, each of the above phenomena corresponds to such a situation.

Definition 0.1: *We say that (F, G, Γ) is* minimal *if and only if, for any such (E, H, φ), we have $E = F$ and φ is an isomorphism.*

As long as (F, G, Γ) is not minimal, we may replace it by any triple $(E, H, \varphi^{-1}(\Delta))$ violating Definition 0.1. In Section 3 we prove that this process stops and that the resulting triple can be chosen canonically. This can then be viewed as the desired algebraic envelope of Γ.

Going on, let $\tilde{G}$ denote the universal covering of G, i.e. consisting of the universal coverings of the individual G_i. Then the commutator morphism of $\tilde{G}$ factors through a unique morphism $[\ ,\ ]^\sim : G \times G \longrightarrow \tilde{G}$. Let $\Gamma' \subset \tilde{G}(F)$ be the closure of the subgroup generated by $[\Gamma, \Gamma]^\sim$. For (E, H, φ) as above, let $\tilde{\varphi} : \tilde{H} \times_E F \longrightarrow \tilde{G}$ be the associated isogeny of universal coverings. The following is the main result of this article.

Main Theorem 0.2:

(a) There exist (E, H, φ) as above such that φ has nowhere vanishing derivative and Γ' is the image under $\tilde{\varphi}$ of an open subgroup of $\tilde{H}(E)$.

(b) The ring E in (a) is uniquely determined, and H and φ are unique up to unique isomorphism.

(c) In particular, if (F, G, Γ) is minimal, then Γ' is open in $\tilde{G}(F)$.

The reader should be aware that we do not assert that Γ is the image of an open subgroup of $H(E)$. Indeed, this can be proved only when the isogeny $\tilde{G} \to G$ is separable.

When φ is an isomorphism, one can view H as a model of G over E. By the classification of semisimple groups, φ must be an isomorphism over F_i unless the root system of G_i possesses roots of different lengths for which the square of the length ratio is equal to the characteristic of F_i. This can happen only in characteristics 2 and 3.

While the formulation of Main Theorem 0.2 was motivated by the peculiarities of the positive characteristic case, it is a pleasant surprise that a single statement covers all kinds of local fields, archimedean and non-archimedean of all characteristics alike. One can view the content of Main Theorem 0.2 as a combination of the field case together with a statement about the interaction between different simple factors. The following consequence means that the algebraic structure of $\tilde{G}$ and F is inherent in the structure of any open compact subgroup as topological group! This can be viewed as a generalization of Weyl's theorem on the algebraicity of compact real Lie groups.

Corollary 0.3: *For each $i = 1, 2$ consider a local field F_i, an absolutely simple simply connected group $\tilde{G}_i$ over F_i, and an open compact subgroup $\tilde{\Gamma}_i \subset \tilde{G}_i(F_i)$. Let $f : \tilde{\Gamma}_1 \xrightarrow{\sim} \tilde{\Gamma}_2$ be an isomorphism of topological groups. Then there exists a unique isomorphism of algebraic groups $\tilde{G}_1 \xrightarrow{\sim} \tilde{G}_2$ over a unique isomorphism of local fields $F_1 \xrightarrow{\sim} F_2$, such that the induced isomorphism $\tilde{G}_1(F_1) \xrightarrow{\sim} \tilde{G}_2(F_2)$ extends f.*

Another special consequence of Main Theorem 0.2, stated in the framework of abstract topological groups and thus with less conceptual ballast, is the following:

Corollary 0.4: *Consider a connected adjoint group G over a local field F and a compact Zariski-dense subgroup $\Gamma \subset G(F)$. Let Γ' denote the closure of the commutator subgroup of Γ. Then there exists a local field E of the same characteristic and the same residue characteristic as F, a connected adjoint group H over E, with universal covering $\pi : \tilde{H} \to H$, and an open compact subgroup $\Delta \subset \tilde{H}(E)$, such that $\Gamma' \cong \pi(\Delta)$ as topological groups.*

Finally, the reduction steps at the beginning of this introduction imply:

Corollary 0.5: *Consider a local field F, a positive integer n, and a compact subgroup $\Gamma \subset \mathrm{GL}_n(F)$. Then there exist closed normal subgroups $\Gamma_3 \subset \Gamma_2 \subset \Gamma_1$ of Γ such that*

(a) Γ/Γ_1 is finite.

(b) Γ_1/Γ_2 is abelian of finite exponent.

(c) There exists a local field E of the same characteristic and the same residue characteristic as F, a connected adjoint group H over E, with universal covering $\pi : \tilde{H} \to H$, and an open compact subgroup $\Delta \subset \tilde{H}(E)$, such that $\Gamma_2/\Gamma_3 \cong \pi(\Delta)$ as topological group.

(d) Γ_3 is a successive extension of $\leq n$ abelian groups.

Trace Characterization: In applying Main Theorem 0.2 it will be desirable to determine the subring E in advance and to have a criterion for φ to be an isomorphism. This can be achieved in most cases using traces of Γ in suitable representations of G. We restrict ourselves here to a few general results; more detailed information can be deduced from the results of Section 3. For any representation ρ of G on an F-module of finite type we let $\mathcal{O}_{\mathrm{tr}(\rho)} \subset F$ be the closure of the subring generated by 1 and by $\mathrm{tr}(\rho(\Gamma))$, and put

$$E_\rho := \left\{ \tfrac{x}{y} \;\middle|\; x, y \in \mathcal{O}_{\mathrm{tr}(\rho)},\ y \in F^* \right\} \subset F.$$

Proposition 0.6: *Let (E, H, φ) be as in Main Theorem 0.2.*

(a) Suppose that F is a field and that ρ is a non-constant irreducible representation occurring as subquotient of the adjoint representation of G. Then we have either $E_\rho = E$, or the characteristic p of F is 2 or 3 and $E_\rho = \{x^p \mid x \in E\}$.

(b) Suppose that F is a field, and that ρ is a subquotient of the adjoint representation of G. Then $E_\rho \subset E$. In particular, if $E_\rho = F$, then $E = F$.

(c) Suppose that $E_\rho = F$ for all nowhere constant fiberwise irreducible representations ρ which occur as subquotients of the adjoint representation of G. Then $E = F$ and φ is an isomorphism.

Related Work: The results of this article are similar, but in some sense complementary, to those of Weisfeiler [13] concerning strong approximation. His main result, in a special case (see [13] Th. 9.1 and Th. 10.2), concerns a finitely generated Zariski dense subgroup Γ of an absolutely simple group over a global field F, and under some additional assumptions he obtains a theorem on simultaneous approximation by Γ at all but a finite, sufficiently large, set S of places of F. The Main Theorem 0.2 above is complementary to that result in that it can be applied to the remaining places. The methods of [13] and the present article can be combined to obtain a strengthening of Weisfeiler's theorem. The author plans to deal with this in a subsequent paper. Sections 2–4 of this article, which apply equally to the local and the global case, have been written already with that application in mind.

Returning to the local case, the motivation for Main Theorem 0.2 originally came from the study of Galois representations associated to function fields. The consequences for Drinfeld modules in generic characteristic are discussed in Pink [10].

Sketch of the Proof: We indicate the method in the following special case of Main Theorem 0.2.

Theorem 0.7: *Consider an absolutely simple connected adjoint group G over a local field F and a compact Zariski-dense subgroup $\Gamma \subset G(F)$. Assume that the adjoint representation of G is irreducible. Then there exists a model H of G over a closed subfield $E \subset F$, such that Γ is an open subgroup of $H(E)$.*

To begin with, let Ad_G denote the adjoint representation of G on its Lie algebra $\mathfrak{g}$. Abbreviate $A := \mathrm{End}_F(\mathfrak{g})$, and let $\mathcal{B} \subset A$ be the closure of the $\mathbb{Z}$-subalgebra that is generated by $\mathrm{Ad}_G(\Gamma)$. Using Burnside's theorem our assumptions imply $F \cdot \mathcal{B} = A$. With arguments mainly from linear algebra we deduce (cf. Section 2):

Lemma 0.8: *$\mathcal{B}$ is an order in a simple algebra $B \subset A$ with center $E \subset F$, such that the natural homomorphism $B \otimes_E F \longrightarrow A$ is an isomorphism.*

By construction we have $\mathrm{Ad}_G(\Gamma) \subset B^*$, which allows us to define a model of G over E, following Vinberg [12]. Namely, let $\underline{B}^*$, resp. $\underline{A}^*$, denote the multiplicative group of B, resp. A, viewed as algebraic group over E, resp. over F. Then $\underline{B}^*$ is a model of $\underline{A}^*$, i.e. we have a natural isomorphism $\underline{B}^* \times_E F \cong \underline{A}^*$. Let H be the Zariski closure of $\mathrm{Ad}_G(\Gamma)$ in $\underline{B}^*$. Then $H \times_E F$ is the Zariski closure of $\mathrm{Ad}_G(\Gamma)$ in $\underline{A}^*$, which is $\mathrm{Ad}_G(G) \cong G$ since Γ is Zariski dense and G is adjoint. By construction we now have $\Gamma \subset H(E)$, and it remains to show that this subgroup is open. Note that replacing (F, G) by (E, H) leaves $\mathcal{B}$ unchanged, so that without loss of generality we may assume that $\mathcal{B}$ is open in A, and we must prove that Γ is open in $G(F)$.

Next select a $\mathcal{B}$-invariant $\mathcal{O}_F$-lattice $\Lambda \subset \mathfrak{g}$. Let π be a uniformizer in $\mathcal{O}_F$, and consider the subgroups

$$\Delta_n := \left\{ g \in G(F) \;\middle|\; (\mathrm{Ad}_G(g) - \mathrm{id})(\Lambda) \subset \pi^n \Lambda \right\}$$

for all integers $n \geq 0$. These principal congruence subgroups form a cofinal system of open compact subgroups of $G(F)$. For all $n \geq m \geq 0$ we have a natural group isomorphism

$$\Delta_n / \Delta_{n+m} \;\cong\; \pi^n \Lambda / \pi^{n+m} \Lambda \,,$$

obtained by truncating the logarithm and exponential series after the first order term (cf. Section 6). Put $\Gamma_n := \Gamma \cap \Delta_n$ for all n, then by construction the subgroup

$$\Gamma_n/\Gamma_{n+m} \ \subset \ \Delta_n/\Delta_{n+m} \ \cong \ \pi^n\Lambda/\pi^{n+m}\Lambda$$

is invariant under $\mathcal{B}$. Fix an integer $r \geq 1$ so that $\pi^r \cdot \mathrm{End}_{\mathcal{O}_F}(\Lambda) \subset \mathcal{B}$, and choose $n \geq 2r$ with $\Gamma_n \neq \Gamma_{n+1}$. Setting $m := 2r$ we deduce

$$\Gamma_n/\Gamma_{n+2r} \ \supset \ \Delta_{n+r}/\Delta_{n+2r} \ .$$

In other words, we have

$$\Gamma_{n+r}/\Gamma_{n+2r} \ = \ \Delta_{n+r}/\Delta_{n+2r} \ .$$

Repeating the argument inductively with $n + (i-1)r$ in place of n, we find

$$\Gamma_{n+ir}/\Gamma_{n+(i+1)r} \ = \ \Delta_{n+ir}/\Delta_{n+(i+1)r}$$

for all $i \geq 1$. This implies $\Gamma_{n+r} = \Delta_{n+r}$, hence Γ is open in $G(F)$, thus finishing the proof of Theorem 0.7.

In the general case of Main Theorem 0.2 all remaining problems are related to inseparable isogenies which are not powers of Frobenius. As a consequence one has to juggle with congruence subgroups with respect to several groups at the same time. Other difficulties arise from the fact that the ring $\mathcal{B}$ may be smaller than an order in a model of $\mathrm{End}_F(\mathfrak{g})$. For an element $\tilde{\gamma} \in \Gamma'$ to generate many new elements under conjugation by Γ, one needs roughly that the logarithm of $\tilde{\gamma}$ is sufficiently far away from the invariant proper subspaces of the Lie algebra. This can be achieved by applying a suitable inseparable isogeny, if necessary, and by raising $\tilde{\gamma}$ to a large power to make it look more toric. For the relevant technical details see Section 7.

Outline of the Article: Section 1 summarizes a number of mostly well-known facts concerning linear algebraic groups over arbitrary fields and their adjoint representations. To make it understandable to a wider audience this paper has been written with an effort to avoid scheme-theoretic machinery as much as possible.

In Section 2 we prove some general results on the image of the group ring of Γ in an algebraic representation ρ of G, where F, G, and Γ are as above. For any semisimple representation one obtains an order in a semisimple central algebra over a suitable subring $E_\rho \subset F$. We also discuss reducible representations in a special case.

In Section 3 the results of Section 2 are applied to the semi-simplification of the adjoint representation of G. The fundamental observation here is that the adjoint representation automatically descends to any model of G over a subring, and that its behavior under isogenies can also be described. Using these facts, we find the candidate for (E, H, φ) in Main Theorem 0.2 and are able to characterize it as in Proposition 0.6.

The study of the adjoint representation is continued in Section 4, where we give a full qualitative characterization of the image of the augmentation ideal of the group ring of Γ. Here the difficulties arise from the fact that the adjoint representation may be far from semisimple in small positive characteristic. The results of this section yield a first order approximation for the action of Γ on small neighborhoods of the identity in $G(F)$, thereby finishing roughly the first half of the proof.

The next two sections set up the technical framework for working with congruence subgroups of $G(F)$. This concerns only the non-archimedean case; the archimedean summands of F will be dealt with by a separate argument in the last section. First, in Section 5 we choose local parameters which are compatible with the action of Γ and with various other maps that must be carried along. Any such choice determines a system of principal congruence subgroups. In Section 6 we discuss the linearization of certain quotients of these by means of the truncated logarithm map.

After all these preparations the proof of Main Theorem 0.2 culminates in Section 7. Having disposed of the archimedean summands of F, we must show that Γ' contains a suitable principal congruence subgroup. The principle here is again to start with a suitably generic element $\tilde{\gamma} \in \Gamma'$ and to conjugate it around by Γ. This is the point where the results of Section 4 play a crucial role. A number of influences have to be balanced out against each other, such as the size of the action of Γ, the choice of $\tilde{\gamma}$, and the presence of non-standard isogenies. This makes the whole argument a relatively delicate matter. However, most of these technical details are necessary only in extreme cases.

The reader willing to avoid certain pathological cases in characteristics 2 and 3 will benefit from substantial technical simplifications throughout the article, except in Section 2. We briefly indicate these. Let us rule out the root systems of type B_n, C_n (for $n \geq 1$), and F_4 in characteristic 2, and type G_2 in characteristic 3. Then in Section 3 any quasi-model is a model, and using the results of Section 2 one easily proves the existence and uniqueness of minimal quasi-models with $E = E_{\alpha^G}$, as in Vinberg [12]. The study in Section 4 can also be cut down significantly, but it cannot be avoided completely when the isogeny $\tilde{G} \to G$ is not separable. In that section and the remaining ones all the special arguments involving the isogeny $G \to H$ and the Frobenius isogeny can be discarded. Altogether, the amount of technical details should decrease by about a third. By contrast, the generality of allowing F to be a finite direct sum of fields introduces no difficulties.

The proofs of the results mentioned in this introduction will be given at the end of section 7.

1. Linear Algebraic Groups: Notations and Well-Known Facts

In this section we summarize a number of mostly well-known facts concerning linear algebraic groups over an arbitrary field F and their adjoint representations. For the fundamentals of linear algebraic groups we refer to Borel [1] and Humphreys [8].

Generalities: For any positive integer n let $\mathrm{GL}_{n,F}$ denote the algebraic group over F of all invertible square matrices of size n. By a linear algebraic group over F we mean a reduced group scheme over F which is isomorphic to a Zariski closed algebraic subgroup of $\mathrm{GL}_{n,F}$ for some n. Important examples of linear algebraic groups are $\mathrm{GL}_{n,F}$ itself, in particular the multiplicative group $\mathbb{G}_{m,F} = \mathrm{GL}_{1,F}$, and the additive group $\mathbb{G}_{a,F}$.

Throughout this article we distinguish between a linear algebraic group G over F and the group of its F-valued points $G(F)$. Among other things G determines the groups of F'-valued points $G(F')$ for any overfield F' of F. Namely, realize G as the subgroup of $\mathrm{GL}_{n,F}$ given by certain equations in the coefficients of $n \times n$-matrices. Then $G(F')$ consists of those invertible $n \times n$-matrices over F' which satisfy the same equations, and this description is independent of the embedding $G \hookrightarrow \mathrm{GL}_{n,F}$.

For any field homomorphism $\tau : F \hookrightarrow F'$ and any linear algebraic group G over F the fiber product $G \times_{F,\tau} F'$ in the sense of schemes defines a linear algebraic group over F'. When τ is the inclusion of a subfield, we abbreviate this as $G \times_F F'$. If $G \subset \mathrm{GL}_{n,F}$, this base extension is then given by the same equations as G, we only "forget" that the coefficients of these equations lie in the subfield $F \subset F'$.

Representations: For any finite dimensional F-vector space V we have the algebraic group of automorphisms $\underline{\mathrm{Aut}}_F(V)$. Namely, any choice of basis identifies V with a standard vector space $F^{\oplus n}$ and $\underline{\mathrm{Aut}}_F(V)$ with $\mathrm{GL}_{n,F}$. A homomorphism of algebraic groups $\rho : G \longrightarrow \underline{\mathrm{Aut}}_F(V)$ is called a representation of G on V, and then, equivalently, V is called a G-module. The representation is called irreducible, resp. the G-module simple, if and only if $V \neq 0$ and it possesses no G-submodule other than 0 and V itself. It is called absolutely irreducible if and only if it is irreducible and the only G-equivariant F-linear endomorphisms of V are the scalars F.

Lie Algebra: The tangent space of G at the identity element 1 is the Lie algebra $\mathrm{Lie}\, G$. Consider the commutator morphism

$$[\ ,\ ]:\ G \times G \longrightarrow G,\ (g,h) \mapsto [g,h] := ghg^{-1}h^{-1}. \tag{1.1}$$

Its total derivative at the identity element yields the Lie bracket $[\ ,\ ]:\ \mathrm{Lie}\, G \times \mathrm{Lie}\, G \to \mathrm{Lie}\, G$. On the other hand consider the conjugation morphism

$$G \times G \longrightarrow G,\ (g,h) \mapsto [g,h] := ghg^{-1}.$$

Its derivative with respect to h, taken at $h = 1$, defines the adjoint representation

$$\mathrm{Ad}_G :\ G \to \underline{\mathrm{Aut}}_F(\mathrm{Lie}\, G).$$

General Notions: The radical $\mathcal{R}(G)$ is the largest solvable connected normal algebraic subgroup of G. The group G is called semisimple if and only if its radical is trivial. The derived group $G^{\mathrm{der}} \subset G$ is the linear algebraic subgroup generated by the image of the commutator morphism (1.1). A connected semisimple group is called adjoint if and only if its adjoint representation is faithful. More generally, if G is connected semisimple, the image of G in the adjoint representation $\mathrm{Ad}_G : G \to \underline{\mathrm{Aut}}_F(\mathrm{Lie}\, G)$ is called the adjoint group G^{ad}. It is an adjoint semisimple group in its own right, although that is not entirely obvious. The notions just explained are, like many others, invariant under base extension. For instance, given G and any field extension $F \subset F'$ we know that G is semisimple (resp. adjoint) if and only if $G \times_F F'$ has the same property.

Central Isogenies: (Cf. Borel-Tits [3] §2.) By definition an isogeny of connected linear algebraic groups $f : G \to H$ is a surjective homomorphism with finite kernel. It is called central if and only if the commutator morphism (1.1) of G factors through a morphism $H \times H \to G$. For example, for any connected semisimple group G the natural homomorphism to its adjoint group $G \to G^{\mathrm{ad}}$ is a central isogeny. It has the universal property that any central isogeny $G \to H$ induces an isomorphism on the adjoint groups $G^{\mathrm{ad}} \xrightarrow{\sim} H^{\mathrm{ad}}$. At the other extreme, a connected semisimple group G is called simply connected if and only if every central isogeny $H \to G$ is an isomorphism. For every connected semisimple group G there exists a simply connected semisimple group $\tilde{G}$ and a central isogeny $\pi : \tilde{G} \longrightarrow G$, both unique up to unique isomorphism. This is called the universal covering of G. By definition the commutator morphism of $\tilde{G}$ factors through a morphism

$$[\ ,\ ]^{\sim} :\ G \times G \longrightarrow \tilde{G}. \tag{1.2}$$

For any subgroup $\Gamma \subset G(F)$ we can therefore define the generalized commutator group as the subgroup of $\tilde{G}(F)$ generated by $[\Gamma, \Gamma]^{\sim}$. Its image in $G(F)$ is, of course, the usual commutator subgroup of Γ.

It is also interesting to look at the derivative of $[\ ,\ ]^{\sim}$ with respect to the second argument. This is a morphism

$$\widetilde{\mathrm{Ad}}_G : G \longrightarrow \underline{\mathrm{Hom}}_F(\mathrm{Lie}\, G, \mathrm{Lie}\, \tilde{G}) \tag{1.3}$$

whose target is the vector space $\mathrm{Hom}_F(\mathrm{Lie}\, G, \mathrm{Lie}\, \tilde{G})$ viewed as an affine algebraic variety over F. This morphism determines the adjoint representation of both G and $\tilde{G}$. For instance, we easily calculate $\mathrm{Ad}_G = \kappa \circ \widetilde{\mathrm{Ad}}_G$, where κ is the morphism

$$\begin{aligned} \kappa :\ \underline{\mathrm{Hom}}_F(\mathrm{Lie}\, G, \mathrm{Lie}\, \tilde{G}) &\longrightarrow \underline{\mathrm{End}}_F(\mathrm{Lie}\, G), \\ f &\mapsto d\pi \circ f + \mathrm{id}. \end{aligned} \tag{1.4}$$

Simple Groups: A connected semisimple group over F is called simple if and only if it is nontrivial and possesses no nontrivial connected proper normal algebraic subgroup. The group G is called absolutely simple if and only if $G \times_F F'$ is simple for every field extension $F \subset F'$. For the most part the study of connected semisimple groups reduces to that of absolutely simple groups. Namely, suppose that G is adjoint or simply connected. Then G is a direct product of simple groups, and each simple factor has the form $\mathcal{R}_{F'/F}H$, where $\mathcal{R}_{F'/F}$ denotes Weil restriction from a finite separable field extension $F \subset F'$ and H is an absolutely simple adjoint group over F'. When F is separably closed, the adjoint or simply connected semisimple groups over F are classified by their root systems. A connected semisimple group is absolutely simple if and only if its root system is irreducible. We shall abbreviate "absolutely simple connected adjoint semisimple" to "absolutely simple adjoint".

Inseparable Isogenies: An isogeny $f : G \longrightarrow H$ is called separable (resp. totally inseparable) if and only if the induced inclusion of function fields $F(H) \hookrightarrow F(G)$ is a separable (resp. totally inseparable) field extension. Equivalently, f is separable if and only if its derivative induces an isomorphism of Lie algebras, and it is totally inseparable if and only if its kernel is supported only in the identity element of G. Note that an isogeny may be both separable and totally inseparable, namely if and only if it is an isomorphism.

Every separable isogeny of connected semisimple groups is central. In the case $\mathrm{char}(F) = 0$ every isogeny is separable and hence central. Suppose that $p := \mathrm{char}(F) > 0$. Then there exist both inseparable central isogenies and non-central ones. Let $\sigma : F \to F$ denote the Frobenius endomorphism $x \mapsto x^p$. For any linear algebraic group G over F and any integer $n \geq 0$ put $(\sigma^n)^*G := G \times_{F,\sigma^n} F$. Then the morphism $G \to G$, defined by $f \mapsto f^{p^n}$ in any coordinate f over F, factors through a unique morphism $\mathrm{Frob}_{p^n} : G \longrightarrow (\sigma^n)^*G$ that makes the following diagram commutative:

(1.5)
$$\begin{tikzcd}
G \arrow[rr] \arrow[dr] \arrow[dd] & & G \arrow[dd] \\
& (\sigma^n)^*G \arrow[ur] \arrow[dl] & \\
\mathbf{Spec}\, F \arrow[rr, "\sigma^n"] & & \mathbf{Spec}\, F
\end{tikzcd}$$

The morphism Frob_{p^n} is a totally inseparable isogeny, called the n^{th} Frobenius isogeny. When G is connected and non-commutative, and $n \geq 1$, this isogeny is not central. The composite of Frobenius isogenies is again a Frobenius isogeny.

Non-standard Isogenies: In a few special cases there exist totally inseparable isogenies between connected semisimple groups which cannot be obtained from central isogenies and Frobenius isogenies. The point is that the Frobenius isogeny Frob_p itself can be factored in a non-trivial way. The resulting isogenies will be called non-standard.

Proposition 1.6: *Let G be an absolutely simple adjoint group over F. Suppose that $p := \mathrm{char}(F)$ is positive and that the root system Φ of G possesses roots of different lengths whose square length ratio is equal to p. Then the Frobenius isogeny Frob_p of G factors through totally inseparable isogenies*

$$G \xrightarrow{\varphi} G^\sharp \xrightarrow{\varphi^\sharp} \sigma^*G,$$

such that neither φ nor $\varphi^\sharp$ is an isomorphism. Here $G^\sharp$ is another absolutely simple adjoint group over F. If $\Phi^\sharp$ is its root system, the possibilities for $(p, \Phi, \Phi^\sharp)$ are listed in the following table.

p	*type of* Φ	*type of* $\Phi^\sharp$
2	B_n $(n \geq 2)$	C_n
2	C_n $(n \geq 2)$	B_n
2	F_4	F_4
3	G_2	G_2

Proof: Suppose first that G splits over F, and fix a split maximal torus $T \subset G$. Let Φ^s, resp. Φ^ℓ denote the set of short, resp. long roots in Φ. Then $\Phi^\sharp := \Phi^\ell \sqcup p \cdot \Phi^s$ is again a root system. Let $T^\sharp$ be the quotient torus of T whose character group is the $\mathbb{Z}$-module generated by $\Phi^\sharp$. Then there exists a split connected adjoint group $G^\sharp$ over F with maximal torus $T^\sharp$ and root system $\Phi^\sharp$. The given isogeny $T \to T^\sharp$ extends to an isogeny $\varphi : G \to G^\sharp$ (e.g., Takeuchi [11] Th. 5.4; the assumption that T splits is sufficient there). The construction is such that the short root spaces in $\mathrm{Lie}\, G$ are annihilated by $d\varphi$, while the long root spaces map isomorphically to the short root spaces in $\mathrm{Lie}\, G^\sharp$.

Repeating this process with $T^\sharp$ the next root system is $p \cdot \Phi$, so we can take $T^{\sharp\sharp} = \sigma^* T$ and $G^{\sharp\sharp} = \sigma^* G$. The composite isogeny $G \to \sigma^* G$ has zero derivative, so it factors through Frob_p ([1] Ch. V Ex. 17.5 (1)). In other words we have $\varphi^\sharp \circ \varphi = \psi \circ \mathrm{Frob}_p$ for some isogeny $\psi : \sigma^* G \longrightarrow \sigma^* G$. By construction ψ is the identity on $\sigma^* T$ and on the root system $p \cdot \Phi$. Thus it is an isomorphism ([1] Ch. V Prop. 22.4). After adjusting $\varphi^\sharp$ by ψ^{-1} we have $\varphi^\sharp \circ \varphi = \mathrm{Frob}_p$, as desired.

When G is not split over F we first apply the above arguments to a split group G_0 of the same type. The Galois cocycle which twists G_0 into G then can be used to twist all of $G_0 \to G_0^\sharp \to \sigma^* G_0$, thus yielding the desired assertion in general.

For an alternative construction of $G^\sharp$ note that the cases listed above are precisely those where the adjoint representation of G possesses two Jordan-Hölder subquotients corresponding to the short resp. the long roots (see below). Let $\mathfrak{k} \subset \mathrm{Lie}\, G$ be the largest G-invariant subspace containing the short root spaces but not the long root spaces. This turns out to be a restricted Lie subalgebra in the sense of [1] Ch. I §3.1, and $G^\sharp$ is nothing but the quotient of G by $\mathfrak{k}$ in the sense of [1] Ch. V Prop. 17.4. (Also, for an explicit discussion of the orthogonal/symplectic case see Borel [1] §23.) $\square$

Classification of Isogenies: All isogenies between connected semisimple groups can be obtained from central isogenies, Frobenius isogenies, and the non-standard isogenies just discussed. We shall make this assertion precise when G is adjoint.

Theorem 1.7: *Let $f : G \to H$ be an isogeny between two absolutely simple adjoint groups over a field F of characteristic p.*

(a) If $p = 0$, then f is an isomorphism.

(b) Suppose that $p > 0$ but that G possesses no non-standard isogenies. Then there exists an integer $n \geq 0$ and an isomorphism $\psi : (\sigma^n)^*G \xrightarrow{\sim} H$ such that $f = \psi \circ \mathrm{Frob}_{p^n}$. If the derivative of f is non-zero, then $n = 0$ and f is an isomorphism.

(c) Suppose that G possesses non-standard isogenies and hence $p > 0$. Then there exists an integer $n \geq 0$ and an isogeny $\psi : (\sigma^n)^*G \longrightarrow H$ with non-vanishing derivative such that $f = \psi \circ \mathrm{Frob}_{p^n}$. Moreover, either ψ is an isomorphism or there exists an isomorphism $\chi : (\sigma^n)^*G^\sharp \xrightarrow{\sim} H$ such that $\psi = \chi \circ \varphi$ where φ is the non-standard isogeny introduced in Proposition 1.6.

Proof: Any homomorphism factors through Frob_p whenever its derivative vanishes ([1] Ch. V Ex. 17.5 (1)). By induction we can therefore reduce ourselves to the case $df \neq 0$. When df is non-zero on all root spaces, then f is central ([1] Ch. V Prop. 22.4). Since both groups are adjoint, f must then be an isomorphism, as desired. Otherwise $\ker(df)$ is a G-invariant non-zero proper subspace of $\mathrm{Lie}\, G$ containing some but not all root spaces. Thus there is a non-standard isogeny $\varphi : G \to G^\sharp$. Since H is adjoint, we easily find that $\ker(df) = \ker(d\varphi)$. By [1] Ch. V Prop. 17.4 it follows that $f = \chi \circ \varphi$ for an isogeny $\chi : G^\sharp \to H$. By construction the derivative of χ induces an isomorphism on the short root spaces, hence an isomorphism of root systems. Thus χ is a central isogeny ([1] Ch. V Prop. 22.4), and therefore again an isomorphism, as desired. $\square$

Let us note the following direct consequence.

Corollary 1.8: *Consider isogenies $G_1 \xleftarrow{\varphi_1} G \xrightarrow{\varphi_2} G_2$ between absolutely simple adjoint groups over a field F. Then one of them factors through the other, i.e. $\varphi_1 = \psi \circ \varphi_2$ for an isogeny $\psi : G_2 \to G_1$, or vice versa.*

For non-adjoint groups we have, by Borel-Tits [3] Props. (2.24) and (2.26):

Proposition 1.9: *Let $\varphi : G \to H$ be an isogeny between connected semisimple groups.*

(a) If G is simply connected, then φ factors uniquely as $G \to \tilde{H} \to H$, where $\tilde{H}$ denotes the universal covering of H.

(b) If H is adjoint, then φ factors uniquely as $G \to G^{\mathrm{ad}} \to H$.

Structure of the Adjoint Representation: Consider an absolutely simple adjoint group G over a field F, with universal covering $\tilde{G}$. Since the commutator morphism (1.1) of $\tilde{G}$ factors through G, so does its adjoint representation. Thus, taking derivatives, the isogeny $\tilde{G} \to G$ induces a G-equivariant linear map between the associated Lie algebras $\tilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$. We denote its kernel by $\mathfrak{z}$, its image by $\bar{\mathfrak{g}}$, and its cokernel by $\mathfrak{z}^*$. In short, we have the exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{z} \arrow[r] & \tilde{\mathfrak{g}} \arrow[rr] \arrow[dr] & & \mathfrak{g} \arrow[r] & \mathfrak{z}^* \arrow[r] & 0 \\
& & & \bar{\mathfrak{g}} \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

It will simplify the exposition to combine $\tilde{\mathfrak{g}}$ and $\mathfrak{g}$ into a single representation.

Proposition 1.10: *There exists a representation $\hat{\rho}$ of G on an F-vector space $\hat{\mathfrak{g}}$ lying in a commutative diagram of G-equivariant homomorphisms, in which all oblique lines are exact:*

$$
\begin{tikzcd}
 & 0 \arrow[dr] & & & & 0 & \\
0 \arrow[dr] & & \mathfrak{z} \arrow[dr] \arrow[dl, equal] & & \mathfrak{z}^* \arrow[ur] \arrow[dr, equal] & & 0 \\
 & \mathfrak{z} \arrow[dr] & & \hat{\mathfrak{g}} \arrow[ur] \arrow[dr, "d\varpi"] & & \mathfrak{z}^* \arrow[ur] & \\
 & & \tilde{\mathfrak{g}} \arrow[ur, "di"] \arrow[dr] & & \mathfrak{g} \arrow[ur] \arrow[dr] & & \\
 & 0 \arrow[ur] & & \bar{\mathfrak{g}} \arrow[ur] \arrow[dr] & & 0 & \\
 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}
$$

Proof: Put $\hat{\mathfrak{g}} := \tilde{\mathfrak{g}} \oplus \mathfrak{z}^*$, and let $di : \tilde{\mathfrak{g}} \hookrightarrow \hat{\mathfrak{g}}$ be the inclusion in the first summand. Let $d\varpi$ be the composite map $\hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \mathfrak{z}^* \twoheadrightarrow \bar{\mathfrak{g}} \oplus \mathfrak{z}^* \cong \mathfrak{g}$, where the last step uses an arbitrary but fixed splitting of F-vector spaces $\mathfrak{z}^* \hookrightarrow \mathfrak{g}$. For any $g \in G$ we set $\hat{\rho}(g) := \mathrm{id} + di \circ \widetilde{\mathrm{Ad}}_G(g) \circ d\varpi$, where $\widetilde{\mathrm{Ad}}_G$ is as in (1.3). A straightforward calculation shows that this defines a group representation. The rest is clear from the construction. $\square$

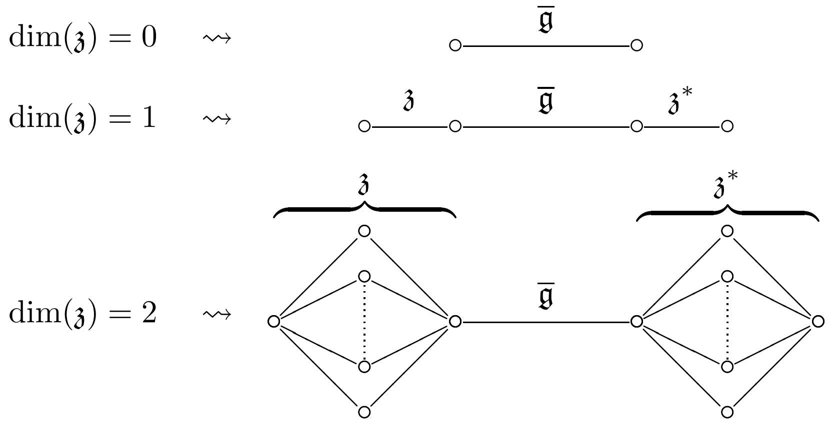
The following proposition classifies all G-submodules of $\hat{\mathfrak{g}}$. Let $p := \mathrm{char}(F)$ and Φ denote the root system of G.

Proposition 1.11:

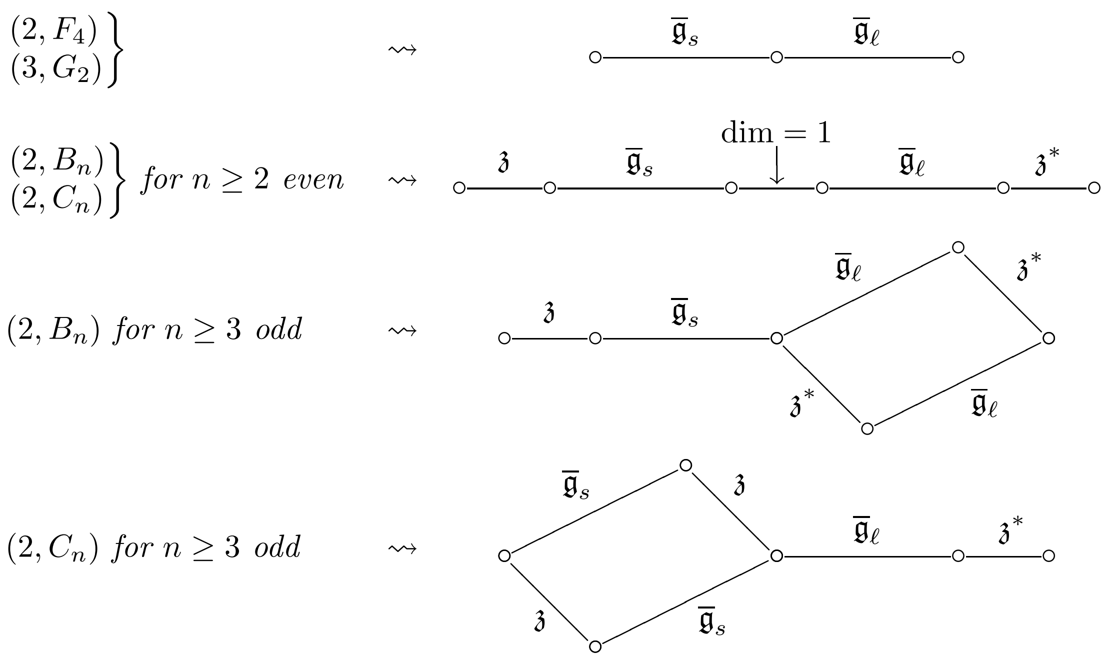
(a) *$\mathfrak{z}$ and $\mathfrak{z}^*$ are constant representations of G of the same dimension. This common dimension is greater than zero if and only if for Φ the index of the root lattice in the weight lattice is divisible by p. It is greater than 1 if and only if $p = 2$ and Φ has type D_n for some even integer n, and in that case the dimension is 2.*

(b) *Suppose that G does not have non-standard isogenies. Then $\bar{\mathfrak{g}}$ is an absolutely irreducible non-constant representation of G. Moreover, it is the unique simple G-submodule of $\mathfrak{g}$ and the unique simple quotient G-module of $\tilde{\mathfrak{g}}$. In other words, the lattice of G-submodules of $\hat{\mathfrak{g}}$ is given by the following graphs, where nodes correspond*

to G-submodules, given in ascending order from left to right:



(c) *Suppose that G possesses non-standard isogenies. Then $\mathfrak{g}$ contains a unique simple G-submodule, denoted $\bar{\mathfrak{g}}_s$, and $\tilde{\mathfrak{g}}$ has a unique simple quotient G-submodule, denoted $\bar{\mathfrak{g}}_\ell$. These two simple subquotients are pairwise inequivalent absolutely irreducible non-constant representations of G. They are the only non-constant simple subquotients in any Jordan-Hölder series of $\hat{\mathfrak{g}}$. The lattice of G-submodules is given by the following graphs, depending on $(p, \text{type of } \Phi)$:*



Proof: (a) is well-known. Most of the remaining assertions are stated and proved explicitly in Hiss [6]; see also Hogeweij [7]. The rest is easily shown by the same arguments. To give a rough sketch: Choose a maximal torus of G. First note that if a G-submodule of $\hat{\mathfrak{g}}$ contains the root space of a root α, then it contains the root spaces for the whole orbit of α under the Weyl group. It follows that in any Jordan-Hölder series of $\hat{\mathfrak{g}}$ there are at most two simple subquotients which possess a non-zero weight, and if there are two, then they must correspond to the set of short roots and the set of long roots, respectively. Next one uses well-known facts about Chevalley bases to determine the Lie bracket between any two root spaces. This information, plus some explicit calculation, suffices to prove that any G-submodule must be among those listed above. To see that the ones in (c) actually exist, consider the derivative $d\varphi : \mathfrak{g} \longrightarrow \mathfrak{g}^\sharp$ of the non-standard isogeny $\varphi : G \to G^\sharp$. Since $d\varphi$ is zero on a root space if and only if that root is long, we can indeed find $\bar{\mathfrak{g}}_s$ and $\bar{\mathfrak{g}}_\ell$ in the kernel, resp. the image of $d\varphi$. The rest is again some explicit calculation. □

The most interesting part of the adjoint representation is $\bar{\mathfrak{g}}$. We denote the representation of G on it by α^G. When G possesses non-standard isogenies, the interesting simple subquotients of $\hat{\mathfrak{g}}$ are $\bar{\mathfrak{g}}_s$ and $\bar{\mathfrak{g}}_\ell$. We denote the representations of G on these spaces by α_s^G and α_ℓ^G. To avoid cumbersome case distinctions we set $\alpha_s^G := \alpha_\ell^G := \alpha^G$ whenever G does not possess non-standard isogenies. The rationale behind this notation is that α_s^G (resp. α_ℓ^G) is always the representation on that simple subquotient of $\hat{\mathfrak{g}}$ which contains copies of the root spaces for all roots of smallest (resp. greatest) possible length.

When $\varphi : G \to G^\sharp$ is the non-standard isogeny of Proposition 1.6 and $\mathfrak{g}^\sharp$ denotes the Lie algebra of $G^\sharp$, the derivative $d\varphi$ induces an isomorphism $\bar{\mathfrak{g}}_\ell \xrightarrow{\sim} \bar{\mathfrak{g}}_s^\sharp$. It follows that $\alpha_\ell^G \cong \alpha_s^{G^\sharp} \circ \varphi$. Furthermore, recall that $(G^\sharp)^\sharp \cong \sigma^* G$ and hence $\mathrm{Lie}(G^\sharp)^\sharp \cong \mathfrak{g} \otimes_{F,\sigma} F$. Thus, by the same token, we obtain an isomorphism $\bar{\mathfrak{g}}_\ell^\sharp \cong \bar{\mathfrak{g}}_s \otimes_{F,\sigma} F$ and hence $\alpha_\ell^{G^\sharp} \circ \varphi \cong \mathrm{Frob}_p \circ \alpha_s^G$.

Image in various representations: We shall need to know the image of G in various subquotient representations of the adjoint representation. In most cases, but not all, it will be enough to have this information for the irreducible subquotients.

Proposition 1.12:

(a) The representation α^G is faithful unless $p = 2$ and Φ has type A_1. In that case there is a canonical isomorphism $\alpha^G(G) \cong \sigma^* \tilde{G}$. In short, we have:

$(p$, *type of* $\Phi)$	$\alpha^G(G)$
$\neq (2, A_1)$	G
$= (2, A_1)$	$\sigma^* \tilde{G}$

(b) Suppose that G possesses non-standard isogenies. Then the images of G under the representations α_s^G and α_ℓ^G are given by the following table:

$(p$, *type of* $\Phi)$	$\alpha_s^G(G)$	$\alpha_\ell^G(G)$
$(2, F_4)$	G	$G^\sharp$
$(3, G_2)$	G	$G^\sharp$
$(2, B_n)$ *for* $n \geq 3$	$\tilde{G}^\sharp$	$G^\sharp$
$(2, C_2) = (2, B_2)$	$\tilde{G}^\sharp$	$\sigma^* \tilde{G}$
$(2, C_n)$ *for* $n \geq 3$	G	$\sigma^* \tilde{G}$

(c) *The representations of G on $\tilde{\mathfrak{g}}$ and on $\mathfrak{g}$ are always faithful.*

(d) *Suppose that $p = 2$ and Φ has type C_n for some $n \geq 2$. Let $\varphi : G \to G^\sharp$ be the non-standard isogeny of Proposition 1.6, and $\tilde{\varphi} : \tilde{G} \to \tilde{G}^\sharp$ the associated isogeny of their universal coverings. Let $\mathfrak{g}_\ell := \mathrm{im}(d\varphi)$ and $\tilde{\mathfrak{g}}_\ell := \mathrm{im}(d\tilde{\varphi})$. Then the image of G in its representation on $\tilde{\mathfrak{g}}_\ell$, resp. on $\mathfrak{g}_\ell$, is $G^\sharp$.*

Proof: (Sketch) This is straightforward to prove, using the same methods as Proposition 1.11. The main point is to look at the derivative of the given representation as a representation of the Lie algebra $\mathfrak{g}$, and to determine which root spaces act trivially. This follows from well-known facts about Chevalley bases as in [6]. The information thus obtained already determines the image of G up to central isogenies. The remaining information results from looking directly at the weights in the given representation. By these methods, one can easily determine the image of G in any given subquotient representation of $\hat{\mathfrak{g}}$. $\square$

Terminology over Semisimple Commutative Rings: Now suppose that F is a finite direct sum of fields $\bigoplus_{i=1}^m F_i$. We assume no relation between the summands F_i; for example, they may have different characteristics. Scheme-theoretically $\mathbf{Spec}\, F$ is the disjoint union $\coprod_{i=1}^m \mathbf{Spec}\, F_i$. Thus an algebraic variety X over F is a disjoint union $\coprod_{i=1}^m X_i$, where each X_i is an algebraic variety of F_i. We say that X_i is the fiber of X over F_i. All concepts concerning algebraic varieties over a field extend to this more general setting. For example, a linear algebraic group G over F is the same as a disjoint union $\coprod_{i=1}^m G_i$ of linear algebraic groups G_i over F_i.

Usually we say that G has a certain property of algebraic groups if and only if each fiber G_i has that property. However, in order to avoid confusion in the case of properties such as "connected", "(absolutely) simple", and others, we shall often say "fiberwise connected" etc. Constructions such as the derived group, the universal covering, the adjoint group of G, and the concepts of homomorphisms and isogenies are also defined fiber by fiber.

An F-module of finite type is the same as a direct sum $V = \bigoplus_{i=1}^m V_i$ of finite dimensional vector spaces V_i over F_i. A representation of G on V thus consists of a representation of each G_i on V_i. More abstractly, the algebra $\mathrm{End}_F(V)$ corresponds to a natural affine algebraic variety over F, denoted $\underline{\mathrm{End}}_F(V)$, which has an algebra structure given by morphisms of varieties over F. Giving a representation of G on V is then the same as giving a homomorphism of linear algebraic groups $G \longrightarrow \underline{\mathrm{Aut}}_F(V) = \underline{\mathrm{End}}_F(V)^*$.

Of particular importance is the adjoint representation Ad_G on the Lie algebra $\mathrm{Lie}\, G = \bigoplus_{i=1}^m \mathrm{Lie}\, G_i$. When G is a fiberwise absolutely simple adjoint group we shall be interested especially in the subquotient representation α_ℓ^G of Ad_G which in every fiber is given by $\alpha_\ell^{G_i}$ defined above.

The group of F-valued points of G is simply $G(F) = \prod_{i=1}^m G_i(F_i)$. When G is fiberwise connected semisimple and $\tilde{G}$ denotes its universal covering, as in (1.2) the commutator induces a morphism $[\ ,\ ]^\sim :\ G \times G \longrightarrow \tilde{G}$. The generalized commutator group of Γ is the subgroup of $\tilde{G}(F)$ defined in the same way as above.

Let $H = \coprod_{j=1}^n H_j$ be a linear algebraic group over another finite direct sum of fields $E = \bigoplus_{j=1}^n E_j$. A ring homomorphism $\tau : E \to F$ is required to map the unit element of

E to that of F, thus making F into an E-algebra. Clearly, giving τ is equivalent to giving a map $\{1, \ldots, m\} \longrightarrow \{1, \ldots, n\},\ i \mapsto j(i)$ and a homomorphism $\tau_i : E_{j(i)} \hookrightarrow F_i$ for every $1 \leq i \leq m$. The base extension of H is then defined as

$$H \times_E F = \coprod_{i=1}^{m} H_{j(i)} \times_{E_{j(i)}, \tau_i} E_i.$$

An important example is the Frobenius isogeny. Let σ be the endomorphism of F which on each simple summand F_i is the identity if $\mathrm{char}(F_i) = 0$, and the Frobenius map $x \mapsto x^p$ if $p = \mathrm{char}(F_i) > 0$. Then we have a canonical isogeny

$$\mathrm{Frob} : G \longrightarrow \sigma^* G = G \times_{F,\sigma} F \tag{1.13}$$

which is the identity in all fibers of characteristic zero, and the Frobenius isogeny in all fibers of positive characteristic.

2. Representations and Associated Rings

Before we begin let us clarify some general terminology. All rings in this article will have a unit element and all homomorphisms of rings are required to map the unit element to the unit element. In particular, any subring of a ring must contain the unit element of the bigger ring, and the unit element must act as the identity on any module. According to Bourbaki [2] §5, no. 1, Déf. 1, a ring A is called semisimple if and only if each left A-module is a direct sum of simple modules. It is simple if and only if it is semisimple, non-zero, and does not possess any two-sided ideals other than $\{0\}$ and itself ([2] §5, no. 2, Déf. 2). Any semisimple ring is a finite direct sum of simple rings ([2] §5, no. 3, Th. 1). The center of a semisimple ring, and in particular any commutative semisimple ring, is therefore a finite direct sum of fields. Actually, any semisimple ring that occurs in this article will turn out to be of finite type as module over its center. In other words, we shall be dealing only with finite direct sums of finite dimensional central simple algebras over fields. However, this will not be entirely obvious from the construction.

In this section and the following ones we fix a commutative semisimple ring F, a connected linear algebraic group G over F, and a fiberwise Zariski dense subgroup $\Gamma \subset G(F)$. As before we let $F = \bigoplus_{i=1}^{m} F_i$ be the decomposition into simple summands, and let G_i denote the fiber of G over F_i. Throughout, we impose one of the following two conditions on F and Γ.

Assumption 2.1:

(a) **Global case:** *Each F_i is a global field, i.e. a finite extension either of $\mathbb{Q}$ or of $\mathbb{F}_p(t)$ for some prime p, and Γ is finitely generated.*

(b) **Local case:** *Each F_i is a local field, and Γ is compact.*

Most of our definitions and theorems will have essentially the same form in both cases. The main difference is that in the local case there will always be an additional topological condition.

Definition 2.2: *Consider a representation ρ of G on an F-module V of finite type.*

(a) $\mathcal{B}_\rho$ is (the closure of, in the local case) the subring of $\mathrm{End}_F(V)$ that is generated by $\rho(\Gamma)$.

(b) $\mathcal{J}_\rho$ is (the closure of, in the local case) the ideal of $\mathcal{B}_\rho$ that is generated by the elements $\rho(\gamma) - \mathrm{id}$ for all $\gamma \in \Gamma$. This is called the augmentation ideal of $\mathcal{B}_\rho$.

The first main result of this section is the following.

Theorem 2.3: *Assume that ρ is fiberwise non-constant absolutely irreducible. We identify F with the scalars in $\mathrm{End}_F(V)$.*

(a) There exists a unique smallest semisimple subring $E_\rho \subset F$ (closed, in the local case) such that:

(i) F is of finite type as module over E_ρ,

(ii) $B_\rho := E_\rho \cdot \mathcal{B}_\rho$ is semisimple with center E_ρ, and

(iii) The natural homomorphism $B_\rho \otimes_{E_\rho} F \longrightarrow \mathrm{End}_F(V)$ is an isomorphism.

(b) Let $\mathcal{O}_{\mathrm{tr}(\rho)} \subset F$ be (the closure of, in the local case) the subring generated by $\mathrm{tr}(\rho(\Gamma))$. Then E_ρ is the total ring of quotients of $\mathcal{O}_{\mathrm{tr}(\rho)}$.

(c) Let $\mathcal{O}_\rho := F \cap \mathcal{B}_\rho$. Then E_ρ is also the total ring of quotients of $\mathcal{O}_\rho$. The subring $\mathcal{O}_\rho \subset E_\rho$ is finitely generated over $\mathbb{Z}$ in the global case, and open compact in the local case. Moreover, $\mathcal{J}_\rho$ and $\mathcal{B}_\rho$ are finite type modules over $\mathcal{O}_\rho$, with finite index in each other.

Note that in the local case (c) implies that $\mathcal{J}_\rho \subset \mathcal{B}_\rho$ are open compact in B_ρ. The proof of Theorem 2.3 will be somewhat lengthy. We begin with the following technical result on semisimple rings. Let us abbreviate $A := \mathrm{End}_F(V)$.

Theorem 2.4: *Consider a subring $B \subset A$ (not necessarily an F-algebra) with the properties $B \cdot F = A$ and $\mathrm{length}_B(V) < \infty$. Let E denote the center of B. Then:*

(a) E is contained in the center F of A.

(b) B is a semisimple ring.

(c) The natural homomorphism $B \otimes_E F \longrightarrow A$ is an isomorphism.

(d) F is of finite type as module over E.

Proof: By definition E commutes with F and B, and thus with $B \cdot F = A$, whence (a).

For (b) we first show that V is a semisimple B-module. Let $V = \bigoplus_{i=1}^m V_i$ be the decomposition according to the decomposition of F into simple summands. It suffices to show that each V_i is a semisimple B-module. Since it has finite length over B, it contains a simple B-submodule $0 \neq W_i \subset V_i$. Consider the submodule $F \cdot W_i = \sum_{x \in F} xW_i$ of V_i. By definition it is stable under $B \cdot F = A$, hence it is equal to V_i. On the other hand, as a sum of simple modules it is semisimple ([2] §3, no. 3, Prop. 7). We now know that V is a faithful semisimple B-module of finite length. By [2] §5, no. 1, Prop. 4, any ring possessing such a module is semisimple. This shows (b).

Next we prove (c). Let $B = \bigoplus_{j=1}^n B_j$ be the decomposition into simple summands and let E_j denote the center of B_j. The inclusion $E \hookrightarrow F$ is then described by a map $\{1, \ldots, m\} \longrightarrow \{1, \ldots, n\}$, $i \mapsto j(i)$ and a homomorphism $\tau_i : E_{j(i)} \hookrightarrow F_i$ for every $1 \leq i \leq m$. Decomposing the homomorphism in (d) according to the simple summands of F, we must show that for every $1 \leq i \leq m$ the natural homomorphism

$$B_{j(i)} \otimes_{E_{j(i)}} F_i \longrightarrow \mathrm{End}_{F_i}(V_i) \tag{2.5}$$

is an isomorphism. Since $B \cdot F = A$, this map is surjective, so its kernel is a proper twosided ideal. But by [2] §7, exerc. 6 (c) the ring $B_{j(i)} \otimes_{E_{j(i)}} F_i$ possesses no nontrivial proper twosided ideals. Hence the homomorphism is also injective, and therefore is an isomorphism. This proves (c).

Finally, since each $\mathrm{End}_{F_i}(V_i)$ has finite dimension over F_i, the isomorphy (2.5) implies that each B_j has finite dimension over E_j. Therefore B is of finite type as E-module. Since V is of finite type over B, it is thus also of finite type over E. Using any F-linear injection $F \hookrightarrow V$ we can now deduce the same for F. This shows (d) and thus finishes the proof of Theorem 2.4. $\square$

Next we note the following algebro-geometric version of Burnside's theorem.

Lemma 2.6: *For every sufficiently large positive integer n the morphism*

$$G \times \ldots \times G \longrightarrow \underline{A} := \underline{\mathrm{End}}_F(V), \quad (g_1, \ldots, g_n) \mapsto \textstyle\sum_{j=1}^n \rho(g_j)$$

is dominant.

Proof: Without loss of generality we may assume that F is a field. Let H denote the image of G in $\underline{A}^*$. Since ρ is nonconstant and G is connected, the Lie algebra of H is a non-zero subspace of A. Since ρ is absolutely irreducible, the action of $(g', g) \in G \times G$ on A by $a \mapsto \rho(g') \cdot a \cdot \rho(g^{-1})$ is also absolutely irreducible. Using the Zariski density of $G(F)$ it follows that for every sufficiently large n there exist g'_j, $g_j \in G(F)$ such that

$$A = \textstyle\sum_{j=1}^n \rho(g'_j) \cdot (\mathrm{Lie}\, H) \cdot \rho(g_j^{-1}).$$

Consider the morphism

$$H \times \ldots \times H \longrightarrow \underline{A}, \quad (h_1, \ldots, h_n) \mapsto \textstyle\sum_{j=1}^n \rho(g'_j) \cdot h_j \cdot \rho(g_j^{-1}).$$

By construction its derivative at $(\mathrm{id}, \ldots, \mathrm{id})$ is surjective. Hence this morphism is dominant, and so is the morphism in the lemma. $\square$

Proof of Theorem 2.3 (a–b): This part of the theorem is essentially due to Vinberg. It does not really depend on Assumption 2.1 and can be proved by the direct argument of [12]. But since we shall need Theorem 2.4 for (c), we might as well use it here, too.

First note that Lemma 2.6 implies that $\mathrm{tr}(\mathcal{B}_\rho) \subset F$ is Zariski dense in the affine line $\mathbb{A}^1_F$. This means that the image of $\mathrm{tr}(\mathcal{B}_\rho)$ and hence of $\mathcal{O}_{\mathrm{tr}(\rho)}$ in any simple summand F_i is infinite. From this one easily deduces that

$$E_{\mathrm{tr}(\rho)} := \left\{ \tfrac{x}{y} \;\middle|\; x, y \in \mathcal{O}_{\mathrm{tr}(\rho)},\ y \in F^* \right\} \;\subset\; F.$$

is semisimple and F is of finite type as module over $E_{\mathrm{tr}(\rho)}$.

Now consider any subring $E_\rho \subset F$ satisfying the conditions in (a). From the isomorphism (a.iii) we deduce that $\mathrm{tr}(\mathcal{B}_\rho) \subset \mathrm{tr}(E_\rho \cdot \mathcal{B}_\rho) = E_\rho$, and hence $\mathcal{O}_{\mathrm{tr}(\rho)}$ and $E_{\mathrm{tr}(\rho)}$ are contained in E_ρ. To prove (a) and (b) it thus remains to show that $E_{\mathrm{tr}(\rho)}$ satisfies the conditions in (a). We already verified (a.i). This, in turn, implies that V is of finite type as module over $B_{\mathrm{tr}(\rho)} := E_{\mathrm{tr}(\rho)} \cdot \mathcal{B}_\rho$. On the other hand, by Burnside's theorem (see, e.g., Curtis-Reiner [4] Th. 3.32), the absolute irreducibility of ρ, and the Zariski density of Γ, we have $B_{\mathrm{tr}(\rho)} \cdot F = A$. Thus, by Theorem 2.4, we deduce that $B_{\mathrm{tr}(\rho)}$ is semisimple and $B_{\mathrm{tr}(\rho)} \otimes_E F \xrightarrow{\sim} A$, where E denotes the center of $B_{\mathrm{tr}(\rho)}$. From this it follows that

$$E \overset{!}{=} \mathrm{tr}(B_{\mathrm{tr}(\rho)}) = \mathrm{tr}(E_{\mathrm{tr}(\rho)} \cdot \mathcal{B}_\rho) = E_{\mathrm{tr}(\rho)} \cdot \mathrm{tr}(\mathcal{B}_\rho) = E_{\mathrm{tr}(\rho)}.$$

Therefore $E_{\mathrm{tr}(\rho)}$ satisfies the conditions in (a), as desired. $\square$

To prove Theorem 2.3 (c) we must construct the ring $B_{\mathrm{tr}(\rho)}$ internally from $\mathcal{B}_\rho$, instead of just imposing the center $E_{\mathrm{tr}(\rho)}$ from the outside. Until the end of the proof we shall drop the subscript $_\rho$, that is, we abbreviate $\mathcal{B} := \mathcal{B}_\rho$, $\mathcal{J} := \mathcal{J}_\rho$, $\mathcal{O}_{\mathrm{tr}} := \mathcal{O}_{\mathrm{tr}(\rho)}$, $E := E_{\mathrm{tr}(\rho)}$, and $B := B_{\mathrm{tr}(\rho)}$.

Lemma 2.7: *There exists an element $b \in \mathcal{J}$ with the following properties:*

(a) It is regular semisimple in A.

(b) In the global case, none of its eigenvalues lies in a finite field.

(c) In the local case, all of its eigenvalues have norm < 1.

Proof: Put

$$b := \textstyle\sum_{j=1}^{n} (\rho(\gamma_j) - \mathrm{id}) \in \mathcal{J}$$

for $\gamma_1, \ldots, \gamma_n \in \Gamma$ and $n \gg 0$. By Lemma 2.6 the elements b thus obtained form a Zariski dense subset of $\underline{A}$. Now condition (a) can be achieved simply because it forbids only a Zariski closed proper subset. The same is true for condition (b), if the eigenvalue is fixed. But any eigenvalue lies in an extension of bounded degree of some F_i. Moreover, the field of constants of any F_i of positive characteristic is itself finite. Thus any eigenvalue in a finite field lies in a finite field of bounded order. Thus there are only finitely many eigenvalues to be ruled out. Therefore condition (b) can be achieved. Finally, condition (c) is automatic if all γ_j lie sufficiently close to the identity of Γ. $\square$

In the following we fix the element b given by Lemma 2.7. We let $\mathcal{R}$ be (the closure of, in the local case) the subring of $\mathrm{End}_F(V)$ that is generated by b. Then

$$K := \{y^{-1}x \mid x \in \mathcal{R},\ y \in \mathcal{R} \cap A^*\}$$

is a commutative semisimple ring, and V is of finite type as module over K.

Lemma 2.8: *We have $B = K \cdot \mathcal{B} = \{y^{-1}x \mid x \in \mathcal{B},\ y \in \mathcal{R} \cap A^*\}$.*

Proof: By construction K is contained in B. We first show that $K \cdot \mathcal{B}$ is a subring. For this consider $x \in \mathcal{B}$ and $r \in \mathcal{R} \cap A^*$. Note that A is of finite type as left K-module! Consider the left K-submodule of A that is generated by the elements xr^i for all $i \in \mathbb{Z}$. As a submodule of a module of finite type, it is itself of finite type. Suppose this submodule is generated by the elements xr^i for all $\ell \leq i \leq k$. Then we have $xr^{\ell-1} \in \sum_{i=\ell}^{k} K \cdot xr^i$. Multiplying by $r^{-\ell}$ on the right hand side we deduce $xr^{-1} \in \sum_{i=0}^{k-\ell} K \cdot xr^i \subset K \cdot \mathcal{B}$. In this way, we have proved that $\mathcal{B} \cdot K \subset K \cdot \mathcal{B}$. This implies that $K \cdot \mathcal{B}$ is a ring.

Now recall that $F \cdot K \cdot \mathcal{B} = F \cdot \mathcal{B} = A$, and that, by construction, V is of finite type as module over $K \cdot \mathcal{B}$. Thus we may apply Theorem 2.4 to this ring. If E' denotes the center of $K \cdot \mathcal{B}$, it follows that $K \cdot \mathcal{B} = E' \cdot \mathcal{B}$. The parts (a) and (b) of Theorem 2.3, which are already proved, now imply that $E \subset E'$. Thus we have $B \subset K \cdot \mathcal{B} \subset B$, as desired. $\square$

Proof of Theorem 2.3 (c): First we consider the local case, which is now rather easy. Observe that $K \cdot \mathcal{J} = K \cdot \mathcal{B} = B$, since $b \in \mathcal{J}$. By construction $\mathcal{R}$ is a compact open subring of K. As B is of finite type as left K-module, the left $\mathcal{R}$-submodules $\mathcal{J} \subset \mathcal{B} \subset B$ are themselves open. On the other hand $\mathcal{J}$ and $\mathcal{B}$ are compact, because Γ is compact. It follows that $F \cap \mathcal{B} = E \cap \mathcal{B}$ is an open compact subring of E, that $\mathcal{J} \subset \mathcal{B}$ are finitely generated modules over $E \cap \mathcal{B}$, and that $\mathcal{B}/\mathcal{J}$ is finite. This proves (c) in the local case.

In the remainder of the proof we consider the global case, which turns out to be more involved. Recall that $\mathcal{O}_{\rm tr}$ denotes the subring of E that is generated by $\operatorname{tr}(\rho(\Gamma))$. Let C be the centralizer of b in B. This is a semisimple ring containing both K and E. Put $\mathcal{C} := C \cap \mathcal{B}$. Our job will be to compare the two rings $\mathcal{C}$ and $\mathcal{O}_{\rm tr}$.

Lemma 2.9:

(a) $\mathcal{O}_{\rm tr}$ *is a finitely generated* $\mathbb{Z}$*-algebra.*

(b) $\mathcal{O}_{\rm tr} \cdot \mathcal{J} \subset \mathcal{O}_{\rm tr} \cdot \mathcal{B}$ *are* $\mathcal{O}_{\rm tr}$*-modules of finite type, with finite index in each other.*

(c) $\mathcal{C}$ *is a finitely generated* $\mathbb{Z}$*-algebra.*

Proof: Choose elements $\gamma_1, \ldots, \gamma_n$ that generate Γ. Choose a system of generators of V over F. Then $\mathcal{O}_{\rm tr}$ is contained in the subring of F that is generated by the coefficients of all the $\gamma_i^{\pm 1}$. Of course, this ring is finitely generated. On the other hand, Assumption 2.1 implies that any subring of a finitely generated subring of F is itself finitely generated. This proves (a).

For (b) choose $x_1, \ldots, x_n \in \mathcal{B}$ such that $B = \sum_{i=1}^n E \cdot x_i$. Then $x \mapsto (\operatorname{tr}(xx_i))_i$ defines an injective homomorphism of E-modules $B \hookrightarrow E^{\oplus n}$. By construction, the image of $\mathcal{O}_{\rm tr} \cdot \mathcal{B}_\rho$ lies in $\mathcal{O}_{\rm tr}^{\oplus n}$. Since $\mathcal{O}_{\rm tr}$ is noetherian, by (a), it follows that $\mathcal{O}_{\rm tr} \cdot \mathcal{J} \subset \mathcal{O}_{\rm tr} \cdot \mathcal{B}$ themselves are finitely generated. Since $b \in \mathcal{J} \cap K^*$, the index must be finite. This proves (b).

From (b) we can deduce that $\mathcal{O}_{\rm tr} \cdot \mathcal{C}$ is finitely generated as module over $\mathcal{O}_{\rm tr}$. Using (a) we find that this is a finitely generated $\mathbb{Z}$-algebra. Again the same follows for the subalgebra $\mathcal{C}$, thus proving (c). $\square$

In order to be able to construct sufficiently many elements of $\mathcal{C}$, we need something like a projector $B \twoheadrightarrow C$. Note that the semisimplicity of b implies that $B = C \oplus [b, B]$.

Lemma 2.10: *There exists an element* $\Pi \in \operatorname{End}_E(B)$ *with the properties:*

(a) $\Pi|_{[b,B]} = 0$*,*

(b) $\Pi|_C$ *is multiplication by some element* $r \in \mathcal{R} \cap K^*$*, and*

(c) $\Pi(\mathcal{B}) \subset \mathcal{C}$*.*

Proof: Recall that B is a left K-module of finite type. The map $\operatorname{ad}_b : x \mapsto [b, x]$ is an endomorphism of this module which is an isomorphism on $[b, B]$, and zero on C. Therefore there is a polynomial $F(X) \in K[X]$ such that $F(\operatorname{ad}_b)$ is zero on $[b, B]$ and the identity on C. Let $r \in \mathcal{R} \cap K^*$ be the common denominator of the coefficients of F. Then the coefficients of $r \cdot F(X)$ lie in $\mathcal{R}$, and $\Pi := r \cdot F(\operatorname{ad}_b)$ has all the desired properties. $\square$

Lemma 2.11: *Consider a place w of C, lying above a non-archimedean place v of E. Suppose that $\mathcal{O}_{\mathrm{tr}}$ contains an element which has a pole at v. Then $\mathcal{C}$ contains an element which has a pole at w.*

Proof: Assume that none of the elements in $\Pi(\mathcal{B})$ has a pole at w. Let Φ denote the composite map

$$B \otimes_E E_v \xrightarrow{\Pi\otimes\mathrm{id}} C \otimes_E E_v \twoheadrightarrow C_w \overset{\ell}{\twoheadrightarrow} E_v,$$

where ℓ is any non-zero E_v-linear form. The assumption implies that $\Phi(\mathcal{B})$ is contained in a bounded subset of E_v. Choose elements $x_1, \ldots, x_n \in \mathcal{B}$ such that $B = \sum_{i=1}^n E \cdot x_i$. Then we have an injective homomorphism of E_v-modules

$$\Psi : B \otimes_E E_v \hookrightarrow E_v^{\oplus n}, \quad x \mapsto (\Phi(xx_i))_i.$$

By construction, the image of $\mathcal{B}$ lies in a bounded subset. Thus $\mathcal{B}$ acts faithfully on some module of finite type over the valuation ring in E_v. It follows that the trace of any element is v-adically integral, contrary to the assumption in the lemma. $\square$

Let $\tilde{\mathcal{C}}$ denote the normalization of the ring $\mathcal{C}$. The preceding lemma has the following crucial consequence.

Lemma 2.12: *We have $\mathcal{O}_{\mathrm{tr}} \subset \tilde{\mathcal{C}}$.*

Proof: Consider any element $x \in \mathcal{O}_{\mathrm{tr}}$. Since $x \in E \subset C$, and $\tilde{\mathcal{C}}$ is a Dedekind ring, it suffices to show that x has no pole at any maximal ideal of $\mathcal{C}$. This is guaranteed by Lemma 2.11. $\square$

Now we can finish the proof of Theorem 2.3 (c). Since $\tilde{\mathcal{C}}/\mathcal{C}$ is finite, so is $\mathcal{O}_{\mathrm{tr}}/(\mathcal{O}_{\mathrm{tr}} \cap \mathcal{C})$. Thus the total ring of quotients of $\mathcal{O}_{\mathrm{tr}} \cap \mathcal{C}$ is again E. The same also follows for the (possibly larger) ring $\mathcal{O}_\rho := F \cap \mathcal{B} = E \cap \mathcal{C}$. As a subring of the finitely generated ring $\mathcal{C}$ (compare Lemma 2.9 (c)) it is itself finitely generated. The remainder of Theorem 2.3 (c) follows from Lemma 2.9 (b). $\square$

Consequences of Theorem 2.3: Let $\tilde{\mathcal{O}}_\rho$ denote the normalization of the ring $\mathcal{O}_\rho$ in Theorem 2.3. The arguments in the proof of Lemma 2.11 give the following characterization of $\tilde{\mathcal{O}}_\rho$.

Corollary 2.13: *Suppose that F is global, and consider a non-archimedean valuation v of E_ρ. Then v corresponds to a maximal ideal of $\tilde{\mathcal{O}}_\rho$ if and only if Γ is v-adically bounded.*

One important application of Theorem 2.3, in particular of its part (a.iii), is the construction of a natural model of $\rho(G)$ over the subring E_ρ. Let $\underline{B}_\rho$ denote the affine algebraic variety over E_ρ corresponding to B_ρ, with its algebra structure given by morphisms of varieties over E_ρ. This is a model of $\underline{A}_\rho$ over E_ρ, i.e. we have a natural isomorphism $\underline{B}_\rho \times_{E_\rho} F \cong \underline{A}_\rho$.

Theorem 2.14: *In the situation of Theorem 2.3 there is a unique algebraic subgroup $G_\rho \subset \underline{B}_\rho^*$ such that*

$$\begin{array}{ccc} \underline{B}_\rho^* \times_{E_\rho} F & \xrightarrow{\sim} & \underline{A}_\rho^* \\ \cup & & \cup \\ G_\rho \times_{E_\rho} F & \xrightarrow{\sim} & \rho(G)\ . \end{array}$$

Under this isomorphism $\rho(\Gamma)$ corresponds to a subgroup of $G_\rho(E_\rho)$.

Proof: By construction $\rho(\Gamma)$ is contained in the subgroup $B_\rho^* \subset A_\rho^*$. Let G_ρ be its Zariski closure in $\underline{B}_\rho^*$. Then $G_\rho \times_{E_\rho} F$ maps isomorphically to the Zariski closure of $\rho(\Gamma)$ in $\underline{A}_\rho^*$. By the Zariski density of Γ, the latter is just $\rho(G)$. This shows the existence of G_ρ, and the uniqueness is obvious. $\square$

We can also compare the rings E_ρ and $\tilde{\mathcal{O}}_\rho$ for suitable different choices of ρ. The following special case will be enough for our purposes.

Proposition 2.15: *Suppose that F is a field, and let ρ and α be two non-constant absolutely irreducible representations of G such that α occurs as subquotient of the Lie algebra of $\rho(G)$. Then*

(a) $E_\alpha \subset E_\rho$, *and*

(b) $\tilde{\mathcal{O}}_\alpha \subset \tilde{\mathcal{O}}_\rho$.

Proof: Let G_ρ be the model of $\rho(G)$ over E_ρ that is given by Theorem 2.14. The assumptions imply that α descends to a representation on a subquotient of the Lie algebra of G_ρ. Hence $\mathrm{tr}(\alpha(\Gamma)) \subset E_\rho$, which implies (a). Assertion (b) follows from (a) together with Corollary 2.13. $\square$

Reducible Representations: Now we turn to the study of $\mathcal{J}_\rho$ for reducible representations. We restrict our attention to the simplest kind of representation which is not completely reducible, namely a non-split extension of two absolutely irreducible representations. Assume that F is a field and consider a short exact sequence of finite dimensional non-zero F-vector spaces

$$0 \to V' \to V \to V'' \to 0.$$

Let ρ be a representation of G on V which stabilizes V'. We assume that the representations ρ', ρ'' induced on V', V'' are absolutely irreducible, and that the sequence does not possess a G-equivariant splitting. Let A_ρ be the stabilizer of V' in $\mathrm{End}_F(V)$. Its radical $\mathrm{Rad}(A_\rho)$ consists of those endomorphisms which annihilate both V' and V''. Thus we have a canonical isomorphism $\mathrm{Rad}(A_\rho) \cong \mathrm{Hom}_F(V'', V')$. Clearly, the subring $\mathcal{B}_\rho$ of Definition 2.2 is contained in A_ρ.

Theorem 2.16: *Suppose that $\rho' \not\cong \rho''$. Then we have*

$$F \cdot \big(\mathrm{Rad}(A_\rho) \cap \mathcal{J}_\rho\big) = \mathrm{Rad}(A_\rho).$$

Furthermore set

$$\mathcal{O} := \begin{cases} \mathcal{O}_{\rho'} := F \cap \mathcal{B}_{\rho'} & \textit{if } \rho'' \textit{ is constant,} \\ \mathcal{O}_{\rho''} := F \cap \mathcal{B}_{\rho''} & \textit{if } \rho' \textit{ is constant,} \\ \mathcal{O}_{\rho'} \cdot \mathcal{O}_{\rho''} & \textit{if both } \rho' \textit{ and } \rho'' \textit{ are non-constant.} \end{cases}$$

Then $\mathrm{Rad}(A_\rho) \cap \mathcal{J}_\rho$ *is a $\mathcal{O}$-module of finite type.*

Proof: If $F \cdot \big(\mathrm{Rad}(A_\rho) \cap \mathcal{J}_\rho\big)$ is non-zero, we study it as a module under left and right multiplication by $\mathcal{B}_\rho \cdot F$. Note that the left action factors through the surjection $\mathcal{B}_\rho \cdot F \twoheadrightarrow \mathcal{B}_{\rho'} \cdot F$. The latter ring is equal to $\mathrm{End}_F(V')$. Indeed, this is obvious when ρ' is the constant representation of dimension 1, otherwise it is just the assertion of Theorem 2.3 (a.iii). Thus $F \cdot \big(\mathrm{Rad}(A_\rho) \cap \mathcal{J}_\rho\big)$ is a non-zero submodule of $\mathrm{Rad}(A_\rho)$ under the left action of $\mathrm{End}_F(V')$. By symmetry, it is also invariant under the right action of $\mathrm{End}_F(V'')$. Since $\mathrm{Rad}(A_\rho)$ is irreducible under the combination of these two actions, the equality follows.

Let us now assume that $\mathrm{Rad}(A_\rho) \cap \mathcal{J}_\rho = 0$, and let $\rho^{\mathrm{ss}} := \rho' \oplus \rho''$ denote the semisimplification of ρ.

Lemma 2.17: *There exists an element in $\mathcal{J}_{\rho^{\mathrm{ss}}}$ which acts as a scalar on V' and as a different scalar on V''.*

Proof: If ρ'' is a constant representation, then ρ' is non-constant and we have $\mathcal{J}_{\rho^{\mathrm{ss}}} \xrightarrow{\sim} \mathcal{J}_{\rho'}$. By Theorem 2.3 (c) there exists a non-zero element in $F \cap \mathcal{J}_{\rho'}$. Then its lift to $\mathcal{J}_{\rho^{\mathrm{ss}}}$ has all the desired properties. When ρ' is constant, the result follows by symmetry.

When both ρ' and ρ'' are non-constant, we can view ρ^{ss} as a nowhere constant absolutely irreducible representation over $F \oplus F$. Then by Theorem 2.3 (c) we can find the desired element unless $E_{\rho^{\mathrm{ss}}} \subset F \oplus F$ is contained in the diagonal. Suppose that this happens. Then $E_{\rho^{\mathrm{ss}}}$ is a subfield of F and $B_{\rho^{\mathrm{ss}}}$ is a simple algebra which maps isomorphically to both $B_{\rho'}$ and $B_{\rho''}$. Thus V' and V'' are isomorphic simple modules over $B_{\rho^{\mathrm{ss}}} \otimes_{E_{\rho^{\mathrm{ss}}}} F$. It follows that ρ' and ρ'' are equivalent F-linear representations of Γ. Since Γ is Zariski dense in G, they are also equivalent as representations of G, contrary to the assumption. $\square$

Continuing the proof of Theorem 2.16, we choose an element as in Lemma 2.17 and lift it to an element $e \in \mathcal{J}_\rho$. By construction the commutator $[e, \mathcal{B}_\rho]$ is contained in $\mathrm{Rad}(A_\rho) \cap \mathcal{J}_\rho$. Since this group was assumed to vanish, it follows that e commutes with $\mathcal{B}_\rho$. Thus it commutes with Γ and, by Zariski density, with G. This implies that the eigenspace decomposition of V under e is G-invariant, i.e. that the extension splits, contrary to the assumption.

Concerning the rest of the theorem, it is clear that $\mathrm{Rad}(A_\rho) \cap \mathcal{J}_\rho$ is an $\mathcal{O}$-module. By Theorem 2.3 (c) and the construction of $\mathcal{O}$ both $\mathcal{O} \cdot \mathcal{B}_{\rho'}$ and $\mathcal{O} \cdot \mathcal{B}_{\rho''}$ are finitely generated $\mathcal{O}$-modules. From the finite generation of $\mathcal{B}_\rho$ as a ring we find that $\mathcal{O} \cdot \mathcal{B}_\rho$ is also finitely generated as module over $\mathcal{O}$. Thus the same holds for the submodule $\mathrm{Rad}(A_\rho) \cap \mathcal{J}_\rho$, as desired. $\square$

3. Minimal Quasi-Models of Semisimple Groups

Let F, G, and Γ be as in the preceding section. From now on we assume that G is fiberwise absolutely simple adjoint. The main question in this section is how far we can reduce the structure constants of G to semisimple subrings $E \subset F$ in such a way that Γ consists of E-rational points. The main tool for this will be the study of the action of Γ in the adjoint representation of G. The results obtained in the process will also lay the groundwork for later sections. In order to deal adequately with the effects of non-standard isogenies, it is best to modify the usual concept of a model of G over a subring.

Definition 3.1: *A* weak quasi-model of (F, G, Γ) *is a triple* (E, H, φ) *where*

(a) E is a semisimple subring of F such that F is of finite type as module over E (and which is closed in the local case),

(b) H is a fiberwise absolutely simple adjoint group over E, and

(c) φ is an isogeny $H \times_E F \longrightarrow G$, such that

(d) Γ is contained in the subgroup $\varphi(H(E)) \subset G(F)$.

Definition 3.2: *A* quasi-model of (F, G, Γ) *is a weak quasi-model* (E, H, φ) *of* (F, G, Γ) *for which the derivative of φ vanishes nowhere.*

Note that a quasi-model is very close to being a model over a subring in the usual sense. Indeed, if the fibers of G do not possess non-standard isogenies, then it follows from Theorem 1.7 that φ in Definition 3.2 is an isomorphism.

Note also that in these definitions the isogeny φ is totally inseparable, because H is adjoint. Therefore the homomorphism $H(E) \hookrightarrow G(F)$ is injective. Thus for any (weak) quasi-model the group Γ is in bijection with the fiberwise Zariski dense subgroup $\varphi^{-1}(\Gamma) \subset H(E)$. We are in a recursive situation since the triple $(E, H, \varphi^{-1}(\Gamma))$ satisfies the same assumptions as (F, G, Γ). For example, if (E', H', φ') is a weak quasi-model of $(E, H, \varphi^{-1}(\Gamma))$, then clearly $(E', H', \varphi \circ \varphi')$ is a weak quasi-model of (F, G, Γ). One should be aware that the composite in this sense of two quasi-models is in general only a weak quasi-model. This is one of the reasons for dealing with the latter at all. On the other hand, every weak quasi-model gives rise to a quasi-model, as follows.

Proposition 3.3: *For any weak quasi-model (E, H, φ) of (F, G, Γ) there exists a ring endomorphism $\tau : F \longrightarrow F$, which on each simple summand F_i is either the identity or a power of Frobenius, and a quasi-model (E_1, H_1, φ_1) of (F, G, Γ), such that $E_1 = \tau(E)$. Clearly, if τ is an isomorphism, then (E, H, φ) is already a quasi-model of (F, G, Γ).*

Proof: Consider a simple summand F_i. If $p := \mathrm{char}(F_i)$ is zero, then $d\varphi$ is already non-zero over F_i, and we can put $\tau|_{F_i} = \mathrm{id}$. Otherwise we know from Theorem 1.7 that over F_i the isogeny φ is the composite of a Frobenius $\mathrm{Frob}_{p^{n_i}}$ for some $n_i \geq 0$ and an isogeny with non-vanishing derivative. In that case we let $\tau|_{F_i}$ be the n_i^{th} Frobenius map $x \mapsto x^{p^{n_i}}$ for all $x \in F_i$. Then by construction φ is the composite of two isogenies

$$H \times_E F \xrightarrow{\psi} \tau^*(H \times_E F) \xrightarrow{\varphi_1} G,$$

where ψ is in each fiber either an isomorphism or a power of Frobenius, and where $d\varphi_1$ vanishes nowhere. By construction we get an isomorphism $E \xrightarrow{\sim} \tau(E) =: E_1 \subset F$. Let H_1 denote the group over E_1 corresponding to H via this isomorphism. Then we have

$$\tau^*(H \times_E F) = (H \times_E F) \times_{F,\tau} F = H \times_{E,\tau} F = H_1 \times_{E_1} F.$$

It is now clear from the construction that (E_1, H_1, φ_1) is a quasi-model of (F, G, Γ). $\square$

The problem of how far the structure constants of G can be reduced is phrased in the following definitions (compare Definition 0.1).

Definition 3.4: *We say that* (F, G, Γ) *is* minimal *if and only if, for every weak quasi-model* (E, H, φ) *of* (F, G, Γ)*, we have* $E = F$ *and* φ *is an isomorphism.*

It is easy to show, though not obvious, that replacing the words "weak quasi-model" by "quasi-model" in Definition 3.4 yields an equivalent definition. Since we shall not make use of this fact, we leave out its proof.

Definition 3.5: *A (weak) quasi-model* (E, H, φ) *of* (F, G, Γ) *is called* minimal *if and only if* $(E, H, \varphi^{-1}(\Gamma))$ *is minimal in its own right.*

Note that this definition is phrased only as a relative minimality condition, not as a universal property vis-à-vis all (weak) quasi-models. (It would have been possible, but awkward, to do so). Thus both the existence and the uniqueness of minimal quasi-models are by no means obvious.

Theorem 3.6: *(a) There exists a minimal quasi-model* (E, H, φ) *of* (F, G, Γ)*.*

(b) The subring $E \subset F$ *in (a) is unique, and* H *and* φ *are determined up to unique isomorphism.*

Proof of Theorem 3.6 (a): We begin by showing that the subring E in a quasi-model cannot become arbitrarily small.

Lemma 3.7: *The number* $\mathrm{length}_E(F)$ *is finite and bounded independently of the quasi-model* (E, H, φ) *of* (F, G, Γ)*.*

Proof: Consider the representation α_ℓ^H of H defined in Section 1. Then there exists a linear representation ρ of G such that $\alpha_\ell^H \otimes_E F = \rho \circ \varphi$. Indeed, consider any simple summand F_i of F. If φ is an isomorphism over F_i, then the corresponding direct summand of ρ is just $\alpha_\ell^{G_i}$. Otherwise, the summand is $\alpha_s^{G_i}$. This gives the desired representation ρ, and since there are only finitely many fibers G_i it also shows that the number of possibilities for the isomorphy class of ρ is finite.

Let $E_\rho \subset F$ denote the subring associated to ρ by Theorem 2.3 (a). Recall that F has finite length as module over E_ρ. On the other hand we can apply the same constructions to $(E, F, \varphi^{-1}(\Gamma))$ and the representation α_ℓ^H, yielding a subring $E_{\alpha_\ell^H} \subset E$. Since the two representations and compact subgroups correspond to each other, we have $E_{\alpha_\ell^H} = E_\rho$, and hence $E_\rho \subset E$. As there are only finitely many possibilities for E_ρ, we deduce that $\mathrm{length}_E(F)$ is bounded universally for all quasi-models of (F, G, Γ). $\square$

Continuing with the proof of (a), we now choose a subring $E_0 \subset F$ such that there exists a quasi-model of (F, G, Γ) of the form (E_0, H_0, φ_0) and such that $\mathrm{length}_{E_0}(F)$ is as large as possible. Afterwards among the quasi-models of the form (E_0, H_0, φ_0) we select one for which the number of simple summands of F where φ_0 is not an isomorphism is as large as possible. I claim that this is a minimal quasi-model.

To show this consider any weak quasi-model (E, H, φ) of $(E_0, H_0, \varphi_0^{-1}(\Gamma))$. Then the composite $(E, H, \varphi_0 \circ \varphi)$ is again a weak quasi-model. Applying Proposition 3.3 to this triple we obtain a ring endomorphism $\tau : F \longrightarrow F$ and a quasi-model (E_1, H_1, φ_1) of (F, G, Γ) such that $E_1 = \tau(E)$. Then we have

$$\mathrm{length}_{E_0}(F) \leq \mathrm{length}_E(F) = \mathrm{length}_{\tau(E)}(\tau(F)) \leq \mathrm{length}_{E_1}(F).$$

Since E_0 was chosen such that $\mathrm{length}_{E_0}(F)$ is maximal among all quasi-models, these inequalities are in fact equalities. This implies $E = E_0$ and that τ is an isomorphism. From Proposition 3.3 it follows that $(E, H, \varphi_0 \circ \varphi)$ is already a quasi-model of (F, G, Γ). Suppose that φ fails to be an isomorphism over some simple summand F_i of F. By the classification of inseparable isogenies in Theorem 1.7 and the fact that the derivative of $\varphi_0 \circ \varphi$ vanishes nowhere the map φ_0 must be an isomorphism over F_i. Thus the number of simple summands of F where $\varphi_0 \circ \varphi$ is not an isomorphism is strictly greater than for φ_0, contradicting the choice of (E_0, H_0, φ_0). Summarizing, we have proved that for any weak quasi-model (E, H, φ) of $(E_0, H_0, \varphi_0^{-1}(\Gamma))$ we have $E_0 = E$ and φ is an isomorphism. In other words $(E_0, H_0, \varphi_0^{-1}(\Gamma))$ is minimal, and hence (E_0, H_0, φ_0) is a minimal quasi-model of (F, G, Γ), as desired. This proves the existence part (a) of Theorem 3.6. $\square$

The proof of the uniqueness part (b) is deferred to the end of this section. The intervening results do not depend on it. Let us only note the following consequence of uniqueness.

Corollary 3.8: *Consider a normal subgroup $\Gamma' \subset \Gamma$ which is also fiberwise Zariski dense in G. If (F, G, Γ) is minimal, then so is (F, G, Γ').*

Proof: Consider any minimal quasi-model (E, H, φ) of (F, G, Γ'). For any $\gamma \in \Gamma$ let $\mathrm{int}(\gamma)$ be the automorphism $g \mapsto \gamma g \gamma^{-1}$ of G. Then $(E, H, \mathrm{int}(\gamma) \circ \varphi)$ is another minimal quasi-model of (F, G, Γ'). By Theorem 3.6 (b) there exists an automorphism ι_γ of H, defined over E, such that $\mathrm{int}(\gamma) \circ \varphi = \varphi \circ \iota_\gamma$. Since φ induces an isomorphism between the groups of outer automorphisms of H and G, we find that ι_γ is an inner automorphism. As H is adjoint, it follows that $\iota_\gamma = \mathrm{int}(\delta)$ for some $\delta \in H(E)$. It follows that $\gamma = \varphi(\delta) \in \varphi(H(E))$, that is, (E, H, φ) is a quasi-model of (F, G, Γ). By minimality of the latter, we have $E = F$ and φ is an isomorphism. Thus $(F, G, \Gamma') \cong (E, H, \varphi^{-1}(\Gamma'))$ is minimal, as desired. $\square$

In the rest of this section we analyze the minimal case. We begin with an easy but useful projection property. Consider a direct summand F' of F; we do not assume that

F' is simple. Let G' be the part of G that lies over F', and let Γ' denote the image of Γ under the projection map $G(F) \longrightarrow G'(F')$. Then the triple (F', G', Γ') satisfies the same assumptions as (F, G, Γ). We say that it is obtained by projection to the summand F'.

Proposition 3.9: *If (F, G, Γ) is minimal, then so is (F', G', Γ').*

Proof: From any weak quasi-model (E', H', φ') of (F', G', Γ') we can construct a weak quasi-model of (F, G, Γ), as follows. Write $F = F' \oplus F''$ and put $E := E' \oplus F''$. Let H be the linear algebraic group over E which coincides with H' over E' and with G over F''. Let $\varphi : H \times_E F \longrightarrow G$ be the isogeny which coincides with φ' over F' and with the identity over F''. Clearly (E, H, φ) is a weak quasi-model of (F, G, Γ). By the minimality of (F, G, Γ) we have $E = F$ and φ is an isomorphism. This implies that $E' = F'$ and φ' is an isomorphism. Hence (F', G', Γ') is minimal, as desired. $\square$

Next we study the rings E_ρ defined in the preceding section for various fiberwise absolutely irreducible representations ρ obtained from the adjoint representation. First, we look at a single irreducible subquotient.

Proposition 3.10: *Suppose that (F, G, Γ) is minimal and that F is a field. Let ρ be a non-constant absolutely irreducible representation of G which occurs as subquotient of the adjoint representation of G. Then $E_\rho = F$ unless*

- $\mathrm{char}(F) = 2$,
- *the root system of G has type C_n for some $n \geq 1$, and*
- *ρ is equivalent to the representation α_ℓ^G defined in Section 1.*

In that case E_ρ is either equal to F or to $F^2 := \{x^2 \mid x \in F\}$.

Proof: From Theorem 2.14 we have a model G_ρ of $\rho(G)$ over E_ρ, such that $\rho(\Gamma)$ corresponds to a subgroup of $G_\rho(E_\rho)$. When ρ is faithful, we obtain a quasi-model of (F, G, Γ) of the form $(E_\rho, G_\rho, \ldots)$. In that case the minimality assumption implies $E_\rho = F$, and we are done. In the general case we have to argue more indirectly.

Lemma 3.11: *Suppose that F is separable over E_ρ. Then there exists an algebraic group H over E_ρ and an isomorphism $\varphi : H \times_{E_\rho} F \xrightarrow{\sim} G$ such that $\Gamma \subset \varphi(H(E_\rho))$.*

Proof: Let H be the Zariski closure of $\Gamma \subset G(F) = (\mathcal{R}_{F/E_\rho} G)(E)$ inside $\mathcal{R}_{F/E_\rho} G$. By the universal property of the Weil restriction the inclusion $H \hookrightarrow \mathcal{R}_{F/E_\rho} G$ corresponds to a homomorphism $\varphi : H \times_{E_\rho} F \longrightarrow G$. The condition on Γ being clear by construction, it remains to prove that φ is an isomorphism. Let $\bar{E}_\rho$ denote a separable closure of E_ρ. Since F is a finite separable extension of E_ρ, we have

$$\big(\mathcal{R}_{F/E_\rho} G\big) \times_{E_\rho} \bar{E}_\rho \cong \prod_\tau G \times_{F,\tau} \bar{E}_\rho$$

where τ runs through $\mathrm{Hom}_{E_\rho}(F, \bar{E}_\rho)$. In particular $\mathcal{R}_{F/E_\rho} G$ is a connected semisimple group and the natural homomorphism

$$\mathcal{R}_{F/E_\rho} G \longrightarrow \mathcal{R}_{F/E_\rho} \rho(G) \cong \mathcal{R}_{F/E_\rho}(G_\rho \times_{E_\rho} F)$$

is a totally inseparable isogeny. By construction the image of H is the subgroup $G_\rho \subset \mathcal{R}_{F/E_\rho}(G_\rho \times_{E_\rho} F)$. It follows that the map $H \longrightarrow G_\rho$ is a totally inseparable isogeny and hence H is an absolutely simple connected semisimple group. It also follows that φ is an isogeny. To show that it is an isomorphism note that we have a closed immersion

$$H \times_{E_\rho} \bar{E}_\rho \hookrightarrow \prod_\tau G \times_{F,\tau} \bar{E}_\rho.$$

The projection to each factor is an isogeny. By Corollary 1.8 one of these isogenies dominates all the others. Since the total map is a monomorphism, one of these isogenies must be an isomorphism. But then each of them is an isomorphism, and hence φ is one, as desired. $\square$

In the case of Lemma 3.11 the triple (E_ρ, H, φ) is a quasi-model of (F, G, Γ), so the minimality assumption implies $E_\rho = F$, as desired.

Lemma 3.12: *Suppose that F is not separable over E_ρ. Set $\sigma : x \mapsto x^p$ denote the Frobenius endomorphism of F, where $p := \mathrm{char}(F)$. Then there exists an absolutely simple adjoint group H over $E := \{x \in F \mid x^p \in E_\rho\}$ and a commutative diagram of isogenies*

$$\begin{array}{ccc} H \times_E F & \xrightarrow{\varphi} & G \\ {\scriptstyle \mathrm{Frob}_p}\big\downarrow & & \big\downarrow \\ \sigma^* H \times_E F & \xrightarrow{\sim} & \rho(G)^{\mathrm{ad}} \end{array}$$

such that $\Gamma \subset \varphi(H(E))$.

Proof: Recall from Assumption 2.1 that F is either a global or a local field. Since F is not separable over E_ρ, it follows that σ^{-1} induces an isomorphism $E_\rho \xrightarrow{\sim} E$. Defining

$$H := (\sigma^{-1})^* G_\rho^{\mathrm{ad}} := G_\rho^{\mathrm{ad}} \times_{E_\rho, \sigma^{-1}} E,$$

we have a canonical isomorphism

$$\sigma^* H = H \times_{E,\sigma} E \cong G_\rho^{\mathrm{ad}} \times_{E_\rho,\mathrm{id}} E,$$

and every E_ρ-valued point of G_ρ^{ad} lifts under Frob_p to an E-valued point of H. By Proposition 1.12 and Theorem 1.7 the Frobenius isogeny of G factors through $\rho(G)^{\mathrm{ad}}$. Consider the composite isogeny

$$H \times_{E,\sigma} F \cong G_\rho^{\mathrm{ad}} \times_{E_\rho,\mathrm{id}} F \xrightarrow{\sim} \rho(G)^{\mathrm{ad}} \longrightarrow \sigma^* G = G \times_{F,\sigma} F.$$

Identifying F through σ^{-1} with its inseparable extension F' of degree p, this corresponds to an isogeny $\varphi : H \times_{E,\mathrm{id}} F' \longrightarrow G \times_{F,\mathrm{id}} F'$ defined over F'. By construction Γ is contained in the image of $H(E)$. Thus φ maps a Zariski dense subgroup of $H(F)$ to $G(F)$. It follows that φ is already defined over F, and we are done. $\square$

In the case of Lemma 3.12 the triple (E, H, φ) is a weak quasi-model of (F, G, Γ). From the minimality assumption we conclude that $E = F$ and that φ is an isomorphism. This means first of all that $E_\rho = F^p$. On the other hand we deduce that $\rho(G)^{\mathrm{ad}} = \sigma^* G$, which by Proposition 1.12 occurs if and only if $p = 2$, the root system of G has type C_n for some $n \geq 1$, and $\rho \cong \alpha_\ell^G$. This finishes the proof of Proposition 3.10. $\square$

Next we compare two irreducible subquotients of the adjoint representation of G.

Proposition 3.13: *Suppose that F is a direct sum of two fields $F_1 \oplus F_2$ and that the projection to each summand $(F_i, G_i, \mathrm{pr}_i(\Gamma))$ is minimal. For each i let ρ_i be a non-constant absolutely irreducible representation of G_i which occurs as subquotient of the adjoint representation of G_i. Put $\rho := \rho_1 \oplus \rho_2$ as representation of G over F. Then we have either*

(a) $E_\rho = E_{\rho_1} \oplus E_{\rho_2}$, or

(b) there exists a quasi-model (E, H, φ) of (F, G, Γ) such that E is a field, φ is an isomorphism, and $\rho = \rho_0 \circ \varphi$ for a representation ρ_0 of H.

Proof: By construction E_ρ is a subring of F whose image in each F_i is equal to E_{ρ_i}. Thus if E_ρ is not a field, we have the case (a). Let us assume that E_ρ is a field; we must then prove (b). Note that the projection maps induce isomorphisms $\pi_i : E_\rho \xrightarrow{\sim} E_{\rho_i}$, and by Proposition 3.10 the latter is equal to F_i or F_i^2. Also recall that by Theorem 2.14 we have an algebraic group G_ρ over E_ρ and an isomorphism

$$G_\rho \times_{E_\rho} F \xrightarrow{\sim} \rho(G)$$

under which $\rho(\Gamma)$ corresponds to a subgroup of $G_\rho(E_\rho)$. For each fiber this amounts to an isomorphism

$$G_\rho \times_{E_\rho, \pi_i} F_i \xrightarrow{\sim} \rho_i(G_i).$$

First suppose that $F_1 \xleftarrow{\sim} E_\rho \xrightarrow{\sim} F_2$. Then the literal analogue of Lemma 3.11 shows everything except that ρ descends to H. But this last assertion follows at once from the fact that by construction ρ comes from a representation of G_ρ and that the isogeny $G \longrightarrow \rho(G)$ descends to an isogeny $H \longrightarrow G_\rho$.

Next consider the case that $F_1^2 \xleftarrow{\sim} E_\rho \xrightarrow{\sim} F_2^2$. This time the literal analogue of Lemma 3.12 shows everything except that ρ descends to H. But now we know that $\rho_i \cong \alpha_\ell^{G_i}$ for both $i = 1, 2$, so ρ descends to $\rho_0 := \alpha_\ell^H$, as desired.

Finally we treat the case $F_1 \xleftarrow{\sim} E_\rho \xrightarrow{\sim} F_2^2$ (the fourth case then follows by symmetry). As in the proof of Lemma 3.12 we have an isomorphism

$$G_\rho^{\mathrm{ad}} \times_{E_\rho, \sigma^{-1} \circ \pi_2} F_2 \xrightarrow{\sim} G_2 \ .$$

Since $\pi_1 : E_\rho \to F_1$ is an isomorphism, we deduce an isogeny

$$\begin{tikzcd}
G_1 \times_{F_1, \sigma^{-1} \circ \pi_2 \circ \pi_1^{-1}} F_2 \arrow[d, "\rho_1^{\mathrm{ad}}"'] & & \\
\rho_1(G_1)^{\mathrm{ad}} \times_{F_1, \sigma^{-1} \circ \pi_2 \circ \pi_1^{-1}} F_2 & G_\rho^{\mathrm{ad}} \times_{E_\rho, \sigma^{-1} \circ \pi_2} F_2 \arrow[l, "\sim"'] \arrow[r, "\sim"] & G_2 \ .
\end{tikzcd}$$

By construction this isogeny maps $\mathrm{pr}_1(\Gamma)$ to $\mathrm{pr}_2(\Gamma)$, so it is part of a quasi-model of $(F_2, G_2, \mathrm{pr}_2(\Gamma))$. By the minimality assumption it must therefore be an isomorphism. On the other hand, by Section 1 the group $\rho_2(G_2)$ is not adjoint, so neither is G_ρ, nor $\rho_1(G_1)$. Since G_2 is adjoint, the isogeny cannot be an isomorphism. This contradiction finishes the proof. $\square$

It is now easy to look at the irreducible subquotients of the adjoint representation of G altogether. Recall that some fibers may have two different interesting irreducible subquotients.

Proposition 3.14: *Consider the algebra*

$$F' := \bigoplus_{i=1}^{m} \begin{cases} F_i \oplus F_i & \textit{if } G_i \textit{ possesses non-standard isogenies,} \\ F_i & \textit{if } G_i \textit{ does not possess non-standard isogenies} \end{cases}$$

over $F = \bigoplus_{i=1}^{m} F_i$*, and the representation of* $G \times_F F'$

$$\rho := \bigoplus_{i=1}^{m} \begin{cases} \alpha_s^{G_i} \oplus \alpha_\ell^{G_i} & \textit{if } G_i \textit{ possesses non-standard isogenies,} \\ \alpha^{G_i} & \textit{if } G_i \textit{ does not possess non-standard isogenies.} \end{cases}$$

Suppose that (F, G, Γ) *is minimal. Then*

$$E_\rho = \bigoplus_{i=1}^{m} \begin{cases} E_{\alpha_s^{G_i}} \oplus E_{\alpha_\ell^{G_i}} & \textit{if } G_i \textit{ possesses non-standard isogenies,} \\ E_{\alpha^{G_i}} & \textit{if } G_i \textit{ does not possess non-standard isogenies.} \end{cases}$$

Proof: By Theorem 2.3 (a) the ring E_ρ is a finite direct sum of fields, and we must show that the decompositions into simple summands of E_ρ and F' correspond to each other. Suppose not. Then there is a simple summand $E_{\rho,0}$ of E_ρ which is not contained in a simple summand of F'. Select two simple summands of F' such that $E_{\rho,0}$ injects into each of them. There are two cases, in each of which we shall establish a contradiction.

Suppose first that these two simple summands of F' lie above two different simple summands F_i, F_j of F. By projecting everything to $F_i \oplus F_j$ and applying Proposition 3.9 we may assume without loss of generality that $F = F_1 \oplus F_2$. Now Proposition 3.13 (b) yields a quasi-model of (F, G, Γ) which contradicts the minimality assumption.

Suppose that the two simple summands of F' lie above the same simple summand F_i of F. By projecting everything to F_i and applying Proposition 3.9 we may assume without loss of generality that F is a field. Then $(F', G \times_F F', \Gamma)$ together with ρ satisfy the conditions of Proposition 3.13. If (E, H, φ) is the quasi-model and ρ_0 the representation provided by Proposition 3.13 (b), we find that by construction ρ_0 is equivalent to both α_s^H and α_ℓ^H. Since these two representations are inequivalent, we have reached a contradiction. $\square$

The preceding results now make it easy to prove the uniqueness of minimal quasi-models.

Proof of Theorem 3.6 (b): Consider two minimal quasi-models (E_1, H_1, φ_1) and (E_2, H_2, φ_2) of (F, G, Γ). We must prove that $E_1 = E_2$ and that there exists a unique isomorphism $\psi : H_1 \xrightarrow{\sim} H_2$, defined over E, such that $\varphi_1 = \varphi_2 \circ \psi$. In fact, the uniqueness of ψ is obvious once it exists.

Let us first assume that F is a field. Then for each $\mu = 1, 2$ the representation α_s^G descends to the representation $\rho_\mu = \alpha_s^{H_\mu}$ or $= \alpha_\ell^{H_\mu}$ of H_μ. Put $\rho := \rho_1 \oplus \rho_2$ as representation of $H_1 \sqcup H_2$ over $E_1 \oplus E_2$. Then the triple $\bigl(E_1 \oplus E_2, H_1 \sqcup H_2, (\varphi_1 \times \varphi_2)^{-1}(\Gamma)\bigr)$ together with ρ satisfies the assumptions of Proposition 3.13. By construction, we have $E_\rho \subset \mathrm{diag}(F) \subset F \oplus F$. Thus we must have the case (b) of Proposition 3.13, i.e. there is a quasi-model (E, H, φ) of $\bigl(E_1 \oplus E_2, H_1 \sqcup H_2, (\varphi_1 \times \varphi_2)^{-1}(\Gamma)\bigr)$ such that E is a field and ρ descends to a representation of H. By projection to each summand and the minimality assumption the induced maps $E \hookrightarrow E_\mu$ and $H \times_E E_\mu \longrightarrow H_\mu$ must be isomorphisms. In particular, $\bigl(E, H, \varphi^{-1}(\varphi_1 \times \varphi_2)^{-1}(\Gamma)\bigr)$ itself is minimal, which by Proposition 3.10 implies that E_ρ is equal to E or to E^2. Since E_ρ is contained in the diagonal $\mathrm{diag}(F)$, it follows easily that $E \subset \mathrm{diag}(F)$ as well. This shows that $E_1 = E_2$ and that the two maps $E \xrightarrow{\sim} E_\mu$ are the same. The isomorphism ψ is obtained from the two isomorphisms $H \times_E E_\mu \longrightarrow H_\mu$.

Now we consider the general case. From Proposition 3.9 and the field case just proved we deduce that the images of E_1, E_2 in any given simple summand F_i of F are equal, and that φ_1 is an isomorphism over F_i if and only if φ_2 has that property. This implies that there is a subquotient representation ρ of the adjoint representation of G such that $\rho \circ \varphi_\mu \cong \alpha_\ell^{H_\mu} \otimes_{E_\mu} F$ for both $\mu = 1, 2$. From Proposition 3.14 it follows that each E_μ is totally inseparable over E_ρ. Since both E_μ have the same image in each F_i, one easily deduces that $E_1 = E_2$. Finally, the isomorphism ψ is constructed by combining the given isomorphisms for all simple summands of $E_1 = E_2$. $\square$

4. The Augmentation Ideal in the Adjoint Representation

We keep the notations and assumptions of the preceding section. The aim of this section is to give a full qualitative characterization of the augmentation ideal $\mathcal{J}_\rho$ defined in Definition 2.2 (b), where ρ is the adjoint representation of one of several groups related to G. Here we assume that (F, G, Γ) is minimal. From Proposition 3.14, together with Theorem 2.3, we already have the best possible result for the semisimplification of the adjoint representation. Thus the problem will be to characterize certain nilpotent elements in $\mathcal{J}_\rho$.

To set up the framework, we fix another fiberwise absolutely simple adjoint group H over F and an isogeny $\varphi : G \to H$ with nowhere vanishing derivative. Let $\tilde{G}$ and $\tilde{H}$ denote their universal coverings. Then we have a commutative diagram of isogenies

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\tilde{\varphi}} & \tilde{H} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\omega} \\ G & \xrightarrow{\varphi} & H\,. \end{array} \tag{4.1}$$

Recall from Proposition 1.10 that the Lie algebras $\tilde{\mathfrak{g}} := \operatorname{Lie}\tilde{G}$ and $\mathfrak{g} := \operatorname{Lie} G$ fit together to a single representation $\hat{\mathfrak{g}}$. Let $\hat{\mathfrak{h}}$ be the analogous representation combining $\tilde{\mathfrak{h}} := \operatorname{Lie}\tilde{H}$ and $\mathfrak{h} := \operatorname{Lie} H$. Then we have a commutative diagram

$$\begin{array}{ccc} \tilde{\mathfrak{g}} & \xrightarrow{d\tilde{\varphi}} & \tilde{\mathfrak{h}} \\ \downarrow & & \downarrow \\ \hat{\mathfrak{g}} & & \hat{\mathfrak{h}} \\ {\scriptstyle d\varpi}\downarrow & & \downarrow \\ \mathfrak{g} & \xrightarrow{d\varphi} & \mathfrak{h}\,. \end{array}$$

Let $\hat{\rho}$ denote the representation of G on $\hat{\mathfrak{h}}$. Let $\sigma : F \to F$ and $\mathrm{Frob} : G \to \sigma^* G$ be as in (1.13). When φ is not an isomorphism, a certain part of the representation $\hat{\rho}$ factors through Frob.

Proposition 4.2: *There is a natural commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{im}(d\tilde{\varphi}) & \longrightarrow & \tilde{\mathfrak{h}} & \longrightarrow & \ker(d\tilde{\varphi}) \otimes_{F,\sigma} F & \longrightarrow & 0 \\ & & \| & & \cap & & \cap & & \\ 0 & \longrightarrow & \operatorname{im}(d\tilde{\varphi}) & \longrightarrow & \hat{\mathfrak{h}} & \longrightarrow & \ker(d\varphi \circ d\varpi) \otimes_{F,\sigma} F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{im}(d\varphi) & \longrightarrow & \mathfrak{h} & \longrightarrow & \ker(d\varphi) \otimes_{F,\sigma} F & \longrightarrow & 0\,. \end{array}$$

Proof: In the first and the third line, the horizontal maps on the right hand side are the derivatives of the homomorphisms $\tilde{H} \longrightarrow \sigma^*\tilde{G}$ and $H \longrightarrow \sigma^* G$, taking into account the natural isomorphisms $\operatorname{Lie}\sigma^* G \cong (\operatorname{Lie} G) \otimes_{F,\sigma} F$ and $\operatorname{Lie}\sigma^*\tilde{G} \cong (\operatorname{Lie}\tilde{G}) \otimes_{F,\sigma} F$. The exactness follows, for instance, by a straightforward case-by-case analysis using the information in Proposition 1.11. The same remarks apply to the second line. For instance, one shows that the term on the left of the second line is the smallest submodule of $\hat{\mathfrak{h}}$ which contains the subquotient $\bar{\mathfrak{g}}_\ell$ if φ is an isomorphism, and the subquotient $\bar{\mathfrak{g}}_s$ otherwise. One also checks easily that the quotient injects into $\hat{\mathfrak{g}} \otimes_{F,\sigma} F$. The rest is left to the reader. $\square$

The following definition collects all the qualitative information about the augmentation ideal $\mathcal{J}_{\hat\rho}$ that is available so far. As before we write G_i for the fiber of G over F_i. Let $\bar{\mathfrak{g}}_i$, $\bar{\mathfrak{g}}_{i,s}$, and $\bar{\mathfrak{g}}_{i,\ell}$ be the representation spaces of α^{G_i}, $\alpha_s^{G_i}$, and $\alpha_\ell^{G_i}$, respectively.

Definition 4.3: *Let $J_{\hat\rho}$ be the set of all $x \in \mathrm{End}_F(\hat{\mathfrak{h}})$ satisfying the following conditions:*

(a) x maps each G-invariant F-submodule of $\hat{\mathfrak{h}}$ into itself,

(b) x annihilates each G-invariant F-subquotient of $\hat{\mathfrak{h}}$ on which G acts trivially,

(c) x maps the subspace $\ker(d\varphi \circ d\varpi) \otimes_{F,\sigma} \sigma(F)$ of $\ker(d\varphi \circ d\varpi) \otimes_{F,\sigma} F$ into itself, and

(d) for each simple summand F_i with $\mathrm{char}(F_i) = 2$ and for which the root system of G_i has type C_n for some $n \geq 1$, the endomorphism of $\bar{\mathfrak{g}}_{i,\ell}$ induced by x lies in $B_{\alpha_\ell^{G_i}}$.

Recall that (F, G, Γ) is assumed to be minimal. Therefore, taking into account Proposition 3.10, this definition shows that $J_{\hat\rho}$ is a $\sigma(F)$-module. Clearly, we have $\mathcal{J}_{\hat\rho} \subset J_{\hat\rho}$. The main result of this section is the following theorem.

Theorem 4.4: *There is a subring $\mathcal{O}' \subset F$ with the following properties:*

(a) $\mathcal{O}'$ is finitely generated in the global case, resp. open compact in the local case,

(b) the total ring of quotients of $\mathcal{O}'$ is F,

(c) $\mathcal{J}_{\hat\rho}$ is a module of finite type over $\sigma(\mathcal{O}')$, and

(d) $\sigma(F) \cdot \mathcal{J}_{\hat\rho} = J_{\hat\rho}$.

In the local case the theorem is equivalent to saying that $\mathcal{J}_{\hat\rho}$ is open compact in $J_{\hat\rho}$.

Proof: First we show (a–c). Let $\mathrm{Rad}(J_{\hat\rho})$ denote the radical of $J_{\hat\rho}$, that is the kernel of the action of $J_{\hat\rho}$ in the semisimplification $\hat\rho^{\mathrm{ss}}$ of $\hat\rho$. Let $\mathcal{O}_{\alpha_\ell^H}$ be as in Theorem 2.3, and let $\tilde{\mathcal{O}}$ be its normalization in F. From Proposition 2.15 we see that for each non-constant irreducible subquotient ρ' of $\hat\rho$, defined over a simple summand F_i of F, the image of $\tilde{\mathcal{O}}$ in F_i is of finite type as module over $\mathcal{O}_{\rho'}$. Therefore there is a subring $\mathcal{O}' \subset \tilde{\mathcal{O}}$ of finite index such that $\mathcal{J}_{\hat\rho}$ modulo $\mathrm{Rad}(J_{\hat\rho})$ is a $\sigma(\mathcal{O}')$-module. By Theorem 2.3 (c) this module is of finite type. In the proof of (d) below we shall show that

$$\sigma(F) \cdot \big(\mathrm{Rad}(J_{\hat\rho})^n \cap \mathcal{J}_{\hat\rho}\big) = \mathrm{Rad}(J_{\hat\rho})^n \tag{4.5}$$

for all $n \geq 0$. On the successive quotients $\mathrm{Rad}(J_{\hat\rho})^n / \mathrm{Rad}(J_{\hat\rho})^{n+1}$ the left and right action of $J_{\hat\rho}$ factors through $J_{\hat\rho}/\mathrm{Rad}(J_{\hat\rho})$. Therefore by downward induction on n we easily deduce that $\mathrm{Rad}(J_{\hat\rho})^n \cap \mathcal{J}_{\hat\rho}$ is a $\sigma(\mathcal{O}')$-module for all n, provided that $\mathcal{O}'$ is made a little smaller at each step when necessary. The fact that this module is finitely generated is proved just as in Theorem 2.16. This shows assertions (a–c) modulo Equation (4.5).

Coming to the proof of (d), we first show that

$$\sigma(F)\cdot\mathcal{J}_{\hat\rho}\equiv J_{\hat\rho}\quad\text{modulo }\operatorname{Rad}(J_{\hat\rho}). \tag{4.6}$$

This assertion concerns only the representation $\hat\rho^{\mathrm{ss}}$. By Definition 4.3 (b) we may discard all those simple subquotients on which G acts trivially. The remaining simple subquotients are precisely those listed in Proposition 3.14, except for the fact that the subquotients $\bar{\mathfrak{g}}_{i,s}$ that do not occur in $\operatorname{im}(d\tilde\varphi)$ are replaced by their pullback via Frobenius $\bar{\mathfrak{g}}_{i,s}\otimes_{F_i,\sigma}F_i$. This fact is accounted for by Definition 4.3 (c). Moreover, Definition 4.3 (d) takes care of the symplectic case of Proposition 3.10. Thus the equation (4.6) follows from Theorem 2.3 and Proposition 3.14.

Next we reduce everything to an equation modulo the square of the radical. Suppose that we know

$$\sigma(F)\cdot\big(\operatorname{Rad}(J_{\hat\rho})\cap\mathcal{J}_{\hat\rho}\big)\equiv\operatorname{Rad}(J_{\hat\rho})\quad\text{modulo }\operatorname{Rad}(J_{\hat\rho})^2. \tag{4.7}$$

By induction on n we find that

$$\sigma(F)\cdot\big(\operatorname{Rad}(J_{\hat\rho})^n\cap\mathcal{J}_{\hat\rho}\big)\equiv\operatorname{Rad}(J_{\hat\rho})^n\quad\text{modulo }\operatorname{Rad}(J_{\hat\rho})^{n+1}$$

for all $n\geq 1$. By (4.6) the same holds for $n=0$. Thus by downward induction on n we deduce (4.5) for all n. For $n=0$ this is just the assertion of Theorem 4.4 (d).

It remains to prove Equation (4.7). For this we first look at all subquotients of $\hat{\mathfrak{h}}$ which are non-trivial extensions of two irreducible representations. The precise form of the result will depend on where in $\hat{\mathfrak{h}}$ this subquotient occurs. The following result will cover all cases.

Lemma 4.8: *Let $0\to V'\to V\to V''\to 0$ be a non-trivial extension of G-modules where V' and V'' are absolutely irreducible. Assume that one of the following two conditions holds:*
(a) V is a subquotient of $\hat{\mathfrak{h}}$ but not of $\ker(d\varphi\circ d\varpi)\otimes_{F,\sigma}F$.
(b) V is a subquotient of $\ker(d\varphi\circ d\varpi)$.
Let ρ denote the representation of G on V, and A_ρ the stabilizer of V' in $\operatorname{End}_F(V)$. Then

$$\sigma(F)\cdot\big(\operatorname{Rad}(A_\rho)\cap\mathcal{J}_\rho\big)=\operatorname{Rad}(A_\rho).$$

Proof of Equation (4.7): Here I advise the gentle reader to view the representation as written in terms of block matrices adapted to a Jordan-Hölder series of $\hat{\mathfrak{h}}$. The part over each F_i has length at most 5, so that scribbling a few small matrices can be of great help in visualizing the argument. First, we see at once that $\operatorname{Rad}(J_{\hat\rho})/\operatorname{Rad}(J_{\hat\rho})^2$ is a direct sum of its images on various subquotients which are extensions of length 2. Consider the decomposition of $\operatorname{Rad}(J_{\hat\rho})/\operatorname{Rad}(J_{\hat\rho})^2$ into isotypic components under the simultaneous left and right action of $J_{\hat\rho}/\operatorname{Rad}(J_{\hat\rho})$. By (4.6) it suffices to prove that $\sigma(F)\cdot\big(\operatorname{Rad}(J_{\hat\rho})\cap\mathcal{J}_{\hat\rho}\big)$ surjects onto each isotypic component.

If the isotypic component arises from exactly one subquotient of $\hat{\mathfrak{h}}$ of length 2, the surjectivity follows directly from Lemma 4.8. Indeed, this is obvious in the case of Lemma 4.8 (a); and in the case (b) the only difference is that one has to apply an extra tensor product $\otimes_{F,\sigma} F$.

Suppose that the isotypic component comes from more than one extension of length 2. By the list in Proposition 1.11 this happens only for a fiber with $\operatorname{char}(F_i) = 2$ and where the root system of G_i has type D_n for some even integer n. In that case we are looking at an extension of $\bar{\mathfrak{g}}_i$ with two copies of the trivial representation of dimension 1. Since that extension does not split even partially, the surjectivity again follows from Lemma 4.8. This proves Equation (4.7) modulo Lemma 4.8. $\square$

Proof of Lemma 4.8: Since the extension is non-trivial, both V' and V'' must be vector spaces over the same simple summand of F. After projecting to that summand and using Proposition 3.9 we may suppose without loss of generality that F is a field. Let ρ', ρ'' denote the representation of G on V', resp. V''. Since the extension is non-trivial and G is semisimple, at least one of ρ', ρ'' is a non-constant representation. As every non-constant irreducible subquotient occurs only once in $\hat{\mathfrak{h}}$ or $\hat{\mathfrak{g}}$, we deduce that $\rho' \not\cong \rho''$. From Theorem 2.16 we now obtain that

$$F \cdot \big(\operatorname{Rad}(A_\rho) \cap \mathcal{J}_\rho\big) = \operatorname{Rad}(A_\rho). \tag{4.9}$$

We need the same equation with F replaced by $\sigma(F)$. Recall that $\operatorname{Rad}(A_\rho) \cap \mathcal{J}_\rho$ is a module over the ring $\mathcal{O}$ defined in Theorem 2.16. Thus, if $\operatorname{Quot}(\mathcal{O}) = F$, we have $\sigma(F) \cdot \mathcal{O} = F$, which directly implies the desired strengthening of (4.9). Thus in the remainder of the proof we assume that $\operatorname{Quot}(\mathcal{O}) \neq F$. Then each of $E_{\rho'}$ and $E_{\rho''}$ (defined whenever the corresponding representation ρ' resp. ρ'' is non-constant) is a proper subfield of F. By classifying the possible cases, the following sublemma extracts some useful common features.

Sublemma 4.10: *If* $\operatorname{Quot}(\mathcal{O}) \neq F$*, we must have:*

(a) $\operatorname{char}(F) = 2$,

(b) $\operatorname{Quot}(\mathcal{O}) = \sigma(F)$,

(c) the group $\rho(G)$ is adjoint, and

(d) the representation ρ does not factor through Frobenius.

Proof: Suppose first that ρ' is constant. Since $\ker(d\varphi \circ d\varpi)$ does not have a non-zero constant representation as a quotient, we must be in the case (b) of Lemma 4.8. Thus ρ'' is a non-constant subquotient of $\hat{\mathfrak{g}}$. Since $E_{\rho''} \neq F$, we must be in the symplectic case of Proposition 3.10, with $\rho'' \cong \alpha_\ell^G$ and $E_{\rho''} = \sigma(F)$. If $\operatorname{rank}(G) = 1$, the second diagram in Proposition 1.11 (b) shows that $V \cong \tilde{\mathfrak{g}}$. Then ρ is faithful, by Proposition 1.12 (c), and all the desired assertions are proved. If $\operatorname{rank}(G) > 1$, the relevant diagram in Proposition 1.11 (c) shows that $V \cong \tilde{\mathfrak{g}}_\ell$ and hence $\rho(G) \cong G^\sharp$ in the notation of Proposition 1.12 (d). Again all assertions are proved in this case.

Suppose that ρ' is non-constant. In each of the cases of Lemma 4.8 this must be a subquotient of $\hat{\mathfrak{g}}$. Therefore we are again in the symplectic case of Proposition 3.10; this time with $\rho' \cong \alpha_\ell^G$ and $E_{\rho'} = \sigma(F)$. If $\mathrm{rank}(G) = 1$, as above we find $V \cong \mathfrak{g}$ and that ρ is faithful, whence the lemma. Otherwise the relevant diagrams in Proposition 1.11 (c) leave two cases. If ρ'' is constant, then $V \cong \mathfrak{g}_\ell$ and hence again $\rho(G) \cong G^\sharp$ in the notation of Proposition 1.12 (d). If ρ'' is non-constant, then φ must be a non-standard isogeny and $V \cong \bar{\mathfrak{h}}$. By Proposition 1.12 (a) the representation of H on $\bar{\mathfrak{h}}$ is faithful, so that again we obtain $\rho(G) \cong G^\sharp$. Thus the sublemma is proved in all cases. $\square$

The rest of the proof of Lemma 4.8 follows the same principle as that of Proposition 3.10. Namely, assuming that Lemma 4.8 is false we shall construct a quasi-model of (F, G, Γ) which violates the minimality condition. The method for constructing a model over a subring follows that of Theorem 2.14, except that now the ring is no longer semisimple.

Sublemma 4.11: *Suppose that the assertion of Lemma 4.8 is false. Set $B_\rho := \sigma(F) \cdot \mathcal{B}_\rho \subset A_\rho$. Then the canonical homomorphism $B_\rho \otimes_{\sigma(F)} F \longrightarrow A_\rho$ is an isomorphism.*

Proof: First we look at the semisimplification of A_ρ. Since ρ' and ρ'' are not equivalent, the image of B_ρ in $A_\rho/\mathrm{Rad}(A_\rho)$ is isomorphic to the direct sum of its images in $\mathrm{End}_F(V')$ and in $\mathrm{End}_F(V'')$. From Sublemma 4.10 (b) we already know that this image is a model of $A_\rho/\mathrm{Rad}(A_\rho)$ over $\sigma(F)$. This is seen directly when ρ' resp. ρ'' is constant; otherwise we use Theorem 2.3 (a.iii).

Next choose an element $e \in \mathcal{B}_\rho$ which acts as two different scalars on V' and V''. Decomposing A_ρ under left and right multiplication by e we easily find that

$$\mathrm{Rad}(B_\rho) := \mathrm{Rad}(A_\rho) \cap B_\rho = \sigma(F) \cdot \big(\mathrm{Rad}(A_\rho) \cap \mathcal{J}_\rho\big).$$

To finish the proof it thus suffices to show that the homomorphism

$$\mathrm{Rad}(B_\rho) \otimes_{\sigma(F)} F \longrightarrow \mathrm{Rad}(A_\rho) \tag{4.12}$$

is an isomorphism. We know already that it is surjective. Let $S \subset \mathrm{End}_F(\mathrm{Rad}(A_\rho))$ denote the $\sigma(F)$-subalgebra generated by left and right multiplication by all elements of B_ρ. Since $F \cdot B_\rho \twoheadrightarrow A_\rho/\mathrm{Rad}(A_\rho)$, we easily find that $F \cdot S = \mathrm{End}_F(\mathrm{Rad}(A_\rho))$. It follows that $\mathrm{Rad}(A_\rho)$ is an S-module of length at most 2. Now by construction $\mathrm{Rad}(B_\rho)$ is a non-zero S-submodule, and by assumption it is a proper submodule. This implies that $\mathrm{length}_S(\mathrm{Rad}(B_\rho)) = 1$ and $\mathrm{length}_S(\mathrm{Rad}(A_\rho)) = 2$. Finally, from Sublemma 4.10 (a) we infer that $\dim_{\sigma(F)}(F) = 2$. This shows that the homomorphism (4.12) is a surjective homomorphism between two S-modules of length 2. Therefore it is an isomorphism, as desired. $\square$

To finish the proof of Lemma 4.8 let us assume that the assertion is false. Then both preceding sublemmas apply. Note that by construction we have $\rho(\Gamma) \subset B_\rho^*$. Therefore we can argue as in Theorem 2.14, obtaining a linear algebraic group G_ρ over $\sigma(F)$ and an isomorphism $G_\rho \times_{\sigma(F)} F \xrightarrow{\sim} \rho(G)$ such that $\rho(\Gamma)$ corresponds to a subgroup of $G_\rho(\sigma(F))$. By Sublemma 4.10 the group G_ρ is adjoint, and the Frobenius isogeny of G factors through an isogeny $\rho(G) \longrightarrow \sigma^* G$ which is not an isomorphism. Thus, following the procedure of Lemma 3.12, we obtain a quasi-model $(F, G_\rho \times_{\sigma(F),\sigma^{-1}} F, \ldots)$ of (F, G, Γ) in which the isogeny is not an isomorphism. This contradicts the minimality of (F, G, Γ). This finishes the proof of Lemma 4.8 and thus of Theorem 4.4. $\square$

Consequences of Theorem 4.4: We formulate two special results that will be needed later on. To fix notations, let τ be the endomorphism of F which on each simple summand F_i is the identity if φ is an isomorphism over F_i, and equal to the Frobenius endomorphism σ otherwise. Clearly the first and the last line in Proposition 4.2 remains exact when σ is replaced by τ. Define $\tau(F)$-submodules $\tilde{W} \subset \tilde{\mathfrak{h}}$ and $W \subset \mathfrak{h}$ so that the following diagrams have exact rows:

$$(4.13)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{im}(d\tilde{\varphi}) & \longrightarrow & \tilde{\mathfrak{h}} & \longrightarrow & \ker(d\tilde{\varphi}) \otimes_{F,\tau} F & \longrightarrow & 0 \\ & & \| & & \cup & & \cup & & \\ 0 & \longrightarrow & \mathrm{im}(d\tilde{\varphi}) & \longrightarrow & \tilde{W} & \longrightarrow & \ker(d\tilde{\varphi}) \otimes_{F,\tau} \tau(F) & \longrightarrow & 0 \end{array}$$

$$(4.14)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{im}(d\varphi) & \longrightarrow & \mathfrak{h} & \longrightarrow & \ker(d\varphi) \otimes_{F,\tau} F & \longrightarrow & 0 \\ & & \| & & \cup & & \cup & & \\ 0 & \longrightarrow & \mathrm{im}(d\varphi) & \longrightarrow & W & \longrightarrow & \ker(d\varphi) \otimes_{F,\tau} \tau(F) & \longrightarrow & 0\,. \end{array}$$

The homomorphism $d\omega$ induces a map $\tilde{W} \to W$, and $J_{\hat{\rho}}$ can be interpreted as a $\tau(F)$-submodule of $\mathrm{Hom}_{\tau(F)}(W, \tilde{W})$. In the following, we let $\mathcal{O}' \subset F$ be the subring given by Theorem 4.4. By a $\tau(\mathcal{O}')$-lattice in a $\tau(F)$-module of finite type we mean a finitely generated $\tau(\mathcal{O}')$-submodule which generates the total space over $\tau(F)$. In the same sense, the main content of Theorem 4.4 is that $\mathcal{J}_{\hat{\rho}}$ is a $\sigma(\mathcal{O}')$-lattice in $J_{\hat{\rho}}$.

For the first special result we consider the quotient module $\tilde{\mathfrak{h}} \twoheadrightarrow \bar{\mathfrak{h}}_\ell$ given by Proposition 1.11, and let $\bar{W}_\ell$ denote the image of $\tilde{W}$ in $\bar{\mathfrak{h}}_\ell$. Let $\tilde{\rho}$ denote the representation of G on $\tilde{\mathfrak{h}}$.

Lemma 4.15: *Let $J_{\tilde{\rho}}$ be the image of $J_{\hat{\rho}}$ in $\mathrm{End}_F(\tilde{\mathfrak{h}})$. Then we have*

$$\big(\mathrm{Hom}_{\tau(F)}(\bar{W}_\ell, \tilde{W}) \cap J_{\tilde{\rho}}\big) \cdot \tau(F) = \mathrm{Hom}_{\tau(F)}(\bar{W}_\ell, \tilde{W}).$$

Proof: Since everything decomposes according to the simple summands of F, we may assume that F is a field. Suppose first that φ is an isomorphism. Then by construction we have $\tilde{W} = \tilde{\mathfrak{h}} = \tilde{\mathfrak{g}}$ and $\bar{W}_\ell = \bar{\mathfrak{h}}_\ell = \bar{\mathfrak{g}}_\ell$, and the latter is the unique irreducible quotient module of the former. Going through Definition 4.3, we find that $\mathrm{Hom}_{\tau(F)}(\bar{W}_\ell, \tilde{W}) \subset J_{\tilde{\rho}}$, except in the symplectic case of Definition 4.3 (d). In that case, the weaker statement of the lemma still follows, using the equality $B_{\alpha^G_\ell} \cdot F = \mathrm{End}_F(\bar{\mathfrak{g}}_\ell)$ of Theorem 2.3 (a.iii).

If φ is not an isomorphism, we have $\tau = \sigma$ and $\bar{W}_\ell \cong \bar{\mathfrak{g}}_s \otimes_{F,\sigma} \sigma(F)$. Going through Definition 4.3 it is straightforward to see that $\mathrm{Hom}_{\sigma(F)}(\bar{W}_\ell, \tilde{W}) \subset J_{\tilde{\rho}}$. This proves Lemma 4.15. $\square$

Since the image of $\mathcal{J}_{\hat{\rho}}$ in $\mathrm{End}_F(\tilde{W})$ is just $\mathcal{J}_{\tilde{\rho}}$, from Theorem 4.4 we now immediately obtain:

Corollary 4.16: $\mathrm{Hom}_{\tau(F)}(\bar{W}_\ell, \tilde{W}) \cap \mathcal{J}_{\tilde{\rho}}$ *generates a* $\tau(\mathcal{O}')$*-lattice in* $\mathrm{Hom}_{\tau(F)}(\bar{W}_\ell, \tilde{W})$.

The second corollary concerns the following quotient module of $\mathfrak{h}$. It suffices to define it fiber by fiber, so let us suppose that F is a field. Then, in the notation of Proposition 1.11, we set

$$\mathfrak{h}_{\ell t} := \begin{cases} \mathfrak{z}^* & \text{if } \mathrm{char}(F) = 2 \text{ and the root system of } H \\ & \qquad \text{has type } C_n \text{ for some } n \geq 1; \text{ otherwise:} \\ d\psi(\mathfrak{h}) & \text{if } H \text{ has a non-standard isogeny } \psi : H \to H^\sharp, \\ \mathfrak{h} & \text{if it does not.} \end{cases} \tag{4.17}$$

Here the index $_\ell$ stands for "long roots" and the index $_t$ for "torus". The reason is this: Recall that the weight 0 subspace of $\mathfrak{h}$ comes from a maximal torus. Let $\theta : \mathfrak{h} \twoheadrightarrow \mathfrak{h}_{\ell t}$ denote the canonical projection. If H does not have non-standard isogenies, then $\mathfrak{h}_{\ell t}$ is the smallest quotient module of $\mathfrak{h}$ such that the weight 0 subspace still injects into $\mathfrak{h}_{\ell t}$. If H has a non-standard isogeny $\psi : H \to H^\sharp$, then $\mathfrak{h}_{\ell t}$ is the smallest quotient module of $d\psi(\mathfrak{h})$ such that the weight 0 subspace of $d\psi(\mathfrak{h})$ injects into $\mathfrak{h}_{\ell t}$. This property will be important later on.

Let $W_{\ell t}$ be the image of W in $\mathfrak{h}_{\ell t}$.

Lemma 4.18: *We have*

$$\mathrm{Hom}_{\tau(F)}(W_{\ell t}, \tilde{W}) \subset J_{\hat{\rho}}.$$

Proof: This is proved in the same way as Lemma 4.15, by going through Definition 4.3 and the cases of Proposition 1.11. □

Using Theorem 4.4 we deduce:

Corollary 4.19: $\mathcal{J}_{\hat{\rho}}$ *contains a* $\tau(\mathcal{O}')$*-lattice in* $\mathrm{Hom}_{\tau(F)}(W_{\ell t}, \tilde{W})$.

5. Local Parameters

In this section and the next two we consider the local case. We also assume that F is non-archimedean, i.e. it is a finite direct sum of non-archimedean local fields. Let G be a linear algebraic group over F. The aim of this section is to set up the framework for studying the profinite structure of congruence subgroups of $G(F)$. In order to speak of principal congruence subgroups we must choose local parameters of G at the identity section. We shall also discuss homomorphisms and commutator maps between various groups, which requires that the choices of local parameters are compatible with each other in a certain sense. The present section is devoted to finding such local parameters for the groups G, $\tilde{G}$, etc. that were discussed in the preceding section. Our arguments will be kept at a rather elementary level, avoiding Bruhat-Tits theory, and all of them work essentially fiber-by-fiber over F.

Let us denote the identity section of G by 1. Let $\hat{\mathcal{O}}_{G,1}$ denote the completion of the affine ring of G with respect to the ideal defining the identity section. If G_i denotes the fiber of G over a simple summand F_i, then $\hat{\mathcal{O}}_{G,1}$ is just the direct sum of the completed local rings $\hat{\mathcal{O}}_{G_i,1}$ of the individual fibers. Moreover we have $\hat{\mathcal{O}}_{G_i,1} \cong F_i[[x_{i,1}, \ldots, x_{i,n_i}]]$ for any system $x_{i,1}, \ldots, x_{i,n_i}$ of local parameters of G_i at 1. Note that the morphism $d : G \times_F G \longrightarrow G,\ (g, h) \mapsto gh^{-1}$ induces an F-algebra homomorphism

$$d^* : \hat{\mathcal{O}}_{G,1} \longrightarrow \hat{\mathcal{O}}_{G,1} \hat{\otimes}_F \hat{\mathcal{O}}_{G,1}\,, \tag{5.1}$$

where $\hat{\otimes}_F$ denotes the completed tensor product. The image of any $x_{i,j}$ is then a power series in the variables $x_{i,k} \otimes 1$ and $1 \otimes x_{i,k}$ for all $1 \leq k \leq n_i$.

Let $\mathcal{O}$ be the maximal compact subring of F. This is, of course, just the direct sum of the valuation rings $\mathcal{O}_i \subset F_i$. We are interested in local parameters for which the above power series have coefficients in $\mathcal{O}$. Giving such local parameters amounts essentially to giving a structure of G over $\mathcal{O}$ in a very small neighborhood of the identity. The most natural terminology for this is that of formal schemes (cf. Hartshorne [5] Ch. II §9). The reader should not feel deterred by our use of formal schemes, since we shall need only their most elementary properties and the arguments below will consist only of easy manipulations of power series.

Definition 5.2: *A* smooth formal model (over $\mathcal{O}$) of G *is a formal scheme* $\mathcal{G} = \mathbf{Spf}\,\mathcal{R}$ *where* $\mathcal{R}$ *is an* $\mathcal{O}$*-subalgebra of* $\hat{\mathcal{O}}_{G,1}$ *such that*

(a) there exists a system $x_{i,1}, \ldots, x_{i,n_i}$ *of local parameters of each* G_i *at* 1 *such that* $\mathcal{R} = \bigoplus_{i=1}^{m} \mathcal{O}_i[[x_{i,1}, \ldots, x_{i,n_i}]]$*, and*

(b) the homomorphism d^* *induces a homomorphism* $\mathcal{R} \longrightarrow \mathcal{R}\hat{\otimes}_{\mathcal{O}}\mathcal{R}$*. In other words, the morphism* d *corresponds to a morphism of formal schemes* $\mathcal{G} \times_{\mathcal{O}} \mathcal{G} \longrightarrow \mathcal{G}$*, which makes* $\mathcal{G}$ *into a smooth formal group scheme over* $\mathcal{O}$*.*

Consider a smooth formal model $\mathcal{G}$ over $\mathcal{O}$ of G. The Lie algebra of $\mathcal{G}$ is the relative tangent space at the identity element. Clearly, in terms of the local parameters of Definition 5.2 (a) we have

$$\mathrm{Lie}\,\mathcal{G} = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n_i} \mathcal{O}_i \cdot \Big(\frac{\partial}{\partial x_{i,j}}\Big|_1 \Big).$$

This is an $\mathcal{O}$-lattice in $\mathrm{Lie}\, G$, that is, a finitely generated $\mathcal{O}$-submodule with $F \cdot (\mathrm{Lie}\,\mathcal{G}) = \mathrm{Lie}\, G$.

Let $\varphi : G \to H$ be a homomorphism of linear algebraic groups over F. If we are given smooth formal models $\mathcal{G} = \mathbf{Spf}\,\mathcal{R}$ of G and $\mathcal{H} = \mathbf{Spf}\,\mathcal{S}$ of H, then φ induces a homomorphism of formal group schemes $\mathcal{G} \to \mathcal{H}$ if and only if its transpose $\varphi^* : \hat{\mathcal{O}}_{H,1} \longrightarrow \hat{\mathcal{O}}_{G,1}$ induces an algebra homomorphism $\mathcal{S} \to \mathcal{R}$. Looking at the derivative we find that a necessary condition for this is that $d\varphi$ maps the lattice $\mathrm{Lie}\,\mathcal{G}$ to the lattice $\mathrm{Lie}\,\mathcal{H}$. The induced homomorphism $\mathcal{G} \to \mathcal{H}$ will be denoted again by φ, whenever it exists.

Let us take up the situation of Section 3. Thus G is now a fiberwise absolutely simple adjoint group over F, and Γ is a fiberwise Zariski dense compact subgroup of $G(F)$. Let $\pi : \tilde{G} \to G$ denote the universal covering of G.

Proposition 5.3: *There exist smooth formal models $\mathcal{G}$ of G and $\tilde{\mathcal{G}}$ of $\tilde{G}$ such that*

(a) π extends to a homomorphism $\tilde{\mathcal{G}} \to \mathcal{G}$, and

(b) for every $\gamma \in \Gamma$ the morphism $[\gamma,\ ]^\sim : G \to \tilde{G}$ of (1.2) extends to a morphism $\mathcal{G} \to \tilde{\mathcal{G}}$.

Note that, as a consequence, the morphism $\pi \circ [\gamma,\ ]^\sim$ extends to a morphism $\mathcal{G} \to \mathcal{G}$, and hence the conjugation action of Γ on G extends to an action on $\mathcal{G}$. Likewise $[\gamma,\ ]^\sim \circ \pi$ defines an extension of the conjugation action on $\tilde{G}$ to $\tilde{\mathcal{G}}$.

Proof: There are several aspects to take care of. First we observe that all fibers over F can be considered separately. Thus, for the purposes of this proof we may and do suppose that F is a field. We use the following "Ansatz": Fix local parameters $\xi_1, \ldots, \xi_n$ of G, and local parameters $\tilde{\xi}_1, \ldots, \tilde{\xi}_n$ of $\tilde{G}$. Fix a uniformizing element $t \in \mathcal{O}$, and put $x_i := t^{-N}\xi_i$ and $\tilde{x}_i := t^{-N}\tilde{\xi}_i$ for all i and some integer N. Let

$$\tilde{\mathcal{G}} := \mathbf{Spf}\,\mathcal{O}[[\tilde{x}_1, \ldots, \tilde{x}_n]],$$

$$\mathcal{G} := \mathbf{Spf}\,\mathcal{O}[[x_1, \ldots, x_n]].$$

We shall show that $\tilde{\mathcal{G}}$ and $\mathcal{G}$ have all the desired properties, provided that the ξ_i and $\tilde{\xi}_i$ satisfy certain conditions that are detailed below, and that N is sufficiently large. We shall also show that the conditions on the ξ_i and $\tilde{\xi}_i$ can indeed be met. Consider the $\mathcal{O}$-lattices

$$\Lambda = \bigoplus_{i=1}^{n} \mathcal{O} \cdot \Big(\frac{\partial}{\partial \xi_i}\Big|_1\Big) \subset \mathrm{Lie}\, G,$$

$$\tilde{\Lambda} = \bigoplus_{i=1}^{n} \mathcal{O} \cdot \Big(\frac{\partial}{\partial \tilde{\xi}_i}\Big|_1\Big) \subset \mathrm{Lie}\, \tilde{G}.$$

If $\tilde{\mathcal{G}}$ and $\mathcal{G}$ are smooth formal models, we clearly have $\mathrm{Lie}\,\tilde{\mathcal{G}} = t^N\tilde{\Lambda}$ and $\mathrm{Lie}\,\mathcal{G} = t^N\Lambda$.

Lemma 5.4: *For any $N \gg 0$, both $\tilde{\mathcal{G}}$ and $\mathcal{G}$ are smooth formal models of $\tilde{G}$ resp. of G.*

Proof: We concentrate on $\mathcal{G}$, the proof for $\tilde{\mathcal{G}}$ being exactly the same. Consider the homomorphism (5.1)

$$d^* : F[[\xi_1, \ldots, \xi_n]] \longrightarrow F[[\xi_1 \otimes 1, \ldots, \xi_n \otimes 1; 1 \otimes \xi_1, \ldots, 1 \otimes \xi_n]].$$

Since the ξ_i are algebraic functions on G, and d^* comes from an algebraic morphism, the image of ξ_i is a power series f_i which is the expansion of an algebraic function. It is known that the coefficients of such a power series are bounded by a linear function of the total degree. For a precise formulation let us write a monomial in the form $\underline{\xi}^{\underline{j}} \otimes \underline{\xi}^{\underline{k}}$ with multi-indices $\underline{j}$ and $\underline{k}$. This term has total degree $|\underline{j} + \underline{k}|$, where $|\ |$ denotes the sum over all entries in a multi-index. Then there exists an integer $M \geq 0$ such that for all i, $\underline{j}$, and $\underline{k}$ the coefficient of $\underline{\xi}^{\underline{j}} \otimes \underline{\xi}^{\underline{k}}$ in f_i has t-adic valuation $\geq -M \cdot (|\underline{j} + \underline{k}| + 1)$. Rewriting everything in terms of the variables x_i, we obtain

$$d^* x_i = t^{-N} \cdot f_i\big(t^N x_1 \otimes 1, \ldots, 1 \otimes t^N x_n\big).$$

Here the coefficient of $\underline{x}^{\underline{j}} \otimes \underline{x}^{\underline{k}}$ has t-adic valuation

$$\geq N \cdot (|\underline{j} + \underline{k}| - 1) - M \cdot (|\underline{j} + \underline{k}| + 1).$$

Taking $N \geq 3M$, this is

$$\begin{aligned} &\geq M \cdot \big(3 \cdot (|\underline{j} + \underline{k}| - 1) - (|\underline{j} + \underline{k}| + 1)\big) \\ &= M \cdot \big(2 \cdot |\underline{j} + \underline{k}| - 4\big) \\ &\geq 0 \end{aligned}$$

provided that $|\underline{j} + \underline{k}| \geq 2$. Thus for all $N \gg 0$, the coefficients of all terms of total degree ≥ 2 lie in $\mathcal{O}$. On the other hand the group axioms imply that

$$f_i(\xi_1 \otimes 1, \ldots, 1 \otimes \xi_n) \equiv \xi_i \otimes 1 - 1 \otimes \xi_i \ \text{ modulo terms of degree} \geq 2.$$

It follows that

$$d^* x_i \equiv x_i \otimes 1 - 1 \otimes x_i \ \text{ modulo terms of degree} \geq 2.$$

Thus all coefficients lie in $\mathcal{O}$, as desired. $\square$

Lemma 5.5: *Suppose that $d\pi : \operatorname{Lie}\tilde{G} \to \operatorname{Lie} G$ maps $\tilde{\Lambda}$ to Λ. Then for any $N \gg 0$ the isogeny π extends to a homomorphism $\tilde{\mathcal{G}} \to \mathcal{G}$.*

Proof: Consider the image of ξ_i under the homomorphism

$$\pi^* : F[[\xi_1, \ldots, \xi_n]] \longrightarrow F[[\tilde{\xi}_1, \ldots, \tilde{\xi}_n]].$$

This is a power series g_i which is the expansion of an algebraic function of the $\tilde{\xi}_j$. The image of x_i is then

$$\pi^* x_i = t^{-N} \cdot g_i\big(t^N \tilde{x}_1, \ldots, t^N \tilde{x}_n\big).$$

The same argument as in the proof of Lemma 5.4 shows that for this series the coefficients of all terms of total degree ≥ 2 lie in $\mathcal{O}$, whenever N is sufficiently large. On the other hand, the coefficients of the linear terms are the same as those on the Lie algebra. Since we already know that $\operatorname{Lie}\tilde{\mathcal{G}} = t^N \tilde{\Lambda}$ and $\operatorname{Lie}\mathcal{G} = t^N \Lambda$, these coefficients lie in $\mathcal{O}$ if and only if $d\pi(\tilde{\Lambda}) \subset \Lambda$. $\square$

Next recall that the derivative of $[\gamma,\ ]^\sim : G \to \tilde{G}$ is the homomorphism $\widetilde{\operatorname{Ad}}_G(\gamma) : \operatorname{Lie} G \longrightarrow \operatorname{Lie}\tilde{G}$ of (1.3). A necessary condition for Proposition 5.3 (b) is that this homomorphism maps Λ to $\tilde{\Lambda}$.

Lemma 5.6: *Suppose that $\widetilde{\operatorname{Ad}}_G(\Gamma)(\Lambda) \subset \tilde{\Lambda} \subset (d\pi)^{-1}(\Lambda)$. Then for any given $N \gg 0$, the morphisms $[\gamma,\ ]^\sim$ extend to morphisms $\mathcal{G} \to \tilde{\mathcal{G}}$ for all $\gamma \in \Gamma$.*

Proof: If we had to deal with only finitely many elements $\gamma \in \Gamma$, we could proceed directly as in the proof of Lemma 5.5. But in order to account for all of Γ at once we must replace Γ by something larger which has the structure of an algebraic variety over $\mathcal{O}$.

Working first over F, recall from (1.3) and (1.4) that $\operatorname{Ad}_G = \kappa \circ \widetilde{\operatorname{Ad}}_G$. Since $\operatorname{Ad}_G(G) \subset \underline{\operatorname{Aut}}_F(\operatorname{Lie} G)$, this shows that $\widetilde{\operatorname{Ad}}_G(G)$ is contained in the open subvariety

$$U := \kappa^{-1}\big(\underline{\operatorname{Aut}}_F(\operatorname{Lie} G)\big) \subset \underline{\operatorname{Hom}}_F(\operatorname{Lie} G, \operatorname{Lie}\tilde{G}).$$

In fact, $\widetilde{\operatorname{Ad}}_G$ must be a closed embedding $G \hookrightarrow U$, because Ad_G is one. Summarizing, we have a commutative diagram

$$\begin{tikzcd} G \arrow[r, hook, "\widetilde{\operatorname{Ad}}_G"] & U \arrow[d, "\kappa"'] \arrow[r, phantom, "\subset"] & \underline{\operatorname{Hom}}_F(\operatorname{Lie} G, \operatorname{Lie}\tilde{G}) \arrow[d, "\kappa"'] \\ & \underline{\operatorname{Aut}}_F(\operatorname{Lie} G) \arrow[r, phantom, "\subset"] & \underline{\operatorname{End}}_F(\operatorname{Lie} G)\ . \end{tikzcd}$$

Next, we repeat these constructions over $\mathcal{O}$, using the chosen lattices. For a start, $\underline{\operatorname{Hom}}_{\mathcal{O}}(\Lambda, \tilde{\Lambda})$ is an affine scheme over $\mathcal{O}$ whose generic fiber is $\underline{\operatorname{Hom}}_F(\operatorname{Lie} G, \operatorname{Lie}\tilde{G})$. The assumption of Lemma 5.5 (a) implies that κ extends to a morphism $\underline{\operatorname{Hom}}_{\mathcal{O}}(\Lambda, \tilde{\Lambda}) \longrightarrow \underline{\operatorname{End}}_{\mathcal{O}}(\Lambda)$. Consider the affine open subscheme $\mathcal{U} := \kappa^{-1}\big(\underline{\operatorname{Aut}}_{\mathcal{O}}(\Lambda)\big) \subset \underline{\operatorname{Hom}}_{\mathcal{O}}(\Lambda, \tilde{\Lambda})$, and let

$\mathcal{H}$ be the Zariski closure of $\widetilde{\mathrm{Ad}}_G(G)$ in $\mathcal{U}$. This is an affine scheme over $\mathcal{O}$, say $\mathcal{H} = \mathbf{Spec}\,\mathcal{B}$ for an $\mathcal{O}$-algebra $\mathcal{B}$. Summarizing, we have another commutative diagram

$$\begin{array}{ccccc} \mathcal{H} & \subset & \mathcal{U} & \subset & \underline{\mathrm{Hom}}_{\mathcal{O}}(\Lambda, \tilde{\Lambda}) \\ & & {\scriptstyle\kappa}\big\downarrow & & {\scriptstyle\kappa}\big\downarrow \\ & & \underline{\mathrm{Aut}}_{\mathcal{O}}(\Lambda) & \subset & \underline{\mathrm{End}}_{\mathcal{O}}(\Lambda)\,. \end{array}$$

These constructions imply that $\widetilde{\mathrm{Ad}}_G(\Gamma) \subset \mathcal{H}(\mathcal{O})$. Indeed, the assumption in Lemma 5.6 shows that $\widetilde{\mathrm{Ad}}_G(\gamma)$ is an $\mathcal{O}$-valued point of $\underline{\mathrm{Hom}}_{\mathcal{O}}(\Lambda, \tilde{\Lambda})$. Therefore $\mathrm{Ad}_G(\gamma)$ stabilizes Λ. Since the same holds for γ^{-1} in place of γ, it follows that $\mathrm{Ad}_G(\gamma) \in \mathrm{Aut}_{\mathcal{O}}(\Lambda)$, and hence $\widetilde{\mathrm{Ad}}_G(\gamma) \in \mathcal{U}(\mathcal{O})$. By the definition of $\mathcal{H}$ this point already lies in $\mathcal{H}(\mathcal{O})$, as desired.

Recall that, by construction, $\widetilde{\mathrm{Ad}}_G$ induces an isomorphism $G \xrightarrow{\sim} \mathcal{H} \times_{\mathcal{O}} F$. In other words $\mathcal{H}$ is a model of G over $\mathcal{O}$. Since we have $\widetilde{\mathrm{Ad}}_G(\Gamma) \subset \mathcal{H}(\mathcal{O})$, to prove the lemma it suffices to show that the generalized commutator morphism $[\ ,\ ]^{\sim} :\ G \times_F G \longrightarrow \tilde{G}$ extends to a morphism $\mathcal{H} \times_{\mathcal{O}} \mathcal{G} \longrightarrow \tilde{\mathcal{G}}$ whenever N is sufficiently large.

With this setup we can now proceed as above. The generalized commutator morphism corresponds to an F-algebra homomorphism

$$F[[\tilde{\xi}_1, \ldots, \tilde{\xi}_n]] \longrightarrow (F \cdot \mathcal{B})[[\xi_1, \ldots, \xi_n]].$$

The image of $\tilde{\xi}_i$ is a power series in $\mathcal{B} \otimes_{\mathcal{O}} \big(F[[\xi_1, \ldots, \xi_n]]\big)$ which represents an algebraic function. Thus, writing the image of $\tilde{x}_i$ as a power series in the x_j, the same argument as in the proof of Lemma 5.4 shows that the coefficients of all terms of total degree ≥ 2 lie in $\mathcal{O}$, whenever $N \gg 0$. For the linear terms it suffices to look at the Lie algebras. Here the integrality follows from

$$\begin{aligned} \mathcal{H} &\subset \underline{\mathrm{Hom}}_{\mathcal{O}}(\Lambda, \tilde{\Lambda}) \\ &= \underline{\mathrm{Hom}}_{\mathcal{O}}(t^N\Lambda, t^N\tilde{\Lambda}) \\ &= \underline{\mathrm{Hom}}_{\mathcal{O}}(\mathrm{Lie}\,\mathcal{G}, \mathrm{Lie}\,\tilde{\mathcal{G}}), \end{aligned}$$

as desired. $\square$

To finish the proof of Proposition 5.3 we must show that the local parameters ξ_i and $\tilde{\xi}_i$ can be chosen in such a way that the conditions of Lemma 5.5 and Lemma 5.6 are satisfied. Since these conditions depend only on Λ and $\tilde{\Lambda}$, we may first select these lattices. Clearly the local parameters can then be chosen accordingly. Thus it remains to prove the following lemma.

Lemma 5.7: *There exist $\mathcal{O}$-lattices $\Lambda \subset \mathrm{Lie}\,G$ and $\tilde{\Lambda} \subset \mathrm{Lie}\,\tilde{G}$, such that*

$$\widetilde{\mathrm{Ad}}_G(\Gamma)(\Lambda)\ \subset\ \tilde{\Lambda}\ \subset\ (d\pi)^{-1}(\Lambda).$$

Proof: Take any $\mathrm{Ad}_G(\Gamma)$-stable lattice $\Lambda \subset \mathrm{Lie}\,G$. Then we have $\widetilde{\mathrm{Ad}}_G(\Gamma)(\Lambda) \subset (d\pi)^{-1}(\Lambda)$, and any lattice $\tilde{\Lambda}$ lying between these two submodules does the job. This finishes the proof of Proposition 5.3. $\square$

Generalization: We shall need to generalize these results by taking into account a further isogeny. Let $\varphi : G \to H$, $\tilde{H}$, etc. be as in (4.1). As in (4.13) ff. let τ be the endomorphism of F which on each simple summand F_i is the identity if φ is an isomorphism over F_i, and equal to the Frobenius endomorphism σ otherwise. Consider the isogeny $G \to \tau^*G$ which is the identity over F_i if φ is an isomorphism over F_i, and the Frobenius isogeny otherwise. By Theorem 1.7 this isogeny factors through φ, and the same holds for the universal coverings. Thus we can enlarge the diagram (4.1) to a commutative diagram of isogenies

$$\begin{array}{ccccc} \tilde{G} & \stackrel{\tilde{\varphi}}{\longrightarrow} & \tilde{H} & \stackrel{\tilde{\psi}}{\longrightarrow} & \tau^*\tilde{G} \\ {\scriptstyle\pi}\Big\downarrow & & {\scriptstyle\omega}\Big\downarrow & & \Big\downarrow{\scriptstyle\tau^*\pi} \\ G & \stackrel{\varphi}{\longrightarrow} & H & \stackrel{\psi}{\longrightarrow} & \tau^*G\,. \end{array} \tag{5.8}$$

The desired generalization of Proposition 5.3 goes as follows.

Proposition 5.9: *There exist smooth formal models $\mathcal{G}$, $\tilde{\mathcal{G}}$, $\mathcal{H}$, $\tilde{\mathcal{H}}$ of G, $\tilde{G}$, H, $\tilde{H}$, such that*

(a) the morphisms π, ω, ψ, and $\tilde{\psi}$ extend to homomorphisms in the following commutative diagram

$$\begin{array}{ccccc} \tilde{\mathcal{G}} & & \tilde{\mathcal{H}} & \stackrel{\tilde{\psi}}{\longrightarrow} & \tau^*\tilde{\mathcal{G}} \\ {\scriptstyle\pi}\Big\downarrow & & {\scriptstyle\omega}\Big\downarrow & & \Big\downarrow{\scriptstyle\tau^*\pi} \\ \mathcal{G} & & \mathcal{H} & \stackrel{\psi}{\longrightarrow} & \tau^*\mathcal{G}\,, \end{array}$$

(b) the derivatives $d\psi : \mathrm{Lie}\,\mathcal{H} \longrightarrow \mathrm{Lie}(\tau^*\mathcal{G})$ and $d\tilde{\psi} : \mathrm{Lie}\,\tilde{\mathcal{H}} \longrightarrow \mathrm{Lie}(\tau^*\tilde{\mathcal{G}})$ have $\mathcal{O}$-torsion free cokernel, and

(c) for every $\gamma \in \Gamma$ the morphisms $[\gamma,\ ]^\sim : G \to \tilde{G}$ and $H \to \tilde{H}$ extend to morphisms $\mathcal{G} \to \tilde{\mathcal{G}}$ and $\mathcal{H} \to \tilde{\mathcal{H}}$.

Proof: As in the proof of Proposition 5.3 we may assume that F is a field. When φ is an isomorphism, we take $\mathcal{G}$ and $\tilde{\mathcal{G}}$ as in Proposition 5.3, and choose the corresponding smooth formal models for $\mathcal{H}$ and $\tilde{\mathcal{H}}$. Then there is nothing new to prove. So let us assume that φ is not an isomorphism, in which case $p := \mathrm{char}(F)$ is positive and $\tau = \sigma$. As before, the important point is to choose suitable lattices in the Lie algebras of all groups in question.

Lemma 5.10: *There exist $\mathcal{O}$-lattices $\Lambda \subset \mathrm{Lie}\,G$, $\tilde{\Lambda} \subset \mathrm{Lie}\,\tilde{G}$, $\mathrm{M} \subset \mathrm{Lie}\,H$, and $\tilde{\mathrm{M}} \subset \mathrm{Lie}\,\tilde{H}$, such that*

(a) the derivatives of π, ω, ψ, and $\tilde{\psi}$ induce homomorphisms

$$\begin{array}{ccccccc} \tilde{\Lambda} & & \tilde{\mathrm{M}} & \stackrel{d\tilde{\psi}}{\longrightarrow} & \tau^*\tilde{\Lambda} & = & \tilde{\Lambda} \otimes_{\mathcal{O},\tau} \mathcal{O} \\ {\scriptstyle d\pi}\Big\downarrow & & {\scriptstyle d\omega}\Big\downarrow & & \Big\downarrow{\scriptstyle\tau^*(d\pi)} & & \\ \Lambda & & \mathrm{M} & \stackrel{d\psi}{\longrightarrow} & \tau^*\Lambda & = & \Lambda \otimes_{\mathcal{O},\tau} \mathcal{O}\,, \end{array}$$

(b) the cokernels of the maps $d\tilde{\psi}$ and $d\psi$ in (a) are $\mathcal{O}$-torsion free, and

(c) $\widetilde{\mathrm{Ad}}_G(\Gamma)(\Lambda) \subset \tilde{\Lambda}$ and $\widetilde{\mathrm{Ad}}_H(\varphi(\Gamma))(\mathrm{M}) \subset \tilde{\mathrm{M}}$.

Proof: The lattices must be chosen inside the F-vector spaces

$$\begin{array}{ccccccc}
\mathrm{Lie}\,\tilde{G} & \quad & \mathrm{Lie}\,\tilde{H} & \xrightarrow{d\tilde{\psi}} & \tau^*(\mathrm{Lie}\,\tilde{G}) & = & (\mathrm{Lie}\,\tilde{G}) \otimes_{F,\tau} F \\
{\scriptstyle d\pi}\big\downarrow & & {\scriptstyle d\omega}\big\downarrow & & \big\downarrow{\scriptstyle \tau^*(d\pi)} & & \\
\mathrm{Lie}\,G & & \mathrm{Lie}\,H & \xrightarrow{d\psi} & \tau^*(\mathrm{Lie}\,G) & = & (\mathrm{Lie}\,G) \otimes_{F,\tau} F\ .
\end{array}$$

Take any $\mathrm{Ad}_G(\Gamma)$-stable lattice $\Lambda \subset \mathrm{Lie}\,G$. Next choose an $\widetilde{\mathrm{Ad}}_H(\varphi(\Gamma))$-stable lattice $\mathrm{M} \subset (d\psi)^{-1}(\tau^*\Lambda)$. As in Lemma 5.7 we can then find a lattice

$$\widetilde{\mathrm{Ad}}_H(\varphi(\Gamma))(\mathrm{M}) \ \subset\ \tilde{\mathrm{M}} \ \subset\ (d\omega)^{-1}(\mathrm{M}).$$

We easily calculate that

$$(d\tilde{\psi})(\tilde{\mathrm{M}}) + \widetilde{\mathrm{Ad}}_G(\Gamma)(\Lambda) \ \subset\ (d\pi)^{-1}(\Lambda),$$

so that we can also find a lattice $\tilde{\Lambda}$ between these two submodules. The lattices thus constructed satisfy all conditions except possibly (b). Consider an integer $r \geq 0$ such that t^r annihilates the $\mathcal{O}$-torsion of both $(\tau^*\Lambda)/d\psi(\mathrm{M})$ and $(\tau^*\tilde{\Lambda})/d\tilde{\psi}(\tilde{\mathrm{M}})$. Then after replacing M by $(d\psi)^{-1}(\tau^*\Lambda) \cap t^{-r}\mathrm{M}$ and $\tilde{\mathrm{M}}$ by $(d\tilde{\psi})^{-1}(\tau^*\tilde{\Lambda}) \cap t^{-r}\tilde{\mathrm{M}}$ one easily checks that all conditions hold. $\square$

The construction of smooth formal models now follows the same Ansatz as in the proof of Proposition 5.3. We choose local parameters $\xi_1, \ldots, \xi_n$ of G such that

$$\Lambda = \bigoplus_{i=1}^{n} \mathcal{O} \cdot \Big(\frac{\partial}{\partial \xi_i}\Big|_1 \Big) \subset \mathrm{Lie}\,G,$$

and local parameters $\tilde{\xi}_i$, η_i, $\tilde{\eta}_i$ of $\tilde{G}$, H, $\tilde{H}$ respectively, which satisfy the analogous relation vis-à-vis the lattices $\tilde{\Lambda}$, M, and $\tilde{\mathrm{M}}$. Next we put $x_i := t^{-N}\xi_i$, $\tilde{x}_i := t^{-N}\tilde{\xi}_i$, $y_i := t^{-pN}\eta_i$, and $\tilde{y}_i := t^{-pN}\tilde{\eta}_i$. Note that the new parameters have a different scaling factor! Finally, we let

$$\begin{aligned}
\tilde{\mathcal{G}} &:= \mathbf{Spf}\,\mathcal{O}[[\tilde{x}_1, \ldots, \tilde{x}_n]], \\
\mathcal{G} &:= \mathbf{Spf}\,\mathcal{O}[[x_1, \ldots, x_n]], \\
\tilde{\mathcal{H}} &:= \mathbf{Spf}\,\mathcal{O}[[\tilde{y}_1, \ldots, \tilde{y}_n]], \\
\mathcal{H} &:= \mathbf{Spf}\,\mathcal{O}[[y_1, \ldots, y_n]].
\end{aligned}$$

Then the proof of Proposition 5.3 already shows everything except the assertions concerning the extensions of ψ and $\tilde{\psi}$. We discuss ψ, the proof for $\tilde{\psi}$ being exactly the same. The transpose of ψ is a homomorphism

$$\psi^* : F[[\xi_1, \ldots, \xi_n]] \otimes_{F,\tau} F \longrightarrow F[[\eta_1, \ldots, \eta_n]].$$

If the image of $\xi_i \otimes 1$ is the power series h_i, the image of $x_i \otimes 1 = (t^{-N}\xi_i) \otimes 1 = \xi_i \otimes t^{-pN}$ is equal to

$$\psi^*(x_i \otimes 1) = t^{-pN} \cdot h_i\big(t^{pN}y_1, \ldots, t^{pN}y_n\big).$$

The same argument as in the proof of Lemma 5.4 shows that for this series the coefficients of all terms of total degree ≥ 2 lie in $\mathcal{O}$, whenever N is sufficiently large. For the linear terms we look at the Lie algebras, which turn out to be $\mathrm{Lie}\,\mathcal{H} = t^{pN}\mathrm{M}$ and $\mathrm{Lie}\,\tau^*\mathcal{G} = \tau^*(\mathrm{Lie}\,\mathcal{G}) = \tau^*(t^N\Lambda) = t^{pN} \cdot \tau^*\Lambda$. Now Lemma 5.10 (a) and (b) implies that $d\psi$ induces a homomomorphism $\mathrm{Lie}\,\mathcal{H} \longrightarrow \mathrm{Lie}\,\tau^*\mathcal{G}$ with $\mathcal{O}$-torsion free cokernel. In particular the linear coefficients are in $\mathcal{O}$, proving Proposition 5.9 (a). At the same time this shows (b), and we are done. $\square$

6. Principal Congruence Subgroups and the Truncated Logarithm Map

Let $\mathcal{O} \subset F$ be as in the preceding section, that is, F is a non-archimedean local commutative semisimple ring and $\mathcal{O}$ is its maximal compact subring. Consider a linear algebraic group G over F, and fix a smooth formal model $\mathcal{G}$ over $\mathcal{O}$ of G. This determines a collection of principal congruence subgroups of $G(F)$. In this section we discuss the linearization of certain quotients of these by means of a truncated logarithm map. Recall that in characteristic zero the logarithm and the exponential series allow us to go back and forth between a Lie group and its Lie algebra. But in arbitrary characteristic we can use only the terms of degree ≤ 1 in these series. This is why we obtain natural isomorphisms only between suitable subquotients. The definition and the properties of these isomorphisms are rather straightforward. We first discuss these things for an arbitrary group G. Later on we specialize to the situation of the preceding sections.

Let $\mathfrak{m}$ denote the radical of $\mathcal{O}$, that is the direct sum of the maximal ideals $\mathfrak{m}_i$ of the local rings $\mathcal{O}_i$.

Definition 6.1: *For any open ideal $\mathfrak{a} \subset \mathfrak{m}$ we set*

$$\mathcal{G}(\mathfrak{a}) := \ker\big(\mathcal{G}(\mathcal{O}) \longrightarrow \mathcal{G}(\mathcal{O}/\mathfrak{a})\big).$$

In other words, this is the set of those points in $\mathcal{G}(\mathcal{O})$ on which all local parameters have values in $\mathfrak{a}$. It is a normal subgroup of $\mathcal{G}(\mathcal{O})$ and open in $G(F)$. When $\mathfrak{a}$ runs through a cofinal system of open ideals, then $\mathcal{G}(\mathfrak{a})$ runs through a cofinal system of neighborhoods of the identity. These groups are called principal congruence subgroups. The following proposition introduces the truncated logarithm map.

Proposition 6.2:

(a) For any open ideals $\mathfrak{a}^2 \subset \mathfrak{b} \subset \mathfrak{a} \subset \mathfrak{m}$ there exists a canonical group isomorphism

$$\log_{\mathfrak{a}/\mathfrak{b}} :\ \mathcal{G}(\mathfrak{a})/\mathcal{G}(\mathfrak{b}) \xrightarrow{\ \sim\ } (\mathrm{Lie}\,\mathcal{G}) \otimes_{\mathcal{O}} (\mathfrak{a}/\mathfrak{b})\ .$$

(b) For any open ideals $\mathfrak{a}^2 \subset \mathfrak{b} \subset \mathfrak{a} \subset \mathfrak{m}$ and $\mathfrak{a}'^2 \subset \mathfrak{b}' \subset \mathfrak{a}' \subset \mathfrak{m}$ such that $\mathfrak{b}' \subset \mathfrak{b}$ and $\mathfrak{a}' \subset \mathfrak{a}$, the following diagram commutes, where the vertical maps are the obvious ones:

$$\begin{tikzcd}
\mathcal{G}(\mathfrak{a}')/\mathcal{G}(\mathfrak{b}') \arrow[r, "\log_{\mathfrak{a}'/\mathfrak{b}'}"] \arrow[d] & (\mathrm{Lie}\,\mathcal{G}) \otimes_{\mathcal{O}} (\mathfrak{a}'/\mathfrak{b}') \arrow[d] \\
\mathcal{G}(\mathfrak{a})/\mathcal{G}(\mathfrak{b}) \arrow[r, "\log_{\mathfrak{a}/\mathfrak{b}}"] & (\mathrm{Lie}\,\mathcal{G}) \otimes_{\mathcal{O}} (\mathfrak{a}/\mathfrak{b})\ .
\end{tikzcd}$$

(c) Let $\mathcal{H}$ be a smooth formal model over $\mathcal{O}$ of another linear algebraic group H. Consider a morphism $\varphi : \mathcal{G} \to \mathcal{H}$, which may or may not be a group homomorphism. Then the following diagram commutes:

$$\begin{tikzcd}
{[g]} \arrow[d] & \in & \mathcal{G}(\mathfrak{a})/\mathcal{G}(\mathfrak{b}) \arrow[r, "\log_{\mathfrak{a}/\mathfrak{b}}"] \arrow[d, "{[\varphi]}"'] & (\mathrm{Lie}\,\mathcal{G}) \otimes_{\mathcal{O}} (\mathfrak{a}/\mathfrak{b}) \arrow[d, "d\varphi\otimes\mathrm{id}"] \\
{[\varphi(g)]} & \in & \mathcal{H}(\mathfrak{a})/\mathcal{H}(\mathfrak{b}) \arrow[r, "\log_{\mathfrak{a}/\mathfrak{b}}"] & (\mathrm{Lie}\,\mathcal{H}) \otimes_{\mathcal{O}} (\mathfrak{a}/\mathfrak{b})\ .
\end{tikzcd}$$

Proof: We may decompose everything according to the simple summands of F. Thus without loss of generality we may assume that F is a field. Then any choice of local parameters $x_1, \ldots, x_n$ of $\mathcal{G}$ determines a homeomorphism

$$\mathcal{G}(\mathfrak{a}) \xrightarrow{\sim} \mathfrak{a}^{\oplus n}, \ g \mapsto (x_1(g), \ldots, x_n(g)).$$

Consider the map

$$\begin{aligned} \mathcal{G}(\mathfrak{a}) &\longrightarrow (\mathrm{Lie}\, \mathcal{G}) \otimes_{\mathcal{O}} (\mathfrak{a}/\mathfrak{b}), \\ g &\mapsto \sum_{i=1}^{n} \Big(\frac{\partial}{\partial x_i}\Big|_1 \Big) \otimes [x_i(g)]. \end{aligned}$$

Since the group structure of $\mathcal{G}$ is given by power series with coefficients in $\mathcal{O}$, all terms of degree ≥ 2 are subsumed in $\mathfrak{a}^2 \subset \mathfrak{b}$. Therefore this is a group homomorphism. Clearly it is surjective with kernel $\mathcal{G}(\mathfrak{b})$, so it induces the desired isomorphism in (a). If we compare this map with that defined by another system of local parameters, the terms of degree ≥ 2 again appear only in $\mathfrak{b}$. Thus one easily calculates that the map is independent of the choice of the x_i. This proves (a). A similar calculation proves (c). Finally, part (b) is obvious from the construction. $\square$

Next we discuss the behavior of truncated logarithm maps under Frobenius isogenies. Let τ be an endomorphism of F which on each F_i is either the identity or the Frobenius endomorphism (provided that $\mathrm{char}(F_i) > 0$). Consider the isogeny $\Phi : G \to \tau^* G$ which over each F_i is the identity or the Frobenius isogeny depending on whether τ is an isomorphism over F_i or not. Clearly Φ extends to a homomorphism $\mathcal{G} \to \tau^*\mathcal{G}$, by the same definition as in (1.5).

Proposition 6.3:

(a) Consider an open ideal $\mathfrak{a}' \subset \mathfrak{m}$ and set $\mathfrak{a} := \tau(\mathfrak{a}')\mathcal{O}$. Then Φ induces an isomorphism

$$\mathcal{G}(\mathfrak{a}') \xrightarrow{\sim} \Phi(G(F)) \cap (\tau^*\mathcal{G})(\mathfrak{a}) .$$

(b) Consider open ideals $\mathfrak{a}'^2 \subset \mathfrak{b}' \subset \mathfrak{a}' \subset \mathfrak{m}$ and set $\mathfrak{a} := \tau(\mathfrak{a}')\mathcal{O}$ and $\mathfrak{b} := \tau(\mathfrak{b}')\mathcal{O}$. Then the following diagram commutes

$$\begin{array}{ccc} \mathcal{G}(\mathfrak{a}')/\mathcal{G}(\mathfrak{b}') & \xrightarrow{\log_{\mathfrak{a}'/\mathfrak{b}'}} & (\mathrm{Lie}\, \mathcal{G}) \otimes_{\mathcal{O}} (\mathfrak{a}'/\mathfrak{b}') \\ {\scriptstyle [\Phi]}\big\downarrow \wr & & \wr \big\downarrow {\scriptstyle \mathrm{id}\otimes\tau} \\ \dfrac{\Phi(G(F)) \cap (\tau^*\mathcal{G})(\mathfrak{a})}{\Phi(G(F)) \cap (\tau^*\mathcal{G})(\mathfrak{b})} & \xrightarrow{\sim} & (\mathrm{Lie}\, \mathcal{G}) \otimes_{\mathcal{O},\tau} \big(\tau(\mathfrak{a}')/\tau(\mathfrak{b}')\big) \\ \cap & & \cap \\ (\tau^*\mathcal{G})(\mathfrak{a})/(\tau^*\mathcal{G})(\mathfrak{b}) & \xrightarrow{\log_{\mathfrak{a}/\mathfrak{b}}} & (\mathrm{Lie}\, \tau^*\mathcal{G}) \otimes_{\mathcal{O}} (\mathfrak{a}/\mathfrak{b}) . \end{array}$$

Proof: As in Proposition 6.2 this is an easy calculation in terms of explicit local parameters.

$\square$

Let us now take up the situation of Proposition 5.9. By Proposition 6.2 we have for all open ideals $\mathfrak{a}^2 \subset \mathfrak{b} \subset \mathfrak{a} \subset \mathfrak{m}$ a commutative diagram

$$\tag{6.4}
\begin{array}{ccc}
\tilde{\mathcal{H}}(\mathfrak{a})/\tilde{\mathcal{H}}(\mathfrak{b}) & \xrightarrow{\log_{\mathfrak{a}/\mathfrak{b}}} & (\mathrm{Lie}\,\tilde{\mathcal{H}}) \otimes_{\mathcal{O}} (\mathfrak{a}/\mathfrak{b}) \\
{\scriptstyle [\omega]}\Big\downarrow & & \Big\downarrow{\scriptstyle d\omega} \\
\mathcal{H}(\mathfrak{a})/\mathcal{H}(\mathfrak{b}) & \xrightarrow{\log_{\mathfrak{a}/\mathfrak{b}}} & (\mathrm{Lie}\,\mathcal{H}) \otimes_{\mathcal{O}} (\mathfrak{a}/\mathfrak{b}) \,.
\end{array}$$

Moreover, the action of any $\gamma \in \Gamma$ yields a commutative diagram

$$\tag{6.5}
\begin{array}{ccc}
\tilde{\mathcal{H}}(\mathfrak{a})/\tilde{\mathcal{H}}(\mathfrak{b}) & \xrightarrow{\log_{\mathfrak{a}/\mathfrak{b}}} & (\mathrm{Lie}\,\tilde{\mathcal{H}}) \otimes_{\mathcal{O}} (\mathfrak{a}/\mathfrak{b}) \\
{\scriptstyle \big[[\varphi(\gamma),\ \ ]^{\sim}\big]}\Big\uparrow & & \Big\uparrow{\scriptstyle \widetilde{\mathrm{Ad}}_H(\varphi(\gamma))} \\
\mathcal{H}(\mathfrak{a})/\mathcal{H}(\mathfrak{b}) & \xrightarrow{\log_{\mathfrak{a}/\mathfrak{b}}} & (\mathrm{Lie}\,\mathcal{H}) \otimes_{\mathcal{O}} (\mathfrak{a}/\mathfrak{b}) \,.
\end{array}$$

Next we want to get a hold on the subquotients of $\varphi(G(F))$ determined by the congruence subgroups of $\mathcal{H}(\mathcal{O})$. Let $\tilde{W} \subset \tilde{\mathfrak{h}}$ and $W \subset \mathfrak{h}$ be as in (4.13) and (4.14). Then $\tilde{\Lambda} := \tilde{W} \cap (\mathrm{Lie}\,\tilde{\mathcal{H}})$ and $\Lambda := W \cap (\mathrm{Lie}\,\mathcal{H})$ are $\tau(\mathcal{O})$-lattices in $\tilde{W}$ resp. in W.

Proposition 6.6: *Consider open ideals $\mathfrak{a}'^2 \subset \mathfrak{b}' \subset \mathfrak{a}' \subset \mathfrak{m}$ and set $\mathfrak{a} := \tau(\mathfrak{a}')\mathcal{O}$ and $\mathfrak{b} := \tau(\mathfrak{b}')\mathcal{O}$. Then we have commutative diagrams*

$$\begin{array}{ccc}
\tilde{\mathcal{H}}(\mathfrak{a})/\tilde{\mathcal{H}}(\mathfrak{b}) & \xrightarrow{\log_{\mathfrak{a}/\mathfrak{b}}} & (\mathrm{Lie}\,\tilde{\mathcal{H}}) \otimes_{\mathcal{O}} (\mathfrak{a}/\mathfrak{b}) \\
\cup & & \cup \\
\dfrac{\tilde{\varphi}(\tilde{G}(F)) \cap \tilde{\mathcal{H}}(\mathfrak{a})}{\tilde{\varphi}(\tilde{G}(F)) \cap \tilde{\mathcal{H}}(\mathfrak{b})} & \hookrightarrow & \tilde{\Lambda} \otimes_{\tau(\mathcal{O})} \big(\tau(\mathfrak{a}')/\tau(\mathfrak{b}')\big)
\end{array}$$

and

$$\begin{array}{ccc}
\mathcal{H}(\mathfrak{a})/\mathcal{H}(\mathfrak{b}) & \xrightarrow{\log_{\mathfrak{a}/\mathfrak{b}}} & (\mathrm{Lie}\,\mathcal{H}) \otimes_{\mathcal{O}} (\mathfrak{a}/\mathfrak{b}) \\
\cup & & \cup \\
\dfrac{\varphi(G(F)) \cap \mathcal{H}(\mathfrak{a})}{\varphi(G(F)) \cap \mathcal{H}(\mathfrak{b})} & \hookrightarrow & \Lambda \otimes_{\tau(\mathcal{O})} \big(\tau(\mathfrak{a}')/\tau(\mathfrak{b}')\big) \,.
\end{array}$$

Proof: The proof is the same for both diagrams. Let us do the second one. Using Proposition 6.3 we obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{H}(\mathfrak{a})/\mathcal{H}(\mathfrak{b}) \arrow[rr, "\log_{\mathfrak{a}/\mathfrak{b}}", "\sim"'] & & (\mathrm{Lie}\,\mathcal{H}) \otimes_{\mathcal{O}} (\mathfrak{a}/\mathfrak{b}) \arrow[dr] & \\
& & & (\mathrm{Lie}\,\sigma^*\mathcal{G}) \otimes_{\mathcal{O}} (\mathfrak{a}/\mathfrak{b}) \\
\dfrac{\varphi(G(F)) \cap \mathcal{H}(\mathfrak{a})}{\varphi(G(F)) \cap \mathcal{H}(\mathfrak{b})} \arrow[uu, hook] \arrow[dr] & & \Lambda \otimes_{\sigma(\mathcal{O})} \big(\sigma(\mathfrak{a}')/\sigma(\mathfrak{b}')\big) \arrow[uu, hook] \arrow[dr] & \\
& \dfrac{\mathrm{Frob}(G(F)) \cap (\sigma^*\mathcal{G})(\mathfrak{a})}{\mathrm{Frob}(G(F)) \cap (\sigma^*\mathcal{G})(\mathfrak{b})} \arrow[rr, "\sim"] & & (\mathrm{Lie}\,\mathcal{G}) \otimes_{\mathcal{O},\sigma} \big(\sigma(\mathfrak{a}')/\sigma(\mathfrak{b}')\big) \,. \arrow[uu, hook]
\end{tikzcd}$$

Here the condition in Proposition 5.9 (b) implies that the four objects on the right hand side form a cartesian subdiagram. Thus we obtain the desired factorization. The map is injective because $\log_{\mathfrak{a}/\mathfrak{b}}$ is an isomorphism. $\square$

Now consider a closed subgroup $\tilde{\Gamma} \subset \tilde{G}(F)$. To analyze the subquotients of $\tilde{\Gamma}$ induced by the filtration of $\tilde{H}(F)$ by principal congruence subgroups we make the following definition.

Definition 6.7: *Consider open ideals $\mathfrak{b}' \subset \mathfrak{a}'$ and $\mathfrak{b} \subset \mathfrak{a}$ as in Proposition 6.6. We say that condition $\mathbf{Full}_{\mathfrak{a}'/\mathfrak{b}'}$ holds if and only if*

$$\log_{\mathfrak{a}/\mathfrak{b}}\Big(\frac{\tilde{\varphi}(\tilde{\Gamma}) \cap \mathcal{H}(\mathfrak{a})}{\tilde{\varphi}(\tilde{\Gamma}) \cap \mathcal{H}(\mathfrak{b})}\Big) = \tilde{\Lambda} \otimes_{\tau(\mathcal{O})} \big(\tau(\mathfrak{a}')/\tau(\mathfrak{b}')\big) .$$

From Proposition 6.6 we deduce the following openness criterion.

Corollary 6.8: *Consider a cofinal system of open ideals $\mathfrak{m} \supset \mathfrak{a}'_0 \supset \mathfrak{a}'_1 \supset \ldots$ satisfying $\mathfrak{a}'_{n+1} \supset {\mathfrak{a}'_n}^2$ for all $n \geq 0$. Set $\mathfrak{a}_0 := \tau(\mathfrak{a}'_0)\mathcal{O}$. Suppose that $\mathbf{Full}_{\mathfrak{a}'_n/\mathfrak{a}'_{n+1}}$ holds for all $n \geq 0$. Then*

$$\tilde{\varphi}(\tilde{G}(F)) \cap \tilde{\mathcal{H}}(\mathfrak{a}_0) \subset \tilde{\varphi}(\tilde{\Gamma}).$$

In particular, $\tilde{\Gamma}$ is open in $G(F)$.

7. The Main Theorem

Let (F, G, Γ) be as in Section 3. We consider the local case, i.e. we make Assumption 2.1 (b). The aim of this section is to relate the concept of minimality of Definition 3.4 with that of topological openness. We begin with the easy part. Let $\pi : \tilde{G} \to G$ be the universal covering of G, and let $\Gamma' \subset \tilde{G}(F)$ denote the closure of the generalized commutator group of Γ.

Proposition 7.1: *If Γ' is open in $\tilde{G}(F)$, then (F, G, Γ) is minimal.*

Proof: Consider any weak quasi-model (E, H, φ) of (F, G, Γ), and let $\tilde{\varphi} : \tilde{H} \times_E F \longrightarrow \tilde{G}$ be the isogeny of universal coverings induced by φ. Applying the universal property of the Weil restriction to $\tilde{\varphi}$ we obtain a homomorphism $\varphi' : \tilde{H} \to \mathcal{R}_{F/E}\tilde{G}$ of algebraic groups over E. Taking E-valued points our assumption implies that the image of the map $\tilde{H}(E) \to (\mathcal{R}_{F/E}\tilde{G})(E) = \tilde{G}(F)$ is open. This can only happen when the morphism φ' is a local isomorphism at some point of $H(E)$. Therefore we must have $E = F$ and that φ itself is an isomorphism. $\square$

The converse can be phrased in slightly greater generality.

Main Theorem 7.2: *Suppose that (F, G, Γ) is minimal. Consider any compact subgroup $\tilde{\Gamma} \subset \tilde{G}(F)$ which is fiberwise Zariski dense and normalized by Γ. Then $\tilde{\Gamma}$ is open in $\tilde{G}(F)$.*

Corollary 7.3: *Suppose that (F, G, Γ) is minimal. Then Γ' is an open subgroup of $\tilde{G}(F)$.*

The proof of the Main Theorem will cover the rest of this section. Throughout we shall assume that (F, G, Γ) is minimal. We begin with a few reduction steps.

Proposition 7.4: *Suppose that each simple summand F_i of F is archimedean. Then*

(a) $F_i = \mathbb{R}$ for all i,

(b) $\tilde{\Gamma} = \tilde{G}(F)$, and

(c) $\tilde{\Gamma}$ is connected.

Proof: Suppose first that F is a field. As $\tilde{\Gamma}$ is compact, it is contained in a compact real form of $\tilde{G}$. By the minimality assumption it follows that $F = \mathbb{R}$. Next, by a theorem of Weyl (cf. [9] Ch. 4, Th. 2.1) any compact subgroup of $\mathrm{GL}_n(\mathbb{R})$ is the group of $\mathbb{R}$-valued points of an algebraic subgroup. Thus the Zariski density of $\tilde{\Gamma}$ implies that $\tilde{\Gamma} = \tilde{G}(F)$. Observe also that, since $\tilde{G}$ is connected, the identity component $\tilde{G}(F)^\circ$ is still Zariski dense. Again by Weyl's theorem it follows that $\tilde{G}(F)^\circ = \tilde{G}(F)$. This proves the proposition when F is a field.

In the general case the projection property Proposition 3.9 proves both (a) and that the map $\tilde{\Gamma} \longrightarrow \tilde{G}_i(F_i)$ is surjective for every i. Look at the map to the adjoint group $\tilde{\Gamma} \longrightarrow G(F) = \prod_{i=1}^m G_i(F_i)$. It is known that each $G_i(F_i)$ is a simple group. Thus the map can fail to be surjective only if the image in $G_i(F_i) \times G_j(F_j)$ is the graph of an isomorphism $G_i(F_i) \cong G_j(F_j)$ for suitable $i \neq j$. Again by Weyl's theorem this isomorphism must be algebraic, contradicting the minimality condition. Thus the total map is surjective. Going back to $\tilde{G}(F)$ by generalized commutators, we find that $\tilde{\Gamma} = \tilde{G}(F)$, proving (b). At last, (c) now follows from the field case together with (b). $\square$

Lemma 7.5: *To prove Main Theorem 7.2 it suffices to consider the case when all F_i are non-archimedean with the same residue characteristic.*

Proof: Let $p_1, \ldots, p_n$ be the pairwise distinct residue characterics occurring in F. Write $F = \bigoplus_{\nu=0}^{n} F_{(\nu)}$, where $F_{(\nu)}$ contains all non-archimedean summands with residue characteristic p_ν if $\nu > 0$, resp. all archimedean summands if $\nu = 0$. Let $\tilde{G}_{(\nu)}$ be the part of $\tilde{G}$ that lies over $F_{(\nu)}$, and $\tilde{\Gamma}_{(\nu)}$ the image of $\tilde{\Gamma}$ in $\tilde{G}_{(\nu)}(F_{(\nu)})$. It is enough to prove that the image of the map

$$\tilde{\Gamma} \longrightarrow \prod_{\nu=0}^{n} \tilde{\Gamma}_{(\nu)} \tag{7.6}$$

is open. For this first note that $\tilde{\Gamma}_{(0)}$ is connected, by Proposition 7.4 (c). Therefore any open subgroup of $\tilde{\Gamma}$ surjects onto $\tilde{\Gamma}_{(0)}$. By the compactness of $\tilde{\Gamma}$ it follows that the identity component $\tilde{\Gamma}^{\circ}$ surjects onto $\tilde{\Gamma}_{(0)}$. Since $\tilde{\Gamma}_{(\nu)}$ is totally disconnected for all $\nu > 0$, the identity component $\tilde{\Gamma}_{(0)}$ lies only in the archimedean summand. This shows already that $\tilde{\Gamma}$ is the direct product of $\tilde{\Gamma}_{(0)}$ with some closed subgroup $\tilde{\Gamma}_{(>0)} \subset \prod_{\nu=1}^{n} \tilde{\Gamma}_{(\nu)}$. On the other hand, observe that we are allowed to replace $\tilde{\Gamma}$ by arbitrarily small open subgroups. Thus after shrinking it we may assume that its image $\tilde{\Gamma}_{(\nu)}$ is a pro-p_ν-group for every $\nu > 0$. As the p_ν are pairwise distinct, the map (7.6) is then bijective. $\square$

In the rest of the proof, we assume that all F_i are non-archimedean with the same residue characteristic p. Sooner or later we have to begin constructing elements of $\tilde{\Gamma}$. The only possible starting point is to choose some sufficiently generic element. We cannot do essentially better than taking $\tilde{\gamma} \in \tilde{\Gamma}$ regular semisimple and sufficiently close to the identity. The last point will be made precise below; apart from that, such an element will be fixed throughout the rest of this section.

Let us first outline the remaining arguments. Let $\mathfrak{m} \subset \mathcal{O} \subset F$ be as in the preceding section. Suppose we are given suitable local parameters for $\tilde{G}$, that is, a smooth formal model $\tilde{\mathcal{G}}$ over $\mathcal{O}$. Then $\tilde{\gamma}$ lies in a unique smallest principal congruence subgroup $\tilde{\mathcal{G}}(\mathfrak{a})$. By the truncated logarithm map of Proposition 6.2 it will correspond to a primitive element $\log(\tilde{\gamma}) \in (\operatorname{Lie}\tilde{\mathcal{G}}) \otimes_{\mathcal{O}} (\mathfrak{a}/\mathfrak{a}^2)$. Under favorable circumstances, the action of Γ generates many new elements from $\log(\tilde{\gamma})$. If

$$(\operatorname{Lie}\tilde{\mathcal{G}}) \otimes_{\mathcal{O}} (\mathfrak{b}/\mathfrak{a}^2) \subset \operatorname{Ad}_{\tilde{G}}(\Gamma) \cdot \log(\tilde{\gamma}) \tag{7.7}$$

for some open ideal $\mathfrak{a}^2 \subset \mathfrak{b} \subset \mathfrak{a}$, it follows that elements of $\tilde{\Gamma}$ fill out the whole quotient group $\tilde{\mathcal{G}}(\mathfrak{b})/\tilde{\mathcal{G}}(\mathfrak{a}^2)$. If we can show that $\mathfrak{b}/\mathfrak{a}^2$ is not too small, we can use the same method to prove inductively that $\tilde{\Gamma}$ fills out the quotient groups $\tilde{\mathcal{G}}(\mathfrak{a}_n)/\tilde{\mathcal{G}}(\mathfrak{a}_{n+1})$ for a cofinal system of open ideals $\mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \ldots$. Then it follows that $\tilde{\Gamma}$ contains $\tilde{\mathcal{G}}(\mathfrak{a}_1)$, and we are done.

The problem with this Ansatz is that the action of Γ on $\operatorname{Lie}\tilde{G}$ may be too small. If Γ does not act irreducibly on $\operatorname{Lie}\tilde{G}$, and $\log(\tilde{\gamma})$ lies too close to a proper invariant subspace, then the subgroup (7.7) will be too small. The only possible remedy regarding $\tilde{\gamma}$ is to replace it by some large power $\tilde{\gamma}^{p^n}$, hoping that its position is then under better control. But in the fibers of $\tilde{G}$ which possess non-standard isogenies that prescription is not always enough. It may be necessary to alter the whole system of principal congruence subgroups and logarithm maps. This can be achieved by choosing an isogeny $\tilde{\varphi} : \tilde{G} \to \tilde{H}$ and working inside $\tilde{H}$. If the choice of $\tilde{\varphi}$ is adapted to the given element $\tilde{\gamma}$, all obstacles will be removed.

The right choices are made as follows. Set $s := \pi(\tilde{\gamma}) \in G(F)$. This element is regular semisimple, so the identity component of its centralizer is a maximal torus $S \subset G$. Let Φ_i be the root system of a fiber G_i with respect to S. Fix a valuation v_i on F_i and extend it to an algebraic closure $\bar{F}_i$. For each $\alpha \in \Phi_i$ we are interested in the eigenvalue $\alpha(s) \in \bar{F}_i$. By choosing $\tilde{\gamma}$ close to the identity, we can make $v_i(\alpha(s) - 1)$ arbitrarily large. In particular we may, and do, assume that

$$v_i\big(\alpha(s) - 1\big) > \begin{cases} 0 & \text{if } \operatorname{char}(F_i) = p, \\ v_i(p) & \text{if } \operatorname{char}(F_i) = 0, \end{cases}$$

for all i and α. This assumption guarantees that

$$v_i\big(\alpha(s^{p^n}) - 1\big) = \begin{cases} p^n \cdot v_i(\alpha(s) - 1) & \text{if } \operatorname{char}(F_i) = p, \\ n \cdot v_i(p) + v_i(\alpha(s) - 1) & \text{if } \operatorname{char}(F_i) = 0. \end{cases} \tag{7.8}$$

We shall not need any further condition on s. Later on we shall see that the "position of $\log(s^{p^n})$" depends mainly on the values (7.8), when n is large.

Next we must choose suitable $\varphi : G \to H$, $\tilde{H}$, etc. as in (4.1). For any choice $T := \varphi(S)$ is a maximal torus of H containing $t := \varphi(s)$. Let H_i denote the fiber of H over F_i, and let Ψ_i be the root system of H_i with respect to T. When H_i possesses non-standard isogenies, let $\Psi_i^\bullet \subset \Psi_i$ denote the set of short roots for $\bullet = s$, resp. long roots for $\bullet = s$, and put

$$v_i^\bullet := \min\big\{v_i(\alpha(t) - 1) \;\big|\; \alpha \in \Psi_i^\bullet\big\}.$$

Lemma 7.9: *Given $\tilde{\gamma}$ as above, the choice of H and $\varphi : G \to H$ can be made such that $v_i^\ell < 2 \cdot v_i^s$ for all fibers H_i which possess non-standard isogenies.*

Proof: If G_i does not have non-standard isogenies, there is nothing to prove. Otherwise let $\Phi_i^\bullet \subset \Phi_i$ denote the set of short roots for $\bullet = s$, resp. long roots for $\bullet = s$, and set

$$u_i^\bullet := \min\big\{v_i(\alpha(s) - 1) \;\big|\; \alpha \in \Phi_i^\bullet\big\}.$$

If $u_i^\ell < 2 \cdot u_i^s$, we can take $H_i = G_i$ and $\varphi = \mathrm{id}$. Otherwise we are forced to take the non-standard isogeny $\varphi : G_i \longrightarrow G_i^\sharp =: H_i$. Recall from Proposition 1.6 that then $\Psi_i^s = \Phi_i^\ell$ and $\Psi_i^\ell = p \cdot \Phi_i^s$. Taking into account the fact that $\operatorname{char}(F_i) = p$ in this non-standard case, we deduce that $v_i^s = u_i^\ell$ and $v_i^\ell = p \cdot u_i^s$. Therefore

$$v_i^\ell = p \cdot u_i^s \leq \frac{p}{2} \cdot u_i^\ell = \frac{p}{2} \cdot v_i^s < 2 \cdot v_i^s,$$

since $p \leq 3 < 4$ whenever there is a non-standard isogeny. This proves the lemma. $\square$

In the rest of the proof $\varphi : G \to H$ etc. will be fixed, subject to the condition in Lemma 7.9. Let $\mathcal{O}$ be the maximal compact subring of F. We fix $\tau : F \to F$ and smooth formal models $\mathcal{G}$, $\tilde{\mathcal{G}}$, $\mathcal{H}$, and $\tilde{\mathcal{H}}$ as in Proposition 5.9. For each $n \geq 0$ let $\mathfrak{a}'_n \subset \mathcal{O}$ be the smallest ideal such that $t^{p^n} \in \mathcal{H}(\mathfrak{a}_n)$, where $\mathfrak{a}_n := \tau(\mathfrak{a}'_n)\mathcal{O}$. As n goes to infinity, $\mathfrak{a}'_n$ runs through a cofinal system of open ideals. Let $W \subset \mathfrak{h}$ be as in (4.14), and consider the $\tau(\mathcal{O})$-lattice $\Lambda := W \cap (\operatorname{Lie}\mathcal{H})$ as in Proposition 6.6. Then

$$\log_{\mathfrak{a}_n/\mathfrak{a}_n^2}(t^{p^n}) \in \Lambda \otimes_{\tau(\mathcal{O})} \big(\tau(\mathfrak{a}'_n)/\tau({\mathfrak{a}'_n}^2)\big). \tag{7.10}$$

Let $\theta : \mathfrak{h} \twoheadrightarrow \mathfrak{h}_{\ell t}$ be as in (4.17). Then $\Lambda_{\ell t} := \theta(\Lambda)$ is a lattice in $W_{\ell t} := \theta(W)$. It is possible that $\theta\big(\log_{\mathfrak{a}_n/\mathfrak{a}_n^2}(t^{p^n})\big)$ is more divisible than the element (7.10). But not too much! Namely, let $\mathfrak{b}'_n$ be the smallest ideal satisfying ${\mathfrak{a}'_n}^2 \subset \mathfrak{b}'_n \subset \mathfrak{a}'_n$ such that

$$\theta\big(\log_{\mathfrak{a}_n/\mathfrak{a}_n^2}(t^{p^n})\big) \in \Lambda_{\ell t} \otimes_{\tau(\mathcal{O})} \big(\tau(\mathfrak{b}'_n)/\tau({\mathfrak{a}'_n}^2)\big), \tag{7.11}$$

and let $\mathfrak{b}_n := \tau(\mathfrak{b}'_n)\mathcal{O}$.

Lemma 7.12: *Fix any open ideal $\mathfrak{c}' \subset \mathcal{O}$. Then ${\mathfrak{a}'_n}^2 \subset \mathfrak{b}'_n\mathfrak{c}'$ for all $n \gg 0$.*

Proof: All fibers over F can be considered separately, hence we may assume that F is a field. For the duration of this proof we shall drop the index $_i$ in Ψ_i, etc. In the case $\operatorname{char}(F) = 0$ the adjoint representation of H is irreducible, so that $W_{\ell t} = W$ and $\Lambda_{\ell t} = \Lambda$. Therefore we have $\mathfrak{b}'_n = \mathfrak{a}'_n$, which is enough since this ideal goes to 0 as $n \to \infty$. So we may assume that $\operatorname{char}(F) = p$.

Let $\tilde{F} \subset \bar{F}$ be the finite extension of F that is generated by the eigenvalues $\alpha(t)$ for all roots $\alpha \in \Psi$. Let $\tilde{\mathcal{O}}$ be the normalization of $\mathcal{O}$ in $\tilde{F}$. We want to replace everything by its base extension to $\tilde{\mathcal{O}}$. For instance, $\operatorname{Lie}\mathcal{H}$ is replaced by $\operatorname{Lie}(\mathcal{H} \times_{\mathcal{O}} \tilde{\mathcal{O}}) = (\operatorname{Lie}\mathcal{H}) \otimes_{\mathcal{O}} \tilde{\mathcal{O}}$, and so on. We find easily that the ideals $\mathfrak{a}'_n\tilde{\mathcal{O}}$ resp. $\mathfrak{b}'_n\tilde{\mathcal{O}}$ have the analogous defining property as $\mathfrak{a}'_n$ and $\mathfrak{b}'_n$. Since we shall not use the minimality assumption in this proof, we may now assume that $\tilde{F} = F$. In other words T splits over F.

Choose a basis $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ of Ψ. Then we have $T = \mathbf{Spec}\, F[\alpha_i^{\pm 1}|_{i=1}^r]$, and $\mathcal{T} := \mathbf{Spf}\, \mathcal{O}[[d^{-1}(\alpha_i - 1)|_{i=1}^r]]$ is a smooth formal model of T for any given $0 \neq d \in \mathcal{O}$. As in Lemma 5.5 we find that the inclusion $T \hookrightarrow H$ extends to a homomorphism $\mathcal{T} \to \mathcal{H}$ whenever d is sufficiently small. Now let $\mathfrak{d} \subset \mathfrak{m}$ be the ideal generated by the elements $\alpha(t) - 1$ for all $\alpha \in \Psi$. As Δ is a basis of Ψ, this is the same as the ideal generated by the elements $\alpha_i(t) - 1$ for $1 \leq i \leq r$. Thus for all $n \gg 0$ we have $t^{p^n} \in \mathcal{T}(d^{-1}\mathfrak{d}^{p^n}) \subset \mathcal{H}(d^{-1}\mathfrak{d}^{p^n})$. Since $t^{p^n} \notin \mathcal{H}(\tau(\mathfrak{m})\mathfrak{a}_n)$ by the definition of $\mathfrak{a}_n$, we deduce that $\tau(\mathfrak{m})\mathfrak{a}_n \subset d^{-1}\mathfrak{d}^{p^n}$ for all $n \gg 0$.

For $\mathfrak{b}_n$ we need a relation in the other direction. Suppose first that H does not have non-standard isogenies. Then θ induces an injective map $\operatorname{Lie} T \hookrightarrow \mathfrak{h}_{\ell t}$, and hence $\operatorname{Lie}\mathcal{T} \hookrightarrow \Lambda_{\ell t}$. Choose an element $0 \neq e \in \mathcal{O}$ which annihilates the torsion of $\Lambda_{\ell t}/\operatorname{Lie}\mathcal{T}$. Then (7.11) implies

$$e \cdot \log_{\mathfrak{a}_n/\mathfrak{a}_n^2}(t^{p^n}) \in \operatorname{Lie}\mathcal{T} \otimes_{\mathcal{O}} (\mathfrak{b}_n/\mathfrak{a}_n^2),$$

where now the logarithm is taken in $\mathcal{T}$. This means that $t^{p^n} \in \mathcal{T}(e^{-1}\mathfrak{b}_n)$. By the definition of $\mathcal{T}$ we obtain the relation $d^{-1}\mathfrak{d}^{p^n} \subset e^{-1}\mathfrak{b}_n$. Altogether we now have the inclusions $\tau(\mathfrak{m})\mathfrak{a}_n \subset d^{-1}\mathfrak{d}^{p^n} \subset e^{-1}\mathfrak{b}_n \subset e^{-1}\mathfrak{a}_n$. This means that $\mathfrak{b}_n$ differs from $\mathfrak{a}_n$ only by a bounded amount. Since $\mathfrak{a}_n$ goes to 0 as $n \to \infty$, the desired assertion follows.

In the presence of non-standard isogenies a slightly modified argument applies. Let $\psi : H \to H^\sharp$ be the isogeny of Proposition 1.6, and set $T^\sharp := \psi(T)$. By the definition of $\mathfrak{h}_{\ell t}$, the image of $\operatorname{Lie} T$ in $\operatorname{Lie} T^\sharp$ injects into $\mathfrak{h}_{\ell t}$. Repeating the above argument with the logarithm of $\psi(t)^{p^n} \in T^\sharp \subset H^\sharp$, we find that $d^{-1}\mathfrak{d}_\ell^{p^n} \subset e^{-1}\mathfrak{b}_n$ for all $n \gg 0$, where $0 \neq e \in \mathcal{O}$ is fixed and $\mathfrak{d}_\ell \subset \mathfrak{m}$ is the ideal generated by the elements $\alpha(\psi(t)) - 1$ for all roots α of $H^\sharp$. Now recall from Proposition 1.6 that the short roots of $H^\sharp$ are in one-to-one correspondence with the long roots of H. Since every root in a root system is an integral linear combination of short roots, we find that $\mathfrak{d}_\ell$ is the ideal generated by $\alpha(t) - 1$ for all long roots α of H. Likewise, $\mathfrak{d}$ is the ideal generated by $\alpha(t) - 1$ for all short roots of H. Thus from the choice in Lemma 7.9 we deduce that $\mathfrak{d}^2 \subsetneqq \mathfrak{d}_\ell$. Altogether we find, in terms of fractional ideals, that

$$\frac{\mathfrak{a}_n^2}{\mathfrak{b}_n} \subset \frac{(\tau(\mathfrak{m})^{-1}d^{-1}\mathfrak{d}^{p^n})^2}{ed^{-1}\mathfrak{d}_\ell^{p^n}} = \frac{1}{ed\tau(\mathfrak{m})^2} \cdot \Big(\frac{\mathfrak{d}^2}{\mathfrak{d}_\ell}\Big)^{p^n},$$

for all $n \gg 0$, where the right hand side goes to 0 as $n \to \infty$. This finishes the proof of Lemma 7.12. $\square$

To prove Main Theorem 7.2 it now remains to combine all the information collected so far. Let $\tilde{W} \subset \tilde{\mathfrak{h}}$ be as in (4.13), and consider the $\tau(\mathcal{O})$-lattice $\tilde{\Lambda} := \tilde{W} \cap (\operatorname{Lie} \tilde{\mathcal{H}})$ as in Proposition 6.6. The following two lemmas do not depend on each other.

Lemma 7.13: *For any open ideal $\mathfrak{d}' \subset \mathcal{O}$ there exists an open ideal $\mathfrak{a}' \subset \mathfrak{d}'$ such that condition $\mathbf{Full}_{\mathfrak{a}'/\mathfrak{a}'\mathfrak{d}'}$ of Definition 6.7 holds.*

Proof: By Corollary 4.19 there exists an open ideal $\mathfrak{c}' \subset \mathcal{O}$ such that

$$\operatorname{Hom}_{\tau(\mathcal{O})}(\Lambda_{\ell t}, \mathfrak{c}\tilde{\Lambda}) \subset \mathcal{J}_{\hat{\rho}},$$

where $\mathfrak{c} := \tau(\mathfrak{c}')\mathcal{O}$. Thus from the definition of $\mathfrak{b}'_n$ we obtain

$$\begin{aligned}
\mathfrak{c} \cdot \big(\tilde{\Lambda} \otimes_{\tau(\mathcal{O})} \big(\tau(\mathfrak{b}'_n)/\tau({\mathfrak{a}'_n}^2)\big)\big) &= \operatorname{Hom}_{\tau(\mathcal{O})}(\Lambda_{\ell t}, \mathfrak{c}\tilde{\Lambda}) \cdot \log_{\mathfrak{a}_n/\mathfrak{a}_n^2}(t^{p^n}) \\
&\subset \mathcal{J}_{\hat{\rho}} \cdot \log_{\mathfrak{a}_n/\mathfrak{a}_n^2}(t^{p^n}) \\
&\subset \tilde{\Lambda} \otimes_{\tau(\mathcal{O})} \big(\tau(\mathfrak{a}'_n)/\tau({\mathfrak{a}'_n}^2)\big)
\end{aligned}$$

for all $n \geq 0$. By Lemma 7.12 we may choose $n \gg 0$ such that ${\mathfrak{a}'_n}^2 \subset \mathfrak{b}'_n\mathfrak{c}'\mathfrak{d}'$. Using (6.5) we find that

$$\tilde{\Lambda} \otimes_{\tau(\mathcal{O})} \big(\tau(\mathfrak{b}'_n\mathfrak{c}')/\tau({\mathfrak{a}'_n}^2)\big) \subset \log_{\mathfrak{a}_n/\mathfrak{a}_n^2}\Big(\frac{\tilde{\varphi}(\tilde{\Gamma}) \cap \mathcal{H}(\mathfrak{a}_n)}{\tilde{\varphi}(\tilde{\Gamma}) \cap \mathcal{H}(\mathfrak{a}_n^2)}\Big).$$

Now Proposition 6.6 implies condition $\mathbf{Full}_{\mathfrak{b}'_n\mathfrak{c}'/{\mathfrak{a}'_n}^2}$. Setting $\mathfrak{a}' := \mathfrak{b}'_n\mathfrak{c}'$, the lemma follows. $\square$

Lemma 7.14: *For any open ideal $\mathfrak{c}' \subset \mathfrak{m}$ there exists an integer $\ell \geq 1$ such that, for any open ideal $\mathfrak{a}' \subset \mathfrak{c}'^{\ell+1}$, the condition $\mathbf{Full}_{\mathfrak{a}'/\mathfrak{a}'\mathfrak{c}'^{\ell}}$ implies $\mathbf{Full}_{\mathfrak{a}'\mathfrak{c}'/\mathfrak{a}'\mathfrak{c}'^{\ell+1}}$.*

Proof: Let $\bar{\Lambda}_\ell$ be the image of $\tilde{\Lambda}$ in $\bar{W}_\ell$. This is a $\tau(\mathcal{O})$-lattice. Let $\mathfrak{c} := \tau(\mathfrak{c}')\mathcal{O}$. Since $\mathfrak{c}' \subset \mathfrak{m}$, Corollary 4.16 implies that

$$\mathrm{Hom}_{\tau(\mathcal{O})}(\bar{\Lambda}_\ell, \mathfrak{c}^\ell \tilde{\Lambda}) \subset \big(\mathrm{Hom}_{\tau(\mathcal{O})}(\bar{\Lambda}_\ell, \mathfrak{c}\tilde{\Lambda}) \cap \mathcal{J}_{\tilde{\rho}}\big) \cdot \tau(\mathcal{O}) \tag{7.15}$$

for any large enough integer $\ell \geq 1$. We shall prove that any such choice of ℓ does the job. Consider any open ideal $\mathfrak{a}'$ as in the lemma, and set $\mathfrak{a} := \tau(\mathfrak{a}')\mathcal{O}$. Then we have a commutative diagram with exact columns:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\log_{\mathfrak{a}\mathfrak{c}^\ell/\mathfrak{a}\mathfrak{c}^{\ell+1}}\Big(\dfrac{\tilde{\varphi}(\tilde{\Gamma}) \cap \mathcal{H}(\mathfrak{a}\mathfrak{c}^\ell)}{\tilde{\varphi}(\tilde{\Gamma}) \cap \mathcal{H}(\mathfrak{a}\mathfrak{c}^{\ell+1})}\Big) & \subset & \tilde{\Lambda} \otimes_{\tau(\mathcal{O})} \big(\tau(\mathfrak{a}'\mathfrak{c}'^\ell)/\tau(\mathfrak{a}'\mathfrak{c}'^{\ell+1})\big) \\
\downarrow & & \downarrow \\
\log_{\mathfrak{a}/\mathfrak{a}\mathfrak{c}^{\ell+1}}\Big(\dfrac{\tilde{\varphi}(\tilde{\Gamma}) \cap \mathcal{H}(\mathfrak{a})}{\tilde{\varphi}(\tilde{\Gamma}) \cap \mathcal{H}(\mathfrak{a}\mathfrak{c}^{\ell+1})}\Big) & \subset & \tilde{\Lambda} \otimes_{\tau(\mathcal{O})} \big(\tau(\mathfrak{a}')/\tau(\mathfrak{a}'\mathfrak{c}'^{\ell+1})\big) \\
\downarrow & & \downarrow \\
\log_{\mathfrak{a}/\mathfrak{a}\mathfrak{c}^{\ell}}\Big(\dfrac{\tilde{\varphi}(\tilde{\Gamma}) \cap \mathcal{H}(\mathfrak{a})}{\tilde{\varphi}(\tilde{\Gamma}) \cap \mathcal{H}(\mathfrak{a}\mathfrak{c}^{\ell})}\Big) & = & \tilde{\Lambda} \otimes_{\tau(\mathcal{O})} \big(\tau(\mathfrak{a}')/\tau(\mathfrak{a}'\mathfrak{c}'^{\ell})\big) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array} \tag{7.16}$$

Note that $\mathrm{Hom}_{\tau(\mathcal{O})}(\bar{\Lambda}_\ell, \mathfrak{c}\tilde{\Lambda}) \cap \mathcal{J}_{\tilde{\rho}}$ annihilates the term on the upper right of Diagram (7.16). Therefore we have

$$\begin{aligned}
&\tilde{\Lambda} \otimes_{\tau(\mathcal{O})} \big(\tau(\mathfrak{a}'\mathfrak{c}'^\ell)/\tau(\mathfrak{a}'\mathfrak{c}'^{\ell+1})\big) \\
&\quad = \mathrm{Hom}_{\tau(\mathcal{O})}(\bar{\Lambda}_\ell, \mathfrak{c}^\ell\tilde{\Lambda}) \cdot \big(\tilde{\Lambda} \otimes_{\tau(\mathcal{O})} \big(\tau(\mathfrak{a}')/\tau(\mathfrak{a}'\mathfrak{c}'^{\ell+1})\big)\big) \\
&\quad \overset{(7.15)}{\subset} \big(\mathrm{Hom}_{\tau(\mathcal{O})}(\bar{\Lambda}_\ell, \mathfrak{c}\tilde{\Lambda}) \cap \mathcal{J}_{\tilde{\rho}}\big) \cdot \tau(\mathcal{O}) \cdot \big(\tilde{\Lambda} \otimes_{\tau(\mathcal{O})} \big(\tau(\mathfrak{a}')/\tau(\mathfrak{a}'\mathfrak{c}'^{\ell+1})\big)\big) \\
&\quad = \big(\mathrm{Hom}_{\tau(\mathcal{O})}(\bar{\Lambda}_\ell, \mathfrak{c}\tilde{\Lambda}) \cap \mathcal{J}_{\tilde{\rho}}\big) \cdot \big(\tilde{\Lambda} \otimes_{\tau(\mathcal{O})} \big(\tau(\mathfrak{a}')/\tau(\mathfrak{a}'\mathfrak{c}'^{\ell+1})\big)\big) \\
&\quad \overset{!}{=} \big(\mathrm{Hom}_{\tau(\mathcal{O})}(\bar{\Lambda}_\ell, \mathfrak{c}\tilde{\Lambda}) \cap \mathcal{J}_{\tilde{\rho}}\big) \cdot \log_{\mathfrak{a}/\mathfrak{a}\mathfrak{c}^{\ell+1}}\Big(\frac{\tilde{\varphi}(\tilde{\Gamma}) \cap \mathcal{H}(\mathfrak{a})}{\tilde{\varphi}(\tilde{\Gamma}) \cap \mathcal{H}(\mathfrak{a}\mathfrak{c}^{\ell+1})}\Big) \\
&\quad \subset \log_{\mathfrak{a}/\mathfrak{a}\mathfrak{c}^{\ell+1}}\Big(\frac{\tilde{\varphi}(\tilde{\Gamma}) \cap \mathcal{H}(\mathfrak{a})}{\tilde{\varphi}(\tilde{\Gamma}) \cap \mathcal{H}(\mathfrak{a}\mathfrak{c}^{\ell+1})}\Big).
\end{aligned}$$

It follows that the first inclusion in Diagram (7.16) is in fact an equality. By the 5-Lemma the same holds for the middle inclusion. In other word, the condition $\mathbf{Full}_{\mathfrak{a}'/\mathfrak{a}'\mathfrak{c}'^{\ell+1}}$ holds. This directly implies the desired condition $\mathbf{Full}_{\mathfrak{a}'\mathfrak{c}'/\mathfrak{a}'\mathfrak{c}'^{\ell+1}}$. □

To finish the proof of Main Theorem 7.2, take any open ideal $\mathfrak{c}' \subset \mathfrak{m}$ and let $\ell \geq 1$ be as in Lemma 7.14. By Lemma 7.13 there exists an open ideal $\mathfrak{a}' \subset \mathfrak{c}'^{\ell+1}$ such that the condition $\mathbf{Full}_{\mathfrak{a}'/\mathfrak{a}'\mathfrak{c}'^{\ell+1}}$ holds. In particular, condition $\mathbf{Full}_{\mathfrak{a}'/\mathfrak{a}'\mathfrak{c}'^{\ell}}$ is true. By induction on $n \geq 0$, using Lemma 7.14, we find that condition $\mathbf{Full}_{\mathfrak{a}'\mathfrak{c}'^{n}/\mathfrak{a}'\mathfrak{c}'^{n+\ell}}$ holds for all $n \geq 0$. In particular we know that $\mathbf{Full}_{\mathfrak{a}'\mathfrak{c}'^{n}/\mathfrak{a}'\mathfrak{c}'^{n+1}}$ is true for all $n \geq 0$. From Corollary 6.8 it follows that $\tilde{\Gamma}$ is open in $G(F)$, as desired. This finishes the proof of Main Theorem 7.2. $\square$

Proof of Main Theorem 0.2: In the situation of Main Theorem 0.2, let (E, H, φ) be a quasi-model of (F, G, Γ) in the sense of Definition 3.1. Let $\tilde{H} \to H$ be the universal covering of H, and $\tilde{\varphi} : \tilde{H} \times_E F \longrightarrow \tilde{G}$ the lift of φ. Let $\Gamma' \subset \tilde{G}(F)$ be the closure of the generalized commutator group of Γ. Then $\tilde{\varphi}^{-1}(\Gamma')$ is the closure of the generalized commutator group of $\varphi^{-1}(\Gamma)$. By Proposition 7.1 and Corollary 7.3 this subgroup is open in $\tilde{H}(E)$ if and only if (E, H, φ) is minimal in the sense of Definition 3.5. The existence and uniqueness of such (E, H, φ) is guaranteed by Theorem 3.6. This proves Main Theorem 0.2. $\square$

Proof of Corollary 0.3: For each $i = 1, 2$ let Γ_i' be the closure of the commutator group of $\tilde{\Gamma}_i$, and G_i the adjoint group of $\tilde{G}_i$. Let $\Gamma \subset G_1(F_1) \times G_2(F_2)$ be the image of the graph of the isomorphism f. Then the closure Γ' of the generalized commutator group of Γ is just the graph of the isomorphism $\Gamma_1' \xrightarrow{\sim} \Gamma_2'$ induced by f. Let (E, H, φ) be a minimal quasi-model of $(F_1 \oplus F_2, G_1 \sqcup G_2, \Gamma)$. By Corollary 7.3 the group Γ' is an open subgroup of $\tilde{\varphi}(\tilde{H}(E))$. Thus the map $\tilde{H}(E) \longrightarrow \tilde{G}_i(F_i)$ is a local isomorphism for each $i = 1, 2$. It follows that both $E \to F_i$ and $\tilde{H} \to \tilde{G}_i$ are isomorphisms. This yields the desired isomorphism in Corollary 0.3. Its uniqueness follows from the uniqueness of minimal quasi-models. $\square$

Proof of Corollary 0.4: As in the introduction write G as a direct product of Weil restrictions $\prod_{i=1}^{m} \mathcal{R}_{F_i/F} G_i$, where each G_i is an absolutely simple adjoint group over a finite separable extension F_i of F. Let G' be the algebraic group over $F' := \bigoplus_{i=1}^{m} F_i$ whose fiber over each F_i is G_i. We can then view Γ as a subgroup of $G'(F')$ and apply Main Theorem 0.2 to (F', G', Γ). Let (E', H', φ') be the resulting quasi-model. As E' is a finite direct sum of local fields of the same characteristic and the same residue characteristic, we can identify each simple summand with a finite separable extension of a fixed local field E. The Weil restriction $H := \mathcal{R}_{E'/E} H'$ is then a connected adjoint group over E, and its universal covering $\tilde{H}$ is the Weil restriction of the universal covering $\tilde{H}'$ of H'. By Main Theorem 0.2 (a) the closure of the commutator subgroup of Γ is isomorphic to the image of some open compact subgroup of $\tilde{H}'(E') \cong \tilde{H}(E)$, as desired. $\square$

Proof of Proposition 0.6: We must show that the characterizations are correct for a minimal quasi-model of (F, G, Γ). In the situation of Proposition 0.6 (a) suppose first that ρ is in the image of $d\varphi : (\operatorname{Lie} H) \otimes_E F \longrightarrow \operatorname{Lie} G$. Then the assertion follows from Proposition 3.10. Otherwise φ is not an isomorphism, so $p := \operatorname{char}(F)$ is equal to 2 or 3, and ρ corresponds to the representation $\operatorname{Frob}_p \circ \alpha_s^H$. Now Proposition 3.10 implies that $E_{\alpha_s^H} = E$, hence $E_\rho = \{x^p \mid x \in E\}$, as desired.

In particular, Proposition 0.6 (a) says that $\operatorname{tr}(\rho(\Gamma)) \subset E$ for every non-constant irreducible subquotient representation of $\operatorname{Lie} G$. Since the same assertion is obvious for any constant representation, it is true for every subquotient of $\operatorname{Lie} G$. This implies Proposition 0.6 (b).

At last, in the situation of Proposition 0.6 (c) we consider the same representation ρ as in the proof of Lemma 3.7. It occurs inside $\operatorname{Lie} G$, but also comes from a representation inside $\operatorname{Lie} H$. Therefore we have $F = E_\rho \subset E$. To prove the isomorphy of φ we can reduce ourselves to the case that F is a field. If φ is not an isomorphism, then $p := \operatorname{char}(F)$ is positive and $\alpha_\ell^G \circ \varphi \cong \operatorname{Frob}_p \circ \alpha_s^H$. Thus $F = E_{\alpha_\ell^G} \subset \{x^p \mid x \in E\} \subsetneqq F$, which is a contradiction. Now everything is proved. $\square$

REFERENCES

[1] A. Borel, Linear Algebraic Groups. GTM **126**, New York etc.: Springer 1991

[2] N. Bourbaki, Algèbre, chap. 8. Paris: Hermann 1958

[3] A. Borel, J. Tits, Compléments à l'Article: "Groupes Réductifs." Publ. Math. IHES **41** (1972), 253–276

[4] C. W. Curtis, I. Reiner, Methods of Representation Theory, vol.I. New York etc.: Wiley 1990

[5] R. Hartshorne, Algebraic Geometry. GTM **52**, New York etc.: Springer 1977

[6] G. Hiss, Die adjungierten Darstellungen der Chevalley-Gruppen. Arch. Math. **42** (1984), 408–416

[7] G. M. D. Hogeweij, Almost Classical Lie Algebras I. Indagationes Math. **44** (1982), 441 - 460

[8] J. E. Humphreys, Linear Algebraic Groups. GTM **21**, New York etc.: Springer 1975, 1981

[9] A. L. Onishchik, E. B. Vinberg (Eds.): Lie groups and Lie algebras III, Encyclopedia of Mathematical Sciences **41**. Berlin etc.: Springer 1994

[10] R. Pink, The Mumford-Tate conjecture for Drinfeld modules. *Publ. RIMS, Kyoto University* **33** (1997), 393–425

[11] M. Takeuchi, A Hyperalgebraic Proof of the Isomorphism and Isogeny Theorems for Reductive Groups. J. Algebra **85** (1983), 179–196

[12] E. B. Vinberg, Rings of definition of dense subgroups of semisimple linear groups. Mathematics of the USSR Izvestija **5** (1971), 45–55

[13] B. Weisfeiler, Strong approximation for Zariski dense subgroups of semi-simple algebraic groups. Annals of Mathematics, **120** (1984), 271–315