the above condition (ii) means that the weight is rational, i.e. defined over \( \mathbb{Q} \). Since both this requirement and condition (vi) above are nontrivial restrictions for Shimura varieties, we want to replace them by weaker conditions. For our purposes it will be enough to assume the same conditions for the image of \( P \) in \( \text{Aut}(W) \), or equivalently in \( \text{Aut}(U) \times \text{Aut}(V) \). Explicitly we replace (ii) and (vi) by the conditions:

(ii') \( \pi \circ h^w: \mathfrak{g}_{m, \mathbb{R}} \to \mathfrak{g}_R \) is a cocharacter of the center of \( G \).

(vi') The center of \( G \) acts on \( U \) and on \( V \) through a torus that is an almost direct product of a \( \mathbb{Q} \)-split torus with a torus of compact type defined over \( \mathbb{Q} \).

Observe that these conditions imply that the weight acts on both \( U \) and \( V \) through a rational scalar character.

It is clear that with these weakened conditions the conclusions of 1.18, except (d), still hold. It is important, however, that (d) does hold for \( M \) an arbitrary irreducible subquotient of \( V \). Note also that the conclusions of 1.17 remain valid. Finally there is a relation with data as in 1.18. By 1.18 (iii) we have \( Z(P) \cap W = 1 \), so \( Z(P) \) is reductive and isomorphic to the kernel of the action of \( Z(G) \) on \( \text{Lie} \ U \oplus \text{Lie} \ V \). Thus \( P_1 = P/Z(P) \) together with the obvious choice for \( \mathcal{H}_1 \) satisfies the conditions of 1.18.
2. Mixed Shimura data

In the first chapter we introduced certain \( P(\mathbb{R}) \cdot U(\mathbb{C}) \)-conjugacy classes of homomorphisms \( S_\mathcal{C} \rightarrow P_\mathcal{C} \) as a natural generalization to non-reductive groups of the conjugacy classes of homomorphisms \( S \rightarrow G_\mathbb{R} \) considered in [D2]. In this chapter we bring in yet another generalization by considering finite coverings of such conjugacy classes. In various places we shall see the usefulness of this generalization (for instance in 3.9, 3.16–17), and indeed its necessity (see 4.11 (example 4.25–26), 7.1–2 or 8.5–6) for a unified formalism of the arithmetic structure of the boundary of Shimura varieties.

Definition 2.1 is a generalization of [D1] 1.5 resp. [D2] 2.1.1.1–3. In 2.3–10 we introduce more notions and some standard examples. In 2.11–12 we study the precise structure of the finite covering mentioned above. The largest part of this chapter, starting with 2.15, deals with properties and constructions related to the unipotent radical. In particular we prove a reduction lemma (2.26), by means of which any mixed Shimura data can (almost) be embedded into a direct product of the usual pure ones (as in [D1] 1.5 or [D2] 2.1.1) together with the standard examples described in this chapter.

2.1. Definition: Let \( P \) be a connected linear algebraic group over \( \mathbb{Q} \). Let \( W \) be its unipotent radical and \( U \) a subgroup of \( W \) that is normal in \( P \). Let \( V := W/U, G := P/W \), and \( \pi : P \rightarrow G \) and \( \pi' : P \rightarrow P/U \) be the canonical projections. Let \( X \) be a left homogeneous space under the subgroup \( P(\mathbb{R}) \cdot U(\mathbb{C}) \cdot C P(\mathbb{C}) \). Let \( h : X \rightarrow \text{Hom}(S_\mathcal{C}, P_\mathcal{C}) \) be a \( P(\mathbb{R}) \cdot U(\mathbb{C}) \)-equivariant map, such that:

(i) Every fibre of \( h \) consists of at most finitely many points.

Furthermore for some (\( \Leftrightarrow \) for all) \( x \in X \) we assume:
(ii) $\tau^\circ h^\times : S^\times \rightarrow (P/U)_G$ is already defined over $\mathbb{R}$.

(iii) $\pi^\circ h^\times \cdot w : G_{m,\mathbb{R}} \rightarrow G_{\mathbb{R}}$ is a cocharacter of the center of $G$.

(iv) $Ad^p \circ h^\times$ induces on $\text{Lie } P$ a rational mixed Hodge structure of type
$$\{(-1,1),(0,0),(1,-1)\} \cup \{(-1,0),(0,-1)\} \cup \{(-1,-1)\}.$$

(v) The weight filtration on $\text{Lie } P$ is given by
$$W_n(\text{Lie } P) = \begin{cases} 0 & \text{if } n < -2 \\ \text{Lie } U & \text{if } n = -2 \\ \text{Lie } W & \text{if } n = -1 \\ \text{Lie } P & \text{if } n \geq 0. \end{cases}$$

(vi) $\text{int}(\pi(h^\times(A)))$ induces a Cartan involution on $G^\text{ad}_{\mathbb{R}}$.

(vii) $G^\text{ad}$ possesses no nontrivial factors of compact type that are defined over $\mathbb{Q}$.

(viii) The center of $G$ acts on $U$ and on $V$ through a torus that is an almost direct product of a $\mathbb{Q}$-split torus with a torus of compact type defined over $\mathbb{Q}$.

**Definition:** We call such a triple $(P,X,h)$ mixed Shimura data. If $W=1$, then we also call it pure Shimura data, or just Shimura data.

**2.2. Remarks:** (a) The data $(P,X,h)$ uniquely determines the rest, in particular $U$ is characterized by (v). Thus the definition is meaningful. For abbreviation we shall mostly write $(P,X)$, since for every pair $(P,X)$ we shall always consider exactly one map $h$.

(b) It is easy to show that condition (iii) is a consequence of (iv).

(c) $X$ consists of finitely many connected components. The map $h : X \rightarrow h(X)$ is a local diffeomorphism, so the complex structure that is given on $h(X)$ by 1.18 (a) induces a complex structure on $X$. 


(d) If \( W=1 \) and \( X\cong h(X) \), then the definition is equivalent to [D1] 1.5 or to [D2] 2.1.1.1-3.

2.3. Definition: Let \((P_1, X_1, h_1)\) and \((P_2, X_2, h_2)\) be mixed Shimura data. A morphism \((P_1, X_1, h_1) \rightarrow (P_2, X_2, h_2)\) consists of a homomorphism \(\varphi: P_1 \rightarrow P_2\) and a \(P_1(\mathbb{R}) \cdot U_1(\mathbb{C})\)-equivariant map \(\psi: X_1 \rightarrow X_2\), such that the following diagram commutes:

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\psi} & X_2 \\
h_1 \downarrow & & \downarrow h_2 \\
\text{Hom}(S_C, P_1, \mathbb{C}) & \xrightarrow{h \cdot \varphi \cdot h} & \text{Hom}(S_C, P_2, \mathbb{C})
\end{array}
\]

We call a morphism an embedding, and write \((P_1, X_1, h_1) \hookrightarrow (P_2, X_2, h_2)\), if both \(\varphi\) and \(\psi\) are injective.

2.4. Proposition: For any morphism \((\varphi, \psi): (P_1, X_1) \rightarrow (P_2, X_2)\) the map \(\psi: X_1 \rightarrow X_2\) is holomorphic. If \((\varphi, \psi)\) is an embedding, then \(\psi\) is a closed embedding.

Proof: By 1.7 the associated map \(h_1(X_1) \rightarrow h_2(X_2)\) is holomorphic. From this the first assertion follows, since \(X_i\) is locally isomorphic to \(h_i(X_i)\) for \(i=1, 2\). The second assertion follows from the fact, that \(P_1(\mathbb{R}) \cdot U_1(\mathbb{C})\) is a closed subgroup of \(P_2(\mathbb{R}) \cdot U_2(\mathbb{C})\). \(\Box\)

2.5. Definition: For any two mixed Shimura data \((P_1, X_1)\) and \((P_2, X_2)\) we define their product as \((P_1, X_1) \times (P_2, X_2) := (P_1 \times P_2, X_1 \times X_2)\) with the obvious map \(X_1 \times X_2 \rightarrow \text{Hom}(S_C, (P_1 \times P_2, \mathbb{C}))\). Clearly this is again mixed Shimura data. In fact this is the direct product in the category of mixed Shimura data, that is for every \((P, X)\) and any two morphisms
(P, X) \to (P_1, X_1) \text{ there exists a unique morphism } (P, X) \to (P_1, X_1) \times (P_2, X_2) \text{ such that the following diagram commutes:}

\[
\begin{array}{ccc}
(P_1, X_1) & \rightarrow & (P_2, X_2) \\
\downarrow \text{pr} & & \downarrow \text{pr} \\
(P, X) & \rightarrow & (P_1, X_1) \times (P_2, X_2) \\
\downarrow \text{pr} & & \downarrow \text{pr} \\
\end{array}
\]

2.6. Example: a torus: Consider a torus T over \( \mathbb{Q} \), a homomorphism \( k: S \to T_\mathbb{Q} \), and a finite set \( \mathcal{Y} \) with a transitive action of \( \pi_0(T(\mathbb{R})) \). With the constant map \( h: \mathcal{Y} \to \{k \} \subset \text{Hom}(S_\mathcal{C}, T_\mathcal{C}) \) we then have mixed Shimura data \( (T, \mathcal{Y}) \). In fact every mixed Shimura data \( (P, X) \), where \( P \) is a torus or where \( X \) is a finite set, is of this form.

2.7. Example: the symplectic group: (Compare [D1] 1.6) Fix an integer \( g \geq 1 \). Let \( V_{2g} \) be a \( \mathbb{Q} \)-vector space of dimension \( 2g \) and \( \Psi: V_{2g} \times V_{2g} \to \mathbb{Q} \) a nondegenerate alternating form. Let \( G_{2g}:=(\text{Sp}_{2g}, \mathbb{Q}) \) be the group of all \( f \in \text{GL}(V_{2g}) \), such that \( \Psi(f(v), f(v')) = \lambda \Psi(v, v') \) for some scalar "multiplier" \( \lambda \). This multiplier yields a surjective homomorphism \( \text{Sp}_{2g}, \mathbb{Q} \to \mathfrak{g}_{m, \mathbb{Q}}, \) whose kernel is \( G_{2g} = (\text{Sp}_{2g}, \mathbb{Q})^{\text{der}} \). Let \( \mathcal{H}_{2g} \) be the set of all homomorphisms \( k: S \to \text{Sp}_{2g}, \mathbb{R} \) that induce a pure Hodge structure of type \( \{(-1, 0), (0, -1)\} \) on \( V_{2g} \), and for which the pairing \( \mathbb{R}^{2g} \times \mathbb{R}^{2g} \to \mathbb{R}, (v, v') \mapsto \Psi(v, k(A)v') \) is symmetric and (positive or negative) definite. Note that this second condition does not depend on the choice of \( A \). It is well-known that \( \text{Sp}_{2g}, \mathbb{Q} \) is a reductive group and operates transitively on \( \mathcal{H}_{2g} \). Moreover \( \mathcal{H}_{2g} \) possesses exactly two connected components. The pair \( (\text{Sp}_{2g}, \mathbb{Q}, \mathcal{H}_{2g}) \) together with the inclusion \( h: \mathcal{H}_{2g} \hookrightarrow \text{Hom}(S_\mathcal{C}, \text{Sp}_{2g}, \mathbb{Q}) \) is pure Shimura data.
For $g=1$ we have $\text{CSp}_{2g} = \text{GL}_{2g}$ and $\mathcal{H}_2$ can be identified in the usual way with the union of the complex upper and lower half planes. This, and more generally the unbounded realization of $\mathcal{H}_{2g}$ will be described in 4.25-26. Later in this chapter we shall give examples with nontrivial $W$, using some general constructions.

2.8 Example: For $g=0$ we have the following analog of $(\text{CSp}_{2g}, \mathcal{H}_{2g})$. Let $G_0 = \mathbb{G}_m$ and $k: \mathbb{Z} \rightarrow \mathcal{O}_0, z \mapsto z \cdot z$. Let $\mathcal{H}_0$ be the set of isomorphisms $\mathcal{O} \rightarrow \mathcal{O}(1)$. Thus an element $\lambda \in \mathcal{H}_0$ corresponds to a unique choice of $\mathcal{A}$ by $\lambda(1) = 2\pi \mathcal{A}$. At present we are only interested in the fact that $\mathcal{H}_0$ consists of exactly two elements, but later the precise definition will become important. Consider the unique transitive operation of $\pi_0(G_0(\mathbb{R})) = \pi_0(\mathbb{R})$ on $\mathcal{H}_0$, and let $h: \mathcal{H}_0 \rightarrow (k) \cdot \text{Hom}(\mathcal{O}_0, \mathcal{O}_0) \cdot \mathcal{O}(1)$ be the constant map. With these definitions $(G_0, \mathcal{H}_0)$ is pure Shimura data, as in 2.6.

For any $g \geq 1$ consider the multiplier map $\psi: \text{CSp}_{2g} \rightarrow \mathbb{G}_m$. Then for every $x \in \mathcal{H}_{2g}$ we have $\psi * h_x = k$. Define a map $\mathcal{H}_{2g} \rightarrow \mathcal{H}_0 \cdot \text{Hom}(\mathcal{O}, \mathcal{O}(1))$ by

$$x \mapsto \text{the unique } \lambda \in \mathcal{H}_0 \text{ such that } \lambda * \mathcal{A} \text{ is a polarization of } \mathcal{V}_{2g}.$$  

Clearly this map is equivariant under $\text{CSp}_{2g}(\mathbb{R}) \rightarrow \mathbb{G}_m(\mathbb{R})$, so we obtain a canonical morphism $(\text{CSp}_{2g}, \mathcal{H}_{2g}) \rightarrow (\mathbb{G}_m, \mathcal{H}_0)$.

2.9 Proposition: Let $(P, X)$ be mixed Shimura data and $P_0$ a normal subgroup of $P$. There exists a quotient mixed Shimura data $(P, X)/P_0$ and a morphism $(P, X) \rightarrow (P, X)/P_0$, unique up to isomorphism, such that every morphism $(P, X) \rightarrow (P', X')$, where the homomorphism $P \rightarrow P'$ factors through $P/P_0$, factors in a unique way through $(P, X)/P_0$. 
\[
\begin{align*}
P & \rightarrow P' \\
(P, X) & \rightarrow (P', X') \\
\downarrow & \quad \Rightarrow \quad \downarrow \\
P/P_0 & \quad \Rightarrow \quad (P, X)/P_0
\end{align*}
\]

**Proof:** The uniqueness follows from the universal property. First we construct \((P, X)/P_0\). Let \(P_1 := P/P_0\) and \(\varphi: P \rightarrow P_1\) be the canonical projection. Then \(W_1 = \varphi(W)\) is the unipotent radical of \(P_1\), and \(U_1 = \varphi(U)\) is normal in \(P_1\). Fixing \(x \in X\) we get a bijection
\[
(P(R) \cdot U(C))/\text{Stab}_{P(R) \cdot U(C)}(x) \longrightarrow X, \ p \mapsto p \cdot x.
\]

Let
\[
X_1 := (P_1(R) \cdot U_1(C))/\varphi(\text{Stab}_{P_1(R) \cdot U_1(C)}(x)), \quad \text{and}
\]

\[
h_1: X_1 \rightarrow \text{Hom}(S, P_1, C), \ [p_1] \mapsto \text{int}(p_1) \circ \varphi \circ h_x.
\]

By construction \((P_1, X_1, h_1)\) satisfies the conditions 2.1, so \((P_1, X_1)\) is mixed Shimura data. Next define an equivariant map \(\psi: X \rightarrow X_1\) by \(p \cdot x \mapsto [\varphi(p)]\), then \((\psi, \psi): (P, X) \rightarrow (P_1, X_1)\) is a morphism of mixed Shimura data. It is now trivial to show that \((P_1, X_1)\) possesses the desired universal property. q.e.d.

**Remark:** Let \((P_1, X_1) := (P, X)/P_0\). By the proof above the map \(X \rightarrow X_1\) is surjective if and only if the homomorphism \(P(R) \cdot U(C) \rightarrow P_1(R) \cdot U_1(C)\) is surjective. While this is not the case in general, it is so if \(P_0\) is unipotent.

**2.10 Example:** Let \((P, X)\) be mixed Shimura data. Since \(P\) acts nontrivially on every subquotient of \(W\), the commutator subgroup \(P^{\text{der}}\) contains the unipotent radical of \(P\). Thus \(P/P^{\text{der}}\) is a torus. The property that \(X = h(X)\) is not invariant under forming quotients, as the
following example shows. Let $(\text{CSp}_{2g}, \mathcal{H}_{2g}) \rightarrow (\mathfrak{g}_{m,0}, \mathcal{K}_0)$ be the morphism defined in 2.8. It is easy to verify that it induces an isomorphism

$$(\text{CSp}_{2g}, \mathcal{H}_{2g})/\text{Sp}_{2g,0} \cong (\mathfrak{g}_{m,0}, \mathcal{K}_0).$$

In this case $\mathcal{H}_{2g} \cong h(\mathcal{H}_{2g})$, but $\mathcal{H}_0 \not\cong h(\mathcal{K}_0)$. This shows one of the advantages of introducing finite coverings in 2.1.

Now let $(P, X, h)$ be mixed Shimura data and $\iota : h(X) \hookrightarrow \text{Hom}(S_C, P_C)$ the inclusion. Then $(P, h(X), \iota)$ is again mixed Shimura data, and there is a canonical morphism $(P, X, h) \rightarrow (P, h(X), \iota)$, or $(P, X) \rightarrow (P, h(X))$. We want to derive a relation between $(P, X)$, $(P, h(X))$ and $(P, X)/P^\text{der}$. This relation will show that the generalization to finite coverings does not yield essentially new objects.

2.11 Proposition: For any mixed Shimura data $(P, X)$ the canonical morphism

$$(P, X) \longrightarrow (P, X)/P^\text{der} \times (P, h(X))$$

is an embedding.

Proof: (Compare [D1] prop. 2.6) Let $T := P/P^\text{der}$, $\psi : P \rightarrow T$ the canonical homomorphism, and $(P, X)/P^\text{der} = (T, Y)$. Fix $x \in X$, and define as abbreviation $A := \text{Stab}_p(R) \cdot U(C)(x)$ and $B := \text{Cent}_p(R) \cdot U(C)(h_x)$. The group $A$ is a subgroup of finite index in $B$. As in the proof of 2.9 we identify $X$ with $(P(R) \cdot U(C))/A$. Since $h(x) = (P(R) \cdot U(C))/B$ we must prove that the map

$$(P(R) \cdot U(C))/A \longrightarrow T(R)/\psi(A) \times (P(R) \cdot U(C))/B$$

is injective. This is equivalent to

$$A = \psi^{-1}(\psi(A)) \cap B.$$
Now $\varphi^{-1}(\varphi(A)) \cap B = A \cdot P_{\text{der}}(R) \cdot U(C) \cap B = A \cdot (P_{\text{der}}(R) \cdot U(C)) \cap B$, because $A$ already lies in $B$. Since $P_{\text{der}}(R) \cdot U(C) \cap B = \text{Cent}_{P_{\text{der}}}(R) \cdot U(C)(h_x)$ the assertion is equivalent to

$$\text{Cent}_{P_{\text{der}}}(R) \cdot U(C)(h_x) \subset A.$$

Now the connected component of the identity of $\text{Cent}_{P_{\text{der}}}(R) \cdot U(C)(h_x)$ is contained in that of $B$, hence in $A$. Thus it is enough to show that $\text{Cent}_{P_{\text{der}}}(R) \cdot U(C)(h_x)$ is connected.

By 1.17 (a) we may now assume $P = G$. Being the centralizer of a torus, the algebraic subgroup $\text{Cent}_{G}(h_x)$ is connected. By 2.1 (vi) its Lie algebra is the subspace of invariants in $(\text{Lie } G)_R$ under $\text{Ad}_G(h(A))$, so $\text{Cent}_{G}(h_x)$ is compact. Since every algebraically connected compact linear group is already topologically connected (see [BT] 14.3), we are done. q.e.d.

2.12. Corollary: Let $(P, X)$ be mixed Shimura data. Every connected component of $X$ is mapped isomorphically to its image in $\text{Hom}(S_G, P_G)$. If $W = 1$, then every connected component of $X$ is a hermitian symmetric domain.

Proof: The first assertion follows directly from the injectivity of $X \hookrightarrow Y \times h(X)$ with a finite set $Y$. The second one follows from the first together with 1.14. q.e.d.

2.13. Covering by irreducible mixed Shimura data: Let $(P, X)$ be mixed Shimura data. Let $P_1$ be a normal subgroup of $P$, defined over $Q$, such that for some (or all) $x \in X$ the homomorphism $h_x$ factors through $P_1 \cdot C$. Then $W$ lies in $P_1$, and $P = P_1 \cdot \pi^{-1}(Z(G))$. Every
$P_1(\mathbb{R}) \cdot U(\mathbb{C})$-orbit $X_1 \subset X$ is the union of some connected components of $X$, since for any $x \in X_1$ there is an isomorphism

$$X_1 \cong P_1(\mathbb{R}) \cdot U(\mathbb{C}) / \text{Stabp}(\mathbb{R}) \cdot U(\mathbb{C})(x) \cong (P_1(\mathbb{R}) \cdot U(\mathbb{C}) \cdot \text{Stabp}(\mathbb{R}) \cdot U(\mathbb{C})(x)) / \text{Stabp}(\mathbb{R}) \cdot U(\mathbb{C})(x),$$

and by 1.17 $P_1(\mathbb{R}) \cdot U(\mathbb{C}) \cdot \text{Stabp}(\mathbb{R}) \cdot U(\mathbb{C})(x)$ is a subgroup of finite index in $P(\mathbb{R}) \cdot U(\mathbb{C})$. With $X_1 \ni x \mapsto h_x: S_0 \to P_1 \cdot C \cdot P \mathbb{C}$ we obtain mixed Shimura data $(P_1, X_1)$ and a canonical embedding $(P_1, X_1) \hookrightarrow (P, X)$.

Writing $X$ as disjoint union of $P_1(\mathbb{R}) \cdot U(\mathbb{C})$-orbits $X_i$ we get a "covering" of $(P, X)$ by these $(P_1, X_i)$. In this sense every mixed Shimura data is covered by others, for which one can impose additional conditions.

In particular call mixed Shimura data $(P, X)$ irreducible if there does not exist a proper normal subgroup of $P$, defined over $\mathbb{Q}$, through which the homomorphism $h_x$ factors for any (⇔ for all) $x \in X$. Any mixed Shimura data possesses a canonical covering by irreducible ones.

2.14 Proposition: Let $(P, X)$ be irreducible mixed Shimura data.

(a) $P$ operates on Lie $U$ through a scalar character $P \to \mathbb{C}_{m,0}$.

(b) There exists a character $P \to \mathbb{C}_{m,0}$, a nondegenerate $P$-equivariant pairing $\Psi: (\text{Lie } V) \otimes (\text{Lie } V) \to \mathbb{Q}$, where $P$ operates on $\mathbb{Q}$ through this character, and for every $x \in X$ a homomorphism $\lambda_x \in \text{Hom}(\mathbb{Q}, \mathbb{Q}(1))$, such that for every $x \in X$ $\lambda_x \cdot \Psi$ is a polarization of the Hodge structure on Lie $V$ defined by $h_x$.

Proof: Without loss of generality we may replace $(P, X)$ by $(P, h(X)) \times Z(P)$, then the conditions of 1.18 hold, in particular the weight is rational. Let $G_1 \subset G$ be the smallest normal subgroup, defined over $\mathbb{Q}$, such that for some (⇔ for all) $x \in X$ the homomorphism $h|_{G_1}$ factors
through $G_1$. Then by the irreducibility of $(P,X)$ we have $G = G_1 \cdot (h_\mathfrak{g} \cdot w)(G_m,0)$. The assertion (a) now follows from the fact that $G_1$ acts trivially on Lie $U$, and that the weight acts on Lie $U$ through a scalar character. For (b) let $M := \text{Lie } V$ and $N, \Psi : M \otimes M \rightarrow N$ as in 1.12. By the remarks in 1.13 we may choose $N$ of dimension 1, and the rest follows. \textit{q.e.d.}

2.15. Structure of the unipotent radical: Consider the short exact sequence $1 \rightarrow U \rightarrow W \rightarrow V \rightarrow 1$. Let $[, ]$ be the Lie bracket in Lie $P$. Since Lie $W$ is mixed of weight $\leq -1$, the subalgebra $[\text{Lie } W, \text{Lie } W] \subset \text{Lie } W$ is mixed of weight $\leq -2$, hence contained in Lie $U$. Furthermore $[\text{Lie } W, \text{Lie } U]$ is mixed of weight $\leq -3$, hence 0, since in Lie $W$ only the weights $-1$ and $-2$ may occur. Thus $W$ is a central extension of the two abelian unipotent groups $V$ and $U$. The commutator on $W$ induces an alternating form $\Psi : V \times V \rightarrow U$, which determines the extension up to isomorphism. In fact, given $U$, $V$ and $\Psi$, one can reconstruct $W$ for instance as follows. Let $W := U \times V$ as variety over $Q$, and define $(u,v) \cdot (u',v') := (u+u'+\frac{1}{2} \Psi(v,v'),v+v')$.

The operation of $P$ by conjugation on $U$ and on $V$ factors through $G$, and the pairing is $G$-equivariant. Interpreting $U$ and $V$ as representations of $G$ we have thus associated to $P$ the data $(G,V,U,\Psi)$. Up to isomorphism $P$ is uniquely determined by this data. In fact observe that the extension $1 \rightarrow W \rightarrow P \rightarrow G \rightarrow 1$ splits in any case, so only the operation of $G$ on $W$ is in question. If $W$ is identified with $U \times V$ as above, then this operation may be defined by $g((u,v)) = (g(u),g(v))$. Since $G$ is reductive and no irreducible representation of $G$ occurs in both $U$ and $V$, this is the unique possible action.
Next we describe how to go back and forth between mixed Shimura data for some group $P$ and that for an extension of $P$ by a unipotent group.

2.16. **Partial inverse of 2.9:** Consider mixed Shimura data $(P, X)$ and an extension $1 \to W_0 \to P_1 \to P \to 1$ of $P$ by a unipotent group $W_0$. We want to study all possible mixed Shimura data $(P_1, X_1)$ with an isomorphism $(P_1, X_1)/W_0 \cong (P, X)$ extending the isomorphism $P_1/W_0 \cong P$. The conditions 2.1 (iv), (v) and (viii) give a necessary condition for the existence of such $(P_1, X_1)$. Observe that the adjoint action of $P_1$ on every abelian subquotient of $W_0$ factors through $P$. Thus the necessary condition says that the Lie algebra of every irreducible subquotient of Lie $W_0$ must be of type $\{(-1,0),(0,-1)\}$ or $\{(-1,-1)\}$ as representation of $G$, and the center of $G$ acts on it through a torus that is an almost direct product of a $\mathbb{Q}$-split torus with a torus of compact type defined over $\mathbb{Q}$.

2.17. **Proposition:** Let $(P, X)$, $W_0$ and $P_1$ be as in 2.16, and assume that Lie $W_0$ satisfies the condition given in 2.16. Then:

(a) There exists mixed Shimura data $(P_1, X_1)$, and a morphism $(P_1, X_1) \to (P, X)$ that extends the given homomorphism $P_1 \to P$, such that $(P_1, X_1)/W_0 \cong (P, X)$. They are uniquely determined up to isomorphism.

(b) For every morphism $(P', X') \to (P, X)$ and every factorization $P' \to P_1 \to P$ there exists exactly one extension $(P', X') \to (P_1, X_1) \to (P, X)$:

$$
\begin{array}{ccc}
P' & \to & P_1 \\
\downarrow & & \downarrow \\
P & \to & (P, X)
\end{array}
$$
Definition: We call \((P_1, X_1)\) a unipotent extension of \((P, X)\). Let \(U_0 := W_0 \cap U_1\) and \(V_0 := W_0 / U_0\), then we have a short exact sequence 
\[1 \to U_0 \to W_0 \to V_0 \to 1,\]
which we call the weight decomposition of \(W_0\).

Proof: Fix a Levi decomposition \(P_1 = W_1 \times G\), this induces a Levi decomposition \(P = W \times G\). Let \(x_0 \in X\) such that \(h_{x_0}\) factors through \(G_{CP}\). By assumption \(h_{x_0}\) induces a rational Hodge structure of type 
\[\{(-1,0), (0,-1), (-1,-1)\}\]
on every irreducible \(G\)-subspace of \(\text{Lie } W_1\). Define \(U_1 \subseteq W_1\) by \(\text{Lie } U_1 = W_2(\text{Lie } W_1)\), this is a subspace defined over \(\mathbb{Q}\). As in 2.15 it follows that \(U_1\) is contained in the center of \(W_1\), so it is a normal subgroup of \(P_1\). Now by definition the conditions 2.1 (iii)-(viii) hold for \(P_1\) and \(h_{x_0}\). Denote the given homomorphism \(P_1 \to P\) by \(\varphi\). If the desired \((P_1, X_1)\) exists, then by 1.17 for every \(x_1 \in X_1\) that is mapped to \(x \in X\) we have 
\[\text{Stab}_{P_1}(R) \cdot U_1(C)(x_1) \cong \text{Stab}_P(R) \cdot U(C)(x)\]
Thus \(X_1\) is isomorphic to
\[X_1 := \{(x, k) \in X \times \text{Hom}(S_\mathbb{C}, P_1, \mathbb{C}) \mid \text{such that } h_x = \varphi \cdot k, \text{ and } n' \cdot k : S_\mathbb{C} \to (P_1 / U_1)_\mathbb{C} \text{ is already defined over } \mathbb{R}\}.

Taking this as definition of \(X_1\), together with the obvious map \(h : X_1 \to \text{Hom}(S_\mathbb{C}, P_1, \mathbb{C}), (x, k) \mapsto k\), all the conditions of 2.1 are satisfied, so we get mixed Shimura data \((P_1, X_1)\) and a morphism \((P_1, X_1) \to (P, X)\), unique up to isomorphism. Comparing this construction with that in 2.9 we find \((P_1, X_1) / W_0 \cong (P, X)\). The universal property of \((P_1, X_1)\) is obvious from its construction. a.e.d.

2.18 The structure of the fibres: Let \((P_1, X_1) \to (P, X)\) be a unipotent extension. Then the map \(\psi : X_1 \to X\) is a \(W_0(R) \cdot (W_0 \cap U_1)(C)\)-torsor, in particular its fibres are connected. Let \(x_1 \in X_1\) and \(x = \psi(x_1)\), then \(x_1\) defines a Hodge structure on \(\text{Lie } W_0\), and as in the proof of 1.16 it follows that the map
\[ W_0(\mathbb{R}) \cdot (W_0 \cap U_1)(\mathbb{C}) \xrightarrow{w_0 \cdot \omega^x_1} \psi^{-1}(x) \]

\[ \text{exp} \]

\[ (\text{Lie } W_0)_0 \cdot (\text{Lie } (W_0 \cap U_1)_0) \cong (\text{Lie } W_0)_0 / F^0(\text{Lie } W_0)_0 \]

is an holomorphic isomorphism.

Consider the constant vector bundle \( W_o \) on \( X_1 \) with fibre \((\text{Lie } W_0)_0\). The operation of \( P_1(\mathbb{R}) \cdot U_1(\mathbb{C}) \) on \((\text{Lie } W_o)_0\) makes it into an equivariant vector bundle. There exists a smooth subbundle \( F^0 W_0 \), whose fibre in \( x_1 \in X_1 \) equals \( F^0(\text{Lie } W_0)_0 \) for the Hodge structure defined by \( x_1 \). This subbundle is invariant under the operation of \( W_0(\mathbb{R}) \cdot (W_0 \cap U_1)(\mathbb{C}) \) if and only if \( W_0 \) is abelian. Then and only then \( F^0 W_0 \subset W_0 \) are pullbacks of vector bundles on \( X \).

Suppose that the short exact sequence \( 1 \to W_o \to P_1 \to P \to 1 \) splits. By 2.17 (b) any splitting yields a section \( \varepsilon: (P, X) \to (P_1, X_1) \). By pullback we obtain vector bundles on \( X \), and the above isomorphism between a fibre \( \psi^{-1}(x) \) and \((\text{Lie } W_o)_0 / F^0(\text{Lie } W_o)_0 \) yields an identification of \( X_1 \to X \) with the holomorphic complex vector bundle \( \varepsilon^*(W_0 / F^0 W_0) \).

Let us apply this to arbitrary mixed Shimura data \((P_1, X_1)\) with \( W_0 = W_1 \). Then the sequence splits, and with 2.12 we have proved:

2.19. Proposition: Let \((P, X)\) be mixed Shimura data. Then every connected component of \( X \) is a holomorphic complex vector bundle on a hermitian symmetric domain.

2.20. Fibre product: Let \((P_1, X_1) \to (P, X)\) be a unipotent extension and \((P_2, X_2) \to (P, X)\) an arbitrary morphism. Let \( P_{12} := P_1 \times_P P_2 \) and \( X_{12} := X_1 \times_X X_2 \) be the fibre products. Since \( P_1 \to P \) is an extension by a unipotent group, \( P_{12} \) is a connected subgroup of \( P_1 \times_P P_2 \) (this would not
be true for an arbitrary morphism \((P_1, X_1) \to (P, X)\). Define \(h_{12}\) by the commutative diagram

\[
\begin{array}{ccc}
X_{12} & \xrightarrow{h_{12}} & \text{Hom}(S_C, P_{12}, C) \\
\downarrow & & \downarrow \\
X_1 \times X_2 & \xrightarrow{h_1 \times h_2} & \text{Hom}(S_C, P_1 \times P_2, C).
\end{array}
\]

Since \(P_{12} \to P_2\) is a unipotent extension, it follows at once that \((P_{12}, X_{12})\) is mixed Shimura data. We have a commutative diagram

\[
\begin{array}{ccc}
(P_{12}, X_{12}) & \longrightarrow & (P_1, X_1) \\
\downarrow & & \downarrow \\
(P_2, X_2) & \longrightarrow & (P, X).
\end{array}
\]

Furthermore \((P_{12}, X_{12})\) possesses the universal property of the fibre product in the category of mixed Shimura data; this is also a trivial consequence of the construction. We write \((P_1, X_1) \times (P, X) (P_2, X_2)\) for \((P_{12}, X_{12})\).

2.21. Operations on the fibre for split unipotent extensions: Consider mixed Shimura data \((P, X)\) and a representation \(W_0\) of \(P\) that satisfies the condition in 2.16. We consider \(W_0\) as an (abelian) unipotent group over \(Q\) and define \(P_1 := W_0 \times P\) with the multiplication \((w, p) \cdot (w', p') = (w \cdot p(w'), p \cdot p')\). By 2.17 we get a canonical morphism \((P_1, X_1) \to (P, X)\) together with a section \((P, X) \to (P_1, X_1)\). For two such representations \(W_0\) and \(W_0'\) and a \(P\)-equivariant homomorphism \(f: W_0 \to W_0'\) we define a homomorphism \(P_1 = W_0 \times P \xrightarrow{f \times \text{id}} W_0' \times P = P_1'\). By 2.17 we get a canonical morphism \((P_1, X_1) \to (P_1', X_1')\) of the corresponding mixed Shimura data. Thus for fixed \((P, X)\) we have a functor from the category of all finite dimensional representations of \(P\) over \(Q\) that satisfy 2.16 to the category of all mixed Shimura data "over" \((P, X)\).
together with a section. It is easily seen that this functor maps direct sums to fibre products (2.20).

For instance the addition \( W_0 \oplus W_0 \rightarrow W_0 \) yields a morphism \( m: (P_1, X_1) \times (P_1, X_1) \rightarrow (P_1, X_1) \), which on the fibre of \( X_1 \rightarrow X \) corresponds to the addition on the corresponding vector bundles of 2.18. Thus \( (P_1, X_1) \) becomes a group object in the category of all Shimura data over \((P, X)\). The endomorphism algebra \( \text{End}_P(W_0) \) also operates on \((P_1, X_1)\) over \((P, X)\). Both these operations are compatible with the “zero” section \( \varepsilon: (P, X) \rightarrow (P_1, X_1) \). There is also the following operation: For any \( w_0 \in W_0(\mathbb{Q}) \) the conjugation by \( w_0 \) induces an isomorphism

\[
\text{int}(w_0): \quad (P_1, X_1) \xrightarrow{\text{int}(w_0) \cdot \varepsilon} (P_1, X_1)
\]

that transforms the zero section \( \varepsilon \) into \( \text{int}(w_0) \cdot \varepsilon \).

2.22. Operations on the fibre in an abelian unipotent extension: Let \( \varphi: (P_1, X_1) \rightarrow (P, X) \) be a unipotent extension with \( W_0 \) abelian. Here we do not assume that the extension splits. Still the operation of \( P_1 \) on \( W_0 \) factors through \( P \), and the vector bundles considered in 2.18 come from canonical vector bundles on \( X \). So the operation of \( W_0(\mathbb{R}) \cdot (W_0 \cap U_1)(\mathbb{C}) \) on \( X_1 \) makes the bundle \( X_1 \rightarrow X \) into a holomorphic torsor under the additive group of this vector bundle \( W_0 \cap U_0 \) on \( X \).

We can express this situation functorially as follows. For more generality let \( (P, X) \rightarrow (P_1, X_1) \) be a morphism such that the operation of \( P \) on \( W_0 \) factors through \( P_1 \). Let \( (P', X') \rightarrow (P_1, X_1) \) be the (splitting) unipotent extension with \( P' = W_0 \times P_1 \). Let \( (P_1, X_1') \) be the fibre product \( (P', X') \times_{(P_1, X_1)} (P_1, X_1) \), then \( P' \times P_1 = (W_0 \times P_1) \times P_1 \). The map

\[
P': \quad (P' \times P_1) = (W_0 \times P_1) \times P_1 \rightarrow W_0 \times P_1
\]
\[ P_1^* \cong W_0 \times P_1 \longrightarrow P_1, \ (w,p_1) \mapsto w \cdot p_1 \]

is a homomorphism, so by 2.17 (b) it extends canonically to a morphism

\[ \mu: (P',X') \times (P_* , X_*) (P_1 , X_1) = (P_1^* , X_1^*) \longrightarrow (P_1 , X_1). \]

2.23. Proposition: This morphism defines an operation of the group object \((P',X')\rightarrow (P_* , X_*)\) on \((P_1 , X_1)\) in the category of all mixed Shimura data over \((P_* , X_*)\). Through this \((P_1 , X_1)\) becomes a \(((P',X')\rightarrow (P_* , X_*))\)-torsor over \((P,X)\).

Proof: The assertion is equivalent to the conjunction of the following two:

(a) The diagram

\[
\begin{array}{c}
(P',X') \times (P_* , X_*) (P_1 , X_1) \xrightarrow{\text{id} \times \mu} (P',X') \times (P_* , X_*) (P_1 , X_1) \\
\mu \downarrow \quad \downarrow \mu
\end{array}
\]

is commutative.

(b) The morphism

\[
(P',X') \times (P_* , X_*) (P_1 , X_1) \xrightarrow{(pr_2, \mu)} (P_1 , X_1) \times (P,X) (P_1 , X_1)
\]

is well-defined and an isomorphism.

Since we are only dealing with unipotent extensions of \((P,X)\), it suffices to prove these assertions for the corresponding homomorphisms \(P' \times P_* \rightarrow P_1\), etc. Consider the diagram in (a). Like for \(P' \times P_* \rightarrow W_0 \times P_1\) there is a similar canonical isomorphism \(P' \times P_* P' \times P_* P_1 \cong W_0 \times W_0 \times P_1\). One finds that an element \((w,w',p_1)\) is mapped to \(w \cdot w' \cdot p_1 \in P_1\) through both corners of the diagram, whence (a). (b) is equivalent to the assertion that the diagram
\[(P_1, X_1) \rightarrow (P_1, X_1) \quad \text{pr}_2 \downarrow \uparrow \text{pr} \]
\[(P_1, X_1) \rightarrow (P, X)\]

is cartesian. This is implied by the fact that the homomorphism \(W \times P_1 \rightarrow P_1 \times P_1\), \((w, p_1) \mapsto (w \cdot p_1, p_1)\) is an isomorphism. \(\text{q.e.d.}\)

2.24 Example: Let \((G_0, H_0)\) be as in 2.8. Let \(U_0=\mathfrak{g}_a=\mathbb{Q}\) with the standard operation of \(G_0=\mathfrak{g}_m, \mathbb{Q}\). Clearly \(U_0\) is pure of type \((-1, -1)\) as representation of \(G_0\), and \(G_0\) is itself \(\mathbb{Q}\)-split, so the construction 2.17 yields a unipotent extension \((P_0, X_0)\) of \((G_0, H_0)\) with \(P_0=U_0 \times G_0\). We can also describe it as follows. Identify \(P_0\) with the subgroup of all matrices of the form \(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\) in \(GL_2, \mathbb{Q}\), \(h(X_0)\) with the set of all homomorphisms \(S(\mathbb{Q}) \rightarrow P_0, \mathbb{Q}\) of the form \(S(\mathbb{Q}) \rightarrow \mathbb{R}, \mathbb{C} \ni z \mapsto (\bar{z}, z)\), and \(X_0\) with \(h(X_0) \times H_0\).

Every connected component of \(X_0\) is isomorphic to \(\mathbb{C}\), and the group structure defined by 2.21 corresponds to that of the additive group \(\mathbb{C}\). As a variant we can also consider the Shimura data \((P_0, h(X_0))\), here \(h(X_0)\) is connected, and \((P_0, h(X_0))\) is a group object over \((\mathfrak{g}_m, \mathbb{Q}, \{k\}) = (G_0, h(X_0))\) (see 2.8). Note that for every \((X_0, X_0) \rightarrow (\mathfrak{g}_m, \mathbb{Q}, H_0)\)-torsor \((P', X') \rightarrow (P, X)\) we have a canonical isomorphism between \(U_0\) and the kernel of the homomorphism \(\pi_0: P' \rightarrow P\).

2.25 Example: Let \((CSp_{2g}, 0, H_{2g})\) be the Shimura data defined in 2.7, and \(V_{2g}\) the standard representation of \(CSp_{2g}, 0\). The construction 2.17 yields mixed Shimura data \((V_{2g}, CSp_{2g}, 0, H_{2g}^*)\). Let \(U_{2g}=\mathfrak{g}_a=\mathbb{Q}\) with the operation of \(CSp_{2g}, 0\) by the multiplier. The given alternating form \(\Psi: V_{2g} \times V_{2g} \rightarrow \mathbb{Q} \rightarrow U_{2g}\) is \(CSp_{2g}, 0\)-equivariant. Let \(1 \rightarrow U_{2g} \rightarrow W_{2g} \rightarrow V_{2g} \rightarrow 1\) be the central extension of unipotent groups defined by \(\Psi\),
and let \( P_{2g} := W_{2g} \times \text{CSP}_{2g, 0} \). By 2.17 we obtain mixed Shimura data \((P_{2g}, X_{2g})\) and morphisms

\[
(P_{2g}, X_{2g}) \rightarrow (V_{2g} \times \text{CSP}_{2g, 0}, \mathcal{H}_{2g}^\circ) \rightarrow (\text{CSP}_{2g, 0}, \mathcal{H}_{2g}).
\]

The unipotent extension \((V_{2g} \times \text{CSP}_{2g, 0}, \mathcal{H}_{2g}^\circ) \rightarrow (\text{CSP}_{2g, 0}, \mathcal{H}_{2g})\) is an example for a group object. Let \((\text{CSP}_{2g, 0}, \mathcal{H}_{2g}) \rightarrow (\mathcal{E}_m, 0, \mathcal{K}_0)\) be the morphism defined in 2.8. Since the operation of \(\text{CSP}_{2g, 0}\) on \(U_{2g}\) factors through \(\mathcal{E}_m, 0\), every isomorphism \(U_{2g} \cong U_0\) gives a structure on \((P_{2g}, X_{2g})\) as a \(((P_0, X_0) \rightarrow (\mathcal{E}_m, 0, \mathcal{K}_0))\)-torsor over \((V_{2g} \times \text{CSP}_{2g, 0}, \mathcal{H}_{2g}^\circ)\).

2.26. Reduction lemma: Let \((P, X)\) be irreducible mixed Shimura data. Then there exist pure Shimura data \((T, \mathcal{Y})\) and \((G, \mathcal{H})\), where \(T\) is a torus and \(h: \mathcal{H} \hookrightarrow \text{Hom}(S, G)\) injective, and a number \(n \geq 0\), such that

(a) If \(V = 1\), then there exists an embedding

\[
(P, X) \hookrightarrow (T, \mathcal{Y}) \times (G, \mathcal{H}) \times \prod_{i=1}^{n} (P_0, h(X_0)),
\]

where \((P_0, h(X_0))\) is the mixed Shimura data defined in 2.24.

(b) If \(2g = \dim(V) > 0\), then there exists a morphism \((P, X) \rightarrow (\mathcal{E}_m, 0, \mathcal{K}_0)\), a \(((P_0, X_0) \rightarrow (\mathcal{E}_m, 0, \mathcal{K}_0))\)-torsor \((P', X') \rightarrow (P, X)\) and an embedding

\[
(P', X') \hookrightarrow (T, \mathcal{Y}) \times (G, \mathcal{H}) \times \prod_{i=1}^{n} (P_{2g}, X_{2g}),
\]

where \((\mathcal{E}_m, 0, \mathcal{K}_0)\), \((P_0, X_0)\), and \((P_{2g}, X_{2g})\) are the mixed Shimura data defined in 2.8, 2.24 and 2.25 respectively.

Proof: (a) By 2.14 (a) \(P\) operates through scalars on \(U\). Let \(P_1 \subset P\) be the kernel of this operation. Let \(\lambda_1, \ldots, \lambda_n\) be a basis for \(\text{Hom}(U, 0)\). For every \(i\) the quotient \(P/P_1 \cdot \text{Ker}(\lambda_i)\) is isomorphic to the group \(P_0\) defined in 2.24. One sees easily that any such isomorphism extends
uniquely to a morphism \((P,X) \rightarrow (P_0, h(X_0))\). By construction the direct product of these morphisms

\[(P, X) \rightarrow \prod_{i=1}^{n} (P_0, h(X_0))\]

is injective on \(U\). Thus the corresponding morphism

\[(P, X) \rightarrow (P, X)/U \times \prod_{i=1}^{n} (P_0, h(X_0))\]

is an embedding. Together with the embedding \((P, X)/U \hookrightarrow (T, Y) \times (G, H) \cong (T, Y) \times (P, h(X))/U\) from 2.11 we get the desired result.

(b) Let \(\Psi: V \times V \rightarrow U\) be pairing induced from the commutator. Note that \(V\) is pure of weight \(-1\), so according to 2.14 (b) we can fix a \(P\)-equivariant pairing \(\tilde{\Psi}: V \times V \rightarrow U_0 = G_{a, Q} = Q\), with a suitable operation of \(P\) on \(Q\), such that for all \(x \in X\) there exists \(\lambda \in \text{Hom}(Q, Q(1))\), such that \(\lambda \ast \tilde{\Psi}\) is a polarization of the Hodge structure on \(V\) defined by \(x\). Define \(U' = U \times Q\) and \(\Psi' = (\Psi, \tilde{\Psi}): V \times V \rightarrow U'\). By 2.15 the triple \((U', V, \Psi')\) defines a central extension \(1 \rightarrow U' \rightarrow W' \rightarrow V \rightarrow 1\), and \(W'/U_0\) is isomorphic to \(W\). Fixing such an isomorphism we let \(P' = W' \times G\), this is then a unipotent extension of \(P\) by \(U_0 = Q\), so we obtain a unipotent extension \((P', X') \rightarrow (P, X)\). Let \(\varphi: P \rightarrow G_{m, Q}\) be the character through which \(P\) operates on \(U_0\). For every \(x \in X\) let \(\lambda_x \in X_0\) be the unique element, such that \(\lambda_x \ast \Psi\) is a polarization of the Hodge structure on \(V\) defined by \(x\). Then \((\varphi, \psi)\) is a morphism \((P, X) \rightarrow (G_{m, Q}, X_0)\), and \((P', X') \rightarrow (P, X)\) is a \((P_0, X_0) \rightarrow (G_{m, Q}, X_0))-\text{torsor in a canonical way.}\)

Since the irreducibility of \((P, X)\) implies that of \((P', X')\), \(P'\) acts through scalars on \(U'\). Consider the set of all \(\mu \in \text{Hom}(U', Q)\) such that for all \(x \in X\) there exists \(\lambda \in \text{Hom}(Q, Q(1))\), so that the pairing \(\lambda \ast \mu \ast \Psi'\) is a polarization of \(V\). This set is open in the usual topology, and nonempty since it contains \(\text{pr}_2: U' \rightarrow U_0 = Q\). Therefore it contains a basis \(\mu_1, \ldots, \mu_n\) of \(\text{Hom}(U', Q)\). Let \(2g = \dim(V)\), and let \((P_{2g}, X_{2g}), W_{2g}, V_{2g}\) be as in 2.25. For every \(i = 1, \ldots, n\) there exists an isomorphism \(V_i = V_{2g}\) such