10. Moduli schemes of abelian varieties

In this chapter we describe an interpretation of some mixed Shimura varieties as moduli schemes, together with the universal family, of abelian varieties or of similar algebraic objects. Given this interpretation, we show that some of the maps 3.4 and 7.17 can be expressed in modular terms: so that, in particular, they are defined over \( \mathbb{Q} \).

We first describe a well-known, suitably rigidified, moduli functor of polarized abelian varieties with level structure. It is representable by a smooth scheme \( \mathfrak{M}_d \) over \( \mathbb{Q} \) (10.1-6). Certain mixed Shimura varieties associated to the mixed Shimura data \( (P_{2g}, X_{2g}) \) defined in 2.25 naturally carry the structure required by this functor (10.7-8). This yields isomorphisms between these mixed Shimura varieties on one side, and \( \mathfrak{M}_{d, \mathbb{C}} \) together with its universal family, on the other side (10.9-10). We show that some of the maps of 3.4 (a) can be described in terms of modular data (10.11-14).

In 10.15-16 we do the same for the mixed Shimura data \( (P_0, X_0) \) defined in 2.24. This is more or less a special case of \( (P_{2g}, X_{2g}) \), where the moduli functor degenerates to one for roots of unity.

The rest of the chapter deals with one special case of the above, but takes into account the toroidal compactification. We describe the Tate curve (10.17-18), and the moduli scheme for generalized elliptic curves with \( d \)-structure (10.19-20), in terms of toroidal compactifications of mixed Shimura varieties. The main result 10.22 is a modular interpretation for the isomorphism 7.17/9.37.

10.1. Abelian schemes: Let \( S \) be a scheme over \( \mathbb{Q} \), and \( A \to S \) an abelian scheme (see [M3] ch.6). We denote the group operation on \( A \) by \( (a, b) \to a + b \). For every positive integer \( d \) let \( [d] : A \to A \) be the morphism
The subgroup scheme \( \ker(\text{id}) \subset A \) is finite and \( \text{étale} \) over \( S \), since \( d \) is invertible over \( S \). Let \( X \rightarrow A \) be a \( \mathbb{G}_m \)-torsor. For every scheme \( T \rightarrow S \) let \( A_T \) be the abelian scheme \( A \times S \) over \( T \), and \( X_T \) the \( \mathbb{G}_m \)-torsor \( X \times S \) over \( A_T \). For every section \( a \in A(T) \) let \( T_a: A_T \rightarrow A_T \) be the morphism \( b \mapsto b + a \). Consider the functor

\[
T \mapsto H(X)(T) = \{ a \in A(T) | T_a \cdot X_T \xrightarrow{=} X_T \}.
\]

The condition \( T_a \cdot X_T \xrightarrow{=} X_T \) is equivalent to the existence of an isomorphism \( f: X_T \rightarrow X_T \) of schemes over \( T \) such that the two diagrams

\[
\begin{array}{ccc}
X_T & \xrightarrow{f} & X_T \\
\downarrow & & \downarrow \\
A_T & \xrightarrow{T_a} & A_T
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mathbb{G}_m \times X_T & \xrightarrow{\text{id} \times f} & \mathbb{G}_m \times X_T \\
\downarrow & & \downarrow \\
X_T & \xrightarrow{f} & X_T
\end{array}
\]

commute, where the vertical arrows in the diagram on the right hand side represent the action of \( \mathbb{G}_m \). This functor is represented by a closed subgroup scheme \( H(X) \subset A \), which is smooth over \( S \). If \( X \) is relatively ample with respect to \( A \rightarrow S \), then \( H(X) \) is finite and \( \text{étale} \) over \( S \).

10.2. The canonical pairing: Let \( a, b \in H(X)(T) \) be sections and \( f \) resp. \( g \) corresponding isomorphisms \( X_T \rightarrow X_T \). Then \( f \circ g \circ f^{-1} \circ g^{-1} \) is an automorphism of the \( \mathbb{G}_m \)-torsor \( X_T \) over \( A_T \), so there exists a unique section \( \alpha \in \mathbb{G}_m(T) \) such that \( f \circ g \circ f^{-1} \circ g^{-1} \) is equal to the multiplication by \( \alpha \). Since \( f \) and \( g \) commute with the \( \mathbb{G}_m \)-action, \( \alpha \) depends only on \( a \) and \( b \). Thus the map

\[
H(X)(T) \times H(X)(T) \rightarrow \mathbb{G}_m(T), \ (a,b) \mapsto e(X)(a,b) := \alpha
\]

is well-defined. This morphism of functors is represented by a unique morphism of schemes \( e(X): H(X) \times_S H(X) \rightarrow \mathbb{G}_m, S \). It is easily seen that this is an alternating bilinear form. If \( X \) is relatively ample with respect to \( A \rightarrow S \), then this is a perfect duality (see [M2] §1. thm. 1).
Let $A = \tilde{A}/\Gamma$ be an abelian variety over $\mathcal{C}$, where $\tilde{A}$ is the universal covering of $A$ in the topological sense, and $E : \Gamma \times \Gamma \to 2\pi A \cdot \mathcal{Z}$ the alternating form that corresponds to the Chern class of $X$ (compare 3.18-19). Let $\Gamma^\times = \{ e \in \Gamma \otimes \mathcal{R} | \forall \gamma e \in \Gamma \ E(e,\gamma e)\in 2\pi A \cdot \mathcal{Z} \}$, then by [M1] §9 prop. p.84 (iii) we have $H(X) = \Gamma^\times / \Gamma$. The canonical pairing $e(X)$ is according to [M1] §24 p.236 given by

$$(\Gamma^\times / \Gamma) \times (\Gamma^\times / \Gamma) \to \mathcal{C}^\times, \ ([(\gamma)],[(\gamma ')]) \mapsto \exp(E(\gamma,\gamma')).$$

10.3. Level structure: Fix integers $g \geq 1$ and $d \geq 1$, and free $\mathbb{Z}/d\mathbb{Z}$-modules $V(d)$ of rank $2g$ and $U(d)$ of rank $1$. Fix also a perfect alternating pairing $\Psi(d) : V(d) \times V(d) \to U(d)$. We consider abelian schemes $A \to S$ of constant relative dimension $g$, together with a $\mathfrak{g}_m$-torsor $X \to A$ such that $H(X) = A[d]$ for the above $d$. Consider the constant finite group schemes $V(d) \times S$ and $U(d) \times S$ over $S$. A symplectic $d$-structure on $X \to A \to S$ consists of an isomorphism $\lambda : V(d) \times S \to A[d]$ and a monomorphism $\mu : U(d) \times S \hookrightarrow \mathfrak{g}_m \cdot S$ such that the following diagram commutes:

$$(V(d) \times S) \times_S (V(d) \times S) \xrightarrow{\Psi(d)} U(d) \times S$$

$$\lambda \times 1 \downarrow \quad \downarrow \mu$$

$$A[d] \times S \to A[d] \quad e(X) \to \mathfrak{g}_m \cdot S$$

Locally in the étale topology on $S$ there always exists a symplectic $d$-structure.

10.4. Normalization of a $\mathfrak{g}_m$-torsor: Let $[-1] : A \to A$ be the inversion $a \mapsto -a$, and $i : A[2] \to A$ the inclusion. The equation $[-1] \circ i = i$ yields a canonical isomorphism $i^* [-1]^* X \cong i^* X$. The $\mathfrak{g}_m$-torsor $X$ is called totally symmetric if there exists an isomorphism $[-1]^* X \cong X$ that agrees with this canonical isomorphism on $A[2]$. In other words $X$ is totally
symmetric, if there exists an isomorphism of schemes $f: X \rightarrow X$ such that

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
A & \xrightarrow{[-1]} & A
\end{array}
\quad
\begin{array}{ccc}
\mathcal{E}_m \times X & \xrightarrow{\text{id} \times f} & \mathcal{E}_m \times X \\
\downarrow & & \downarrow \\
X \times_A A[2] & \xrightarrow{\text{pr}_1 \downarrow \text{pr}_1} & X
\end{array}
\]

are commutative. If $S$ is the spectrum of an algebraically closed field, the remark in [M2] §2 p.307 says that every totally symmetric $\mathcal{E}_m$-torsor is uniquely determined by its Chern class.

Let $e: S \rightarrow A$ be the zero section. A normalized $\mathcal{E}_m$-torsor is a $\mathcal{E}_m$-torsor $X \rightarrow A$ together with an isomorphism $e^*X \cong \mathcal{E}_m, S$, i.e. with a trivialization along the zero section. The trivialization is uniquely determined by its restriction to the "identity"-section in $\mathcal{E}_m, S$. Thus it is equivalent to a morphism of schemes $\tau: S \rightarrow X$ such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\tau} & S \\
\downarrow & & \downarrow \\
A & \xrightarrow{\sigma} & A
\end{array}
\]

commutes. In the following we shall be interested in normalized totally symmetric $\mathcal{E}_m$-torsors $(X, \tau)$ on $A \rightarrow S$. By the above remark such an object is uniquely determined up to isomorphism by the relative Chern class of $X$.

10.5. A moduli problem: Let $g$, $d$, $V[d]$, $U[d]$ and $\Psi[d]$ be as in 10.3. We suppose that $d$ is even and at least 4. For every scheme $S$ over $\text{Spec}(\mathbb{Q})$ let
\[ \mathcal{M}_d(S) := \{ (X, \tau) \text{ is a relatively ample normalized totally symmetric } \]
\[ \text{6}_m \text{-torsor on } A \to S \text{ such that } H(X) = A[d], \text{ and } \]
\[ (\lambda, \mu) \text{ a symplectic } d \text{-structure on } X \to A \to S. \]

In the category of complex spaces we define \( \mathcal{M}_d(S) \) analogously. The following theorem is well-known.

10.6. **Theorem:** The functor \( \mathcal{M}_d \) on the category of schemes is representable by a smooth quasiprojective scheme \( \mathcal{M}_d \) over \( \mathbb{Q} \).

**Proof:** The functor \( A_{g, d\mathfrak{g}, d} \) of [M3] thm. 7.9 is representable by a quasiprojective scheme over \( \text{Spec}(\mathbb{Z}) \). Our functor \( \mathcal{M}_d \) is isomorphic to an open and closed subfunctor of \( A_{g, d\mathfrak{g}, d} \times \text{Spec}(\mathbb{Q}) \), hence \( \mathcal{M}_d \) is representable by a quasiprojective scheme \( \mathcal{M}_d \) over \( \mathbb{Q} \). Since \( d \geq 4 \) an object \((A, X, \tau, \lambda, \mu)\) possesses no nontrivial automorphisms, hence \( \mathcal{M}_d \) is smooth. \( \text{q.e.d.} \)

10.7. **Construction of the moduli scheme as Shimura variety:** Consider the mixed Shimura data \( (P, X) = (P_{2g}, X_{2g}) \) defined in 2.25. We shall describe a natural structure as in 10.5 on certain mixed Shimura varieties associated to \((P_{2g}, X_{2g})\).

As explained in 2.15, we may assume that \( W = U \times V \) with the group operation \( (u, v) \cdot (u', v') = (u + u' + \frac{1}{2} \Psi(v, v'), v + v') \) and the \( G = \text{Sp}_{2g} \text{-action} \)
\[ g((u, v)) = (g(u), g(v)). \]
Choose \( \mathbb{Z} \)-structures on \( U \) and \( V \) such that \( \Psi \) induces a unimodular pairing \( V(\mathbb{Z}) \times V(\mathbb{Z}) \to \mathbb{U}(\mathbb{Z}) \). Since \( G \) acts faithfully on \( V \) this also gives a \( \mathbb{Z} \)-structure on \( G \). For every even positive integer \( d \) we define the open compact subgroups
\[ K_f = K_f(d) := \{ g \in G(\hat{\mathbb{Z}}) \mid \text{g = 1 mod d} \}. \]
\[ K_f^W = K_f^W(d) := (d \cdot U(\mathbb{Z})) \times (d \cdot V(\mathbb{Z})). \]

Since \( d \) is even, the definition of the group operation on \( W \) implies that \( K_f^W \) is indeed a subgroup of \( W(A_f) \). Clearly it is normalized by \( K_f \). Thus \( K_f^P := K_f^W \cap K_f \) is an open compact subgroup of \( P(A_f) \). From now on we assume \( d \geq 4 \). Let \( K_f^U := U(A_f) \cap K_f^W \) and \( K_f^V := K_f^W / K_f^U \), and write

\[
M = M(d) := M_C^{K_f}(\text{CSp}_{2g}, 0, \mathcal{H}_{2g})
\]
\[
M_V = M_V(d) := M_C^{K_f^V}(\text{CSp}_{2g}, 0, \mathcal{H}_{2g})
\]
\[
M_W = M_W(d) := M_C^{K_f^P}(P_{2g}, \mathcal{H}_{2g}),
\]

(see 9.25). The projections

\[
(P_{2g}, \mathcal{H}_{2g}) \rightarrow (V_{2g} \times \text{CSp}_{2g}, 0, \mathcal{H}_{2g}) \rightarrow (\text{CSp}_{2g}, 0, \mathcal{H}_{2g})
\]

induce holomorphic maps \( M_W \rightarrow M_V \rightarrow M \).

By 3.14, \( M_V \rightarrow M \) is a family of abelian varieties of relative dimension \( g \). Remember from 2.25 the structure on \( (P_{2g}, \mathcal{H}_{2g}) \rightarrow (V_{2g} \times \text{CSp}_{2g}, 0, \mathcal{H}_{2g}) \) as a \((\text{CSp}_{2g}, 0, \mathcal{H}_{2g})\)-torsor. The image of \( K_f \) in \( \mathcal{E}_m(A_f) \) is

\[ K_f^* = K_f^*(d) := \{ t \in \mathcal{E}_m(A_f) \mid t \equiv 1 \text{ mod } d \}, \]

so by 3.12 (b), \( M_W \rightarrow M_V \) is a torsor under the group scheme \( M_C^{K_f^U \times K_f}(P_0, \mathcal{X}_0) \rightarrow M_C^{K_f^*(\mathcal{E}_m, 0, \mathcal{X}_0)} \). As in 3.17, under the isomorphism

\[
M_C^{K_f^U \times K_f^*(P_0, \mathcal{X}_0)} \leftrightarrow \mathcal{E}_m \times M_C^{K_f^*(\mathcal{E}_m, 0, \mathcal{X}_0)}
\]

defined in 3.16, this turns \( M_W \rightarrow M_V \) into a \( \mathcal{E}_m \)-torsor. By 3.21 its inverse is relatively ample with respect to \( M_V \rightarrow M \). From now on we replace this \( \mathcal{E}_m \)-action by its twist under the automorphism \( \mathcal{E}_m \rightarrow \mathcal{E}_m, z \mapsto z^{-1} \). This corresponds to replacing the \( \mathcal{E}_m \)-torsor \( M_W \rightarrow M_V \) by its inverse, but as a scheme the "new" \( M_W \) is equal to the "old" one. With this new structure, it is relatively ample.
Let \( e: (\text{CSP}_{2g,0}, X_{2g}) \to (V_{2g} \times \text{CSP}_{2g,0}, X_{2g}) \) and \( \tau: (\text{CSP}_{2g,0}, X_{2g}) \to (P_{2g}, X_{2g}) \) be the morphisms that correspond to the splittings \( \text{CSP}_{2g,0} \to V_{2g} \times \text{CSP}_{2g,0} \) and \( \text{CSP}_{2g,0} \to P_{2g} \). They induce morphisms \([e]\) and \([\tau]\) such that the following diagram commutes.

\[
\begin{array}{ccc}
M & \xrightarrow{[\tau]} & M_V \\
\downarrow{[e]} & \searrow{id} & \downarrow{\iota} \\
M_W & \xrightarrow{\kappa} & M
\end{array}
\]

Since \([e]\) is the zero section, this defines a normalization of the \( \mathfrak{g}_m \)-torsor \( M_W \to M_V \).

Define \( V(d) := V(Z)/d \cdot V(Z) \cong V(\tilde{Z})/K_f^Y \) and \( U(d) := U(Z)/d \cdot U(Z) \cong U(\tilde{Z})/K_f^U \). Since \( \mathfrak{T}: V(Z) \times V(Z) \to U(Z) \) is unimodular, it induces a perfect alternating pairing \( \mathfrak{V}(d): V(d) \times V(d) \to U(d) \). The group \( V(\tilde{Z}) \) normalizes \( K_f^Y \times K_f^U \) in \( V(A_f) \times G(A_f) \). Thus 3.15 gives an isomorphism \( \lambda: V(d) \times M \to M_V(d) \) of group schemes over \( M \). Likewise \( U(\tilde{Z}) \) normalizes \( K_f^U \times K_f^* \) in \( P_0(A_f) \), so 3.15 yields a monomorphism \( U(d) \times M_c(K_f^0(\mathfrak{g}_m, 0, X_0)) \to M_c(K_f^U \times K_f^*(P_0, X_0)). \) Under the above identifications we get a monomorphism \( \mu: U(d) \times M_c \to \mathfrak{g}_m \times M \).

**10.8 Proposition:** The quintuple \( (M_W, M_V, [\tau], \lambda, \mu) \) is an element of \( \mathfrak{M}_d(M) \), and hence corresponds to a unique morphism \( M \to \mathfrak{M}_d, \mathfrak{C} \).

**Proof:** We have to show:

(a) \( M_W \to M_V \) is totally symmetric.

(b) \( H(M_W) = M_Y(d) \).

(c) The pair \( (\lambda, \mu) \) is a symplectic d-structure on \( M_W \to M_V \to M \).

It suffices to verify these assertions fibrewise and in the category of complex spaces. Fix a point \( ((y, g_f)) \in M(\mathfrak{C}) = M_c^K_f(\text{CSP}_{2g,0}, X_{2g})(\mathfrak{C}) \). Since \( \text{CSP}_{2g}(A_f) = G(A_f) = G(\mathfrak{Q}) \cdot G(\tilde{Z}) \), we may assume that \( (y, g_f) \in X_{2g} \times G(\tilde{Z}) \). Let
$\Gamma_{Y} = d \cdot V(Z)$ and $\Gamma_{W} = (d \cdot U(Z)) \cdot (d \cdot V(Z))$, then the fibres of $M_{W}$ and $M_{Y}$ over $[(y,g_{y})]$ are isomorphic to $X = \Gamma_{W} \setminus W(R) \cdot U(C)$, respectively $A = \Gamma_{Y} \setminus V(R)$.

For (a) we consider the homomorphism $W \rightarrow W$, $(u,v) \mapsto (u,-v)$. It maps $K_{f}^{W}$ to itself and commutes with the action of $G$, so it induces an automorphism $\tau$ of the Shimura data $(P_{2g}, \chi_{2g})$, and hence an automorphism $[\tau]$ of $M_{W}$. It operates on $X$ by the same formula $(u,v) \mapsto (u,-v)$, and it commutes with the action of $U(C)$ on $X$ and therefore with its structure as $C^{\times}$-torsor. Every point in $A[2]$ is of the form $[v]$ for $v \in \frac{1}{2} \cdot V(Z)$. For such $v$ we have $(0,2v) \in \Gamma_{W}$, hence for any $u \in U(C)$

$$(0,2v) \cdot (u,v) = (u + \frac{1}{2} \cdot \Psi(2v,-v), 2v - v) = (u,v),$$

since $\Psi(v,v) = 0$. This shows $[u]([v,u,v]) = [u,-v] = [(u,v)]$, so $[\tau]$ is equal to the identity over $A[2]$. By 10.4 this proves (a).

Since we have inverted the $C_{m}$-torsor $M_{W} \rightarrow M_{Y}$, 3.19 implies that the Chern class of $X \rightarrow A$ now corresponds to the alternating pairing $+\Psi: \Gamma_{Y} \times \Gamma_{Y} \rightarrow \Gamma_{U} = U(Q) \cap \Gamma_{W}$. Via the isomorphism 3.16, we get the equation $E = \frac{1}{d} \cdot \lambda_{\Psi}(y) \cdot \Psi$. It is verified at once that $(\Gamma_{Y})^{\times} = V(Z)$, whence $K(X) = V(Z)/\Gamma_{Y} = A[d]$, which implies (b).

By 10.2 this also shows that $e(X): A[d] \times A[d] \rightarrow C^{\times}$ is given by

$$[[v],[v']] \mapsto \exp\left(\frac{1}{d} \cdot \lambda_{\Psi}(y) \cdot \Psi(v,v')\right).$$

By 3.15, $\lambda$ is given by

$$V[d] = V(Z)/d \cdot V(Z) \rightarrow V(Z)/d \cdot V(Z) = V(Z)/d \cdot V(Z) = A[d],$$

$$[v_{f}] \mapsto [-g_{f}(v_{f})],$$

and by 3.15 and 3.16, $\mu$ is given by

$$U[d] = U(Z)/d \cdot U(Z) \rightarrow U(Z)/d \cdot U(Z) = U(Z)/d \cdot U(Z) = C^{\times},$$

$$[u_{f}] \mapsto [-g_{f}(u_{f})] = [u] \mapsto \exp(-\frac{1}{d} \cdot \lambda_{\Psi}(y)(u)).$$
The commutativity of the diagram 10.3 is therefore equivalent to the identity

$$\Psi(-g\mu(v_1),-g\mu(v_1')) = \Psi(v_1,v_1') \mod d\cdot U(\mathbb{C})$$

This congruence follows from the bilinearity and the $G$-equivariance of the pairing $\Psi$. This proves (c). q.e.d.

10.3 Proposition: The morphism $M \to \mathbb{P}_{d,C}$ of 10.8 is an isomorphism.

Proof: Both are smooth varieties over $\mathbb{C}$, so it suffices to prove the bijectivity on their $\mathbb{C}$-valued points. Consider $(A,X,\tau,\lambda,\mu) \in \mathbb{P}_{d,C}$. Write $A = \tilde{A}/\Gamma$, where $\tilde{A}$ is the universal covering of $A$, and let $E: \Gamma \times \Gamma \to 2\pi A \cdot Z$ be the pairing associated to $X$. Let $\lambda_0$ be an isomorphism $Z \to 2\pi A \cdot Z$. Then $\Gamma^+ \cdot \frac{1}{d} \Gamma$ is in $\Gamma \otimes \mathbb{Q}$, and the pairing $\Gamma^+ \times \Gamma^+ \to Z$ induced by $d^2 \lambda_0^{-1} \cdot E$ is unimodular. Thus there exists an isomorphism $\Psi: \Gamma \to \Gamma_V$ such that this pairing corresponds to the pairing $\Psi: V(Z) \times V(Z) \to U(Z) = Z$. The complex structure on $A = V(\mathbb{R})/\Gamma_V$ then corresponds to a unique action of $S$ on $V(\mathbb{R})$. By [M1] 52 cor. p.18 we have $E(A \cdot v,A \cdot v') = E(v,v')$ for all $v, v' \in V(\mathbb{R})$, hence this action factors through a unique homomorphism $h: S \to G_\mathbb{R}$. Since $X$ is ample, $-\lambda_0 \cdot \Psi$ is a polarization of the Hodge structure on $V$ defined by $h$, hence $h \in H_{2g}$. The given isomorphism $\lambda: V[d] \to A[d] = V[d]$ is symplectic, so there exists $g_\mu \in G(\mathbb{Z})$ such that $\lambda$ is induced by conjugation with $g_\mu$. We have already chosen an isomorphism between $A$ and the fibre of $M_V \to M$ over the point $[(h,g_\mu)]$. The Chern classes of the two normalized totally symmetric $\mathbb{C}^\times$-torsors correspond to each other, so by 10.4 they are isomorphic. Since $\mu$ is uniquely determined by $\lambda$ and the commutativity of the diagram 10.3, the $d$-structures also correspond. Thus $(A,X,\tau,\lambda,\mu)$ is in the image of the morphism $M \to \mathbb{P}_{d,C}$, and the surjectivity is proved.
Every other isomorphism $\Gamma \to \Gamma', \varphi$, such that $E$ corresponds to the pairing $\Psi: V(Z) \times V(Z) \to U(Z) \cong Z$, must be of the form $\text{int}(g) \cdot \varphi$ for some $g \in G(Z)$. The homomorphism $h$ must then be replaced by $\text{int}(g) \cdot h$ and $g_f$ by $g \cdot g_f \mod K_f$. But $[(\text{int}(g) \cdot h, g \cdot g_f)] = [(h, g_f)]$ in $M$, this proves the injectivity. \text{q.e.d.}

10.10. Corollary: Let $X_d \to A_d \to \mathbb{M}_d$ be the universal family, these are smooth quasiprojective schemes over $Q$. The data in 10.8 determines isomorphisms $M_W(d) \to X_d, \mathcal{C}$, $M_V(d) \to A_d, \mathcal{C}$ and $M(d) \to \mathbb{M}_d, \mathcal{C}$, compatible with all described structures.

\textbf{Proof:} This follows from 10.9 and the universal property of $\mathbb{M}_d$. \text{q.e.d.}

We now study the relation of the modular interpretation of $M_W$ with some of the maps of 3.4 (a).

10.11. Proposition: Let $d > 4$ be an even integer. Any element $w_f \in K_W^W(2)$ normalizes $K_P^P(d)$, and the automorphism of the scheme $X_d, \mathcal{C}$ corresponding to $w_f: M_W \to M_W$ can be described uniquely in terms of modular data.

\textbf{Proof:} Let $(u_f, v_f) = w_f$, and consider $w_f = (u_f, v_f) \in K_W^W(d)$. We have $[w_f, w_f] = \Psi(v_f, v_f) \in U(A_f)$, and since $v_f \in 2\cdot V(\mathbb{Z})$ and $v_f \in 2d \cdot V(\mathbb{Z})$, it follows that $\Psi(v_f, v_f) \in 2d \cdot U(\mathbb{Z}) \subset K_f^U$. Thus $w_f$ normalizes $K_f^W(d)$. Next let $g_f \in K_f(d)$, then

$$[w_f, g_f] = w_f \cdot g_f(w_f)^{-1} = (u_f, v_f) \cdot (g_f(u_f), g_f(v_f))^{-1}$$

$$= (u_f, v_f) \cdot (-g_f(u_f), -g_f(v_f))$$

$$= (u_f - g_f(u_f), -\frac{1}{2} \cdot \Psi(v_f, g_f(v_f)), v_f - g_f(v_f)).$$
Since \( g_f \equiv 1 \pmod{d} \) and \( v_f \in 2 \cdot V(\mathbb{Z}) \), we have \( v_f - g_f(v_f) \in 2d \cdot V(\mathbb{Z}) \) and \( u_f - g_f(u_f) \in 2d \cdot U(\mathbb{Z}) \). Since \( \Psi \) is alternating, we have

\[
\frac{1}{2} \Psi(v_f, g_f(v_f)) = \frac{1}{2} \Psi(v_f, g_f(v_f) - v_f) \in \frac{1}{2} 
\cdot 2d \cdot U(\mathbb{Z}) \subset d \cdot U(\mathbb{Z}).
\]

Thus \( [w_f, g_f] \in K_f^W(d) \), which proves that \( w_f \) normalizes \( K_f^P(d) \).

For the second assertion we write \( w_f = (u_f, 0) \cdot (0, 2v_f) \) with \( u_f \in 2 \cdot U(\mathbb{Z}) \) and \( v_f \in V(\mathbb{Z}) \). If \( v_f = 0 \), then \( \cdot w_f \) operates on the \( \mathbb{C}^\times \)-torser \( M_W \rightarrow M_V \) by multiplication by \( \mu([-u_f]) \in \mathbb{C}^\times \), so this map is given in modular terms. From now on we assume \( v_f = 0 \).

As in the proof of 10.8 consider a fibre \( X = \Gamma_W \backslash W(\mathbb{R}) \cdot U(\mathbb{C}) \) of \( M_W \rightarrow M \) over \( -[(y, g)] \), where \( g \in G(\mathbb{Z}) \). By 3.15 \( \cdot w_f \) operates on \( X \) by \( [w] \mapsto [w^{-1} \cdot w'] \) for any \( w \in W(\mathbb{Q}) \) such that \( g_f^{-1}(w) = w_f \mod \ker^W_f \).

Let \( v \in V(\mathbb{Z}) \) such that \( g_f^{-1}(v) = v_f \mod d \cdot V(\mathbb{Z}) \), then we can for instance choose \( w = (0, 2v) \). The isomorphism \( \cdot w_f \) then induces the translation \( [v] \mapsto [v + 2v] \) on \( A = \Gamma_\gamma \backslash V(\mathbb{R}) \), and since \( U \) is in the center of \( W \), it commutes with the action of \( \mathbb{C}^\times \). Now \( [v] \in H(X) \), and every holomorphic isomorphism of \( \mathbb{C}^\times \)-torsors \( X \rightarrow X \) over the translation \( T[{-v}] \) is of the form \( f : [w'] \mapsto [(u, v)^{-1} \cdot w'] \) for some \( u \in U(\mathbb{C}) \). The isomorphism \( [u] : X \rightarrow X, [(u', v')] \mapsto [(u', -v')] \) in the proof of 10.8 is the unique isomorphism \( X \rightarrow X \) over the inversion map on \( A \), which induces the identity map over the zero element of \( A \). For arbitrary \( u \in U(\mathbb{C}) \) we have

\[
[u]((u, v)) \cdot (u, v)^{-1} = (u, -v) \cdot (-u, -v) = (0, -2v) = (0, 2v)^{-1}.
\]

Therefore \( [u] \cdot f^{-1} \cdot [u] \cdot f = \cdot w_f \) on \( X \), and this isomorphism does not depend on the choice of \( u \). This shows that \( \cdot w_f \) is uniquely determined by the modular data (compare [M2] §2 p.319 remark 3). q.e.d.
10.12. **Lemma:** Let \( g_f \in G(\mathcal{A}_f) \). Then for every sufficiently divisible even integer \( d \gg 4 \) there exists an even integer \( d' \gg 4 \) such that

\[
K_P^f(d') \subseteq g_f^{-1}K_P^f(d)g_f^{-1} \subseteq K_P^f(2).
\]

**Proof:** This follows immediately from the fact that the \( K_P^f(d) \) are cofinal in the system of all open compact subgroups of \( \mathcal{P}(\mathcal{A}_f) \). \( \text{q.e.d.} \)

10.13. **Proposition:** Let \( g_f, d \) and \( d' \) as in 10.12, and define \( d'' := d \cdot d' \). The morphism \( X_d, \mathcal{C} \to X_d, \mathcal{C} \) corresponding to the map \([ \cdot g_f]: M_W(d'') \to M_W(d)\) can be described uniquely in terms of modular data.

**Proof:** By the inclusions \( K_P^f(d'') \subseteq K_P^f(d') \subseteq g_f^{-1}K_P^f(d)g_f^{-1} \) the map \([ \cdot g_f]: M_W(d'') \to M_W(d)\) is defined. The fibre over a point \([y, g_f, g_f] \in \mathcal{M}(d'')(\mathcal{C})\) is mapped to the fibre over \([y, g_f, g_f] \in \mathcal{M}(d)(\mathcal{C})\). Let \( \Gamma''_W := W(\mathbb{Q}) \cap g_f(K_P^W(d'')) \) and \( \Gamma_W := W(\mathbb{Q}) \cap g_f(K_P^W(d)) \), then with the identification of 3.13, \([ \cdot g_f] \) is on these fibres given by

\[
X'' := \Gamma''_W \backslash W(\mathbb{R}) \cdot \mathcal{U}(\mathcal{C}) \longrightarrow X := \Gamma_W \backslash W(\mathbb{R}) \cdot \mathcal{U}(\mathcal{C}), \ [w] \mapsto [w].
\]

Consider the isomorphism

\[
g_f(k_P^W(d))/K_P^W(d'') \xrightarrow{g_f(1)} (g_f \cdot g_f)(k_P^W(d))/g_f(k_P^W(d'')) = \Gamma_W/\Gamma''_W.
\]

Let \([w_f] \in g_f(k_P^W(d))/K_P^W(d'')\), and \([w] \) its image in \( \Gamma_W/\Gamma''_W \). Since by assumption \( g_f(k_P^W(d)) \subseteq K_P^W(2) \), it follows, as in the proof of 10.11, that \([ \cdot w_f] \) is on \( X'' \) given by \([w_f] \mapsto [w^{-1} \cdot w']\). Hence \([ \cdot g_f] \) induces an isomorphism \( X''/g_f(k_P^W(d)) \to X \). By 10.11 this quotient is uniquely determined by the modular data on \( X'' \). It remains to show that the other structures on \( X \) are determined by those on \( X'' \).

According to 3.16 we have the identification

\[
(U(\mathbb{Q}) \cap \Gamma''_W) \backslash \mathcal{U}(\mathcal{C}) \to \mathcal{C}^x, \ [u] \mapsto \exp(\frac{1}{d} \cdot \lambda(\gamma)(u)),
\]
and since we have inverted the $\mathcal{C}^x$-torsor $\mathcal{M}_W \to \mathcal{M}_V$, this group operates on $X$ by $[w] \to [u^{-1} \cdot w]$. The analogous formula holds for $X$, and the identifications lie in a commutative diagram

$$
\begin{array}{c}
(U(Q) \cap \Gamma_W^u) \cup \mathcal{U}(C) \\
\downarrow \quad \downarrow \\
(U(Q) \cap \Gamma_W^u) \cup \mathcal{U}(C) \\
\end{array} \quad \xrightarrow{\tau} \quad \mathcal{C}^x
$$

which is made such that the projection $X'' \to X$ is equivariant under the two actions. Hence the $\mathcal{C}^x$-action on $X$ comes from that on $X''$. Since $g_f \in G(A_f)$, the map $[-g_f]$ commutes with the section $[\tau]$, so the normalization of the $\mathcal{C}^x$-torsor $X''$ induces that of $X$. Let $\Gamma_V$ and $\Gamma_V''$ be the images of $\Gamma_W$ and $\Gamma_W''$ in $V(Q)$, and let $A := \Gamma_V \setminus V(R)$ and $A'' := \Gamma_V'' \setminus V(R)$. It remains to show that the symplectic $d'$-structure on $A$ comes from the symplectic $d''$-structure on $A''$.

The symplectic $d''$-structure on $A''$ is uniquely determined by the isomorphism $\chi: V[d''] \to A''[d'']$. Under the canonical identification

$$A''[d''] = (\frac{1}{d''} \cdot \Gamma_V'') \cap \Gamma_V = g_f'(V(\tilde{Z}))/g_f'(d'' \cdot V(\tilde{Z}))$$

this isomorphism is by 3.15 of the form

$$V[d''] \cdot V(\tilde{Z})/d'' \cdot V(\tilde{Z}) \cong g_f'(V(\tilde{Z}))/g_f'(d'' \cdot V(\tilde{Z})), \quad [v_f] \mapsto [g_f(V(\tilde{Z}))].$$

The analogous formula holds for $A$. Since $d'' = d \cdot d'$, and by assumption $d' \cdot V(\tilde{Z}) \subseteq g_f(d \cdot V(\tilde{Z}))$, the subgroup $\frac{1}{d''} \cdot \Gamma_V''$ is already contained in $\Gamma_V$. Thus the isogeny $A'' \to A$ factors through the endomorphism $[d]: A'' \to A''$. So let us consider the new isogeny

$$A'' = \Gamma_V'' \setminus V(R) \to A = \Gamma_V \setminus V(R), \quad [v] \mapsto [\frac{1}{d} \cdot v].$$

The inverse image of $A[d]$ under this map is $\Gamma_V \cap \Gamma_V'' = (g_f \cdot g_f)(d \cdot V(\tilde{Z})) / g_f(d' \cdot V(\tilde{Z}))$. Since by assumption $g_f(d \cdot V(\tilde{Z})) \subseteq 2 \cdot V(\tilde{Z})$, we have $(g_f \cdot g_f)(d \cdot V(\tilde{Z})) \subseteq g_f(V(\tilde{Z}))$, so this inverse image lies in $A''[d'']$. For
the $d''$-structure on $A''$ and the $d$-structure on $A'$ we therefore have
the commutative diagram

$$
\begin{array}{c}
\text{[d-gf(v_f)]} \in \text{V}[d''] \longrightarrow \ A''[d''] \hookrightarrow A'' \ni [v] \\
\downarrow \quad \downarrow \quad \downarrow \\
[v_f] \in \text{V}([Z]) \mathrel{\overset{\text{id}}{\longrightarrow}} \text{V}[d] \longrightarrow \ A[d] \hookrightarrow A \ni \frac{1}{d} \cdot v.
\end{array}
$$

This implies the assertion. \textit{q.e.d.}

10.14 \textbf{Corollary}: The automorphism of $X_{d,c}$ in 10.11, and the
morphism $X_{d''},c \to X_{d,c}$ in 10.13, both descend to morphisms of schemes
over $Q$.

\textbf{Proof}: By 10.11, 10.13, and the universal property of $X_d \to \mathcal{M}_d$.
\textit{q.e.d.}

\textbf{Remark}: $P_{2g}(A_f)$ is generated by $k^w_f(2)$ together with $\text{CSp}_{2g}(A_f)$.
Thus, for our purposes, the corollary holds for sufficiently many maps.

We now want to obtain the same results for the mixed Shimura
data $(P_0,X_0)$ defined in 2.24. This is, mutatis mutandum, a special case
of the above.

10.15 \textbf{A moduli scheme for roots of unity}: For any positive integer
d, let $\mu_{d,0} \subset \text{G}_{m,0}$ be the kernel of the homomorphism $t \mapsto t^d$. Let
$\mathcal{M}^0_{d} \subset \text{G}_{m,0}$ be the reduced closed subscheme $\mu_{d,0} \mathrel{\cup_{d'}} \mu_{d',d} \mu_{d',0}$ of all
primitive $d$th roots of unity. For any scheme $S$ over $Q$, it is equivalent
to give an $S$-valued point on $\mathcal{M}^0_d$, or to give an isomorphism $(\mathbb{Z}/d-\mathbb{Z}) \times S$
$\cong \mu_{d,S}$. In other words, $\mathcal{M}^0_d$ represents the functor

$$
\text{Isom}((\mathbb{Z}/d-\mathbb{Z}) \times \text{Spec}(Q), \mu_{d,0}).
$$
Consider the mixed Shimura data \((P_0, X_0)\), defined in 2.24. Let

\[ K_{f}^{U}(d) := d \cdot \mathbb{Z} \subset A_f = U_0(A_f) \text{ and} \]
\[ K_f(d) := \{ t_f \in \mathcal{E}_m(\mathbb{Z}) | t_f = 1 \mod d \}, \]

then \( K_f(d) := K_f^{U}(d) \times K_f(d) \) is an open compact subgroup of \( P_0(A_f) \). Every \( K_f(d) \) is a normal subgroup of \( K_f^{U}(1) \). Put

\[ M^0(d) := M_c^{K_f^{U}(d)}(\mathcal{E}_m, 0, \mathcal{H}_0) \text{ and} \]
\[ M^0_U(d) := M_c^{K_f}(P_0, X_0). \]

The canonical isomorphism \( M^0_U(d)(\mathbb{C}) \cong \mathbb{C} \times M^0(d)(\mathbb{C}) \) defined in 3.16 gives an isomorphism \( M^0_U(d) \rightarrow \mathcal{E}_m \times M^0(d) \). Since \( K_f^{U}(1)/K_f^{U}(d) = \mathbb{Z}/d \cdot \mathbb{Z} \cong \mathbb{Z}/d \cdot \mathbb{Z} \), \( 3.15 \) defines a monomorphism \( \mathbb{Z}/d \cdot \mathbb{Z} \times M^0(d) \hookrightarrow M^0_U(d) \rightarrow \mathcal{E}_m \times M^0(d) \), whence an isomorphism

\[ \mathbb{Z}/d \cdot \mathbb{Z} \times M^0(d) \cong \mu_d \times M^0(d). \]

By the universal property of \( \mathcal{M}_d \), this defines a morphism \( M^0(d) \rightarrow \mathcal{M}_d^0, \mathbb{C} \).

Defining \( \mathcal{X}_d^0 := \mathcal{E}_m, 0 \times \mathcal{M}_d^0 \), we in turn get a morphism \( M^0_U(d) \rightarrow \mathcal{X}_d^0, \mathbb{C} \).

10.16. Proposition: (a) The morphisms \( M^0(d) \rightarrow \mathcal{M}_d^0, \mathbb{C} \) and \( M^0_U(d) \rightarrow \mathcal{X}_d^0, \mathbb{C} \) are isomorphisms.

(b) Let \( p_f \in K_f^{U}(1) \mathcal{U}_{m}(A_f), \) and \( d', d \) such that the map \( [\cdot p_f] : M^0_U(d') \rightarrow M^0_U(d) \) of \( 3.4 \) (a) is defined. Then with the identifications in (a) this map corresponds to a morphism \( \mathcal{X}_d^0 \rightarrow \mathcal{X}_{d'}^0 \) over \( \mathbb{Q} \).

Proof: (a) Let \( e : (\mathcal{E}_m, 0, \mathcal{H}_0) \hookrightarrow (P_0, X_0) \) be the given section, then the isomorphism \( M^0_U(d)(\mathbb{C}) \cong \mathbb{C} \times M^0(d)(\mathbb{C}) \) of 3.16 is given by

\[ [(u, e(y), e(t_f))] \mapsto (\exp(t_f^d \cdot \lambda_y(u)), I(y, t_f)) \]

for \( u \in \mathbb{U}(\mathbb{C}), \) \( y \in \mathcal{H}_0, \) and \( t_f \in \mathcal{E}_m(\mathbb{Z}) \). Using 3.15 it follows that the isomorphism \( \mathbb{Z}/d \cdot \mathbb{Z} \times M^0(d) \cong \mu_d \times M^0(d) \) is given by
\((u_l, [(y, t)]) \mapsto (\exp^{1/d}_y(\lambda_y(u)))^{1/t}, [(y, t)])\)

for \([u] \in \mathbb{Z}/d \cdot \mathbb{Z}\), \(y \in \mathcal{H}_0\), and \(t \in \mathcal{E}_m(\mathbb{Z})\). Clearly every isomorphism \(\mathbb{Z}/d \cdot \mathbb{Z} = \mu_d(\mathbb{C})\) occurs in precisely one fibre over \(\mathcal{M}^0(d)(\mathbb{C})\), so the map \(\mathcal{M}^0(d)(\mathbb{C}) \to \mathcal{M}_d^0(\mathbb{C})\) is a bijection. Since both are normal reduced schemes, \(\mathcal{M}^0(d) \to \mathcal{M}_d^0(\mathbb{C})\) must be an isomorphism. This implies (a).

(b) If \(p_f = u_f \in \mathbb{Z}/K_f^1(1)\), then by the definition 3.15 the map \([-u_f]: \mathcal{M}_d^0(\mathbb{C}) \to \mathcal{M}^0_d(d)\) corresponds to the automorphism \((t, \zeta) \mapsto (t, \zeta u_f, \zeta)\) of \(\mathcal{E}_m, 0 \times \mathcal{M}_d^0\). Next if \(p_f = e(t)\) with \(t \in \mathcal{E}_m(\mathbb{Z})\), the formulas in (a) show that the map \([e(t)]: \mathcal{M}_d^0(\mathbb{C}) \to \mathcal{M}_d^0_d(\mathbb{C})\) corresponds to the automorphism \((t, \zeta) \mapsto (t, \zeta e(t))\) of \(\mathcal{E}_m, 0 \times \mathcal{M}_d^0_d\). It remains to consider the case \(p_f = e(t)\) with \(t \in \mathcal{Q}^0 \subset \mathcal{E}_m(\mathbb{Q})\). Since \(e(t) \cdot K_f^1(\mathbb{C}) \cdot e(t)^{-1} = K_f^1(\mathbb{C}) \cdot (t, d)\), \(d'\) must be a multiple of \(t \cdot d\). On \(\mathcal{E} \times \mathcal{M}_d^0(\mathbb{C})\) the map \([e(t)]: \mathcal{M}_d^0(d') \to \mathcal{M}_d^0(\mathbb{C})\) is given by

\[
(z, [(y, t)]) = [(\lambda_y^{-1}(d' \cdot \log(z)) \cdot e(y), e(t))] \\
\mapsto [(\lambda_y^{-1}(d' \cdot \log(z)) \cdot e(y), e(tp) - e(t))] \\
= [(e(tp) - e(t)^{-1})^{-1}(d' \cdot \log(z)) \cdot e(y), e(t)] \\
= [(t^{-1}(\lambda_y^{-1}(d' \cdot \log(z))) \cdot e(y), e(t)]) \\
= (\exp^{1/d}_y(\lambda_y(\frac{1}{d} - \lambda_y^{-1}(d' \cdot \log(z))))], [(y, t)]) \\
= (z^{d''}, [(y, t)])
\]

where \(d'' = \frac{d}{d - t} \in \mathbb{Z}\). Thus it corresponds to the morphism

\(\mathcal{E}_m, 0 \times \mathcal{M}_d^0 \to \mathcal{E}_m, 0 \times \mathcal{M}_d^0, (t, \zeta) \mapsto (e^{d''}, e^{d''/d})\).

In each of the three cases the morphism \(\mathcal{X}_d^0 \to \mathcal{X}_d^0\) is defined over \(\mathbb{Q}\), as desired. \emph{q.e.d.}

For the rest of this chapter we shall do something similar for the compactification of a mixed Shimura variety. In a very special case, we
shall give a modular interpretation for the isomorphism 7.17. The following construction is well-known.

10.17. The Tate curve as a torus embedding: Consider the projection \( \text{pr}_1: \mathbb{G}_m^2, \mathbb{G}_m \to \mathbb{G}_m, \mathbb{G}_m \). We may consider this as the (relative) commutative group scheme obtained from \( \mathbb{G}_m, \mathbb{G}_m \to \text{Spec}(\mathbb{Q}) \) by base change \( \times \mathbb{Q} \mathbb{G}_m, \mathbb{G}_m \).

The torus embedding associated to \( \sigma: \mathbb{R}^{20} \subset \mathbb{R} = Y_*(\mathbb{G}_m, \mathbb{Q}) \) is just \( \mathbb{G}_m, \mathbb{Q} \hookrightarrow \mathbb{A}_\mathbb{Q}^1 \). Let \( J \) be the partial cone decomposition of \( \mathbb{R}^{20} = Y_*(\mathbb{G}_m, \mathbb{Q}) \) that consists of the cones \( \mathbb{R}^{20} \cdot (1, n) + \mathbb{R}^{20} \cdot (1, n+1) \) for all \( n \in \mathbb{Z} \), and their faces; as illustrated in the diagram:

\[
\begin{array}{c}
\text{(1,0)} \\
\text{(1,1)} \\
\text{(1,-1)} \\
\text{(0,0)} \\
y
\end{array}
\]

Let \( J^c \subset J \) be the subset of all cones of dimension \( \leq 1 \). Write \( Z:=(\mathbb{G}_m^2, \mathbb{Q})_J \) and \( Z^c:=(\mathbb{G}_m^2, \mathbb{Q})_{J^c} \).

By the criterion 5.4 for the functoriality of torus embeddings, the projection \( \mathbb{G}_m^2, \mathbb{Q} \to \mathbb{G}_m, \mathbb{Q} \) extends to a morphism \( Z \to \mathbb{A}_\mathbb{Q}^1 \). The inverse image of \( \mathbb{G}_m, \mathbb{Q} \subset \mathbb{A}_\mathbb{Q}^1 \) is just \( \mathbb{G}_m^2, \mathbb{Q} \), but the fibre over \( 0 \) is an infinite sequence of projective lines, with the point \( 0 \) of each component glued to the point \( \infty \) of the next one. It is easily checked that the morphism \( Z \to \mathbb{A}_\mathbb{Q}^1 \) is flat. The open subscheme, on which it is smooth, is just \( Z^c \).

The structure of commutative group scheme on \( \text{pr}_1: \mathbb{G}_m^2, \mathbb{Q} \to \mathbb{G}_m, \mathbb{Q} \) is given by the identity section, the inversion, and the group operation. These are

\[
\begin{align*}
\mathbb{G}_m, \mathbb{Q} & \to \mathbb{G}_m^2, \mathbb{Q}, & s & \mapsto (s, 1), \\
\mathbb{G}_m^2, \mathbb{Q} & \to \mathbb{G}_m^2, \mathbb{Q}, & (s, t) & \mapsto (s, t^{-1}), \\
\mathbb{G}_m^2, \mathbb{Q} \times \mathbb{G}_m^2, \mathbb{Q} & \to \mathbb{G}_m^2, \mathbb{Q}, & (s, t_1, t_2) & \mapsto (s, t_1 t_2)
\end{align*}
\]
respectively. We claim that they extend to morphisms
\[ A^1_0 \rightarrow \mathbb{Z}^r, \mathbb{Z}^r \rightarrow \mathbb{Z}^r, \text{ and } \mathbb{Z}^r \times A^1_0 \mathbb{Z}^r \rightarrow \mathbb{Z}^r \text{ and } \mathbb{Z}^r \times A^1_0 \mathbb{Z} \rightarrow \mathbb{Z} \]
respectively. In fact, for the first two morphisms this is a direct application of 5.4. It is easily checked that the embedding
\[ \mathcal{E}^3_{m,0} = \mathcal{E}^2_{m,0} \times \mathcal{E}^2_{m,0} \xrightarrow{c} \mathbb{Z}^r \times A^1_0 \mathbb{Z} \]
is the torus embedding with respect to the partial cone decomposition of \( \mathbb{R}^3 \), consisting of the cones
\[ \mathbb{R}^{20}.(1,m,n) \oplus \mathbb{R}^{20}.(1,m,n+1) \]
for all \( m,n \in \mathbb{Z} \), and their faces. Since such a cone maps to the cone
\[ \mathbb{R}^{20}.(1,m+n) \oplus \mathbb{R}^{20}.(1,m+n+1) \in \mathcal{T}, \]
by functoriality we get a morphism \( \mathbb{Z}^r \times A^1_0 \mathbb{Z} \rightarrow \mathbb{Z} \). The same argument applies to \( \mathbb{Z}^r \times A^1_0 \mathbb{Z}^r \rightarrow \mathbb{Z}^r \). Since \( \mathcal{E}^2_{m,0} \) is dense in \( \mathbb{Z}^r \), the group axioms for \( pr_1: \mathcal{E}^2_{m,0} \rightarrow \mathcal{E}^2_{m,0} \) extend, so \( \mathbb{Z}^r \rightarrow A^1_0 \mathbb{Z} \) is a commutative group scheme. Likewise, the morphism \( \mathbb{Z}^r \times A^1_0 \mathbb{Z} \rightarrow \mathbb{Z} \) defines a group action. \( \mathbb{Z}^r \rightarrow A^1_0 \mathbb{Z} \) is the Néron-model of \( \mathcal{E}_{m,0} \), but we will not need this fact.

Consider the action of \( a \in \mathbb{Z} \) on \( \mathcal{E}^2_{m,0} \) through \( (s,t) \mapsto (s,sa-t) \). Clearly \( \mathcal{T} \) is invariant under these substitutions, and \( \mathcal{Z} \) cyclically and transitively permutes the irreducible components of the special fibre. The identity component of the special fibre is again \( \mathcal{E}_{m,0} \), so we get a canonical group isomorphism between the special fibre of \( \mathbb{Z}^r \rightarrow A^1_0 \mathbb{Z} \) and \( \mathbb{Z} \times \mathcal{E}_{m,0} \).

Let \( \mathcal{G} \) be the formal completion of the special fibre in \( \mathbb{Z} \). For any positive integer \( d \) we can form the quotient \( \mathcal{G}/d \cdot \mathbb{Z} \). This is formally proper and flat of relative dimension 1 over the completion of \( A^1_0 \mathbb{Z} \) in \( \{0\} \), so this formal scheme is algebraizable, yielding a proper flat scheme \( E \rightarrow \text{Spec } \mathbb{Q}[s] \). Letting \( E^{\text{CE}} \) be the open subscheme which is smooth,
over $\text{Spec } \mathbb{Q}[[s]]$, we obtain the structure of a commutative group scheme on $E^r \to \text{Spec } \mathbb{Q}[[s]]$, and an action $E^r \times_{\text{Spec } \mathbb{Q}[[s]]} E \to E$. This is a generalized elliptic curve in the sense of [DR] II.1.12. It is easy to show that $E \to \text{Spec } \mathbb{Q}[[s]]$ is the universal deformation of the closed fibre as a generalized elliptic curve (compare [DR] III.1.2).

The kernel of the multiplication by $d$ on $\mathcal{G}/d \cdot \mathcal{Z}$ is the formal scheme associated to $(\mathcal{Z}/d \cdot \mathcal{Z}) \times \mu_d$, so $E[d]$ is canonically isomorphic to $(\mathcal{Z}/d \cdot \mathcal{Z}) \times \mu_d \times \text{Spec } \mathbb{Q}[[s]] \to \text{Spec } \mathbb{Q}[[s]]$.

A $d$-structure on $E \to \text{Spec } \mathbb{Q}[[s]]$, in the sense of [DR] IV.2.3, is an isomorphism $(\mathcal{Z}/d \cdot \mathcal{Z})^2 = E[d]$. Thus giving a $d$-structure is in our case equivalent to giving an isomorphism $\mathcal{Z}/d \cdot \mathcal{Z} \cong \mu_d$. In particular, if $M_d^0$ is as in 10.15, there is a canonical $d$-structure on $E \times M_d^0 \to \text{Spec } \mathbb{Q}[[s]] \times M_d^0$. Again, this is the universal deformation of the closed fibre as a generalized elliptic curve with $d$-structure.

10.18. The Tate curve in terms of mixed Shimura varieties: Consider the mixed Shimura data $(P_0, X_0)$ defined in 2.24. Let $(\tilde{P}_0, \tilde{X}_0)$ be the fibre product of $(P_0, X_0)$ with itself over $(\mathcal{E}_m, 0, \mathcal{H}_0)$; we have canonically $\tilde{P}_0 = (U_0 \times U_0) \times \mathcal{E}_m, 0$. Consider the open compact subgroups

$$
\begin{align*}
K_f^{U_0} &:= d \cdot \tilde{Z} \subset A_f = U_0(A_f), \\
K_f^{\mathcal{E}_m} &:= \{ t_f \in \tilde{Z} \mid t_f \equiv 1 \mod d \}, \\
K_f^{P_0} &:= K_f^{U_0} \times K_f^{\mathcal{E}_m} \subset P_0(A_f), \text{ and} \\
K_f^{\tilde{P}_0} &:= (K_f^{U_0} \times K_f^{U_0}) \times K_f^{\mathcal{E}_m} \subset \tilde{P}_0(A_f).
\end{align*}
$$

With the identification of 10.15 we have canonical isomorphisms

$$
\begin{align*}
M_d^0 := M_{\mathcal{E}_m}^{K_f^{P_0}}(P_0, X_0) &\cong \mathcal{E}_m \times M_d^0, \\
\tilde{M}_d^0 := M_{\mathcal{E}_m}^{K_f^{\tilde{P}_0}}(\tilde{P}_0, \tilde{X}_0) &\cong \mathcal{E}_m^2 \times M_d^0.
\end{align*}
$$
By 6.8-9, the partial cone decompositions $\mathcal{T}^*C\mathcal{T}$ of 10.17 correspond to unique $K^0_\mathcal{P}$-admissible partial cone decompositions $\mathcal{B}_0^0 \subseteq \mathcal{X}_0$ for $(\overline{\mathcal{P}_0}, \overline{\mathcal{X}_0})$. Thus if $\overline{\mathcal{X}_0}$ denotes the cone decomposition for $(\mathcal{P}_0, \mathcal{X}_0)$ considered in 6.9, we get canonical isomorphisms

$$
\overline{\mathcal{M}}_0^{\mathcal{P}_0} := M_\mathcal{C}^{K^0_\mathcal{P}}(\overline{\mathcal{P}_0}, \overline{\mathcal{X}_0}, \overline{\mathcal{B}_0}) \cong \mathbb{Z}^r \times M_\mathcal{D}_0, \mathcal{C}
$$

$$
\overline{\mathcal{M}}_1^{\mathcal{P}_0} := M_\mathcal{C}^{K^0_\mathcal{P}}(\overline{\mathcal{P}_0}, \overline{\mathcal{X}_0}, \overline{\mathcal{B}_0}) \cong \mathbb{Z} \times \mathbb{L}_\mathcal{D}_0, \mathcal{C}
$$

$$
\overline{\mathcal{M}}_0^{\mathcal{P}_0} := M_\mathcal{C}^{K^0_\mathcal{P}}(\overline{\mathcal{P}_0}, \overline{\mathcal{X}_0}, \overline{\mathcal{B}_0}) \cong A^1 \times M_0, \mathcal{C}.
$$

Recall (2.21, 3.12 (a)) that the group structure on $\overline{\mathcal{M}}_0 \to \overline{\mathcal{M}}_0$ is given in terms of maps 3.4 (b). We can directly translate 10.17: the condition of 6.25 (b) holds for each of the necessary maps. The action of $\mathbb{Z}$ on $\mathbb{Z} \times M_0, \mathcal{C}$ corresponds to the action on $\overline{\mathcal{M}}_0$ induced by the maps $[\varphi_a]$, where $\varphi_a$ is the automorphism of $(\overline{\mathcal{P}_0}, \overline{\mathcal{X}_0})$ given by

$$
\overline{\mathcal{P}_0} = (U_0 \times U_0) \times \mathcal{B}_m, \mathcal{C} \ni (u_1, u_2, t) \mapsto (u_1, a \cdot u_1 + u_2, t).
$$

By 10.15 the canonical isomorphism $\mathbb{Z} / \mathcal{d} \mathbb{Z} \cong \mu_\mathcal{D}$ over $M_0, \mathcal{C}$ can be expressed in terms of maps 3.4 (a). Thus 10.17 shows that, over the formal completion of $\overline{\mathcal{M}}_0$ along the boundary $\overline{\mathcal{M}}_0 \setminus M_0$, we have a canonical generalized elliptic curve with $\mathcal{D}$-structure, and this structure can be defined purely in terms of maps 6.25.

We now "glue" the objects just constructed into the boundary of other objects.

10.19. A certain toroidal compactification: Consider the Shimura data $(\text{GL}_2, 0, \mathcal{X}_2)$ defined in 2.7, and its unipotent extension $(\mathcal{P}, \mathcal{X}) := (\mathcal{V}_2 \times \text{GL}_2, 0, \mathcal{X}_2)$, defined in 2.25. Here $\mathcal{V}_2$ is just the standard 2-dimensional representation of $\text{GL}_2, 0$. For every positive integer $\mathcal{d}$ we let
$K_f = K_f(d) := \{ g \in \text{GL}_2(\mathbb{Z}) | g \equiv 1 \mod d \}$, \\
$K_f^{\vee} = K_f^{\vee}(d) := d \cdot \mathbb{Z}^2$, and \\
$K_f^P = K_f^P(d) := K_f^{\vee} \ltimes K_f$.

and define

$M := M_{c_2}^{K_f}(\text{GL}_2, \mathbb{H}_2,)$,

$M_V := M_{c_2}^{K_f}(V_2 \times \text{GL}_2, \mathbb{H}_2)$.

As explained in 4.25, we identify the mixed Shimura data $(P_0, X_0)$ with a rational boundary component of $(\text{GL}_2, \mathbb{H}_2)$. Every proper rational boundary component is $\text{GL}_2(\mathbb{Q})$-conjugate to this one. The rational boundary component of $(V_2 \times \text{GL}_2, \mathbb{H}_2)$ associated to $(P_0, X_0)$ is isomorphic to $(\hat{P}_0, \hat{X}_0)$. Embedding all our groups into $\text{GL}_3, \mathbb{Q}$ we have

$$p = \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{bmatrix}, \quad \hat{p}_0 = \begin{bmatrix} * & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad q = \begin{bmatrix} * & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $Q$ is the normalizer of $\hat{P}_0$. We identify the unipotent radical of $\hat{P}_0$ with $G_2^{\mathbb{Q}}$ by

$$(x, y) \longleftrightarrow \begin{bmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The cone decompositions of 10.18 correspond canonically to $K_f^P(1)$-admissible cone decompositions for $(P, X)$. In fact, since $P(A_f) = Q(Q) \cdot K_f^P(1)$, there are precisely two conjugacy classes of $(P_1, \mathbb{P}_f)$ modulo the actions 6.4 (ii) and (iii). One of these corresponds to $P_1 = P$, for which there is nothing to specify. Since the other is represented by $(\hat{P}_0, 1)$, any cone decomposition $\mathcal{A}$ for $(P, X)$ is determined by $\mathcal{A}(\hat{P}_0, \hat{X}_0)$. By 4.26, the cone $C^*(X^0, \hat{P}_0)$ corresponds to $\{0\} \cup \mathbb{R}^{10} \times \mathbb{R}$ under the above identification. Moreover, $\Gamma := \text{Stab}_Q(Q)(X^0) \cap \hat{P}_0(A_f) \cdot K_f^P(1)$ acts on this through

$$(x, y) \mapsto b \cdot (x, x + y + ax) \text{ for } a \in \mathbb{Z} \text{ and } b \in Q^{10}.$$
Thus there are unique $K_f^P(1)$-admissible partial cone decompositions $\delta^f \subset \delta$ for $(P,X)$ with $\delta^f(\tilde{p}_0, \tilde{x}_0) = \delta^f_0$ and $\delta(\tilde{p}_0, \tilde{x}_0) = \delta_0$, and the latter is complete. There exists a unique complete $K_f(1)$-admissible cone decomposition for $(GL_2, \mathcal{O}, \mathcal{H}_2)$, which we denote by $\overline{\delta}$. Of course, all these cone decompositions remain admissible for $K_f^P = K_f^P(d)$, resp. $K_f = K_f(d)$. We define

$$\overline{M}_V := M^K_C(V_2 \times GL_2, \mathcal{O}, \mathcal{H}_2, \delta),$$
$$\overline{M}_V^f := M^K_C(V_2 \times GL_2, \mathcal{O}, \mathcal{H}_2, \delta^f),$$
and

$$\overline{M} := M^K_C(GL_2, \mathcal{O}, \mathcal{H}_2, \overline{\delta}) = M^K_C(GL_2, \mathcal{O}, \mathcal{H}_2)^*.$$

(By 9.39 (b) we know that $\overline{M}_V$ is a projective variety for every $d$, but we shall not need this fact.)

10.20. The moduli scheme of generalized elliptic curves in terms of mixed Shimura varieties: From now on we assume that $d \geq 3$. Then the fibres of $M_V \to M$ are elliptic curves. Consider the boundary stratum $M^K_C^{\text{fm}}(\mathcal{H}_m, 0, \mathcal{H}_0)$ of $\overline{M}$, associated to $(P_0, X_0)$ and $p_f = 1$ (see 6.3). This is canonically identified with the unique proper boundary stratum of $\overline{M}_U^0$, and 9.37 yields an isomorphism between its formal neighborhoods in $\overline{M}$ and $\overline{M}_U^0$ respectively. Likewise, for the rational boundary component $(\tilde{p}_0, \tilde{x}_0)$ of $(P, X)$, and $p_f = 1$, the group $\Delta_1$ of 7.3 acts on $\overline{M}_U^0$ just like the group $d \cdot \mathbb{Z}$ in 10.18. Thus, by 7.17, the inverse image of $M^K_C^{\text{fm}}(\mathcal{H}_m, 0, \mathcal{H}_0)$ under $M_V \to \overline{M}$ is isomorphic to $(\overline{M}_U^0 \setminus \overline{M}_U^0)/d \cdot \mathbb{Z}$, and 9.37 yields an isomorphism between its formal neighborhood in $\overline{M}_V$ and the quotient by $d \cdot \mathbb{Z}$ of the formal neighborhood of $\overline{M}_U^0 \setminus \overline{M}_U$ in $\overline{M}_U^0$.

Recall (10.7-8) that the structure of $M_V \to M$ as an elliptic curve with $d$-structure can be defined purely in terms of the maps 3.4. As in 10.18, we find that this structure extends and makes $\overline{M}_V \to \overline{M}$ into a generalized elliptic curve with $d$-structure.
Now let $\mathbb{M}_d$ be the moduli scheme of generalized elliptic curves over $\mathbb{Q}$ with $d$-structure. By [DR] IV.2.9, it is smooth, projective, of dimension 1 over $\mathbb{Q}$. Consider the moduli scheme $\mathbb{M}_d$ in 10.6 for $g=1$. One easily checks that, for $g=1$, the data $(A,X,t,\lambda,\mu)$ in 10.5 is determined up to unique isomorphism by $(A,\lambda)$. Thus the forgetful functor induces an open embedding $\mathbb{M}_d \hookrightarrow \mathbb{M}_d, \text{ and the image is Zariski-dense.}$ By the universal property of $\mathbb{M}_d$, the structures defined above induce a morphism $\mathbb{M} \rightarrow \mathbb{M}_d, \text{ This morphism extends the isomorphism } M \rightarrow \mathbb{M}_d, \text{ of 10.9. It is easy to see that } \mathbb{M} \rightarrow \mathbb{M}_d, \text{ is an isomorphism (for instance because } \mathbb{M} \text{ is also proper and smooth). If } E_d \rightarrow \mathbb{M}_d, \text{ is the universal family, then this also induces an isomorphism } \mathbb{M}_V \cong E_d, \text{.}

10.21. Summary of 10.17-20: To tie everything up, consider the generalized elliptic curve with $d$-structure, constructed in 10.17. By the universal property of $\mathbb{M}_d$, it defines a morphism $\text{Spec } \mathbb{Q}[[s]] \times M_0^0 \rightarrow \mathbb{M}_d$. Since the former is a universal deformation, this gives an isomorphism of the completion of $\mathbb{M}_d$ along $M_0^0 \hookrightarrow \mathbb{M}_d$.

In summary, the different objects and morphisms we have constructed form the following diagram.
Here $\mathcal{F}(\cdot)$ denotes formal completion along a closed subscheme that was specified above. All diagonal arrows are isomorphisms. These, and the arrows in the foreground, are all defined in terms of modular data, and the latter descend to $\mathcal{O}$. The arrows in the background have been defined in terms of maps 6.25. By the universal property of the moduli schemes involved, the whole diagram is commutative! We have thus proved the following proposition.

**10.22. Proposition:** With the identifications $\overline{M} \cong \overline{\Pi}_d,\mathcal{C}$ of 10.20, and $\overline{M}^0_U \cong A^1 \times \overline{M}^0_d,\mathcal{C}$ of 10.18, the isomorphism 9.37 of the formal completions of $\overline{M}$ and $\overline{M}^0_U$ corresponds to an isomorphism of the formal completions of $\overline{\Pi}_d$ and $A^1 \times \overline{M}^0_d$, that is defined over $\mathcal{O}$. 