On Weil restriction of reductive groups and a theorem of Prasad

by

Richard Pink

Departement Mathematik
ETH Zentrum
CH-8092 Zürich
Switzerland
pink@math.ethz.ch

June 19, 2000

Abstract

Let G be a connected simple semisimple algebraic group over a local field F of arbitrary characteristic. In a previous article by the author the Zariski dense compact subgroups of $G(F)$ were classified. In the present paper this information is used to give another proof of a theorem of Prasad [8] (also proved by Margulis [3]) which asserts that, if G is isotropic, every non-discrete closed subgroup of finite covolume contains the image of $\tilde{G}(F)$, where $\tilde{G}$ denotes the universal covering of G. This result played a central role in Prasad's proof of strong approximation. The present proof relies on some basic properties of Weil restrictions over possibly inseparable field extensions, which are also proved here.[1]

[1]MR 2000 classification: primary: 20G25, secondary: 14L15

Acknowledgements: The author would like to thank Gopal Prasad and Marc Burger for interesting conversations, and especially to the former for suggesting a combination of the methods of [8] and [5] to obtain another proof of strong approximation in arbitrary characteristic.

1 Weil restriction of linear algebraic groups

Let F be a field and F' a subfield such that $[F/F'] < \infty$. In this section we discuss some properties of the Weil restriction $\mathcal{R}_{F/F'}G$ where G is a linear algebraic group over F. We are interested particularly in the case that F/F' is inseparable, where the Weil restriction involves some infinitesimal aspects. Thus the natural setting is that of group schemes. We assume that G is a connected affine group scheme of finite type that is smooth over F. The smoothness condition is equivalent to saying that G is reduced and "defined over F" in the terminology of [11] Ch.11.

Throughout, we will speak of a scheme over a ring R when we really mean a scheme over $\mathbf{Spec}\,R$. Similarly, for any ring homomorphism $R' \to R$ and any scheme X' over R' we will abbreviate $X' \times_{R'} R := X' \times_{\mathbf{Spec}\,R'} \mathbf{Spec}\,R$. The basic facts on Weil restrictions that we need are summarized in [4] Appendix 2–3. Throughout the following we abbreviate

$$G' := \mathcal{R}_{F/F'}G.$$

By [4] A.3.2, A.3.7 this is a connected smooth affine group scheme over F'. The universal property of the Weil restriction identifies $G'(F')$ with $G(F)$.

Next, we fix an algebraic closure E' of F' and abbreviate $E := F \otimes_{F'} E'$. With $\Sigma := \mathrm{Hom}_{F'}(F, E')$ there is then a unique decomposition $E = \bigoplus_{\sigma\in\Sigma} E_\sigma$, where each E_σ is a local ring with residue field E' and the composite map $F \to E_\sigma \twoheadrightarrow E'$ is equal to σ. The Weil restriction from any finite dimensional commutative E'-algebra down to E' is defined, and by [4] A.2.7–8 we have natural isomorphisms

$$\begin{aligned} G' \times_{F'} E' &\cong \mathcal{R}_{E/E'}(G \times_F E) \\ &= \mathcal{R}_{E/E'}\left(\coprod_{\sigma\in\Sigma} G \times_F E_\sigma\right) \\ &\cong \prod_{\sigma\in\Sigma} G_\sigma \end{aligned} \tag{1.1}$$

with

$$G_\sigma := \mathcal{R}_{E_\sigma/E'}(G \times_F E_\sigma).$$

These isomorphisms are functorial in G and equivariant under $\mathrm{Aut}(E'/F')$, which acts on the right hand side by permuting the factors according to its action on Σ. Next, for every $\sigma \in \Sigma$ we fix a filtration of E_σ by ideals

$$E_\sigma \supsetneqq I_{\sigma,1} \supsetneqq \ldots \supsetneqq I_{\sigma,q-1} \supsetneqq I_{\sigma,q} = 0$$

with subquotients of length 1. Here q is the degree of the inseparable part of F/F'. We also choose a basis of every successive subquotient. For every $1 \leq i \leq q$ there is a natural homomorphism

$$G_\sigma = \mathcal{R}_{E_\sigma/E'}(G \times_F E_\sigma) \longrightarrow \mathcal{R}_{(E_\sigma/I_{\sigma,i})/E'}\big(G \times_F (E_\sigma/I_{\sigma,i})\big).$$

Let $G_{\sigma,i}$ denote its kernel. By [4] A.3.5 we find that each $G_{\sigma,i}$ is smooth over F' and there are canonical isomorphisms

$$G_\sigma/G_{\sigma,1} \cong G \times_{F,\sigma} E' \tag{1.2}$$

and

$$G_{\sigma,i}/G_{\sigma,i+1} \cong \operatorname{Lie} G \otimes_{F,\sigma} \mathbb{G}_{a,E'} \tag{1.3}$$

for all $1 \leq i \leq q-1$, where $\mathbb{G}_a$ denotes the additive group of dimension 1. Moreover, this description is functorial in G. Namely, let H be another smooth group scheme over F and define $H' := \mathcal{R}_{F/F'}H$, H_σ and $H_{\sigma,i}$ in the obvious way. Then any homomorphism $\varphi\colon H \to G$ induces homomorphisms $\mathcal{R}_{F/F'}\varphi\colon H' \to G'$, $H_\sigma \to G_\sigma$ and $H_{\sigma,i} \to G_{\sigma,i}$ and the resulting homomorphisms on subquotients are just

$$\varphi \times \mathrm{id}\colon H \times_{F,\sigma} E' \longrightarrow G \times_{F,\sigma} E' \tag{1.4}$$

and

$$d\varphi \otimes \mathrm{id}\colon \operatorname{Lie} H \otimes_{F,\sigma} \mathbb{G}_{a,E'} \longrightarrow \operatorname{Lie} G \otimes_{F,\sigma} \mathbb{G}_{a,E'}. \tag{1.5}$$

Recall that an *isogeny* of algebraic groups is a surjective homomorphism with finite kernel. An isogeny φ is *separable* if and only if its derivative $d\varphi$ is an isomorphism.

Proposition 1.6 *Let $\varphi\colon H \to G$ be a homomorphism of connected smooth linear algebraic groups over F.*

(a) If F/F' is separable, then $\mathcal{R}_{F/F'}\varphi\colon H' \to G'$ is an isogeny if and only if φ is an isogeny.

(b) If F/F' is inseparable, then $\mathcal{R}_{F/F'}\varphi\colon H' \to G'$ is an isogeny if and only if φ is a separable isogeny.

Proof. In the separable case we have $E' \xrightarrow{\sim} E_\sigma$, and assertion (a) follows directly from the decomposition 1.1 and the functoriality 1.4. So assume that F/F' is inseparable, i.e., that $q > 1$. First note that $\dim H' = [F/F'] \cdot \dim H$ and $\dim G' = [F/F'] \cdot \dim G$, by the successive extension above or by [4] A.3.3. Thus if either φ or $\mathcal{R}_{F/F'}\varphi$ is an isogeny, we must have $\dim H = \dim G$.

If $\mathcal{R}_{F/F'}\varphi$ is an isogeny, its kernel is finite; hence so is the kernel of its restriction $H_{\sigma,q-1} \to G_{\sigma,q-1}$. By 1.5 this means that $d\varphi$ is injective. For dimension reasons it follows that $d\varphi$ is an isomorphism; hence φ is a separable isogeny, as desired.

Conversely, suppose that φ is a separable isogeny. Then all the homomorphisms on subquotients 1.4 and 1.5 induced by $\mathcal{R}_{F/F'}\varphi$ are surjective. Using the snake lemma inductively one deduces that $\mathcal{R}_{F/F'}\varphi$ itself is surjective. For dimension reasons it is therefore an isogeny, as desired. **q.e.d.**

Theorem 1.7 *If G is reductive and F' infinite, then $G'(F')$ is Zariski dense in G'.*

Proof. If F/F' is separable, the isomorphism 1.1 shows that G' is reductive. In that case the assertion is well-known: see [11] Cor.13.3.12 (i). We will adapt the argument to the general case.

Assume first that $G = T$ is a torus. Choose a finite separable extension F_1/F which splits T, and fix an isomorphism $\mathbb{G}_{m,F_1}^n \xrightarrow{\sim} T \times_F F_1$, where $\mathbb{G}_m$ denotes the multiplicative group of dimension 1. Combining this with the norm map yields a surjective homomorphism

$$\mathcal{R}_{F_1/F}\mathbb{G}_{m,F_1}^n \longrightarrow \mathcal{R}_{F_1/F}(T \times_F F_1) \xrightarrow{\mathrm{Nm}} T.$$

Since F_1/F is separable, this morphism is smooth. By [4] A.2.4, A.2.12 it induces a smooth homomorphism

$$\mathcal{R}_{F_1/F'}\mathbb{G}_{m,F_1}^n \cong \mathcal{R}_{F/F'}\mathcal{R}_{F_1/F}\mathbb{G}_{m,F_1}^n \longrightarrow \mathcal{R}_{F/F'}T.$$

In particular, this morphism is dominant. On the other hand we have an open embedding $\mathbb{G}_{m,F_1}^n \hookrightarrow \mathbb{A}_{F_1}^n$ and hence, by [4] A.2.11, an open embedding $\mathcal{R}_{F_1/F'}\mathbb{G}_{m,F_1}^n \hookrightarrow \mathcal{R}_{F_1/F'}\mathbb{A}_{F_1}^n$. It is trivial to show that $\mathcal{R}_{F_1/F'}\mathbb{A}_{F_1}^n \cong \mathbb{A}_{F'}^{nd}$, where $d = [F_1/F']$. It follows that the F'-rational points in $\mathcal{R}_{F_1/F'}\mathbb{G}_{m,F_1}^n$ are Zariski dense, and so their images form a Zariski dense set of F'-rational points in $\mathcal{R}_{F/F'}T$, proving the theorem in this case.

If G is arbitrary let T be a maximal torus of G. As $\mathcal{R}_{F/F'}T$ is commutative, it possesses a unique maximal torus T', which is smooth over F' by [11] Thm.13.3.6.

Lemma 1.8 *$\mathcal{R}_{F/F'}T$ is the centralizer of T' in G'.*

Proof. If F/F' is separable, this follows from the fact that $\mathcal{R}_{F/F'}T$ is a maximal torus of G'. So assume that F/F' is inseparable of characteristic p. Since $(\mathcal{R}_{F/F'}T)/T'$ is unipotent, we have $T' = (\mathcal{R}_{F/F'}T)^{p^n}$ for suitable $n \gg 0$. As T' is smooth and the rational points of $\mathcal{R}_{F/F'}T$ are Zariski dense, the centralizer of T' is equal to the centralizer of $(\mathcal{R}_{F/F'}T)(F')^{p^n}$. Note that the universal property of the Weil restriction identifies $(\mathcal{R}_{F/F'}T)(F')$ with $T(F)$.

Consider a scheme S' over F' and an S'-valued point $\varphi'\colon S' \to G'$. Via the universal property of the Weil restriction φ' corresponds to an $S' \times_{F'} F$-valued point $\varphi\colon S' \times_{F'} F \to G$. We have seen that φ' factors through the centralizer of T' if and only if it commutes with $(\mathcal{R}_{F/F'}T)(F')^{p^n}$. This is equivalent to saying that φ commutes with $T(F)^{p^n}$. As T is a torus and F infinite, the subgroup $T(F)^{p^n}$ is Zariski dense in T. The condition therefore amounts to saying that φ factors through the centralizer of T. But this centralizer is equal to T. Therefore, translated back to G', the condition says that φ' factors through $\mathcal{R}_{F/F'}T$. This proves the lemma. **q.e.d.**

By Lemma 1.8 the subgroup $\mathcal{R}_{F/F'}T$ is the centralizer of a maximal torus of G', i.e., it is a Cartan subgroup of G'. Thus [11] Cor.13.3.12 implies that $G'(F')$ is Zariski dense in G', proving Theorem 1.7. **q.e.d.**

Remark 1.9 If F' is a non-discrete complete normed field, Theorem 1.7 is true for arbitrary connected smooth algebraic groups G. This is an easy consequence of the implicit function theorem.

Next we turn to simple groups. To fix ideas, a smooth linear algebraic group over a field will be called *simple* if it is non-trivial and possesses no non-trivial proper connected smooth normal algebraic subgroup. It is called *absolutely simple* if it remains simple over the algebraic closure of the base field.

If G is simply connected semisimple and simple over F, it is isomorphic to $\mathcal{R}_{F_1/F}G_1$ for an absolutely simple simply connected semisimple group G_1 over some finite separable extension F_1/F (cf. [11] Ex.16.2.9). From [4] A.2.4 we then deduce that $G' \cong \mathcal{R}_{F_1/F'}G_1$. In this way questions about G' can be reduced to the case that G is absolutely simple.

Theorem 1.10 *Assume that G is simply connected semisimple and simple over F. Then G' is simple over F'.*

Proof. By the above remarks we may assume that G is absolutely simple. Consider a non-trivial connected smooth normal algebraic subgroup $H' \subset G'$. Let

$$\bar{H}' \subset \prod_{\sigma\in\Sigma} G \times_{F,\sigma} E' \tag{1.11}$$

denote the image of $H' \times_{F'} E'$ under the composite of the natural maps

$$G' \times_{F'} E' \stackrel{1.1}{\cong} \prod_{\sigma\in\Sigma} G_\sigma \longrightarrow\!\!\!\!\rightarrow \prod_{\sigma\in\Sigma} G_\sigma/G_{\sigma,1} \stackrel{1.2}{\cong} \prod_{\sigma\in\Sigma} G \times_{F,\sigma} E'.$$

Since H' is non-trivial and "defined over F'", by [11] Cor.12.4.3 we have $\bar{H}' \neq 1$. Since $H' \subset G'$ is a connected normal subgroup, so is $\bar{H}'$ in 1.11. It is therefore equal to the product of some of the factors on the right hand side. As $\bar{H}'$ is non-trivial, it contains at least one of these factors. But by construction it is also invariant under $\mathrm{Aut}(E'/F')$, which permutes the factors transitively. We deduce that the inclusion 1.11 is in fact an equality. Now the following lemma implies that $H' \times_{F'} E' = G' \times_{F'} E'$; and hence $H' = G'$, as desired. **q.e.d.**

Lemma 1.12 *In the situation of Theorem 1.10, every normal algebraic subgroup $H \subset G' \times_{F'} E'$ which surjects to $\prod_{\sigma\in\Sigma} G \times_{F,\sigma} E'$ is equal to $G' \times_{F'} E'$.*

Proof. Using descending induction on i we will prove that $G_{\sigma,i} \subset H$ for all $\sigma \in \Sigma$ and $1 \leq i \leq q$. For $i = q$ the assertion is obvious, because $G_{\sigma,q} = 1$. Let us assume the inclusion for $G_{\sigma,i+1}$ and abbreviate

$$\mathrm{gr}_i\, H_\sigma := \frac{H \cap G_{\sigma,i}}{G_{\sigma,i+1}} \subset \frac{G_{\sigma,i}}{G_{\sigma,i+1}} \stackrel{1.3}{\cong} \mathrm{Lie}\, G \otimes_{F,\sigma} \mathbb{G}_{a,E'}. \tag{1.13}$$

By functoriality of the isomorphism 1.3, the conjugation action of $G'(E')$ on $G_{\sigma,i}$ corresponds to the adjoint representation of $G \times_{F,\sigma} E'$ on the right hand side. As H is a normal subgroup, all commutators between H and $G_{\sigma,i}$ must lie in H. It follows that

$$(\mathrm{Ad}_h - \mathrm{id})(\mathrm{Lie}\, G) \otimes_{F,\sigma} \mathbb{G}_{a,E'} \subset \mathrm{gr}_i\, H_\sigma \tag{1.14}$$

for every $h \in H(E')$. Since G is simply connected, it is known that the space of coinvariants of its adjoint representation is trivial (cf. [1], [2], or [5] Prop.1.11). On the other hand E' is algebraically closed, so by assumption $H(E')$ maps to a Zariski dense subgroup of $G \times_{F,\sigma} E'$. Thus, as h varies, the subgroups in 1.14 generate $\mathrm{Lie}\, G \otimes_{F,\sigma} \mathbb{G}_{a,E'}$. The inclusion in 1.13 is therefore an equality, and so we have $G_{\sigma,i} \subset H$.

At the end of the induction we have $G_{\sigma,1} \subset H$ for all $\sigma \in \Sigma$. Combining this with the fact that H surjects to $\prod_{\sigma\in\Sigma} G_\sigma/G_{\sigma,1}$, we finally deduce $H = G' \times_{F'} E'$, as desired. This proves Lemma 1.12 and thereby finishes the proof of Theorem 1.10. **q.e.d.**

Remark 1.15 The analogue of Theorem 1.10 fails if G is not simply connected and both F/F' and the universal central extension $\pi : \tilde{G} \to G$ are inseparable. The reason is that by Proposition 1.6 (b) the homomorphism $\mathcal{R}_{F/F'}\varphi \colon \mathcal{R}_{F/F'}\tilde{G} \to G'$ is not surjective, so its image is a subgroup that makes G' not simple.

Corollary 1.16 *If G is semisimple and simply connected, then G' is perfect.*

Proof. We may assume that G is simple. Then G is connected and non-commutative; hence so is G'. The commutator group of G' is therefore non-trivial connected and normal, and by [11] Cor.2.2.8 it is "defined over F" and thus smooth. By Theorem 1.10 it is therefore equal to G', as desired. **q.e.d.**

Theorem 1.17 *If G is simple isotropic and simply connected and F is infinite, then G' is generated by split tori.*

Proof. By assumption there exists a closed embedding $\mathbb{G}_{m,F'} \times_{F'} F \cong \mathbb{G}_{m,F} \hookrightarrow G$. The homomorphism $\mathbb{G}_{m,F'} \to G'$ corresponding to it by the universal property of the Weil restriction is again non-trivial; hence G' contains a non-trivial split torus. The algebraic subgroup of G' that is generated by all split tori in G' is therefore non-trivial. By construction it is normalized by $G'(F')$, so by Theorem 1.7 it is normal in G'. Being generated by smooth connected subgroups, it is itself smooth and connected by [11] Prop.2.2.6 (iii). By Theorem 1.10 it is therefore equal to G', as desired. **q.e.d.**

2 Main results

In the following we consider a connected semisimple group G over a local field F. Let $\pi\colon \tilde{G} \to G$ denote its universal central extension. The commutator pairing $\tilde{G} \times \tilde{G} \to \tilde{G}$ factors through a unique morphism

$$[\ ,\ ]^{\sim}\colon G \times G \to \tilde{G}.$$

For any closed subgroup $\Gamma \subset G(F)$ we let $\tilde{\Gamma}'$ denote the closure of the subgroup of $\tilde{G}(F)$ that is generated by the set of generalized commutators $[\Gamma, \Gamma]^{\sim}$.

Theorem 2.1 *Let F be a local field, and let G be an isotropic connected simple semisimple group over F. Let $\Gamma \subset G(F)$ be a non-discrete closed subgroup whose covolume for any invariant measure is finite. Then $\tilde{\Gamma}'$ is open in $\tilde{G}(F)$.*

Before proving this, we note the following consequence (cf. [8], [3]).

Corollary 2.2 *Under the assumptions of Theorem 2.1 we have $\tilde{\Gamma}' = \tilde{G}(F)$. In particular, Γ contains $\pi\big(\tilde{G}(F)\big)$.*

Proof. Since $G(F)$ is not compact and Γ is a subgroup of finite covolume, this subgroup is not compact. Thus $\tilde{\Gamma}'$ is normalized by an unbounded subgroup of $G(F)$, and it is open in $\tilde{G}(F)$ by Theorem 2.1. As in [6] Thm.2.2 one deduces from this that $\tilde{\Gamma}'$ is unbounded. Let $\tilde{G}(F)^+$ denote the subgroup of $\tilde{G}(F)$ that is generated by the rational points of the unipotent radicals of all rational parabolic subgroups. The Kneser-Tits conjecture, which is proved in this case (see [7] Thm. 7.6 or [10]), asserts that $\tilde{G}(F)^+ = \tilde{G}(F)$. On the other hand, a theorem of Tits [9] states that every unbounded open subgroup of $\tilde{G}(F)^+$ is equal to $\tilde{G}(F)^+$. Altogether this implies $\tilde{\Gamma}' = \tilde{G}(F)$, as desired. **q.e.d.**

Proof of Theorem 2.1. In the case $\mathrm{char}(F) = 0$ the proof in [8] §2 cannot be improved. It covers in particular the archimedean case. We will give a unified proof in the non-archimedean case, beginning with a few reductions.

Let Γ^{ad} denote the image of Γ in the adjoint group G^{ad} of G. Then $\tilde{\Gamma}'$ depends only on Γ^{ad}. On the other hand, all the assumptions in 2.1 are still satisfied for $\Gamma^{\mathrm{ad}} \subset G^{\mathrm{ad}}(F)$. Namely, since the homomorphism $G(F) \to G^{\mathrm{ad}}(F)$ is proper with finite kernel, the subgroup Γ^{ad} is still non-discrete and closed. On the other hand, as the image of $G(F)$ in $G^{\mathrm{ad}}(F)$ is cocompact, the covolume of Γ^{ad} in $G^{\mathrm{ad}}(F)$ is again finite. To prove the theorem, we may therefore replace G by G^{ad} and Γ by Γ^{ad}. In other words, we may assume that G is adjoint.

Next, since G is connected simple and adjoint, it is isomorphic to $\mathcal{R}_{F_1/F}G_1$ for some absolutely simple connected adjoint group G_1 over a finite separable extension F_1/F. If $\tilde{G}_1$ denotes the universal covering of G_1, we then have $\tilde{G} \cong \mathcal{R}_{F_1/F}\tilde{G}_1$. By the definition of Weil restriction we have $G(F) \cong G_1(F_1)$ and $\tilde{G}(F) \cong \tilde{G}_1(F_1)$; and since G is isotropic, so is G_1. Thus after replacing F by F_1 and G by G_1 we may assume that G is absolutely simple.

For the next preparations note that F is non-archimedean, so $G(F)$ possesses an open compact subgroup. Its intersection with Γ is an open compact subgroup of Γ; let us call it Δ. Let $\tilde{\Delta}'$ denote the closure of the subgroup of $\tilde{G}(F)$ that is generated by the set of generalized commutators $[\Delta, \Delta]^{\sim}$.

We will study the relation between these subgroups and various Weil restrictions of G. Consider any closed subfield $F' \subset F$ such that $[F/F']$ is finite. Note that in the case $\mathrm{char}(F) = 0$ there is a unique smallest such F', namely the closure of $\mathbb{Q}$. But in positive characteristic the extension F/F' may be arbitrarily large and, what is worse, it may be inseparable.

Set $G' := \mathcal{R}_{F/F'}G$ and $\tilde{G}' := \mathcal{R}_{F/F'}\tilde{G}$, and let $\pi' \colon \tilde{G}' \to G'$ be the homomorphism induced by π. From Proposition 1.6 we know that π' is not necessarily an isogeny. Identifying $G(F)$ with $G'(F')$ via the universal property of the Weil restriction, we can view Γ as a non-discrete closed subgroup of finite covolume of $G'(F')$. Similarly, we can view $\tilde{\Delta}'$ as a subgroup of $\tilde{G}'(F')$.

Lemma 2.3 *$\tilde{\Delta}'$ is Zariski dense in $\tilde{G}'$.*

Proof. Let $H' \subset G'$ and $\tilde{H}' \subset \tilde{G}'$ be the Zariski closures of Δ and $\tilde{\Delta}'$, respectively. By [11] Lemma 11.2.4 (ii) these groups are "defined over F'", i.e., smooth over F'. The intersection of Δ with the identity component of H' is open in Δ and thus again an open compact subgroup of Γ. After shrinking Δ we may therefore assume that H' is connected. For any $\gamma \in \Gamma$ the subgroup $\gamma\Delta\gamma^{-1}$ is again an open compact subgroup of Γ, so it is commensurable with Δ. Thus $\gamma H' \gamma^{-1}$ is commensurable with H'. Since H' is connected, they must be equal; hence H' is normalized by Γ. It is therefore also normalized by the Zariski closure of Γ.

Under the assumptions of 2.1, a theorem of Wang [12] implies that the Zariski closure of Γ in G' contains all split tori of G'. Thus, in particular, it contains the images under π' of all split tori in $\tilde{G}'$. Since G is simple isotropic, so is $\tilde{G}$; hence by Theorem 1.17 these tori generate $\tilde{G}'$. It follows that H' is normalized by the image of $\tilde{G}'$. By construction $\tilde{H}'$ is the algebraic subgroup of $\tilde{G}'$ that is generated by the image of the connected variety $H' \times_{F'} H'$ under $[\ ,\ ]^{\sim}$. It is therefore connected and normalized by $\tilde{G}'$.

Since Γ is non-discrete, the group Δ is not finite, and so H' is non-trivial. Let H denote the image of $H' \times_{F'} F$ under the canonical adjunction morphism $G' \times_{F'} F \to G$. By construction H is just the Zariski closure of Δ in G, so by the above arguments in the case $F' = F$ it is normalized by the image of $\tilde{G}$. But $\pi \colon \tilde{G} \to G$ is surjective, so H is a non-trivial connected normal subgroup of G. As G is absolutely simple, this implies $H = G$. As G is perfect, it follows that $\tilde{H}' \times_{F'} F$ surjects to G.

All in all we now deduce that $\tilde{H}'$ is a non-trivial connected smooth normal algebraic subgroup of $\tilde{G}'$. By Theorem 1.10 this implies $\tilde{H}' = \tilde{G}'$, as desired.

q.e.d.

Note that Lemma 2.3 in the case $F' = F$ says that $\tilde{\Delta}'$ is Zariski dense in $\tilde{G}$. In particular Δ is compact and Zariski dense in G, so we can apply [5] Main

Theorem 0.2. It follows that there exists a closed subfield $E \subset F$ such that $[F/E]$ is finite, an absolutely simple and simply connected semisimple algebraic group $\tilde{H}$ over E, and an isogeny $\tilde{\varphi}\colon \tilde{H} \times_E F \to \tilde{G}$ with non-vanishing derivative, such that $\tilde{\Delta}'$ is the image under $\tilde{\varphi}$ of an open subgroup of $\tilde{H}(E)$.

Lemma 2.4 $E = F$.

Proof. Via the universal property of the Weil restriction the isogeny $\tilde{\varphi}$ corresponds to a homomorphism $\tilde{\varphi}'\colon \tilde{H} \to \mathcal{R}_{F/E}\tilde{G}$, which satisfies

$$\tilde{\Delta}' \subset \tilde{\varphi}'(\tilde{H}(E)) \subset (\mathcal{R}_{F/E}\tilde{G})(E) = \tilde{G}(F).$$

By Lemma 2.3 in the case $F' = E$ we know that $\tilde{\Delta}'$ is Zariski dense in $\mathcal{R}_{F/E}\tilde{G}$. It follows that $\tilde{\varphi}'$ is dominant. This implies

$$\dim \tilde{H} \geq \dim \mathcal{R}_{F/E}\tilde{G} = [F/E] \cdot \dim \tilde{G} = [F/E] \cdot \dim \tilde{H};$$

hence $[F/E] = 1$, as desired. **q.e.d.**

Lemma 2.5 *$\tilde{\varphi}$ is an isomorphism.*

Proof. As $\tilde{\varphi}$ is an isogeny between simply connected groups, it is an isomorphism if and only if it is separable. In characteristic zero this is automatically the case. (Since $d\tilde{\varphi} \neq 0$, this is actually true whenever $\mathrm{char}(F) \neq 2, 3$ (cf. [5] Thm.1.7), but we do not need that fact.) So for the rest of the proof we may suppose that $p := \mathrm{char}(F)$ is positive. Set $F' := \{x^p \mid x \in F\}$; then F/F' is an inseparable extension of degree p. Consider the induced homomorphism

$$\tilde{\psi} := \mathcal{R}_{F/F'}\tilde{\varphi}\colon \mathcal{R}_{F/F'}\tilde{H} \longrightarrow \mathcal{R}_{F/F'}\tilde{G}.$$

By construction it satisfies

$$\begin{array}{ccccc} \tilde{\Delta}' & \subset & \tilde{\psi}\big((\mathcal{R}_{F/F'}\tilde{H})(F')\big) & \subset & (\mathcal{R}_{F/F'}\tilde{G})(F') \\ & & \| & & \| \\ & & \tilde{\varphi}\big(\tilde{H}(F)\big) & \subset & \tilde{G}(F). \end{array}$$

Since $\tilde{\Delta}'$ is Zariski dense in $\mathcal{R}_{F/F'}\tilde{G}$ by Lemma 2.3, we deduce that $\tilde{\psi}$ is dominant. So for dimension reasons it is an isogeny. Proposition 1.6 (b) now shows that $\tilde{\varphi}$ is separable, as desired. **q.e.d.**

Combining Lemmas 2.4 and 2.5, we now deduce that $\tilde{\Delta}'$ is open in $\tilde{G}(F)$. Thus $\tilde{\Gamma}'$ is open in $\tilde{G}(F)$, completing the proof of Theorem 2.1. **q.e.d.**

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