

Arithmetical Compactification of Mixed Shimura Varieties

Richard Pink

For Mohamed Elham Mohamed Sheiry,
who was the first to teach me real mathematics.

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Chapter 0

Introduction

Let $\mathcal{H} \subset \mathbb{C}$ denote the so-called “complex upper half plane”, i.e. the set of all $z \in \mathbb{C}$ with (strictly) positive imaginary part. The map

$$\mathrm{SL}_2(\mathbb{Z}) \times \mathcal{H} \rightarrow \mathcal{H}, \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) \mapsto \frac{az+b}{cz+d}$$

defines a holomorphic left operation of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$. For any subgroup of finite index $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, the quotient $\Gamma \backslash \mathcal{H}$ inherits the structure of a Riemann surface. This Riemann surface is not compact, but by adjoining a finite number of additional points, the so-called cusps, it can be embedded in a compact Riemann surface (see [Sh] ch.1). In particular, there is a smooth projective algebraic curve $\overline{X}$ over $\mathbb{C}$, a non-empty Zariski-open subset $X \subset \overline{X}$, and an isomorphism $X(\mathbb{C}) \cong \Gamma \backslash \mathcal{H}$, all of this unique up to isomorphism.

Let us now assume that Γ is a congruence subgroup. This signifies that Γ contains the subgroup

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1, b \equiv c \equiv 0 \pmod{N} \right\}$$

for some integer N . Then $\Gamma \backslash \mathcal{H}$ is a moduli space for elliptic curves with a certain level structure (which depends on Γ). For instance, if $\Gamma = \Gamma(N)$, then $\Gamma \backslash \mathcal{H}$ is in bijection with the set of all isomorphism classes of pairs (E, x, y) , where E is an elliptic curve over $\mathbb{C}$ and $\{x, y\}$ is a basis of the subgroup $E[N]$ of all N -torsion points of E , such that $e_N(x, y) = e^{2\pi i/N}$. Here e_N denotes the canonical pairing $E[N] \times E[N] \rightarrow \mu_N$. The moduli problem associated to Γ can be described in algebraic terms, and it can be shown in a purely algebraic way that there exists a corresponding (coarse or fine) moduli scheme M over a certain explicit number field K . In the case $\Gamma = \Gamma(N)$ we have $K = \mathbb{Q}(e^{2\pi i/N}) \subset \mathbb{C}$, while for arbitrary $\Gamma \supset \Gamma(N)$ any subfield of $\mathbb{Q}(e^{2\pi i/N})$ may occur.

This modular interpretation of $\Gamma \backslash \mathcal{H}$ determines an isomorphism $X \cong M \times_K \mathbb{C}$. In other words, it defines a “model” of X over K . Let $\overline{M}$ be the (up to isomorphism) unique non-singular projective curve over K that contains M as an open dense subset. Then the above isomorphism extends to an isomorphism $\overline{X} \cong \overline{M} \times_K \mathbb{C}$, that is, the model of X extends to one of $\overline{X}$. Thus the following problem arises naturally: Describe the structure of this model on, and near, the cusps $\overline{X} \setminus X$.

An answer could have the following form. Let D be the (unique) reduced closed subscheme of $\overline{M}$ with support in $\overline{M} \setminus M$. Via the isomorphism

$$D(\overline{\mathbb{Q}}) = D(\mathbb{C}) \cong \overline{X}(\mathbb{C}) \setminus X(\mathbb{C}) = \overline{X}(\overline{\mathbb{Q}}) \setminus X(\overline{\mathbb{Q}})$$

induced from the above, the model defines an action of $\text{Gal}(\overline{\mathbb{Q}}/K)$ on the cusps. Conversely this action determines the structure of D . More abstractly, it would be desirable to have an explicit scheme D_1 over K and an explicit isomorphism $D \cong D_1$. This latter formulation generalizes to the infinitesimal neighborhoods: one would like to have an explicit formal scheme $\mathcal{D}_1$ over K , and an explicit isomorphism of $\mathcal{D}_1$ with the formal completion of $\overline{M}$ along D .

In the case under consideration, such descriptions follow from a modular interpretation of $\overline{X}(\mathbb{C})$. This involves a degenerate version of elliptic curves with N -structure, the so-called generalized elliptic curves. For the formal completion, the Tate-curve can be used (see [DR], and 10.17–22 below).

In these notes we carry out the same program for arbitrary Shimura varieties. To explain the complications arising in the general case, fix a hermitian symmetric domain $\mathcal{H}$ of non-compact type. Also fix a semisimple algebraic group G over $\mathbb{Q}$, and an isomorphism from $G(\mathbb{R})^0$ to the group of all holomorphic automorphisms of $\mathcal{H}$. Let Γ be an arithmetic subgroup of $G(\mathbb{Q}) \cap G(\mathbb{R})^0$, then $\Gamma \backslash \mathcal{H}$ again carries a natural structure of normal complex space. If Γ is sufficiently small, then $\Gamma \backslash \mathcal{H}$ is already a complex manifold. For simplicity we assume this to be the case.

In general, $\Gamma \backslash \mathcal{H}$ is not compact. It possesses different possible compactifications as a normal complex space, with different advantages. Only in the case $G \cong PGL_{2,\mathbb{Q}}$ they happen to coincide. Logically and historically, the Baily-Borel compactification comes first. Its nice properties are that it depends on no extra data, and that it is uniquely characterized as the minimal normal compactification. In any case, it can be defined intrinsically and is a projective variety. It possesses a natural stratification in terms of quotients $\Gamma' \backslash \mathcal{H}'$ for certain other hermitian symmetric domains $\mathcal{H}'$ (the so-called boundary components), and arithmetic subgroups Γ' of $\text{Aut}(\mathcal{H}')$ (see [BB], [AMRT], and 6.2–3 below).

Unfortunately, the Baily-Borel compactification may have serious singularities along the boundary. A nice resolution of these singularities is the toroidal compactification. It is defined locally as a torus embedding. The cone decompositions involved in this definition are not canonical, so there are different toroidal compactifications associated to different systems of cone decompositions. However, for suitable choices the resulting complex space is projective and smooth, and the boundary is the union of smooth divisors with at most normal crossings. The toroidal compactification dominates the Baily-Borel compactification, and possesses a natural stratification compatible with that of the Baily-Borel compactification. But the strata now have a more general form: they are torus-torsors over families of abelian varieties over the strata of the Baily-Borel compactification.

These compactifications carry a natural algebraic structure, and induce an algebraic structure on $\Gamma \backslash \mathcal{H}$. The next step is to ask for a model of these algebraic varieties over a number field. Here one gets stronger results if one considers not only $\Gamma \backslash \mathcal{H}$, but a certain finite disjoint union of such quotients. For this, G is replaced by a central extension with a torus, $\mathcal{H}$ by a finite disjoint union of hermitian symmetric domains with a transitive holomorphic action of $G(\mathbb{R})$, and instead of $\Gamma \backslash \mathcal{H}$ one considers a double quotient $G(\mathbb{Q}) \backslash \mathcal{H} \times (G(\mathbb{A}_f)/K_f)$. Here $\mathbb{A}_f$ denotes the finite adeles of $\mathbb{Q}$, K_f is an open compact subgroup of $G(\mathbb{A}_f)$, and $G(\mathbb{Q})$ acts on both factors. This double quotient is indeed of the desired form, and constitutes the set of complex points of a Shimura variety. Let us denote the associated

algebraic variety over $\mathbb{C}$ by $M_{\mathbb{C}}$, the Baily-Borel compactification by $\overline{M}_{\mathbb{C}}$, and the toroidal compactification associated to a system Σ of cone decompositions by $M_{\Sigma, \mathbb{C}}$.

The advantage of this adelic definition is that one gets a model over a number field that is independent of K_f , called the reflex field. It is determined by G , $\mathcal{H}$, and a little extra data concerning the center of G . Some Shimura varieties have a natural interpretation as moduli spaces of abelian varieties with some extra structure. For instance, generalizing the special case considered above, the group of symplectic similitudes $CSp_{2g, \mathbb{Q}}$ gives rise to the Siegel modular variety parametrizing principally polarized abelian varieties of dimension g with level structure. Such a modular interpretation induces a model M of $M_{\mathbb{C}}$ over the reflex field. However, not every Shimura variety has such a purely algebraic modular interpretation. Nevertheless, following Shimura the models defined by modular interpretation can be uniquely characterized in an intrinsic way (see [D1]). Taking this characterization as a definition, one arrives at the so-called canonical model of an arbitrary Shimura variety. Its uniqueness is not too hard to prove ([D1]), but its existence in the most general case has been established only recently (see [Mi1]).

The first of the questions we have to examine is: Under what condition does the canonical model extend to a model of $\overline{M}_{\mathbb{C}}$ or of $M_{\Sigma, \mathbb{C}}$? For $\overline{M}_{\mathbb{C}}$ it always extends uniquely to a model $\overline{M}$, since the Baily-Borel compactification can be defined intrinsically. For the toroidal compactification the answer depends on Σ , and is linked with the other questions. In any case, there is up to isomorphism at most one extension M_{Σ} , so we do not get into trouble if we directly turn to the next question: Which model is induced on the boundary?

In the case of the Baily-Borel compactification, the stratification of the boundary suggests a natural guess: the boundary of the extended canonical model $\overline{M}$ should possess a stratification in terms of canonical models of other Shimura varieties. This indeed turns out to be the case (see our first main theorem 12.3). The complete answer includes an explicit description of the Shimura varieties that occur at the boundary (see 6.3).

For the toroidal compactification one has to work much harder to get the analogous results. The chief reason for this is that, although the boundary strata are fibred over (other) Shimura varieties, they are more general objects. Thus, in order to be able to describe the boundary within the same framework, one has to generalize the concept of Shimura variety.

A close look at the group theoretical data involved in the definition of the toroidal compactification (see [AMRT] ch.III) suggests that the desired generalization involves an extension of a reductive group (associated to Shimura variety) by a unipotent group of a certain type. Such groups arise naturally as normal subgroups of maximal parabolic subgroups of our original group G . Along another line of thought, note that every Shimura variety, even if it does not possess a (known) algebraic modular interpretation, at least parametrizes, as a complex manifold, suitable variations of polarized pure Hodge structures or combinations thereof. If the group is not reductive, these “combinations” can no longer be direct sums, but should be mixed Hodge structures! These considerations naturally lead to a concept of mixed Shimura varieties that parametrize variations of mixed Hodge structures, all of whose pure constituents are polarizable.

In the chapters 1–3 we develop a theory of mixed Shimura varieties. We consider mixed Shimura varieties in their own right, not only insofar as they occur in the boundary of

other Shimura varieties. Nevertheless the desired properties (variation of mixed Hodge structures, i.e. Griffiths' transversality, and polarizability of all pure constituents) imply that they also are torus-torsors over families of abelian varieties over usual (henceforth called pure) Shimura varieties (see 3.12ff). They have a rich structure; in particular the group operations, torus actions, etc. on the fibres can all be described in terms of abstractly defined morphisms of Shimura varieties. Moreover, for pure Shimura varieties that possess an interpretation as moduli schemes of polarized abelian varieties, the universal families (of abelian varieties, and of polarizing line bundles) themselves can be described as mixed Shimura varieties (see 10.7). This phenomenon can be viewed as a modular interpretation for these mixed Shimura varieties. A mixed Shimura variety is pure when the associated algebraic group is reductive. The treatment of chapter 1 follows [D2] § 1.

The next problem we have to deal with is how to make mixed Shimura varieties algebraic. Like for pure Shimura varieties, this problem can be solved by suitable projective compactifications. Such compactifications are also interesting in connection with the modular interpretation: one can describe degenerations of abelian varieties in terms of compactifications of mixed Shimura varieties. As, for instance, [M4] suggests, these compactifications should be toroidal. In the special case of elliptic curves we indeed give such description (see 10.17-22).

All in all, we see that toroidal compactifications should be constructed not only for pure, but for arbitrary mixed Shimura varieties. This is done in the chapter 4 and 6, which rely heavily on [AMRT]. Chapter 5 contains an assortment of results about torus embeddings, which will be needed throughout the remaining chapters. This construction is not more complicated than for pure Shimura varieties, on the contrary the general setting accounts for a certain coherence. The essential new ingredient is an explicit way of associating certain mixed Shimura varieties to a given (possibly pure) one. These are closely related with what in the usual theory is called rational boundary components, which is why we give them the same name. The rational boundary components are in general "more mixed" than the original mixed Shimura variety. The toroidal compactification is then constructed as in [AMRT]. First one considers for every rational boundary component a torus embedding, with respect to a certain cone decomposition and the torus that acts naturally on the unipotent fibre. If the cone decompositions have been chosen compatibly, these partial compactifications can be glued together to form the desired toroidal compactification.

We carry out this construction with an eye toward the question for the model of the boundary. Since we eventually want to describe a stratification of the boundary in terms of other mixed Shimura varieties, in particular in adelic language, it is desirable that the construction already reflects this. Unfortunately this adds (mainly notational) complexity to the constructions of the chapters 6–9 and 12. The benefit is that we directly get a stratification of the toroidal compactification in terms of the mixed Shimura varieties occurring as rational boundary components, and quotients thereof. This stratification is studied in chapter 7.

In order to algebraize mixed Shimura varieties and their compactifications, we have to find compactifications that are projective. In other words we have to construct ample line bundles. While for pure Shimura varieties (in [AMRT] ch.IV §2) such ample line bundles can be described in terms of "piecewise linear strictly convex rational functions" on the cone decompositions, the presence of abelian varieties in the fibres of mixed Shimura varieties

necessitates a more invariant construction (see 8.15, 9.39). As it turns out, one can use the torus-torsor structure on mixed Shimura varieties to turn certain toroidal compactifications of mixed Shimura varieties into line bundles. (In the case of a modular interpretation this has been mentioned above.) In other words, certain morphisms $M'_{\Sigma',\mathbb{C}} \rightarrow M_{\Sigma,\mathbb{C}}$ of toroidal compactifications of mixed Shimura varieties “are” line bundles in a canonical way (see 8.6).

In chapter 8 we derive a criterion for such a line bundle to be ample (along the unipotent fibres). What remains to achieve the algebraization, is the construction of cone decompositions satisfying this criterion. This is done in chapter 9. We prove that for every mixed Shimura variety $M_{\mathbb{C}}$, there exists a mixed Shimura variety $M'_{\mathbb{C}}$ and cone decomposition Σ, Σ' , so that $M_{\Sigma,\mathbb{C}}$ is compact, $M'_{\Sigma',\mathbb{C}} \rightarrow M_{\Sigma,\mathbb{C}}$ is a line bundle, and some combination with the canonical sheaf is ample. Moreover, Σ can be chosen such that $M_{\Sigma,\mathbb{C}}$ is smooth, the complement $M_{\Sigma,\mathbb{C}} \setminus M_{\mathbb{C}}$ is a union of smooth divisors with a most normal crossings, and such Σ can be made arbitrarily fine. In particular, both $M_{\mathbb{C}}$ and such $M_{\Sigma,\mathbb{C}}$ are algebraic varieties. We also prove that “most” other toroidal compactifications $M_{\Sigma,\mathbb{C}}$ are algebraic varieties, and that all of them are algebraic spaces in the sense of [K].

Following the same program as above, we next have to define canonical models for mixed Shimura varieties. This is done in the chapters 10–11. The definition and the proof of uniqueness are exactly as for pure Shimura varieties. The modular interpretation yields a canonical model for the mixed Shimura varieties associated to the moduli problem of polarized abelian varieties with level structure. Here, as in [D1], the main theorem of Shimura and Taniyama about abelian varieties with complex multiplication plays the essential role. Using this, along the lines of [D1], the existence of canonical models in general is reduced to the case of pure Shimura varieties, where it is known by [Mi1].

Having developed the theory of mixed Shimura varieties thus far, we are now in the position to formulate and to prove the main result for the toroidal compactification of a mixed Shimura variety. Recall that chapter 7 gives a stratification in terms of other mixed Shimura varieties. By the results of chapter 9, this stratification is in fact algebraic. Just as for the Baily-Borel compactification, the main theorem (12.4) asserts that (if the cone decompositions satisfies a certain arithmeticity condition, see 6.4) the canonical model extends to the toroidal compactification, and that this stratification descends to the reflex field.

Since the neighborhoods of a stratum in the toroidal compactification are so nice, without much extra effort we get an even better assertion. By 7.17 some neighborhood of a stratum is essentially isomorphic to a neighborhood of a stratum in a simple torus embedding of a rational boundary component. By 9.37, this description can be algebraized, yielding an isomorphism between the respective formal completions (over $\mathbb{C}$). The last part of our main theorem (12.4 (c)) asserts that this isomorphism of formal schemes also descends to the reflex field. These results constitute a complete answer to the questions put forward above.

The main theorem is proved in chapter 12, in the following way. First, using the moduli scheme of generalized elliptic curves (see 10.20), we prove it in the special case of elliptic modular curves (associated to the group $GL_{2,\mathbb{Q}}$). (In particular, as mentioned above, the universal family of generalized elliptic curves “is” the canonical model of a certain toroidal

compactification of a mixed Shimura variety.) We then embed twists of such modular curves into arbitrary mixed Shimura variety (like Hirzebruch Zagier cycles in Hilbert modular surfaces). Finally a density argument, much as in the intrinsic characterization of canonical models, but this time for the formal neighborhood of a boundary component, extends the result to the general case.

The result of these notes have several applications. Quite immediate are the consequences for q -expansions of (certain) automorphic forms. Consider, for a given mixed Shimura variety M , a space of sections of a vector bundle F constructed out of sheaves of differential forms, possibly with logarithmic poles at infinity, or with some other boundary condition. Our description of the formal neighborhood $\mathcal{M}$ of a boundary stratum M_1 implies a similar description for the pullback $\mathcal{F}$ of F . Moreover, since $\mathcal{M}$ has an explicit description in terms of a torus embedding, there is a canonical decomposition of $\Gamma(\mathcal{M}, \mathcal{F})$ as a product of $\Gamma(M_1, F_\nu)$ for certain explicit sheaves F_ν on M_1 . The map

$$\Gamma(M_\Sigma, F) \rightarrow \Gamma(\mathcal{M}, F) \rightarrow \prod_{\nu} \Gamma(M_1, F_\nu)$$

now associates to a section of F its “ q -expansion coefficients.” These “coefficients” are analog of Jacobi modular forms. Our results immediately imply a q -expansion principle for rationality of such automorphic forms over the reflex field: A section over $\mathbb{C}$ descends to the reflex field if and only its q -expansion coefficients do so at a suitable set of boundary components (see 12.18–20). In a similar form, for the Baily-Borel compactification, this result has been obtained by Harris [Ha1], [Ha2] for arbitrary vector-valued automorphic forms.

In another direction these results have consequences for the ℓ -adic cohomology of (mixed) Shimura varieties. The explicit description of formal neighborhoods of the boundary should make it possible to describe the pure constituents of, say $\mathcal{H}_{\acute{e}t}^*(M_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ in terms of some pure cohomology (e.g. intersection cohomology, cuspidal cohomology, or the like) of its boundary components (including itself) with values in some other explicit ℓ -adic sheaves. The author plans to take this up in a sequel to these notes. This would of course be a prerequisite for studying the mixed motives that occur in $\mathcal{H}_{\acute{e}t}^*(M_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$.

Two more technical remarks are in order. In the explanations above we have tacitly used the following slight generalization of the definition of a pure Shimura variety. In the usual definition, $\mathcal{H}$ is a $G(\mathbb{R})$ -conjugacy class of certain homomorphisms $\mathbb{C}^\times \rightarrow G(\mathbb{R})$. The generalization consists in allowing a finite $G(\mathbb{R})$ -equivariant covering $\mathcal{X} \rightarrow \mathcal{H}$, with $G(\mathbb{R})$ acting transitively on $\mathcal{X}$. This generalization makes a difference for the field of definition of a connected component of a Shimura variety, and is necessary in order to stratify a normal compactification in the same framework. From the point of view of global abelian class field theory, it is in fact the more natural choice.

The other remark concerns questions of sign. In order to avoid a source of errors, we have found it useful to work without any globally fixed choice of a square root of -1 . In fact, complex analysis by no means depends on such a choice! We hope to have been successful in this.

As explained above, our results rely on the modular interpretation of certain (mixed) Shimura varieties. This modular approach, of course, achieves much more: it can be carried

out not only over a number field, but over suitable rings of finite type over $\mathbb{Z}$. This has been done in several instances: Deligne and Rapoport [DR] for elliptic modular curves, Rapoport [Rap] for Hilbert modular varieties, and Chai [C] and Faltings [F2] for the Siegel modular variety. A modular interpretation of certain mixed Shimura varieties (in terms of “1-motifs”) has been described in Brylinski’s thesis [Br]. In these cases, a part of the results of this thesis have been known.

The toroidal compactification of pure Shimura varieties has been treated over $\mathbb{C}$ in [AMRT] and [N]. That it descends to $\overline{\mathbb{Q}}$ has been proved in [F1], using a rigidity argument. After the present author had obtained his results, the following two manuscripts came to his attention: (a) a survey article by Milne [Mi2] that independently suggests the concept of mixed Shimura variety along a similar line of ideas; and (b) a manuscript by Harris [Ha3] containing similar results for pure Shimura varieties.

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“Just follow that line forever,” said the Mathemagician, “and when you reach the end, turn left. There you’ll find the land of Infinity, where the tallest, the shortest, the smallest, and the most and the least of everything are kept.” ... Up he went — very quickly at first — then more slowly — then in a little while even more slowly than that — and finally, after many minutes of climbing up the endless stairway, one weary foot was barely able to follow the other. Milo suddenly realized that with all his efforts he was no closer to the top than when he began, and not a great deal further from the bottom.

Norton Juster, *The phantom tollbooth*

L’arithmétique ça fatigue, ça énerve.

Eugène Ionesco, *La leçon*

0.1 Notations and conventions

General: The set of connected components of a topological space X is denoted by $\pi_0(X)$. For any two sets X, Y , the set-theoretic difference is denoted by $X \setminus Y$. This is not to be confused with elementwise difference or with left quotient: If $(A, +)$ is an abelian group, $a \in A$ and $X, Y \subset A$, then the elementwise sum or difference is denoted by $X + Y, X - Y, a + X, a - X$, etc. Likewise for an arbitrary group $(G, \cdot)$.

Fields and schemes: If $T \rightarrow S$ is a morphism of schemes, and X a scheme over S , then we write $X_T := X \times_S T$. If $S = \text{Spec}(K)$ and $T = \text{Spec}(L)$ for a field extension L/K , then we write this also in the form $X_L = X \times_K L$. If Y_L is a subscheme of X_L which is derived from a subscheme of X by base extension, then by abuse of terminology we say that it is defined over K . An “L-subobject” (e.g. “L-subgroup”, “L-parabolic subgroup”, “L-simple subgroup”) of an object X is a subscheme of X_L that possesses the indicated property.

Let K/L be a field extension and X a scheme over L . By a model of X over L we mean a scheme X_0 over K together with an isomorphism $X_{0,L} \cong X$ over L . The Weil restriction of X with respect to L/K is denoted by $\mathcal{R}_{L/K}X$. If X is a group scheme (general, reductive, semisimple, or a torus), then so is $\mathcal{R}_{L/K}X$.

$\mathbb{R}$ is the field of real numbers, $\mathbb{C}$ a fixed algebraic closure of $\mathbb{R}$. The symbol $\sqrt{-1}$ denotes an arbitrary solution in $\mathbb{C}$ of the equation $z^2 + 1 = 0$. We do not fix a particular choice of $\sqrt{-1}$. Wherever this symbol occurs, it will be clear that the whole assertion is independent of such a choice. In the expression $2\pi\sqrt{-1}$, the symbol π denotes the unique positive real number for which $2\pi\sqrt{-1}$ is a generator of kernel of the exponential function $\exp: \mathbb{C} \rightarrow \mathbb{C}^\times$ (Clearly this does not depend on $\sqrt{-1}$).

If X is a variety over a field K that is contained in $\mathbb{R}$, then we call X of compact type if and only if $X(\mathbb{R})$ is compact.

All number fields K are considered as subfields of $\mathbb{C}$, in other words they are given together with a distinguished embedding into $\mathbb{C}$. We denote by $\overline{K}$ the algebraic closure of K in $\mathbb{C}$, so that the absolute Galois group $\text{Gal}(\overline{K}/K)$ is a quotient of $\text{Aut}(\mathbb{C})$. The ring of adeles of K is denoted by $\mathbb{A}_K$, the subring of finite adeles by $\mathbb{A}_{K,f}$, and we abbreviate $\mathbb{A} := \mathbb{A}_{\mathbb{Q}}$ and $\mathbb{A}_f := \mathbb{A}_{\mathbb{Q},f}$.

Groups: Let G be a group and $g, g' \in G$. The left action of G by conjugation on itself is denoted by $\text{int}_G(g): g' \mapsto g \cdot g' \cdot g^{-1}$. The commutator is $(g, g') := g \cdot g' \cdot g^{-1} \cdot g'^{-1}$. An almost direct (almost semidirect) product in a category of groups in any finite quotient of the usual direct (semidirect) product.

If A is a topological or algebraic group, then A^0 is the connected component of the identity of A . The center of A is denoted by $Z(A)$, this is a closed, but not necessarily connected subgroup. If A is a topological group, then $\pi_0(A)$ is isomorphic to A/A^0 .

Let G be a linear algebraic group over a field K . By a representation of G we always mean a finite dimensional representation over K . If $L \subset \text{Lie } G$ is a K -sub Lie algebra, then $\exp L$ denotes the unique connected K -subgroup of G with Lie algebra L .

By a reductive group over a field K we always mean a connected reductive linear algebraic group over K . In particular, every semisimple group and every torus will be connected. If G is a reductive group, then $G^{der} = [G, G]$ and $G^{ad} = G/Z(G)$ denote its derived, resp. adjoint group. The adjoint representation of G on $\text{Lie } G$ will be denoted by Ad_G , its derivative (the induced left action of $\text{Lie } G$ on itself) by ad_G .

Let P be a connected linear algebraic group over K , and U its unipotent radical. A Levi-decomposition is a decomposition of P into a semidirect product $P = U \rtimes G$. Any two Levi-decompositions are conjugate under $U(K)$. Also, let $\mathcal{H}$ be a reductive group over K , and $\varphi_1, \varphi_2 : \mathcal{H} \rightarrow P$ two homomorphisms so that the composites $\mathcal{H} \rightarrow P \rightrightarrows P/U$ are equal. Then φ_1 and φ_2 are conjugate under $U(K)$.

Group actions: If a group G acts from the left hand side on a set X , we denote by $G \backslash X$ the set of all G -orbits in X . Likewise for $X/\mathcal{H}$ if a group $\mathcal{H}$ acts from the right hand side, and consequently $G \backslash X/\mathcal{H}$ if two such actions are given. Fortunately we shall not need actions from above or below. A left homogeneous space under G is a set X with a transitive left action of G . If G is a real Lie-group, then such X is a C^∞ -manifold in a canonical way.

Let Γ be a discrete group acting on a locally compact (hence Hausdorff) topological space X . By definition Γ acts properly discontinuously on X if and only if for any two compact subsets K and K' of X the set $\{\gamma \in \Gamma \mid \gamma K \cap K' \neq \emptyset\}$ is finite. An equivalent condition is that any two points in X possess neighborhoods U and V such that the set $\{\gamma \in \Gamma \mid \gamma U \cap V \neq \emptyset\}$ is finite. If this condition holds, then $\Gamma \backslash X$, endowed with the quotient topology, is again Hausdorff (even locally compact) and locally isomorphic to a quotient of X by a finite group. In particular the stabilizer of any point is finite.

Contrary to the usual terminology we also say that the group Γ acts properly discontinuously if the action factors through a quotient $\bar{\Gamma}$ which acts properly discontinuously in the usual sense. The point is that the kernel of the action is allowed to be infinite.

Arithmetic subgroups: Let G be a linear algebraic group over $\mathbb{Q}$. A subgroup of $G(\mathbb{Q})$ is called a congruence subgroup if and only if it is of the form $G(\mathbb{Q}) \cap K_f$ for some open compact subgroup $K_f \subset G(\mathbb{A}_f)$. A subgroup of $G(\mathbb{Q})$ is called an arithmetic subgroup if and only if it is commensurable with some (hence with every) congruence subgroup.

According to [B] 17.1 an element $g \in GL_n(\mathbb{Q})$ is called neat, if the subgroup of $\overline{\mathbb{Q}}^\times$ that is generated by the eigenvalues of g is torsion free. If G is a linear algebraic group, then an element $g \in G(\mathbb{Q})$ is called neat if its image in some faithful representation of G is neat. It is easy to check that this condition then holds for every, not necessarily faithful representation of G (see [B] 17.3). A subgroup of $G(\mathbb{Q})$ is called neat if all its elements are neat. In particular such a group is torsion free. Every subgroup of a neat subgroup is neat, and the image of a neat subgroup under a homomorphism $G \rightarrow \mathcal{H}$ is again neat. Finally every sufficiently small congruence subgroup of $G(\mathbb{Q})$ is neat (by [B] 17.4, or 0.6 below).

Neatness and adelic groups: We want to extend the notion of neatness to subgroups $K_f \subset G(\mathbb{A}_f)$, with the same functorial properties, and so that $G(\mathbb{Q}) \cap K_f$ is neat whenever K_f is neat. First consider an element $g_f = (g_p)_p \in GL_n(\mathbb{A}_f)$. For every p let Γ_p be the subgroup of $\overline{\mathbb{Q}}_p^\times$ generated by all eigenvalues of g_p . Let $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ be some embedding,

and consider the torsion part $(\overline{\mathbb{Q}}^\times \cap \Gamma_p)_{tors}$. Since every subgroup of $\overline{\mathbb{Q}}^\times$ consisting of roots of unity is normalized by $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, this group does not depend on the choice of the embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. We now define g_f to be neat if and only if

$$\bigcap_p (\overline{\mathbb{Q}}^\times \cap \Gamma_p)_{tors} = \{1\}.$$

As before, if G is a linear algebraic group, we call an element $g_f \in G(\mathbb{A}_f)$ neat if its image in some faithful representation of G is neat. If ρ is a faithful representation of G , and ρ' another representation of G , then the group Γ_p above for ρ is the same as that for $\rho \oplus \rho'$, and contains that for ρ' . Thus if g_f is neat, then its image in any representation of G is neat. A subgroup of $G(\mathbb{A}_f)$ is called neat if all its elements are neat. Clearly a subgroup of a neat subgroup is again neat. If $\mathcal{H}$ is a linear algebraic subgroup of G , and $K_f \subset G(\mathbb{A}_f)$ is neat, then $\mathcal{H}(\mathbb{A}_f) \cap K_f$ is neat. If $\varphi : G \rightarrow \mathcal{H}$ is a homomorphism and $K_f \subset G(\mathbb{A}_f)$ is neat, then $\varphi(K_f)$ is neat, since we can consider any faithful representation of $\mathcal{H}$ as a representation of G . The neatness property is invariant under all automorphisms of G , and all inner automorphisms of $G(\mathbb{A}_f)$. From the definition it is clear that if $K_f \subset G(\mathbb{A}_f)$ is neat, then $G(\mathbb{Q}) \cap K_f$ is neat in the usual sense.

Finally fix an integer $d \geq 3$, and let $K_f(d) \subset GL_n(\hat{\mathbb{Z}})$ be the subgroup of all elements that act trivially on $\hat{\mathbb{Z}}^n/d \cdot \hat{\mathbb{Z}}^n$. We shall show that $K_f(d)$ is neat. It follows that every sufficiently small open compact subgroup of a linear algebraic group is neat. The proof of the neatness of $K_f(d)$ is standard. Note that d is divisible by some prime power $p^i \geq 3$. Consider an element $g_p \in GL_n(\mathbb{Z}_p)$ that acts trivially on $\mathbb{Z}_p^n/p^i \cdot \mathbb{Z}_p^n$, then it suffices to show that the group Γ_p above does not contain a nontrivial root of unity. By assumption every eigenvalue of g_p in $\overline{\mathbb{Z}}_p$ (the algebraic closure) is congruent to 1 modulo $p^i \cdot \overline{\mathbb{Z}}_p$. Thus the same follows for every element of Γ_p , and we have to show that every root of unity $\zeta \in \overline{\mathbb{Z}}_p$, that is congruent to 1 modulo $p^i \cdot \overline{\mathbb{Z}}_p$, is equal to 1. Since the residue class of ζ in $\overline{\mathbb{F}}_p^\times$ is 1, the order of ζ must be a power of p . Suppose this order is p^j with $j \geq 1$, then the following inequalities for the p-adic valuation

$$v_p(1 - \zeta) \geq v_p(p^i) = i \cdot v_p(p) = i \cdot p^{j-1} \cdot (p-1) \cdot v_p(1 - \zeta) > 0$$

imply $i \cdot (p-1) \leq i \cdot p^{j-1} \cdot (p-1) \leq 1$, which contradicts the assumption that $p^i \geq 3$. Thus $\zeta = 1$, as desired.

List of frequently used symbols

$\mathbb{A}, \mathbb{A}_f, \mathbb{A}_K, \mathbb{A}_{K,f}$	adele ring	0.2
Ad_G, ad_G	adjoint operation	0.3
$A[d]$	group of d -division points	10.1
$B(\mathcal{X}^0), B(\mathcal{X}^0, P_1)$	set of commutators	8.11
$\beta(P_1, \mathcal{X}_1, P'_1, \mathcal{X}'_1, p_f), \bar{\beta}(P_1, \mathcal{X}_1, P'_1, \mathcal{X}'_1, p_f)$	certain maps in $\mathcal{U}, \bar{\mathcal{U}}$	6.11, 6.15
$\mathbb{C}$	set of complex numbers	0.2
$\mathcal{C}(P, \mathcal{X})$	conical complex	4.24
$C(\mathcal{X}^0, P_1)$	open cone associated to boundary component	4.15
$C^*(\mathcal{X}^0, P_1)$	union of all boundary components of $C(\mathcal{X}^0, P_1)$	4.22
$(CS_{P_{2g}}, \mathcal{H}_{2g})$	Shimura data for the symplectic group	2.7
Δ_1	normalizer of a boundary stratum in the Baily-Borel compactification	6.3, 6.18, 7.3
e_σ	splitting associated to a cone σ	5.11, 8.5
$\exp L$	subgroup associated to sub-Lie algebra	0.3
$E(P, \mathcal{X})$	reflex field	11.1
$e(X)$	canonical pairing $H(X) \times H(X) \rightarrow \mathbb{G}_m$	10.2
$\mathcal{E}(\mathcal{X}^0, P'_1)$	set of splittings	9.2
$F^P M_{\mathbb{C}}$	Hodge filtration	1.1
G	semisimple part P/W	2.1
G^{ad}, G^{der}	adjoint, derived group	0.3
$(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0), (\mathbb{G}_{m, \mathbb{Q}}, h(\mathcal{H}_0))$	standard Shimura data	2.8
h	homomorphism $\mathbb{S}_{\mathbb{C}} \rightarrow P_{\mathbb{C}}$	1.4
h	equivariant map $\mathcal{X} \rightarrow \text{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$	2.1
H_0, h_0, h_∞	reference group and homomorphisms	4.3
$H(X)$	group of translations normalizing X	10.1
$\mathcal{H}_W$	conjugacy class of $h : \mathbb{S}_{\mathbb{C}} \rightarrow P_{\mathbb{C}}$	1.6
im	imaginary part	4.14
int_G	interior automorphism	0.3
$\text{int}(p)$	interior automorphism of $(P, \mathcal{X})$	3.5
$K_f(d), K_f^P(d), K_f^U(d), K_f^W(d)$	special open compact subgroups	10.7, 10.15
$\mathcal{L}_P$	canonical invertible sheaf on $M^{K_f}(P, \mathcal{X})^*(\mathbb{C})$	8.1
$\mathcal{L}_{\mathcal{X}}$	invertible sheaf occuring in q -expansion	12.18
λ	isomorphism $\mathbb{Z} \rightarrow \mathbb{Z}(1)$	2.8, 3.16, 8.5
$M^{p,q}$	Hodge decomposition	1.1
$M(n)$	Tate twist	1.11
$M^{K_f}(P, \mathcal{X})(\mathbb{C}), M_{\mathbb{C}}^{K_f}(P, \mathcal{X}), M^{K_f}(P, \mathcal{X})$	(mixed) Shimura variety	3.1, 9.25, 11.5
$M^{K_f}(P, \mathcal{X})^*(\mathbb{C}), M_{\mathbb{C}}^{K_f}(P, \mathcal{X})^*, M^{K_f}(P, \mathcal{X})^*$	Baily-Borel compactification	6.2, 9.25, 12.3
$M^{K_f}(P, \mathcal{X})(\mathbb{C}), M_{\mathbb{C}}^{K_f}(P, \mathcal{X}), M^{K_f}(P, \mathcal{X})$		

	toroidal compactification	6.24, 9.25, 9.34, 12.4–5
$M^{K_f}(P, \mathcal{X})^+(\mathbb{C})$	union of $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ with all boundary strata of codimension 1	8.2
$M(d), M_V(d), M_W(d), M^0(d), M_U^0(d)$	special mixed Shimura varieties	10.7, 10.15
$\mathcal{M}_d$	moduli scheme of abelian varieties	10.6, 11.16
$\underline{M}_d^0, \underline{X}_d^0$	“moduli scheme” of roots of unity	10.15
μ	canonical cocharacter of $\mathbb{S}$	1.3
ord_T	homomorphism $T(\mathbb{C}) \rightarrow Y_*(T)_{\mathbb{R}}$	5.8
$\omega_{X/S}^T$	sheaf of invariant differentials	5.26
$\omega[\text{dlog}], \omega_{X_{\Sigma}}[\text{dlog}], \omega_{X_{\Sigma}}/S[\text{dlog}]$	sheaf of differentials with at most logarithmic poles along the boundary	5.26, 8.1
$(P, \mathcal{X}), (P, \mathcal{X}, h)$	(mixed) Shimura data	2.1
$(P, \mathcal{X})/P_0$	quotient mixed Shimura data	2.9
$(P_1, \mathcal{X}_1) \times_{(P, \mathcal{X})} (P_2, \mathcal{X}_2)$	fibre product	2.20
$(P_0, \mathcal{X}_0)$	unipotent extension of $(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$	2.24
$(P_{2g}, \mathcal{X}_{2g})$	unipotent extension of $(CS_{P_{2g}}, \mathcal{H}_{2g})$	2.25
$(P_{[\sigma]}, \mathcal{X}_{[\sigma]})$	mixed Shimura data associated to $[\sigma]$	7.1
π	the positive real number	0.2
π, π'	projections $P \rightarrow G = P/W, P \rightarrow P/U$	2.1
$\pi_0(X)$	set of connected components	0.1
π_{σ}	canonical projection $T_{\sigma} \rightarrow \tilde{T}_{\sigma}$	5.2
$\pi_{[\sigma]}$	projection $(P, \mathcal{X}) \rightarrow (P_{[\sigma]}, \mathcal{X}_{[\sigma]})$	7.1
$[\pi]^*$	projection to Baily-Borel compactification	6.24, 9.25, 12.4
Q	admissible $\mathbb{Q}$ -parabolic subgroup	4.5
$\mathfrak{R}_{L/K} X$	Weil restriction	0.2
$\mathbb{R}$	set of real numbers	0.2
$\mathbb{S}, \mathbb{S}^1$	Deligne torus	1.3
$\Sigma, \Sigma(\mathcal{X}^0, P_1, p_f)$	admissible (partial) cone decomposition	6.4
Σ^0	subset of all cones along the unipotent fibre	6.5
$\Sigma_{ (P_1, \mathcal{X}_1)}$	restriction to a boundary component	6.5
$\Sigma_{[\sigma]}$	induced cone decomposition	7.7
$\Sigma(f)$	cone decomposition associated to a piecewise linear convex rational function	5.20
$\Sigma_{\Sigma}^+, \Sigma'_{\Sigma}, \Sigma_{\Sigma}$	cone decompositions associated to Σ	9.12
$ \Sigma $	support of a cone decomposition	5.1
σ^0	interior of a cone	5.1
σ_0	standard cone	5.10, 8.5, 9.1
$\check{\sigma}$	dual cone	5.1
$[\sigma]$	double coset of cones	7.1
Σ	locally polyhedral subset	9.8
T_{Σ}	torus embedding	5.3
T_{σ}	affine torus embedding	5.2

$\tilde{T}_\sigma$	orbit in a torus embedding	5.2
U	weight -2 subgroup of P	2.1
$\mathcal{U}, \mathcal{U}(P_1, \mathcal{X}_1, p_f)$	covering of $M^{K_f}(P, \mathcal{X})(\mathbb{C})$	6.10
$\bar{\mathcal{U}}, \bar{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$	covering of $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$	6.13
V	weight -1 subquotient of P	2.1
$V[d], U[d]$	standard $\mathbb{Z}/d\mathbb{Z}$ -modules	10.3, 10.7
w	“weight” of $\mathbb{S}$	1.3
W	unipotent radical of P	2.1
$W_n M$	weight filtration	1.1
$X_d \rightarrow A_d \rightarrow \underline{\mathcal{M}}_d$	universal family of abelian varieties	10.10, 11.16
X_Σ	relative torus embedding	5.5
$X^*(T)$	character group of a torus T	5.2
$\mathcal{X}^+$	open subset of $\mathcal{X}$ that maps to $\mathcal{X}_1$	4.11
$\mathcal{X}^*$	union of hermitain symmetric domain with all its rational boundary components	6.2
$Y_*(T)$	cocharacter group of torus T	5.2
$\mathbb{Z}(n)$	Tate Hodge structure	1.11
$\sim$	equivalence relation on $\mathcal{U}$ or $\bar{\mathcal{U}}$	6.10, 6.16
$\sqrt{-1}$	the complex number	0.2
$[,]$	commutator	0.3
$[\cdot p_f], [\varphi]$	morphisms of mixed Shimura varieties	3.4, 6.2, 6.25, 9.25, 12.3, 12.4
$[\cdot p_f]^* \Sigma, \varphi^* \Sigma$	pullback of admissible cone decomposition	6.5

Chapter 1

Equivariant families of mixed Hodge structures

Usual Shimura varieties can be viewed analytically as moduli spaces for variations of certain polarized pure Hodge structures, or combinations thereof. Thus, in analogy to [D2] §1.1, a natural starting point for us is to study how certain variations of mixed Hodge structures can be expressed in group theoretic terms. For most of the chapter we consider the case of rational weights, that is where the weight filtration is defined over $\mathbb{Q}$. In 1.3–1.8 we describe equivariant families of rational mixed Hodge structures: these form naturally a complex manifold (1.7). Then we analyze which of these satisfy the two requirements: (a) Griffiths’ transversality (1.9–10), and (b) polarizability of all pure constituents (1.11–12). To avoid some pathologies we replace the group in question by a smaller one (1.13–17). Thus we are lead naturally to the group theoretic data described in 1.18. Since the rationality of the weight is a true restriction for usual Shimura varieties, it is desirable to drop this requirement. It turns out that to define mixed Shimura varieties it is enough to assume that the weight acts rationally on the “mixed” part of the data (see 3.3 (a)). The modification 1.19 of the data of 1.18 achieves this.

1.1. Hodge structures: (compare [D3] §§2.1 and 2.3) Let M be a finite dimensional $\mathbb{Q}$ -vector space. A pure Hodge structure of weight $n \in \mathbb{Z}$ on M is a decomposition $M_{\mathbb{C}} = \bigoplus_{p+q=n} M^{p,q}$ into $\mathbb{C}$ -vector spaces, such that for all $p, q \in \mathbb{Z}$ with $p + q = n$ one has $\overline{(M^{p,q})} = M^{q,p}$. The associated (descending) Hodge filtration on $M_{\mathbb{C}}$ is defined by $F^p M_{\mathbb{C}} := \bigoplus_{p' \geq p} M^{p',q}$. It determines the Hodge structure uniquely, because $M^{p,q} = F^p M_{\mathbb{C}} \cap \overline{(F^q M_{\mathbb{C}})}$.

A rational mixed Hodge structure on M consists of an ascending, exhausting, separated filtration $\{W_n M\}_{n \in \mathbb{Z}}$ of M by $\mathbb{Q}$ -vector spaces, called weight filtration, together with a descending, exhausting, separated filtration $\{F^p M_{\mathbb{C}}\}_{p \in \mathbb{Z}}$ of $M_{\mathbb{C}}$, called Hodge filtration, such that for all $n \in \mathbb{Z}$ the Hodge filtration induces a pure Hodge structure of weight n on $Gr_n^W M := W_n M / W_{n-1} M$. A pure Hodge structure of weight n is considered a special case of a mixed Hodge structure by defining the weight filtration as $W_{n'} M = M$ for $n' \geq n$, and $W_{n'} M = 0$ for $n' < n$.

The Hodge numbers are defined as $h^{p,q} := \dim_{\mathbb{C}}(Gr_{p+q}^W M)^{p,q}$. They satisfy $h^{p,q} = h^{q,p}$, almost all $h^{p,q}$ are zero, and their sum is equal to the dimension of M . If $A \subset \mathbb{Z} \oplus \mathbb{Z}$ is an

arbitrary subset, then we call a Hodge structure of type A, if $h^{p,q} = 0$ for all $p, q \notin A$. The weights that occur in a Hodge structure are the numbers $p + q$ for all pairs (p, q) , for which $h^{p,q} \neq 0$. The notions “of weight $\leq n$ ” and “of weight $\geq n$ ” are defined in the obvious way.

A morphism of rational mixed Hodge structures is a homomorphism $f : M \rightarrow M'$, such that $f(W_n M) \subset W_n M'$ and $f(F^p M_{\mathbb{C}}) \subset F^p M'_{\mathbb{C}}$ for all $n, p \in \mathbb{Z}$. By [D3] 2.3.5 (i) the rational mixed Hodge structures form an abelian category with these morphisms. Given rational mixed Hodge structures on M_1 and M_2 , there are canonical rational mixed Hodge structures on $M_1 \oplus M_2$, the dual $(M_1)^{\vee}$, on $\text{Hom}(M_1, M_2)$, and on $M_1 \otimes M_2$ (see [D3] 1.1.6, 1.1.12 and 2.1.11).

A mixed Hodge structure on M splits over $\mathbb{R}$, if there exists a decomposition $M_{\mathbb{C}} = \bigoplus_{p,q} M^{p,q}$, such that $W_n M_{\mathbb{C}} = \bigoplus_{p+q \leq n} M^{p,q}$, $F^p M_{\mathbb{C}} = \bigoplus_{p' \geq p} M^{p',q}$, and $\overline{(M^{p,q})} = M^{p,q}$. This decomposition is then uniquely determined by these properties. Every pure Hodge structure splits over $\mathbb{R}$, but not every mixed Hodge structure does. If one weakens the requirements, however, one can still associate to every mixed Hodge structure a canonical decomposition $M_{\mathbb{C}} = \bigoplus_{p,q} M^{p,q}$, as in the following proposition.

1.2. Proposition: Fix a rational mixed Hodge structure on M .

- (a) There exists a decomposition $M_{\mathbb{C}} = \bigoplus_{p,q} M^{p,q}$, such that $W_n M_{\mathbb{C}} = \bigoplus_{p+q \leq n} M^{p,q}$ and $F^p M_{\mathbb{C}} = \bigoplus_{p' \geq p} M^{p',q}$.
- (b) The Hodge structure is uniquely determined by any such decomposition.
- (c) There exists a unique decomposition as in (a), which also satisfies

$$\overline{(M^{p,q})} \equiv M^{p,q} \bmod \bigoplus_{\substack{p' < p, \\ q' < q}} M^{p',q'}.$$

Proof. (a) follows from (c), and (b) is obvious. For the existence in (c) see [D3] 1.2.11, for the uniqueness [CK] 2.2. **q.e.d.**

Remark. In general there exist different decompositions satisfying (a).

1.3. The Deligne-torus: (compare [D2] 1.1.1. Attention: Our conversation is opposite to that in [D1] 1.3 and [D3] 2.1.5.1.) Consider the torus $\mathbb{S} := \mathcal{R}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$. Over $\mathbb{C}$ it is canonically isomorphic to $\mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$, but the action of complex conjugation is twisted by the automorphism c that interchanges the two factors. In particular $\mathbb{S}(\mathbb{R}) = \mathbb{C}^{\times}$ corresponds to the points of the form $(z, \bar{z})$ with $z \in \mathbb{C}^{\times}$. While the character group of $\mathbb{G}_{m,\mathbb{C}}$ in $\mathbb{Z}$ in the standard way, we identify the character group of $\mathbb{S}$ with $\mathbb{Z} \oplus \mathbb{Z}$ such that the character (p, q) maps $z \in \mathbb{S}(\mathbb{R}) = \mathbb{C}^{\times}$ to $z^{-p} \cdot (\bar{z})^{-q} \in \mathbb{C}^{\times}$. Under this identification the complex conjugation still operates on $\mathbb{Z} \oplus \mathbb{Z}$ by interchanging the two factors. Consider the homomorphisms $w : \mathbb{G}_{m,\mathbb{R}} \hookrightarrow \mathbb{S}$, $\mathbb{R}^{\times} \ni t \mapsto t \in \mathbb{C}^{\times}$; $\mu : \mathbb{G}_{m,\mathbb{C}} \rightarrow \mathbb{S}_{\mathbb{C}}$, $\mathbb{C}^{\times} \ni z \mapsto (z, 1) \in \mathbb{C}^{\times} \times \mathbb{C}^{\times} = \mathbb{S}(\mathbb{C})$; and the norm $N : \mathbb{S} \rightarrow \mathbb{G}_{m,\mathbb{R}}$, $\mathbb{S}(\mathbb{R}) = \mathbb{C}^{\times} \ni z \mapsto z \cdot \bar{z} \in \mathbb{R}^{\times}$. The kernel $\mathbb{S}^1$ of N is anisotropic over $\mathbb{R}$, and we have a short exact sequence $1 \rightarrow \mathbb{S}^1 \rightarrow \mathbb{S} \rightarrow \mathbb{G}_{m,\mathbb{R}} \rightarrow 1$.

Let M be a finite dimensional $\mathbb{Q}$ -vector space. The choice of a representation $k : \mathbb{S}_{\mathbb{C}} \rightarrow GL(M_{\mathbb{C}})$ is equivalent to the choice of a decomposition $M_{\mathbb{C}} = \bigoplus_{p,q} M^{p,q}$, where $M^{p,q}$ is the eigenspace in $M_{\mathbb{C}}$ to the character (p, q) . Like in 1.1 we call $W_n M_{\mathbb{C}} = \bigoplus_{p+q \leq n} M^{p,q}$

and $F^p M_{\mathbb{C}} = \bigoplus_{p' \geq p} M^{p',q}$ the associated weight filtration, respectively Hodge filtration, and define the notions “of type A”, pure, etc. in the same way. These notions coincide with those of 1.1, if the filtrations are those of a rational mixed Hodge structure on M . We next study the question, under which condition on k this is the case.

1.4. Proposition: Let P be a connected linear algebraic group over $\mathbb{Q}$. Let W be the unipotent radical of P , let $G := P/W$ and $\pi : P \rightarrow G$ the canonical projection. Let $h : \mathbb{S}_{\mathbb{C}} \rightarrow P_{\mathbb{C}}$ be a homomorphism, such that the following conditions are satisfied:

- (i) $\pi \circ h : \mathbb{S}_{\mathbb{C}} \rightarrow G_{\mathbb{C}}$ is already defined over $\mathbb{R}$.
- (ii) $\pi \circ h \circ w : \mathbb{G}_{m,\mathbb{R}} \rightarrow G_{\mathbb{R}}$ is a cocharacter of the center of P/W , that is already defined over $\mathbb{Q}$.
- (iii) Under the weight filtration on $\text{Lie } P$ defined by $\text{Ad } p \circ h$ we have $W_{-1}(\text{Lie } P) = \text{Lie } W$.

Then:

- (a) For every representation $\rho : P \rightarrow GL(M)$, defined over $\mathbb{Q}$ the homomorphism $\rho \circ h : \mathbb{S}_{\mathbb{C}} \rightarrow GL(M_{\mathbb{C}})$ induces a rational mixed Hodge structure on M .
- (b) The weight filtration on M is stable under P .
- (c) For any $p \in P(\mathbb{R}) \cdot W(\mathbb{C})$ the assertions (a) and (b) also hold for $\text{int}(p) \circ h$ in place of h . The weight filtration and the Hodge numbers in any representation are the same for $\text{int}(p) \circ h$ and for h .

Proof. Assertion (c) follows directly from (a) and (b), because the conditions (i) to (iii) are invariant under conjugation by an element of $P(\mathbb{R}) \cdot W(\mathbb{C})$. For (a) and (b) we first use induction to prove the statement: The weight filtration is defined over $\mathbb{Q}$ and stable under P .

In the case $\dim M = 0$ there is nothing to prove. Otherwise there exists a short exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ of representations of P , with M'' nontrivial and irreducible. Let M_W be the space of W -coinvariants of M , then M'' is a quotient of M_W . By definition M_W is completely reducible under P , so there exists a P -invariant subspace $N \subset M$ with $N + M'' = M$ and $M'' \cong \text{im}(N) \subset M_W$. If N is properly contained in M , then the assertion follows from the formula $W_n M_{\mathbb{C}} = W_n M'_{\mathbb{C}} + W_n N_{\mathbb{C}}$ for all $n \in \mathbb{Z}$ and the inductive assumption. Otherwise we have $M'' = M_W$. The differential of ρ induces a P -equivariant linear map $(\text{Lie } W) \otimes M \rightarrow m$, whose cokernel is precisely the space of $(\text{Lie } W)$ -coinvariants, so again equal to M'' . Thus we obtain a surjective P -equivariant map $(\text{Lie } W) \otimes M \rightarrow M'$. Let n be the highest weight that occurs in M . Since by assumption $\text{Lie } W$ is of weight ≤ -1 , M' must be of weight $\leq n - 1$. Thus the weight n does not occur in M' , hence it must occur in M'' . By condition (ii) every irreducible representation is pure of some weight, so M'' must be pure of weight n . For the weight filtration we thus get: $W_m M_{\mathbb{C}} = W_m M'_{\mathbb{C}}$ for $m < n$, and $W_m M_{\mathbb{C}} = M_{\mathbb{C}}$ for $m \geq n$. Therefore the assertion follows from the inductive assumption for M' .

With this (b) is proved. For (a) it remains to show that for every $n \in \mathbb{Z}$ the Hodge filtration induces a pure Hodge structure of weight n on $Gr_n^W M := W_n M / W_{n-1} M$. Since $Gr_n^W M$, as we have just proved, is again a representation of P over $\mathbb{Q}$, it suffices to show: If M is already pure of weight n , then the decomposition $M_{\mathbb{C}} = \bigoplus_{p+q=n} M^{p,q}$ is a pure Hodge structure of weight n . Let us again consider the map $(\text{Lie } W) \otimes M \rightarrow M$ of above.

The left hand side is mixed of weight $\leq n - 1$, the right hand side pure of weight n . Thus this map must be zero, which means that W operates trivially on M . By (i) the operation of $\mathbb{S}_{\mathbb{C}}$ on $M_{\mathbb{C}}$ is therefore defined over $\mathbb{R}$, so $\overline{(M^{p,q})} = M^{p,q}$ for all p, q with $p + q = n$. **q.e.d.**

1.5. Propositionen: Let M be a finite dimensional $\mathbb{Q}$ -vector space. A representation $k : \mathbb{S}_{\mathbb{C}} \rightarrow GL(M_{\mathbb{C}})$ defines a rational mixed Hodge structure on M , if and only if there exists a connected linear algebraic group P over $\mathbb{Q}$, a representation $\rho : P \rightarrow GL(M)$ defined over $\mathbb{Q}$, and a homomorphism $h : \mathbb{S}_{\mathbb{C}} \rightarrow P_{\mathbb{C}}$, such that $k = \rho \circ h$ and the conditions in 1.4 are satisfied.

Proof. One implication is already contained in 1.4. So let us assume that k induces a rational mixed Hodge structure on M . Let $P \subset GL(M)$ be the stabilizer of the weight filtration, this is a $\mathbb{Q}$ -parabolic, hence connected subgroup of $GL(M)$. Let $\rho : P \hookrightarrow GL(M)$ be the inclusion, then k factors in exactly one way through a homomorphism $h : \mathbb{S}_{\mathbb{C}} \rightarrow P_{\mathbb{C}}$. By assumption $Gr_n^W M$ is a pure Hodge structure of weight n for every $n \in \mathbb{Z}$. Thus for all $p, q \in \mathbb{Z}$ with $p + q = n$ the space $\overline{(Gr_n^W M)^{p,q}} = (Gr_n^W)^{p,q}$ is the eigenspace in $Gr_n^W M$ associated to the character $c((p, q)) = (p, q)$, so the operation of $\mathbb{S}$ on $Gr_n^W M_{\mathbb{C}}$ is already defined over $\mathbb{R}$. Since furthermore $\mathbb{G}_{m, \mathbb{R}} \subset \mathbb{S}$ acts by scalars on $Gr_n^W M_{\mathbb{R}}$, its operation is given by cocharacter $\mathbb{G}_{m, \mathbb{Q}} \rightarrow Z(GL(Gr_n^W M))$ that is defined over $\mathbb{Q}$. The properties 1.4 (i) and (ii) now follow from the isomorphism $P/W \cong \prod_N GL(Gr_N^W M)$.

Finally $\text{Lie } P$ is the successive extension of $\text{Hom}(Gr_n^W M, Gr_m^W M)$ for all $n \geq m$ and $\text{Lie } W$ the successive extension of $\text{Hom}(Gr_n^W, Gr_m^W M)$ for all $n > m$. In $\text{Hom}(Gr_n^W M, Gr_m^W M)_{\mathbb{C}}$ as representation of $\mathbb{S}_{\mathbb{C}}$ only characters (p, q) occur with $p + q = m - n$, so $\text{Lie } W$ contains precisely all eigenspaces in $\text{Lie } P$ associated to the characters (p, q) with $p + q \leq -1$. This implies 1.4 (iii). **q.e.d.**

1.6. Equivariant families of Hodge structures: Generalizing [D2] 1.1.11 and 1.1.12 we want to parametrize Hodge structures by a homogeneous space under a real Lie group. Furthermore we want this group to come from a linear algebraic group over $\mathbb{Q}$. Finally the homogeneous space should be described in terms of homomorphisms $\mathbb{S}_{\mathbb{C}} \rightarrow P_{\mathbb{C}}$ and should carry a canonical complex structure.

More concretely let P be a connected linear algebraic group over $\mathbb{Q}$. In a crucial case (see 4.7) P will be strongly related to a parabolic subgroup of another group, whence the unusual notation. Let W be the unipotent radical of P , and H_W a $P(\mathbb{R}) \cdot W(\mathbb{C})$ -conjugacy class in $\text{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$. We assume that for one, hence by 1.4 (c) for all $h \in H_W$ the conditions in 1.4 hold. We call a representation M of P pure of weight n , mixed of weight $\leq n$, of type $A \subset \mathbb{Z} \oplus \mathbb{Z}$, etc. if and only if for some ($\Leftrightarrow$ by 1.4 (c) for all) $h \in H_W$ the induced Hodge structure on M has the respective property. Note that by 1.4 (ii) $\text{Lie } G = \text{Lie } P/W$ is pure of weight 0, so by 1.4 (iii) $\text{Lie } P$ is mixed of weight ≤ 0 . The following propositions 1.7, 1.10 and 1.12 are generalizations of [D2] 1.1.14.

1.7. Proposition: Let P and $\mathcal{H}_W$ be as in 1.6, and let M be a faithful representation of P . Let ϕ be the obvious map

$$\mathcal{H}_W \rightarrow \{\text{rational mixed Hodge structures on } M.\}$$

(a) There exists a unique structure on $\phi(\mathcal{H}_W)$ as a complex manifold, such that the Hodge filtration on $M_{\mathbb{C}}$ depends analytically on $\phi(h) \in \phi(\mathcal{H}_W)$. This structure is $P(\mathbb{R}) \cdot W(\mathbb{C})$ -invariant, and $W(\mathbb{C})$ acts analytically on $\phi(\mathcal{H}_W)$.

(b) For any other representation M' of P the analogous map

$$\phi' : \mathcal{H}_W \rightarrow \{\text{rational mixed Hodge structures on } M'\}$$

factors through $\phi(\mathcal{H}_W)$. The Hodge filtration on M' varies analytically with $\phi(h) \in \phi(\mathcal{H}_W)$.

(c) If in addition M' is faithful, then $\phi(\mathcal{H}_W)$ and $\phi'(\mathcal{H}_W)$ are canonically isomorphic and the isomorphism is compatible with the complex structure.

Proof. (a) Since the weight filtration on M is constant, the Hodge filtration gives an injective map $\phi(\mathcal{H}_W) \hookrightarrow \text{Grass}(M)(\mathbb{C})$ to a certain Grassmann variety. To prove the first assertion we have to show that its image is a locally closed analytic subvariety. Fix $h \in \mathcal{H}_W$ and let $C(h)$ be the centralizer of h in $P(\mathbb{R}) \cdot W(\mathbb{C})$, then ϕ factors through

$$\mathcal{H}_W \cong P(\mathbb{R}) \cdot W(\mathbb{C})/C(h) \rightarrow P(\mathbb{C})/\exp F^0(\text{Lie } P)_{\mathbb{C}} \hookrightarrow \text{Grass}(M)(\mathbb{C}).$$

The map on the right hand side is a closed embedding of complex manifolds. The complex structure in the middle is $P(\mathbb{C})$ -invariant, and $P(\mathbb{C})$ operates analytically. Thus it is enough to show that the image of $\mathcal{H}_W$ is an open subset of $P(\mathbb{C})/\exp F^0(\text{Lie } P)_{\mathbb{C}}$. Let $L := (\text{Lie } P)_{\mathbb{R}} + (\text{Lie } W)_{\mathbb{C}} \subset (\text{Lie } P)_{\mathbb{C}}$, and let $L^{0,0} \subset L$ be the eigenspace corresponding to the trivial operation of $\text{Ad}_P \circ h$. Then on the tangent space at h the map is given by

$$T_{\mathcal{H}_W, h} \cong L/L^{0,0} \rightarrow (\text{Lie } P)_{\mathbb{C}}/F^0(\text{Lie } P)_{\mathbb{C}},$$

and we have to show that this map is surjective. This is equivalent to the equality $(\text{Lie } P)_{\mathbb{C}} = L + F^0(\text{Lie } P)_{\mathbb{C}}$. Since both sides of this contain $(\text{Lie } W)_{\mathbb{C}}$, it suffices to show the equality $(\text{Lie } G)_{\mathbb{C}} = (\text{Lie } G)_{\mathbb{R}} + F^0(\text{Lie } G)_{\mathbb{C}}$. But $\text{Lie } G$ carries a pure Hodge structure of weight 0, so this equality follows from $(\text{Lie } G)_{\mathbb{R}} \cap F^0(\text{Lie } G)_{\mathbb{C}} \subset (\text{Lie } G)^{0,0}$ and dimension count.

(b) follows directly from (c), applied to $M \oplus M'$ in the place of M' . By part (b) of the following lemma the fibres of ϕ do not depend on the choice of M , so the two maps ϕ and ϕ' have identical fibres. This implies the first assertion of (c). Finally the proof shows that the complex structure on $\phi(\mathcal{H}_W)$ can be defined independently on M . **q.e.d.**

1.8. Lemma: Let P , $\mathcal{H}_W$, M and ϕ be as in 1.7, and let $h \in \mathcal{H}_W$.

(a) The projection $\pi : P \rightarrow G$ induces an isomorphism

$$\text{Cent}_{P(\mathbb{R}) \cdot W(\mathbb{C})}(h) \rightarrow \text{Cent}_{G(\mathbb{R})}(\pi \circ h).$$

(b) We have

$$\text{Stab}_{P(\mathbb{R}) \cdot W(\mathbb{C})}(\phi(h)) = \exp(F^0(\text{Lie } W)_{\mathbb{C}}) \rtimes \text{Cent}_{P(\mathbb{R}) \cdot W(\mathbb{C})}(h).$$

Proof. (a) By 1.4 (ii) and (iii) the space of invariants in $\text{Lie } W_{\mathbb{C}}$ under $(\text{Ad}|_W) \circ h \circ w$ is the zero subspace. Hence h defines a canonical Levi decomposition $P_{\mathbb{C}} = W_{\mathbb{C}} \rtimes \text{Cent}_{P_{\mathbb{C}}}(h \circ w)$. Since moreover $\pi \circ h$ is by 1.4 (ii) already defined over $\mathbb{R}$, we have a corresponding decomposition

$$P(\mathbb{R}) \cdot W(\mathbb{C}) = W(\mathbb{C}) \rtimes \text{Cent}_{P(\mathbb{R}) \cdot W(\mathbb{C})}(h \circ w).$$

Thus $\text{Cent}_{P(\mathbb{R}) \cdot W(\mathbb{C})}(h \circ W) \cong G(\mathbb{R})$, so the centralizers of h in both these groups are isomorphic.

(b) First suppose that $W = 1$. Then there exists a unique decomposition $M = \bigoplus_{n \in \mathbb{Z}} M_n$ into representations M_n of $P = G$ that are pure of weight n . For all p, q we therefore have

$$M^{p,q} = F^p M_{\mathbb{C}} \cap \overline{F^q m_{\mathbb{C}}} \cap W_{p+q} M_{\mathbb{C}},$$

so the decomposition into eigenspaces under $\rho \circ h$ is already uniquely determined by $\phi(h)$. Hence we have $\text{Stab}_{P(\mathbb{R})}(\phi(h)) = \text{Cent}_{P(\mathbb{R})}(h)$. In the general case this together with (a) implies

$$\text{Cent}_{P(\mathbb{R}) \cdot W(\mathbb{R})}(h) \subset \text{Stab}_{P(\mathbb{R}) \cdot W(\mathbb{C})}(\phi(h)) \subset W(\mathbb{C}) \rtimes \text{Cent}_{P(\mathbb{R}) \cdot W(\mathbb{C})}(h),$$

so it only remains to show that $\text{Stab}_{W(\mathbb{C})}(\phi(h)) = \exp(F^0(\text{Lie } W)_{\mathbb{C}})$. This follows from

$$\text{Stab}_{(\text{Lie } W)_{\mathbb{C}}}(\phi(h)) = (\text{Lie } W)_{\mathbb{C}} \cap F^0(\text{End}(M))_{\mathbb{C}} = F^0(\text{Lie } W)_{\mathbb{C}}.$$

q.e.d.

1.9. Variation of Hodge structures: Let X be a complex manifold. A variation of rational mixed Hodge structures over X consists of a local system of finite dimensional $\mathbb{Q}$ -vector spaces, together with a rational mixed Hodge structure on every fibre, such that:

(a) The weight filtration is locally constant and the Hodge filtration varies holomorphically. Let $\mathcal{M}$ be the locally free coherent $\mathcal{O}_X$ -sheaf associated to the vector bundle, and let $\mathcal{F}^p \mathcal{M}$ be the coherent subsheaves for all $p \in \mathbb{Z}$ that induce the Hodge filtration in each stalk.

(b) Transversality: For every $p \in \mathbb{Z}$ the canonical connection $\nabla : \mathcal{M} \rightarrow \mathcal{M} \otimes_{\mathcal{O}_X} \Omega_X^1$ maps the subsheaf $F^p \mathcal{M}$ to $F^{p-1} \mathcal{M} \otimes_{\mathcal{O}_X} \Omega_X^1$.

1.10. Proposition: Let $P, \mathcal{H}_W, M$ and ϕ be as in 1.7. Then we have a variation of rational mixed Hodge structures on M over $\phi(\mathcal{H}_W)$, if and only if for one ($\Leftrightarrow$ for all) $h \in \mathcal{H}_W$ the Hodge structure on $\text{Lie } P$ is of type

$$\{(-1, 1), (0, 0), (1, -1)\} \cup \{(-1, 0), (0, -1)\} \cup \{(-1, -1)\}.$$

Proof. With the notation of the proof of 1.7 the transversality condition in the point $\phi(h)$ is equivalent to

$$\text{im}(d\Phi) \subset F^{-1} \text{End}(M)_{\mathbb{C}} / F^0 \text{End}(M)_{\mathbb{C}}.$$

Since this image equals $(\text{Lie } P)_{\mathbb{C}} / F^0(\text{Lie } P)_{\mathbb{C}}$, this means $F^{-1}(\text{Lie } P)_{\mathbb{C}} = (\text{Lie } P)_{\mathbb{C}}$. In other words the Hodge numbers $h^{p,q}$ must vanish for all $p, q \in \mathbb{Z}$ with $q < -1$. If they vanish, then so do all $h^{p,q}$ with $p < -1$. Since $\text{Lie } P$ is mixed of weight ≤ 0 , one easily finds the condition equivalent to the stated type restriction. **q.e.d.**

1.11 Polarization of pure Hodge structures: Let $\sqrt{-1}$ be a primitive fourth root of unity in $\mathbb{C}$, then $\mathbb{Z}(1) := 2\pi\sqrt{-1} \cdot \mathbb{Z}$ is a submodule of $\mathbb{C}$ that is independent of the choice of $\sqrt{-1}$. For all $n \in \mathbb{Z}$ and any $\mathbb{Z}$ -module M define $M(n) := M \otimes \mathbb{Z}(1)^{\otimes n}$. By the given embedding $\mathbb{Z}(1) \hookrightarrow \mathbb{C}$ we get a canonical isomorphism $M(n)_{\mathbb{C}} \cong M_{\mathbb{C}}$. The Tate Hodge

structure is the $\mathbb{Q}$ -vector space $\mathbb{Q}(1)$ with the pure Hodge structure of type $(-1, -1)$. For every $n \in \mathbb{Z}$ we get a pure Hodge structure of type $(-n, -n)$ on $\mathbb{Q}(n) = \mathbb{Q}(1)^{\otimes n}$, and again a canonical isomorphism $\mathbb{Q}(n)_{\mathbb{C}} \cong \mathbb{C}$.

Let M be a $\mathbb{Q}$ -vector space and $k : \mathbb{S} \rightarrow GL(M_{\mathbb{R}})$ a representation that induces a pure Hodge structure of weight n on M . Let $\Psi : M \otimes M \rightarrow \mathbb{Q}(-n)$ be a morphism of Hodge structures. Choose a primitive fourth root of unity $\sqrt{-1} \in \mathbb{C}^{\times} = \mathbb{S}(\mathbb{R})$, and consider the bilinear form

$$M_{\mathbb{R}} \times M_{\mathbb{R}} \rightarrow \mathbb{Q}(-n)_{\mathbb{C}} = \mathbb{C}, \quad (m, m') \mapsto (2\pi\sqrt{-1})^n \cdot \Psi(m, k(\sqrt{-1})m').$$

Claim: It does not depend on the choice of $\sqrt{-1}$, and takes values in $\mathbb{R}$.

Proof. Since M is pure of weight n , we have $k(-\sqrt{-1}) = k(-1) \cdot k(\sqrt{-1}) = (-1)^n \cdot k(\sqrt{-1})$. Thus on replacing $\sqrt{-1}$ by $-\sqrt{-1}$ the bilinear form is multiplied by $(-1)^n \cdot (-1)^n = 1$. So it does not depend on the choice of $\sqrt{-1}$. Also note that $k(\sqrt{-1})$ lies in $GL(M)(\mathbb{R})$, so $\Psi(m, k(\sqrt{-1})m')$ lies in $\mathbb{Q}(-n)_{\mathbb{R}} = (2\pi\sqrt{-1})^{-n} \cdot \mathbb{R} \subset \mathbb{C}$. This implies the second assertion. **q.e.d.**

Having shown this the following definition is meaningful. A polarization of a rational Hodge structure pure of weight n is a morphism of Hodge structures $\Psi : M \otimes M \rightarrow \mathbb{Q}(-n)$, such that the bilinear form

$$M_{\mathbb{R}} \times M_{\mathbb{R}} \rightarrow \mathbb{R}, \quad (m, m') \mapsto (2\pi\sqrt{-1})^n \cdot \Psi(m, k(\sqrt{-1})m')$$

is symmetric and positive definite. Note that this pairing is symmetric if and only if n is even and Ψ symmetric, or n odd and Ψ alternating.

1.12 Proposition: Let P and $\mathcal{H}_W$ be as in 1.6. Let G_1 be a normal subgroup of $G = P/W$, defined over $\mathbb{Q}$, which for one ($\Leftrightarrow$ for all) $h \in \mathcal{H}_W$ contains the image of $h(\mathbb{S}^1)$. The following assertions are equivalent:

(a) $\text{int}(\pi(h\sqrt{-1}))$ is a Cartan involution on G_1 for ($\Leftrightarrow$ for all) $h \in \mathcal{H}_W$. In particular the center of G_1 is compact, hence splits over a CM -field.

(b) For every representation M of P which is pure of some weight n there exist

- a representation N of P , defined over $\mathbb{Q}$, which factors through G/G_1 and is pure of type (n, n) ,

— a P -equivariant nondegenerate pairing $\Psi : M \otimes M \rightarrow N$, and

— for every $h \in \mathcal{H}_W$ a morphism of rational Hodge structures $\lambda_h : N \rightarrow \mathbb{Q}(-n)$,

such that for every $h \in \mathcal{H}_W$ the pairing $\lambda_h \circ \Psi : M \otimes M \rightarrow \mathbb{Q}(-n)$ is a polarization for the Hodge structure on M defined by h .

Proof. Both conditions depend only on P/W , so we may assume that $P = G$. Fix a pure representation $\rho : G \rightarrow GL(M)$. Fix $h_0 \in \mathcal{H}_W$ and a choice of $\sqrt{-1}$, then we shall show that condition (b) for the given representation M is equivalent to the following one:

(c) There exists a $G_{1,\mathbb{R}}$ -invariant pairing $\Psi' : M_{\mathbb{R}} \times M_{\mathbb{R}} \rightarrow \mathbb{R}$, defined over $\mathbb{R}$, such that $(m, m') \mapsto \Psi'(m, \rho \circ h_0(\sqrt{-1})m')$ is symmetric and positive definite.

Clearly (b) implies (c) with $\Psi' : (2\pi\sqrt{-1})^n \cdot \lambda_{h_0} \circ \Psi$. Conversely assume (c). Note first that since $(\rho \circ h_0(\sqrt{-1}))^2 = \rho \circ h_0(-1) = (-1)^n$, the pairing Ψ' is symmetric if n is even,

alternating if n is odd. So let $N := (S^2 M)_{G_1}$ if n is even, respectively $(\bigwedge^2 M)_{G_1}$ if n is odd, where $(\cdot)_{G_1}$ denotes the G_1 -coinvariants. Let $\Psi : M \otimes M \rightarrow N$ be the canonical pairing, then by construction Ψ' factors through N , hence $\Psi' = \mu_0 \circ \Psi$ for some homomorphism $\mu_0 : N_{\mathbb{R}} \rightarrow \mathbb{R}$. Since G_1 is a normal subgroup of G , N is a G -invariant quotient of $M \otimes M$, hence necessarily pure of weight $2n$. Since moreover G_1 operates trivially on N , it is already pure of type (n, n) . Since by assumption Ψ' is nontrivial, so is Ψ .

By the construction of N for every $\mu \in \text{Hom}(N_{\mathbb{R}}, \mathbb{R})$ the pairing

$$M_{\mathbb{R}} \times M_{\mathbb{R}} \rightarrow \mathbb{R}, (m, m') \mapsto \mu \circ \Psi(m, \rho \circ h_0(\sqrt{-1})m')$$

is automatically symmetric. Thus the set of all μ , such that this pairing is symmetric and positive definite, is open in the usual topology. Furthermore the existence of μ_0 shows that this set is nonempty. Since the homomorphisms that are defined over $\mathbb{Q}$ are dense, we can replace μ_0 by a homomorphism $N \rightarrow \mathbb{Q}$ that is defined over $\mathbb{Q}$. Let $\lambda_{h_0} := (2\pi\sqrt{-1})^{-1} \cdot \mu_0 : N \rightarrow \mathbb{Q}(-n)$, this is automatically a morphism of rational Hodge structures, and by construction $\lambda_{h_0} \circ \Psi$ is a polarization for the Hodge structure on M defined by h_0 .

Now let $g \in G(\mathbb{R})$ and $h = \text{int}(g) \circ h_0$. Denote by $\tau G \rightarrow GL(N)$ the representation of G on N . Then

$$\mu_0 \circ \tau(g^{-1}) \circ \Psi(m, \rho \circ h(\sqrt{-1})m') = \mu_0 \circ \Psi(\rho(g^{-1})m, \rho \circ h_0(\sqrt{-1}) \cdot \rho(g^{-1})m'),$$

so after replacing μ_0 by $\mu_0 \circ \tau(g)^{-1}$, condition (c) holds for h in place of h_0 . Since the above definitions of N and Ψ do not depend on h_0 , the desired λ_h exists for every $h \in \mathcal{H}_W$. Thus (c) implies (b).

By [D2] 1.1.15 condition (c) holds for all M if and only if $\text{int}(h_0(\sqrt{-1}))$ is a Cartan involution on $G_{1, \mathbb{R}}$. Clearly this second property is invariant under conjugation by $G(\mathbb{R})$. It also implies that $Z(G_1)(\mathbb{R})$ is compact, so the torus $Z(G_1)^0$ splits over a CM -field. Thus (c) is equivalent to (a). **q.e.d.**

1.13. Restriction of the group: In the situation of 1.12, we encounter the following phenomenon. The quotient G/G_1 is entirely arbitrary, but it contributes almost nothing to the family of Hodge structures. In particular the semisimple part of G/G_1 is ballast. On the other hand, N is an arbitrary representation of G/G_1 , so the G -orbit generated by λ_h may be disturbingly large. For our purposes we want to have that this orbit consists only of multiples of λ_h , at least for irreducible M . By 1.12, this clearly holds if G/G_1 is a $\mathbb{Q}$ -split torus. Thus, to solve both problems at the same time, we impose the following condition on P . Let G_2 be the smallest normal subgroup of G , defined over $\mathbb{Q}$, which contains the image $\pi \circ h(\mathbb{S})$ for one ($\Leftrightarrow$ for all) $h \in \mathcal{H}_W$. This group essentially determines the family of Hodge structures. We require:

(*) G/G_2 is an almost direct product of a $\mathbb{Q}$ -split torus with a torus of compact type defined over $\mathbb{Q}$.

Consider the situation of 1.12, with G_1 the smallest normal $\mathbb{Q}$ -subgroup of G such that G/G_1 is a $\mathbb{Q}$ -split torus. Clearly the condition (*) above together with 1.12 (a) is equivalent to: (a') $\text{int}(\pi(h(\sqrt{-1})))$ induces a Cartan involution on $G_{\mathbb{R}}^{ad}$, and G^{ad} possesses

no nontrivial factors of compact type that are defined over $\mathbb{Q}$. Furthermore the center of G is an almost direct product of a $\mathbb{Q}$ -split torus with a torus of compact type defined over $\mathbb{Q}$.

Assume that this condition holds. Then in 1.12 (b), N is a direct sum of onedimensional representations of G . If M is irreducible, then the space of G_1 -coinvariants of $M \otimes M$ has dimension ≤ 1 , so we can choose N to have dimension 1. If M is arbitrary, but $\dim(G/G_1) \leq 1$, then G operates by scalars on N through a character that depends only on n . From the proof of 1.12 we see that again we can choose N to have dimension 1. Observe that if $n \neq 0$, the G -orbit generated by λ_h always has dimension ≥ 1 . Moreover, as in the examples 2.7–8 below, the different λ_h need not have the same sign.

1.14. Proposition: Suppose that $W = 1$ and the conditions of 1.10 and 1.12 are satisfied. Then every connected component of $\mathcal{H}_W \cong \phi(\mathcal{H}_W)$ is a hermitian symmetric domain.

Proof. [D2] 1.1.14 (iii).

q.e.d.

1.15 Replacing by smaller orbit: In general the map ϕ from proposition 1.7 is not injective. We can, however, replace $\mathcal{H}_W$ by an orbit under a subgroup of $P(\mathbb{R}) \cdot W(\mathbb{C})$, and still get the same image under ϕ . Which subgroup we can choose is seen from 1.2 (c). In particular, if we want a subgroup of the form $P(\mathbb{R}) \cdot U(\mathbb{C})$, where $U \subset W$ is a subgroup defined over $\mathbb{Q}$, in general there is only the following possibility. Although in general the resulting map will still not be injective, it will be injective in the case that is of interest to us, namely when the condition of 1.10 is satisfied.

Explicitly let P and $\mathcal{H}_W$ be as in 1.6. Let $U \subset W$ be the unique connected subgroup, defined over $\mathbb{Q}$, such that $\text{Lie } U = W_{-2}(\text{Lie } W)$. By 1.4 (c) it does not depend on $h \in \mathcal{H}_W$. Let π' be the canonical projection $P \rightarrow P/U$. Let $V := W/U$, then $\text{Lie } V$ is pure of weight -1 and -2 , V must be abelian. Consider instead of 1.4 (i) the following stronger condition for $h \in \text{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$:

$$(i') \pi' \circ h : \mathbb{S}_{\mathbb{C}} \rightarrow (P/U)_{\mathbb{C}} \text{ is already defined over } \mathbb{R}.$$

Finally let

$$\mathcal{H} := \{h \in \mathcal{H}_W | h \text{ satisfies } (i')\}$$

1.16 Proposition: With the notation on 1.15 we have: (a) $\mathcal{H}$ is a nonempty $P(\mathbb{R}) \cdot U(\mathbb{C})$ -orbit in $\text{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$.

(b) For M and ϕ as in 1.7 we have $\phi(\mathcal{H}) = \phi(\mathcal{H}_W)$.

(c) If moreover $F^0(\text{Lie } U)_{\mathbb{C}} = 0$, then $\phi(\mathcal{H}) \cong \mathcal{H}$.

Proof. Since $\text{Lie } V$ is pure of weight -1 , we have

$$F^0(\text{Lie } V)_{\mathbb{C}} \cap F^0(\text{Lie } V)_{\mathbb{C}} = (\text{Lie } V)_{\mathbb{R}} \cap F^0(\text{Lie } v)_{\mathbb{C}} = 0,$$

and by dimension count we get $(\text{Lie } V)_{\mathbb{C}} = (\text{Lie } V)_{\mathbb{R}} \otimes F^0(\text{Lie } v)_{\mathbb{C}} = 0$. This implies on the one hand $(\text{Lie } W)_{\mathbb{C}} = ((\text{Lie } W)_{\mathbb{R}} + (\text{Lie } U)_{\mathbb{C}}) + F^0(\text{Lie } W)_{\mathbb{C}}$, that is

$$W(\mathbb{C}) = (W(\mathbb{R}) \cdot U(\mathbb{C})) \cdot \exp(F^0(\text{Lie } W)_{\mathbb{C}}),$$

on the other hand

$$((\mathrm{Lie} W)_{\mathbb{R}} + (\mathrm{Lie} U)_{\mathbb{C}}) \cap F^0(\mathrm{Lie} W)_{\mathbb{C}} = (\mathrm{Lie} U)_{\mathbb{C}} \cap F^0(\mathrm{Lie} W)_{\mathbb{C}} = (\mathrm{Lie} U)_{\mathbb{C}}.$$

that is

$$(W(\mathbb{R}) \cdot U(\mathbb{C})) \cap \exp(F^0(\mathrm{Lie} W)_{\mathbb{C}}) = \exp(F^0(\mathrm{Lie} U)_{\mathbb{C}}).$$

As $\mathrm{Lie} W$ is of weight ≤ -1 , there exists no nontrivial group of W on which P operates trivially. Thus the variety of all Levi decompositions of P is a principal homogeneous space under W .

Let us first prove (b). We have to show that for every $h \in \mathcal{H}_W$ there exists $p \in \mathrm{Stab}_{P(\mathbb{R}) \cdot W(\mathbb{C})}(\phi(h))$, such that $\mathrm{int}(p) \circ h \in \mathcal{H}$, that is $\pi' \circ \mathrm{int}(p) \circ h : \mathbb{S} \rightarrow (P/U)_{\mathbb{C}}$ is defined over $\mathbb{R}$. As in the proof of 1.8 (a) consider the Levi decomposition of $P_{\mathbb{C}}$ defined by h . Since there also exists a Levi decomposition that is defined over $\mathbb{R}$, there exists $w_0 \in W(\mathbb{C})$ such that $\mathrm{int}(w_0) \circ h \circ w$, hence also $\mathrm{int}(w_0) \circ h$ is defined over $\mathbb{R}$. Write $w_0 = w' \cdot w''$ with $w' \in W(\mathbb{R}) \cdot U(\mathbb{C})$ and $w'' \in \exp(F^0(\mathrm{Lie} W)_{\mathbb{C}})$. Then $\pi' \circ \mathrm{int}(w'') \circ h = \pi' \circ \mathrm{int}(w')^{-1} \circ (\mathrm{int}(w_0) \circ h)$ is defined over $\mathbb{R}$, and by 1.8 (b) we have $\phi(\mathrm{int}(w'') \circ h) = \phi(h)$. Thus the assertion follows with $p = w''$.

This also proves that $\mathcal{H}$ is nonempty. (a) now follows from the stronger statement:

(*) For any $h \in \mathcal{H}$ and $p \in P(\mathbb{R}) \cdot W(\mathbb{R})$, we have $\mathrm{int}(p) \circ h \in \mathcal{H}$ if and only if $p \in P(\mathbb{R}) \cdot U(\mathbb{C})$.

To prove this we may assume without loss of generality that U is trivial. Writing $p = p' \cdot w'$ with $p' \in P(\mathbb{R})$ and $w' \in W(\mathbb{C})$ we find that $\mathrm{int}(p) \circ h$ is defined over $\mathbb{R}$ if and only if $\mathrm{int}(w') \circ h$ is defined over $\mathbb{R}$. That again is true if and only if the Levi decomposition of $P_{\mathbb{C}}$ that is defined by $\mathrm{int}(w') \circ h \circ w$ is defined over $\mathbb{R}$. By the above remark the map

$$w' \mapsto (\text{the Levi decomposition of } P_{\mathbb{C}} \text{ defined by } \mathrm{int}(w') \circ h \circ w)$$

is an algebraic isomorphism of W to the variety of Levi decompositions of P , and is defined over $\mathbb{R}$ since $h \circ w$ is defined over $\mathbb{R}$. Thus the Levi decomposition associated to $\mathrm{int}(w') \circ h \circ w$ is defined over $\mathbb{R}$ if and only if $w' \in W(\mathbb{R})$, as desired.

For (c) consider an element $h \in \mathcal{H}$. The fibre $(\phi|_{\mathcal{H}})^{-1}(\phi(h))$ is the $\mathrm{Stab}_{P(\mathbb{R}) \cdot U(\mathbb{C})}(\phi(h))$ -orbit in $\mathcal{H}$ that is generated by h . If $F^0(\mathrm{Lie} U)_{\mathbb{C}} = 0$, then by part (b) of the following lemma this fibre consists only of h . Thus $\phi|_{\mathcal{H}}$ is a bijection. **q.e.d.**

1.17. Lemma: Let $P, \mathcal{H}$, etc. be as in 1.15, and let $h \in \mathcal{H}$. Then (a) $\mathrm{Cent}_{P(\mathbb{R}) \cdot U(\mathbb{C})}(h) = \mathrm{Cent}_{P(\mathbb{R}) \cdot W(\mathbb{C})}(h) \cong \mathrm{Cent}_{G(\mathbb{R})}(\pi \circ h)$, and

(b) $\mathrm{Stab}_{P(\mathbb{R}) \cdot U(\mathbb{C})}(\phi(h)) = \exp(F^0(\mathrm{Lie} U)_{\mathbb{C}}) \rtimes \mathrm{Cent}_{P(\mathbb{R}) \cdot U(\mathbb{C})}(h)$.

Proof. (a) By 1.8 (a) it suffices to show that $\mathrm{Cent}_{P(\mathbb{R}) \cdot W(\mathbb{C})}(h)$ is contained in $P(\mathbb{R}) \cdot U(\mathbb{C})$. This follows from the assertion (*) in the proof of 1.16.

(b) The assertions (a) and 1.8 (b) imply

$$\begin{aligned} \mathrm{Stab}_{P(\mathbb{R}) \cdot U(\mathbb{C})}(\phi(h)) &= (P(\mathbb{R}) \cdot U(\mathbb{C})) \cap \mathrm{Stab}_{P(\mathbb{R}) \cdot W(\mathbb{C})}(\phi(h)) \\ &= (P(\mathbb{R}) \cdot U(\mathbb{C})) \cap \exp(F^0(\mathrm{Lie} W)_{\mathbb{C}}) \rtimes \mathrm{Cent}_{P(\mathbb{R}) \cdot W(\mathbb{C})}(h) \\ &= ((P(\mathbb{R}) \cdot U(\mathbb{C})) \cap \exp(F^0(\mathrm{Lie} W)_{\mathbb{C}})) \rtimes \mathrm{Cent}_{P(\mathbb{R}) \cdot W(\mathbb{C})}(h), \end{aligned}$$

and in the proof of 1.16 we showed

$$\begin{aligned} (P(\mathbb{R}) \cdot U(\mathbb{C})) \cap (\exp(F^0(\mathrm{Lie} W)_{\mathbb{C}})) &= (P(\mathbb{R}) \cdot U(\mathbb{C})) \cap (\exp(F^0(\mathrm{Lie} W)_{\mathbb{C}})) \\ &= \exp(F^0(\mathrm{Lie} U)_{\mathbb{C}}) \rtimes \mathrm{Cent}_{P(\mathbb{R}) \cdot U(\mathbb{C})}(h). \end{aligned}$$

q.e.d.

1.18 Summary Putting together the results of 1.4, 1.7, 1.10, 1.12, 1.13 and 1.16 we get the following, albeit somewhat complicated picture. Let P be a connected linear algebraic group over $\mathbb{Q}$. Let W be the unipotent radical of P , and $U \subset W$ a subgroup that is normal in P . Let $V := W/U$, $G := P/W$ and $\pi : P \rightarrow G$ and $\pi' : P \rightarrow P/U$ be the canonical projections. Let $\mathcal{H}$ be a $P(\mathbb{R}) \cdot U(\mathbb{C})$ -orbit in $\mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$, such that for some ($\Leftrightarrow$ for all) $h \in \mathcal{H}$:

- (i') $\pi' \circ h : \mathbb{S}_{\mathbb{C}} \rightarrow (P/U)_{\mathbb{C}}$ is already defined over $\mathbb{R}$.
- (ii) $\pi \circ h \circ w : \mathbb{G}_{m, \mathbb{R}} \rightarrow G_{\mathbb{R}}$ is a cocharacter of the center of G that is defined over $\mathbb{Q}$.
- (iii) Under the weight filtration on $\mathrm{Lie} P$ that is defined by $\mathrm{Ad}_P \circ h$ we have $W_{-1}(\mathrm{Lie} P) = \mathrm{Lie} W$ and $W_{-2}(\mathrm{Lie} P) = \mathrm{Lie} U$.
- (iv) The Hodge structure on $\mathrm{Lie} P$ is of type

$$\{(-1, 1), (0, 0), (1, -1)\} \cup \{(-1, 0), (0, -1)\} \cup \{(-1, -1)\}.$$

- (v) $\mathrm{int}(\pi(h(\sqrt{-1})))$ induces a Cartan involution on $G_{\mathbb{R}}^{ad}$, and G^{ad} possesses no nontrivial factors of compact type that are defined over $\mathbb{Q}$.
- (vi) The center of G is an almost direct product of a $\mathbb{Q}$ -split torus with a torus of compact type defined over $\mathbb{Q}$.

Then

- (a) $\mathcal{H}$ possesses a canonical $P(\mathbb{R}) \cdot U(\mathbb{C})$ -invariant complex structure (1.7 and 1.16).
- (b) For every representation M of P we have a variation of rational mixed Hodge structures (1.7 and 1.10).
- (c) The operation of $P(\mathbb{R}) \cdot U(\mathbb{C})$ on $\mathcal{H}$ can be extended canonically to an operation of $P(\mathbb{R}) \cdot W(\mathbb{C})$. The complex structure is still invariant under this bigger group, and $W(\mathbb{C})$ operates analytically on $\mathcal{H}$ (1.7 and 1.16).
- (d) For every irreducible representation M of P that is pure of some weight n there exists a one dimensional representation of P on $\mathbb{Q}(-n)$, and a P -equivariant bilinear form $\Psi : M \times M \rightarrow \mathbb{Q}(-n)$, such that for all $h \in \mathcal{H}_W$ either Ψ or $-\Psi$ is a polarization of the corresponding Hodge structure on M (1.12 and 1.13).
- (e) If $W = 1$, then every connected component of $\mathcal{H}$ is a hermitian symmetric domain (1.14).

Remark. We shall see later (2.19) that every connected component of $\mathcal{H}$ is (noncanonically) isomorphic to a holomorphic vector bundle on a hermitian symmetric domain.

1.19. Generalization to arbitrary weights: The cocharacter $\pi \circ h \circ w$ above can be considered as a weight, since it corresponds to the weight of the Hodge structure induced

on any representation of G . In this sense the above condition (ii) means that the weight is rational, i.e. defined over $\mathbb{Q}$. Since both this requirement and condition (vi) above are nontrivial restrictions for Shimura varieties, we want to replace them by weaker conditions. For our purposes it will be enough to assume the same conditions for the image of P in $\text{Aut}(W)$, or equivalently in $\text{Aut}(U) \times \text{Aut}(V)$. Explicitly we replace (ii) and (vi) by the conditions:

(ii') $\pi \circ h \circ w : \mathbb{G}_{m,\mathbb{R}} \rightarrow G_{\mathbb{R}}$ is a cocharacter of the center of G .

(vi') The center of G acts on U and on V through a torus that is an almost direct product of a $\mathbb{Q}$ -split torus with a torus of compact type defined over $\mathbb{Q}$.

Observe that these conditions imply that the weight acts on both U and V through a rational scalar character.

It is clear that with these weakened conditions the conclusions of 1.18, except (d), still hold. It is important, however, that (d) does hold for M an arbitrary irreducible subquotient of V . Note also that the conclusion of 1.17 remain valid. Finally there is a relation with data as in 1.18. By 1.18 (iii) we have $Z(P) \cap W = 1$, so $Z(P)$ is reductive and isomorphic to the kernel of the action of $Z(G)$ on $\text{Lie } U \otimes \text{Lie } V$. Thus $P_1 := P/Z(P)$ together with the obvious choice for $\mathcal{H}_1$ satisfies the conditions of 1.18.

Chapter 2

Mixed Shimura data

In the first chapter we introduced certain $P(\mathbb{R}) \cdot U(\mathbb{C})$ -conjugacy classes of homomorphisms $\mathbb{S}_{\mathbb{C}} \rightarrow P_{\mathbb{C}}$ as a natural generalization to nonreductive groups of the conjugacy classes of homomorphisms $\mathbb{S} \rightarrow G_{\mathbb{R}}$ considered in [D2]. In this chapter we bring in yet another generalization by considering finite coverings of such conjugacy classes. In various places we shall see the usefulness of this generalization (for instance in 3.9, 3.16–17), and indeed its necessity (see 4.11 (example 4.25–26), 7.1–2 or 8.5–6) for a unified formalism of the arithmetic structure of the boundary of Shimura varieties.

Definition 2.1 is a generalization of [D1] 1.5 resp. [D2] 2.1.1.1–3. In 2.3–10 we introduce more notions and some standard examples. In 2.11–12 we study the precise structure of the finite covering mentioned above. The largest part of this chapter, starting with 2.15, deals with properties and constructions related to the unipoint radical. In particular we prove a reduction lemma (2.26), by means of which any mixed Shimura data can (almost) be embedded into a direct product of usual pure ones (as in [D1] 1.5 or [D2] 2.1.1) together with the standard examples described in this chapter.

2.1 Definiton: Let P be a connected linear algebraic group over $\mathbb{Q}$. Let W be its unipoint radical and U a subgroup of W that is normal in P . Let $V := W/U$, $G := P/W$, and $\pi : P \rightarrow G$ and $\pi' : P \rightarrow P/U$ be the canonical projections. Let $\mathcal{X}$ be a left homogeneous space under the subgroup $P(\mathbb{R}) \cdot U(\mathbb{C}) \subset P(\mathbb{C})$. Let $h : \mathcal{X} \rightarrow \text{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$ be a $P(\mathbb{R}) \cdot U(\mathbb{C})$ -equivariant map, such that:

(i) Every fibre of h consists of at most finitely many points.

Furthermore for some ($\Leftrightarrow$ for all) $x \in \mathcal{X}$ we assume:

(ii) $\pi' \circ h_x : \mathbb{S}_{\mathbb{C}} \rightarrow (P/U)_{\mathbb{C}}$ is already defined over $\mathbb{R}$.

(iii) $\pi \circ h_x \circ w : G_{m, \mathbb{R}} \rightarrow G_{\mathbb{R}}$ is a cocharacter of the center of G .

(iv) $\text{Ad}_P \circ h_x$ induces on $\text{Lie } P$ a rational mixed Hodge structure of type

$$\{(-1, 1), (0, 0), (1, -1)\} \cup \{(-1, 0), (0, -1)\} \cup \{(-1, -1)\}.$$

(v) The weight filtration on $\mathrm{Lie} P$ is given by

$$W_n(\mathrm{Lie} P) = \begin{cases} 0 & \text{if } n < -2, \\ \mathrm{Lie} U & \text{if } n = -2, \\ \mathrm{Lie} V & \text{if } n = -1, \\ \mathrm{Lie} W & \text{if } n \geq 0. \end{cases}$$

(vi) $\mathrm{int}(\pi(h_x(\sqrt{-1})))$ induces a Cartan involution on $G_{\mathbb{R}}^{ad}$.

(vii) G^{ad} possesses no nontrivial factors of compact type that are defined over $\mathbb{Q}$.

(viii) The center of G acts on U and on V through a torus that is an almost direct product of a $\mathbb{Q}$ -split torus with a torus of compact type defined over $\mathbb{Q}$.

Definition: We call such a triple $(P, \mathcal{X}, h)$ mixed Shimura data. If $W = 1$, then we also call it pure Shimura data, or just Shimura data.

Remarks: (a) The data $(P, \mathcal{X}, h)$ uniquely determines the rest, in particular U is characterized by (v). Thus the definition is meaningful. For abbreviation we shall mostly write $(P, \mathcal{X})$, since for every pair $(P, \mathcal{X})$, we shall always consider exactly one map h .

(b) It is easy to show that condition (iii) is a consequence of (iv).

(c) $\mathcal{X}$ consists of finitely many connected components. The map $h : \mathcal{X} \rightarrow h(\mathcal{X})$ is a local diffeomorphism, so the complex structure that is given on $h(\mathcal{X})$ by 1.18 (a) induces a complex structure on $\mathcal{X}$.

(d) If $W = 1$ and $\mathcal{X} \cong h(\mathcal{X})$, then the definition is equivalent to [D1] 1.5 or to [D2] 2.1.1.1–3.

2.3. Definition: Let $(P_1, \mathcal{X}_1, h_1)$ and $(P_2, \mathcal{X}_2, h_2)$ be mixed Shimura data. A morphism $(P_1, \mathcal{X}_1, h_1) \rightarrow (P_2, \mathcal{X}_2, h_2)$ consists of a homomorphism $\phi : P_1 \rightarrow P_2$ and a $P_1(\mathbb{R}) \cdot U_1(\mathbb{C})$ -equivariant map $\Psi : \mathcal{X}_1 \rightarrow \mathcal{X}_2$, such that following diagram commutes:

$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{\psi} & \mathcal{X}_2 \\ h_1 \downarrow & & \downarrow h_2 \\ \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{1,\mathbb{C}}) & \xrightarrow{h \mapsto \phi \circ h} & \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{2,\mathbb{C}}) \end{array}$$

We call a morphism an embedding, and write $(P_1, \mathcal{X}_1, h_1) \hookrightarrow (P_2, \mathcal{X}_2, h_2)$, if both ϕ and Ψ are injective.

2.4 Proposition: For any morphism $(\phi, \Psi) : (P_1, \mathcal{X}_1) \rightarrow (P_2, \mathcal{X}_2)$ the map $\Psi : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ is holomorphic. If (ϕ, Ψ) is an embedding, then Ψ is a closed embedding.

Proof. By 1.7 the associated map $h_1(\mathcal{X}_1) \rightarrow h_2(\mathcal{X}_2)$ is holomorphic. From this the first assertion follows, since $\mathcal{X}_i$ is locally isomorphic to $h_i(\mathcal{X}_i)$ for $i = 1, 2$. The second assertion follows from the fact, that $P_1(\mathbb{R}) \cdot U_1(\mathbb{C})$ is a closed subgroup of $P_2(\mathbb{R}) \cdot U_2(\mathbb{C})$. **q.e.d.**

2.5. Definition: For any two mixed Shimura data $(P_1, \mathcal{X}_1)$ and $(P_2, \mathcal{X}_2)$ we define their product as $(P_1, \mathcal{X}_1) \times (P_2, \mathcal{X}_2) := (P_1 \times P_2, \mathcal{X}_1 \times \mathcal{X}_2)$ with the obvious map $\mathcal{X}_1 \times \mathcal{X}_2 \rightarrow \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, (P_1 \times P_2)_{\mathbb{C}})$. Clearly this is again mixed Shimura data. In fact this is the direct product in the category of mixed Shimura data, that is for every $(P, \mathcal{X})$ and any two

morphism $(P, \mathcal{X}) \rightarrow (P_i, \mathcal{X}_i)$ there exists a unique morphism $(P, \mathcal{X}) \rightarrow (P_1, \mathcal{X}_1) \times (P_2, \mathcal{X}_2)$ such that the following diagram commutes:

$$\begin{array}{ccc}
 & (P_1, \mathcal{X}_1) & \\
 \nearrow & & \nwarrow^{pr_1} \\
 (P, \mathcal{X}) & \longrightarrow & (P_1, \mathcal{X}_1) \times (P_2, \mathcal{X}_2) \\
 \searrow & & \swarrow_{pr_2} \\
 & (P_2, \mathcal{X}_2) &
 \end{array}$$

2.6. Example: Consider a torus T over $\mathbb{Q}$, a homomorphism $k : \mathbb{S} \rightarrow T_{\mathbb{R}}$, and a finite set $\mathcal{Y}$ with a transitive action of $\pi_0(T(\mathbb{R}))$. With the constant map $h : \mathcal{Y} \rightarrow \{k\} \subset \text{Hom}(\mathbb{S}_{\mathbb{C}}, T_{\mathbb{C}})$ we then have mixed Shimura data $(T, \mathcal{Y})$. In fact every mixed Shimura data $(P, \mathcal{X})$, where P is a torus or where $\mathcal{X}$ is a finite set, is of this form.

2.7. Example: the symplectic group: (Compare [D1] 1.6) Fix an integer $g \geq 1$. Let V_{2g} be a $\mathbb{Q}$ -vector space of dimension $2g$ and $\Psi : V_{2g} \times V_{2g} \rightarrow \mathbb{Q}$ a nondegenerate alternating form. Let $G_{2g} := CS_{P_{2g}, \mathbb{Q}}$ be the group of all $f \in GL(V_{2g})$, such that $\Psi(f(v), f(v')) = \lambda \cdot \Psi(v, v')$ for some scalar “multiplier” λ . This multiplier yields a surjective homomorphism $CS_{P_{2g}, \mathbb{Q}} \rightarrow \mathbb{G}_{m, \mathbb{Q}}$, whose kernel is $S_{P_{2g}, \mathbb{Q}} = (CS_{P_{2g}, \mathbb{Q}})^{der}$. Let $\mathcal{H}_{2g}$ be the set of all homomorphisms $k : \mathbb{S} \rightarrow CS_{P_{2g}, \mathbb{R}}$ that induces a pure Hodge structure of type $\{(-1, 0), (0, -1)\}$ on V_{2g} , and for which the pairing $\mathbb{R}^{2g} \times \mathbb{R}^{2g} \rightarrow \mathbb{R}$, $(v, v') \mapsto \Psi(v, k(\sqrt{-1})v')$ is symmetric and (positive or negative) definite. Note that this second condition does not depend on the choice of $\sqrt{-1}$. It is wellknown that $CS_{P_{2g}, \mathbb{Q}}$ is a reductive group and operates transitively on $\mathcal{H}_{2g}$. Moreover $\mathcal{H}_{2g}$ possesses exactly two connected components. The pair $(CS_{P_{2g}, \mathbb{Q}}, \mathcal{H}_{2g})$ together with the inclusion $h : \mathcal{H}_{2g} \hookrightarrow \text{Hom}(\mathbb{S}_{\mathbb{C}}, CS_{P_{2g}, \mathbb{Q}})$ is pure Shimura data.

For $g = 1$ we have $CS_{P_{2g}, \mathbb{Q}} = GL_{2, \mathbb{Q}}$ and $\mathcal{H}_2$ can be identified in the usual way with the union of the complex upper and lower half planes. This, and more generally the unbounded realization of $\mathcal{H}_{2g}$ will be described in 4.25–26. Later in this chapter we shall give examples with nontrivial W , using some general constructions.

2.8. Example: For $g = 0$ we have the following analog of $(CS_{P_{2g}, \mathbb{Q}}, \mathcal{H}_{2g})$. Let $G_0 := \mathbb{G}_{m, \mathbb{Q}}$ and $k : \mathbb{S} \rightarrow G_{0, \mathbb{R}}, z \mapsto z \cdot \bar{z}$. Let $\mathcal{H}_0$ be the set of isomorphism $\mathbb{Z} \rightarrow \mathbb{Z}(1)$. Thus an element $\lambda \in \mathcal{H}_0$ corresponds to a unique choice of $\sqrt{-1}$ by $\lambda(1) = 2\pi\sqrt{-1}$. At present we are only interested in the fact that $\mathcal{H}_0$ consists exactly two elements, but later the precise definition will become important. Consider the unique transitive operation of $\pi_0(G_0(\mathbb{R})) = \pi_0(\mathbb{R}^{\times})$ of $\mathcal{H}_0$, and let $h : \mathcal{H}_0 \rightarrow \{k\} \subset \text{Hom}(\mathbb{S}_{\mathbb{C}}, G_{0, \mathbb{C}})$ be the constant map. With these definitions $(G_0, \mathcal{H}_0)$ is pure Shimura data, as in 2.6.

For any $g \geq 1$ consider the multiplier map $\phi : CS_{P_{2g}, \mathbb{Q}} \rightarrow \mathbb{G}_{m, \mathbb{Q}}$. Then for every $x \in \mathcal{H}_{2g}$ we have $\phi \circ h_x = k$. Define a map $\mathcal{H}_{2g} \rightarrow \mathcal{H}_0 \subset \text{Hom}(\mathbb{Q}, \mathbb{Q}(1))$ by

$$x \mapsto \text{the unique } \lambda \in \mathcal{H}_0 \text{ such that } \lambda \circ \Psi \text{ is a polarization of } V_{2g}.$$

Clearly this map is equivariant under $CS_{P_{2g}}(\mathbb{R}) \rightarrow \mathbb{G}_m(\mathbb{R})$, so we obtain a canonical morphism $(CS_{P_{2g}, \mathbb{Q}}, \mathcal{H}_{2g}) \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$.

2.9. Proposition: Let $(P, \mathcal{X})$ be mixed Shimura data and P_0 a normal subgroup of P . There exists a quotient mixed Shimura data $(P, \mathcal{X})/P_0$ and a morphism $(P, \mathcal{X}) \rightarrow (P, \mathcal{X})/P_0$, unique up to isomorphism, such that every morphism $(P, \mathcal{X}) \rightarrow (P', \mathcal{X}')$, where the homomorphism $P \rightarrow P'$ factors through P/P_0 , factors in a unique way through $(P, \mathcal{X})/P_0$:

$$\begin{array}{ccc} P & \longrightarrow & P' \\ \downarrow & \nearrow & \\ P/P_0 & & \end{array} \implies \begin{array}{ccc} (P, \mathcal{X}) & \longrightarrow & (P', \mathcal{X}') \\ \downarrow & \nearrow & \\ (P, \mathcal{X})/P_0 & & \end{array}$$

Proof. The uniqueness follows from the universal property. First we construct $(P, \mathcal{X})/P_0$. Let $P_1 := P/P_0$ and $\phi : P \rightarrow P_1$ be the canonical projection. Then $W_1 = \phi(W)$ is the unipotent radical of P_1 , and $U_1 := \phi(U)$ is normal in P_1 . Fixing $x \in \mathcal{X}$ we get a bijection

$$(P(\mathbb{R}) \cdot U(\mathbb{C})/\text{Stab}_{P(\mathbb{R}) \cdot U(\mathbb{C})}(x)) \xrightarrow{\sim} \mathcal{X}, \quad p \mapsto p \cdot x.$$

Let

$$\mathcal{X}_1 := (P_1(\mathbb{R}) \cdot U_1(\mathbb{C}))/\phi(\text{Stab}_{P(\mathbb{R}) \cdot U(\mathbb{C})}(x)), \text{ and}$$

$$h_1 : \mathcal{X}_1 \rightarrow \text{Hom}(\mathbb{S}_{\mathbb{C}}, P_{1, \mathbb{C}}), \quad [p_1] \mapsto \text{int}(p_1) \circ \phi \circ h_x.$$

By construction $(P_1, \mathcal{X}_1, h_1)$ satisfies the conditions 2.1, so $(P_1, \mathcal{X}_1)$ is mixed Shimura data. Next define an equivariant map $\Psi : \mathcal{X} \rightarrow \mathcal{X}_1$ by $p \cdot x \mapsto [\phi(p)]$, then $(\phi, \Psi) : (P, \mathcal{X}) \rightarrow (P_1, \mathcal{X}_1)$ is a morphism of mixed Shimura data. It is now trivial to show that $(P_1, \mathcal{X}_1)$ possesses the desired universal property. **q.e.d.**

Remark. Let $(P_1, \mathcal{X}_1) := (P, \mathcal{X})/P_0$. By the proof above the map $\mathcal{X} \rightarrow \mathcal{X}_1$ is surjective if and only if the homomorphism $P(\mathbb{R}) \cdot U(\mathbb{C}) \rightarrow P_1(\mathbb{R}) \cdot U_1(\mathbb{C})$ is surjective. While this is not the case in general, it is so if P_0 is unipotent.

2.10 Example: Let $(P, \mathcal{X})$ be mixed Shimura data. Since P acts nontrivially on every subquotient of W , the commutator subgroup P^{der} contains the unipotent radical of P . Thus P/P^{der} is a torus. The property that $\mathcal{X} \cong h(\mathcal{X})$ is not invariant under forming quotients, as the following example shows. Let $(CS_{P_{2g, \mathbb{Q}}}, \mathcal{H}_{2g}) \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$ be the morphism defined in 2.8. It is easy to verify that it induces an isomorphism

$$(CS_{P_{2g, \mathbb{Q}}}, \mathcal{H}_{2g})/S_{P_{2g, \mathbb{Q}}} \xrightarrow{\sim} (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0).$$

In this case $\mathcal{H}_{2g} \cong h(\mathcal{H}_{2g})$, but $\mathcal{H}_0 \not\cong h(\mathcal{H}_0)$. This shows one of the advantages of introducing finite coverings in 2.1.

Now let $(P, \mathcal{X}, h)$ be mixed Shimura data and $\iota : h(\mathcal{X}) \hookrightarrow \text{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$ the inclusion. Then $(P, h(\mathcal{X}), \iota)$ is again mixed Shimura data, and there is a canonical morphism $(P, \mathcal{X}, h) \rightarrow (P, h(\mathcal{X}), \iota)$, or $(P, \mathcal{X}) \rightarrow (P, h(\mathcal{X}))$. We want to derive a relation between $(P, \mathcal{X})$, $(P, h(\mathcal{X}))$ and $(P, \mathcal{X})/P^{der}$. This relation will show that the generalization to finite covering does not yield essentially new objects.

2.11. Proposition: For any mixed Shimura data $(P, \mathcal{X})$ the canonical morphism

$$(P, \mathcal{X}) \longrightarrow (P, \mathcal{X})/P^{der} \times (P, h(\mathcal{X}))$$

is an embedding.

Proof. (Compare [D1] prop. 2.6) Let $T := P/P^{der}$, $\phi : P \rightarrow T$ the canonical homomorphism, and $(P, \mathcal{X})/P^{der} = (T, \mathcal{Y})$. Fix $x \in \mathcal{X}$, and define as abbreviation $A := \text{Stab}_{P(\mathbb{R}) \cdot U(\mathbb{C})}(x)$ and $B := \text{Cent}_{P(\mathbb{R}) \cdot U(\mathbb{C})}(h_x)$. The group A is a subgroup of finite index in B . As in the proof of 2.9 we identify $\mathcal{X}$ with $(P(\mathbb{R}) \cdot U(\mathbb{C}))/A$. Since $h(\mathcal{X}) \cong (P(\mathbb{R}) \cdot U(\mathbb{C}))/B$ we must prove that the map

$$(P(\mathbb{R}) \cdot U(\mathbb{C}))/A \rightarrow T(\mathbb{R})/\phi(A) \times (P(\mathbb{R}) \cdot U(\mathbb{C}))/B$$

is injective. This is equivalent to

$$A = \phi^{-1}(\phi(A)) \cap B.$$

Now $\phi^{-1}(\phi(A)) \cap B = A \cdot P^{der}(\mathbb{R}) \cdot U(\mathbb{C}) \cap B = A \cdot (P^{der}(\mathbb{R}) \cdot U(\mathbb{C}) \cap B)$, because A already lies in B . Since $P^{der}(\mathbb{R}) \cdot U(\mathbb{C}) \cap B = \text{Cent}_{P^{der}(\mathbb{R}) \cdot U(\mathbb{C})}(h_x)$ the assertion is equivalent to

$$\text{Cent}_{P^{der}(\mathbb{R}) \cdot U(\mathbb{C})}(h_x) \subset A.$$

Now the connected component of identity of $\text{Cent}_{P^{der}(\mathbb{R}) \cdot U(\mathbb{C})}(h_x)$ is contained in that of B , hence in A . Thus it is enough to show that $\text{Cent}_{P^{der}(\mathbb{R}) \cdot U(\mathbb{C})}(h_x)$ is connected.

By 1.17 (a) we may now assume $P = G$. Being the centralizer of a torus, the algebraic subgroup $\text{Cent}_{G^{der}}(h_x)$ is connected. By 2.1 (vi) its Lie algebra is the subspace of invariants in $(\text{Lie } G)_{\mathbb{R}}$ under $\text{Ad}_G(h(\sqrt{-1}))$, so $\text{Cent}_{G^{der}(\mathbb{R})}(h_x)$ is compact. Since every algebraically connected compact linear group is already topologically connected (see [BT] 14.3), we are done. **q.e.d.**

2.12. Corollary: Let $(P, \mathcal{X})$ be mixed Shimura data. Every connected component of $\mathcal{X}$ is mapped isomorphically to its image in $\text{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$. If $W = 1$, then every connected component of $\mathcal{X}$ is a hermitian symmetric domain.

Proof. The first assertion follows directly from the injection of $\mathcal{X} \hookrightarrow \mathcal{Y} \times h(\mathcal{X})$ with a finite set $\mathcal{Y}$. The second one follows from the first together with 1.14. **q.e.d.**

2.13. Covering by irreducible mixed Shimura data: Let $(P, \mathcal{X})$ be mixed Shimura data. Let P_1 be a normal subgroup of P , defined over $\mathbb{Q}$, such that for some ($\Leftrightarrow$ for all) $x \in \mathcal{X}$ the homomorphism h_x factors through $P_{1, \mathbb{C}}$. Then W lies in P_1 , and $P = P_1 \cdot \pi^{-1}(Z(G))$. Every $P_1(\mathbb{R}) \cdot U(\mathbb{C})$ -orbit $\mathcal{X}_1 \subset \mathcal{X}$ is the union of some connected components of $\mathcal{X}$, since for any $x \in \mathcal{X}_1$ there is an isomorphism

$$\begin{aligned} \mathcal{X}_1 &\cong P_1(\mathbb{R}) \cdot U(\mathbb{C}) / \text{Stab}_{P_1(\mathbb{R}) \cdot U(\mathbb{C})}(x) \\ &\cong (P_1(\mathbb{R}) \cdot U(\mathbb{C}) \cdot \text{Stab}_{P(\mathbb{R}) \cdot U(\mathbb{C})}(x)) / \text{Stab}_{P(\mathbb{R}) \cdot U(\mathbb{C})}(x), \end{aligned}$$

and by 1.17 $P_1(\mathbb{R}) \cdot U(\mathbb{C}) \cdot \text{Stab}_{P(\mathbb{R}) \cdot U(\mathbb{C})}(x)$ is a subgroup of finite index in $P(\mathbb{R}) \cdot U(\mathbb{C})$. With $\mathcal{X}_1 \ni x \mapsto h_x : \mathbb{S}_{\mathbb{C}} \rightarrow P_{1, \mathbb{C}} \subset P_{\mathbb{C}}$ we obtain mixed Shimura data $(P_1, \mathcal{X}_1)$ and a canonical embedding $(P_1, \mathcal{X}_1) \hookrightarrow (P, \mathcal{X})$.

Writing $\mathcal{X}$ as a disjoint union of $P_1(\mathbb{R}) \cdot U(\mathbb{C})$ -orbits $\mathcal{X}_i$ we get a “covering” of $(P, \mathcal{X})$ by these $(P_1, \mathcal{X}_i)$. In this sense every mixed Shimura data is covered by others, for which one can impose additional conditions.

In particular call mixed Shimura data $(P, \mathcal{X})$ irreducible if there does exist a proper normal subgroup of P , defined over $\mathbb{Q}$, through which the homomorphism h_x factors for any ($\Leftrightarrow$ for all) $x \in \mathcal{X}$. Any mixed Shimura data possesses a canonical covering by irreducible ones.

2.14. Proposition: Let $(P, \mathcal{X})$ be irreducible mixed Shimura data.

- (a) P operates on $\text{Lie } U$ through a scalar character $P \rightarrow G_{m, \mathbb{Q}}$.
- (b) There exists a character $P \rightarrow G_{m, \mathbb{Q}}$, a nondegenerate P -equivariant pairing $\Psi : (\text{Lie } V) \otimes (\text{Lie } V) \rightarrow \mathbb{Q}$, where P operates on $\mathbb{Q}$ through this character, and for every $x \in \mathcal{X}$ a homomorphism $\lambda_x \in \text{Hom}(\mathbb{Q}, \mathbb{Q}(1))$, such that for every $x \in \mathcal{X}$ $\lambda_x \circ \Psi$ is a polarization of the Hodge structure on $\text{Lie } V$ defined by h_x .

Proof. Without loss of generality we may replace $(P, \mathcal{X})$ by $(P, h(\mathcal{X}))/Z(P)$, then the conditions of 1.18 hold, in particular the weight is rational. Let $G_1 \subset G$ be the smallest normal subgroup, defined over $\mathbb{Q}$, such that for some ($\Leftrightarrow$ for all) $x \in \mathcal{X}$ the homomorphism $h|_{\mathbb{S}^1}$ factors through G_1 . Then by the irreducibility of $(P, \mathcal{X})$ we have $G = G_1 \cdot (h_x \circ w)(\mathbb{G}_{m, \mathbb{Q}})$. The assertion (a) follows from the fact that G_1 acts trivially on $\text{Lie } U$, and that the weight acts on $\text{Lie } U$ through a scalar character. For (b) let $M := \text{Lie } V$ and N , $\Psi : M \otimes M \rightarrow N$ as in 1.12. By the remarks in 1.13 we may choose N of dimension 1, and the rest follows. **q.e.d.**

2.15 Structure of the unipotent radical: Consider the short exact sequence $1 \rightarrow U \rightarrow W \rightarrow V \rightarrow 1$. Let $[\cdot, \cdot]$ be the Lie bracket in $\text{Lie } P$. Since $\text{Lie } W$ is mixed of weight ≤ -1 , the subalgebra $[\text{Lie } W, \text{Lie } W] \subset \text{Lie } W$ is mixed of weight ≤ -2 , hence contained in $\text{Lie } U$. Furthermore $[\text{Lie } W, \text{Lie } U]$ is mixed of weight ≤ -3 , hence 0, since in $\text{Lie } W$ only the weight -1 and -2 may occur. Thus W is a central extension of the two abelian unipotent groups V and U . The commutator on W induces an alternating form $\Psi : V \times V \rightarrow U$, which determines the extension up to isomorphism. In fact, given U , V and Ψ , one can reconstruct W for instance as follows. Let $W := U \times V$ as a variety over $\mathbb{Q}$, and define $(u, v) \cdot (u', v') := (u + u' + \frac{1}{2} \cdot \Psi(v, v'), v + v')$.

The operation of P by conjugation on U and on V factors through G , and the pairing in G -equivariant. Interpreting U and V as representations of G we have thus associated to P the data (G, V, U, Ψ) . Up to isomorphism P is uniquely determined by this data. In fact observe that the extension $1 \rightarrow W \rightarrow P \rightarrow G \rightarrow 1$ splits in any case, so only the operation of G on W is in question. If W is identified with $U \times V$ as above, then this operation may be defined by $g((u, v)) = (g(u), g(v))$. Since G is reductive and no irreducible representation of G occurs in both U and V , this is the unique possible action.

Next we describe how to go back and forth between mixed Shimura data for some group P and that for an extension of P by a unipotent group.

2.16. Partial inverse of 2.9: Consider mixed Shimura data $(P, \mathcal{X})$ and an extension $1 \rightarrow W_0 \rightarrow P_1 \rightarrow P \rightarrow 1$ of P by a unipotent group W_0 . We want to study all possible

mixed Shimura data $(P_1, \mathcal{X}_1)$ with an isomorphism $(P_1, \mathcal{X}_1)/W_0 \cong (P, \mathcal{X})$ extending the isomorphism $P_1/W_0 \cong P$. The conditions 2.1 (iv), (v) and (viii) give a necessary condition for the existence of such $(P_1, \mathcal{X}_1)$. Observe that the adjoint action of P_1 on every abelian subquotient of W_0 factors through P . Thus the necessary condition says that the Lie Algebra of every irreducible subquotient of $\text{Lie } W_0$ must be type $\{(-1, 0), (0, -1)\}$ or $\{(-1, -1)\}$ as representation of G , and the center of G acts on it through a torus that is an almost direct product of a $\mathbb{Q}$ -split torus with a torus of compact type defined over $\mathbb{Q}$.

2.17. Proposition: Let $(P, \mathcal{X})$, W_0 and P_1 be as in 2.16, and assume that $\text{Lie } W_0$ satisfies the condition given in 2.16. Then:

- (a) There exists mixed Shimura data $(P_1, \mathcal{X}_1)$ and a morphism $(P_1, \mathcal{X}_1) \rightarrow (P, \mathcal{X})$ that extends the given homomorphism $P_1 \rightarrow P$, such that $(P_1, \mathcal{X}_1)/W_0 \cong (P, \mathcal{X})$. They are uniquely determined up to isomorphism.
- (b) For every morphism $(P', \mathcal{X}') \rightarrow (P, \mathcal{X})$ and every factorization $P' \rightarrow P_1 \rightarrow P$ there exists exactly one extension $(P', \mathcal{X}') \rightarrow (P_1, \mathcal{X}_1) \rightarrow (P, \mathcal{X})$:

$$\begin{array}{ccc} P' & \longrightarrow & P_1 \\ & \searrow & \downarrow \\ & & P \end{array} \implies \begin{array}{ccc} (P', \mathcal{X}') & \longrightarrow & (P_1, \mathcal{X}_1) \\ & \searrow & \downarrow \\ & & (P, \mathcal{X}) \end{array}$$

Definition: We call $(P_1, \mathcal{X}_1)$ a unipotent extension of $(P, \mathcal{X})$. Let $U_0 := W_0 \cap U_1$ and $V_0 := W_0/U_0$, then we have a short exact sequence $1 \rightarrow U_0 \rightarrow W_0 \rightarrow V_0 \rightarrow 1$, which we call the weight decomposition of W_0 .

Proof. Fix a Levi decomposition $P_1 = W_1 \rtimes G$, this induces a Levi decomposition $P = W \rtimes G$. Let $x_0 \in \mathcal{X}$ such that h_{x_0} factors through $G \subset P$. By assumption h_{x_0} induces a rational Hodge structure of type $\{(-1, 0), (0, -1), (-1, -1)\}$ on every irreducible G -subspace of $\text{Lie } W_1$. Define $U_1 \subset W_1$ by $\text{Lie } U_1 = W_{-2}(\text{Lie } W_1)$, this is a subspace defined over $\mathbb{Q}$. As in 2.15 it follows that U_1 is contained in the center of W_1 , so it is a normal subgroup of P_1 . Now by definition the conditions 2.1 (iii)–(viii) hold for P_1 and h_{x_0} . Denote the given homomorphism $P_1 \rightarrow P$ by ϕ . If desired $(P_1, \mathcal{X}_1)$ exists, then by 1.17 for every $x_1 \in \mathcal{X}_1$ that is mapped to $x \in \mathcal{X}$ we have $\text{Stab}_{P_1(\mathbb{R}) \cdot U_1(\mathbb{C})}(x_1) \cong \text{Stab}_{P(\mathbb{R}) \cdot U(\mathbb{C})}(x)$. Thus $\mathcal{X}_1$ is isomorphic to

$$\mathcal{X}_1 := \{(x, k) \in \mathcal{X} \times \text{Hom}(\mathbb{S}_{\mathbb{C}}, P_{1, \mathbb{C}}) \text{ such that } h_x = \phi \circ k, \text{ and}$$

$$\pi' \circ k : \mathbb{S}_{\mathbb{C}} \rightarrow (P_1/U_1)_{\mathbb{C}} \text{ is already defined over } \mathbb{R}\}.$$

Taking this as definition of $\mathcal{X}_1$, together with the obvious map $h : \mathcal{X}_1 \rightarrow \text{Hom}(\mathbb{S}_{\mathbb{C}}, P_{1, \mathbb{C}})$, $(x, k) \mapsto k$, all the conditions of 2.1 are satisfied, so we get mixed Shimura data $(P_1, \mathcal{X}_1)$ and a morphism $(P_1, \mathcal{X}_1) \rightarrow (P, \mathcal{X})$, unique up to the isomorphism. Comparing this construction with that in 2.9 we find $(P_1, \mathcal{X}_1)/W_0 \cong (P, \mathcal{X})$. The universal property of $(P_1, \mathcal{X}_1)$ is obvious from its construction. **q.e.d.**

2.18. The structure of the fibres: Let $(P_1, \mathcal{X}_1) \rightarrow (P, \mathcal{X})$ be a unipotent extension. Then the map $\Psi : \mathcal{X}_1 \rightarrow \mathcal{X}$ is a $W_0(\mathbb{R}) \cdot (W_0 \cap U_1)(\mathbb{C})$ -torsor, in particular its fibres are

connected. Let $x_1 \in \mathcal{X}_1$ and $x := \Psi(x_1)$, then x_1 defines a Hodge structure on $\mathrm{Lie} W_0$, and as in the proof of 1.16 it follows that the map

$$\begin{array}{ccc} W_0(\mathbb{R}) \cdot (W_0 \cap U_1)(\mathbb{C}) & \xrightarrow{w_0 \mapsto w_0 \cdot x_1} & \Psi^{-1}(x) \\ \wr \uparrow \exp & & \uparrow \\ (\mathrm{Lie} W_0)_{\mathbb{R}} + \mathrm{Lie} W_0 \cap U_1(\mathbb{C}) & \cong & (\mathrm{Lie} W_0)_{\mathbb{C}} / F^0(\mathrm{Lie} W_0)_{\mathbb{C}} \end{array}$$

is an holomorphic isomorphism.

Consider the constant vector bundle $\mathcal{W}_0$ on $\mathcal{X}_1$ with fibre $(\mathrm{Lie} W_0)_{\mathbb{C}}$. The operation of $P_1(\mathbb{R}) \cdot U_1(\mathbb{C})$ on $(\mathrm{Lie} W_0)_{\mathbb{C}}$ makes it into an equivariant vector bundle. There exists a smooth subbundle $\mathcal{F}^0 \mathcal{W}_0$, whose fibre in $x_1 \in \mathcal{X}_1$ equals $F^0(\mathrm{Lie} W_0)_{\mathbb{C}}$ for the Hodge structure defined by x_1 . This subbundle is invariant under the operation of $W_0(\mathbb{R}) \cdot (W_0 \cap U_1)(\mathbb{C})$ if and only if W_0 is abelian. Then and only then $\mathcal{F}^0 \mathcal{W}_0 \subset \mathcal{W}_0$ are pullbacks of vector bundles on $\mathcal{X}$.

Suppose that the short exact sequence $1 \rightarrow W_0 \rightarrow P_1 \rightarrow P \rightarrow 1$ splits. By 2.17 (b) any splitting yields a section $\varepsilon : (P, \mathcal{X}) \rightarrow (P_1, \mathcal{X}_1)$. By pullback we obtain vector bundles on $\mathcal{X}$, and the above isomorphism between a fibre $\Psi^{-1}(x)$ and $(\mathrm{Lie} W_0)_{\mathbb{C}} / F^0(\mathrm{Lie} W_0)_{\mathbb{C}}$ yields an identification of $\mathcal{X}_1 \rightarrow \mathcal{X}$ with the holomorphic complex vector bundle $\varepsilon^*(\mathcal{W}_0 / \mathcal{F}^0 \mathcal{W}_0)$.

Let us apply this to arbitrary mixed Shimura data $(P_1, \mathcal{X}_1)$ with $W_0 = W_1$. Then the sequence splits, and with 2.12 we have proved:

2.19. Proposition: Let $(P, \mathcal{X})$ be mixed Shimura data. Then every connected component of $\mathcal{X}$ is a holomorphic complex vector bundle on a hermitian symmetric domain.

2.20. Fibre product: Let $(P_1, \mathcal{X}_1) \rightarrow (P, \mathcal{X})$ be a unipotent extension and $(P_2, \mathcal{X}_2) \rightarrow (P, \mathcal{X})$ an arbitrary morphism. Let $P_{12} := P_1 \times_P P_2$ and $\mathcal{X}_{12} := \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_2$ be the fibre products. Since $P_1 \rightarrow P$ is an extension by a unipotent group, P_{12} is a connected subgroup of $P_1 \times P_2$ (this would not be true for an arbitrary morphism $(P_1, \mathcal{X}_1) \rightarrow (P, \mathcal{X})$). Define h_{12} by the commutative diagram

$$\begin{array}{ccc} \mathcal{X}_{12} & \xrightarrow{h_{12}} & \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{12, \mathbb{C}}) \\ \downarrow & & \downarrow \\ \mathcal{X}_1 \times \mathcal{X}_2 & \xrightarrow{h_1 \times h_2} & \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{1, \mathbb{C}} \times P_{2, \mathbb{C}}). \end{array}$$

Since $P_{12} \rightarrow P_2$ is a unipotent extension, it follows at once $(P_{12}, \mathcal{X}_{12})$ is mixed Shimura data. We have a commutative diagram

$$\begin{array}{ccc} (P_{12}, \mathcal{X}_{12}) & \longrightarrow & (P_1, \mathcal{X}_1) \\ \downarrow & & \downarrow \\ (P_2, \mathcal{X}_2) & \longrightarrow & (P, \mathcal{X}) \end{array}$$

Furthermore $(P_{12}, \mathcal{X}_{12})$ possesses the universal property of the fibre product in the category of mixed Shimura data; this is also a trivial consequence of the construction. We write $(P_1, \mathcal{X}_1) \times_{(P, \mathcal{X})} (P_2, \mathcal{X}_2)$ for $(P_{12}, \mathcal{X}_{12})$.

2.21. Operations on the fibre for split unipotent extensions: Consider mixed Shimura data $(P, \mathcal{X})$ and a representation W_0 of P that satisfies the condition in 2.16. We consider W_0 as an (abelian) unipotent group over $\mathbb{Q}$ and define $P_1 := W_0 \rtimes P$ with the

multiplication $(w, p) \cdot (w', p') = (w \cdot p(w'), p \cdot p')$. By 2.17 we get a canonical morphism $(P_1, \mathcal{X}_1) \rightarrow (P, \mathcal{X})$ together with a section $(P, \mathcal{X}) \rightarrow (P_1, \mathcal{X}_1)$. For two such representations W_0 and W'_0 and a P -equivariant homomorphism $f: W_0 \rightarrow W'_0$ we define a homomorphism $P_1 = W_0 \rtimes P \xrightarrow{f \times id} W'_0 \rtimes P = P'_1$. By 2.17 we get a canonical morphism $(P_1, \mathcal{X}_1) \rightarrow (P'_1, \mathcal{X}'_1)$ of the corresponding mixed Shimura data. Thus for fixed $(P, \mathcal{X})$ we have a functor from the category of all dimensional representations of P over $\mathbb{Q}$ that satisfy 2.16 to the category of all mixed Shimura data “over” $(P, \mathcal{X})$ together with a section. It is easily seen that this functor maps direct sums to fibre products (2.20).

For instance the addition $W_0 \otimes W_0 \rightarrow W_0$ yields a morphism $m: (P_1, \mathcal{X}_1) \times_{(P, \mathcal{X})} (P_1, \mathcal{X}_1) \rightarrow (P_1, \mathcal{X}_1)$, which on the fibre of $\mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_1 \rightarrow \mathcal{X}$ corresponds to the addition on the corresponding vector bundle of 2.18. Thus $(P_1, \mathcal{X}_1)$ becomes a group object in the category of all Shimura data over $(P, \mathcal{X})$. The endomorphism algebra $\text{End}_P(W_0)$ also operates on $(P_1, \mathcal{X}_1)$ over $(P, \mathcal{X})$. Both these operations are compatible with the “zero” section $\varepsilon: (P, \mathcal{X}) \rightarrow (P_1, \mathcal{X}_1)$. There is also the following operation: For any $w_0 \in W_0(\mathbb{Q})$ the conjugation by w_0 induces an isomorphism

$$\begin{array}{ccc} \text{int}(w_0): (P_1, \mathcal{X}_1) & \xrightarrow{(\text{int}(w_0), (w_0 \cdot))} & (P_1, \mathcal{X}_1) \\ & \searrow & \swarrow \\ & (P, \mathcal{X}) & \end{array}$$

that transforms the zero section ε into $\text{int}(w_0) \circ \varepsilon$.

2.22. Operations on the fibre in an abelian unipotent extension: Let $\phi: (P_1, \mathcal{X}_1) \rightarrow (P, \mathcal{X})$ be a unipotent extension with W_0 abelian. Here we do not assume that the extension splits. Still the operation of P_1 on W_0 factors through P , and the vector bundles considered in 2.18 come from canonical vector bundles on $\mathcal{X}$. So the operation of $W_0(\mathbb{R}) \cdot (W_0 \cap U_1)(\mathbb{C})$ on $\mathcal{X}_1$ makes the bundle $\mathcal{X}_1 \rightarrow \mathcal{X}$ into a holomorphic torsor under the additive group of this vector bundle $W_0/\mathcal{F}^0 W_0$ on $\mathcal{X}$.

We can express this situation functorially as follows. For more generality let $(P, \mathcal{X}) \rightarrow (P_*, \mathcal{X}_*)$ be a morphism such that the operation of P on W_0 factors through P_* . Let $(P', \mathcal{X}') \rightarrow (P_*, \mathcal{X}_*)$ be the (splitting) unipotent extension with $P' = W_0 \rtimes P_*$. Let $(P'_1, \mathcal{X}'_1)$ be the fibre product $(P', \mathcal{X}') \times_{(P_*, \mathcal{X}_*)} (P_1, \mathcal{X}_1)$, then $P'_1 = P' \times_{P_*} P_1 = (W_0 \rtimes P_*) \times_{P_*} P_1 \cong W_0 \rtimes P_1$.

The map

$$P'_1 \cong W_0 \rtimes P_1 \rightarrow P_1, (w, p_1) \mapsto w \cdot p_1$$

is a homomorphism, so by 2.17 (b) it extends canonically to a morphism

$$\mu: (P', \mathcal{X}') \times_{(P_*, \mathcal{X}_*)} (P_1, \mathcal{X}_1) \rightarrow (P_1, \mathcal{X}_1).$$

2.23. Proposition: This morphism defines an operation of the group object $(P', \mathcal{X}') \rightarrow (P_*, \mathcal{X}_*)$ on $(P_1, \mathcal{X}_1)$ in the category of all mixed Shimura data over $(P_*, \mathcal{X}_*)$. Through this $(P_1, \mathcal{X}_1)$ becomes a $(P', \mathcal{X}') \rightarrow (P_*, \mathcal{X}_*)$ -torsor over $(P, \mathcal{X})$.

Proof. The assertion is equivalent to the conjunction of the following two:

(a) The diagram

$$\begin{array}{ccc}
(P', \mathcal{X}') \times_{(P_*, \mathcal{X}_*)} (P', \mathcal{X}') \times_{(P_*, \mathcal{X}_*)} (P_1, \mathcal{X}_1) & \xrightarrow{id \times \mu} & (P', \mathcal{X}') \times_{(P_*, \mathcal{X}_*)} (P_1, \mathcal{X}_1) \\
\downarrow m \times id & & \downarrow \mu \\
(P', \mathcal{X}') \times_{(P_*, \mathcal{X}_*)} (P_1, \mathcal{X}_1) & \xrightarrow{\mu} & (P_1, \mathcal{X}_1)
\end{array}$$

is commutative.

(b) The morphism

$$(P', \mathcal{X}') \times_{(P_*, \mathcal{X}_*)} (P_1, \mathcal{X}_1) \xrightarrow{(pr_2, \mu)} (P_1, \mathcal{X}_1) \times_{(P, \mathcal{X})} (P_1, \mathcal{X}_1)$$

is well-defined and an isomorphism.

Since we are only dealing with unipotent extensions of $(P, \mathcal{X})$, it suffices to prove these assertions for the corresponding homomorphisms $P' \times_{P_*} P_1 \rightarrow P_1$, etc. Consider the diagram in (a). Like for $P' \times_{P_*} P_1 \cong W_0 \rtimes P_1$ there is a similar canonical isomorphism $P' \times_{P_*} P' \times_{P_*} P_1 \cong W_0 \rtimes W_0 \rtimes P_1$. One finds that an element (w, w', p_1) is mapped to $w \cdot w' \cdot p_1 \in P_1$ through both corners of the diagram, whence (a). (b) is equivalent to the assertion that the diagram

$$\begin{array}{ccc}
(P'_1, \mathcal{X}'_1) & \xrightarrow{\mu} & (P_1, \mathcal{X}_1) \\
pr_2 \downarrow & & \downarrow \phi \\
(P_1, \mathcal{X}_1) & \xrightarrow{\phi} & (P, \mathcal{X})
\end{array}$$

is cartesian. This is implied by the fact that the homomorphism $W_0 \rtimes P_1 \rightarrow P_1 \times_P P_1$, $(w, p_1) \mapsto (w \cdot p_1, p_1)$ is an isomorphism. **q.e.d.**

2.24. Example: Let $(G_0, \mathcal{H}_0)$ be as in 2.8. let $U_0 = \mathbb{G}_a = \mathbb{Q}$ with the standard operation of $G_0 = \mathbb{G}_{m, \mathbb{Q}}$. Clearly U_0 is pure of type $(-1, -1)$ as representation of G_0 , and G_0 is itself $\mathbb{Q}$ -split, so the construction 2.17 yields a unipotent extension $(P_0, \mathcal{X}_0)$ of $(G_0, \mathcal{H}_0)$ with $P_0 = U_0 \rtimes G_0$. We can also describe it as follows. Identify P_0 with the subgroup of all matrices of the form $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ in $GL_{2, \mathbb{Q}}$, $h(\mathcal{X}_0)$ with the set of all homomorphisms $\mathbb{S}_{\mathbb{C}} \rightarrow P_{0, \mathbb{C}}$ of the form $\mathbb{S}(\mathbb{C}) \supset \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \ni z \mapsto \begin{pmatrix} z\bar{z} & * \\ 0 & 1 \end{pmatrix}$, and $\mathcal{X}_0$ with $h(\mathcal{X}_0) \times \mathcal{H}_0$.

Every connected component of $\mathcal{X}_0$ is isomorphic to $\mathbb{C}$, and the group structure defined by 2.21 corresponds to that of the additive group $\mathbb{C}$. As a variant we can also consider the Shimura data $P_0, h(\mathcal{X}_0)$, here $h(\mathcal{X}_0)$ is connected, and $(P_0, h(\mathcal{X}_0))$ is a group object over $(\mathbb{G}_{m, \mathbb{Q}}, \{k\}) = (G_0, h(\mathcal{H}_0))$ (see 2.8). Note that for every $((P_0, \mathcal{X}_0) \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0))$ -torsor $(P', \mathcal{X}') \rightarrow (P, \mathcal{X})$ we have a canonical isomorphism between U_0 and the kernel of the homomorphism $\pi_0 : P' \rightarrow P$.

2.25. Example: Let $(CS_{P_{2g, \mathbb{Q}}}, \mathcal{H}_{2g})$ be the Shimura data defined in 2.7, and V_{2g} the standard representation of $CS_{P_{2g, \mathbb{Q}}}$. The construction 2.17 yields mixed Shimura data $(V_{2g} \rtimes CS_{P_{2g, \mathbb{Q}}}, \mathcal{H}'_{2g})$. Let $U_{2g} := \mathbb{G}_a = \mathbb{Q}$ with the operation of $CS_{P_{2g, \mathbb{Q}}}$ by the multiplier. The given alternating form $\Psi : V_{2g} \times V_{2g} \rightarrow \mathbb{Q} = U_{2g}$ is $CS_{P_{2g, \mathbb{Q}}}$ -equivariant. Let $1 \rightarrow U_{2g} \rightarrow W_{2g} \rightarrow V_{2g} \rightarrow 1$ be the central extension of unipotent groups defined by Ψ , and let $P_{2g} := W_{2g} \rtimes CS_{P_{2g, \mathbb{Q}}}$. By 2.17 we obtain mixed Shimura data $(P_{2g}, \mathcal{X}_{2g})$ and morphisms

$$(P_{2g}, \mathcal{X}_{2g}) \rightarrow (V_{2g} \rtimes CS_{P_{2g, \mathbb{Q}}}, \mathcal{H}'_{2g}) \rightarrow (CS_{P_{2g, \mathbb{Q}}}, \mathcal{H}_{2g}).$$

The unipotent extension $(V_{2g} \rtimes CS_{P_{2g}, \mathbb{Q}}, \mathcal{H}'_{2g}) \rightarrow (CS_{P_{2g}, \mathbb{Q}}, \mathcal{H}_{2g})$ is an example for a group object. Let $(CS_{P_{2g}, \mathbb{Q}}, \mathcal{H}_{2g}) \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$ be the morphism defined in 2.8. Since the operation of $CS_{P_{2g}, \mathbb{Q}}$ on U_{2g} factors through $\mathbb{G}_{m, \mathbb{Q}}$, every isomorphism $U_{2g} \cong U_0$ gives a structure on $(P_{2g}, \mathcal{X}_{2g})$ as a $((P_0, \mathcal{X}_0) \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0))$ -torsor over $(V_{2g} \rtimes CS_{P_{2g}, \mathbb{Q}}, \mathcal{H}'_{2g})$.

2.26. Reduction lemma: Let $(P, \mathcal{X})$ be irreducible mixed Shimura data. Then there exists pure Shimura data $(T, \mathcal{Y})$ and $(G, \mathcal{H})$, where T is a torus and $h : \mathcal{H} \hookrightarrow \text{Hom}(\mathbb{S}, G_{\mathbb{C}})$ injective, and a number $n \geq 0$ such that

(a) If $V = 1$, then there exists an embedding

$$(P, \mathcal{X}) \hookrightarrow (T, \mathcal{Y}) \times (G, \mathcal{H}) \times \prod_{i=1}^n (P_0, h(\mathcal{X}_0)),$$

where $(P_0, h(\mathcal{X}_0))$ is the mixed Shimura data defined in 2.24.

(b) If $2g = \dim(V) > 0$, then there exists a morphism $(P, \mathcal{X}) \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$, a $((P_0, \mathcal{X}_0) \rightarrow \mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$ -torsor $(P', \mathcal{X}') \rightarrow (P, \mathcal{X})$ and an embedding

$$(P', \mathcal{X}') \hookrightarrow (T, \mathcal{Y}) \times (G, \mathcal{H}) \times \prod_{i=1}^n (P_{2g}, \mathcal{X}_{2g}),$$

where $(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$, $(P_0, \mathcal{X}_0)$, and $(P_{2g}, \mathcal{X}_{2g})$ are mixed Shimura data defined in 2.8, 2.24 and 2.25 respectively.

Proof. (a) By 2.14 (a) P operates through scalars on U . Let $P_1 \subset P$ be the kernel of this operation. Let $\lambda_1, \dots, \lambda_n$ be a basis for $\text{Hom}(U, \mathbb{Q})$. For every i the quotient $P/P_1 \cdot \text{Ker}(\lambda_i)$ is isomorphic to the group P_0 defined in 2.24. One sees easily that any such isomorphism extends uniquely to a morphism $(P, \mathcal{X}) \rightarrow (P_0, h(\mathcal{X}_0))$. By construction the direct product of these morphisms

$$(P, \mathcal{X}) \rightarrow \prod_{i=1}^n (P_0, h(\mathcal{X}_0))$$

is injective on U . Thus the corresponding morphism

$$(P, \mathcal{X}) \rightarrow (P, \mathcal{X})/U \times \prod_{i=1}^n (P_0, h(\mathcal{X}_0))$$

is an embedding. Together with the embedding $(P, \mathcal{X})/U \hookrightarrow (T, \mathcal{Y}) \times (G, \mathcal{H}) := (T, \mathcal{Y}) \times (P, h(\mathcal{X}))/U$ from 2.11 we get the desired result.

(b) Let $\Psi : V \times V \rightarrow U$ be pairing induced from the commutator. Note that V is pure of weight -1 , so according to 2.14 (b) we can fix a P -equivariant pairing $\Phi : V \times V \rightarrow U_0 := \mathbb{G}_{a, \mathbb{Q}} = \mathbb{Q}$ with a suitable operation of P on $\mathbb{Q}$, such that for all $x \in \mathcal{X}$ there exists $\lambda \in \text{Hom}(\mathbb{Q}, \mathbb{Q}(1))$, such that $\lambda \circ \Phi$ is a polarization of the Hodge structure on V defined by x . Define $U' := U \oplus \mathbb{Q}$ and $\Psi' := (\Psi, \Phi) : V \times V \rightarrow U'$. By 2.15 the triple (U', V, Ψ') defines a central extension $1 \rightarrow U' \rightarrow W' \rightarrow V \rightarrow 1$, and W'/U_0 is isomorphic to W . Fixing such an isomorphism we let $P' := W' \rtimes G$, this is then a unipotent extension of P by $U_0 = \mathbb{Q}$,

so we obtain a unipotent extension $(P', \mathcal{X}') \rightarrow (P, \mathcal{X})$. Let $\phi : P \rightarrow \mathbb{G}_{m, \mathbb{Q}}$ be the character through which P operates on U_0 . For every $x \in \mathcal{X}$ let $\lambda_x \in \mathcal{H}_0$ be the unique element, such that $\lambda_x \circ \Phi$ is a polarization of the Hodge structure on V defined by x . Then (ϕ, Ψ) is a morphism $(P, \mathcal{X}) \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$, and $(P', \mathcal{X}') \rightarrow (P, \mathcal{X})$ is a $((P_0, \mathcal{X}_0) \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0))$ -torsor in a canonical way.

Since the irreducibility of $(P, \mathcal{X})$ implies that of $(P', \mathcal{X}')$, P' acts through scalars on U' . Consider the set of all $\mu \in \text{Hom}(U', \mathbb{Q})$ such that for all $x \in \mathcal{X}'$ there exists $\lambda \in \text{Hom}(\mathbb{Q}, \mathbb{Q}(1))$, so that the pairing $\lambda \circ \mu \circ \Psi'$ is a polarization of V . This set is open in the usual topology, and nonempty since it contains $\text{pr}_2 : U' \rightarrow U_0 = \mathbb{Q}$. Therefore it contains a basis $\mu_1 \dots \mu_n$ of $\text{Hom}(U', \mathbb{Q})$. Let $2g = \dim(V)$, and let $(P_{2g}, \mathcal{X}_{2g})$, W_{2g} , and V_{2g} be as in 2.25. For every $i = 1, \dots, n$ there exists an isomorphism $V \cong V_{2g}$ such that the pairing $\mu_i \circ \Psi'$ corresponds to the pairing on V_{2g} fixed in 2.7. Thus there is a homomorphism $W' \rightarrow W_{2g}$ that induces an isomorphism $V \cong V_{2g}$ and whose kernel on U' is equal to the kernel on μ_j . By the definition of $(P_{2g}, \mathcal{X}_{2g})$ this gives a morphism $(P', \mathcal{X}') \rightarrow (P_{2g}, \mathcal{X}_{2g})$. Taking together these n morphisms we obtain a morphism of Shimura data

$$(P', \mathcal{X}') \rightarrow \prod_{i=1}^n (P_{2g}, \mathcal{X}_{2g}),$$

which by construction is injective on the unipotent radical W' . Hence the corresponding morphism

$$(P', \mathcal{X}') \rightarrow (P', \mathcal{X}')/W' \times \prod_{i=1}^n (P_{2g}, \mathcal{X}_{2g})$$

is an embedding. Together with the embedding $(P', \mathcal{X}')/W' \hookrightarrow (T, \mathcal{Y}) \times (G, \mathcal{H}) := (T, \mathcal{Y}) \times (P', h(\mathcal{X}'))/W'$ from 2.11 we obtain the desired result. **q.e.d.**

Chapter 3

Mixed Shimura varieties

In this chapter we begin with the definition of mixed Shimura varieties, and show (in 3.3) that these objects are nice complex spaces. Then we go on to study the morphisms and constructions induced by those considered in chapter 2. Again most of the chapter (starting from 3.10) is devoted to the study of the unipotent fibres. It is remarkable that, under very mild conditions, the group objects and actions of 2.21—23 yield the same structures for the associated mixed Shimura varieties (see 3.10—12). We prove that the unipotent fibre are torus-torsors over abelian varieties (3.14 and 3.22 (a)). In an important special case this torus is canonically isomorphic to $\mathbb{C}^\times$ (3.16), in which case we are interested in the associated line bundle (3.17). In 3.21 we give an ampleness criterion for this line bundle.

3.1. Definition: Let $(P, \mathcal{X})$ be mixed Shimura data, and K_f an open compact subgroup of $P(\mathbb{A}_f)$. We define the corresponding mixed Shimura variety, as

$$M^{K_f}(P, \mathcal{X})(\mathbb{C}) := P(\mathbb{Q}) \backslash \mathcal{X} \times (P(\mathbb{A}_f)/K_f),$$

where $P(\mathbb{Q})$ acts on both factors from the left hand side. If P is reductive, we also speak of pure Shimura variety, or just Shimura variety. The notation is chosen to be compatible with later chapters.

3.2. Elementary properties: Since K_f is open in $P(\mathbb{A}_f)$, the set $P(\mathbb{A}_f)/K_f$ is a discrete topological space. We endow $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ with the quotient topology. The set of connected components $\pi_0(M^{K_f}(P, \mathcal{X})(\mathbb{C}))$ is a quotient of $\pi_0(P(\mathbb{Q}) \backslash P(\mathbb{R}) \cdot U(\mathbb{C}) \times P(\mathbb{A}_f)/K_f) = \pi_0(P(\mathbb{Q}) \backslash P(\mathbb{A})/K_f)$, hence a finite discrete set (see [G] thm. 5). For every system of representatives $\{(\mathcal{X}^0, p_f)\}$, where $\mathcal{X}^0$ is a connected component of $\mathcal{X}$ and $p_f \in P(\mathbb{A}_f)$, we have

$$M^{K_f}(P, \mathcal{X})(\mathbb{C}) \cong \coprod \Gamma(p_f) \backslash \mathcal{X}^0,$$

where $\Gamma(p_f) = P(\mathbb{Q}) \cap p_f \cdot K_f \cdot p_f^{-1}$ is a congruence subgroup of $P(\mathbb{Q})$ depending on p_f and K_f . If P is a torus, then $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ is a finite set.

3.3. Proposition: (a) For any pair $(\mathcal{X}^0, p_f)$ the group $\Gamma(p_f)$ operates properly discontinuously on $\mathcal{X}^0$. There is a canonical structure of a normal complex space on $M^{K_f}(P, \mathcal{X})(\mathbb{C})$, whose singularities are at most quotient singularities by finite groups.

(b) If K_f is neat, then $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ is a complex manifold.

Proof. (a) By 2.1 (viii) $P/P^{der} \cdot Z(P)$ is an almost direct product of a $\mathbb{Q}$ -split torus with a torus of compact type defined over $\mathbb{Q}$. Thus the image of $\Gamma(p_f)$ in $(P/P^{der} \cdot Z(P))(\mathbb{Q})$ is finite, so it suffices to prove the assertion for $\Gamma := \Gamma(p_f) \cap (P^{der} \cdot Z(P))(\mathbb{Q})$. Now $Z(P)(\mathbb{R})^0$ acts trivially on $\mathcal{X}^0$, so in view of the convention 0.4 we may replace $(P, \mathcal{X})$ by $(P, \mathcal{X})/Z(P)$. Then Γ is an arithmetic subgroup of $P^{der}(\mathbb{Q})$. In particular it is a discrete subgroup of $P^{der}(\mathbb{R})$, so it operates properly discontinuously on $P_\infty := P^{der}(\mathbb{R})^0 \cdot U(\mathbb{C})$. For any $x \in \mathcal{X}^0$ we have a homomorphism $\mathcal{X}^0 \cong P_\infty/K_\infty$ with $K_\infty := \text{Stab}_{P_\infty}(x)$. But by 1.17 (a) and 2.1 (vi) K_∞ is compact, so Γ also operates properly discontinuously on $\mathcal{X}^0$, as desired. The complex structure on $\Gamma(p_f) \backslash \mathcal{X}^0$ comes from the canonical $P(\mathbb{R}) \cdot U(\mathbb{C})$ -invariant complex structure on $\mathcal{X}$.

(b) If K_f is neat, then so is $\Gamma(p_f)$, hence it is already contained in $(P^{der} \cdot Z(P))(\mathbb{Q})$. Moreover its image in $(P/Z(P))(\mathbb{Q})$ is again neat, so in particular torsion free. By the argument in (a) the stabilizer of any point $x \in \mathcal{X}^0$ in this image is finite, so it must be trivial. Thus $\Gamma(p_f) \backslash \mathcal{X}^0$ is a complex manifold, as desired. **q.e.d.**

3.4. Morphisms on $M^{K_f}(P, \mathcal{X})(\mathbb{C})$: (a) Let $(P, \mathcal{X})$ be mixed Shimura data $p_f \in P(\mathbb{A}_f)$, and $K_f, K'_f \subset P(\mathbb{A}_f)$ open compact subgroups, such that $K'_f \subset p_f \cdot K_f \cdot p_f^{-1}$. Then the map

$$[\cdot p_f] = [\cdot p_f]_{K'_f, K_f} : M^{K_f}(P, \mathcal{X})(\mathbb{C}) \rightarrow (M^{K_f}(P, \mathcal{X})(\mathbb{C})),$$

$$[(x, p'_f)] \mapsto [(x, p'_f \cdot p_f)]$$

is well-defined. It is holomorphic, closed, finite, and surjective. For $K'_f = p_f \cdot K_f \cdot p_f^{-1}$ it is an isomorphism.

(b) Let $\phi : (P_1, \mathcal{X}_1) \rightarrow (P_2, \mathcal{X}_2)$ be a morphism of mixed Shimura data, and $K_f^1 \subset P_1(\mathbb{A}_f)$ and $K_f^2 \subset P_2(\mathbb{A}_f)$ open compact subgroups. If $\phi(K_f^1) \subset K_f^2$, then the canonical map

$$[\phi] = [\phi]_{K_f^1, K_f^2} : M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C}) \rightarrow M^{K_f^2}(P_2, \mathcal{X}_2)(\mathbb{C}),$$

$$[(x_1, p_f^1)] \mapsto [(\phi(x_1), \phi(p_f^1))]$$

is well-defined. By 2.4 and 3.3 it is holomorphic. If ϕ is an isomorphism and $\phi(K_f^1) = K_f^2$, then $[\phi]$ is an isomorphism.

3.5. Functoriality: The morphisms in 3.4 have the following functorial properties:

(a) Let $(P, \mathcal{X})$ be mixed Shimura data, $p_f, p'_f \in P(\mathbb{A}_f)$, and $K_f, K'_f, K''_f \subset P(\mathbb{A}_f)$ open compact subgroups, such that $K'_f \subset p_f \cdot K_f \cdot p_f^{-1}$ and $K''_f \subset p'_f \cdot K'_f \cdot (p'_f)^{-1}$. Then $K''_f \subset (p'_f \cdot p_f) \cdot K_f \cdot (p'_f \cdot p_f)^{-1}$ and

$$[\cdot (p'_f \cdot p_f)]_{K''_f, K_f} = [\cdot p_f]_{K'_f, K_f} \circ [\cdot p'_f]_{K''_f, K'_f}.$$

(b) Let $\phi_1 : (P_1, \mathcal{X}_1) \rightarrow (P_2, \mathcal{X}_2)$ and $\phi_2 : (P_2, \mathcal{X}_2) \rightarrow (P_3, \mathcal{X}_3)$ be morphisms and $K_f^1 \subset P_1(\mathbb{A}_f)$, $K_f^2 \subset P_2(\mathbb{A}_f)$, and $K_f^3 \subset P_3(\mathbb{A}_f)$, open compact subgroups, such that $\phi_1(K_f^1) \subset K_f^2$ and $\phi_2(K_f^2) \subset K_f^3$. Then $(\phi_2 \circ \phi_1)(K_f^1) \subset K_f^3$ and

$$[\phi_2 \circ \phi_1]_{K_f^1, K_f^3} = [\phi_2]_{K_f^2, K_f^3} \circ [\phi_1]_{K_f^1, K_f^2}.$$

(c) Let $\phi : (P_1, \mathcal{X}_1) \rightarrow (P_2, \mathcal{X}_2)$ be a morphism, $p_f \in P_1(\mathbb{A}_f)$, and $K_f^1, K_f^{1'} \subset P_1(\mathbb{A}_f)$ and $K_f^2, K_f^{2'} \subset P_2(\mathbb{A}_f)$ open compact subgroups, such that $\phi(K_f^1) \subset K_f^2$, $\phi(K_f^{1'}) \subset K_f^{2'}$, $K_f^1 \subset p_f \cdot K_f^1 \cdot p_f^{-1}$ and $K_f^{2'} \subset \phi(p_f) \cdot K_f^2 \cdot \phi(p_f)^{-1}$. Then

$$[\phi]_{K_f^1, K_f^2} \circ [\cdot p_f]_{K_f^{1'}, K_f^1} = [\cdot \phi(p_f)]_{K_f^{2'}, K_f^2} \circ [\phi]_{K_f^{1'}, K_f^{2'}}.$$

(d) Let $(P, \mathcal{X})$ be mixed Shimura data, K_f an open compact subgroup of $P(\mathbb{A}_f)$, and $k_f \in K_f$. Then $[\cdot k_f]_{K_f, K_f}$ is the identity on $M^{K_f}(P, \mathcal{X})(\mathbb{C})$.

(e) Let $(P, \mathcal{X})$ be mixed Shimura data, $p \in P(\mathbb{Q})$ and $K_f, K_f' \subset P(\mathbb{A}_f)$ open compact subgroups, such that $K_f' \subset p \cdot K_f \cdot p^{-1}$. Denote by $\text{int}(p^{-1}) : (P, \mathcal{X}) \rightarrow (P, \mathcal{X})$ the morphism defined by $P \rightarrow P$, $p' \mapsto p^{-1} \cdot p' \cdot p$ and $\mathcal{X} \rightarrow \mathcal{X}$, $x \mapsto p^{-1} \cdot x$. Then the map $[\text{int}(p^{-1})]_{K_f', K_f}$ is defined and equal to $[\cdot p]_{K_f', K_f}$.

3.6. The projective system: For $p_f = 1$ and $K_f' \subset K_f$ the maps of 3.4 give the canonical maps

$$[id] = [id]_{K_f', K_f} : M^{K_f}(P, \mathcal{X})(\mathbb{C}) \rightarrow M^{K_f}(P, \mathcal{X})(\mathbb{C}),$$

$$[(x, p_f')] \mapsto [(x, p_f')].$$

With these maps the $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ form a projective system of complex spaces, indexed by the system of all open compact subgroups of $P(\mathbb{A}_f)$. The morphisms 3.4 (a) define an operation of $P(\mathbb{A}_f)$ from the right hand side on this projective system. A morphism of mixed Shimura data induces an equivariant morphism of projective systems with this right- $P(\mathbb{A}_f)$ -operation.

3.7. Lemma: Let $(P, \mathcal{X})$ be mixed Shimura data, let Γ_Z be an arithmetic subgroup of $\{z \in Z(P)(\mathbb{Q}) \mid z|_{\mathcal{X}} = id\}$, and $\bar{\Gamma}_Z$ be its closure in $P(\mathbb{A}_f)$.

(a) For any open compact subgroup $K_f \subset P(\mathbb{A}_f)$ the projection $M^{K_f}(P, \mathcal{X})(\mathbb{C}) \rightarrow M^{\bar{\Gamma}_Z \cdot K_f}(P, \mathcal{X})(\mathbb{C})$ is an isomorphism.

(b) Let A be a compact subgroup of $P(\mathbb{A}_f)$. Then the obvious map

$$P(\mathbb{Q}) \backslash \mathcal{X} \times (P(\mathbb{A}_f) / \bar{\Gamma}_Z \cdot A) \longrightarrow \varprojlim M^{K_f}(P, \mathcal{X})(\mathbb{C})$$

is bijective, where the limit is taken over all open compact subgroups $K_f \subset P(\mathbb{A}_f)$ that contain A .

Proof. (Compare [D1] 1.15.2)(a) follows from the isomorphisms

$$P(\mathbb{Q}) \backslash \mathcal{X} \times (P(\mathbb{A}_f) / K_f) \cong P(\mathbb{Q}) \backslash \mathcal{X} \times (\Gamma_Z \backslash P(\mathbb{A}_f) / K_f)$$

and

$$\Gamma_Z \backslash P(\mathbb{A}_f) / K_f \cong P(\mathbb{A}_f) / \Gamma_Z \cdot K_f \cong P(\mathbb{A}_f) / \bar{\Gamma}_Z \cdot K_f.$$

For (b) let $\{K_f^\alpha\}$ be the system of all open subgroups that contain A . An element of the inverse limit is a system of elements $[(x_\alpha, p_f^\alpha)] \in M^{K_f^\alpha}(P, \mathcal{X})(\mathbb{C})$, such that for all $K_f^\beta \subset K_f^\alpha$

the image of $[(x_\beta, p_f^\beta)]$ in $M^{K_f^\alpha}(P, \mathcal{X})(\mathbb{C})$ is equal to $[(x_\alpha, p_f^\alpha)]$. Thus in particular the coset $P(\mathbb{Q}) \cdot x_\alpha$ does not depend on α , so we may replace the representatives by others, for which all x_α are equal to a fixed $x \in \mathcal{X}$. Having done this, let $B := \text{Stab}_{P(\mathbb{Q})}(x)$, then we can still replace every p_f^α by an arbitrary element of the double coset $B \cdot p_f^\alpha \cdot K_f^\alpha$. Since $b \cdot p_f^\alpha \cdot K_f^\alpha$ is open and compact for every $b \in B$, this double coset is closed. For all $K_f^\beta \subset K_f^\alpha$ we have $B \cdot p_f^\beta \cdot K_f^\beta \subset B \cdot p_f^\alpha \cdot K_f^\alpha$, so $(B \cdot p_f^\beta \cdot K_f^\beta) \cap p_f^\alpha \cdot K_f^\alpha$ is nonempty and compact. Keeping α fixed, these sets form a decreasing system of nonempty compact subsets of $P(\mathbb{A}_f)$. Thus their intersection must be nonempty. This shows that there exists a common representative for all $[(x_\alpha, p_f^\alpha)]$, whence the desired surjective.

For the injectivity we may assume that A contains $\bar{\Gamma}_Z$. Let x and B be as before, and fix $p_f \in P(\mathbb{A}_f)$, then we have to prove that $\bigcap_\alpha B \cdot p_f \cdot K_f^\alpha = B \cdot p_f \cdot A$. By the proof of 3.3 (a) Γ_Z is of finite index in $B \cap p_f \cdot K_f^\alpha \cdot p_f^{-1}$, so for all sufficiently small K_f^α we have $B \cap p_f \cdot K_f^\alpha \cdot p_f^{-1} = B \cap p_f \cdot A \cdot p_f^{-1}$. Fix such an α , then it follows:

$$\begin{aligned}
\bigcap_\beta B \cdot p_f \cdot K_f^\beta &= \bigcup_{b \in B} \bigcup_\beta (B \cdot p_f \cdot K_f^\beta) \cap (b \cdot p_f \cdot K_f^\alpha) \\
&= \bigcup_{b \in B} \bigcap_\beta b \cdot (B \cap p_f \cdot K_f^\alpha \cdot p_f^{-1}) \cdot p_f \cdot K_f^\beta \\
&= B \cdot \bigcap_\beta (B \cap p_f \cdot A \cdot p_f^{-1}) \cdot p_f \cdot K_f^\beta \\
&= B \cdot \bigcap_\beta (B \cdot p_f \cdot K_f^\beta) \cap (p_f \cdot K_f^\beta) \\
&= B \cdot \bigcap_\beta p_f \cdot K_f^\beta \\
&= B \cdot p_f \cdot A,
\end{aligned}$$

as desired.

q.e.d.

3.8. Proposition (compare [D1] 1.15): Let $(P_1, \mathcal{X}_1) \hookrightarrow (P_2, \mathcal{X}_2)$ be an embedding, and $K_f^1 \subset P_1(\mathbb{A}_f)$, $K_f^2 \subset P_2(\mathbb{A}_f)$ open compact subgroups such that $K_f^1 = \phi^{-1}(K_f^2)$.

- (a) The map $[\phi] : M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C}) \rightarrow M^{K_f^2}(P_2, \mathcal{X}_2)(\mathbb{C})$ is finite and closed.
- (b) For every K_f^1 there exists K_f^2 such that this map is a closed embedding.

Proof. (a) Since $Z(P_1)$ is mapped to $Z(P_2)$, we may as in the proof of 3.3 (a) reduce to the case where these subgroups are trivial. Since $M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C})$ has only finitely many connected components, it is enough to consider one such component. On it the map is given as $\Gamma_1 \backslash \mathcal{X}_1^0 \rightarrow \Gamma_2 \backslash \mathcal{X}_2^0$, where $\mathcal{X}_i^0$ are connected components of $\mathcal{X}_i$ and Γ_i arithmetic subgroups of $P_i(\mathbb{Q})$ with $\Gamma_1 = P_1(\mathbb{Q}) \cap \Gamma_2$. Let $x \in \mathcal{X}_1^0$ and write $\mathcal{X}_1^0 \cong P_\infty^i / K_\infty^i$ as in the proof of 3.3 (a), for $i = 1, 2$.

As in [Rag] 10.15 let $\rho : P_2^{der} \rightarrow GL_{n, \mathbb{Q}}$ be a representation and $v \in \mathbb{Q}^n$, such that P_1^{der} is the stabilizer of v in P_2^{der} . Consider the continuous left- P_∞^2 -invariant map

$$\Phi : P_\infty^2 \rightarrow \mathbb{C}^n, \quad p \mapsto \rho(p)v.$$

By construction $\Phi^{-1}(m) = P_\infty^1$. Since $\rho(\Gamma_2)$ stabilizes some $\mathbb{Z}$ -lattice in $\mathbb{Q}^n$, the set $\Phi(\Gamma_2)$ is a discrete subset of $\mathbb{C}^n$. Like in [loc. cit.] it follows that $\Gamma_2 \cdot P_\infty^1$ is closed in P_∞^2 and contains P_∞^1 as an open and closed subset. Consequently we have a closed embedding $\Gamma_1/P_\infty^1 \hookrightarrow \Gamma_2 \backslash P_\infty^2$. Since K_∞^2 is compact, the composition $\Gamma_1/P_\infty^1 \hookrightarrow \Gamma_2 \backslash P_\infty^2 \rightarrow \Gamma_2 \backslash P_\infty^2 / K_\infty^2$ is also closed and thus also the map $[\phi]$.

Let $p, p' \in P_\infty^1$, $\gamma \in \Gamma_2$ and $k \in K_\infty^2$, such that $p' = \gamma \cdot p \cdot k$. Then $\gamma^{-1} \cdot p' = p \cdot k$, whence $\Phi(\gamma^{-1}) = \Phi(\gamma^{-1} \cdot p') \in \Phi(p \cdot K_\infty^1) = \rho(p) \cdot \Phi(K_\infty^1)$. The image of p' in $\Gamma_1 \backslash P_\infty^1 / K_\infty^1$ depends only on $\Phi(\gamma^{-1})$ and $\Phi(k)$. But as $\Phi(K_\infty^1)$ is compact, the set $\Phi(\Gamma_2) \cap \rho(p) \cdot \Phi(K_\infty^1)$ is finite. Hence the fibre over $[p] \in \Gamma_2 \backslash P_\infty^2 / K_\infty^2$ is finite.

(b) (Compare [D1] 1.15) Without loss of generality we may assume K_f^1 and K_f^2 neat. By (a) it suffices to prove the injectivity. Fix K_f^1 and let $X(K_f^2)$ be the difference kernel of the two maps

$$[\phi] \circ pr_i : (M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C})^2 \rightarrow M^{K_f^2}(P_2, \mathcal{X}_2)(\mathbb{C}).$$

This is a closed analytic subvariety of $M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C}) \times M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C})$, finite and étale over both factors, which decreases with K_f^2 . Thus $X(K_f^2)$ becomes constant for $K_f^2 \supset K_f^1$ sufficiently small. Hence it suffices to show the injectivity in the projective limit. By 3.7 we have to show that the map

$$P_1(\mathbb{Q}) \backslash \mathcal{X}_1 \times (P_1(\mathbb{A}_f) / K_f^1) \rightarrow P_2(\mathbb{Q}) \backslash \mathcal{X}_2 \times (P_2(\mathbb{A}_f) / \overline{\Gamma_{Z_2}} \cdot K_f^1)$$

is injective, where Γ_{Z_2} is an arbitrary arithmetic subgroup of $\{z_2 \in Z(P_2)(\mathbb{Q}) \mid z_{2|\mathcal{X}_2} = id\}$. Write $Z(P_2)^0$ is an almost direct product $(Z(P_2) \cap P_1)^0 \cdot T$ for some torus T , and choose $\Gamma_{Z_2} = \Gamma_{Z_1} \cdot \Gamma_T$ with $\Gamma_{Z_1} \subset P_1(\mathbb{Q}) \cap K_f^1$ and $\Gamma_T \subset T(\mathbb{Q}) \cap T(\mathbb{R})^0$ such that $\Gamma_T \cap P_1(\mathbb{Q}) = \{1\}$. Then we have $\overline{\Gamma_{Z_2}} \cdot P_1(\mathbb{A}_f) \cong \overline{\Gamma_T} \times P_1(\mathbb{A}_f)$. Using this relation it is straightforward to explicitly verify the injectivity. **q.e.d.**

3.9. Proposition (compare [D1] 2.5): Let $(P, \mathcal{X})$ be mixed Shimura data with P^{der}/W simply connected. Let $(T, \mathcal{Y}) = (P, \mathcal{X})/P^{der}$, and $\phi : P \rightarrow T$ the canonical projection. Then for every open compact subgroup $K_f \subset P(\mathbb{A}_f)$ the image $\phi(K_f) \subset T(\mathbb{A}_f)$ is again open and compact, and the map $[\phi] : M^{K_f}(P, \mathcal{X})(\mathbb{C}) \rightarrow M^{\phi(K_f)}(T, \mathcal{Y})(\mathbb{C})$ induces a bijection of the connected components

$$\pi_0(M^{K_f}(P, \mathcal{X})(\mathbb{C})) \xrightarrow{\sim} M^{\phi(K_f)}(T, \mathcal{Y})(\mathbb{C}).$$

Proof. Since K_f is compact, its image $\phi(K_f)$ is also compact. Its openness follows as in the proof of [D1] 2.4. Fix $x \in \mathcal{X}$ and let A be its stabilizer in $P(\mathbb{R}) \cdot U(\mathbb{C})$, then $\mathcal{X} \cong P(\mathbb{R}) \cdot U(\mathbb{C})/A$. We have

$$\begin{aligned} \pi_0(M^{K_f}(P, \mathcal{X})(\mathbb{C})) &= \pi_0(P(\mathbb{Q}) \backslash \mathcal{X} \times (P(\mathbb{A}_f) / K_f)) \\ &= \pi_0(P(\mathbb{Q}) \backslash (P(\mathbb{R}) \cdot U(\mathbb{C}) / A) \times (P(\mathbb{A}_f) / K_f)) \\ &= \pi_0(P(\mathbb{Q}) \backslash P(\mathbb{A}) / (A \times K_f)) \\ &= \pi_0(P(\mathbb{Q}) \backslash P(\mathbb{A}) / K_f) / \pi_0(A). \end{aligned}$$

Since $\pi_0(P(\mathbb{Q}) \backslash P(\mathbb{A}) / K_f) = \pi_0(P(\mathbb{Q}) \backslash P(\mathbb{A}) / (P(\mathbb{R})^0 \times K_f))$, and $P(\mathbb{R})^0 \times K_f$ is open in $P(\mathbb{A})$, we have $\pi_0(P(\mathbb{Q}) \backslash P(\mathbb{A}) / K_f) = \pi_0((P(\mathbb{Q}) \cdot P(\mathbb{R})^0) \backslash P(\mathbb{A}) / K_f)$. Since $W(\mathbb{Q}) \cdot W(\mathbb{R})$

is dense in $W(\mathbb{A})$, this set is equal to $\pi_0(G(\mathbb{Q}) \backslash G(\mathbb{A}) / \pi(K_f))$. Thus we may assume without loss of generality that $P = G$. In this case [D1] 2.5 asserts that the canonical map $\pi_0(G(\mathbb{Q}) \backslash G(\mathbb{A}) / \pi(K_f)) \rightarrow \pi_0(T(\mathbb{Q}) \backslash T(\mathbb{A}) / \phi(K_f))$ is bijective. Hence

$$\begin{aligned} \pi_0(M^{K_f}(P, \mathcal{X})(\mathbb{C})) &\cong \pi_0(P(\mathbb{Q}) \backslash P(\mathbb{A}) / K_f) / \pi_0(A) \\ &\xrightarrow{\sim} \pi_0(T(\mathbb{Q}) \backslash T(\mathbb{A}) / \phi(K_f)) / \pi_0(A) \\ &= \pi_0(T(\mathbb{Q}) \backslash T(\mathbb{A}) / \phi(K_f)) / \pi_0(\phi(A)), \end{aligned}$$

and the latter is equal to $K^{\phi(K_f)}(T, \mathcal{Y})(\mathbb{C})$, since by the construction of $(P, \mathcal{X}) / P^{der}$ (see 2.9) $\mathcal{Y} \cong \pi_0(T(\mathbb{R}) / \phi(A))$. **q.e.d.**

3.10. Fibre product: Let $(\phi_1, \Psi_1) : (P_1, \mathcal{X}_1) \rightarrow (P, \mathcal{X})$ be a unipotent extension by W_0 , $(\phi_2, \Psi_2) : (P_2, \mathcal{X}_2) \rightarrow (P, \mathcal{X})$ an arbitrary morphism, and let $(P_{12}, \mathcal{X}_{12})$ be the fibre product defined in 2.20. Let $K_f^1 \subset P_1(\mathbb{A}_f)$ and $K_f^2 \subset P_2(\mathbb{A}_f)$ be open compact subgroups, such that $\phi_2(K_f^2) \subset K_f := \phi_1(K_f^1) \subset P(\mathbb{A}_f)$. Since P_1 is isomorphic to $W_0 \times P$ as an algebraic variety, K_f is also open and compact. Let $K_f^{12} := K_f^1 \times_{K_f} K_f^2$ be the fibre product, this is an open compact subgroup of $P_{12}(\mathbb{A}_f)$. By 2.20 and 3.5 we now have a commutative diagram

$$\begin{array}{ccc} M^{K_f^{12}}(P_{12}, \mathcal{X}_{12})(\mathbb{C}) & \longrightarrow & M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C}) \\ \downarrow & & \downarrow \\ M^{K_f^2}(P_2, \mathcal{X}_2)(\mathbb{C}) & \longrightarrow & M^{K_f}(P, \mathcal{X})(\mathbb{C}) \end{array}$$

This diagram induces a holomorphic map to the fibre product:

$$M^{K_f^{12}}(P_{12}, \mathcal{X}_{12})(\mathbb{C}) \rightarrow M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C}) \times_{M^{K_f}(P, \mathcal{X})(\mathbb{C})} M^{K_f^2}(P_2, \mathcal{X}_2)(\mathbb{C}).$$

3.11. Lemma: In the situation of 3.10, assume that K_f is neat, and that the homomorphism

$$\{z_2 \in Z(P_2)(\mathbb{Q}) \mid z_{2|\mathcal{X}_2} = id\} \cap K_f^2 \longrightarrow \{z \in Z(P)(\mathbb{Q}) \mid z|_{\mathcal{X}} = id\} \cap K_f$$

is surjective. Then the map in 3.10 is an isomorphism.

Remark. If K_f is neat, then the group $\{z \in Z(P)(\mathbb{Q}) \mid z|_{\mathcal{X}} = id\} \cap K_f$ is contained in $T(\mathbb{Q})$, where T is the Zariski closure of any sufficiently small arithmetic subgroup of $Z(P)(\mathbb{Q})$. Thus the second condition follows from the first if $Z(P)$ is an almost direct product of a $\mathbb{Q}$ -split torus with a torus of compact type defined over $\mathbb{Q}$.

Proof. Since $P_1 \rightarrow P$ is a unipotent extension, the homomorphisms $P_1(\mathbb{A}_f) \rightarrow P(\mathbb{A}_f)$, $P_{12}(\mathbb{A}_f) \rightarrow P_2(\mathbb{A}_f)$, $P_1(\mathbb{Q}) \rightarrow P(\mathbb{Q})$, and $P_{12}(\mathbb{Q}) \rightarrow P_2(\mathbb{Q})$ are surjective, and the homomorphisms $P_{12}(\mathbb{A}_f) \rightarrow P_1(\mathbb{A}_f) \times_{P(\mathbb{A}_f)} P_2(\mathbb{A}_f)$ and $P_{12}(\mathbb{Q}) \rightarrow P_1(\mathbb{Q}) \times_{P(\mathbb{Q})} P_2(\mathbb{Q})$ are isomorphisms. Furthermore the maps $\mathcal{X}_1 \rightarrow \mathcal{X}$ and $\mathcal{X}_{12} \rightarrow \mathcal{X}_2$ are surjective, and by the construction 2.20 of $P_{12}, \mathcal{X}_{12}$ we have $\mathcal{X}_{12} \cong \mathcal{X}_1 \times_{\mathcal{X}} \mathcal{X}_2$. First we show the surjectivity. Consider points $[(x_i, p_f^i)] \in P_i(\mathbb{Q}) \backslash \mathcal{X}_i \times (P_i(\mathbb{A}_f) / K_f^i)$ for $i = 1, 2$ which map to the same point in $P(\mathbb{Q}) \backslash \mathcal{X} \times (P(\mathbb{A}_f) / K_f)$. Then there exists $p \in P(\mathbb{Q})$ and $k_f \in K_f$ such that

$p \cdot \Psi_1(x_1) = \Psi_2(x_2)$ and $p \cdot \phi_1(p_f^1) \cdot k_f = \phi_2(p_f^2)$. Lift p and k_f to $p_1 \in P_1(\mathbb{Q})$, resp. $k_f^1 \in K_f^1$, and replace (x_1, p_f^1) by $p_1 \cdot (x_1, p_f^1 \cdot k_f^1)$. Then we have $\Psi_1(x_1) = \Psi_2(x_2)$ and $\phi_1(p_f^1) = \phi_2(p_f^2)$. Letting $x_{12} := (x_1, x_2)$ and $p_f^{12} := (p_f^1, p_f^2)$ we find that $[(x_{12}, p_f^{12})]$ maps to $[(x_1, p_f^1)], [(x_2, p_f^2)]$ in $M^{K_f^{12}}(P_{12}, \mathcal{X}_{12})(\mathbb{C})$. Thus the map is surjective.

For the injective consider $x_{12} = (x_1, x_2)$, $x'_{12} = (x'_1, x'_2) \in \mathcal{X}_{12}$ and $p_f^{12} = (p_f^1, p_f^2)$, $p_f^{12'} = (p_f^{1'}, p_f^{2'}) \in P_{12}(\mathbb{A}_f)$, such that $[(x_{12}, p_f^{12})]$ and $[(x'_{12}, p_f^{12'})]$ have the same image in $P_i(\mathbb{Q}) \backslash \mathcal{X}_i \times (P_i(\mathbb{A}_f)/K_f^i)$ for both $i = 1, 2$. Then there exists $p_i \in P_i(\mathbb{Q})$ and $k_f^i \in K_f^i$ such that $p_i \cdot x_i = x'_i$ and $p_i \cdot p_f^i \cdot k_f^i = p_f^{i'}$. Lift p_2 and k_f^2 to elements $p_{12} \in P_{12}(\mathbb{Q})$, resp. $k_f^{12} \in K_f^{12}$, and replace (x_{12}, p_f^{12}) by $p_{12} \cdot (x_{12}, p_f^{12} \cdot k_f^{12})$. Then we have $x_2 = x'_2$ and $p_f^2 = p_f^{2'}$. This implies $\Psi_1(x_1) = \Psi_2(x_2) = \Psi_2(x'_2) = \Psi_1(x'_1)$ and likewise $\phi_1(p_f^1) = \phi_1(p_f^{1'})$. Let $x := \Psi_1(x_1)$ and $p_f := \phi_1(p_f^1)$, then these equalities show that $\phi_1(p_1)$ lies in $\text{Stab}_{P(\mathbb{Q})}(x)$ as well as in $p_f \cdot K_f \cdot p_f^{-1}$. Since K_f is neat, the intersection of these two groups is contained in $Z(P)(\mathbb{Q})$. So this element is central and fixes one point of $\mathcal{X}$, hence it acts trivially on the whole of $\mathcal{X}$. Thus we get $\phi_1(p_1) \in \{z \in Z(P)(\mathbb{Q}) \mid z|_{\mathcal{X}} = id\} \cap K_f$. By assumption there exists $z_2 \in Z(P_2)(\mathbb{Q}) \cap K_f^2$ such that $z_2|_{\mathcal{X}_2} = id$ and $\phi_2(z_2) = \phi_1(p_1)$. Define $p'_{12} := (p_1, z_2) \in P_{12}(\mathbb{Q})$ and $k_f^{12'} := (k_f^1, z_2^{-1})$, the latter is an element of K_f^{12} since $\phi_1(k_f^1) = \phi_1(p_f^1)^{-1} \cdot \phi_1(p_1)^{-1} \cdot \phi_1(p_f^{1'}) = p_f^{-1} \cdot \phi_2(z_2)^{-1} \cdot p_f = \phi_2(z_2^{-1})$. Then one easily verifies that $p'_{12} \cdot (x_{12}, p_f^{12} \cdot k_f^{12'}) = (x'_{12}, p_f^{12'})$, as desired. **q.e.d.**

3.12. Corollary: (a) (Compare 2.21) Let $(P', \mathcal{X}') \rightarrow (P, \mathcal{X})$ be a unipotent extension with a fixed splitting $P' \cong W_0 \rtimes P$. Suppose that W_0 is abelian, and let $K'_f \subset P'(\mathbb{A}_f)$ be an open compact subgroup of the form $K_f^W \rtimes K_f$. If K_f is neat, then

$$M^{K'_f}(P', \mathcal{X}')(\mathbb{C}) \longrightarrow M^{K_f}(P, \mathcal{X})(\mathbb{C})$$

is a smooth holomorphic family of abelian complex Lie groups.

(b) (Compare 2.22) Let $\phi : (P_1, \mathcal{X}_1) \rightarrow (P, \mathcal{X})$ be a unipotent extension by an abelian group W_0 . Let $\phi_* : (P, \mathcal{X}) \rightarrow (P_*, \mathcal{X}_*)$ be a morphism, such that the operation of P on W_0 factors through P_* , and such that $Z(P_*)$ is an almost direct product of a $\mathbb{Q}$ -split torus with a torus of compact type defined over $\mathbb{Q}$ (by 2.1. (viii) this is no big restriction). Let $(P', \mathcal{X}') \rightarrow (P_*, \mathcal{X}_*)$ be the splitting unipotent extension with $P' = W_0 \rtimes P_*$. Let $K_f^1 \subset P_1(\mathbb{A}_f)$ be a neat open compact subgroup, $K_f := \phi(K_f^1)$, and K_f^W the image of $(\{z_1 \in Z(P_1)(\mathbb{Q}) \mid z_1|_{\mathcal{X}_1} = id\} \rtimes W_0(\mathbb{A}_f)) \cap K_f^1$ under the projection $Z(P_1) \times W_0 \rightarrow W_0$. Furthermore let $K_f^* \subset P_*(\mathbb{A}_f)$ be a neat open compact subgroup which normalizes K_f^W , such that $\phi_*(K_f) \subset K_f^*$. Let $K'_f := K_f^W \rtimes K_f^*$. Then

$$M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C}) \longrightarrow M^{K'_f}(P, \mathcal{X})(\mathbb{C}).$$

is in a canonical way a holomorphic torsor under the family of groups $M^{K'_f}(P', \mathcal{X}')(\mathbb{C}) \longrightarrow M^{K_f^*}(P_*, \mathcal{X}_*)(\mathbb{C})$.

Proof. (a) Consider the maps “zero section”, “inverse” and “group operation” on the mixed Shimura varieties induced by the corresponding morphisms of 2.21. Of course the desired relations between these maps follow directly by functoriality. The only problem is that

the group operation induces only a map $M^{K_f^W \rtimes K_f}(W_0 \rtimes (P', \mathcal{X}'))(\mathbb{C}) \longrightarrow M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C})$. Let $T \subset Z(P)$ be the Zariski closure on any sufficiently small arithmetic subgroup of $Z(P)(\mathbb{Q})$. Since $(P', \mathcal{X}') \rightarrow (P, \mathcal{X})$ is a unipotent extension and K_f is neat, it follows that the group $\{z \in Z(P)(\mathbb{Q}) \mid z|_{\mathcal{X}} = id\} \cap K_f$ is contained in $T(\mathbb{Q})$. By 2.1 (viii) T operates trivially on W_0 , so under the semidirect product decomposition $\{z \in Z(P)(\mathbb{Q}) \mid z|_{\mathcal{X}} = id\} \cap K_f$ lies in $\{z' \in Z(P')(\mathbb{Q}) \mid z'|_{\mathcal{X}'} = id\} \cap K'_f$. Thus the conditions of 3.11 are satisfied, so $M^{K_f^W \rtimes K_f}(W_0 \rtimes (P', \mathcal{X}'))(\mathbb{C})$ is isomorphic to the fibre product of $M^{K_f}(P', \mathcal{X}')(\mathbb{C})$ with itself over $M^{K_f}(P, \mathcal{X})(\mathbb{C})$. This yields the desired group structure. Since K_f is neat, the map $M^{K'_f}(P', \mathcal{X}')(\mathbb{C}) \rightarrow M^{K_f}(P, \mathcal{X})(\mathbb{C})$ is locally isomorphic to the projection $\mathcal{X}' \rightarrow \mathcal{X}$, whence the smoothness.

(b) Let $A := (\{z_1 \in Z(P_1)(\mathbb{Q}) \mid z_1|_{\mathcal{X}_1} = id\} \times W_0(\mathbb{A}_f)) \cap K_f^1$, then $pr_1(A)$ is an arithmetic subgroup of $\{z_1 \in Z(P_1)(\mathbb{Q}) \mid z_1|_{\mathcal{X}_1} = id\}$. By 3.7 (a) we may therefore replace K_f^1 by $pr_1(A) \cdot K_f^1$ without changing the mixed Shimura variety $M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C})$. After this modification $A = (\{z_1 \in Z(P_1)(\mathbb{Q}) \mid z_1|_{\mathcal{X}_1} = id\} \times W_0(\mathbb{A}_f) \cap K_f^1)$, so we have $K_f^W = W_0(\mathbb{A}_f) \cap K_f^1$. As in (a) the morphism $\mu : (P', \mathcal{X}') \times_{(P_*, \mathcal{X}_*)} (P_1, \mathcal{X}_1) \rightarrow (P_1, \mathcal{X}_1)$ of 2.22 induces the desired group action. Note that here the second condition of 3.11 trivially holds. It remains to prove that this defines a torsor. One easily verifies that the isomorphism $(pr_2, \mu) : (P', \mathcal{X}') \times_{(P_*, \mathcal{X}_*)} (P_1, \mathcal{X}_1) \rightarrow (P_1, \mathcal{X}_1) \times_{(P, \mathcal{X})} (P_1, \mathcal{X}_1)$ in the proof of 2.23 induces an isomorphism of the associated mixed Shimura varieties. It thus remains to show that the mixed Shimura variety associated to $(P_1, \mathcal{X}_1) \times_{(P, \mathcal{X})} (P_1, \mathcal{X}_1)$ is isomorphic to the fibre product of $M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C})$ with itself over $M^{K_f}(P, \mathcal{X})(\mathbb{C})$. This follows from 3.11 if we can show that the homomorphism $\{z_1 \in Z(P_1)(\mathbb{Q}) \mid z_1|_{\mathcal{X}_1} = id\} \cap K_f^1 \rightarrow \{z \in Z(P)(\mathbb{Q}) \mid z|_{\mathcal{X}} = id\} \cap K_f$ is surjective. But with our hypothesis about A this is proved as in (a). **q.e.d.**

3.13. The fibre of a unipotent extension: Let $(\phi, \Psi) : (P_1, \mathcal{X}_1) \rightarrow (P, \mathcal{X})$ be a unipotent extension. Let $K_f^1 \subset P_1(\mathbb{A}_f)$ be an open compact subgroup and $K_f := \phi(K_f^1) \subset P(\mathbb{A}_f)$. Fix $x_1 \in \mathcal{X}_1$, $p_{1,f} \in P_1(\mathbb{A}_f)$, $x := \Psi(x_1)$ and $p_f := \phi(p_{1,f})$. We want to describe the fibre of the map $[\phi] : M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C}) \rightarrow M^{K_f}(P, \mathcal{X})(\mathbb{C})$ explicitly. Since $W_0(\mathbb{Q})$ is dense in $W_0(\mathbb{A}_f)$, the fibre over $[(x, p_f)]$ is equal to

$$\begin{aligned} [\phi]^{-1}([(x, p_f)]) &\cong P_1(\mathbb{Q}) \backslash P_1(\mathbb{Q}) \cdot (\Psi^{-1}(x) \times \phi^{-1}(p_f) \cdot K_f^1 / K_f^1) \\ &\cong \text{Stab}_{P_1(\mathbb{Q})}(x) \backslash (\Psi^{-1}(x) \times \phi^{-1}(p_f) \cdot K_f^1 / K_f^1) \\ &\cong (\text{Stab}_{P_1(\mathbb{Q})}(x) \cap p_{1,f} \cdot K_f^1 \cdot p_{1,f}^{-1}) \backslash \Psi^{-1}(x) \times p_{1,f} \cdot K_f^1 / K_f^1 \\ &\cong (\text{Stab}_{P_1(\mathbb{Q})}(x) \cap p_{1,f} \cdot K_f^1 \cdot p_{1,f}^{-1}) \backslash \Psi^{-1}(x). \end{aligned}$$

Since $\Psi^{-1}(x)$ is an orbit under $W_0(\mathbb{R}) \cdot (W_0 \cap U_1)(\mathbb{C})$ and this group is connected, all fibres of $[\phi]$ are connected. Now suppose that K_f is neat. Then it follows as in the proof of 3.11:

$$\text{Stab}_{P_1(\mathbb{Q})}(x) \cap p_{1,f} \cdot K_f^1 \cdot p_{1,f}^{-1} \subset \{z_1 \in Z(P_1)(\mathbb{Q}) \mid z_1|_{\mathcal{X}_1} = id\} \times W_0(\mathbb{Q}).$$

Let Γ be the image of

$$(\{z_1 \in Z(P_1)(\mathbb{Q}) \mid z_1|_{\mathcal{X}_1} = id\} \times W_0(\mathbb{Q})) \cap p_{1,f} \cdot K_f^1 \cdot p_{1,f}^{-1}$$

under the projection $Z(P_1) \times W_0 \rightarrow W_0$. For any splitting $e : (P, \mathcal{X}) \rightarrow (P_1, \mathcal{X}_1)$, the map $[w] \mapsto [(w \cdot e(x), p_f)]$ is an isomorphism of the fibre over $[(x, p_f)]$ with $\Gamma \backslash W_0(\mathbb{R}) \cdot (W_0 \cap U_1)(\mathbb{C})$. If there exists no splitting, then there are still such isomorphisms fibrewise and noncanonically. If $Z(P)$ is an almost direct product of a $\mathbb{Q}$ -split torus with a torus of compact type defined over $\mathbb{Q}$, then we already have

$$\Gamma \cong W_0(\mathbb{Q}) \cap p_{1,f} \cdot K_f^1 \cdot p_{1,f}^{-1}.$$

3.14. The group structure on the fibre: Consider the situation on 3.12 (a). The fibre of $M^{K'_f}(P', \mathcal{X}')(\mathbb{C}) \rightarrow M^{K_f}(P, \mathcal{X})(\mathbb{C})$ is $\Gamma \backslash W_0(\mathbb{R}) \cdot W_0 \cap U_1(\mathbb{C})$, where Γ is the image of $(\{z' \in Z(P')(\mathbb{Q}) \mid z'|_{\mathcal{X}'} = id\} \times W_0(\mathbb{Q})) \cap p_f(K_f^W)$ in $W_0(\mathbb{Q})$. This is an arithmetic subgroup of $W_0(\mathbb{Q})$, and $W_0(\mathbb{R}) \cdot (W_0 \cap U_1)(\mathbb{C})$ is an abelian Lie group. The definition of the group multiplication 2.21 together with 3.12 (a) implies that the group structure on the fibre is the natural one on $\Gamma \backslash W_0(\mathbb{R}) \cdot (W_0 \cap U_1)(\mathbb{C})$. The complex structure can be described by writing this quotient in the form $\Gamma \backslash W_0(\mathbb{C}) / \exp(F^0(\text{Lie } W_0)_{\mathbb{C}})$ following 2.18.

Let $1 \rightarrow U_0 \rightarrow W_0 \rightarrow V_0 \rightarrow 1$ be the weight decomposition of W_0 , and $1 \rightarrow \Gamma_U \rightarrow \Gamma \rightarrow \Gamma_V \rightarrow 1$ the corresponding short exact sequence of arithmetic subgroups. Then we have a short exact sequence

$$1 \rightarrow \Gamma_U \backslash U_0(\mathbb{C}) \rightarrow \Gamma \backslash W_0(\mathbb{R}) \cdot U_0(\mathbb{C}) \rightarrow \Gamma_V \backslash V_0(\mathbb{R}) \rightarrow 1,$$

or, to indicate the complex structure:

$$1 \rightarrow \Gamma_U \backslash U_0(\mathbb{C}) \rightarrow \Gamma \backslash W_0(\mathbb{C}) / \exp(F^0(\text{Lie } W_0)_{\mathbb{C}}) \rightarrow \Gamma_V \backslash V_0(\mathbb{C}) / \exp(F^0(\text{Lie } V_0)_{\mathbb{C}}) \rightarrow 1.$$

Since Γ_U is a lattice in $U_0(\mathbb{R})$, we have $U_0(\mathbb{C}) \cong \Gamma_U \otimes \mathbb{C}$, so $\Gamma_U \backslash U_0(\mathbb{C}) \cong \Gamma_U \otimes (\mathbb{Z} \backslash \mathbb{C}) \cong \Gamma_U \otimes \mathbb{C}^\times$. Thus this is an algebraic torus over $\mathbb{C}$. Since Γ_V is a lattice in $V_0(\mathbb{R})$, the group $\Gamma_V \backslash V_0(\mathbb{R}) \cong \Gamma_V \otimes (\mathbb{Z} \backslash \mathbb{R})$ is a compact complex torus. We shall see in 3.22 that it is an abelian variety.

3.15. Torsion points on the fibre: Consider a unipotent extension $(\phi, \Psi) : (P', \mathcal{X}') \rightarrow (P, \mathcal{X})$ by W_0 , and neat open compact subgroups $K_f = \Psi(K'_f)$. Let $K_f^W := W_0(\mathbb{A}_f) \cap K'_f$ and $K_f^{W,+}$ be the normalizer of K'_f in $W_0(\mathbb{A}_f)$. Since P' operates nontrivially on every subquotient of W_0 , $K_f^{W,+}$ is also an open compact subgroup of $W_0(\mathbb{A}_f)$, so $K_f^{W,+}/K_f^W$ is a finite group. For every $w_f \in K_f^{W,+}$ the map $[\cdot w_f]_{K'_f, K'_f}$ is well-defined, and this defines an operation of $K_f^{W,+}$ from the right hand side on $M^{K'_f}(P', \mathcal{X}')(\mathbb{C})$ over $M^{K_f}(P, \mathcal{X})(\mathbb{C})$. By 3.5 (d) this operation factors through $K_f^{W,+}/K_f^W$. On a fibre $\Gamma \backslash \Psi^{-1}(x)$ as in 3.13 this operation is given by

$$[\cdot w_f]([x']) = [w^{-1} \cdot x']$$

for any $w \in W_0(\mathbb{Q})$ such that $p_f'^{-1} \cdot w \cdot p'_f \equiv w_f \pmod{K_f^W}$. Clearly this is a free action if $\Gamma = W_0(\mathbb{Q}) \cap p'_f \cdot K'_f \cdot p_f'^{-1}$, for instance if $Z(P)$ is an almost direct product of a $\mathbb{Q}$ -split torus with a torus of compact type defined over $\mathbb{Q}$.

Let us apply this in the situation of 3.12 (a). By the proof of 3.12 (a) the action is free. Using the given splitting $e : (P, \mathcal{X}) \hookrightarrow (P', \mathcal{X}')$ we may associate to each $[w_f] \in K_f^{W,+}/K_f^W$ the section $[\cdot w_f] \circ [e] : M^{K_f}(P, \mathcal{X})(\mathbb{C}) \rightarrow M^{K'_f}(P', \mathcal{X}')(\mathbb{C})$.

This definition induces a holomorphic monomorphism of groups over $M^{K_f}(P, \mathcal{X})(\mathbb{C})$

$$(K_f^{W,+}/K_f^W) \times M^{K_f}(P, \mathcal{X})(\mathbb{C}) \longrightarrow M^{K'_f}(P', \mathcal{X}')(\mathbb{C}).$$

On the fibre this monomorphism is given by

$$K_f^{W,+}/K_f^W \xrightarrow[\sim]{p_f \cdot} p_f(K_f^{W,+})/p_f(K_f^W) \xleftarrow[\sim]{-[id]} (W_0(\mathbb{Q}) \cap p_f(K_f^{W,+}))/\Gamma$$

$$\downarrow$$

$$\Gamma \backslash W_0(\mathbb{R}) \cdot (W_0 \cap U')(\mathbb{C}).$$

If for example $K_f^{W,+}$ consists of all $w_f \in W_0(\mathbb{A}_f)$ such that $d \cdot w_f \in K_f^W$, then this defines a isomorphism of $K_f^{W,+}/K_f^W$ with the group of all d -division points on every fibre.

3.16. Example: Let $(P_0, \mathcal{X}_0) \rightarrow (\mathbb{G}_{m,\mathbb{Q}}, \mathcal{H}_0)$ be the unipotent extension defined in 2.24, and let $K_f \subset \mathbb{G}_m(\mathbb{A}_f)$ and $K_f^U \subset U_0(\mathbb{A}_f)$ be open compact subgroups. Since $\mathcal{H}_0$ consists of two points, and $K_f \subset \hat{\mathbb{Z}}^\times$, for every $y \in \mathcal{H}_0$ the intersection $\text{Stab}_{\mathbb{Q}^\times}(y) \cap K_f$ is trivial. Thus in this case the group operation 3.12 (a) is defined without restriction on K_f . We want to define a canonical isomorphism

$$M^{K_f^U \rtimes K_f}(P_0, \mathcal{X}_0)(\mathbb{C}) \longrightarrow \mathbb{C}^\times \times M^{K_f}(\mathbb{G}_{m,\mathbb{Q}}, \mathcal{H}_0)(\mathbb{C}).$$

Since $K_f \subset \hat{\mathbb{Z}}^\times$ and $\mathbb{G}_m(\mathbb{A}_f) = \mathbb{Q}^\times \cdot \hat{\mathbb{Z}}^\times$, the embedding $\mathcal{H}_0 \times \hat{\mathbb{Z}}^\times \hookrightarrow \mathcal{H}_0 \times \mathbb{G}_m(\mathbb{A}_f)$ induces an isomorphism

$$\{\pm 1\} \backslash \mathcal{H}_0 \times (\hat{\mathbb{Z}}^\times / K_f) \xrightarrow{\sim} M^{K_f}(\mathbb{G}_{m,\mathbb{Q}}, \mathcal{H}_0)(\mathbb{C}).$$

By definition U_0 is $\mathbb{Q}$ interpreted as unipotent group. Thus $U_0(\mathbb{Q}) = \mathbb{Q}$ is dense in $U_0(\mathbb{A}_f) = \mathbb{A}_f$, and similarly we have $P_0(\mathbb{A}_f) = P_0(\mathbb{Q}) \cdot (K_f^U \rtimes \hat{\mathbb{Z}}^\times)$. Let d be the unique positive rational number such that $U_0(\mathbb{Q}) \cap K_f^U = d \cdot \mathbb{Z}$. The embedding $\mathcal{X}_0 \times \hat{\mathbb{Z}}^\times \hookrightarrow \mathcal{X}_0 \times P_0(\mathbb{A}_f)$ induces an isomorphism

$$((d \cdot \mathbb{Z}) \rtimes \{\pm 1\}) \backslash \mathcal{X}_0 \times (\hat{\mathbb{Z}}^\times / K_f) \xrightarrow{\sim} M^{K_f^U \rtimes K_f}(P_0, \mathcal{X}_0)(\mathbb{C}).$$

The projection $P_0 \rightarrow \mathbb{G}_{m,\mathbb{Q}}$ and the given section $\iota : \mathbb{G}_{m,\mathbb{Q}} \rightarrow P_0$ induce a canonical isomorphism

$$\mathcal{X}_0 = h(\mathcal{X}_0) \times \mathcal{H}_0 \cong U_0(\mathbb{C}) \times \mathcal{H}_0 = \mathbb{C} \times \mathcal{H}_0,$$

and the operation of $P_0(\mathbb{R}) \cdot U_0(\mathbb{C}) = \mathbb{C} \rtimes \mathbb{R}^\times \ni p = (u, x)$ on $\mathbb{C}$ is given by the affine automorphism $\alpha_p : z \mapsto u + x \cdot z$. As $\{\pm 1\}$ operates nontrivially on $\mathbb{C}$, it is possible to decompose $((d \cdot \mathbb{Z}) \rtimes \{\pm 1\}) \backslash \mathbb{C} \times \mathcal{H}_0 \times (\hat{\mathbb{Z}}^\times / K_f)$ canonically into a product. But it turns out that this is the same problem as the choice of a square root of -1 , a choice that is necessary to apply the exponential function in the situation.

Recall the definition $\mathcal{H}_0 := \text{Isom}(\mathbb{Z}, \mathbb{Z}(1))$ of 2.8. For every $\lambda \in \mathcal{H}_0$ and $u \in \mathbb{C}$ we get an element $\lambda(u) \in \mathbb{Q}(1) \otimes \mathbb{C} = \mathbb{C}$. Let $\exp : \mathbb{C} \rightarrow \mathbb{C}^\times$ be the exponential function, and consider the map

$$\mathbb{C} \times \mathcal{H}_0 \longrightarrow \mathbb{C}^\times \times \mathcal{H}_0, (z, y) \mapsto (\exp(\frac{1}{d} \cdot \lambda(z)), y).$$

Since $((-1) \cdot \lambda)((-1) \cdot z) = \lambda(z)$, this map induces a holomorphic isomorphism

$$\begin{aligned} M^{K_f^U \rtimes K_f}(P_0, \mathcal{X}_0)(\mathbb{C}) &\cong ((d \cdot \mathbb{Z}) \rtimes \{\pm 1\}) \backslash \mathbb{C} \times \mathcal{H}_0 \times (\hat{\mathbb{Z}}^\times / K_f) \\ &\longrightarrow \mathbb{C}^\times \times \{\pm 1\} \backslash \mathcal{H}_0 \times (\hat{\mathbb{Z}}^\times / K_f) \\ &\cong \mathbb{C}^\times \times M^{K_f}(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)(\mathbb{C}). \end{aligned}$$

It is easily verified that this isomorphism is compatible with the group structure.

Remark. We have seen that a canonical isomorphism can be obtained only because $\mathcal{H}_0$ consists of two points, and because they correspond to the two choices of $\sqrt{-1}$. This was the most important reason for our somewhat strange looking definition of $\mathcal{H}_0$ in 2.8.

3.17. $\mathbb{C}^\times$ -Torsors: Consider the situation of 3.12 (b) with $(P', \mathcal{X}') \rightarrow (P_* \mathbb{C}_*)$ taken to be the unipotent extension $(P_0, \mathcal{X}_0) \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$. Via the isomorphism 3.16 $M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C})$ becomes a holomorphic $\mathbb{C}^\times$ -torsor over $M^{K_f}(P, \mathcal{X})(\mathbb{C})$. Under the bijective correspondence between holomorphic $\mathbb{C}^\times$ -torsors and line bundles we therefore get a canonical line bundle on $M^{K_f}(P, \mathcal{X})(\mathbb{C})$. (We shall discuss later (5.10 and 8.5–6) how this line bundle can be constructed explicitly as a torus embedding by adding the zero section) First we shall study the restriction of the $\mathbb{C}^\times$ -torsor to a fibre under a unipotent extension. We shall show particular that under certain conditions the associated line bundle defines a polarization on the fibre.

3.18. Restriction of a torsor to a fibre: Let $P_1 = W_0 \rtimes P$ with a arbitrary W_0 and let $\phi : (P_1, \mathcal{X}_1) \rightarrow (P, \mathcal{X})$ be the associated unipotent extension. Let $1 \rightarrow U_0 \rightarrow W_0 \rightarrow V_0 \rightarrow 1$ be the weight decomposition of W_0 . Let $K_f^1 \subset P_1(\mathbb{A}_f)$ be neat, and fix $[(x, p_f)] \in M^{\phi(K_f^1)}(P, \mathcal{X})(\mathbb{C})$. By 3.13 the fibre over $[x, p_f]$ is canonically isomorphic to $\Gamma_W \backslash W_0(\mathbb{R}) \cdot U_0(\mathbb{C})$, where Γ_W is a certain arithmetic subgroup of $W_0(\mathbb{Q})$. Let $1 \rightarrow \Gamma_U \rightarrow \Gamma_W \rightarrow \Gamma_V \rightarrow 1$ be the corresponding short exact sequence of arithmetic subgroups. By 3.12 $\Gamma_W \backslash W_0(\mathbb{R}) \cdot U_0(\mathbb{C})$ is a holomorphic $\Gamma_U \backslash U_0(\mathbb{C})$ -torsor over $\Gamma_V \backslash V_0(\mathbb{R})$.

Let $\underline{U}_0(\mathbb{C})$ be the coherent sheaf on $\Gamma_V \backslash V_0(\mathbb{R})$ of all holomorphic maps to $U_0(\mathbb{C})$, and $\underline{\Gamma}_U$ the constant subsheaf with values in Γ_U . Consider the short exact sequence on $\Gamma_V \backslash V_0(\mathbb{R})$:

$$1 \rightarrow \underline{\Gamma}_U \rightarrow \underline{U}_0(\mathbb{C}) \rightarrow \underline{\Gamma}_U \backslash \underline{U}_0(\mathbb{C}) \rightarrow 1.$$

The isomorphism class of the torsor in question corresponds to a cohomology class in $\mathcal{H}^1(\Gamma_V \backslash V_0(\mathbb{R}), \underline{\Gamma}_U \backslash \underline{U}_0(\mathbb{C}))$. We want to determine its image in $\mathcal{H}^2(\Gamma_V \backslash V_0(\mathbb{R}), \underline{\Gamma}_U)$ under the boundary homomorphism.

Since $\Gamma_V \backslash V_0(\mathbb{R})$ is a $K(\Gamma_V, 1)$, there is a canonical isomorphism $\mathcal{H}^2(\Gamma_V \backslash V_0(\mathbb{R}), \underline{\Gamma}_U) \cong \mathcal{H}^2(\Gamma_V, \Gamma_U)$. Let $\text{Alt}^2(\Gamma_V, \Gamma_U)$ be the group of all alternating forms $\Gamma_V \times \Gamma_V \rightarrow \Gamma_U$. The cocycle condition implies directly that $\text{Alt}^2(\Gamma_V, \Gamma_U)$ is contained in the group of 2-cocycles $Z^2(\Gamma_V, \Gamma_U)$. By the lemma in [M1] §2 p.16 this inclusion induces an isomorphism

$Alt^2(\Gamma_V, \Gamma_U) \rightarrow \mathcal{H}^2(\Gamma_V, \Gamma_U)$. The cohomology class thus corresponds canonically to an alternating form. But we already have a canonical alternating form, namely that induced by the commutator pairing $\Psi : V_0 \times V_0 \rightarrow U_0$.

3.19. Lemma: The cohomology class of this torsor corresponds to the alternating form $-\Psi : \Gamma_V \times \Gamma_V \rightarrow \Gamma_U$.

Proof. We follow the proof in [M1] §2. Let $\phi : W_0 \rightarrow V_0 = W_0/U_0$ be the given projection, and choose a holomorphic section $s : V_0 \rightarrow W_0$. The 1-cocycle that represents the $\Gamma_U \backslash U_0(\mathbb{C})$ -bundle on $\Gamma_V \backslash V_0(\mathbb{R})$ comes from the map

$$\Gamma_W \times V_0(\mathbb{R}) \rightarrow U_0(\mathbb{R}), (\omega, v) \mapsto f_\omega(v) := \omega \cdot s(v) \cdot s(\phi(\omega) \cdot v).$$

The corresponding 2-cocycle on Γ_V with values in Γ_U is induced from the map

$$\Gamma_W \times \Gamma_W \rightarrow \Gamma_U,$$

$$(\omega_1, \omega_2) \mapsto F(\omega_1, \omega_2) := f_{\omega_2}(\phi(\omega_1)) \cdot f_{\omega_1 \omega_2}(1)^{-1} \cdot f_{\omega_1}(1).$$

An elementary calculation now shows

$$F(\omega_1, \omega_2) = [\omega_2, \omega_1] = -\Psi(\phi(\omega_1), \phi(\omega_2)),$$

whence the assertion. **q.e.d.**

3.20. Construction of a polarization: Let $(P_2, \mathcal{X}_2) \rightarrow (P, \mathcal{X})$ be a unipotent extension by W_0 , and let $1 \rightarrow U_0 \rightarrow W_0 \rightarrow V_0 \rightarrow 1$ be the weight decomposition of W_0 . Let $(P_1, \mathcal{X}_1) := (P_2, \mathcal{X}_2)/U_0$, and suppose that $(P_2, \mathcal{X}_2) \rightarrow (P_1, \mathcal{X}_1)$ is given as a $(P_0, \mathcal{X}_0) \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$ -torsor with respect to a morphism $(\phi, \Psi) : (P, \mathcal{X}) \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$. In particular we are given an isomorphism $U_0 \cong \mathbb{Q}$, and for every $x \in \mathcal{X}$ we have an element $\lambda_{\Psi(x)} \in Isom(\mathbb{Z}, \mathbb{Z}(1))$. Let us assume that for every $x \in \mathcal{X}$ the pairing

$$\lambda_{\Psi(x)} \circ \Psi : V_0 \times V_0 \rightarrow U_0 \cong \mathbb{Q} \rightarrow \mathbb{Q}(1)$$

is a polarization of the Hodge structure on V_0 defined by x . Then $\Psi : V_0 \times V_0 \rightarrow U_0$ is necessarily surjective. Let $K_f^2 \subset P_2(\mathbb{A}_f)$ be a neat open compact subgroup, K_f^1 its image in $P_1(\mathbb{A}_f)$, and K_f its image in $P(\mathbb{A}_f)$. Then by 3.17

$$M^{K_f^2}(P_2, \mathcal{X}_2)(\mathbb{C}) \longrightarrow M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C})$$

is a $\mathbb{C}^\times$ -torsor. We are interested in its restriction to the fibres of

$$M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C}) \longrightarrow M^{K_f}(P, \mathcal{X})(\mathbb{C}).$$

3.21. Proposition: In 3.20 the inverse of the associated line bundle is ample on every fibre of $M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C}) \rightarrow M^{K_f}(P, \mathcal{X})(\mathbb{C})$.

Proof. The complex structure on the fibre is given by 3.14. Let $\rho : P \rightarrow GL(V_0)$ be the operation of P on V_0 . Since V_0 is pure of weight -1 , for every $x \in \mathcal{X}$ the complex

structure on $V_0(\mathbb{R})$ is given by $\mathbb{C} \times V_0(\mathbb{R}) \rightarrow V_0(\mathbb{R})$, $(z, v) \mapsto \rho \circ h_x(z)(v)$. By 3.16 and 3.19 the Chern class of the line bundle is represented by the alternating form

$$E : \Gamma_V \times \Gamma_V \rightarrow 2\pi\sqrt{-1} \cdot \mathbb{Z}, \quad (v, v') \mapsto -\frac{1}{d} \cdot \lambda_{\Psi(x)} \circ \Psi(v, v').$$

The unique hermitian form $V_0(\mathbb{R}) \times V_0(\mathbb{R}) \rightarrow \mathbb{Q}(1) \otimes \mathbb{C} = \mathbb{C}$ with respect to the complex structure defined by x , whose imaginary part is equal to E , is given by

$$V_0(\mathbb{R}) \times V_0(\mathbb{R}) \rightarrow \mathbb{C}, \quad (v, v') \mapsto E(v, v') + \sqrt{-1} \cdot E(v, \rho \circ h_x(\sqrt{-1})v').$$

Note that as in 1.11 this formula does not depend on the choice of $\sqrt{-1}$. The real part of this hermitian form is

$$V_0(\mathbb{R}) \times V_0(\mathbb{R}) \rightarrow \mathbb{R}, \quad (v, v') \mapsto -\frac{1}{d} \cdot \sqrt{-1} \cdot \lambda_{\Psi(X)} \circ \Psi(v, \rho \circ h_x(\sqrt{-1})v'),$$

so the assumption about polarization implies that this hermitian form is negative definite. The theorem of Lefschetz (see [M1] §3 p. 29) now implies that the inverse of the line bundle is ample. **q.e.d.**

3.22. Further constructions (not needed in the sequel): (a) It is now easy to prove that the compact complex torus in 3.14 is an abelian variety. In fact, consider a unipotent extension $(P_1, \mathcal{X}_1) \rightarrow (P, \mathcal{X})$ with $P_1 = V_0 \rtimes P$ and V_0 pure of weight -1 . Without loss of generality one may assume that $(P, \mathcal{X})$ is irreducible. Then with 2.14 (b) one can, as in the proof of 2.26 (b), construct $(P_2, \mathcal{X}_2)$ such that the assumptions of 3.20 are satisfied. Thus by 3.21 the fibre $\Gamma_V \backslash V_0(\mathbb{C}) / \exp(F^0(\text{Lie } V_0)_{\mathbb{C}})$ is an abelian variety. This argument will in fact be part of the proof of the algebraicity of an arbitrary mixed Shimura variety. See chapter 10.

(b) As a variant of the construction in (a) it is a nice exercise to construct the dual abelian variety together with the $\mathbb{C}^\times$ -torsor associated to the Poincaré-bundle. For this let $U_0 := \mathbb{Q}$, $V'_0 := \text{Hom}(V_0, U_0)$, and let W_0 be the central extension of $V_0 \oplus V'_0$ by U_0 with the commutator pairing $((v, \ell), (v', \ell')) \mapsto \ell(v') - \ell'(v)$. Assume that $(P, \mathcal{X})$ is irreducible, and let P act on U_0 through the character given by 2.14 (b). Then for suitable choices of open compact subgroups the fibre of the mixed Shimura variety associated to $V'_0 \rtimes P$ is the dual abelian variety, and $W_0 \rtimes P$ yields the Poincaré-bundle.

(c) One can find a lot of other structures in the unipotent fibres. For instance fix $(P, \mathcal{X})$ with $Z(P)$ an almost direct product of a $\mathbb{Q}$ -split torus with a torus of compact type defined over $\mathbb{Q}$, and a neat open compact subgroup $K_f \subset P(\mathbb{A}_f)$. Then as in 2.21 we get for every suitable representation W_0 of P and open compact subgroup $K_f^W \subset W_0(\mathbb{A}_f)$ a holomorphic family of complex Lie groups over $M^{K_f}(P, \mathcal{X})(\mathbb{C})$. Clearly this functor preserves exact sequences. Also the endomorphism algebra $\{\phi \in \text{End}_P(W_0) \mid \phi(K_f^W) = K_f^W\}$ acts on the fibres. With 3.15 one can define level structures. It is nice that all these constructions can be carried out within the framework of mixed Shimura varieties. We shall apply some of them in chapter 10 to describe a modular interpretation for the mixed Shimura variety associated to $(P_{2g}, \mathcal{X}_{2g})$ as defined in 2.24–25.

Chapter 4

Rational boundary components

We now turn to the basic ingredient of our desired compactification, namely what we shall call rational boundary components. This chapter deals only with group theoretic data, so it depends logically only on chapters 1 and 2. The central observation is that boundary components encode a relation between a given equivariant family of mixed Hodge structures as in 2.1, and another one, which is in some sense “more mixed” (see 4.12).

We begin with some facts for a certain class of parabolic subgroups of P , if $(P, \mathcal{X})$ is mixed Shimura data. This culminates in 4.6, which gives an equivalent characterization of this class through certain homomorphisms from a “reference group” H_0 introduced in 4.3. In 4.7–11 this is then used to construct certain other mixed Shimura data associated to a given $(P, \mathcal{X})$ and an admissible $\mathbb{Q}$ -parabolic subgroup Q . These we call rational boundary components in (a somewhat loose) analogy to the usual terminology (see [AMRT] ch.III §3, and the remark 4.11 (i)). In 4.12–15 we generalize the realization of a hermitian symmetric domain as a Siegel domain of the third kind, as carried out in [AMRT] ch.III §4. In 4.16–23 we study various functional properties, in particular the hereditary behavior of rational boundary components. At the end of the chapter we illustrate the general theory with some examples.

4.1. Cocharacters and parabolic subgroups: Let G be a connected linear algebraic group over a field k . A subgroup P of G , defined over k is called a k -parabolic subgroup, if G/P is a projective variety (see [BR] 4.1).

First we suppose that G is reductive. Let P be a k -parabolic subgroup of G , $B \subset P$ a minimal k -parabolic subgroup, and $T \subset B$ a maximal k -split torus. Let Φ be the corresponding k -root system and $\Delta \subset \Phi$ the simple roots defined by B . Then $\text{Lie } P = \text{Lie } T \oplus \bigoplus_{\alpha \in \Psi} \mathfrak{g}_\alpha$ for some subset $\Psi \subset \Phi$. There exists a subset $\Sigma \subset \Delta$ such that Ψ is the set of all nonnegative linear combinations of elements of Δ together with all nonpositive linear combinations of elements of Σ (see [B–Lie] ch.VI 1.7 prop. 20). Let λ_α be the fundamental weight associated to $\alpha \in \Delta$, then

$$\Psi = \{\beta \in \Phi \mid \forall \alpha \in \Delta - \Sigma \langle \lambda_\alpha, \beta \rangle \geq 0\}.$$

Let $\lambda := \sum_{\alpha \in \Delta - \Sigma} \lambda_\alpha$, then this weight corresponds to a unique cocharacter $\mathbb{G}_{m,k} \rightarrow T$. The Lie algebra of a Levi component of P that contains T is generated by the centralizer

of T (this contains $\text{Lie } T$) and all $\mathfrak{g}_\beta$ for which $\langle \lambda, \beta \rangle = 0$. Thus all these β are orthogonal to λ , hence λ does not depend on the choice of B . Furthermore all maximal k -split tori of P are conjugate under P , so $\lambda : \mathbb{G}_{m,k} \rightarrow P$ is uniquely determined by P up to conjugation by P . Conversely P is uniquely determined by λ , because $\text{Lie } P$ is the direct sum of all nonnegative weight spaces in $\text{Lie } G$ under $\text{Ad}_G \circ \lambda$. Note that λ factors through the subgroup $P \cap G^{\text{der}} \subset G$.

Let W be the unipotent radical of P and $\pi : P \rightarrow P/W$ the canonical projection. Then the composite homomorphism $\pi \circ \lambda : \mathbb{G}_{m,k} \rightarrow P/W$ is a cocharacter of the center of P/W , and is uniquely determined by P . Every lifting of this cocharacter to P occurs as λ . Now let L/k be a field extension and $\lambda_L : \mathbb{G}_{m,L} \rightarrow P_L$ the cocharacter, defined over L , that is associated to P_L as an L -parabolic subgroup.

Claim. : $\pi \circ \lambda_L = \pi \circ \lambda$.

Indeed, choose a maximal L -split torus S_L in P_L that contains T_L . Furthermore choose a minimal L -parabolic subgroup of G_L that contains S_L and is contained in B_L . With these choices we have to prove that $\lambda_L = \lambda$. Let Φ_L be the corresponding L -root system, Δ_L the set of L -simple roots, and $\Sigma_L \subset \Delta_L$ the subset associated to P_L in the same way. Let $\text{Res}_{T \subset S} : \Phi_L \rightarrow \Phi \cup \{0\}$ be the restriction map, then the image of Δ_L lies in $\Delta \cup \{0\}$. By definition an λ -root $\alpha \in \Delta_L$ lies in Σ_L if and only if $\text{Res}_{T \subset S}(\alpha) \in \Sigma \cup \{0\}$. Thus for all $\alpha \in \Delta_L$ we have $\langle \lambda, \text{Res}_{T \subset S}(\alpha) \rangle = 0$ if $\alpha \in \Sigma_L$, and $= 1$ else. This implies $\lambda_L = \lambda$, as asserted.

This shows that the P -conjugacy class of λ does not depend on the field over which P is considered. In particular it implies that the cocharacter $\pi \circ \lambda : \mathbb{G}_{m,k} \rightarrow P/W$ is defined over a subfield of k whenever P is defined over that field.

Now let P be a connected linear algebraic group, which we do not assume to be reductive. Let W be its unipotent radical. Every parabolic subgroup Q of P contains W . Let $\lambda : \mathbb{G}_{m,k} \rightarrow Q/W$ be the cocharacter of above for Q/W as parabolic subgroup of P/W . Suppose that we are given a cocharacter μ of the center of P/W , and let w be a lifting of $\mu \cdot \lambda$ to Q . The Q -conjugacy class of w is uniquely determined by Q and μ . If $\text{Lie } W$ has only nonnegative weights under $\text{Ad } p \circ w$, then we again have that $\text{Lie } Q$ is the direct sum of all nonnegative weight spaces in $\text{Lie } P$ under $\text{Ad}_G \circ w$.

4.2. Application: some special mixed Hodge structures: Let B be the Borel subgroup of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ of $GL_{2,\mathbb{Q}}$. Let $\lambda : \mathbb{G}_{m,\mathbb{Q}} \rightarrow B$ as in 4.1. Explicitly λ has the form $t \mapsto \begin{pmatrix} t & * \\ 0 & t^{-1} \end{pmatrix}$. Let $\mu : \mathbb{G}_{m,\mathbb{Q}} \rightarrow GL(M)$, $t \mapsto t \cdot \text{id}$. For every representation $\nu : \mathbb{G}_m \rightarrow GL(M)$ let $W_n^{(\nu)}$ be the filtration on M by negative weights under ν , that is $W_n^{(\nu)} M$ is the direct sum of the eigenspaces under ν associated to the weights $-m$ for all $m \leq n$. In particular consider the standard representation $M := \mathbb{Q}^2$ of $GL_{2,\mathbb{Q}}$ and its subspace $M' := \begin{pmatrix} * \\ 0 \end{pmatrix}$. Then we get explicitly

$$W_n^{(\mu)} M = \begin{cases} 0 & \text{for } n \leq -2, \\ M & \text{for } n \geq -1. \end{cases} \quad W_n^{(\mu \cdot \lambda)} M = \begin{cases} 0 & \text{for } n \leq -2, \\ M' & \text{for } n = -1, -2, \\ M & \text{for } n \geq -1. \end{cases}$$

Now consider a rational mixed Hodge structure on M , which possesses one of these filtrations as weight filtration, and for which the only nontrivial step in the Hodge filtration is

the term $F^0 M_{\mathbb{C}}$. With the weight filtration $W^{(\mu)}$ this Hodge structure must be pure of type $\{(-1, 0), (0, -1)\}$, and $F^0 M_{\mathbb{C}}$ may be an arbitrary one dimensional subspace of $M_{\mathbb{C}}$ that is not defined over $\mathbb{R}$. With the other weight filtration it must be mixed of type $\{(0, 0), (-1, -1)\}$ and $F^0 M_{\mathbb{C}}$ may be an arbitrary one dimensional subspace of $M'_{\mathbb{C}}$ that is different from $M'_{\mathbb{C}}$. Thus we can make the following observation: For every pure Hodge structure $(W^{(\mu)}, F^{\cdot})$ on M of type $\{(-1, 0), (0, -1)\}$ the pair $(W^{(\mu, \lambda)}, F^{\cdot})$ is a mixed Hodge structure of type $\{(0, 0), (-1, -1)\}$.

The Hodge structure $(W^{(\mu)}, F^{\cdot})$ corresponds to a unique homomorphism $\mathbb{S} \rightarrow GL_{2, \mathbb{R}}$. For the Hodge structure $(W^{(\mu, \lambda)}, F^{\cdot})$ the decomposition $M_{\mathbb{C}} = M'_{\mathbb{C}} \oplus F^0 M_{\mathbb{C}}$ shows that there exists only one decomposition as in 1.2 (a). Thus this Hodge structure corresponds to a unique homomorphism $\mathbb{S}_{\mathbb{C}} \rightarrow B_{\mathbb{C}}$. In this way we have associated to each homomorphism $\mathbb{S} \rightarrow GL_{2, \mathbb{R}}$, which induces a pure Hodge structure of type $\{(-1, 0), (0, -1)\}$ on M , a canonical homomorphism $\mathbb{S}_{\mathbb{C}} \rightarrow B_{\mathbb{C}}$ that induces a mixed Hodge structure of type $\{(0, 0), (-1, -1)\}$. The restriction of these two homomorphisms to $\mathbb{G}_{m, \mathbb{R}} \subset \mathbb{S}$ are precisely the cocharacter μ , *resp.* $\mu \cdot \lambda$.

All pure Hodge structures on M of type $\{(-1, 0), (0, -1)\}$ are conjugate under $GL_2(\mathbb{R})$. Note that the space of these Hodge structures possesses two connected components. The Iwasawa decomposition implies that they are already conjugate under $B(\mathbb{R})$. Clearly the map $(W^{(\mu)}, F^{\cdot}) \mapsto (W^{(\mu, \lambda)}, F^{\cdot})$ is equivariant under $B(\mathbb{R})$. Let us also observe that both Hodge structures induce the same Hodge structure on $\Lambda^2 M$, namely of type $(-1, -1)$. This corresponds to the fact that $\mathbb{S}$ operates on $\Lambda^2 M$ in the same way through both homomorphisms.

4.3. The reference group H_0 : Now we introduce our analog of the group “ $U^1 \times SL(2, \mathbb{R})$ ” considered in [AMRT] ch.III p.174ff. It embodies some of the features of Shimura data, but is defined only over $\mathbb{R}$. It will be used to study certain parabolic subgroups of P for mixed Shimura data $(P, \mathcal{X})$. Consider the reductive group over $\mathbb{R}$

$$H_0 := \{(z, \alpha) \in \mathbb{S} \times GL_{2, \mathbb{R}} \mid z \cdot \bar{z} = \det(\alpha)\}.$$

Let $B_0 \subset H_0$ be the Borel subgroup of all elements of the form $(*, (\begin{smallmatrix} x & * \\ 0 & * \end{smallmatrix}))$. Let $M_0 = \mathbb{R}^2$ and $N_0 (= \mathbb{C} \text{ as an } \mathbb{R}\text{-vector space})$ be the representations of H_0 over $\mathbb{R}$ that come from the two dimensional standard representation of $GL_{2, \mathbb{R}}$, respectively of $\mathbb{S}$. Consider an arbitrary homomorphism $h_0 : \mathbb{S} \rightarrow H_0$ that induces on both M_0 and N_0 a pure Hodge structure of type $\{(-1, 0), (0, -1)\}$. Clearly such a homomorphism exists. Let $h_{\infty} : \mathbb{S}_{\mathbb{C}} \rightarrow \mathbb{S}_{\mathbb{C}} \times B_{\mathbb{C}}$ be the unique homomorphism that induces the same Hodge structure on N_0 and the mixed Hodge structure described in 4.2 on M_0 . Since $\mathbb{S}$ acts on $\Lambda^2 M_0$ in the same way through h_0 and through h_{∞} , the homomorphism h_{∞} factors through $H_{0, \mathbb{C}}$. Let U_0 be the unipotent radical of B_0 , then modulo U_0 h_{∞} is defined over $\mathbb{R}$.

One easily sees that $H_0(\mathbb{R})$ is connected. Hence $H_0(\mathbb{R})$ does not operate transitively on the set of all h_0 . But still $B(\mathbb{R})$ operates transitively and the map $h_0 \mapsto h_{\infty}$ is equivariant under this group. From now on we fix one h_0 and the corresponding h_{∞} . We also define $\lambda_1 : \mathbb{G}_{m, \mathbb{R}} \rightarrow H_0^{der} \cap B_0$ such that $h_{\infty} \circ w = (h_0 \circ w) \cdot \lambda_1$.

4.4. Lemma: Let $(P, \mathcal{X})$ be mixed Shimura data and $\omega : H_{0, \mathbb{C}} \rightarrow P_{\mathbb{C}}$ a homomorphism such that $\pi \circ \omega$ is defined over $\mathbb{R}$. Suppose that we are given an $x \in \mathcal{X}$ such that $h_x = \omega \circ h_0$.

Consider the operation of H_0 on $(\mathrm{Lie} U)_{\mathbb{R}}$, $(\mathrm{Lie} V)_{\mathbb{R}}$ and $(\mathrm{Lie} G)_{\mathbb{R}}$. We want to compare the “new” mixed Hodge structures on these spaces defined by $\mathrm{Ad} p \circ \omega \circ h_{\infty}$ with the “old” mixed Hodge structures defined by $\mathrm{Ad} p \circ h_x = \mathrm{Ad} p \circ \omega \circ h_0$.

- (a) $(\mathrm{Lie} U)_{\mathbb{R}|H_0}$ is a direct sum of copies of $\Lambda^2 M_0$. In the particular $\mathrm{Lie} U$ remains pure of type $(-1, -1)$.
- (b) $(\mathrm{Lie} V)_{\mathbb{R}|H_0}$ is a direct sum of copies of N_0 and M_0 . In particular $\mathrm{Lie} V$ becomes mixed of type $\{(0, 0), (-1, 0), (0, -1), (-1, -1)\}$.
- (c) $(\mathrm{Lie} G)_{\mathbb{R}|H_0}$ is a direct sum of irreducible components occurring in $(N_0)^{\vee} \otimes N_0$, $(N_0)^{\vee} \otimes M_0$ or in $(M_0)^{\vee} \otimes M_0$. In the particular $\mathrm{Lie} G$ becomes mixed of type

$$\{(1, 1), (1, 0), (0, 1), (1, -1), (0, 0), (-1, 1), (0, -1), (-1, 0), (-1, -1)\}.$$

Proof. (Compare [AMRT] ch.III p.182 prop. 3) Every irreducible representation of H_0 over $\mathbb{C}$ is isomorphic to $S^n M_{0,\mathbb{C}} \otimes \mathcal{X}$, where n is a nonnegative integer and $\mathcal{X}$ a character of $H_{0,\mathbb{C}}$. Since the composite homomorphism $\mathbb{S}_{\mathbb{C}} \xrightarrow{h_0} H_{0,\mathbb{C}} \rightarrow (H_0/H_0^{\mathrm{der}})_{\mathbb{C}}$ is an isomorphism, the character group of $H_{0,\mathbb{C}}$ is isomorphic to that of $\mathbb{S}_{\mathbb{C}}$. Let (a, b) be the pair of integers representing the character $\mathcal{X} \circ h_0$, then $S_n M_{0,\mathbb{C}} \otimes \mathcal{X}$ is of type $\{(a, b - n), (a - 1, b - n + 1), \dots, (a - n, b)\}$ under $\mathcal{X} \circ h_0$. By assumption $(\mathrm{Lie} U)_{\mathbb{R}}$ is pure of type $(-1, -1)$, $(\mathrm{Lie} V)_{\mathbb{R}}$ of type $\{(-1, 0), (0, -1)\}$ and $(\mathrm{Lie} G)_{\mathbb{R}}$ of type $\{(-1, 1), (0, 0), (1, -1)\}$ under $\mathrm{Ad} p \circ \omega \circ h_0$. This implies the assertion about the representation of H_0 . The type under $\mathrm{Ad} p \circ \omega \circ h_{\infty}$ follows from this and from 4.2 and 4.3. **q.e.d.**

4.5. Definition: Let $(P, \mathcal{X})$ be mixed Shimura data. Let $G := P/W$, then every $\mathbb{Q}$ -parabolic subgroup of P is the inverse image of a $\mathbb{Q}$ -parabolic subgroup of G^{ad} . Let $G^{\mathrm{ad}} = G_1 \times \dots \times G_r$ be the decomposition into $\mathbb{Q}$ -simple factors. Let $Q_i \subset G_i$ be a $\mathbb{Q}$ -parabolic subgroup for every i , and let $Q \subset P$ be the inverse image of $Q_1 \times \dots \times Q_r$. We call Q an admissible $\mathbb{Q}$ -parabolic subgroup of P , if every Q_i is either equal to G_i or a maximal proper $\mathbb{Q}$ -parabolic subgroup of G_i .

4.6. Proposition: Let $(P, \mathcal{X})$ be mixed Shimura data and Q a $\mathbb{Q}$ -parabolic subgroup of P . As in 2.1 let π' be the projection $P \rightarrow P/U$. Then the following assertions are equivalent:

- (a) Q is an admissible $\mathbb{Q}$ -parabolic subgroup of P .
- (b) For every $x \in \mathcal{X}$ there exists a unique homomorphism $\omega = \omega_x : H_{0,\mathbb{C}} \rightarrow P_{\mathbb{C}}$ such that:
 - (i) $\pi' \circ \omega : H_{0,\mathbb{C}} \rightarrow (P/U)_{\mathbb{C}}$ is already defined over $\mathbb{R}$.
 - (ii) $h_x = \omega \circ h_0$.
 - (iii) $\omega \circ h_{\infty} \circ w : \mathbb{G}_{m,\mathbb{C}} \rightarrow Q_{\mathbb{C}}$ is of the form $\mu \cdot \lambda$ as in 4.1 with $\mu = h_x \circ w$, and $(\mathrm{Lie} Q)_{\mathbb{C}}$ is the direct sum of all nonnegative weight spaces in $(\mathrm{Lie} P)_{\mathbb{C}}$ under $\mathrm{Ad} p \circ \omega \circ h_{\infty} \circ w$ ($w : \mathbb{G}_{m,\mathbb{R}} \hookrightarrow \mathbb{S}$ is defined in 1.3).
- (c) There exists an $x \in \mathcal{X}$ and a homomorphism $\omega : H_{0,\mathbb{C}} \rightarrow P_{\mathbb{C}}$ such that the three conditions in (b) are satisfied.

Proof. (a) $\Rightarrow$ (b): First assume $W = 1$. Let H_1 be the subgroup $\mathbb{S}^1 \times SL_{2,\mathbb{R}} \subset H_0$ and $\mathcal{X}^0$ the connected component of $\mathcal{X}$ that contains x . By [AMRT] ch.III p.220 no.2 the

parabolic subgroup Q corresponds to a unique rational boundary component of $\mathcal{X}^0$, in the terminology of [loc.cit.]. Therefore by [AMRT] p.199 thm. 1 (v) and p.205 thm. 2 there exists a unique homomorphism $\omega_1 : H_1 \rightarrow G_{\mathbb{R}}^{ad}$, such that $h_x|_{\mathbb{S}^1} = \omega_1 \circ (h_0|_{\mathbb{S}^1})$, and $(\text{Lie } Q)_{\mathbb{R}}$ is the direct sum of all nonnegative weight spaces under $\text{Ad } p \circ \omega_1 \circ \lambda_1$. Thus we have a commutative diagram

$$\begin{array}{ccccc}
 & & H_1 & \xrightarrow{\omega_1} & G_{\mathbb{R}}^{ad} \\
 & \nearrow h_0|_{\mathbb{S}^1} & \downarrow & & \uparrow \\
 \mathbb{S}^1 & & H_0 & & G_{\mathbb{R}} \\
 \downarrow & \nearrow h_0 & & \nearrow h_x & \\
 \mathbb{S} & & & &
 \end{array}$$

and our aim is to find a homomorphism $\omega : H_0 \rightarrow G_{\mathbb{R}}$ which completes it to a commutative diagram. Since H_0 is generated by $h_0(\mathbb{S})$ and H_1 , such ω is uniquely determined such if it exists. Moreover then the conditions (i) and (ii), and the second part of condition (iii) are automatically satisfied. The first part of (iii) is equivalent to the assertion that $\omega_1 \circ \lambda_1$ is a λ as in 4.1. This follows from the explicit description of the $\mathbb{R}$ -root system of $(P/W)^{ad}$ in [AMRT] ch.III and the construction of ω_1 in the proof of [loc. cit.] p.199 thm. 1 (v). It thus remains only to construct some ω , with which the diagram stays commutative.

First we shall lift ω_1 to a homomorphism $\tilde{\omega}_1 : H_1 \rightarrow G$. Let us write $h_0|_{\mathbb{S}^1} = (e_1, e_2)$, where $e_1 : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is the identity and e_2 is a homomorphism $\mathbb{S}^1 \rightarrow SL_{2,\mathbb{R}}$. Since $SL_{2,\mathbb{R}}$ is simply connected, we can lift ω_1 uniquely on the second factor to a homomorphism $\tilde{\omega}_1 : SL_{2,\mathbb{R}} \rightarrow G_{\mathbb{R}}^{der}$. On the first factor we then define the lifting by $z \mapsto h_0(z) \cdot \tilde{\omega}_1(e_2(z))^{-1}$. With this definition we obtain another commutative diagram

$$\begin{array}{ccccc}
 & & H_1 & & \\
 & \nearrow h_0|_{\mathbb{S}^1} & \downarrow & \searrow \tilde{\omega}_1 & \\
 \mathbb{S}^1 & & H_0 & & G_{\mathbb{R}} \\
 \downarrow & \nearrow h_0 & & \nearrow h_x & \\
 \mathbb{S} & & & &
 \end{array}$$

Consider the homomorphism

$$\mathbb{G}_{m,\mathbb{R}} \times H_1 = \mathbb{G}_{m,\mathbb{R}} \times \mathbb{S}^1 \times SL_{2,\mathbb{R}} \rightarrow H_0, (t, z, g) \mapsto (t \cdot z, t \cdot g).$$

This is an isogeny of degree 2, whose kernel is $\{\pm(1, 1, id)\}$. Since by 2.1 (iii) the image of $h_x|_{\mathbb{G}_{m,\mathbb{R}}}$ lies in the center of $G_{\mathbb{R}}$, we can define a homomorphism

$$\tilde{\omega} : \mathbb{G}_{m,\mathbb{R}} \times \mathbb{S}^1 \times SL_{2,\mathbb{R}} \rightarrow G_{\mathbb{R}}, (t, z, g) \mapsto h_x(t) \cdot \tilde{\omega}_1((z, g)).$$

Now the desired homomorphism ω exists if and only if $\tilde{\omega}$ factors through the above isogeny. But this follows from the calculation

$$\begin{aligned}
 \tilde{\omega}((-1, -1, -id)) &= h_x(-1) \cdot \tilde{\omega}_1((-1, -id)) \\
 &= h_x(-1) \cdot \tilde{\omega}_1(h_0(-1)) \\
 &= h_x(-1) \cdot h_x(-1) \\
 &= 1.
 \end{aligned}$$

Thus the assertion is proved in the case $W = 1$.

Now let P be arbitrary. Then we already have a unique homomorphism $H_0 \rightarrow G_{\mathbb{R}}$. Note that $h_x \circ w$ determines a unique Levi decomposition of $P_{\mathbb{C}}$, which is defined over $\mathbb{R}$ modulo $U_{\mathbb{C}}$. The desired homomorphism $\omega : H_{0,\mathbb{C}} \rightarrow P_{\mathbb{C}}$ must map into this Levi component, since the image of $h_0 \circ w$ lies in the center of H_0 . Thus it is uniquely determined and we can define it directly using this Levi decomposition. Condition (i) then follows from the same property of the decomposition. Condition (ii) holds by construction. Likewise (iii) follows after we have shown that $\text{Lie } W$ is of weight ≥ 0 under $\text{Ad}_P \circ \omega \circ h_{\infty} \circ w$. But this follows from 4.4 (a) and (b). This finishes the proof of the implication (a) $\Rightarrow$ (b).

The implication (b) $\Rightarrow$ (c) is trivial. For (c) $\Rightarrow$ (a) we may assume P is to be semisimple after having divided by the radical of P as in 2.9. Then the assertion follows from [AMRT] ch.III, in particular p.174 cor., p.180 prop. 2 and the construction of the standard parabolic subgroups of P in [loc. cit.] ch.III §3. **q.e.d.**

4.7. A canonical subgroup of an admissible $\mathbb{Q}$ -parabolic subgroup: We now want to use the data given by 4.6 to construct certain other mixed Shimura data associated to a given $(P, \mathcal{X})$ and an admissible $\mathbb{Q}$ -parabolic subgroup Q . The first step is to define the group involved. Let $(P, \mathcal{X})$, Q , x and ω_x be as in 4.6. We define P_1 to be the smallest normal subgroup of Q , defined over $\mathbb{Q}$, such that $\omega_x \circ h_{\infty} : \mathbb{S} \rightarrow Q_{\mathbb{C}}$ factors through $P_{1,\mathbb{C}}$.

Claim. (a) $\omega_x \circ h_{\infty} : \mathbb{S} \rightarrow Q_{\mathbb{C}}$ does not depend on the choice of h_0 .

(b) P_1 depends only on $(P, \mathcal{X})$ and on Q .

(c) The map $\mathcal{X} \rightarrow \text{Hom}(\mathbb{S}_{\mathbb{C}}, P_{1,\mathbb{C}})$, $x \mapsto \omega_x \circ h_{\infty}$ is $Q(\mathbb{R}) \cdot U(\mathbb{C})$ -equivariant.

Proof. Fix $x \in \mathcal{X}$. By 4.2 every other h'_0 is of the form $\text{int}(b) \circ h_0$ for some $b \in B(\mathbb{R})$, and then the corresponding h'_{∞} equals $\text{int}(b) \circ h_{\infty}$. The homomorphism $\omega'_x := \omega_x \circ \text{int}(b^{-1})$ satisfies the conditions of 4.6 with h_0 replaced by h'_0 . The assertion (a) now follows from the equation $\omega'_x \circ h'_{\infty} = \omega_x \circ \text{int}(b^{-1}) \circ \text{int}(b) \circ h_{\infty} = \omega_x \circ h_{\infty}$. In particular P_1 does not depend on the choice of h_0 . By the Iwasawa decomposition for G the group $Q(\mathbb{R}) \cdot U(\mathbb{C})$ already operates transitively on $\mathcal{X}$. Let $q \in Q(\mathbb{R}) \cdot U(\mathbb{C})$, then $\text{int}(q) \circ \omega_x$ satisfies the same conditions of 4.6 as $\omega_{q \cdot x}$. This implies (b) and (c), since q normalizes $P_{1,\mathbb{C}}$. **q.e.d.**

4.8. Lemma: Consider the situation of 4.7. Let W_1 be the unipotent radical of P_1 . Then there exists a subgroup $U_1 \subset W_1$, defined over $\mathbb{Q}$, normal in Q , and depending only on $(P, \mathcal{X})$ and Q , such that $(P_1, \mathcal{X}_1)$, W_1 , U_1 together with the homomorphism $\omega_x \circ h_{\infty} : \mathbb{S}_{\mathbb{C}} \rightarrow P_{1,\mathbb{C}}$ satisfy the conditions 2.1 (ii)–(v).

Proof. Conditions 2.1 (iii) and (iv) do not depend on the choice of U_1 and are easily proved. In fact, (iii) is a direct consequence of 4.6 (iii) and the same assertion for $(P, \mathcal{X})$. For (iv) observe that $\text{Lie } P_1$ is contained in $\text{Lie } Q$, and the latter is of weight ≤ 0 by 4.6 (iii), so the condition follows from 4.4.

Next we have to construct U_1 . Before we can do this, however, we have to prove that $\text{Ad}_Q \circ \omega_x \circ h_{\infty}$ induces a rational mixed Hodge structure on $\text{Lie } Q$, and that its weight filtration is stable under Q . Then it is also independent of x . To prove this we may replace $(P, \mathcal{X})$ by $(P, \mathcal{X})/Z(P)$, then the homomorphism $\pi \circ h_x \circ w : \mathbb{G}_{m,\mathbb{R}} \rightarrow G_{\mathbb{R}}$ of 2.1 (iii) is already defined over $\mathbb{Q}$. It suffices to show the conditions 1.4 (i)–(iii) for the group P_1 and the homomorphism $\omega_x \circ h_{\infty}$. Denote the unipotent radical of Q by W_Q .

In the case $W = 1$ we show the conditions a fortiori for Q instead of P_1 . By 4.6 (b) (i) ω_x is already defined over $\mathbb{R}$. Since $\text{Lie } U_0$ is of weight -2 under $\text{Ad}_{B_0} \circ h_\infty$, it is mapped to W_Q under ω_x . Now h_∞ is defined over $\mathbb{R}$ modulo U_0 , hence modulo W_Q the homomorphism $\omega_x \circ h_\infty$ is defined over $\mathbb{R}$. This is condition 1.4 (i). For 1.4 (ii) we have to show that modulo W_Q the homomorphism $\omega_x \circ h_\infty \circ w$ is central and defined over $\mathbb{Q}$. By 4.6 (b) (iii) this follows from the corresponding assertion for $h_x \circ w$ (see above and 2.1 (iii) for $(P, \mathcal{X})$) and for λ (see 4.1). For 1.4 (iii) we have to prove that $\text{Lie } W_Q$ is of weight ≤ -1 under $\text{Ad}_Q \circ \omega_x \circ h_\infty$. But this follows again from 4.6 (b) (iii), since the centralizer of $w_x \circ h_\infty \circ w$ is a Levi component of Q .

Let us now consider the general case. The conditions 1.4 (i) and (ii) carry over directly, so it remains to study $\text{Lie } W_Q$. Condition 1.4 (iii) follows from P_1 if we can prove the equality $\text{Lie } W_1 = W_{-1}(\text{Lie } W_Q)$. We may fix a splitting $P = W \rtimes G$ and assume without loss of generality that h_x , hence also ω_x , factors through $G_{\mathbb{R}}$. Since we have already proved the conditions of 1.4 for $Q/W \cong G \cap Q$, we have a rational mixed Hodge structure on $\text{Lie } Q$ that is stable under $G \cap Q$. By 4.6 (b) (iii) $\text{Lie } W_Q$ is of weight ≤ 0 . Thus $[\text{Lie } W_Q, W_{-1}(\text{Lie } W_Q)]$ is contained in $W_{-1}(\text{Lie } W_Q)$, so $W_{-1}(\text{Lie } W_Q)$ is an ideal of $\text{Lie } W_Q$. Since it is already stable under $G \cap Q$, it is already an ideal of $\text{Lie } Q$. Thus $W_Q^1 := \exp(W_{-1}(\text{Lie } W_Q)) \subset W_Q$ is a normal subgroup of Q . By the preceding paragraph we have $W \cdot W_Q^1 = W_Q$, hence W_Q/W_Q^1 is isomorphic to $W/(W \cap W_Q^1)$. By 4.4 (a) U must be contained in W_Q^1 , so W_Q/W_Q^1 is isomorphic to a quotient of V . Thus it is abelian, so the adjoint operation of Q on $\text{Lie}(W_Q/W_Q^1)$ factors through Q/W_Q . Moreover 4.4 (b) implies that $\text{Lie}(W_Q/W_Q^1)$ is pure of type $(0, 0)$. This means that the image of $\omega_x \circ h_\infty$ lies in the kernel of the adjoint representation of Q on $\text{Lie}(W_Q/W_Q^1)$. By definition P_1 must also lie in this kernel, so it operates trivially on W_Q/W_Q^1 . We have now proved that the Levi component of the group $(P_1 \cdot W_Q)/W_Q^1$ operates trivially on the unipotent radical. But such a group possesses only one Levi decomposition, which is a direct product. Again from the definition of P_1 it follows that $(P_1 \cdot W_Q^1)/W_Q^1$ is the unique Levi component of $(P_1 \cdot W_Q)/W_Q^1$, whence $W_1 \subset W_Q^1$, as desired. The reverse inclusion follows from the fact that P_1 operates nontrivially on every simple subquotient of W_Q^1 .

If we now define $U_1 := \exp(W_{-2}(\text{Lie } W_1))$, then by the above this subgroup is normal in Q and independent of x . Condition 2.1 (v) holds by construction and by the result in the preceding paragraph. Finally 2.1 (ii) follows with the same argument as above: Since by 4.4 (a) $U \subset U_1$, we may without loss of generality assume that $U = 1$. Then by 4.6 (b) (i) ω_x is already defined over $\mathbb{R}$. Since $\text{Lie } U_0$ is of weight -2 , it is mapped to U_1 under ω_x . Because h_∞ is defined over $\mathbb{R}$ modulo U_0 , it follows that modulo U_1 the homomorphism $\omega_x \circ h_\infty$ is also defined over $\mathbb{R}$. **q.e.d.**

4.9. Lemma: Consider the situation of 4.8. Let $P_{11} \subset P_1 \cap P^{\text{der}}$ be the smallest normal subgroup such that $(P_1 \cap P^{\text{der}})/P_{11}$ is a $\mathbb{Q}$ -split torus.

Let $\tilde{G}_{1,\mathbb{C}}$ be the centralizer of $\omega_x \circ h_\infty \circ w$ in $P_{1,\mathbb{C}}$, and let $\tilde{G}_{11,\mathbb{C}} := \tilde{G}_{1,\mathbb{C}} \cap P_{11,\mathbb{C}}$. These Levi components of $P_{1,\mathbb{C}}$ resp. of $P_{11,\mathbb{C}}$. Then

(a) $\tilde{G}_{11,\mathbb{C}}$ commutes with $\omega_x(H_{0,\mathbb{C}}^{\text{der}})$ in $P_{\mathbb{C}}$. Thus $\tilde{G}_{11,\mathbb{C}} \cdot \omega_x(H_{0,\mathbb{C}}^{\text{der}})$ is a reductive subgroup of $P_{\mathbb{C}}$.

(b) The subset $\tilde{G}_{1,\mathbb{C}} \cdot \omega_x(H_{0,\mathbb{C}}^{der})$ is a reductive subgroup of $P_{\mathbb{C}}$. It contains $\tilde{G}_{11,\mathbb{C}} \cdot \omega_x(H_{0,\mathbb{C}}^{der})$ as a normal subgroup, and it contains $\omega_x(H_{0,\mathbb{C}})$ as a not necessarily normal subgroup. Its semisimple part is the same as that of $\tilde{G}_{11,\mathbb{C}} \cdot \omega_x(H_{0,\mathbb{C}}^{der})$.

(c) If ω_x is defined over $\mathbb{R}$, then each of the groups $\tilde{G}_{11,\mathbb{C}}$, $\tilde{G}_{11,\mathbb{C}} \cdot \omega_x(H_{0,\mathbb{C}}^{der})$ and $\tilde{G}_{1,\mathbb{C}} \cdot \omega_x(H_{0,\mathbb{C}}^{der})$ is defined over $\mathbb{R}$.

Proof. Since all the groups mentioned lie inside the centralizer of $h_x \circ w$ in $P_{\mathbb{C}}$, which is a Levi component of $P_{\mathbb{C}}$, both (a) and (b) follow directly from the corresponding assertion for $(P, \mathcal{X})/W$ instead of $(P, \mathcal{X})$. The same holds for (c), since this Levi component is defined over $\mathbb{R}$ if ω_x is defined over $\mathbb{R}$. Thus we may assume that $W = 1$, so ω_x is defined over $\mathbb{R}$.

(a) Since U_0 is of weight -2 under $\text{Ad} \circ h_{\infty}$, it is mapped to $U_{1,\mathbb{R}}$ under ω_x . Since $(\text{Lie } U_1)_{\mathbb{R}}|_{\text{Ad}_P \circ \omega_x \circ h_{\infty}}$ is pure of type $(-1, -1)$, $\mathbb{S}$ operates through scalars on $(\text{Lie } U_1)_{\mathbb{R}}$. Hence by definition P_1 also operates through scalars on $\text{Lie } U_1$. Thus P_{11} operates trivially on U_1 , and in particular $\tilde{G}_{11,\mathbb{C}}$ commutes with $\omega_x(U_{0,\mathbb{C}})$. By construction $\tilde{G}_{11,\mathbb{C}}$ also commutes with the image of $\omega_x \circ h_{\infty} \circ w$. Now the equation $h_{\infty} \circ w = (h_0 \circ w) \cdot \lambda_1$ of 4.3, and the fact that the image of $\omega_x \circ h_0 \circ w = h_x \circ w$ lies in the center of P , imply that $\tilde{G}_{11,\mathbb{C}}$ commutes with the image of $\omega_x \circ \lambda_1$. Hence it commutes with the Borel subgroup $\omega_x(H_0^{der} \cap B_0)_{\mathbb{C}}$ of $\omega_x(H_{0,\mathbb{C}}^{der})$. This implies, for instance by the lemma 1 of [B-Lie] ch.VIII §11.1, that $\tilde{G}_{11,\mathbb{C}}$ commutes with the whole of $\omega_x(H_{0,\mathbb{C}}^{der})$, as desired.

(b) It is easy to see that the assertions are invariant under replacing $(P, \mathcal{X})$ by $(P, \mathcal{X})/Z(P)$, so we may assume that P is semisimple. Then by the definition of P_1 we have $P_{1,\mathbb{R}} = P_{11,\mathbb{R}} \cdot \omega_x \circ h_{\infty} \circ w(\mathbb{G}_{m,\mathbb{R}})$, whence $\tilde{G}_{1,\mathbb{C}} = \tilde{G}_{11,\mathbb{C}} \cdot \omega_x \circ h_{\infty} \circ w(\mathbb{G}_{m,\mathbb{R}})$. Since the center of $P = G$ is trivial, we have $w_x \circ h_0 \circ w = h_x \circ w = 1$, hence by 4.3 $\omega_x \circ h_{\infty} \circ w = (\omega_x \circ h_0 \circ w) \cdot (\omega_x \circ \lambda_1) = \omega_x \circ \lambda_1$. But λ_1 factors through $H_{0,\mathbb{C}}^{der}$, so we even get $\tilde{G}_{1,\mathbb{C}} \cdot \omega_x(H_{0,\mathbb{C}}^{der}) = \tilde{G}_{11,\mathbb{C}} \cdot \omega_x(H_{0,\mathbb{C}}^{der})$. Thus the desired assertions are obvious, except that it contains $\omega_x(H_{0,\mathbb{C}})$. But by definition $\tilde{G}_{1,\mathbb{C}}$ contains the image of $\omega_x \circ h_{\infty}$, so this follows from the fact that H_0 is generated by H_0^{der} together with the image of h_{∞} .

(c) Let $b \in (H_0^{der} \cap B_0)(\mathbb{C})$ such that $\text{int}(b) \circ h_{\infty} : \mathbb{S}_{\mathbb{C}} \rightarrow H_{0,\mathbb{C}}$ is defined over $\mathbb{R}$. Then by definition both $\text{int}(\omega_x(b))(\tilde{G}_{1,\mathbb{C}})$ and $\text{int}(\omega_x(b))(\tilde{G}_{11,\mathbb{C}})$ are defined over $\mathbb{R}$. Since $\omega_x(b)$ lies in $\omega_x(H_0^{der})(\mathbb{C})$, the assertion follows for the groups $\tilde{G}_{1,\mathbb{C}} \cdot \omega_x(H_{0,\mathbb{C}}^{der})$ and $\tilde{G}_{11,\mathbb{C}} \cdot \omega_x(H_{0,\mathbb{C}}^{der})$. For $\tilde{G}_{11,\mathbb{C}}$ it follows since by (a) this subgroup commutes with $\omega_x(b)$. **q.e.d.**

4.10. Corollary: In 4.8 the conditions 2.1 (vi)–(viii) are also satisfied.

Proof. For 2.1 (vi) and (vii) we may assume that P is semisimple. Consider the group $\tilde{G}_{11,\mathbb{R}}$ of 4.9, which by 4.9 (c) is defined over $\mathbb{R}$. By 4.9 (b) it is a normal subgroup of $\tilde{G}_{1,\mathbb{C}} \cdot \omega_x(H_{0,\mathbb{C}})$, so in particular it is normalized by the image of $\omega_x \circ h_0 = h_x$. The condition 2.1 (vi) for $(P, \mathcal{X})$ thus implies that $\text{int}(h_x(\sqrt{-1}))$ induces a Cartan involution on $\tilde{G}_{11,\mathbb{R}}$. By definition $h_0(\sqrt{-1}) \equiv h_{\infty}(\sqrt{-1})$ modulo H_0^{der} , so $\text{int}(\omega_x \circ h_{\infty}(\sqrt{-1}))$ also induces a Cartan involution on $\tilde{G}_{11,\mathbb{R}}$. This implies 2.1 (vi) for P_1 . By the definition of P_1 the image of $\omega_x \circ h_{\infty}(\mathbb{S}_{\mathbb{C}})$ in any $\mathbb{Q}$ -direct factor of $(P_1/W_1)^{ad}$ is nontrivial, so it also implies 2.1 (vii).

To prove 2.1 (viii) we may replace $(P, \mathcal{X})$ by $(P, \mathcal{X})/Z(P)$. Then the center of G is an almost direct product of a $\mathbb{Q}$ -split torus with a torus of a compact type defined over $\mathbb{Q}$. It suffices to prove the same for the center of P_1/W_1 , and to do this we may assume that $W = 1$. Then we may again divide by $Z(P)$, after which we find ourselves in the situation

above. Since P_1/P_{11} is a $\mathbb{Q}$ -split torus, it suffices to study the center of P_{11}/W_1 . By the above $\text{int}(\omega_x \circ h_\infty(\sqrt{-1}))$ induces a Cartan involution $(P_{11}/W_1)_{\mathbb{R}}$, but since the image of $\omega_x \circ h_\infty(\sqrt{-1})$ in $(P_1/W_1)(\mathbb{R})$ commutes with the center of $(P_{11}/W_1)_{\mathbb{R}}$, the latter is already of compact type. **q.e.d.**

4.11. Rational boundary components: Consider the situation of 4.8. Let $q \in Q(\mathbb{R})^0 \cdot U(\mathbb{C})$, then by 4.7 (c) we have $\omega_{q \cdot x} \circ h_\infty = \text{int}(q) \circ \omega_x \circ h_\infty$. Since $Q(\mathbb{R})^0 \cdot U(\mathbb{C})$ induces only inner automorphisms on $P_1(\mathbb{R})/W_1(\mathbb{R})$, there exists $p_1 \in P_1(\mathbb{R})$ such that $\omega_{q \cdot x} \circ h_\infty \equiv \text{int}(p_1) \circ \omega_x \circ h_\infty$ modulo $W_1(\mathbb{C})$. But the Levi decomposition of $P_{1,\mathbb{C}}$ defined by $\omega_{q \cdot x} \circ h_\infty$ and by $\omega_x \circ h_\infty$ are modulo U_1 both defined over $\mathbb{R}$, and any two such Levi decompositions are conjugate under $W_1(\mathbb{R}) \cdot U_1(\mathbb{C})$. Hence we may even find $p_1 \in P_1(\mathbb{R}) \cdot U_1(\mathbb{C})$ such that $\omega_{q \cdot x} \circ h_\infty = \text{int}(p_1) \circ \omega_x \circ h_\infty$. Since $Q(\mathbb{R})^0 \cdot U(\mathbb{C}) \cdot x$ is a connected component of $\mathcal{X}$, this shows that the map $\mathcal{X} \rightarrow \text{Hom}(\mathbb{S}_{\mathbb{C}}, P_{1,\mathbb{C}})$, $x \mapsto \omega_x \circ h_\infty$ maps every connected component of $\mathcal{X}$ to a $P_1(\mathbb{R}) \cdot U_1(\mathbb{C})$ -orbit.

Through $\pi_0(P_1(\mathbb{R}) \cdot U_1(\mathbb{C})) = \pi_0(P_1(\mathbb{R})) \xrightarrow{[inclusion]} \pi_0(P(\mathbb{R}))$ the group $P_1(\mathbb{R}) \cdot U_1(\mathbb{C})$ operates on $\pi_0(\mathcal{X})$ in a canonical way. Consider the $Q(\mathbb{R}) \cdot U(\mathbb{C})$ -equivariant map

$$\mathcal{X} \rightarrow \pi_0(\mathcal{X}) \times \text{Hom}(\mathbb{S}_{\mathbb{C}}, P_{1,\mathbb{C}}), \quad x \mapsto ([x], \omega_x \circ h_\infty),$$

where we denote by $[x]$ the connected component of $\mathcal{X}$ that contains x . Let $\mathcal{X}_1$ be a $P_1(\mathbb{R}) \cdot U_1(\mathbb{C})$ -orbit in $\pi_0(\mathcal{X}) \times \text{Hom}(\mathbb{S}_{\mathbb{C}}, P_{1,\mathbb{C}})$ that contains an element of the form $([x], \omega_x \circ h_\infty)$. The image of the above map is contained in finitely many such $\mathcal{X}_1$. Let $\mathcal{X}^+ \subset \mathcal{X}$ be the inverse image of $\mathcal{X}_1$. From 4.8, 4.10 and the definition of P_1 we now get:

$(P_1, \mathcal{X}_1)$ is irreducible mixed Shimura data. It depends only on $(P, \mathcal{X})$, Q and the choice of a $P_1(\mathbb{R})$ -orbit in $\pi_0(\mathcal{X})$. There is a canonical $P_1(\mathbb{R}) \cdot U(\mathbb{C})$ -equivariant map $\mathcal{X}^+ \rightarrow \mathcal{X}_1$, which induces a bijection on the connected components $\pi_0(\mathcal{X}^+) \xrightarrow{\sim} \pi_0(\mathcal{X}_1)$. We call $(P_1, \mathcal{X}_1)$ a rational boundary component of $(P, \mathcal{X})$. We say that $(P_1, \mathcal{X}_1)$ is a proper rational boundary component if Q is a proper parabolic subgroup of P . Otherwise we call $(P_1, \mathcal{X}_1)$ an improper boundary component. Note that $(P, \mathcal{X})$ is a rational boundary component of itself if and only if it is irreducible.

Remarks: (i) This name is somewhat misleading, but it was chosen for the following reason. Let $W = 1$, then the pure Shimura data $(P_1, \mathcal{X}_1)/W_1$ corresponds to some boundary components in the Baily-Borel compactification of $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ (see 6.2–3), while $(P_1, \mathcal{X}_1)$ really corresponds to a formal neighborhood of this. For arbitrary P we have the following situation. The boundary strata in a toroidal compactification of $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ correspond to a mixed Shimura varieties associated to certain quotients $(P_1, \mathcal{X}_1)/U_2$ for $U_2 \subset U_1$, and again $(P_1, \mathcal{X}_1)$ itself corresponds to a formal neighborhood of this (see 7.3 and 7.17).

(ii) With our definition a rational boundary component is always irreducible. This is not strictly necessary, since other choices for P_1 , which differ from our definition only in the center of P_1/W_1 , also lead to mixed Shimura data. For our purposes, however, the irreducibility requirement seems the most practical.

4.12. Proposition Consider mixed Shimura data $(P, \mathcal{X})$ and a rational boundary component $(P_1, \mathcal{X}_1)$ of $(P, \mathcal{X})$. Let M be a representation of P . Let $x \in \mathcal{X}$ and x_1 its image

in $\mathcal{X}_1$. By 1.4 (a) both x and x_1 define a rational mixed Hodge structure on M . The Hodge filtration of these two Hodge structures coincide.

Remarks: (i) By 4.6 (iii) the weight filtration of the Hodge structure defined by x_1 depends only on $(P, \mathcal{X})$ and Q . The proposition thus shows how the “new” Hodge structure can be obtained from the “old” one. By 1.16 this Hodge structure is described by a unique homomorphism $\mathbb{S}_{\mathbb{C}} \rightarrow P_{1,\mathbb{C}}$. Thus with this result one could define $(P_1, \mathcal{X}_1)$ and the map $\mathcal{X}^+ \rightarrow \mathcal{X}_1$ directly.

(ii) In particular the proposition implies that the “old” Hodge filtration together with the weight filtration defined by Q define a mixed Hodge structure. According to M. Rapoport this fact has already been known for reductive P . I could, however, not find a reference.

Proof. Let ω_x be as in 4.6, then the two Hodge filtrations are defined by $\omega_x \circ h_0$ and by $\omega_x \circ h_{\infty}$. Thus it suffices to show the analogous statement for every representation of $H_{0,\mathbb{C}}$. As in the proof of 4.4 every irreducible representation of $H_{0,\mathbb{C}}$ is of the form $S^n M_{0,\mathbb{C}} \otimes \chi$ for a nonnegative integer n and a linear character χ of $H_{0,\mathbb{C}}$. In particular every irreducible representation occurs in some $M_{0,\mathbb{C}^{\otimes n}} \otimes \chi$. Since h_{∞} and h_0 do not differ modulo H_0^{der} , the desired assertion holds for every one dimensional representation. By the definition of h_{∞} in 4.3 it also holds for $M_{0,\mathbb{C}}$. Since the Hodge filtration on a tensor product is given by the formula $F^p(M_1 \otimes M_2) = \sum_{p_1+p_2=p} F^{p_1} M_1 \otimes F^{p_2} M_2$, in which the weight filtration does not occur, the assertion also follows for $M_{0,\mathbb{C}^{\otimes n}} \otimes \chi$. **q.e.d.**

4.13. Corollary: The map $\mathcal{X}^+ \rightarrow \mathcal{X}_1$ is injective and holomorphic.

Proof. Let M be a faithful representation of P . By 4.12 the Hodge structure on M defined by $x \in \mathcal{X}^+$ is already uniquely determined by the image of x in $\mathcal{X}_1$. By 1.16 and 2.12 the injectivity follows on each connected component of $\mathcal{X}^+$. By construction the map is bijective on the set of connected components, hence it itself is injective. The holomorphy follows from 1.7, 1.16 and 4.12. **q.e.d.**

4.14. Imaginary part: Let $(P, \mathcal{X})$ be mixed Shimura data. As in 1.11 let $U(\mathbb{R})(-1) = (2\pi\sqrt{-1})^{-1} \cdot U(\mathbb{R})$ inside the $\mathbb{C}$ -vector space $U(\mathbb{C})$. This notation for the imaginary part may seem somewhat arbitrary, but we shall see in 6.6 why the Tate twist by -1 is the natural choice. Now every element of $P(\mathbb{R}) \cdot U(\mathbb{C})$ can be written uniquely in the form $u \cdot p$ with $u \in U(\mathbb{R})(-1)$ and $p \in P(\mathbb{R})$. Since $P(\mathbb{R})U(\mathbb{C})$ operates transitively on $\mathcal{X}$, and for some $x_0 \in \mathcal{X}$ the homomorphism h_{x_0} is defined over $\mathbb{R}$, for every $x \in \mathcal{X}$ there exists an element $u_x \in U(\mathbb{R})(-1)$ such that $\text{int}(u_x^{-1}) \circ h_x$ is defined over $\mathbb{R}$. By 1.17 (a) this element is uniquely determined. Thus we have a canonical map

$$\text{im} : \mathcal{X} \rightarrow U(\mathbb{R})(-1), \quad x \mapsto \text{im}(x) := u_x,$$

which we call the projection to the imaginary part. Let $P(\mathbb{R}) \cdot U(\mathbb{C})$ act on $U(\mathbb{R})(-1)$ through the isomorphism $U(\mathbb{R})(-1) \rightarrow P(\mathbb{R}) \cdot U(\mathbb{C})/P(\mathbb{R})$, $u \mapsto [u]$. In other words we let $P(\mathbb{R})$ act by conjugation, and $U(\mathbb{R})(-1)$ by translation on itself. Clearly im is equivariant and surjective. Moreover this construction is functorial. In particular, if P is a normal subgroup of a connected group Q , then $Q(\mathbb{R})^0 \cdot U(\mathbb{C})$ operates again equivariantly on $\mathcal{X}$ and

on $U(\mathbb{R})(-1)$. Also, for a morphism $(P, \mathcal{X}) \rightarrow (P', \mathcal{X}')$ we have a $P(\mathbb{R}) \cdot U(\mathbb{C})$ -equivariant commutative diagram:

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{X}' \\ \downarrow & & \downarrow \\ U(\mathbb{R})(-1) & \longrightarrow & U'(\mathbb{R})(-1). \end{array}$$

4.15. Proposition: Consider mixed Shimura data $(P, \mathcal{X})$ and a rational boundary component $(P_1, \mathcal{X}_1)$ of $(P, \mathcal{X})$. Let $\mathcal{X}^+ \subset \mathcal{X}$ be as in 4.11. Let $\mathcal{X}^0$ be a connected component of $\mathcal{X}^+$, and $\mathcal{X}_1^0$ the corresponding connected component of $\mathcal{X}_1$.

- (a) The map $\mathcal{X}^+ \rightarrow \mathcal{X}_1$ is an open embedding.
- (b) The image of $\mathcal{X}^0$ in $\mathcal{X}_1^0$ is the inverse image of an open convex cone $C : +C(\mathcal{X}^0, P_1) \subset U_1(\mathbb{R})(-1)$ under the map $\text{im}_{\mathcal{X}_1^0} : \mathcal{X}_1^0 \rightarrow U_1(\mathbb{R})(-1)$ defined in 4.14.
- (c) This cone is an orbit in $U_1(\mathbb{R})(-1)$ under translation by $U(\mathbb{R})(-1)$ and conjugation by $Q(\mathbb{R})^0$. It is also invariant under translation by $(U_1 \cap W)(\mathbb{R})(-1)$.
- (d) Modulo $(U_1 \cap W)(\mathbb{R})(-1)$ the cone C is a nondegenerate homogeneous selfadjoint cone (in the sense of [AMRT] ch.II p.57 §1.1).

Remark. In general $C(\mathcal{X}^0, P_1)$ does depend on the connected component $\mathcal{X}^0$. See for instance 4.26.

Proof. In the case $W = 1$ these are the assertions of [AMRT] ch.III p.227 thm. 1 and the lemma p.236. In the general case let $(P', \mathcal{X}') := (P, \mathcal{X})/W$ and $(P'_1, \mathcal{X}'_1) := (P_1, \mathcal{X}_1)/(W \cap W_1)$, then in the obvious way $(P'_1, \mathcal{X}'_1)$ is a rational boundary component of $(P', \mathcal{X}')$. The maps of 4.11 and 4.14 yield a commutative equivariant diagram

$$\begin{array}{ccccc} \mathcal{X}^0 & \hookrightarrow & \mathcal{X}_1^0 & \xrightarrow{\text{im}} & U_1(\mathbb{R})(-1) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{X}'^0 & \hookrightarrow & \mathcal{X}'_1{}^0 & \xrightarrow{\text{im}} & U'_1(\mathbb{R})(-1) \end{array}$$

Let us first prove that the image of $\mathcal{X}^0$ in $\mathcal{X}_1^0$ is stable under $W(\mathbb{R}) \cdot (W \cap U_1)(\mathbb{C})$. For this consider a faithful representation M on P . By 1.17 (b) the set of mixed Hodge structures on M defined by $\mathcal{X}^0$ is invariant under $W(\mathbb{C})$, hence also under $W(\mathbb{R}) \cdot (W \cap U_1)(\mathbb{C})$. By 4.12 the map $\mathcal{X}^0 \rightarrow \mathcal{X}_1^0$ is equivariant under this group, which proves the desired assertion.

By definition we have $\mathcal{X}_1^0 \cong \mathcal{X}_1^0/(W \cap W_1)(\mathbb{R}) \cdot (W \cap U_1)(\mathbb{C})$ and we know that the image of $\mathcal{X}^0$ is invariant under $(W \cap W_1)(\mathbb{R}) \cdot (W \cap U_1)(\mathbb{C})$. This shows that left hand square in the above diagram is cartesian. Let $C(\mathcal{X}^0, P_1)$ be the inverse image of $C(\mathcal{X}'^0, P'_1)$ under the projection $U_1(\mathbb{R})(-1) \rightarrow (U_1/(W \cap U_1))(\mathbb{R})(-1) = U'_1(\mathbb{R})(-1)$. Then all the assertions, except the first statement of (c), follow directly from the same assertions for $(P', \mathcal{X}')$. Finally we already know that $(Q/W)(\mathbb{R})^0$ acts transitively on $\mathcal{X}'^0$, and $W(\mathbb{R}) \cdot U(\mathbb{C})$ acts transitively on the fibres of the projection $\mathcal{X}^0 \rightarrow \mathcal{X}'^0$. Thus $Q(\mathbb{R})^0 \cdot U(\mathbb{C}) = U(\mathbb{R})(-1) \rtimes Q(\mathbb{R})^0$ acts transitively on $\mathcal{X}^0$, whence also on $C(\mathcal{X}^0, P_1)$. As explained in 4.14, $Q(\mathbb{R})^0$ acts by conjugation, and $U(\mathbb{R})(-1)$ by translation on $U_1(\mathbb{R})(-1)$. This finishes the proof of (c). **q.e.d.**

4.16. Functoriality: Let $(\phi, \psi) : (P, \mathcal{X}) \rightarrow (P', \mathcal{X}')$ be a morphism of mixed Shimura data and Q an admissible $\mathbb{Q}$ -parabolic subgroup of P . Let $x \in \mathcal{X}$ and $\omega_x : H_{0, \mathbb{C}} \rightarrow P_{\mathbb{C}}$ as in

4.6. Let $Q'_\mathbb{C}$ be the unique connected subgroup of $P'_\mathbb{C}$ whose Lie algebra is the direct sum of all nonnegative weight spaces in $(\text{Lie } P')_\mathbb{C}$ under $\text{Ad}_{P'} \circ \phi \circ \omega_x \circ h_\infty \circ w$. It follows from 4.4 (a) and (b) that $Q'_\mathbb{C}$ contains the unipotent radical of $P'_\mathbb{C}$, so this is a parabolic subgroup. By construction it contains $\phi(Q_\mathbb{C})$. The cocharacter $\phi \circ \omega_x \circ h_\infty \circ w$ is modulo the unipotent radical of Q independent of x and h_0 and defined over $\mathbb{Q}$. Thus $Q'_\mathbb{C}$ depends only on Q and is defined over $\mathbb{Q}$. By 4.6 Q' is an admissible $\mathbb{Q}$ -parabolic subgroup of P' .

Let $P_1 \subset Q$ and $P'_1 \subset Q'$ be defined as in 4.7. By definition ϕ induces a homomorphism $\phi_1 := \phi|_{P_1} : P_1 \rightarrow P'_1$. Let $\mathcal{X}_1$ and $\mathcal{X}^+$ be as in 4.11. Let $(\mathcal{X}')^+ \subset \mathcal{X}'$ be the $P'_1(\mathbb{R}) \cdot Q'(\mathbb{R})^0 \cdot U'(\mathbb{C})$ -orbit in $\mathcal{X}'$ that is generated by $\phi(\mathcal{X}^+)$, and $\mathcal{X}'_1$ the corresponding $P'_1(\mathbb{R}) \cdot U'_1(\mathbb{C})$ -orbit in $\pi_0(\mathcal{X}') \times \text{Hom}(\mathbb{S}_\mathbb{C}, P'_{1,\mathbb{C}})$. Thus we have associated to $(P_1, \mathcal{X}_1)$ a canonical rational boundary component $(P'_1, \mathcal{X}'_1)$ of $(P', \mathcal{X}')$. Furthermore ϕ_1 extends uniquely to a morphism of mixed Shimura data $(\phi_1, \psi_1) : (P_1, \mathcal{X}_1) \rightarrow (P'_1, \mathcal{X}'_1)$, such that the following diagram is commutative and $P_1(\mathbb{R}) \cdot U(\mathbb{C})$ -equivariant:

$$\begin{array}{ccc} \mathcal{X}^+ & \xrightarrow{\psi} & (\mathcal{X}')^+ \\ \downarrow & & \downarrow \\ \mathcal{X}_1 & \xrightarrow{\psi_1} & \mathcal{X}'_1 \end{array}$$

Consider connected components $\mathcal{X}^0$ of $\mathcal{X}^+$ and $(\mathcal{X}')^0$ of $\mathcal{X}'$ such that $\mathcal{X}^0$ is mapped to $(\mathcal{X}')^0$. Let $C(\mathcal{X}^0, P_1) \subset U_1(\mathbb{R})(-1)$ and $C((\mathcal{X}')^0, P'_1) \subset U'_1(\mathbb{R})(-1)$ be the corresponding cones according to 4.15. Then $C(\mathcal{X}^0, P_1)$ is mapped to $C((\mathcal{X}')^0, P'_1)$ under ϕ .

Let (ϕ, ψ) be an automorphism of $(P, \mathcal{X})$, and $(P_1, \mathcal{X}_1)$ a rational boundary component of $(P, \mathcal{X})$. Then denote by $(\phi(P_1), \psi(\mathcal{X}_1))$ the unique rational boundary component $(P'_1, \mathcal{X}'_1)$ of above. This applies especially to interior automorphisms $(\text{int}(p), p \cdot)$ for $p \in P(\mathbb{Q})$.

4.17. Hereditary properties: Let $(P, \mathcal{X})$ be mixed Shimura data, $(P_1, \mathcal{X}_1)$ a rational boundary component of $(P, \mathcal{X})$, and $(P_2, \mathcal{X}_2)$ a rational boundary component of $(P_1, \mathcal{X}_1)$. We want to show $(P_2, \mathcal{X}_2)$ can be identified canonically with a rational boundary component of $(P, \mathcal{X})$.

Let $Q_1 \subset P$ be the parabolic subgroup belonging to $(P_1, \mathcal{X}_1)$, and $Q_{12} \subset P_1$ that belonging to $(P_2, \mathcal{X}_2)$. Let $\mathcal{X} \supset \mathcal{X}^+ \hookrightarrow \mathcal{X}_1$ and $\mathcal{X}_1 \supset \mathcal{X}_1^+ \hookrightarrow \mathcal{X}_2$ be as in 4.11, and let $\mathcal{X} \supset \mathcal{X}^{++}$ be the inverse image of $\mathcal{X}_1^+$ in $\mathcal{X}^+$. Let $x \in \mathcal{X}^{++}$, x_1 its image in $\mathcal{X}_1^+$, and x_2 its image in $\mathcal{X}_2$. Let $h_x : \mathbb{S}_\mathbb{C} \rightarrow P_\mathbb{C}$, $h_{x_1} : \mathbb{S}_\mathbb{C} \rightarrow P_{1,\mathbb{C}}$ and $h_{x_2} : \mathbb{S}_\mathbb{C} \rightarrow P_{2,\mathbb{C}}$ be the corresponding homomorphisms. Let $\omega_x : H_{0,\mathbb{C}} \rightarrow P_\mathbb{C}$ and $\omega_{x_1} : H_{0,\mathbb{C}} \rightarrow P_{1,\mathbb{C}}$ be as in 4.6. Then we have $h_x = \omega_x \circ h_0$, $h_{x_1} = \omega_x \circ h_\infty = \omega_{x_1} \circ h_0$ and $h_{x_2} = \omega_{x_1} \circ h_\infty$.

Consider the following reductive group over $\mathbb{R}$

$$H_{00} := \{(z, \alpha, \beta) \in \mathbb{S} \times (GL_{2,\mathbb{R}})^2 \mid z \cdot \bar{z} = \det(\alpha) = \det(\beta)\},$$

and the homomorphisms $\phi_1, \phi_2, \phi_3 : H_{0,\mathbb{C}} \rightarrow H_{00,\mathbb{C}}$ that are defined by $\phi_1(z, \alpha) := (z, pr_2 \circ h_0(z), \alpha)$, $\phi_2(z, \alpha) := (z, \alpha, pr_2 \circ h_\infty(z))$ and $\phi_3(z, \alpha) := (z, \alpha, \alpha)$. Observe the relations $\phi_1 \circ h_\infty = \phi_2 \circ h_0$ and $\phi_2 \circ h_\infty = \phi_3 \circ h_\infty$. The homomorphisms $\phi_1, \phi_2|_{H_0^{der}}$ and ϕ_3 are defined over $\mathbb{R}$.

4.18. Lemma: In the situation of 4.17 there exists a unique homomorphism $\Omega_x : H_{00,\mathbb{C}} \rightarrow P_\mathbb{C}$ such that:

- (i) $\pi' \circ \Omega_x : H_{00, \mathbb{C}} \rightarrow (P/U)_{\mathbb{C}}$ is defined over $\mathbb{R}$.
- (ii) $\Omega_x \circ \phi_1 = \omega_x$.
- (iii) $\Omega_x \circ \phi_2 = \omega_{x_1}$.

Proof. We shall construct Ω_x explicitly. The map

$$\begin{aligned} H_0 \times H_0^{der} &\rightarrow H_{00}, ((z, \alpha), (1, \beta)) \mapsto \phi_1(z, \alpha) \cdot \phi_2(1, \beta) \\ &= (z, pr_2 \circ h_0(z) \cdot \beta, \alpha) \end{aligned}$$

is an isomorphism of algebraic varieties. This and the conditions (ii) and (iii) imply that Ω_x must be given as follows:

$$\begin{aligned} H_{00, \mathbb{C}} &\xleftarrow{\sim} (H_0 \times H_0^{der})_{\mathbb{C}} \longrightarrow P_{\mathbb{C}}, \\ (z, \alpha, \beta) &\mapsto ((z, \beta), (1, pr_2 \circ h_0(z^{-1}) \cdot \alpha)) \mapsto \omega_x(z, \beta) \cdot \omega_{x_1}(1, pr_2 \circ h_0(z^{-1}) \cdot \alpha). \end{aligned}$$

With this definition (ii) is automatic. For (iii) we have

$$\begin{aligned} \Omega_x \circ \phi_2(z, \alpha) &= \omega_z(z, pr_2 \circ h_{\infty}(z)) \cdot \omega_{x_1}(1, pr_2 \circ h_0(z^{-1}) \cdot \alpha) \\ &= \omega_x \circ h_{\infty}(z) \cdot \omega_{x_1} \circ h_0(z^{-1}) \cdot \omega_{x_1}(z, \alpha) \\ &= h_{x_1}(z) \cdot h_{x_1}(z^{-1}) \cdot \omega_{x_1}(z, \alpha) \\ &= \omega_{x_1}(z, \alpha), \end{aligned}$$

as desired. For (i) note that by assumption $\pi' \circ \omega_x$ is defined over $\mathbb{R}$. Thus it suffices to show that $\omega_{x_1|H_0^{der}}$ is defined over $\mathbb{R}$. Now there exists an element $u_1 \in U_1(\mathbb{C})$ such that $\text{int}(u_1^{-1}) \circ \omega_{x_1}$ is defined over $\mathbb{R}$. By 4.4 (a) $\omega_{x_1}(H_0^{der})$ commutes with U_1 . Therefore $\omega_{x_1|H_0^{der}}$ itself is defined over $\mathbb{R}$. Thus we have shown that the map Ω_x satisfies the conditions (i) to (iii), but we still have to prove that it is a homomorphism.

To do this first assume that $W = 1$. For any $t \in \mathbb{G}_{m, \mathbb{R}} \subset \mathbb{S}$ the element $\Omega_x(t, t, t) = \Omega_x \circ \phi_1 \circ h_0 \circ w(t) = h_x \circ w(t)$ lies in the center of P . Thus every such element commutes with $\Omega_x(\mathbb{S}^1 \times (SL_{2, \mathbb{R}})^2)$, so that it suffices to prove that $\Omega_x|_{\mathbb{S}^1 \times (SL_{2, \mathbb{R}})^2}$ is a homomorphism. Since this group is connected, it is enough to prove this for the composite of this map with each of the projections $P \rightarrow P/P^{der}$ and $P \rightarrow P^{ad}$. The first case follows from condition (ii). The second follows from [AMRT] ch.III p.201 prop. 1 and p.207 cor. Now let W be arbitrary. Then $h_x \circ w$ defines a Levi decomposition of $P_{\mathbb{C}}$. The homomorphism ω_x respects this decomposition, hence so does $h_{x_1} = \omega_x \circ h_{\infty}$. This in turn implies that ω_{x_1} respects this decomposition as well, hence so does Ω_x . Thus the assertion follows from that in the case $W = 1$. **q.e.d.**

4.19. Lemma: Consider the situation of 4.17.

- (a) Let $\omega'_x := \Omega_x \circ \phi_3 : H_{0, \mathbb{C}} \rightarrow P_{\mathbb{C}}$. There exists a unique admissible $\mathbb{Q}$ -parabolic subgroup $Q_2 \subset P$ such that the conditions 4.6 (b) hold for Q_2 and ω'_x . It depends only on $(P_1, \mathcal{X}_1)$ and $(P_2, \mathcal{X}_2)$.
- (b) $Q_{12} = P_1 \cap Q_2$, and $Q_1 = (Q_1 \cap Q_2) \cdot P_1$.
- (c) Let $(P_3, \mathcal{X}_3)$ be the unique rational boundary component of $(P, \mathcal{X})$ that belongs to Q_2 and for which x is mapped to $\mathcal{X}_3$. Then $P_2 = P_3$ as subgroups of P , and there exists a unique

isomorphism $\mathcal{X}_2 \rightarrow \mathcal{X}_3$, which together with the identity on P_2 yields an isomorphism of mixed Shimura data $(P_2, \mathcal{X}_2) \xrightarrow{\sim} (P_3, \mathcal{X}_3)$ such that the following diagram is commutative:

$$\begin{array}{ccccc} \mathcal{X}^{++} & \hookrightarrow & \mathcal{X}_1^+ & \hookrightarrow & \mathcal{X}_2 \\ \parallel & & & & \downarrow \wr \\ \mathcal{X}^{++} & \hookrightarrow & & & \mathcal{X}_3. \end{array}$$

Proof. (a) Define $Q_{2,\mathbb{C}} \subset P_{\mathbb{C}}$ by the properties 4.6 (b). By 4.4 (a) and (b) this is a parabolic subgroup. We want to show that it is defined over $\mathbb{Q}$. By assumption $\omega_x(U_{0,\mathbb{C}})$ is contained in $U_{1,\mathbb{C}}$ and $\omega_{x_1}(U_{0,\mathbb{C}})$ in $U_{2,\mathbb{C}}$. Since by 4.4 (a) $U_{1,\mathbb{C}}$ is contained in $U_{2,\mathbb{C}}$, we get

$$\omega'_x(U_{0,\mathbb{C}}) \subset \Omega_x(\{1\} \times \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}^2) \omega_x(U_{0,\mathbb{C}}) \cdot \omega_{x_1}(U_{0,\mathbb{C}}) \subset U_{2,\mathbb{C}}.$$

Furthermore we have

$$\omega'_x \circ h_\infty = \Omega_x \circ \phi_3 \circ h_\infty = \Omega_x \circ \phi_2 \circ h_\infty = \omega_{x_1} \circ h_\infty = h_{x_2}.$$

Thus $U_{2,\mathbb{C}}$ lies in the unipotent radical of $Q_{2,\mathbb{C}}$ and $\omega'_x \circ h_\infty$ is defined over $\mathbb{Q}$ modulo U_2 . Hence $Q_{2,\mathbb{C}}$ is already defined over $\mathbb{Q}$. Now 4.6 implies that Q_2 is an admissible $\mathbb{Q}$ -parabolic subgroup of P that does not depend on x . This proves (a).

(b) By 4.6 (b) $(\text{Lie } Q_2)_{\mathbb{C}}$ is the direct sum of all nonnegative weight spaces in $(\text{Lie } P)_{\mathbb{C}}$ under $\omega'_x \circ h_\infty \circ w$. Likewise $(\text{Lie } Q_{12})_{\mathbb{C}}$ is the direct sum of all nonnegative weight spaces in $(\text{Lie } P_1)_{\mathbb{C}}$ under $\omega_{x_1} \circ h_\infty \circ w$. As we have just seen, these cocharacters are equal, whence the first equality. For the second it suffices to prove the inclusion $Q_1 \subset Q_2 \cdot P_1$. Observe that, by the definition 4.7 of P_1 , the adjoint action of this group on $(\text{Lie } Q_1)/(\text{Lie } P_1)$ is trivial. In particular, this quotient is purely of type $(0,0)$ under $\omega_{x_1} \circ h_\infty$. By the definition of Q_2 , this implies $\text{Lie } Q_1 \subset \text{Lie } Q_2 + \text{Lie } P_1$, whence $Q_1 \subset Q_2 \cdot P_1$.

(c) The second assertion follows from the first and the relation $\omega'_x \circ h_\infty = \omega_{x_1} \circ h_\infty$. For the equation $P_2 = P_3$ we can replace P by $(P/W)^{ad}$ without loss of generality, so we may assume P to be semisimple and adjoint. In this case the assertion follows from [AMRT] ch.III thm. 3 p.240. **q.e.d.**

4.20. Corollary: Every rational boundary component of a rational boundary component of $(P, \mathcal{X})$ is itself in a canonical way a rational boundary component of $(P, \mathcal{X})$.

Proof. 4.18 and 4.19. **q.e.d.**

Remark. (i) This corollary implies that the natural order relation between rational boundary components is indeed a partial order.

(ii) There exists only finitely many $P(\mathbb{Q})$ -conjugacy classes of rational boundary components. This follows at once from the corresponding fact for admissible $\mathbb{Q}$ -parabolic subgroups and the construction in 4.11. Observe also that since $Q_1(\mathbb{R})$ acts transitively on $\pi_0(\mathcal{X})$, the conjugacy class of $(P_1, \mathcal{X}_1)$ is already determined by P_1 or Q_1 alone. If $(P/W)^{ad}$ is $\mathbb{Q}$ -simple, then by the explicit description of admissible $\mathbb{Q}$ -parabolic subgroups in [AMRT] ch.III or [BB] the conjugacy classes are in total order. In general there still exists unique minimal, resp. maximal conjugacy classes. The maximal conjugacy class consists of course of all irreducible components of $(P, \mathcal{X})$.

(iii) Fix a rational boundary component $(P_1, \mathcal{X}_1)$ of $(P, \mathcal{X})$, then two rational boundary components of $(P_1, \mathcal{X}_1)$ are conjugate under $P(\mathbb{Q})$ if and only if they are conjugate under $P_1(\mathbb{Q})$. In fact, it suffices to check this assertion in the case where $(P/W)^{ad}$ is $\mathbb{Q}$ -simple. Then by the explicit description of the $\mathbb{Q}$ -root system of $(P_1/W_1)^{ad}$ in [AMRT] ch.III §3.5 p.220 this is also $\mathbb{Q}$ -simple. But the total order relation is of course preserved.

4.21. Proposition: Consider the situation of 4.17. Let $\mathcal{X}^0 \subset \mathcal{X}^{++}$, $\mathcal{X}_1^0 \subset \mathcal{X}_1^+$ and $\mathcal{X}_2^0 \subset \mathcal{X}_2$ be connected components that are mapped to each other by the canonical inclusion $\mathcal{X}^{++} \hookrightarrow \mathcal{X}_1^+ \hookrightarrow \mathcal{X}_2$. Recall that $U_1 \subset U_2$ (by 4.4 (a)). Then:

- (a) $C(\mathcal{X}_1^0, P_2) = C(\mathcal{X}^0, P_2) + U_1(\mathbb{R})(-1) = C(\mathcal{X}^0, P_2) + (W_1 \cap U_2)(\mathbb{R})(-1)$.
- (b) $C(\mathcal{X}^0, P_1)$ is contained in the closure of $C(\mathcal{X}^0, P_2)$.
- (c) Let $(P_3, \mathcal{X}_3)$ be another rational boundary component of $(P, \mathcal{X})$ that lies “between” $(P_2, \mathcal{X}_2)$ and $(P, \mathcal{X})$. Then $C(\mathcal{X}^0, P_1)$ and $C(\mathcal{X}^0, P_3)$ have a point in common if and only if $(P_3, \mathcal{X}_3) = (P_1, \mathcal{X}_1)$.

Proof. By 4.15 (b) we have $\mathcal{X}_1^0 = U_1(\mathbb{R})(-1) \cdot \text{image}(\mathcal{X}^0)$, which implies the equation $C(\mathcal{X}_1^0, P_2) = C(\mathcal{X}^0, P_2) + U_1(\mathbb{R})(-1)$. The second equation of (a) follows from 4.15 (c). This proves (a).

If $W = 1$, then the assertion (b) and (c) follow from [AMRT] ch.III p.240 thm. 3. In general, the rational boundary component of $(P, \mathcal{X})/W$ corresponding to $(P_1, \mathcal{X}_1)$ is canonically isomorphic to $(P_1, \mathcal{X}_1)/(W_1 \cap W)$. Moreover, $C(\mathcal{X}^0/W, P_1/(W_1 \cap W))$ is just the image of $C(\mathcal{X}^0, P_1)$, and by 4.15 (c), the latter is the full inverse image of the former under the projection $U_1(\mathbb{R})(-1) \rightarrow (U_1/U_1 \cap W)(\mathbb{R})(-1)$. The same holds for $(P_2, \mathcal{X}_2)$, and (b) and (c) follow in the general case. **q.e.d.**

4.22. Definition-Proposition: Let $(P_1, \mathcal{X}_1)$ be a rational boundary component of $(P, \mathcal{X})$ and $\mathcal{X}^0$ a connected component of $\mathcal{X}^+$. We define $C^*(\mathcal{X}^0, P_1) \subset U_1(\mathbb{R})(-1)$ to be the union of the cones $C(\mathcal{X}^0, P_2)$ for all rational boundary components $(P_2, \mathcal{X}_2)$ between $(P, \mathcal{X})$ and $(P_1, \mathcal{X}_1)$, including $(P, \mathcal{X})$ and $(P_1, \mathcal{X}_1)$.

- (a) $C^*(\mathcal{X}^0, P_1)$ is a convex cone.

Fix a rational boundary component $(P_2, \mathcal{X}_2)$ between $(P_1, \mathcal{X}_1)$ and $(P, \mathcal{X})$. Then

- (b) $C^*(\mathcal{X}_2^0, P_1) = C^*(\mathcal{X}^0, P_1) + U_2(\mathbb{R})(-1)$, and
- (c) $C^*(\mathcal{X}^0, P_2) = C^*(\mathcal{X}^0, P_1) \cap U_2(\mathbb{R})(-1) = C^*(\mathcal{X}^0, P_1) \cap (U_1 \cap W_2)(\mathbb{R})(-1)$.

Proof. (a) Since $C^*(\mathcal{X}^0, P_1)$ is a union of cones, it remains to prove the convexity. This follows from the stronger Claim: Let $u_2 \in C(\mathcal{X}^0, P_2) \subset C^*(\mathcal{X}^0, P_1)$ and $u'_2 \in C(\mathcal{X}^0, P'_2) \subset C^*(\mathcal{X}^0, P_1)$, then the open line segment $\{\lambda \cdot u_2 + (1 - \lambda) \cdot u'_2 \mid 0 < \lambda < 1\}$ is contained in $C(\mathcal{X}^0, P_3)$ for a unique rational boundary component of both $(P_2, \mathcal{X}_2)$ and $(P'_2, \mathcal{X}'_2)$ such that $(P_1, \mathcal{X}_1)$ is a rational boundary component of $(P_3, \mathcal{X}_3)$. If $W = 1$, then this follows from [AMRT] ch.II together with ch.III p.240 thm. 3. The general case follows from this, together with the second assertion of 4.15 (c).

(b) The inclusion “ $\subset$ ” follows from 4.21 (a). For the other inclusion let $u_1 \in C^*(\mathcal{X}^0, P_1)$ and $u_2 \in C(\mathcal{X}^0, P_2)$. By the claim above, $u_1 + u_2 \in C(\mathcal{X}^0, P_3)$ for a rational boundary component $(P_3, \mathcal{X}_3)$ between $(P_1, \mathcal{X}_1)$ and $(P_2, \mathcal{X}_2)$. Thus $u_1 \in C(\mathcal{X}^0, P_3) + U_2(\mathbb{R})(-1) = C(\mathcal{X}_2^0, P_3) \subset C^*(\mathcal{X}_2^0, P_1)$ by 4.21 (a), as desired.

(c) The inclusion “ $\subset$ ” are obvious. Let $u_1 \in C(\mathcal{X}^0, P_3) \cap (U_1 \cap W_2)(\mathbb{R})(-1)$ for some $(P_3, \mathcal{X}_3)$ between $(P_1, \mathcal{X}_1)$ and $(P, \mathcal{X})$. Let $u_2 \in C(\mathcal{X}^0, P_2)$, then by the claim above, $u_1 + u_2 \in C(\mathcal{X}^0, P_4)$ for some $(P_4, \mathcal{X}_4)$ between $(P_1, \mathcal{X}_1)$ and $(P_2, \mathcal{X}_2)$, and also between $(P_1, \mathcal{X}_1)$ and $(P_3, \mathcal{X}_3)$. In particular, we have $C(\mathcal{X}^0, P_4) \cap (U_4 \cap W_2)(\mathbb{R})(-1) \neq \emptyset$. By 4.21 (a), this implies that $C(\mathcal{X}_2^0, P_4) = C(\mathcal{X}^0, P_4) + (U_4 \cap W_2)(\mathbb{R})(-1)$ contains the origin, so by 4.15 we must have $(P_4, \mathcal{X}_4) = (P_2, \mathcal{X}_2)$. Hence $(P_3, \mathcal{X}_3)$ is between $(P_2, \mathcal{X}_2)$ and $(P, \mathcal{X})$, and $C(\mathcal{X}^0, P_3) \subset C^*(\mathcal{X}^0, P_2)$, as desired. **q.e.d.**

4.23. Proposition: Let $(P_1, \mathcal{X}_1)$ be a rational boundary component of $(P, \mathcal{X})$ and $(P_i, \mathcal{X}_i)$ for $i = 2, 3$ rational boundary component between $(P_1, \mathcal{X}_1)$ and $(P, \mathcal{X})$. Let $(P, \mathcal{X}) \hookrightarrow (P', \mathcal{X}')$ be an embedding, and $(P'_i, \mathcal{X}'_i)$ be the corresponding rational boundary components of $(P', \mathcal{X}')$. Then $(P_2, \mathcal{X}_2)$ is a rational boundary component of $(P_3, \mathcal{X}_3)$ if and only if $(P'_2, \mathcal{X}'_2)$ is a rational boundary component of $(P'_3, \mathcal{X}'_3)$.

Proof. Fix $u_i \in C(\mathcal{X}_0, P_i)$, then by the proof of 4.22, $u_2 + u_3 \in C(\mathcal{X}^0, P_4)$ for some $(P_4, \mathcal{X}_4)$ between $(P_1, \mathcal{X}_1)$ on one hand, and both $(P_2, \mathcal{X}_2)$, $(P_3, \mathcal{X}_3)$ on the other hand. Thus it suffices to show that $(P_4, \mathcal{X}_4) = (P_3, \mathcal{X}_3)$ if and only if $(P'_4, \mathcal{X}'_4) = (P'_3, \mathcal{X}'_3)$. Now, by definition, $(P_4, \mathcal{X}_4) = (P_3, \mathcal{X}_3)$ if and only if the corresponding homomorphism $\omega : H_{0, \mathbb{C}} \rightarrow P_{3, \mathbb{C}}$ of 4.6, resp. 4.11, is trivial on $H_{0, \mathbb{C}}^{der}$. By the injectivity of $P_3 \hookrightarrow P'_3$, the equivalence follows. **q.e.d.**

4.24. The conical complex: The convex cone $C^*(\mathcal{X}^0, P_1)$ considered in the previous lemma is in general neither open nor closed in $U_1(\mathbb{R})(-1)$. By 4.21 it is stratified by the boundary components lying in between. With respect to the $\mathbb{Q}$ -structure $U_1(\mathbb{R})(-1) = U_1(\mathbb{Q})(-1) \otimes \mathbb{R}$ each such stratum is an open convex cone in a subspace of $U_1(\mathbb{R})(-1)$ that is defined over $\mathbb{Q}$. Define

$$\mathcal{C}(P, \mathcal{X}) := \coprod_{(\mathcal{X}^0, P_1)} C^*(\mathcal{X}^0, P_1) / \sim,$$

whence $\sim$ is the equivalence relation generated by the graph of all embeddings $C^*(\mathcal{X}^0, P_1) \hookrightarrow C^*(\mathcal{X}^0, P_2)$, where $(P_2, \mathcal{X}_2)$ is a rational boundary component of $(P_1, \mathcal{X}_1)$ and $(P_1, \mathcal{X}_1)$ one of $(P, \mathcal{X})$. As a set $\mathcal{C}(P, \mathcal{X})$ is the disjoint union of all $C(\mathcal{X}^0, P_1)$, but endowed with the quotient topology the $C(\mathcal{X}^0, P_1)$ form a locally closed stratification of $\mathcal{C}(P, \mathcal{X})$. The closure of any $C(\mathcal{X}^0, P_1)$ is isomorphic to $C^*(\mathcal{X}^0, P_1)$. This is the conical complex associated to $(P, \mathcal{X})$.

By 4.16 the conical complex is functorial in $(P, \mathcal{X})$. In particular the automorphisms of $(P, \mathcal{X})$ act from the left hand side on $\mathcal{C}(P, \mathcal{X})$. There is also the following relation with the conical complex for a rational boundary component $(P_1, \mathcal{X}_1)$ of $(P, \mathcal{X})$. Consider the sets $C^*(\mathcal{X}^0, P_2)$ for all rational boundary components $(P_2, \mathcal{X}_2)$ of $(P_1, \mathcal{X}_1)$. We can consider their union $\mathcal{C}(P, \mathcal{X})$, but by 4.22 (b) we can also consider their union in $\mathcal{C}(P_1, \mathcal{X}_1)$. These two are isomorphic. In $\mathcal{C}(P, \mathcal{X})$ they form a closed, in $\mathcal{C}(P_1, \mathcal{X}_1)$ and open subset.

In the rest of this chapter we shall describe some examples.

4.25. Example: Let $(P, \mathcal{X})$ be the pure Shimura data $(CSp_{2g, \mathbb{Q}}, \mathcal{H}_{2g})$ defined in 2.7. Let $V' \subset V := V_{2g}$ be a totally isotropic subspace with respect to the alternating form Ψ ,

and let $Q := \text{Stab}_P(V')$. For $V' \neq 0$ this is a maximal proper $\mathbb{Q}$ -parabolic subgroup, and every such is obtained in this way. Thus $V' \mapsto Q$ defines a bijection between the totally Ψ -isotropic subspaces of V and the admissible $\mathbb{Q}$ -parabolic subgroups of $CSp_{2g, \mathbb{Q}}$. We shall now keep V' fixed.

Let $(P_1, \mathcal{X}_1)$ be the associated rational boundary component. The Hodge structure on V defined by an element $x \in \mathcal{H}_{2g}$ is pure of the type $\{(-1, 0), (0, -1)\}$, and $F^0 V_{\mathbb{C}}$ is a maximal totally Ψ -isotropic subspace of $V_{\mathbb{C}}$. Let $x_1 \in \mathcal{X}_1$ be the image of x , then the rational mixed Hodge on V that is defined by x_1 has the weight filtration

$$W_n V = \begin{cases} 0 & \text{if } n < -3, \\ V' & \text{if } n = -2, \\ (V')^{\perp} & \text{if } n = -1, \\ V & \text{if } n \geq 0. \end{cases}$$

and by 4.12 the same Hodge filtration as the pure Hodge structure. The decomposition of $V_{\mathbb{C}}$ into eigenspaces under h_x is $V_{\mathbb{C}} = F^0 V_{\mathbb{C}} \oplus \overline{F^0 V_{\mathbb{C}}}$. By 1.2 the decomposition into eigenspaces under h_{x_1} turns out to be $V = V^{-1, -1} \oplus V^{-1, 0} \oplus V^{0, -1} \oplus V^{0, 0}$, where

$$\begin{aligned} V^{-1, -1} &= V'_{\mathbb{C}}, \\ V^{0, -1} &= (V')_{\mathbb{C}}^{\perp} \cap F^0 V_{\mathbb{C}}, \\ V^{-1, 0} &= \overline{V^{0, -1}}, \text{ and} \\ V^{0, 0} &= F^0 V_{\mathbb{C}} \cap (\overline{F^0 V_{\mathbb{C}}} + V'_{\mathbb{C}}). \end{aligned}$$

This implies

$$P_1 = \{q \in Q \mid q \text{ acts trivially on } V/(V')^{\perp}\}, \text{ and}$$

$$U_1 = \{q \in Q \mid q \text{ acts trivially on } V/V' \text{ and on } (V')^{\perp}\}.$$

The map $u_1 \mapsto \Psi(., (\text{id} - u_1)(.))$ identifies U_1 with the vector space of all symmetric bilinear forms on $V/(V')^{\perp}$. Clearly this isomorphism is Q -equivariant under the natural operation of Q on U_1 and on $V/(V')^{\perp}$. It is easy to check that Q maps onto $GL(V/(V')^{\perp})$.

In the case $\dim(V') = 1$ it turns out that $(P_1, \mathcal{X}_1)$ is isomorphic to the mixed Shimura data $(P_{2(g-1)}, \mathcal{X}_{2(g-1)})$ defined in 2.25. At the other extreme, where V' is a maximal totally isotropic subspace of V , $(P_1, \mathcal{X}_1)$ is isomorphic to a $\frac{g(g+1)}{2}$ -fold fibre product of $(P_0, \mathcal{X}_0)$ with itself over $(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$.

4.26. Continuation of 4.25: The cone: Let $\mathcal{H}^0$ be the connected component of $\mathcal{H}_{2g}$ that contains x . Let $\lambda: \mathbb{Q} \xrightarrow{\sim} \mathbb{Q}(1)$ be the isomorphism defined by $\mathcal{H}^0$ according to 2.8, and let $C := C(\mathcal{H}^0, P_1)$ and $C^*(\mathcal{H}^0, P_1)$. Then $\lambda(C)$ is an open cone in $U_1(\mathbb{R})$.

Claim: (a) $\lambda(C)$ is the cone of all positive definite symmetric bilinear forms on $(V/(V')^{\perp})_{\mathbb{R}}$.

(b) $\lambda(C^*)$ is the cone of all positive semi-definite symmetric bilinear forms on $(V/(V')^{\perp})_{\mathbb{R}}$, whose degenerate subspace is defined over $\mathbb{Q}$.

Consequence: In general the cone in $U_1(\mathbb{R})(-1)$ does depend on the connected component of $\mathcal{X}$. Indeed in the case under consideration, λ depends on $\mathcal{H}^0$ but by $\lambda(C)$ does not.

Proof. Fix a choice for $\sqrt{-1}$ such that $\lambda(1) = 2\pi\sqrt{-1}$. By 1.11 and 2.8, the pairing $\Psi(., h_x(\sqrt{-1})(.))$ on $V_{\mathbb{R}}$ is symmetric and positive definite. Hence the associated hermitian form $(v, v') \mapsto \Psi(v, h_x(\sqrt{-1})(\bar{v}'))$ on $V_{\mathbb{C}}$ is also positive definite. By 1.3, and since $h_x(\sqrt{-1})$ is defined over $\mathbb{R}$, we have $h_x(\sqrt{-1})(\bar{v}) = \sqrt{-1} \cdot \bar{v}$ for every $v \in F^0 V_{\mathbb{C}}$. Now let $u_1 \in C$ be the imaginary part of x_1 . By 4.12, it is characterized by the fact that $u_1^{-1} V^{0,0}$ is defined over $\mathbb{R}$. Let B be the symmetric form associated to u_1 by the identification in 4.25. For any $0 \neq v \in V_{\mathbb{R}}$ such that $u_1 v \in V^{0,0}$, we get

$$\begin{aligned} 0 < \Psi(u_1 v, h_x(\sqrt{-1})(\overline{u_1 v})) &= \Psi(u_1 v, \sqrt{-1} \cdot u_1^{-1} v) \\ &= \Psi(v, \sqrt{-1} \cdot u_1^{-2} v) \\ &= \Psi(v, \sqrt{-1} \cdot (u_1^{-2} - \text{id})v) \\ &= 2 \cdot B([v], \sqrt{-1} \cdot [v]) \\ &= \frac{1}{\pi} \cdot \lambda \circ B([v], [v]), \end{aligned}$$

where we used $\overline{u_1} = u_1^{-1}$, and the fact that Ψ is alternating. This shows that B is positive definite, as desired. Now $GL(V/(V')^{\perp})(\mathbb{R})$ transitively permutes all positive definite symmetric bilinear forms, whence (a). The assertion (b) follows from the fact that the boundary components between $(P, \mathcal{X})$ and $(P_1, \mathcal{X}_1)$ correspond bijectively to the subspaces $V'' \subset V'$ defined over $\mathbb{Q}$. **q.e.d.**

4.27. Example: In the case $W/U \neq 1$ the situation is somewhat more complicated. Consider the following special case. Let $(P, \mathcal{X})$ be the mixed Shimura data $(P_{2g}, \mathcal{X}_{2g})$ defined in 2.25. for $g = 1$. Let $(P_1, \mathcal{X}_1)$ be a proper rational boundary component of $(P, \mathcal{X})$. By 4.25 we can view $(P, \mathcal{X})$ itself as rational boundary component of $(CSp_{2g, \mathbb{Q}}, \mathcal{H}_{2g})$ for $g = 2$. Thus, for a suitable choice of coordinates, U_1 is isomorphic to the space of all symmetric 2×2 -matrices, and Q acts on this space by $A \mapsto B \cdot A \cdot {}^t B$ for $B \in \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. The image of $U_1 \cap W$ is the subspace of all matrices of the form $\begin{pmatrix} * & * \\ * & 0 \end{pmatrix}$, the image of U the subspace of all matrices of the form $\begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}$. In other coordinates, U_1 corresponds to $\mathbb{Q}^3$, and Q acts on this through all matrices of the form

$$\begin{pmatrix} st & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s/t \end{pmatrix} \cdot \begin{pmatrix} 1 & 2a & a^2 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$$

for $s, t \in \mathbb{G}_{m, \mathbb{Q}}$ and $a \in \mathbb{G}_{a, \mathbb{Q}}$. We see in particular that the unipotent radical of Q may act through quadratic terms on U_1 .

The cone can be determined as follows. In 4.26 the cone $\lambda(C)$ was the set of all positive definite real symmetric 2×2 -matrices. Thus 4.21 (a) implies that in our case the cone $\lambda(C(\mathcal{X}^0, P_1))$ is the set of all matrices of the form $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ with $a, b \in \mathbb{R}$ and $c \in \mathbb{R}^{>0}$. Moreover by 4.21, $\lambda(C^*(\mathcal{X}^0, P_1))$ is the union of $\lambda(C(\mathcal{X}^0, P_1))$ with the set of all matrices of the form $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ with $a \in \mathbb{R}$.

Chapter 5

Torus embeddings

This chapter contains some results about torus embeddings that will be needed later. In 5.1–5 we begin by recalling standard facts, most of which can be found for instance in [KKMS] ch.I or in [AMRT] ch.I. In 5.7 we derive a density result of concerning the formal completion along a stratum. In 5.9 we describe the topology of a torus embedding of complex spaces. Starting from 5.10 we study equivariant line bundles on relative torus embeddings. In particular we show how they can themselves be described as torus embeddings. The main result is the ampleness criterion 5.19, which generalizes [KKMS] ch.I §3 thm. 13 to the relative case. In 5.20–25 we deal with the construction of cone decomposition that satisfy the ampleness condition. The chapter ends with a study of canonical sheaves on torus embeddings in 5.26–27.

5.1. Cone decompositions: Let M be a finite dimensional $\mathbb{Q}$ -vector space. A subset $\sigma \subset M_{\mathbb{R}}$ is called a convex rational polyhedral cone if it can be described in one of the following two equivalent ways:

- (i) There exists finitely many $\ell_i \in M^{\vee}$, $i \in I$ (note that the ℓ_i are defined over $\mathbb{Q}$), such that $\sigma = \{x \in M_{\mathbb{R}} \mid \forall i \in I \langle \ell_i, x \rangle \geq 0\}$.
- (ii) There exists finitely many $x_i \in M$, $i \in I$ (note that the x_i are defined over $\mathbb{Q}$), such that $\sigma = \{\sum_{i \in I} \lambda_i \cdot x_i \mid \forall i \in I \lambda_i \in \mathbb{R}^{\geq 0}\}$.

Let $\sigma = \{x \in M_{\mathbb{R}} \mid \forall i \in I \langle \ell_i, x \rangle \geq 0\}$ be a convex rational polyhedral cone. A face of σ is a subset of the form $\{x \in \sigma \mid \forall i \in J \langle \ell_i, x \rangle = 0\}$ for some subset $J \subset I$. A face is again a convex rational polyhedral cone. The interior of σ is the subset σ^0 of all points that do lie in a proper face of σ . This is the topological interior of σ within the $\mathbb{R}$ -subspace generated by σ , but not in general in $M_{\mathbb{R}}$. These definitions are independent of the chosen representation of σ . Any convex rational polyhedral cone is the disjoint union of the interiors of all its faces. Observe that the intersection of any two convex rational polyhedral cones in $M_{\mathbb{R}}$ is again a convex rational polyhedral cone.

For any convex rational polyhedral cone σ in $M_{\mathbb{R}}$ let $\check{\sigma} := \{\ell \in M_{\mathbb{R}}^{\vee} \mid \forall x \in \sigma \langle \ell, x \rangle \geq 0\}$. This is a convex rational polyhedral cone in $M_{\mathbb{R}}^{\vee}$, called the dual cone. Clearly $\check{\check{\sigma}} = \sigma$. If $\tau \subset \sigma$, then $\check{\tau} \supset \check{\sigma}$. The dual cone $\check{\sigma}$ contains a basis of $M^{\vee}$ if and only if σ does not contain a nontrivial linear subspace. Every such convex rational polyhedral cone σ possesses the cone $\{0\}$ as one of its faces. Note that $\{0\}^{\vee} = M_{\mathbb{R}}^{\vee}$.

A rational partial polyhedral decomposition of $M_{\mathbb{R}}$ is a collection Σ of a convex rational polyhedral cones, such that the following conditions hold:

- (i) Any face of any $\sigma \in \Sigma$ is also contained in Σ .
- (ii) The intersection of any two $\sigma, \tau \in \Sigma$ is a face both of σ and of τ .
- (iii) $\{0\} \in \Sigma$.

The conditions (ii) and (iii) imply that $\{0\}$ is a face of every $\sigma \in \Sigma$, so by (i) σ contains a nontrivial linear subspace. Conversely if Σ satisfies (i) and (ii), and some $\sigma \in \Sigma$ possesses no nontrivial linear subspace, then (iii) follows. Let $|\Sigma| := \bigcap_{\sigma \in \Sigma} \sigma$. If $C \subset M_{\mathbb{R}}$ is a subset which contains $|\Sigma|$, then we call Σ a rational partial polyhedral decomposition of C . If moreover $C = |\Sigma|$, then we call Σ a (complete) rational polyhedral decomposition of C . If Σ_1 and Σ_2 are rational polyhedral decompositions, then so is $\Sigma_3 := \{\sigma_1 \cap \sigma_2 \mid \sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2\}$. If both Σ_1 and Σ_2 are finite or complete rational polyhedral decompositions of C , then Σ_3 has the same property. If $|\Sigma_1| = |\Sigma_2|$, and every $\sigma \in \Sigma_1$ is contained in some $\tau \in \Sigma_2$, then Σ_1 is called refinement of Σ_2 . Equivalently Σ_1 is a refinement of Σ_2 if and only if $|\Sigma_1| = |\Sigma_2|$ and every $\tau \in \Sigma_2$ is the union of all $\sigma \in \Sigma_1$ that are contained in τ .

5.2. Affine torus embeddings: Let k be a field of characteristic zero, and T a split algebraic torus over k , that is $T \cong (\mathbb{G}_{m,k})^{\dim(T)}$. Let $X^*(T) := \text{Hom}(T, \mathbb{G}_{m,k})$ be its character group and $Y_*(T) := \text{Hom}(\mathbb{G}_{m,k}, T)$ its cocharacter group. The bilinear map

$$X^*(T) \times Y_*(T) \rightarrow \text{Hom}(\mathbb{G}_{m,k}, \mathbb{G}_{m,k}) = \mathbb{Z}, (\chi, \lambda) \mapsto \deg(\chi \circ \lambda)$$

is a perfect pairing. The affine coordinate ring of T is canonically isomorphic to the group ring $k[X^*(T)]$. Let σ be a convex rational polyhedral cone in $Y_*(T)_{\mathbb{R}}$ that does not contain a nontrivial linear subspace. Define $T_{\sigma} := \text{Spec } k[\check{\sigma} \cap X^*(T)]$, this is a normal affine variety over k . Since $\check{\sigma} \cap X^*(T)$ contains a $\mathbb{Q}$ -basis of $X^*(T)_{\mathbb{Q}}$, it can easily be shown that it contains a $\mathbb{Z}$ -basis of $X^*(T)$, so T_{σ} contains T as an open dense subvariety. Moreover the action of T by multiplication on itself extends uniquely to an action on T_{σ} .

Call σ smooth (with respect to the lattice $Y_*(T) \subset Y_*(T)_{\mathbb{Q}}$) if the semigroup $\sigma \cap Y_*(T)$ can be generated by a subset of a $\mathbb{Z}$ -basis of $Y_*(T)$. The variety T_{σ} is smooth if and only if σ is smooth. As an example, take $T = \mathbb{G}_{m,k}$ and $\sigma = \mathbb{R}^{\geq 0} \subset \mathbb{R} = Y_*(\mathbb{G}_{m,k})_{\mathbb{R}}$, then one easily sees that $T_{\sigma} \cong \mathbb{A}_k^1$ with the obvious embedding of $\mathbb{G}_{m,k}$. In general the embedding $T \hookrightarrow T_{\sigma}$ is smooth if and only if it is isomorphic to the embedding $(\mathbb{G}_{m,k})^{\dim(T)} \hookrightarrow (\mathbb{A}_k^1)^r \times (\mathbb{G}_{m,k})^{\dim(T)-r}$ for some integer r . Then the complement $T_{\sigma} \setminus T$ is a union of smooth divisors with only normal crossings.

There is a unique closed T -orbit $\tilde{T}_{\sigma}$ in T_{σ} . Its ideal in $k[\check{\sigma} \cap X^*(T)]$ is generated by $\{\chi \in X^*(T) \mid \forall \lambda \in \sigma^0 \langle \lambda, \chi \rangle > 0\}$. thus it is canonically isomorphic to the torus with character group $X^*(T) \cap (\mathbb{R} \cdot \sigma)^{\vee} = \{\chi \in X^*(T) \mid \forall \lambda \in \sigma \langle \lambda, \chi \rangle = 0\}$ and cocharacter group $(Y_*(T) + \mathbb{R} \cdot \sigma) / \mathbb{R} \cdot \sigma$. The inclusion of semigroups $X^*(T) \cap (\mathbb{R} \cdot \sigma)^{\vee} \hookrightarrow X^*(T) \cap \check{\sigma}$ defines a morphism $\pi_{\sigma} : T_{\sigma} \rightarrow \tilde{T}_{\sigma}$, the canonical projection. In particular $\tilde{T}_{\sigma}$ is canonically isomorphic to a quotient of T .

If τ is a face of σ , then T_{τ} is an open subvariety of T_{σ} . The T -orbits in T_{σ} are just the $\tilde{T}_{\tau}$ for the faces τ of σ , and the unique open orbit $T \subset T_{\sigma}$ equals $\tilde{T}_{\{0\}}$. The closure of $\tilde{T}_{\tau}$ in T_{σ} is the union of all $\tilde{T}_{\rho}$ such that τ is a face of ρ and ρ a face of σ . By the above

$\tilde{T}_\tau$ is the torus with cocharacter group $(Y_*(T) + \mathbb{R} \cdot \tau)/\mathbb{R} \cdot \tau$, so let us consider the convex rational polyhedral cone $\sigma_\tau := \sigma \bmod \mathbb{R} \cdot \tau$. It defines a torus embedding $(\tilde{T}_\tau)_{\sigma_\tau}$ of $\tilde{T}_\tau$, and one easily verifies that $\tilde{T}_{\tau, \sigma_\tau}$ is isomorphic to the unique reduced closure subscheme of T_σ with support the closure of $\tilde{T}_\tau$ in T_σ . Moreover the canonical projection $T_\tau \rightarrow \tilde{T}_\tau$ extends uniquely to a T -equivariant morphism $T_\sigma \rightarrow (\tilde{T}_\tau)_{\sigma_\tau}$.

5.3. General torus embeddings: Let Σ be a rational polyhedral decomposition of $Y_*(T)_\mathbb{R}$. Then the T_σ for $\sigma \in \Sigma$ can be glued together canonically to a normal scheme T_Σ . This contains every T_σ as an open dense subscheme. It is of finite type over k if and only if Σ is finite. It is proper if and only if Σ is finite and $|\Sigma| = Y_*(T)_\mathbb{R}$. It is smooth and the complement is a union of smooth divisors with only normal crossings, if and only if all $\sigma \in \Sigma$ are smooth. Then we call Σ smooth (with respect to the lattice $Y_*(T) \subset Y_*(T)_\mathbb{Q}$).

The action of T extends to T_Σ , and the T -orbits are in one-to-one correspondence with the $\sigma \in \Sigma$. Thus we have a bijection between all T -variant subsets $S \subset T_\Sigma$ and all subsets Σ_1 of Σ . Clearly S is open in T_Σ if and only if every face of a cone in Σ_1 is also in Σ_1 , or equivalently in $\bigcup_{\sigma \in \Sigma_1} \sigma^0$ is relatively closed in $|\Sigma|$. Dually it follows that S is closed in T_Σ if and only if $\bigcup_{\sigma \in \Sigma_1} \sigma^0$ is relatively open in $|\Sigma|$.

For any $\sigma \in \Sigma$ the closure of $\tilde{T}_\sigma$ in T_Σ is the union of all $\tilde{T}_\tau$ such that $\tau \in \Sigma$ and σ is a face of τ . Fix σ , then

$$\Sigma_\sigma := \{\tau \bmod \mathbb{R} \cdot \sigma \mid \tau \in \Sigma \text{ such that } \sigma \text{ is a face of } \tau\}.$$

is a rational partial polyhedral decomposition of $Y_*(\tilde{T}_\sigma)_\mathbb{R} = (Y_*(T) + \mathbb{R} \cdot \sigma)/\mathbb{R} \cdot \sigma$. The unique reduced closed subscheme of T_Σ with support the closure of $\tilde{T}_\sigma$ is canonically isomorphic to $(\tilde{T}_\sigma)_{\Sigma_\sigma}$. Let

$$\Sigma_1 := \{\rho \in \Sigma \mid \exists \tau \in \Sigma \text{ such that } \sigma \text{ and } \rho \text{ are faces of } \tau\}.$$

Then the canonical projection extends to a morphism $\pi_\sigma : T_{\Sigma_1} \rightarrow (\tilde{T}_\sigma)_{\Sigma_\sigma}$.

5.4. Functoriality and isogenies: Let $\phi : T_1 \rightarrow T_2$ be a homomorphism of tori. By $\lambda \mapsto \phi \circ \lambda$ it induces a linear map $\phi_* : Y_*(T_1) \rightarrow Y_*(T_2)$. Let Σ_1 and Σ_2 be rational partial polyhedral decompositions of $Y_*(T_1)_\mathbb{R}$, resp. $Y_*(T_2)_\mathbb{R}$. Then ϕ extends to a morphism $T_{1, \Sigma_1} \rightarrow T_{2, \Sigma_2}$ if and only if for every $\sigma_1 \in \Sigma_1$ there exists $\sigma_2 \in \Sigma_2$ such that $\phi_*(\sigma_1) \subset \sigma_2$.

Let T be a torus and Γ a discrete group acting from the left hand side on T by group automorphisms. Then Γ acts from the left hand side on $Y_*(T)$, and we call a rational partial polyhedral decomposition Γ -invariant if $\sigma \in \Sigma$ and $\gamma \in \Gamma$ implies $\gamma_*(\sigma) \in \Sigma$. The action of Γ on T extends to T_Σ if and only if Σ is Γ -invariant.

If $\phi : T_1 \rightarrow T_2$ is an isogeny, then ϕ_* induces an isomorphism $Y_*(T_1)_\mathbb{R} \rightarrow Y_*(T_2)_\mathbb{R}$. Let Σ be a rational partial polyhedral decomposition of this vector space, then we get a morphism $T_{1, \Sigma} \rightarrow T_{2, \Sigma}$. The kernel of ϕ acts by translations on T_1 , and T_2 is just the quotient of T_1 by this action. Since $T_1 \hookrightarrow T_{1, \Sigma}$ is an equivariant embedding, the action of $\ker(\phi)$ extends to $T_{1, \Sigma}$, and one easily verifies (on the affine rings of all T_σ) that $T_{2, \Sigma}$ is the scheme-theoretic quotient of $T_{1, \Sigma}$ by $\ker(\phi)$. This shows that torus embeddings commute with isogenies.

5.5. The relative case: Let T be as before, S a scheme over k , and $X \rightarrow S$ a T -torsor. Let Σ be a rational partial polyhedral decomposition of $Y_*(T)_{\mathbb{R}}$. Locally for the Zariski topology on S there exist sections in X . Every local section Σ defines a trivialization of X , hence a T -equivariant embedding $X \hookrightarrow X_{\Sigma}$ over S . Since the difference of any two local sections is a unique morphism to T , and the embeddings are T -equivariant, we get a canonical isomorphism of the two respective embeddings. Thus there exists up to isomorphism a unique global T -equivariant embedding $X \hookrightarrow X_{\Sigma}$ that is locally on S isomorphic to $T_{\Sigma} \times S$.

All of the above notions are constructions, in particular the canonical projection, extend to the relative case. We have functoriality even in the following general situation. Let $X_i \rightarrow S_i$ be a T_i -torsor and Σ_i a rational partial polyhedral decomposition of $Y_*(T_i)_{\mathbb{R}}$ for $i = 1, 2$. Let $\phi : T_1 \rightarrow T_2$ be a homomorphism, and suppose we are given an equivariant commutative diagram of morphisms

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ S_1 & \longrightarrow & S_2 \end{array}$$

Suppose furthermore that for every $\sigma_1 \in \Sigma_1$ there exists $\sigma_2 \in \Sigma_2$ such that $\phi_*(\sigma_1) \subset \sigma_2$. Then the morphism $X_1 \rightarrow X_2$ extends uniquely to a morphism $X_{1,\Sigma_1} \rightarrow X_{2,\Sigma_2}$. In fact, this is clear from the construction if $S_1 = S_2$ or if $T_1 = T_2$. In general consider the T_2 -torsor $X_2 \times_{S_2} S_1 \rightarrow S_1$, then the composite $X_{1,\Sigma_1} \rightarrow (X_2 \times_{S_2} S_1)_{\Sigma_2} \rightarrow X_{2,\Sigma_2}$ is the desired morphism.

The same constructions go through if T itself is only given as a locally constant family of k -split tori over S , and the rational partial polyhedral decomposition is given fibrewise and locally constant over S . Let us also mention another generalization, where T is not a split torus. In this case $Y_*(T) := \text{Hom}(\mathbb{G}_{m,\bar{k}}, T_{\bar{k}})$, where $\bar{k}$ is the algebraic closure of k . The Galois group $\text{Gal}(\bar{k}/k)$ operates on this through a finite quotient, and by functoriality all constructions descend to k if and only if the rational partial polyhedral decomposition is invariant under $\text{Gal}(\bar{k}/k)$.

5.6. Completion along an orbit: For simplicity assume that σ is of top dimension in 5.2, i.e. that $\mathbb{R} \cdot \sigma = Y_*(T)_{\mathbb{R}}$. Writing $\Lambda_+ := \check{\sigma} \cap X^*(T)$, the coordinate ring of T_{σ} is $k[\Lambda_+]$, and the ideal of the unique closed orbit is generated by all nonzero elements of Λ_+ . Thus the completion of $k[\Lambda_+]$ with respect to this ideal can be described as

$$k[[\Lambda_+]] = \varprojlim k[\Lambda_+]/A \cdot k[\Lambda_+],$$

where A runs through all subsets $A \subset \Lambda_+$ such that $A + \Lambda_+ = \Lambda_+$ and $\Lambda_+ \setminus A$ is finite. The formal completion $\hat{T}_{\sigma}$ of T_{σ} along the closed orbit is just the formal spectrum of $k[[\Lambda_+]]$. For relative torus embeddings, and hence in particular if σ is not of top dimension, we get similar formulas.

Let $T' \hookrightarrow T'_{\sigma'}$ be another affine torus embedding, with σ' of top dimension. By 5.4, a homomorphism $\phi : T' \rightarrow T$ extends to a morphism $T'_{\sigma'} \rightarrow T_{\sigma}$ if and only if $\phi_*(\sigma') \subset \sigma$. Clearly, the closed orbit $\tilde{T}'_{\sigma'}$ is mapped to the closed orbit $\tilde{T}_{\sigma}$ if and only if $\phi_*(\sigma'^0) \subset \sigma^0$.

In this case we get a morphism of formal schemes $\hat{T}'_{\sigma'} \rightarrow \hat{T}_{\sigma}$, corresponding to a continuous homomorphism $k[[\Lambda_+]] \rightarrow k[[\Lambda'_+]]$.

By [EGA-I] 5.4.2, a morphism of schemes $f : X \rightarrow Y$ is called scheme-theoretically dominant, if the associated homomorphism $\mathcal{O}_Y \rightarrow f_*(\mathcal{O}_X)$ is injective. We extend this notion to formal schemes in the obvious way: namely, a morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of formal schemes is scheme-theoretically dominant if the associated homomorphism $\mathcal{O}_{\mathcal{Y}} \rightarrow f_*(\mathcal{O}_{\mathcal{X}})$ is injective.

5.7. Proposition: Let $T \hookrightarrow T_{\sigma}$ be an affine torus embedding, with σ of top dimension. Let $X \rightarrow S$ be a T -torsor, and denote by $\hat{X}_{\sigma}$ the formal completion of X_{σ} along the closed orbit. For every i in some index set I consider the same situation again, i.e. a T_i -torsor $X_i \rightarrow S_i$, σ_i of top dimension, and the formal completion $\hat{X}_{i,\sigma_i}$. For every $i \in I$, let $\phi_i : T_i \rightarrow T$ be a homomorphism, and $(\Psi_i, \psi_i : (X_i, S_i) \rightarrow (X, S))$ a T_i -equivariant morphism. We assume:

- (i) $\phi_{i,*}(\sigma_i^0) \subset \sigma^0$ for every $i \in I$. In particular, by 5.6 we get induced morphisms $\hat{\Psi}_i : \hat{X}_{i,\sigma_i} \rightarrow \hat{X}_{\sigma}$.
- (ii) $\bigcup_{i \in I} \phi_{i,*}(\sigma_i^0)$ is dense in σ^0 (in the usual topology).
- (iii) Every $\psi_i : S_i \rightarrow S$ is scheme-theoretically dominant.

Then the morphism

$$\prod_{i \in I} \hat{\Psi}_i : \prod_{i \in I} \hat{X}_{i,\sigma_i} \longrightarrow \hat{X}_{\sigma}$$

is scheme-theoretically dominant.

Proof. Since the assertion is local in S , we may assume that S is affine, say $S = \mathbf{Spec}(R)$, and that $X = T \times S = \mathbf{Spec}(R[X^*(T)])$. We may also replace each X_i by a scheme over X_i . In particular, we may replace S_i by open affine coverings, so that $X_i \cong T_i \times S_i = \mathbf{Spec}(R_i[X^*(T_i)])$. By assumption, ψ_i corresponds to an injection $\psi_i^* : R \hookrightarrow R_i$, and by equivariance, $\Psi_i^* : R[X^*(T)] \rightarrow R_i[X^*(T_i)]$ extends ψ_i^* and is given by $X^*(T) \ni \chi \mapsto \phi_i^* \cdot \xi_i(\chi)$ for some group homomorphism $\xi_i : X^*(T) \rightarrow R_i^*$. Writing $\Lambda_+ := \check{\sigma}_i \cap X^*(T)$ as in 5.6, and $\Lambda_{i,+} = \check{\sigma}_i \cap X^*(T_i)$, we have to prove the injectivity of the homomorphism

$$\prod_{i \in I} \Psi_i^* : R[[\Lambda_+]] \longrightarrow \prod_{i \in I} R_i[[\Lambda_{i,+}]].$$

For every $i \in I$ denote by $\sigma_{i,j}$ all 1-dimensional convex rational polyhedral cones in $Y_*(T_i)_{\mathbb{R}}$ with $\sigma_{i,j}^0 \subset \sigma_i^0$. For all (i,j) , let $T_{i,j} \subset T_i$ be the unique subtorus with $Y_*(T_{i,j})_{\mathbb{R}} = \mathbb{R} \cdot \sigma_{i,j}$. Since the union $\bigcup_j \sigma_{i,j}^0$ is dense in σ_i^0 , the pairs $(T_{i,j}, \sigma_{i,j})$ together with the morphisms $(\Psi_i|_{T_{i,j}} \times \psi_i, \psi_i)$ satisfy the same assumptions. Thus we may without loss of generality assume that all T_i have dimension 1, or even that all $T_i = \mathbb{G}_{m,k}$ and $\sigma_i = \mathbb{R}^{\geq 0}$, so $\Lambda_{i,+} = \mathbb{Z}^{\geq 0}$. In the latter case the set of all ϕ_i^* can be viewed as a subset $\Phi \subset \sigma^0 \cap Y_*(T)$ such that $\mathbb{R}^{>0} \cdot \Phi$ is dense in σ^0 .

By 5.6, $R[[\Lambda_+]] = \varprojlim R[\Lambda_+]/A \cdot R[\Lambda_+]$, where A runs through all subsets $A \subset \Lambda_+$ such that $A + \Lambda_+ = \Lambda_+$ and $\Lambda_+ \setminus A$ is finite. Let $\mathcal{A}$ be the collection of all such subsets A , for which the kernel of the above homomorphism is contained in the ideal $A \cdot R[[\Lambda_+]]$. We have to prove that $\mathcal{A}$ is cofinal, i.e. that $\bigcap_{A \in \mathcal{A}} A = \emptyset$. Fix any $i_0 \in I$, then for any $x > 0$ there

are only finitely many $\lambda \in \Lambda_+$ with $\phi_{i_0}^* \leq x$. Thus it suffices to prove: For every $A \in \mathcal{A}$ there exists $a_0 \in A$ with $\phi_{i_0}^*(a_0) \leq \phi_{i_0}^*(a)$ for all $a \in A$, and such that $A' := A \setminus \{a_0\} \in \mathcal{A}$. Note that here the assumption $A \in \mathcal{A}$ means that the kernel of the homomorphism

$$R[[\Lambda_+]]/A' \cdot R[[\Lambda_+]] \longrightarrow \prod_{i \in I} R_i[[\mathbb{Z}^{\geq 0}]]/\phi_i^*(A') \cdot R_i[[\mathbb{Z}^{\geq 0}]]$$

is contained in $A \cdot R[[\Lambda_+]]/A' \cdot R[[\Lambda_+]] \cong [a_0] \cdot R$. Since $\Psi_i^*(a_0) = \phi_i^*(a_0) \cdot \xi_i(a_0)$ with $\xi_i(a_0) \in R_i^\times$, we certainly have $A' \in \mathcal{A}$ if there exists $i \in I$ such that $\phi_i^*(a_0) < \phi_i^*(a)$ for all $a_0 \neq a \in A$. In other words, it suffices to prove: For all nonempty subsets $A \subset \Lambda_+$ there exists $a_0 \in A$ and $\phi_i^* \in \Phi$, such that $\phi_i^*(a_0) \leq \phi_i^*(a)$ and $\phi_i^*(a_0) < \phi_i^*(a)$ for all $a_0 \neq a \in A$.

To show this, let $x := \min(\phi_{i_0}^*(A))$ and $K := \{\lambda \in \sigma \mid \phi_{i_0}^*(\lambda) = x\}$, then $A \cap K$ is a nonempty finite set. We take a_0 to be any extreme point of the convex closure of $A \cap K$, this is necessarily a point in A . By construction, $\sigma = \mathbb{R}^{\geq 0} \cdot K$, and there exists $\varepsilon \in \mathbb{Q}^{>0}$ so that $A \setminus K \subset \mathbb{R}^{\geq 1+\varepsilon} \cdot K$. Thus a_0 is also an extreme point of the convex closure of $A \cup \mathbb{R}^{\geq 1+\varepsilon} \cdot K$. If we let $\tau \subset X^*(T)_{\mathbb{R}}$ the convex rational polyhedral cone generated by

$$\{\lambda - a_0 \mid \lambda \in A \cup (1 + \varepsilon) \cdot K\},$$

this means that τ does not contain a nontrivial linear subspace. Now by construction $\check{\sigma} \subset \tau$, whence $\check{\tau} \subset \sigma$. But $\check{\tau}^0$ is open in $Y_*(T)_{\mathbb{R}}$, so by assumption $\Phi \cap \check{\tau}^0$ is non-empty. Any ϕ_i^* in this set satisfies the desired condition. **q.e.d.**

5.8. Torus embeddings of complex spaces: Let T be a torus over $\mathbb{C}$, S a complex space (see [BPV] ch.I §8), and $X \rightarrow S$ an analytic $T(\mathbb{C})$ -torsor. Again $X \rightarrow S$ is locally trivial, so for every rational partial polyhedral decomposition of $Y_*(T)_{\mathbb{R}}$ the construction of 5.5 yields a canonical $T(\mathbb{C})$ -equivariant embedding $X \hookrightarrow X_{\Sigma}$ of complex spaces, that is locally on S isomorphic to $T(\mathbb{C}) \times S \hookrightarrow T_{\Sigma}(\mathbb{C}) \times S$. If S is normal, then X_{Σ} is normal. Again all the other notions and constructions make sense in this case.

Note the obvious canonical isomorphism $T(\mathbb{C}) \cong Y_*(T) \otimes \mathbb{C}^\times$. The continuous homomorphism

$$\text{ord} = \text{ord}_T : T(\mathbb{C}) = Y_*(T) \otimes \mathbb{C}^\times \rightarrow Y_*(T) \otimes \mathbb{R}, \quad \lambda \otimes z \mapsto \lambda \otimes (-\log |z|)$$

(note the minus-sign) identifies $Y_*(T)_{\mathbb{R}}$ with the quotient of $T(\mathbb{C})$ by its unique maximal compact subgroup. Clearly ord_T is functorial in T . More generally let $X \rightarrow S$ be an analytic $T(\mathbb{C})$ -torsor over a complex space S . Then the quotient of X by the maximal compact subgroup of $T(\mathbb{C})$ is a real analytic $Y_*(T)_{\mathbb{R}}$ -torsor on S . It can be C^∞ -trivialized (using a partition of 1), and any trivialization induces a projection $X \rightarrow Y_*(T)_{\mathbb{R}}$ that we again denote by $\text{ord} = \text{ord}_X$. Since any two trivializations differ by a C^∞ -section in $Y_*(T)_{\mathbb{R}}$, which can be lifted to $T(\mathbb{C})$, this map ord_X is unique up to translation by C^∞ -sections in $T(\mathbb{C})$.

Let Σ be a rational partial polyhedral decomposition of $Y_*(T)_{\mathbb{R}}$. The following proposition determines the topology on X_{Σ} .

5.9. Proposition: For every $\sigma \in \Sigma$, and all subsets $V \subset Y_*(T)_\mathbb{R}$ and $U \subset \tilde{X}_\sigma$ consider the following subset of X_Σ :

$$\coprod_{\tau \text{ face of } \sigma} \pi_\tau(\text{ord}^{-1}(V + \sigma) \cap \pi_\sigma^{-1}(U)).$$

(a) For V and U open, this set is open. Moreover these sets form a base for the topology on X_Σ .

(b) For V and U compact, this set is compact.

Proof. For fixed σ , apart from the openness, resp. the compactness, we have to show that the sets in (a) contain a base for the neighborhoods in X_Σ of every point in $\tilde{X}_\sigma$. First we reduce the assertions to a special case. The set in question is contained in X_σ , which is open in X_Σ , so without loss of generality we may assume $X_\Sigma = X_\sigma$. Let T' be the subtorus of T whose cocharacter group is $\mathbb{R} \cdot \sigma \cap Y_*(T)$. Then via the canonical projection $\pi_\sigma : X_\sigma \rightarrow \tilde{T}_\sigma$ the embedding $X \hookrightarrow X_\sigma$ can be viewed as the relative torus embedding associated to $T' \hookrightarrow T'_\sigma$. After replacing T by T' we may therefore assume that $\mathbb{R} \cdot \sigma = Y_*(T)_\mathbb{R}$. Now all assertions are local in S , so we may also assume that the T -torsor $X \rightarrow S$ is trivial. Moreover the topology on X_Σ is invariant under translation by C^∞ -sections $S \rightarrow T(\mathbb{C})$, so we may assume that the map ord_X comes from the corresponding map on $T(\mathbb{C})$. Then the set in question is the product of a subset of $T_\sigma(\mathbb{C})$ with the subset $U \subset \tilde{X}_\sigma = S$. Since $X_\sigma \cong T_\sigma(\mathbb{C}) \times S$ carries the product topology, this shows that it suffices to treat the case $X = T$. Then by $\mathbb{R} \cdot \sigma = Y_*(T)_\mathbb{R}$ the stratum $\tilde{T}_\sigma(\mathbb{C})$ is just a point, so we have to show that the subset

$$\coprod_{\tau \text{ face of } \sigma} \pi_\tau(\text{ord}^{-1}(V + \sigma)).$$

of $T_\sigma(\mathbb{C})$ is open if V is open, compact if V is compact, and for V open they form a base for the neighborhoods of $\tilde{T}_\sigma(\mathbb{C})$ in $T_\sigma(\mathbb{C})$.

For V open we clearly have $V + \sigma = V + \sigma^0 = \bigcup_{\lambda \in V} \lambda + \sigma^0$, hence it suffices to prove the assertions for $V = V + \sigma = \lambda + \sigma^0$ for all $\lambda \in Y_*(T)_\mathbb{R}$. If V is compact, the T -equivariance of the torus embedding yields

$$\coprod_{\tau \text{ face of } \sigma} \pi_\tau(\text{ord}^{-1}(V + \sigma)) = \text{ord}^{-1}(V) \cdot \coprod_{\tau \text{ face of } \sigma} \pi_\tau(\text{ord}^{-1}(\sigma)),$$

where $\text{ord}^{-1}(V)$ is a compact subset of $T(\mathbb{C})$. Thus for the compactness we may assume $V = V + \sigma = \lambda + \sigma$ for some $\lambda \in Y_*(T)_\mathbb{R}$.

Choose nonzero generators $X_1, \dots, X_n$ of the semigroup $\check{\sigma} \cap X^*(T)$. They define a closed embedding $T_\sigma = \mathbf{Spec} \mathbb{C}[\check{\sigma} \cap X^*(T)] \hookrightarrow \mathbb{A}_\mathbb{C}^n$ that maps $\tilde{T}_\sigma$ to $\{0\}$. The relation $\check{\sigma} = \sigma$ implies that for all $\lambda \in Y_*(T)_\mathbb{R}$

$$\lambda + \sigma = \{y \in Y_*(T)_\mathbb{R} \mid \forall i = 1, \dots, n \langle X_i, y \rangle \geq \langle X_i, \lambda \rangle\}.$$

Hence

$$\text{ord}^{-1}(\lambda + \sigma) = \{t \in T(\mathbb{C}) \mid \forall i = 1, \dots, n |X_i(t)| \leq e^{-\langle X_i, \lambda \rangle}\},$$

and for every face τ of σ

$$\pi_\tau(\text{ord}^{-1}(\lambda + \sigma)) = \{t \in \tilde{T}_\sigma(\mathbb{C}) \mid \forall i = 1, \dots, n \mid |X_i(t)| \leq e^{-\langle X_i, \lambda \rangle}\},$$

(note that in the last formula some $X_i(t)$ may be zero). Thus the set in question is the intersection of $T_\sigma(\mathbb{C})$ with the compact subset

$$\{\xi \in \mathbb{A}^n(\mathbb{C}) \mid \forall i = 1, \dots, n \mid |\xi_i| \leq e^{-\langle X_i, \lambda \rangle}\}$$

of $\mathbb{A}^n(\mathbb{C})$ hence itself compact. This shows (b). For $\lambda + \sigma^0$ instead of $\lambda + \sigma$ above we have strict inequalities, so by the same argument the set is open. Finally since σ does not contain a nontrivial linear subspace, λ can be chosen to make all $\langle X_i, \lambda \rangle$ arbitrarily large. Therefore the sets form a base of neighborhoods of $\{0\}$ in $\mathbb{A}^n(\mathbb{C})$, whence the corresponding assertion for $T_\sigma(\mathbb{C})$. **q.e.d.**

5.10. Line bundles as torus embeddings: Let S be a scheme of finite type over k , and $\pi : X \rightarrow S$ a $\mathbb{G}_m$ -torsor. The cone $\sigma_0 := \mathbb{R}^{\geq 0} \subset Y_*(\mathbb{G}_m)_{\mathbb{R}}$ defines a relative torus embedding $X \hookrightarrow X_{\sigma_0}$ over S . Clearly $X_{\sigma_0} \rightarrow S$ is just the line bundle associated to the torsor $X \rightarrow S$. Denote X_{σ_0} by L and the projection $L \rightarrow S$ again by π . Let $\mathcal{L}$ be the invertible sheaf on S of all local sections in L . Then $\pi_*(\mathcal{O}_L)$ is canonically isomorphic to the sheaf of $\mathcal{O}_S$ -algebras $\bigoplus_{n \geq 0} \mathcal{L}^{\otimes(-n)}$. Indeed, for any open $U \subset S$ we have canonically

$$\begin{aligned} \text{Hom}_{\mathcal{O}_U\text{-algebra}}(\pi_*(\mathcal{O}_L), \mathcal{O}_U) &\cong \text{Mor}_S(U, L) \cong \Gamma(U, \mathcal{L}) \cong \\ &\cong \text{Hom}_{\mathcal{O}_U}(\mathcal{L}^{\otimes(-1)}, \mathcal{O}_U) \cong \text{Hom}_{\mathcal{O}_U\text{-algebra}}(\bigoplus_{n \geq 0} \mathcal{L}^{\otimes(-n)}, \mathcal{O}_U). \end{aligned}$$

For $i = 1, \dots, n$ let $a_i \in \mathbb{Z}$ and $X_i \rightarrow S$ be a $\mathbb{G}_m$ -torsor. Consider the homomorphism

$$\phi : \mathbb{G}_m^n \rightarrow \mathbb{G}_m, (t_1, \dots, t_n) \mapsto t_1^{a_1} \cdots t_n^{a_n}$$

The action of $\mathbb{G}_m^n$ on

$$(X_1 \times_S \dots \times_S X_n) / \ker(\phi)$$

factors through ϕ , and endows this quotient with the structure of a $\mathbb{G}_m$ -torsor. It is easy to see that this $\mathbb{G}_m$ -torsor is canonically isomorphic to the tensor product $X_1^{\otimes a_1} \otimes \dots \otimes X_n^{\otimes a_n}$. In fact, let x_i be local sections of X_i , then the map

$$X_1 \times_S \dots \times_S X_n \rightarrow X_1^{\otimes a_1} \otimes \dots \otimes X_n^{\otimes a_n}, (x_1, \dots, x_n) \mapsto x_1^{a_1} \otimes \dots \otimes x_n^{a_n}$$

is ϕ -equivariant and factors through $(X_1 \times_S \dots \times_S X_n) / \ker(\phi)$. This yields a convenient way to describe tensor products of $\mathbb{G}_m$ -torsors. In particular the inverse of a $\mathbb{G}_m$ -torsor is just the same scheme $X \rightarrow S$ with the opposite $\mathbb{G}_m$ -action. For any integer $n \geq 1$ $X^{\otimes n} \cong X / \mu_n$ with the $\mathbb{G}_m$ -action through the quotient $\mathbb{G}_m \rightrightarrows \mathbb{G}_m / \mu_n \xrightarrow{\sim} \mathbb{G}_m, t \mapsto t^n$. Clearly the same results hold if S is a complex space, T a torus over $\mathbb{C}$ and $X \rightarrow S$ a $T(\mathbb{C})$ -torsor.

5.11. Equivariant line bundles on torus embeddings: Let $1 \rightarrow \mathbb{G}_m \rightarrow T' \xrightarrow{\pi} T \rightarrow 1$ be a short exact sequence of split tori over k . Then T' is a $\mathbb{G}_m$ -torsor on T ,

equivariant under the action of T' . Note that an equivariant trivialization of this torsor is equivalent to a splitting of the above short exact sequence. Let Σ be a rational partial polyhedral decomposition of $Y_*(T)_{\mathbb{R}}$. We want to study how the $\mathbb{G}_m$ -torsor $T' \rightarrow T$ can be extended T' -equivariantly to T_{Σ} .

Since T_{σ} is affine for every $\sigma \in \Sigma$, the extension must be trivializable over T_{σ} . Thus there exists a splitting $e_{\sigma} : T \rightarrow T'$ such that the extension of the $\mathbb{G}_m$ -torsor to T_{σ} is isomorphic to the torus embedding $T'_{e_{\sigma*}(\sigma)}$. Note that e_{σ} is not uniquely determined by $e_{\sigma*}(\sigma)$ unless $\mathbb{R} \cdot \sigma = Y_*(T)_{\mathbb{R}}$. Globally the extension must be the torus embedding T'_{Σ^+} for some rational partial polyhedral decomposition Σ^+ of $Y_*(T')_{\mathbb{R}}$. The necessary and sufficient condition on Σ^+ is that π_* induces a bijection $\Sigma^+ \rightarrow \Sigma$, $\sigma' \mapsto \pi_*(\sigma')$, such that for every $\sigma' \in \Sigma^+$ there exists a splitting $e_{\sigma'} : T \rightarrow T'$ such that $\sigma' = e_{\sigma'*} \circ \pi_*(\sigma')$.

Let Σ^+ be such a decomposition. Consider the convex rational polyhedral cone $\sigma_0 := \mathbb{R}_{\geq 0} \subset Y_*(\mathbb{G}_m)_{\mathbb{R}} \hookrightarrow Y_*(T')_{\mathbb{R}}$, and define $\Sigma' := \{\sigma', \sigma' + \sigma_0 \mid \sigma' \in \Sigma^+\}$. One easily verifies locally that $T'_{\Sigma^+} \hookrightarrow T'_{\Sigma'}$ is the relative torus embedding over T_{Σ} associated to the torus $\mathbb{G}_m$ and the cone σ_0 . Thus by 5.10 $T'_{\Sigma'}$ is the line bundle on T_{Σ} associated to the $\mathbb{G}_m$ -torsor T'_{Σ^*} .

Let S be a scheme of finite type over k , $X' \rightarrow S$ a T' -torsor and $X : +X'/\mathbb{G}_m$ the associated T -torsor on S . Let $X'_{\Sigma'} \rightarrow X_{\Sigma}$ be the associated relative torus embeddings, this is a line bundle. Any splitting $e : T \rightarrow T'$ determines an isomorphism $\mathbb{G}_m \cong T'/e(T)$, through which $X'/e(T)$ becomes a $\mathbb{G}_m$ -torsor. Denote by $\mathcal{L}$ the invertible sheaf on X_{Σ} associated to $X'_{\Sigma'}$. We keep these notations for the rest of this chapter. The following proposition determines the direct image of $\mathcal{L}$ under the projection $f : X_{\Sigma} \rightarrow S$.

5.12. Proposition: (a) In the situation of 5.11 $f_*\mathcal{L}$ is canonically isomorphic to the direct sum of the invertible sheaves associated to the line bundles

$$(X'_{\sigma_0})/e(T) \rightarrow S$$

for all splittings $e : T \rightarrow T'$ that extend to a morphism $T_{\Sigma} \rightarrow T'_{\Sigma'}$.

(b) The same assertion also holds if T is a torus over $\mathbb{C}$, $X \rightarrow S$ an analytic $T(\mathbb{C})$ -torsor over a complex space S , and Σ a finite and complete rational polyhedral decomposition of $Y_*(T)_{\mathbb{R}}$.

Proof. For any splitting $e : T_{\Sigma} \rightarrow T'_{\Sigma'}$, let $T' \rightarrow T \times (T'/e(T))$ be the canonical map $t' \mapsto (\pi(t'), [t'])$. It is clearly an isomorphism, and its inverse extends to a morphism

$$\begin{array}{ccc} T \times (T'/e(T)) & \xleftarrow{\sim} & T' \\ \downarrow & & \downarrow \\ T_{\Sigma} \times (T'_{\sigma_0}/e(T)) & \longrightarrow & T'_{\Sigma'} \end{array}$$

Indeed, if we decompose $T' \cong T \times \mathbb{G}_m$ via the splitting e , the torus embedding on the bottom left is covered by affines $T_{\sigma} \times \mathbb{A}^1 = T'_{\sigma+\sigma_0}$. Since by assumption $\sigma \subset e_{\sigma}(\sigma) + \sigma_0$, we also have $\sigma + \sigma_0 \subset e_{\sigma}(\sigma) + \sigma_0$, hence the morphism extends to $T'_{\sigma+\sigma_0} \rightarrow T'_{e_{\sigma}(\sigma)+\sigma_0} \subset T'_{\Sigma'}$, as desired. This implies that the analogous morphism of the torsors also extends:

$$\begin{array}{ccc} X \times (X'/e(T)) & \xleftarrow{\sim} & X' \\ \downarrow & & \downarrow \\ X_{\Sigma} \times (X'_{\sigma_0}/e(T)) & \longrightarrow & X'_{\Sigma'} \end{array}$$

since it does so locally. Thus for any open $U \subset S$ every local section $y : U \rightarrow (X'_{\sigma_0})/e(T)$ yields a section

$$X_{\Sigma} \times_S U \xrightarrow{\text{id} \times y} X_{\Sigma} \times (X'_{\sigma_0}/e(T)) \longrightarrow X'_{\Sigma'},$$

i.e. an element of $\Gamma(X_{\Sigma} \times_S U, \mathcal{L}) = \Gamma(U, f_* \mathcal{L})$. This defines a canonical homomorphism from the invertible sheaf associated to $(X'_{\sigma_0}/e(T))$ to $f_* \mathcal{L}$. Taking the direct sum over all splittings $e : T_{\Sigma} \rightarrow T'_{\Sigma'}$, we get the desired canonical homomorphism.

It remains to show that in both cases (a) and (b) this is an isomorphism. Since this is a local property, we may assume that X' is a trivial torsor, which then reduces to the case $X' = T'$, resp. $X' = T'(\mathbb{C})$. In case (b) $T_{\Sigma}(\mathbb{C})$ is compact by assumption, hence the holomorphic sections of $\mathcal{L}$ coincide with the algebraic sections. Thus for the rest of the proof we may stay in the algebraic category.

Now $\Gamma(T_{\Sigma}, \mathcal{L})$ is just the intersection of all $\Gamma(T_{\sigma}, \mathcal{L})$ inside $\Gamma(T, \mathcal{L})$. Since likewise any splitting $T \rightarrow T'$ extends to $T_{\Sigma} \rightarrow T'_{\Sigma'}$ if and only if it extends to $T_{\sigma} \rightarrow T'_{e_{\sigma}(\sigma) + \sigma_0}$ for every $\sigma \in \Sigma$, we are reduced to the case $T_{\Sigma} = T_{\sigma}$. After using the splitting e_{σ} to write $T' \cong \mathbb{G}_m \times T$, we may assume that $T' = \mathbb{G}_m \times T$ and the embedding is $T'_{\sigma_0 \times \sigma} = \mathbb{A}^1 \times T_{\sigma}$. Then we have canonical isomorphisms

$$\Gamma(T_{\sigma}, \mathcal{L}) \cong \text{Mor}_{T_{\sigma}}(T_{\sigma}, \mathbb{A}^1 \times T_{\sigma}) \cong \text{Mor}(T_{\sigma}, \mathbb{A}^1) \cong \Gamma(T_{\sigma}, \mathcal{O}_{T_{\sigma}}) \cong k[X^*(T) \cap \check{\sigma}].$$

Any splitting $e : T \rightarrow \mathbb{G}_m \times T$ is of the form $t \mapsto (\chi(t), t)$ for some $\chi \in X^*(T)$, so $e : Y_*(T) \rightarrow \mathbb{Z} \oplus Y_*(T)$ is given by $\lambda \mapsto (\langle \chi, \lambda \rangle, \lambda)$. It extends to a morphism $T_{\sigma} \rightarrow \mathbb{A}^1 \times T_{\sigma}$ if and only if $e_*(\sigma) \subset \sigma_0 \times \sigma$, i.e. if $\langle \chi, \lambda \rangle \geq 0$ for all $\lambda \in \sigma$, which is equivalent to $\chi \in \check{\sigma}$. Hence the map under consideration goes from $\bigoplus_{\chi \in X^*(T) \cap \check{\sigma}} k \cdot \chi = k[X^*(T) \cap \check{\sigma}]$ to itself, and it suffices to show that it is the identity map.

The section associated to e is determined by the following diagram

$$\begin{array}{ccccccc} T & \xrightarrow{(\text{id}, 1)} & T \times (\mathbb{G}_m \times T)/e(T) & \xleftarrow{\sim} & \mathbb{G}_m \times T & \xrightarrow{pr_1} & \mathbb{G}_m \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ T_{\sigma} & \xrightarrow{(\text{id}, 1)} & T_{\sigma} \times (\mathbb{A}^1 \times T)/e(T) & \longrightarrow & \mathbb{A}^1 \times T_{\sigma} & \xrightarrow{pr_1} & \mathbb{A}^1 \end{array}$$

On the level of cocharacter groups this diagram reads

$$\begin{array}{ccccccc} Y_*(T) & \xrightarrow{(\text{id}, 0)} & Y_*(T) \oplus (\mathbb{Z} \oplus Y_*(T))/e_*(Y_*(T)) & \xleftarrow{\sim} & \mathbb{Z} \oplus Y_*(T) & \xrightarrow{pr_1} & \mathbb{Z} \\ \lambda & \longmapsto & (\lambda, [(0, 0)]) = (\lambda, [(\langle \chi, \lambda \rangle, \lambda)]) & & (\langle \chi, \lambda \rangle, \lambda) & \longmapsto & \langle \chi, \lambda \rangle \end{array}$$

This shows that the image of the section associated to e is indeed the cocharacter χ , as desired. **q.e.d.**

5.13. Example: In the situation of 5.11 assume that $T' = \mathbb{G}_m \times T$ and $\Sigma' = \{\{0\} \times \sigma, \sigma_0 \times \sigma \mid \sigma \in \Sigma\}$, in which case $\mathcal{L}$ is just the structure sheaf of X_{Σ} . Then $f_* \mathcal{L}$ is the direct sum of the invertible sheaves associated to the $\mathbb{G}_m$ -torsors $X/\ker(\chi) \rightarrow S$, for all $\chi \in \text{Hom}(T, \mathbb{G}_m) = X^*(T)$ that take nonpositive values on $|\Sigma|$. In fact, any splitting $e : T \hookrightarrow T'$ is of the form $e(t) = (\chi^{-1}(t), t)$ for $\chi \in X^*(T)$, and it extends to $T_{\Sigma} \hookrightarrow T'_{\Sigma'} = \mathbb{A}^1 \times T_{\Sigma}$ if and only if χ^{-1} is nonnegative on $|\Sigma|$. By 5.12 (a), $f_* \mathcal{O}_X$ is the direct sum of the invertible sheaves associated to the line bundles $(X'_{\sigma_0}/e(T)) \rightarrow S$ for all such χ . But

$(X'_{\sigma_0}/e(T) = (\mathbb{A}^1 \times X)/e(T) \cong X/\ker(\chi)$, with the $\mathbb{G}_m$ -structure given by the commutative diagram

$$\begin{array}{ccc} \mathbb{G}_m \times T & \longrightarrow & (\mathbb{G}_m \times T)/e(T) \\ \uparrow & & \wr \\ T & \xrightarrow{\chi} & \mathbb{G}_m \end{array}$$

In particular, if $T_\Sigma = T_\sigma$ is an affine torus embedding, then the χ in question are precisely those with $-\chi \in \check{\sigma}$. Assume moreover that σ is of top dimension, and denote by $\hat{f}: \hat{X}_\sigma \rightarrow S$ the formal completion of X_σ along the closed orbit, as in 5.7. By the description in 5.6, $\hat{f}_* \mathcal{O}_{\hat{X}_\sigma}$ is canonically isomorphic to the direct product of the above invertible sheaves.

5.14. Proposition: In the situation of 5.12 (a) or (b) assume that Σ is a finite and complete rational polyhedral decomposition of $Y_*(T)_\mathbb{R}$, and that $Y_*(T)_\mathbb{R} \setminus |\Sigma'|$ is convex. For every $\sigma \in \Sigma$ with $\mathbb{R} \cdot \sigma = Y_*(T)_\mathbb{R}$ let $e_\sigma: T \rightarrow T'$ be the splitting defined in 5.11, and $\mathcal{M}_\sigma$ the invertible sheaf on S associated to the line bundle $(X'_{\sigma_0}/e_\sigma(T) \rightarrow S$.

(a) For every such σ $\mathcal{M}_\sigma$ is a direct summand of $f_* \mathcal{L}$ as in 5.12.

(b) For every $n \geq 1$ and every direct summand $\mathcal{M}$ of $f_*(\mathcal{L}^{\otimes n})$ according to 5.12 there exist integers $b \geq 1$ and $a_\sigma \geq 0$, for all σ as above, such that $b \cdot n = \sum_\sigma a_\sigma$ and

$$\mathcal{M}^{\otimes b} \cong \bigotimes_\sigma \mathcal{M}_\sigma^{\otimes a_\sigma}.$$

Remark. In general b cannot be chosen equal to 1, although it can be bounded independently of $\mathcal{M}$ and n . This is related to the fact that on an arbitrary (non-smooth) complete torus embedding there exist ample invertible sheaves that are not very ample.

Proof. After choosing a splitting we may assume that $T' = \mathbb{G}_m \times T$. For every σ of top dimension write $e_\sigma = (\chi_\sigma, \text{id})$ for $\chi_\sigma \in X^*(T)$. Recall (5.11) that for such σ χ_σ is uniquely determined by σ and Σ' . Consider arbitrary $a_\sigma \in \mathbb{Z}$. By 5.10 $\bigotimes_\sigma \mathcal{M}_\sigma^{\otimes a_\sigma}$ is the invertible sheaf associated to the $\mathbb{G}_m$ -torsor $(X' \times_S \dots \times_S X')/\ker(\phi)$, where ϕ is the composite homomorphism

$$\Pi_\sigma T' \rightarrow \Pi_\sigma T'/e_\sigma(T) \xleftarrow{\sim} \Pi_\sigma \mathbb{G}_m \rightarrow \mathbb{G}_m,$$

$$((x_\sigma, t_\sigma)) \mapsto ([x_\sigma, t_\sigma]), ([x_\sigma, 1]) \mapsto (x_\sigma) \mapsto \Pi x_\sigma^{a_\sigma}.$$

An easy calculation shows that the composite homomorphism

$$\mathbb{G}_m \times T = T' \xrightarrow{\text{diag}} \Pi_\sigma T' \xrightarrow{\phi} \mathbb{G}_m$$

is given by $\psi: (x, t) \mapsto x^a \cdot \chi^*(t)^{-1}$, where $\chi^* := \sum_\sigma a_\sigma \cdot \chi_\sigma$ and $a := \sum_\sigma a_\sigma$. This shows that the composite map

$$X' \xrightarrow{\text{diag}} X' \times_S \dots \times_S X' \rightarrow (X' \times_S \dots \times_S X')/\ker(\phi)$$

induces a T' -equivariant isomorphism

$$X'/\ker(\psi) \xrightarrow{\sim} (X' \times_S \dots \times_S X')/\ker(\phi).$$

Thus $\bigotimes_\sigma \mathcal{M}_\sigma^{\otimes a_\sigma}$ is the invertible sheaf associated to the $\mathbb{G}_m$ -torsor $X'/\ker(\psi) \rightarrow S$.

By 5.10 $\mathcal{L}^{\otimes n}$ is the invertible sheaf associated to the line bundle $X'_{\Sigma}/\mu_n \rightarrow X_{\Sigma}$. Since canonically $Y_*(T'/\mu_n) = Y_*((\mathbb{G}_m/\mu_n) \times T) = \frac{1}{n} \cdot \mathbb{Z} \oplus Y_*(T)$, every splitting $e : T \rightarrow T'/\mu_n$ is on the cocharacter group given as (χ, id) for some $\chi \in \frac{1}{n} \cdot X^*(T)$. An easy calculation shows that the composite homomorphism

$$T' = \mathbb{G}_m \times T \mathbf{1} ((\mathbb{G}_m/\mu_n) \times T)/e(T) \xleftarrow{\sim} \mathbb{G}_m$$

is given by $\xi : (x, t) \mapsto x^n \cdot (\chi^n)(t)^{-1}$. Thus $\mathcal{M}$ is the invertible sheaf associated to the $\mathbb{G}_m$ -torsor $X'/\ker(\xi) \rightarrow S$ for some $\chi \in \frac{1}{n} \cdot X^*(T)$. For any integer b it follows that $\mathcal{M}^{\otimes b}$ is the invertible sheaf associated to the $\mathbb{G}_m$ -torsor $X'/\ker(\xi^b) \rightarrow S$. Thus for (b) we have to prove that for any $\chi \in \frac{1}{n} \cdot X^*(T)$ such that the section $(\chi, \text{id}) : T \rightarrow T'/\mu_n$ extends to a morphism $T - \Sigma \rightarrow T'_{\Sigma}/\mu_n$, there exist integers $b \geq 1$ and $a_{\sigma} \geq 0$ such that $b \cdot n = \sum_{\sigma} a_{\sigma}$ and $\psi = \xi^b$. This last equality is equivalent to $b \cdot n \cdot \chi = \sum a_{\sigma} \cdot \chi_{\sigma}$. Let $\alpha_{\sigma} := \frac{a_{\sigma}}{b \cdot n}$ for all σ , then these must be nonnegative rational numbers with $\sum \alpha_{\sigma} = 1$ and $\chi = \sum \alpha_{\sigma} \cdot \chi_{\sigma}$. Since we can make b arbitrarily large, it suffices to find such $\alpha_{\sigma} \in \mathbb{Q}^{\geq 0}$. An easy argument shows that if such elements exist in $\mathbb{R}^{\geq 0}$, then one can also find them in $\mathbb{Q}^{\geq 0}$. Thus we are reduced to the assertion: χ lies in the convex closure of the χ_{σ} .

Let us analyze the condition that a splitting extends to T_{Σ} . Let $f : Y_*(T)_{\mathbb{R}} \rightarrow \mathbb{R}$ be the unique function such that for all $\sigma \in \Sigma$ and all $\lambda \in \sigma$ $e_{\sigma*}(\lambda) = (f(\lambda), \lambda)$. This function is continuous, coincides with χ_{σ} for each σ of top dimension, and has integer values on $Y_*(T)$. Moreover by definition

$$|\Sigma'| = \{(f(\lambda) + \alpha, \lambda) \mid \lambda \in Y_*(T)_{\mathbb{R}}, \alpha \in \mathbb{R}^{\geq 0}\},$$

hence the convexity of $Y_*(T)_{\mathbb{R}} \setminus |\Sigma'|$ is equivalent to the convexity of f (in the sense of [KKMS] footnote p.27), i.e. for all $\lambda, \lambda' \in Y_*(T)_{\mathbb{R}}$ and $\alpha \in [0, 1]$ $f(\alpha\lambda + (1 - \alpha)\lambda') \geq \alpha f(\lambda) + (1 - \alpha)f(\lambda')$. Consider the set

$$K := \{x \in X^*(T)_{\mathbb{R}} \mid \forall \lambda \in Y_*(T)_{\mathbb{R}} \langle x, \lambda \rangle \geq f(\lambda)\}.$$

Let $\chi \in \frac{1}{n} \cdot X^*(T)$, then by 5.4 the splitting $(\chi, \text{id}) : T \rightarrow T'/\mu_n$ extends to a morphism $T_{\Sigma} \rightarrow T'_{\Sigma}/\mu_n$ if and only if for all $\sigma \in \Sigma$ $(\chi, \text{id})(\sigma) \subset e_{\sigma*}(\sigma) + \sigma_0$. But this is equivalent to $\forall \lambda \in Y_*(T)_{\mathbb{R}} \langle \chi, \lambda \rangle \geq f(\lambda)$, so the splitting extends if and only if $\chi \in K$. Thus the assertion (a) is equivalent to $\chi_{\sigma} \in K$ for all $\sigma \in \Sigma$ of top dimension, and (b) is equivalent to the statement: Every $\chi \in K \cap \frac{1}{n} \cdot X^*(T)$ lies in the convex closure of these χ_{σ} . Both are implied by the following lemma. Observe that up to now we have not used any of the assumptions on Σ' . **q.e.d.**

5.15. Lemma: Let $K \subset X^*(T)_{\mathbb{R}}$ and $\chi_{\sigma} \in X^*(T)$ for all $\sigma \in \Sigma$ with $\mathbb{R} \cdot \sigma = Y_*(T)_{\mathbb{R}}$ as in the proof of 5.14. Then K is the convex closure of these χ_{σ} .

Proof. Let us first check that the χ_{σ} lie in K . For any $\lambda \in Y_*(T)_{\mathbb{R}}$ we have to show that $\langle \chi_{\sigma}, \lambda \rangle \geq f(\lambda)$. Choose any $\lambda' \in \sigma^0$, then σ^0 is open in $Y_*(T)_{\mathbb{R}}$, hence there exists $0 < \alpha < 1$ such that $\alpha\lambda + (1 - \alpha)\lambda' \in \sigma^0$. The convexity of f implies $f(\alpha\lambda + (1 - \alpha)\lambda') \geq \alpha \cdot f(\lambda) + (1 - \alpha) \cdot f(\lambda')$. But f coincides with $\langle \chi_{\sigma} \rangle$ on σ , so it follows $\langle \chi_{\sigma}, \alpha\lambda + (1 - \alpha)\lambda' \rangle \geq \alpha \cdot f(\lambda) + (1 - \alpha) \cdot \langle \chi_{\sigma}, \lambda' \rangle$. This implies $\langle \chi_{\sigma}, \lambda \rangle \geq f(\lambda)$, as desired.

Next K is an intersection of closed half spaces, hence it is closed and convex. We claim that K is compact. To prove this it suffices to show that it is bounded in any fixed euclidean metric on $X^*(T)_{\mathbb{R}}$. Assume it is not bounded, and fix $x \in K$. For every $\alpha \geq 0$ let $B_{\alpha}(x)$ be the open ball of radius α around x . Let $S := \partial B_1(x)$, then for every $\alpha \geq 1$ the intersection of S with the convex closure of $\{x\} \cup (K \setminus B_{\alpha}(x))$ is a nonempty closed subset, which decreases as α increases. By the compactness of S their intersection is nonempty, hence K contains some half-line $x + \mathbb{R}^{\geq 0} \cdot x'$ for $x' \in S$. Let $\lambda \in Y_*(T)_{\mathbb{R}}$ be such that $\langle x', \lambda \rangle < 0$, then it follows that $\langle x + \alpha x', \lambda \rangle \geq f(\lambda)$ for all $\alpha \in \mathbb{R}^{\geq 0}$, which is clearly a contradiction. Thus K is compact.

Recall that an element x of a closed convex subset A of a finite dimensional $\mathbb{R}$ -vector space is called an extreme point of A if it cannot be written in the form $\frac{1}{2} \cdot (x' + x'')$ with $x', x'' \in A$ and $x' \neq x \neq x''$. By the Krein-Milman theorem every compact convex set A is the convex closure of its extreme points. Thus in our case it suffices to prove that the extreme points of K are just the χ_{σ} .

Fix an extreme point $x_0 \in K$ and let

$$A := \{\lambda \in Y_*(T)_{\mathbb{R}} \mid \langle x_0, \lambda \rangle = f(\lambda)\}.$$

This is a convex subset. Indeed, let $\lambda, \lambda' \in A$, $0 \leq \alpha \leq 1$ and $\lambda'' := \alpha\lambda + (1 - \alpha)\lambda'$. Then the sequence of inequalities $f(\lambda'') \leq \langle x_0, \lambda'' \rangle = \alpha \cdot \langle x_0, \lambda \rangle + (1 - \alpha) \cdot \langle x_0, \lambda' \rangle = \alpha \cdot f(\lambda) + (1 - \alpha) \cdot f(\lambda') \leq f(\lambda'')$ shows that $f(\lambda'') = \langle x_0, \lambda'' \rangle$, hence $\lambda'' \in A$, as desired. Clearly $A \cap \sigma$ is a convex rational polyhedral cone for every $\sigma \in \Sigma$. Since A is convex, it is itself a convex rational polyhedral cone.

We want to show that $\mathbb{R} \cdot A = Y_*(T)_{\mathbb{R}}$. Let $\{\tau_i\} \subset \Sigma$ be the finite subset of all τ with $\dim(\mathbb{R} \cdot \tau) = 1$. Choose $\lambda_i \in \tau_i^0$ for every i , then we have $K = \{x \in X^*(T)_{\mathbb{R}} \mid \forall i \langle x, \lambda_i \rangle \geq f(\lambda_i)\}$. Indeed, every $\lambda \in Y_*(T)_{\mathbb{R}}$ lies in some $\sigma \in \Sigma$, hence is a nonnegative linear combination of the λ_i such that τ_i is the face of σ , and f is linear on σ . Let I be the set of all i such that $\langle x_0, \lambda_i \rangle = f(\lambda_i)$, then by definition $L := \{x \in X^*(T)_{\mathbb{R}} \mid \forall i \notin I \langle x, \lambda_i \rangle \geq f(\lambda_i)\}$ is a neighborhood of x_0 in $X^*(T)_{\mathbb{R}}$. Now K contains the subset

$$L \cap \{x \in X^*(T)_{\mathbb{R}} \mid \forall i \in I \langle x, \lambda_i \rangle = f(\lambda_i)\} = L \cap (x_0 + \{x \in X^*(T)_{\mathbb{R}} \mid \forall i \in I \langle x, \lambda_i \rangle = 0\}),$$

so since x_0 is an extreme point of K , the linear subspace $\{x \in X^*(T)_{\mathbb{R}} \mid \forall i \in I \langle x, \lambda_i \rangle = 0\}$ is reduced to a point. In other words the set $\{\lambda_i \mid i \in I\}$ generates $Y_*(T)_{\mathbb{R}}$. But by definition it is a subset of A , whence $\mathbb{R} \cdot A = Y_*(T)_{\mathbb{R}}$.

Finally it now follows that there exists $\sigma \in \Sigma$ with $\mathbb{R} \cdot \sigma = Y_*(T)_{\mathbb{R}}$ and $\sigma^0 \cap A^0 \neq \emptyset$. Fix $\lambda \in \sigma^0 \cap A^0$, then f coincides with both χ_{σ} and x_0 in a neighborhood of λ . This implies $x_0 = \chi_{\sigma}$, as desired. **q.e.d.**

Next we study ampleness criteria in our setting. It turns out that the property “anti-ample” has a more natural geometric meaning.

5.16. Ample line bundles: In the situation of 5.12 assume that $|\Sigma|$ is convex, for instance $|\Sigma| = Y_*(T)_{\mathbb{R}}$. We call Σ' strictly convex if

(i) $|\Sigma'|$ is convex and

(ii) Σ^+ consists of all cones $|\Sigma'| \cap \ker(\langle x, \rangle)$ for all $x \in X^*(T)_{\mathbb{Q}}$ such that $\langle x, \rangle$ is nonnegative on $|\Sigma'|$, and restricts to the identity on $\mathbb{R} \cdot \sigma_0$.

Note that if these conditions are satisfied, then Σ^+ , hence also Σ' and Σ , are uniquely determined by $|\Sigma'|$. If Σ is finite, then condition (i) means that $|\Sigma'|$ is a convex rational polyhedral cone, and (ii) means that Σ^+ consists of all faces of $|\Sigma'|$ that do not contain σ_0 .

Assume now that Σ is finite, and $|\Sigma|$ convex. Then the invertible sheaf $\mathcal{L}^{\otimes(-1)}$ is relatively ample with respect to $f : X_{\Sigma} \rightarrow S$ if and only if Σ' is strictly convex. Indeed this is a reformulation of [KKMS] ch.I §3 thm.13 under slightly stronger assumptions. But let us briefly indicate another, somewhat amusing proof, which exhibits the relationship between ampleness and convexity within the setting of torus embeddings. Without loss of generality let $X = T$. If Σ' is strictly convex, then we have a torus embedding $T' \hookrightarrow T'_{|\Sigma'|}$, hence a commutative diagram

$$\begin{array}{ccccc} T_{\Sigma} & \xleftarrow{\text{zero section}} & T'_{\Sigma'} & \xleftarrow{\text{open}} & T'_{\Sigma^+} \\ \downarrow & & \downarrow & & \parallel \\ S \cong \tilde{T}'_{|\Sigma'|} & \xleftarrow{\text{closed orbit}} & T'_{|\Sigma'|} & \xleftarrow{\text{open}} & T'_{\Sigma^+} \end{array}$$

Observe that the morphism $T'_{\Sigma^+} \rightarrow T'_{|\Sigma'|}$, which exists if $|\Sigma'|$ is convex, is an open embedding if and only if Σ' is strictly convex. Grauert's ampleness criterion ([EGA-II] thm. 8.9.1) now implies that $\mathcal{L}^{\otimes(-1)}$ is ample on T_{Σ} (note that the convention of [EGA-II] 1.7.8 concerning the relation between $T'_{\Sigma'}$ and $\mathcal{L}$ is the opposite of ours). Conversely, if $\mathcal{L}^{\otimes(-1)}$ is ample on T_{Σ} , we have the same diagram with $T'_{|\Sigma'|}$ replaced by the affine cone associated to $\mathcal{L}$ by [EGA-II] 8.8.2. But by definition the T' -action extends to this cone, so by [EGA-II] 8.8.6 (ii) and [KKMS] ch.I thm. 1' this cone is isomorphic to T'_{τ} for some convex rational polyhedral cone $\tau \subset Y_*(T')_{\mathbb{R}}$. One easily verifies that $\tau = |\Sigma'|$, and the strict convexity of Σ' follows.

For the proof of theorem 5.19 we shall need the following lemmata.

5.17. Lemma: Let $f : X \rightarrow Y$ be the proper morphism of schemes of finite type over k . Let $\mathcal{L}$ be an invertible sheaf on X , $\mathcal{L}_i$ for $i = 1, \dots, n$ invertible sheaves on Y such that

- (i) $\mathcal{L}$ is very ample with respect to f (in the sense of [EGA-II] 4.4.2).
- (ii) For all i $\mathcal{L}_i$ is ample on Y .
- (iii) $f_*\mathcal{L} \cong \bigoplus_{i=1}^n \mathcal{L}_i$.

Then $\mathcal{L}$ is ample on X .

Proof. By [EGA-II] 4.6.11 and 4.4.10 there exists a nonnegative integer k such that $\mathcal{L} \otimes f^*\mathcal{L}_1^{\otimes k}$ is very ample. By [EGA-II] 4.5.10 it suffices to prove that $\mathcal{L}^{\otimes(M+1)}$ is very ample for some large positive integer M , to be chosen below. Since

$$\mathcal{L}^{\otimes(M+1)} \cong (\mathcal{L} \otimes f^*\mathcal{L}_1^{\otimes k}) \otimes (\mathcal{L}^{\otimes M} \otimes f^*\mathcal{L}_1^{\otimes(-k)}),$$

by [EGA-II] 4.4.8 it suffices to choose M such that the sheaf $\mathcal{L}^{\otimes M} \otimes f^*\mathcal{L}_1^{\otimes(-k)}$ is generated by global sections. Since $\mathcal{L}$ is relatively very ample, the canonical homomorphism $f^*f_*\mathcal{L} \rightarrow \mathcal{L}$ is surjective. Thus there is a surjective homomorphism $f^*(f_*\mathcal{L})^{\otimes M} \rightarrow \mathcal{L}^{\otimes M}$, so it suffices

to have $(f_*\mathcal{L})^{\otimes M} \otimes \mathcal{L}_1^{\otimes(-k)}$ generated by global sections on Y . But $(f_*\mathcal{L})^{\otimes M}$ is a direct sum of tensor products of M copies of different $\mathcal{L}_i$, so M must satisfy: For all integers m_i such that $m_1 \geq -k$, $m_i \geq 0$ for $i = 2, \dots, n$, and $M = \sum_{i=1}^n m_i$, the sheaf $\bigotimes_{i=1}^n \mathcal{L}_i^{\otimes m_i}$ is generated by global sections.

To produce such an M note that there exists a positive integer m_0 such that for all $m \geq m_0$ and all $i, j \in \{1, \dots, n\}$ the sheaves $\mathcal{L}_i \otimes \mathcal{L}_j^{\otimes m}$, $\mathcal{L}_1^{\otimes(-1)} \otimes \mathcal{L}_j^{\otimes m}$, $\mathcal{L}_j^{\otimes m}$ are generated by global sections ([EGA-II] 4.5.5). Any tensor product of such sheaves is also generated by global sections. Now let $\underline{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n$ such that $m_1 \geq -k$, $\forall i = 2, \dots, n$ $m_i \geq 0$, and $M = \sum_{i=1}^n m_i$. Write $\underline{m} = m_0 \cdot \underline{a} + \underline{b}$ with $\underline{a}, \underline{b} \in \mathbb{Z}^n$ such that $\forall i = 1, \dots, n$ $a_i \geq 0$, $-k \leq b_1 < m_0$, and $\forall i = 2, \dots, n$ $0 \leq b_i < m_0$. If $\sum_{i=1}^n a_i \geq \sum_{i=1}^n |b_i|$, then we can write $\underline{m}$ as a sum of n -tuples of the form $m_0 \cdot \underline{e}_i$, $m_0 \cdot \underline{e}_i + \underline{e}_j$ and $m_0 \cdot \underline{e}_i - \underline{e}_1$, where $\underline{e}_i$ is the i^{th} standard basis vector. So in that case the sheaf is generated by global sections. Note that

$$N := (n-1) \cdot (m_0 - 1) = \max\{k, m_0 - 1\} \geq \sum_{i=1}^n |b_i|.$$

Thus for $M := (m_0 + 1) \cdot N$ we get

$$\sum_{i=1}^n a_i = \frac{1}{m_0} \cdot (M - \sum_{i=1}^n b_i) \geq \frac{1}{m_0} \cdot (M - N) = N \geq \sum_{i=1}^n |b_i|,$$

as desired. q.e.d.

5.18. Lemma: Let $f : X \rightarrow Y$ be a holomorphic map of compact normal complex spaces, and $\mathcal{L}$ and $\mathcal{M}$ invertible sheaves on X, Y respectively, such that $\mathcal{M}$ is ample, and $\mathcal{L}$ is ample on every fibre of f . Then there exists a positive integer n such that $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes n}$ is ample. In particular X is projective.

Proof. By the direct image theorem (see [GR] ch.10 §4.6) $f_*\mathcal{L}$ is a coherent sheaf. By the ampleness of $\mathcal{M}$ there exists a positive integer n such that $(f_*\mathcal{L}) \otimes \mathcal{M}^{\otimes(n-1)}$ is generated by global sections. We shall prove the ampleness of $\mathcal{L} \otimes f^*\mathcal{M}^{\otimes n}$ for every such n . By Grauert's ampleness criterion (see [BPV] ch.I thm. 19.3) we have to show that for every irreducible closed analytic subset $Z \subset X$ of strictly positive dimension there exists an integer $m \geq 1$ such that $(\mathcal{L} \otimes f^*\mathcal{M}^{\otimes n})^{\otimes m}|_Z$ possesses a nonzero section which has at least one zero. If Z is contained in a fibre of f , then $(\mathcal{L} \otimes f^*\mathcal{M}^{\otimes n})|_Z$ is isomorphic to $\mathcal{L}|_Z$, so this follows from the ampleness of $\mathcal{L}$ on the fibres. Otherwise by the proper mapping theorem (see [GR] ch.10 §6.1) $f(Z)$ is a closed analytic subset of Y , of strictly positive dimension. Thus by the ampleness of $\mathcal{M}$ there exists an integer $m \geq 1$ and a nonzero section s of $\mathcal{M}^{\otimes m}|_{f(Z)}$ that has at least one zero. Let t be a section of $(f_*\mathcal{L}) \otimes \mathcal{M}^{\otimes(n-1)}$ that does not vanish identically on $f(Z)$. Denoting the pullback to Z of these sections by the same letters, it follows that $t^{\otimes m} \otimes s$ is a nonzero section of $(\mathcal{L} \otimes f^*\mathcal{M}^{\otimes n})|_Z$, whose zero locus is non-empty since it contains that of s . This is the desired section. q.e.d.

5.19. Theorem: In the situation of 5.12 (a) or (b) assume that Σ is a finite and complete rational polyhedral decomposition of $Y_*(T)_{\mathbb{R}}$. In the case (b) suppose in addition

that S is compact. For every section $e : T \rightarrow T'$ let $\mathcal{M}_e$ be the invertible sheaf on S associated to the line bundle $(X'_{\sigma_0})/e(T) \rightarrow S$. Then the following assertions are equivalent:

- (a) $\mathcal{L}^{\otimes(-1)}$ is ample on X_Σ .
- (b) Σ' is strictly convex, and $\mathcal{M}_{e_\sigma}^{\otimes(-1)}$ is ample on S for all $\sigma \in \Sigma$ with $\mathbb{R} \cdot \sigma = Y_*(T)_\mathbb{R}$.

Proof. Let us first reduce the analytic case to the algebraic case. For this, if S is a compact space, it suffices to prove that S and X_Σ are projective whenever (a) or (b) holds. If (b) holds, then there exists an ample line bundle on S , and a line bundle on X_Σ that is ample in every fibre over S . By 5.18 there exists also an absolutely ample line bundle on X_Σ , so both S and X_Σ are projective. If (a) holds, then X_Σ is projective. Choose any $\sigma \in \Sigma$ with $\mathbb{R} \cdot \sigma = Y_*(T)_\mathbb{R}$, then $\tilde{X}_\sigma = S$ is also projective, as desired. For the rest of the proof S is a scheme over k .

Assume (b), then for every $n \geq 1$ $f_*(\mathcal{L}^{\otimes(-n)})$ is a direct sum of ample invertible sheaves on S . In fact, this is a consequence of 5.14 (b), the ampleness of the $\mathcal{M}_{e_\sigma}^{\otimes(-1)}$, and the fact that a tensor product of ample invertible sheaves is again ample ([EGA-II] 4.5.7). Thus if we choose n so that $\mathcal{L}^{\otimes(-n)}$ is relatively very ample for $f : X_\Sigma \rightarrow S$, then by 5.17 it is already absolutely ample, whence (a). Conversely assume (a). Then the strict convexity of Σ' follows from 5.16. For any $\sigma \in \Sigma$ with $\mathbb{R} \cdot \sigma = Y_*(T)_\mathbb{R}$ we have a section $i_\sigma : S \cong \tilde{X}_\sigma \hookrightarrow X_\sigma$, and it follows that $i_\sigma^* \mathcal{L}^{\otimes(-1)}$ is ample. Now $\mathcal{L}|_{\tilde{X}_\sigma}$ is the invertible sheaf associated to the $\mathbb{G}_m$ -torsor $X'/e_\sigma(T)$ on $\tilde{X}_\sigma \cong X/T$, hence $i_\sigma^* \mathcal{L}$ is isomorphic to $\mathcal{M}_{e_\sigma}$. Thus $\mathcal{M}_{e_\sigma}^{\otimes(-1)}$ is ample, which proves (b). **q.e.d.**

5.20. Construction of projective refinements: We now want to deal with the construction of rational polyhedral cone decompositions that are strictly convex in the sense of 5.16. Our main point of interest is to construct refinements of given cone decompositions, subject to strict convexity and/or smoothness conditions.

Consider a short exact sequence of finite dimensional $\mathbb{Q}$ -vector spaces $0 \rightarrow \mathbb{Q} \rightarrow M' \xrightarrow{\pi} M \rightarrow 0$, and let $\sigma_0 := \mathbb{R}^{\geq 0} \times \{0\}$. Let $\sigma \subset M_\mathbb{R}$ be a convex rational polyhedral cone, and consider finite rational partial polyhedral decompositions $\Sigma, \Sigma^+, \Sigma'$ as in 5.11, such that $|\Sigma| = \sigma$. If Σ' is strictly convex, then the definition 5.16 shows that Σ^+ , hence also Σ and Σ' are uniquely determined by $|\Sigma'|$. Conversely let σ' be a convex rational polyhedral cone in $M'_\mathbb{R}$ such that $\sigma' \cap \mathbb{R} \cdot \sigma_0 = \sigma_0$ and $\pi(\sigma') = \sigma$. Let Σ^+ be the set of all faces of σ' that do not contain σ_0 , and $\Sigma := \{\pi(\sigma^+) \mid \sigma^+ \in \Sigma^+\}$ and $\Sigma' := \{\sigma^+, \sigma^+ + \sigma_0 \mid \sigma^+ \in \Sigma^+\}$, then it is easy to see that Σ' is strictly convex with $|\Sigma'| = \sigma'$ and $|\Sigma| = \sigma$.

Next we fix a splitting $M' = \mathbb{Q} \times M$. For any Σ as above there exists a unique function $f : \sigma \rightarrow \mathbb{R}$ such that

$$\Sigma = \{(t, x) \in \mathbb{R} \times M_\mathbb{R} \mid t + f(x) \geq 0\}.$$

In fact, if $e = (\chi, \text{id})$ is the splitting $M \rightarrow M'$ associated to some $\tau \in \Sigma$, then f coincides with $-\chi$ on τ (This function is the negative of the function considered in the proof of 5.14). Clearly f is continuous, linear on every $\tau \in \Sigma$, takes rational values on $\sigma \cap M$, and is convex in the sense that $f(\alpha x + (1 - \alpha)y) \geq \alpha f(x) + (1 - \alpha)f(y)$ for all $x, y \in M_\mathbb{R}$ and $\alpha \in [0, 1]$. Conversely suppose that we are given a convex continuous piecewise linear function $f : \sigma \rightarrow \mathbb{R}$, such that $f(\alpha \cdot x) = \alpha \cdot f(x)$ for all $x \in \sigma$ and $\alpha \in \mathbb{R}^{\geq 0}$, and which

takes rational values on $\sigma \cap M$. We call such a function a piecewise linear convex rational function. Then the above formula defines a convex rational polyhedral cone Σ with the described properties. Denote by $\Sigma(f)$ the corresponding rational polyhedral decomposition of σ , it consists of all maximal subcones of σ on which f is linear.

In the following propositions we shall use this description to construct certain refinements of cone decompositions. The general idea in these constructions is the induction over the number and the dimension of the cones in the given decomposition. Thus in the induction step we assume that the construction is already carried out on every proper face of a convex rational polyhedral cone σ , and we extend it to all of σ .

5.21. Proposition: Let $\sigma \subset M_{\mathbb{R}}$ be a convex rational polyhedral cone that does not contain a nontrivial linear subspace, and Σ a finite rational polyhedral decomposition with $|\Sigma| = \sigma$. Let $f_{\sigma} : \sigma \setminus \sigma^0 \rightarrow \mathbb{R}$ be a function such that for every proper face ρ of σ , the restriction $f_{\sigma}|_{\rho}$ is a piecewise linear convex rational function, and $\Sigma(f_{\sigma}|_{\rho})$ is a refinement of $\{\pi \in \Sigma \mid \pi \subset \rho\}$. Then there exists a piecewise linear convex rational function $f : \sigma \rightarrow \mathbb{R}$, which coincides with f_{σ} on $\sigma \setminus \sigma^0$, such that $\Sigma(f)$ is a refinement of Σ .

Remark. The assumption on σ is not necessary. By induction the proposition implies that every finite rational polyhedral decomposition Σ , such that $|\Sigma|$ is convex, possesses a refinement which admits a strictly convex Σ' as in 5.16.

Proof. Let f be any piecewise linear convex rational function whose restriction to $\sigma \setminus \sigma^0$ is f_{σ} . That there exists such a function is obvious if $\dim(\mathbb{R} \cdot \sigma) \leq 1$. Otherwise σ is the convex closure of $\sigma \setminus \sigma^0$, so there exists a unique smallest convex function which is $\geq f_{\sigma}$ on $\sigma \setminus \sigma^0$, and it is easy to see that its restriction to $\sigma \setminus \sigma^0$ is f_{σ} . After replacing Σ by its refinement $\{\tau \cap \pi \mid \pi \in \Sigma, \tau \in \Sigma(f)\}$ we may assume that $\Sigma(f_{\sigma}|_{\rho}) = \{\tau \in \Sigma \mid \tau \subset \rho\}$ for every proper face ρ of σ .

Next let $x \in \sigma^0$ and $\mathcal{T}$ be the set of all faces of $\rho + \mathbb{R}^{\geq 0} \cdot x$ for all proper faces ρ of σ . This is a rational polyhedral decomposition of σ that contains every proper face of σ , and every $\tau \in \mathcal{T}$ is contained in $\rho \cup \sigma^0$ for some proper face ρ of σ . After replacing Σ by the refinement $\{\tau \cap \pi \mid \pi \in \Sigma, \tau \in \mathcal{T}\}$ we may assume the same property for Σ .

Since $\mathcal{T}$ contains every proper face of σ , we still have $\Sigma(f_{\sigma}|_{\rho}) = \{\tau \in \Sigma \mid \tau \subset \rho\}$ for every proper face ρ of σ .

Now fix $\tau \in \Sigma$. We want to find a piecewise linear convex rational function $f_{\tau} : \sigma \rightarrow \mathbb{R}$, which coincides with f_{σ} on $\sigma \setminus \sigma^0$, such that $\tau \in \Sigma(f_{\tau})$. Let ρ be a proper face of σ such that $\tau \subset \rho \cup \sigma^0$. By assumption $\rho \cap \tau \in \Sigma(f_{\sigma}|_{\rho})$, so there exists a linear form: $\ell : M \rightarrow \mathbb{Q}$ such that $\ell = f_{\sigma}$ on $\rho \cap \tau$, and $\ell > f_{\sigma}$ on $\rho \setminus \tau$. Let $m : M \rightarrow \mathbb{Q}$ be a linear form which vanishes on ρ , and is strictly positive on $\sigma \setminus \rho$. Since we may replace ℓ by $\ell + \alpha \cdot m$ for arbitrarily large $\alpha \in \mathbb{Q}^{>0}$, we can choose ℓ such that $\ell > f_{\sigma}$ on $\sigma \setminus (\rho \cup \sigma^0)$. Let $f_{\tau} : \sigma \rightarrow \mathbb{R}$ be the unique smallest convex function, which is $\geq f_{\sigma}$ on $\sigma \setminus \sigma^0$, and $\geq \ell$ on τ . Since by assumption $\tau \subset \rho \cup \sigma^0$, we have $f_{\sigma} = \ell$ on $(\sigma \setminus \sigma^0) \cap \tau$. Thus the restriction of f_{τ} for $\sigma \setminus \sigma^0$ is f_{σ} , and $f_{\tau} > \ell$ on $\sigma \setminus \tau$. Since $f_{\tau} = \ell$ on τ , we have $\tau \in \Sigma(f_{\tau})$, as desired.

Finally let $f = \text{card}(\Sigma)^{-1} \cdot \sum_{\tau \in \Sigma} f_{\tau}$ be the average of all f_{τ} . This is again a piecewise linear convex rational function and coincides with f_{σ} on $\sigma \setminus \sigma^0$. Moreover it is easy to check that $\Sigma(f)$ consists of all intersections of elements of the different $\Sigma(f_{\tau})$. In particular $\Sigma(f)$

is a refinement of every $\Sigma(f_\tau)$, hence every τ is the union of cones in $\Sigma(f)$. But this means that $\Sigma(f)$ is a refinement of Σ , as desired. **q.e.d.**

5.22. Lemma: Let $\sigma \subset M_{\mathbb{R}}$ be a convex rational polyhedral cone, $f : \sigma \rightarrow \mathbb{R}$ a piecewise linear convex rational function, and $g : \sigma \rightarrow \mathbb{R}$ a function whose restriction to every $\rho \in \Sigma(f)$ is piecewise linear convex rational. Then for every sufficiently small $\varepsilon \in \mathbb{Q}^{>0}$ the function $f + \varepsilon \cdot g$ is again piecewise linear convex rational, and $\Sigma(f + \varepsilon \cdot g) = \bigcup_{\rho \in \Sigma(f)} \Sigma(g|_{\rho})$.

Proof. Let $\mathbb{R}^{\geq 0} \cdot x_i$ be the finitely many one dimensional cones in $\bigcup_{\rho \in \Sigma(f)} \Sigma(g|_{\rho})$. Let $\rho \in \Sigma(f)$ with $\mathbb{R} \cdot \rho = \mathbb{R} \cdot \sigma$, and $\pi \in \Sigma(g|_{\rho})$ with $\mathbb{R} \cdot \pi = \mathbb{R} \cdot \rho$. Let ℓ and m be linear functions on M with $\ell|_{\rho} = f|_{\rho}$ and $m|_{\pi} = g|_{\pi}$. Then $(\ell + \varepsilon \cdot m)(x_i) = (f + \varepsilon \cdot g)(x_i)$ for every $x_i \in \pi$. But for every $x_i \notin \pi$ we have either $\ell(x_i) > f(x_i)$, or $\ell(x_i) = f(x_i)$ and $m(x_i) > g(x_i)$. Thus for any sufficiently small $\varepsilon \in \mathbb{Q}^{>0}$ we have $(\ell + \varepsilon \cdot m)(x_i) > (f + \varepsilon \cdot g)(x_i)$ for all $x_i \notin \pi$. If we have this for every π , then $f + \varepsilon \cdot g$ is convex, it is automatically a piecewise linear convex rational function, and π is the subset of σ on which $\ell + \varepsilon \cdot m$ coincides with $f + \varepsilon \cdot g$. Since every cone in $\bigcup_{\rho \in \Sigma(f)} \Sigma(g|_{\rho})$ is a face such a π , the assertion follows. **q.e.d.**

5.23. Proposition: Consider the situation of 5.21. Assume that in addition we are given a lattice $M_{\mathbb{Z}} \subset M$, and that for every proper face τ on σ , every $\pi \in \Sigma(f_0|_{\tau})$ is smooth in the sense of 5.2. Then the function f in 5.21 can be chosen such that every $\pi \in \Sigma(f)$ is smooth.

Remark. By induction the proposition implies that every finite rational polyhedral decomposition Σ , such that $|\Sigma|$ is convex, possesses a smooth refinement which admits a strictly convex Σ' as in 5.16.

Proof. Suppose first that Σ consists of the faces of σ . In this case we just have a function $f_0 : \sigma \setminus \sigma^0 \rightarrow \mathbb{R}$, whose restriction to every proper face τ of σ is a piecewise linear convex rational function, such that every $\pi \in \Sigma(f_0|_{\tau})$ is smooth. The assertion is that there exists a piecewise linear convex rational function $f : \sigma \rightarrow \mathbb{R}$, which coincides with f_0 on $\sigma \setminus \sigma^0$, such that every $\pi \in \Sigma(f)$ is smooth. This is precisely what is shown in the proof of [KKMS] ch.I §2 thm. 11 on pp.33–35.

In general let f be as in 5.21. By 5.22 it suffices to construct a function $g : \sigma \rightarrow \mathbb{R}$, which is zero on $\sigma \setminus \sigma^0$, and piecewise linear convex rational on every $\rho \in \Sigma(f)$, such that every $\pi \in \Sigma(g|_{\rho})$ is smooth. Let $\mathcal{T}$ be a rational partial polyhedral decomposition contained in $\Sigma(f)$, which contains every $\rho \in \Sigma(f)$ that lies in a proper face of σ . We shall define g on $|\mathcal{T}|$ by induction over $\mathcal{T}$. First let $\mathcal{T}$ consist of all $\rho \in \Sigma(f)$ that lie in a proper face of σ , and define g to be zero on $\sigma \setminus \sigma^0$. By assumption this function satisfies the requirements. For the induction step choose $\rho \in \Sigma(f) \setminus \mathcal{T}$ of minimal dimension. Then g is already defined on $\rho \setminus \rho^0$, and the function $g|_{\rho \setminus \rho^0}$ satisfies the assumptions of the special case above. Thus we can extend g to a function on $|\mathcal{T}| \cup \rho$ with the desired properties. **q.e.d.**

5.24. The barycentric subdivision: Let $M_{\mathbb{Z}}$ be a lattice in M , and $\sigma \subset M_{\mathbb{R}}$ a convex rational polyhedral cone that is smooth with respect to $M_{\mathbb{Z}}$. Then by definition $\sigma = \sum_{i=1}^d \mathbb{R}^{\geq 0} \cdot x_i$ for certain $x_i \in M_{\mathbb{Z}}$ which are part of some basis of $M_{\mathbb{Z}}$. The $\mathbb{R}^{\geq 0} \cdot x_i$

are the one dimensional faces of σ , and the x_i are uniquely determined up to permutation. The barycentric subdivision of σ is then defined as follows.

For every nonempty subset $S \subset \{1, \dots, d\}$ let $x_S := \sum_{i \in S} x_i$. For every $0 \leq r \leq d$ consider a sequence $\underline{S}$ of subsets $\emptyset \subsetneq S_1 \subsetneq \dots \subsetneq S_r \subset \{1, \dots, d\}$, and let $\sigma_{\underline{S}} := \sum_{i=1}^r \mathbb{R}^{\geq 0} \cdot x_{S_i}$. It is easy to check that the set Σ of all $\sigma_{\underline{S}}$ is a rational polyhedral decomposition of σ (compare [Sp] §3.3 thm. 9). This is the barycentric subdivision of σ . By construction it depends only on σ and the lattice $M_{\mathbb{Z}}$. Moreover every $\sigma_{\underline{S}}$ is smooth, and for any face τ of σ the decomposition $\{\sigma_{\underline{S}} \in \Sigma \mid \sigma_{\underline{S}} \subset \tau\}$ is the barycentric subdivision of τ .

The barycentric subdivision has the following property, which is why we are interested in it. Consider $\sigma_{\underline{S}} \in \Sigma$. Every face of $\sigma_{\underline{S}}$ is of the form $\sigma_{\underline{S}'}$ for a subsequence $\underline{S}'$ of $\underline{S}$. We saw above that every automorphism of $M_{\mathbb{Z}} \cap \mathbb{R} \cdot \sigma$ that fixes σ comes from a permutation of the x_i . This implies that no two distinct faces of the same $\sigma_{\underline{S}}$ are conjugate under such an automorphism. More generally let τ_1, τ_2 be the two faces of σ , and ϕ an isomorphism $M_{\mathbb{Z}} \cap \mathbb{R} \cdot \tau_2 \rightarrow M_{\mathbb{Z}} \cap \mathbb{R} \cdot \tau_1$ that identifies τ_1 with τ_2 . If $\sigma_{\underline{S}'}$ is a face of $\sigma_{\underline{S}}$ that is contained in τ_1 , and $\phi(\sigma_{\underline{S}'})$ is a face of $\sigma_{\underline{S}}$, then again we must have $\phi(\sigma_{\underline{S}'}) = \sigma_{\underline{S}'}$.

5.25. Lemma: In 5.24 suppose that we are given a function $f_0 : \sigma \setminus \sigma^0 \rightarrow \mathbb{R}$, whose restriction to every proper face ρ of σ is a piecewise linear convex function, such that $\Sigma(f_0|_{\rho})$ is the barycentric subdivision of ρ . Then there exists a piecewise linear convex rational function $f : \sigma \rightarrow \mathbb{R}$, which coincides with f_0 on $\sigma \setminus \sigma^0$, such that $\Sigma(f)$ is the barycentric subdivision of σ .

Proof. let f be the smallest convex function on σ with $f(x_1 + \dots + x_d) = C$, and which is $\geq f_0$ on $\sigma \setminus \sigma^0$. If C is a sufficiently large positive rational number, then f has the desired properties. **q.e.d.**

5.26. Canonical sheaves on torus embeddings: Let $\pi : X \rightarrow S$ be a torsor under a torus T of dimension d , and $j : X \hookrightarrow X_{\Sigma}$ a relative torus embedding. What we shall do will be equally valid in the algebraic and the analytic category. Let $\omega_{X_{\Sigma}/S}[\text{dlog}]$ be the sheaf of meromorphic relative differential d -forms on X_{Σ} over S with at most simple poles along $X_{\Sigma} \setminus X$. It is a coherent subsheaf of $j_*\omega_{X/S}$, called the relative canonical sheaf with logarithmic poles along $X_{\Sigma} \setminus X$. Similarly if S is smooth and equidimensional of dimension d' over the field k , let $\omega_{X_{\Sigma}}[\text{dlog}]$ be the sheaf of meromorphic relative differential $(d + d')$ -forms on X_{Σ} over k with at most simple poles along $X_{\Sigma} \setminus X$. It is a coherent subsheaf of $j_*\omega_X := j_*\omega_{X/k}$, called the canonical sheaf with logarithmic poles along $X_{\Sigma} \setminus X$. Observe that these definitions depend only on the embedding $X \hookrightarrow X_{\Sigma}$ and the morphism $\bar{\pi} : X_{\Sigma} \rightarrow S$ and not on either T or the fact that $X \hookrightarrow X_{\Sigma}$ is given as a torus embedding. The second definition does not even depend on S . Note also the canonical isomorphism

$$\omega_{X_{\Sigma}}[\text{dlog}] \cong \omega_{X_{\Sigma}/S}[\text{dlog}] \otimes \bar{\pi}^*\omega_S.$$

We are also interested in the T -invariant differentials on X . By definition the sheaf of T -invariant relative differential d -forms $\omega_{X/S}^T$ is the difference kernel of the two homomorphisms of $\mathcal{O}_S$ -sheaves

$$pr_2^*, \mu^* : \pi_*\omega_{X/S} \rightarrow (\pi \circ pr_2)_*\omega_{T \times X/T \times S},$$

where $\mu : T \times X \rightarrow X$ denote the T -action. It is an invertible coherent sheaf on S , and the canonical map $\pi^* \pi_* \omega_{X/S} \rightarrow \omega_{X/S}$ induces an isomorphism $\pi^* \omega_{X/S}^T \rightarrow \omega_{X/S}$. Let $i : \omega_{X/S}^T \rightarrow \pi^* \omega_{X/S}$ be the inclusion. The invariance property implies that for any open $U \subset S$ and any morphism $t : U \rightarrow T$ the following diagram is commutative

$$\begin{array}{ccc} \omega_{X/S}^T & \xrightarrow{i} & \pi^* \omega_{X/S} \\ & \searrow i & \downarrow [t \cdot]^* \\ & & \pi_* \omega_{X/S}, \end{array}$$

i.e. $[t \cdot]^*$ induces the identity on $\omega_{X/S}^T$

5.27. Proposition: Let $[1] : S \rightarrow T$ be the constant morphism $s \mapsto 1$. There exist canonical isomorphisms

$$\begin{aligned} \omega_{X_\Sigma/S}[\text{dlog}] &\cong \bar{\pi}^* \omega_{X/S}^T, \text{ and} \\ \omega_{X/S}^T &\cong [1]^* \omega_T. \end{aligned}$$

If Σ is finite and complete, they induce a canonical isomorphism

$$\bar{\pi}_* \omega_{X_\Sigma/S}[\text{dlog}] \cong [1]^* \omega_T.$$

Proof. For the first isomorphism note that $j_* j^* \bar{\pi}^* \omega_{X/S}^T \cong j_* \pi^* \omega_{X/S}^T \cong j_* \omega_{X/S}$, so $\bar{\pi}^* \omega_{X/S}^T$ is like $\omega_{X_\Sigma/S}[\text{dlog}]$ canonically isomorphic to a subsheaf of $j_* \omega_{X/S}$. Thus we have to show that their images are equal. Since this is a local question, we may assume that $X = T \times S$, which then reduces to the case $X = T$. Let ω be a nonzero T -invariant d -form on T , we then have to show that $\omega_{T_\Sigma}[\text{dlog}]$ is generated by ω . Now it is well-known that ω has precisely a simple pole at every stratum $\tilde{T}_\sigma$ of codimension 1. Indeed, for any such σ the embedding $T \hookrightarrow T_\sigma$ is isomorphic to the embedding $\mathbb{G}_m^d \hookrightarrow \mathbb{G}_m^{d-1} \times \mathbb{A}^1$, and if $T_1, \dots, T_d$ are the d coordinate functions, then ω corresponds to the differential $\frac{dT_1}{T_1} \wedge \dots \wedge \frac{dT_d}{T_d}$, which clearly has a pole of order 1 along $\mathbb{G}_m^{d-1} \times \{0\}$. This shows that $\omega_{T_\Sigma}[\text{dlog}] = \omega \cdot \mathcal{O}_{T_\Sigma}$, as desired.

To construct a canonical isomorphism $\omega_{X/S}^T \cong [1]^* \omega_T$ we begin with the case $X = T \times S$. In that case we define it as the composite homomorphism

$$\omega_{X \times S/S}^T \xrightarrow{i} \pi_* \omega_{T \times S/S} \longrightarrow \pi_* [1]_* [1]^* \omega_T = [1]^* \omega_T,$$

which is well-known to be an isomorphism. For arbitrary X we shall show that there exists a unique homomorphism $\omega_{X/S}^T \rightarrow [1]^* \omega_T$ which for every local trivialization over an open $U \subset S$ becomes equal to the homomorphism above. Then it is clearly an isomorphism, as desired. Now such a homomorphism exists if and only if for every open $U \subset S$ and every morphism $t : U \rightarrow T$ the diagram

$$\begin{array}{ccc} \omega_{T \times U/U}^T & & \\ [t \cdot]^* \downarrow & \searrow & \\ \omega_{T \times U/U}^T & \longrightarrow & [1]^* \omega_{T|U} \end{array}$$

commutes. But by the last remark in 5.16 $[t\cdot]^*$ induces the identity on $\omega_{T\times U/U}^T$, as desired.

Together these maps yields a canonical isomorphism $\omega_{X_\Sigma/S}[\mathrm{dlog}] \cong \bar{\pi}^*[1]^*\omega_T$, whence an isomorphism $\bar{\pi}_*\omega_{X_\Sigma/S}[\mathrm{dlog}] \cong \bar{\pi}_*\bar{\pi}^*[1]^*\omega_T$. But for Σ finite and complete, $X_\Sigma \rightarrow S$ is a locally trivial bundle with fibre T_Σ which is proper, connected and reduced so the canonical map $[1]^*\omega_T \rightarrow \bar{\pi}_*\bar{\pi}^*[1]^*\omega_T$ is an isomorphism. This gives the last isomorphism. **q.e.d.**

Chapter 6

Toroidal compactification

In this chapter we construct (partial or complete) toroidal compactifications of mixed Shimura varieties. For our purposes it is best to carry out this construction within the framework of the objects $M^{K_f}(P, \mathcal{X})(\mathbb{C})$, instead of considering just one connected component at a time. This, and the fact that we consider mixed Shimura varieties as opposed to pure ones, makes it necessary to redo as a whole the construction of the toroidal compactification in [AMRT]. The general line of the proof is the same as there, and of course we take resort to the reduction theoretic results [AMRT] pp.116, 157 and 258–9, see 6.2, 6.19 and 6.22 below.

We begin in 6.1–2 by describing the Baily-Borel compactification within our framework. For this we use an explicit description of the Satake-topology that is more or less contained in [AMRT] ch.III §6 (see 6.2 below). It is striking that the Satake-topology can be described in a way completely analogous to the topology on a torus embedding, as described in 5.9.

In 6.4–5 we state the requirements for the systems of cone decompositions, on which our toroidal compactification will depend. Then, in 6.6–9, we study torus embeddings “along the unipotent fibre”: these are just the torus embeddings with respect to a torsor structure as described in 3.12. To define the general toroidal compactification of $M^{K_f}(P, \mathcal{X})(\mathbb{C})$, we begin by considering a certain “big” infinite disjoint union $\mathcal{U}$ of open subsets of certain other mixed Shimura varieties, associated to all rational boundary components of $(P, \mathcal{X})$. In any particular case, it would suffice to consider only finitely many components of $\mathcal{U}$, but with our choice, $\mathcal{U}$ has a much nicer functorial behavior. We show how $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ can be identified with the quotient of $\mathcal{U}$ by an explicit equivalence relation $\sim$ (6.10–12). Then we use the torus embeddings along the unipotent fibre to define an open embedding $\mathcal{U} \hookrightarrow \overline{\mathcal{U}}$ (6.13). In 6.14–17 we extend $\sim$ to an equivalence relation on $\overline{\mathcal{U}}$. The study of the quotient $\overline{\mathcal{U}}/\sim$, which will be the desired toroidal compactification, is then carried out in two steps.

First we consider the action of a certain arithmetic subgroups on some subset of $\overline{\mathcal{U}}$. The main result 6.20 is that this action is properly discontinuous. Then in 6.21 we relate $\overline{\mathcal{U}}$ to the Baily-Borel compactification of $M^{\pi(K_f)}((P, \mathcal{X})/W)(\mathbb{C})$, and in 6.22 we prove that this action is in some sense locally equivalent to the whole equivalence relation. As in consequence the graph $\sim$ is closed (6.23), so $\overline{\mathcal{U}}/\sim$ is a Hausdorff space, and locally isomorphic to quotients of subsets of $\overline{\mathcal{U}}$ by finite groups. It is therefore a normal complex space. This is the desired toroidal compactification (6.24). We finish the chapter by describing the functorial behavior

of this compactification (6.25), and by giving smoothness (6.26), resp. compactness (6.27) criterion.

6.1. Cores: Let $(P, \mathcal{X})$ be mixed Shimura data, $Q \subset P$ an admissible $\mathbb{Q}$ -parabolic subgroup, and $(P_1, \mathcal{X}_1)$ a corresponding rational boundary component of $(P, \mathcal{X})$. Let $\mathcal{X}^0 \subset \mathcal{X}^+$ be a connected component. Choose a $\mathbb{Z}$ -lattice $U_1(\mathbb{Z})(-1)$ in $U_1(\mathbb{Q})(-1)$, and let D_0 be the convex closure of $U_1(\mathbb{Z})(-1) \cap C(\mathcal{X}^0, P_1)$. A subset $D \subset C(\mathcal{X}^0, P_1)$ is called a core if there exists numbers $\Lambda, \mu \in \mathbb{R}^{>0}$ such that $\Lambda \cdot D_0 \subset D \subset \mu \cdot D_0$. Clearly this definition is independent of the choice of the lattice (Compare [AMRT] ch.II §5.1). If D is a core, then so are the following sets, as is easy to see: its closure $\overline{D}$, its interior D^0 , and the sets $\mathbb{R}^{\geq 1} \cdot D$ and $D + U_1(\mathbb{R})(-1)$. Moreover by choosing a small enough lattice it becomes clear that the set $D + C(\mathcal{X}^0, P_1)$ is also a core, and that $C(\mathcal{X}^0, P_1) = \mathbb{R}^{>0} \cdot D$.

6.2. The Baily-Borel compactification: Let $(P, \mathcal{X})$ be pure Shimura data, and K_f an open compact subgroup of $P(\mathbb{A}_f)$. Since every connected component of $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ is a quotient of a hermitian symmetric domain by an arithmetic group, it possesses a canonical minimal compactification as a normal complex space, the (Satake-)Baily-Borel compactification (see [BB] or [AMRT]). We want to reformulate the main points of its construction in the language developed in the previous chapters.

First one defines

$$\mathcal{X}^* := \coprod \mathcal{X}_1/W_1,$$

where the disjoint union is extended over all rational boundary components $(P_1, \mathcal{X}_1)$ of $(P, \mathcal{X})$, and we write $(P_1, \mathcal{X}_1)/W_1$ as $(P_1/W_1, \mathcal{X}_1/W_1)$. Note that each $\mathcal{X}_1/W_1$ is a finite disjoint union of hermitian symmetric domains, and that $\mathcal{X}$ embeds canonically into $\mathcal{X}^*$ since it is the disjoint union of all $\mathcal{X}_1$ for all improper boundary components. Then one defines a certain topology on $\mathcal{X}^*$, called the Satake-topology. We will not give its original definition (cf. [AMRT] ch.III §6.1 p.257), but an equivalent one. For every connected component $\mathcal{X}^0$ of $\mathcal{X}$, and for every $(P_1, \mathcal{X}_1)$ such that $\mathcal{X}^0$ maps to $\mathcal{X}_1$, let ψ_1 be the projection $\mathcal{X}_1 \rightarrow \mathcal{X}_1/W_1$, and fix a convex core $D \subset C(\mathcal{X}^0, P_1)$. Note that the inverse image under ψ_1 of any subset of $C(\mathcal{X}^0, P_1)$ lies in the subset $\mathcal{X}^0 \subset \mathcal{X}_1$. For any subset $\mathcal{Y} \subset \mathcal{X}_1/W_1$ and every positive number $\Lambda \in \mathbb{R}^{>0}$ consider the subset

$$\coprod \psi_2(\text{im}^{-1}(\Lambda \cdot D) \cap \psi_1^{-1}(\mathcal{Y})) \subset \mathcal{X}^*,$$

where the disjoint union is extended over all rational boundary components $(P_2, \mathcal{X}_2)$ between $(P, \mathcal{X})$ and $(P_1, \mathcal{X}_1)$. Let $x_1 \in \mathcal{X}_1/W_1$, then if $\mathcal{Y}$ runs through all neighborhoods of x_1 in $\mathcal{X}_1/W_1$ and Λ through $\mathbb{R}^{>0}$, these sets form a fundamental system of neighborhoods of x_1 in the Satake-topology on $\mathcal{X}^*$.

Indeed, the equivalence of this definition of the Satake-topology with the usual one is only a slight strengthening of [AMRT] ch.III p.259 thm 3, and its proof is almost the same. Let us give a few indications as to how the proof of [loc. cit.] can be adapted: Take an arithmetic subgroup Γ of $P(\mathbb{Q})$ and let $\Gamma_x := \text{Stab}_\Gamma(x)$. Then for any Siegel set $\mathfrak{S}$ (see [loc. cit.]) the set of all $\gamma \in \Gamma$ such that $\gamma \cdot x \in \mathfrak{S}$ is a finite union of Γ_x -cosets. Without loss of generality we may assume that D is a standard core, and $\mathcal{Y}$ is stable under Γ_x , then our set above is Γ_x -invariant. It now easily follows from the usual definition of the

Satake-topology that there exists finitely many Siegel sets $\mathfrak{S}_j$ with $x \in \overline{\mathfrak{S}_j}$, such that our sets form a fundamental system of neighborhoods of x if and only if they do so on every $\overline{\mathfrak{S}_j}$. But the proof of the lemma on p.263 in [AMRT] gives an explicit description of the topology on the closure of a Siegel set. Using the maps ψ_i above for a certain rational boundary components $(P_i, \mathcal{X}_i)$ between $(P, \mathcal{X})$ and $(P_1, \mathcal{X}_1)$, a fundamental system of neighborhoods of x in $\overline{\mathfrak{S}_j}$ can be written as the union of $\psi_i(S_{n,t})$ for the $S_{n,t}$ as in the cited lemma. The rest follows as in [loc. cit.] p.265.

The Baily-Borel compactification can now be defined as the double quotient

$$M^{K_f}(P, \mathcal{X})^*(\mathbb{C}) := P(\mathbb{Q}) \backslash \mathcal{X}^* \times (P(\mathbb{A}_f)/K_f),$$

where as in 3.1 $P(\mathbb{Q})$ operates naturally on both factors from the left hand side. It is endowed with the quotient topology, and by a fundamental result of Baily and Borel it is then a compact Hausdorff space (see [AMRT] ch.II p.258 thm. 2). It contains $M^{K_f}(P, \mathcal{X})^*(\mathbb{C})$ as an open dense subset. The obvious stratification of $\mathcal{X}^*$ by hermitian symmetric domains induces a stratification of $M^{K_f}(P, \mathcal{X})^*(\mathbb{C})$ by finitely many locally closed subsets of the form $\Gamma_Q \backslash (\mathcal{X}_1/W_1)^0$, where $(\mathcal{X}_1/W_1)^0$ is a connected component of $\mathcal{X}_1/W_1$, Q the admissible $\mathbb{Q}$ -parabolic subgroup of P associated to P_1 and Γ_Q an arithmetic group of $Q(\mathbb{Q})$. By [BB] thm. 10.4 there exists a unique structure of a projective normal complex space on $M^{K_f}(P, \mathcal{X})^*(\mathbb{C})$, whose restriction to each stratum is isomorphic to that induced by the complex structure on the hermitian symmetric domain. The morphisms 3.4 (a) and (b) extend to holomorphic maps of the Baily-Borel compactification, if the respective groups are reductive.

6.3. Stratification of the Baily-Borel compactification: We want to describe a stratification of $M^{K_f}(P, \mathcal{X})^*(\mathbb{C})$ by pure Shimura varieties associated to the rational boundary components of $(P, \mathcal{X})$. For this fix a representative $(P_1, \mathcal{X}_1)$ for every $P(\mathbb{Q})$ -conjugacy class of rational boundary components of $(P, \mathcal{X})$. Let Q be the admissible $\mathbb{Q}$ -parabolic subgroup of P associated to P_1 , then the conjugacy class of $(P_1, \mathcal{X}_1)$ contributes to $M^{K_f}(P, \mathcal{X})^*(\mathbb{C})$ the subset

$$\text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}_1) \backslash (\mathcal{X}_1/W_1) \times (P(\mathbb{A}_f)/K_f).$$

For fixed $(P_1, \mathcal{X}_1)$ choose a set of representatives p_f of the double quotient

$$\text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}_1) \cdot P_1(\mathbb{A}_f) \backslash P(\mathbb{A}_f)/K_f.$$

Then we easily see that $\text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}_1) \backslash (\mathcal{X}_1/W_1) \times (P(\mathbb{A}_f)/K_f)$ is the disjoint union over these p_f of

$$(\text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}_1) \cap (P_1(\mathbb{A}_f) \cdot p_f \cdot K_f \cdot p_f^{-1})) \backslash (\mathcal{X}_1/W_1) \times (P_1(\mathbb{A}_f) \cdot p_f \cdot K_f/K_f).$$

Here observe that $\text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}_1) \cap (P_1(\mathbb{A}_f) \cdot p_f \cdot K_f \cdot p_f^{-1})$ is a subgroup of $Q(\mathbb{Q})$, that contains $P_1(\mathbb{Q})$ as a normal subgroup, so we can define

$$\Lambda_1 := \text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}_1) \cap (P_1(\mathbb{A}_f) \cdot p_f \cdot K_f \cdot p_f^{-1}) / P_1(\mathbb{Q}).$$

Next with $K_f^1 := P_1(\mathbb{A}_f) \cap p_f \cdot K_f \cdot p_f^{-1}$ the map

$$P_1(\mathbb{A}_f)/K_f^1 \rightarrow P_1(\mathbb{A}_f) \cdot p_f \cdot K_f/K_f, [p_{1,f}] \mapsto [p_{1,f} \cdot p_f]$$

is bijective. This gives an identification of the above set with

$$\begin{aligned} \Lambda_1 \backslash (P_1(\mathbb{Q}) \backslash (\mathcal{X}_1/W_1) \times (P_1(\mathbb{A}_f)/K_f^1)) \\ \cong \Lambda_1 \backslash P_1(\mathbb{Q}) \backslash (\mathcal{X}_1/W_1) \times (P_1(\mathbb{A}_f)/W_1(\mathbb{A}_f) \cdot K_f^1) \\ \cong \Lambda_1 \backslash M^{\pi_1(K_f^1)}((P_1, \mathcal{X}_1)/W_1)(\mathbb{C}). \end{aligned}$$

By definition Λ_1 is an arithmetic subgroup of $(Q/P_1)(\mathbb{Q})$. Since P_1/W_1 is an almost direct factor of Q/W_1 , a subgroup of finite index of Λ_1 can be lifted to the centralizer of P_1/W_1 in Q , hence acts trivially on $M^{\pi_1(K_f^1)}((P_1, \mathcal{X}_1)/W_1)(\mathbb{C})$. Thus Λ_1 acts through a finite quotient.

All in all we now have a stratification of $M^{K_f}(P, \mathcal{X})^*(\mathbb{C})$ by finitely many quotients of pure Shimura varieties by finite groups. Of course the connected components of these form precisely the stratification mentioned in 6.2. Thus they are locally closed analytic subsets, and the embedding is a holomorphic map.

6.4. Admissible cone decompositions: Fix mixed Shimura data $(P, \mathcal{X})$ and consider the set $\mathcal{C}(P, \mathcal{X}) \times P(\mathbb{A}_f)$. Like in the definition 3.1 of mixed Shimura varieties we are interested in two commuting operations on this set: $P(\mathbb{Q})$ operates from the left on both factors, namely by 4.23 resp. by left multiplication on the second factor, and $P(\mathbb{A}_f)$ acts from the right by multiplication in the second factor. We shall also need the left action of $P(\mathbb{A}_f)$ by left multiplication on the second factor. We write “ p ” for the former action, and “ p_f ” for the latter: the presence or absence of an index “ f ” will distinguish the two left actions. Each of these actions induces one subsets of $\mathcal{C}(P, \mathcal{X}) \times P(\mathbb{A}_f)$. For any rational boundary component $(P_1, \mathcal{X}_1)$ of $(P, \mathcal{X})$, any connected component $\mathcal{X}^0$ of $\mathcal{X}^+$, and any $p_f \in P(\mathbb{A}_f)$ consider the subset $C^*(\mathcal{X}^0, P_1) \times \{p_f\} \subset \mathcal{C}(P, \mathcal{X}) \times P(\mathbb{A}_f)$. These subsets form a covering, that is invariant under the above group actions. In the obvious way (“forget p_f ”) we can interpret $C^*(\mathcal{X}^0, P_1) \times \{p_f\}$ as a cone in the vector space $U_1(\mathbb{Q})(-1) \otimes \mathbb{R}$.

Now consider a collection Σ of subsets of $\mathcal{C}(P, \mathcal{X}) \times P(\mathbb{A}_f)$, such that any $\sigma \in \Sigma$ is contained in some $C^*(\mathcal{X}^0, P_1) \times \{p_f\}$. Define $\Sigma(\mathcal{X}^0, P_1, p_f) := \{\sigma \in \Sigma \mid \sigma \subset C^*(\mathcal{X}^0, P_1) \times \{p_f\}\}$. Let $K_f \subset P(\mathbb{A}_f)$ be an open compact subgroup. We call Σ a K_f -admissible partial cone decomposition for $(P, \mathcal{X})$ if the following conditions hold:

- (i) For all $\mathcal{X}^0$, $(P_1, \mathcal{X}_1)$ and p_f , $\Sigma(\mathcal{X}^0, P_1, p_f)$ is a rational partial polyhedral decomposition of $C^*(\mathcal{X}^0, P_1) \times \{p_f\}$.
- (ii) Σ is invariant under right multiplication of K_f on the second factor of $\mathcal{C}(P, \mathcal{X}) \times P(\mathbb{A}_f)$.
- (iii) Σ is invariant under the left action of $P(\mathbb{Q})$ on both factors of $\mathcal{C}(P, \mathcal{X}) \times P(\mathbb{A}_f)$.
- (iv) For all $(P_1, \mathcal{X}_1)$ the set $\bigcup_{p_f \in P(\mathbb{A}_f)} \Sigma(\mathcal{X}^0, P_1, p_f)$ is invariant under left multiplication of $P_1(\mathbb{A}_f)$ on $P(\mathbb{A}_f)$.

We call Σ finite if in addition the following condition holds:

- (v) The double quotient $P(\mathbb{Q}) \backslash \Sigma / K_f$ is finite, i.e. there exist only finitely many convex rational polyhedral cones in Σ modulo the two actions (ii) and (iii).

We call a K_f -admissible complete cone decomposition for $(P, \mathcal{X})$ if it is finite and the following strengthening of (i) holds:

(i') For all $\mathcal{X}^0$, $(P_1, \mathcal{X}_1)$ and p_f , $\Sigma(\mathcal{X}^0, P_1, p_f)$ is a complete rational polyhedral decomposition of $C^*(\mathcal{X}^0, P_1) \times \{p_f\}$.

Let Σ a K_f -admissible partial cone decomposition for $(P, \mathcal{X})$. For every $\mathcal{X}^0$, $(P_1, \mathcal{X}_1)$ and p_f let $\Gamma_U \subset U_1(\mathbb{Q})$ be the image of

$$(\{z \in Z(P)(\mathbb{Q}) \mid z|_{\mathcal{X}} = \text{id}\} \times U_1(\mathbb{Q})) \cap p_f \cdot K_f \cdot p_f^{-1}$$

under the projection $Z(P) \times U_1 \rightarrow U_1$. We call Σ smooth with respect to K_f if and only if for all $\mathcal{X}^0$, $(P_1, \mathcal{X}_1)$ and p_f the rational partial polyhedral decomposition $\Sigma(\mathcal{X}^0, P_1, p_f)$ is smooth with respect to the lattice $\Gamma_U(-1) \subset U_1(\mathbb{Q})(-1)$.

Remarks. (a) Conditions (i'), (ii), (iii) and (v) together are equivalent to admissible decompositions in the sense of [AMRT] ch.III p.252 for every connected component of $M^{K_f}(P, \mathcal{X})(\mathbb{C})$. Condition (iv) is the ‘‘arithmeticity’’ condition, and is not necessary to define the toroidal compactification over $\mathbb{C}$. But it is precisely the condition needed in order to be able to describe the natural stratification of the toroidal compactification in terms of other mixed Shimura varieties. Thus for our main purpose, the construction of the canonical model for the toroidal compactification, it is the essential new ingredient. (However, in general it is not strictly necessary condition to be able to define the toroidal compactification over the reflex field.)

(b) As in 6.3 one can find finitely many rational boundary components $(P_1, \mathcal{X}_1)$ and finitely many $p_f \in P(\mathbb{A}_f)$ such that Σ is the union of all left $P(\mathbb{Q})$ - and right K_f -conjugates of the $\Sigma(\mathcal{X}^0, P_1, p_f)$ for these $(P_1, \mathcal{X}_1)$ and p_f . Thus the finiteness of Σ is equivalent to the finiteness of every quotient $\Gamma_Q \backslash \Sigma(\mathcal{X}^0, P_1, p_f)$, where Q is the admissible $\mathbb{Q}$ -parabolic subgroup of P associated to P_1 , and Γ_Q is an arithmetic subgroup of $Q(\mathbb{Q})$.

(c) The study of the existence of admissible cone decompositions will be postponed to chapter 9, where we shall in fact construct a special type of these.

6.5. Operations on admissible cone decompositions: Let $(P, \mathcal{X})$ be mixed Shimura data, K_f an open compact subgroup of $P(\mathbb{A}_f)$, and Σ a K_f -admissible partial cone decomposition for $(P, \mathcal{X})$.

(a) For $p_f \in P(\mathbb{A}_f)$ let $[\cdot p_f]^* \Sigma$ be the set of all ‘‘cones’’ $\{(u, p'_f \cdot p_f^{-1}) \mid (u, p'_f) \in \sigma\}$ for all $\sigma \in \Sigma$. This is a $p_f \cdot K_f \cdot p_f^{-1}$ -admissible partial cone decomposition for $(P, \mathcal{X})$. It is finite, respectively complete if and only if Σ is finite, resp. complete.

(b) Let $\phi : (P, \mathcal{X}) \rightarrow (P, \mathcal{X})$ be an automorphism. It induces an automorphism of $\mathcal{C}(P, \mathcal{X}) \times P(\mathbb{A}_f)$ that we again denote by ϕ . Let $\phi^* \Sigma := \{\phi^{-1}(\sigma) \mid \sigma \in \Sigma\}$. This is a $\phi^{-1}(K_f)$ -admissible partial cone decomposition for $(P, \mathcal{X})$, and finite, resp. complete if and only if Σ has the same property.

(c) Let $(P_1, \mathcal{X}_1)$ be a rational boundary component of $(P, \mathcal{X})$. We define a $(P_1(\mathbb{A}_f) \cap K_f)$ -admissible partial cone decomposition for $(P_1, \mathcal{X}_1)$, denoted by $\Sigma|_{(P_1, \mathcal{X}_1)}$, according to the rule

$$(\Sigma|_{(P_1, \mathcal{X}_1)})(\mathcal{X}_1^0, P_2, p_{1,f}) = \Sigma(\mathcal{X}^0, P_2, p_{1,f})$$

for all $p_{1,f} \in P_1(\mathbb{A}_f)$, every rational boundary component $(P_2, \mathcal{X}_2)$ of $(P_1, \mathcal{X}_1)$, and every pair of corresponding connected components $\mathcal{X}^0 \subset \mathcal{X}$ and $\mathcal{X}_1^0 \subset \mathcal{X}_1$ such that $\mathcal{X}^0 \hookrightarrow \mathcal{X}_1^0 \hookrightarrow \mathcal{X}_2$. By 4.22 (b) this indeed defines a $(P_1(\mathbb{A}_f) \cap K_f)$ -admissible partial cone decomposition for $(P_1, \mathcal{X}_1)$. Note that $\Sigma|_{(P_1, \mathcal{X}_1)}$ inherits in general neither finiteness nor completeness.

(d) Define Σ^0 as the set of all $\sigma \in \Sigma$ such that $\sigma \subset C(\mathcal{X}^0, P_1) \times P(\mathbb{A}_f)$ for some improper rational boundary component $(P_1, \mathcal{X}_1)$. We say that Σ is concentrated in the unipotent fibre if $\Sigma = \Sigma^0$.

6.6. Torus embedding along the unipotent fibre: Let $(P, \mathcal{X})$ be mixed Shimura data and K_f an open compact subgroup of $P(\mathbb{A}_f)$. Let $(P', \mathcal{X}') := (P, \mathcal{X})/U$ and K'_f the image of K_f in $P'(\mathbb{A}_f)$. Let Σ be a K_f -admissible partial cone decomposition for $(P, \mathcal{X})$ that is concentrated in the unipotent fibre. We want to define a canonical torus embedding associated to Σ along the fibres of $[\pi'] : M^{K_f}(P, \mathcal{X})(\mathbb{C}) \rightarrow M^{K'_f}(P', \mathcal{X}')(\mathbb{C})$.

Let us first assume that K_f is neat. Let $\mathcal{X}^0$ be a connected component of $\mathcal{X}$, let $x \in \mathcal{X}^0$ and $p_f \in P(\mathbb{A}_f)$. By 3.14 the fibre over $[\pi']([(x, p_f)])$ is isomorphic to $\Gamma_U \backslash U(\mathbb{C})$, where Γ_U is the image of

$$(\{z \in Z(P)(\mathbb{Q}) \mid z|_{\mathcal{X}} = \text{id}\} \times U(\mathbb{Q})) \cap p_f \cdot K_f \cdot p_f^{-1}$$

under the projection $Z(P) \times U \rightarrow U$. This is an algebraic torus, whose cocharacter group $\text{Hom}_{\mathbb{C}}(\mathbb{C}^\times, \Gamma_U \backslash U(\mathbb{C}))$ is canonically isomorphic to $\text{Hom}(\mathbb{Z}(1), \Gamma_U) = \Gamma_U(-1)$. Now the rational partial polyhedral decomposition $\Sigma(\mathcal{X}^0, P, p_f)$ of $U(\mathbb{R})(-1) = \Gamma_U(-1)_{\mathbb{R}}$ defines a torus embedding of this fibre. By the functoriality of torus embeddings, conditions 6.4 (ii) and (iii) ensure that this embedding is independent of the choice of x and p_f , up to isomorphism. Moreover these torus embeddings form a holomorphic family of torus embeddings over $M^{K'_f}(P', \mathcal{X}')(\mathbb{C})$. Denote the total space of this embedding by $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$. It is a normal complex space.

Since K_f is neat, $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ is smooth. Thus by definition 6.4 and by 5.3 $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is smooth if and only if Σ is smooth with respect to K_f . If we no longer require K_f to be neat, we can construct an embedding as follows. Choose a open normal subgroup K_f^* of K_f which is neat. Clearly the action of K_f^*/K_f through the morphism 3.4 (a) extends to $M^{K_f^*}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ (this is a special case of the functoriality 6.7 (a) below), so we may define $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ as a quotient of $M^{K_f^*}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ by this action. By 5.4 this definition does not depend on the choice of K_f^* .

6.7. Functoriality: The morphism of 3.4 extend to torus embeddings along the unipotent fibre in the following cases. This follows easily from the definition 6.6 and from 5.3–5 and 5.8.

(a) Let $(P, \mathcal{X})$ be mixed Shimura data, $p_f \in P(\mathbb{A}_f)$, and $K_f, K'_f \subset P(\mathbb{A}_f)$ open compact subgroups, such that $K'_f \subset p_f \cdot K_f \cdot p_f^{-1}$. Let Σ be a K_f -admissible and Σ' be a K'_f -admissible partial cone decomposition for $(P, \mathcal{X})$, both concentrated in the unipotent fibre. Suppose that for every $\sigma' \in \Sigma'$ there exists $\sigma \in \Sigma$ such that $\sigma' \subset \{(u, p'_f \cdot p_f^{-1}) \mid (u, p'_f) \in \sigma\}$, or equivalently $\tau \in [\cdot p_f]^* \Sigma$ such that $\sigma' \subset \tau$. Then the map of 3.4 (a) extends uniquely to a holomorphic map

$$[\cdot p_f] = [\cdot p_f]_{K'_f, K_f} : M^{K'_f}(P, \mathcal{X}, \Sigma')(\mathbb{C}) \rightarrow M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}).$$

It is an isomorphism if $K'_f = p_f \cdot K_f \cdot p_f^{-1}$ and $\Sigma' = [\cdot p_f]^* \Sigma$. If $K'_f = K_f$ and $\Sigma' \subset \Sigma$, then it is an open embedding. In that case the complement of the image is a closed analytic subset. If $\Sigma' = \Sigma$, $K'_f = K_f$ and $p_f \in K_f$, then $[\cdot p_f]$ is the identity map. If $\Sigma' = \Sigma$ and K'_f is a normal subgroup of K_f , then the map $[\cdot 1]_{K'_f, K_f}$ identifies $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ with the quotient of $M^{K'_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ by the finite group K'_f/K_f , acting from the right hand side through the morphisms $[\cdot k_f]$ on $M^{K'_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$.

(b) Consider a morphism $\phi : (P_1, \mathcal{X}_1) \rightarrow (P_2, \mathcal{X}_2)$ and open compact subgroups $K_f^1 \subset P_1(\mathbb{A}_f)$ and $K_f^2 \subset P_2(\mathbb{A}_f)$ such that $\phi(K_f^1) \subset K_f^2$. Let Σ_1 be a K_f^1 -admissible partial cone decomposition for $(P_1, \mathcal{X}_1)$ and Σ_2 a K_f^2 -admissible partial cone decomposition for $(P_2, \mathcal{X}_2)$, each concentrated in the unipotent fibre. Suppose that for every $\sigma_1 \in \Sigma_1$ there exists $\sigma_2 \in \Sigma_2$ such that $\phi(\sigma_1) \subset \sigma_2$. Then the map of 3.4 (b) extends to a holomorphic map

$$[\phi] = [\phi]_{K_f^1, K_f^2} : M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1)(\mathbb{C}) \rightarrow M^{K_f^2}(P_2, \mathcal{X}_2, \Sigma_2)(\mathbb{C}).$$

If $\phi : (P, \mathcal{X}) \rightarrow (P, \mathcal{X})$ is an automorphism, $\phi(K_f^1) = K_f^2$ and $\Sigma_1 = \phi^* \Sigma_2$, then $[\phi]$ is an isomorphism.

6.8. Reinterpretation of 6.6: To illustrate the “arithmeticity condition” 6.4 (iv) we want to interpret the torus embedding along the unipotent fibre in a different way. We shall show that this torus embedding corresponds to an embedding of a certain standard torus. Let $(P, \mathcal{X})$, $(P', \mathcal{X}')$, K_f neat, K'_f and Σ be as in 6.6. Suppose this time that $(P, \mathcal{X})$ is irreducible. Then by 2.14 (a) P acts through the scalar character $\phi : P \rightarrow \mathbb{G}_{m, \mathbb{Q}}$ on U . Let $(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$ be as in 2.8, and let $(P_*, \mathcal{X}_*) \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$ be the unipotent extension with $P_* = U \rtimes \mathbb{G}_{m, \mathbb{Q}}$, with $\mathbb{G}_{m, \mathbb{Q}}$ acting on U through the character ϕ . In a canonical way, $(P, \mathcal{X}) \rightarrow (P', \mathcal{X}')$ is a $((P_*, h(\mathcal{X}_*)) \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, h(\mathcal{H}_0)))$ -torsor. It can be made (noncanonically) into a $((P_*, \mathcal{X}_*) \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0))$ -torsor if and only if the morphism $((P', \mathcal{X}') \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, h(\mathcal{H}_0)))$ factors through $(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$. Let K_f^U be the image of $(\{z \in Z(P)(\mathbb{Q}) \mid z|_{\mathcal{X}} = \text{id}\} \times U(\mathbb{A}_f)) \cap K_f$ under the projection $Z(P) \times U \rightarrow U$, and $K_f^* := K_f^U \rtimes \phi(K_f)$.

By condition 6.4 (iv), the rational partial polyhedral decomposition $\Sigma(\mathcal{X}^0, P, p_f)$ of $U(\mathbb{R})(-1)$ does not depend on p_f at all. Note that since $P(\mathbb{Q})$ operates by scalars on $U(\mathbb{R})(-1)$, this decomposition is stable under all those $p \in P(\mathbb{Q})$ that act through positive scalars on $U(\mathbb{R})(-1)$. If there exists a $p \in P(\mathbb{Q})$ that stabilizes a connected component and acts through a negative scalar, then $\Sigma(\mathcal{X}^0, P, p_f)$ is also independent of $\mathcal{X}^0$, and invariant under multiplication by -1 . In this case Σ is just the pullback of a K_f^* -admissible partial cone decomposition Σ_* for $(P_*, h(\mathcal{X}_*))$. Otherwise the morphism $(P', \mathcal{X}') \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, h(\mathcal{H}_0))$ factors through $(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$, and Σ is the pullback of a K_f^* -admissible partial cone decomposition Σ_* for $(P_*, \mathcal{X}_*)$.

Now by 3.12 (b), $M^{K_f}(P, \mathcal{X})(\mathbb{C}) \rightarrow M^{K'_f}(P', \mathcal{X}')(\mathbb{C})$ is a torsor under $M^{K_f^*}(P_*, \mathcal{X}_*)(\mathbb{C}) \rightarrow M^{\phi(K_f)}(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)(\mathbb{C})$, resp. the same with $\mathcal{X}_*$ and $\mathcal{H}_0$ replaced by $h(\mathcal{X}_*)$, $h(\mathcal{H}_0)$. So by 5.5 the torus embedding

$$M^{K_f^*}(P_*, \mathcal{X}_*)(\mathbb{C}) \hookrightarrow M^{K_f^*}(P_*, \mathcal{X}_*, \Sigma_*)(\mathbb{C}),$$

resp. the same with $h(\mathcal{X}_*)$ and $h(\mathcal{H}_0)$, also defines a torus embedding of $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ along the fibres over $M^{K'_f}(P', \mathcal{X}')(\mathbb{C})$. One easily verifies (fibrewise) that this is isomorphic

to that defined in 6.6. Next we describe the torus embedding in the fundamental case $(P_*, \mathcal{X}_*)$.

Remark. Usually the morphism $(P', \mathcal{X}') \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, h(\mathcal{H}_0))$ does factor through $(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$. In fact, 4.15 (d) implies this whenever $(P, \mathcal{X})$ is a proper rational boundary component of some other mixed Shimura data.

6.9. The case of a standard torus: We first show that giving a K_f^* -admissible cone decomposition Σ_* for $(P_*, \mathcal{X}_*)$ is equivalent to giving a rational partial polyhedral decomposition Σ_U of $U(\mathbb{R})$. Consider a connected component $\mathcal{X}_*^0$ of $\mathcal{X}_*$, and denote its image in $\mathcal{H}_0 = \text{Isom}(\mathbb{Z}, \mathbb{Z}(1))$ by λ . After a twist by -1 it corresponds to an isomorphism $\lambda : U(\mathbb{R})(-1) \rightarrow U(\mathbb{R})$. For any $p_{*,f} \in P_*(\mathbb{A}_f)$, $\Sigma_*(\mathcal{X}_*^0, P_*, p_{*,f})$ is a rational partial polyhedral decomposition of $U(\mathbb{R})(-1)$ that, by 6.4 (iv), depends only on $\mathcal{X}_*^0$. Clearly $\Sigma_U := \lambda(\Sigma_*(\mathcal{X}_*^0, P_*, p_{*,f}))$ is a rational partial polyhedral decomposition of $U(\mathbb{R})$ that is independent of $\mathcal{X}_*^0$. The converse is obvious, since $(P_*, \mathcal{X}_*)$ possesses no rational boundary components except itself.

Now let $\Gamma_U := U(\mathbb{Q}) \cap K_f^U$, and consider the unipotent extension $(P_0, \mathcal{X}_0) \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$ defined in 2.24. Our aim is to prove:

Claim. The isomorphism of 3.16 induces canonical isomorphisms in the commutative diagram

$$\begin{array}{ccccc} M^{K_f^*}(P_*, \mathcal{X}_*)(\mathbb{C}) & \xrightarrow{\sim} & \mathcal{C}^\times \otimes \Gamma_U & \times & M^{\phi(K_f)}(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)(\mathbb{C}) \\ \cap & & \cap & & \parallel \\ M^{K_f^*}(P_*, \mathcal{X}_*, \Sigma_*)(\mathbb{C}) & \xrightarrow{\sim} & (\mathcal{C}^\times \otimes \Gamma_U)_{\Sigma_U} & \times & M^{\phi(K_f)}(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)(\mathbb{C}) \end{array}$$

Proof. Any $\mathbb{Z}$ -basis of Γ_U determines an isomorphism between $(P_*, \mathcal{X}_*)$ and the $\dim(U)$ -fold fibre product of $(P_0, \mathcal{X}_0)$ with itself over $(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$. Using the canonical identification $U \cong U_0 \otimes \Gamma_U$, we thus get the isomorphism in the first line. Take $\mathcal{X}_*^0$ and λ as above, and $t_f \in \hat{\mathbb{Z}}^\times$, $u \in U_0(\mathbb{C}) = \mathbb{C}$, and $\gamma \in \Gamma_U$. If $e : (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0) \hookrightarrow (P_*, \mathcal{X}_*)$ denotes the given splitting, then this isomorphism is explicitly given as

$$[(u \otimes \gamma) \cdot e(\lambda), e(t_f)] \mapsto (\exp(\lambda(u)) \otimes \gamma, [(\lambda, t_f)]).$$

By 6.6, the cocharacter group of the torus on the left is $\Gamma_U(-1) = \lambda^{-1}(\Gamma_U)$, and the torus embedding is that with respect to the decomposition $\Sigma_*(\mathcal{X}_*^0, P_*, p_{*,f}) = \lambda^{-1}(\Sigma_U)$. By the explicit formula for the isomorphism, the isomorphism in the second line follows. **q.e.d.**

As an example of this take $(P_*, \mathcal{X}_*) = (P_0, \mathcal{X}_0)$ and $\Sigma_U \{\mathbb{R}^{\geq 0}, \{0\}\}$. Then the torus embedding $\Sigma_0 := \Sigma_*$ corresponds to $\mathbb{C}^\times \hookrightarrow \mathbb{C}$, so we get

$$M^{K_f^0}(P_0, \mathcal{X}_0, \Sigma_0)(\mathbb{C}) \cong \mathbb{C} \times M^{\phi(K_f)}(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)(\mathbb{C}).$$

6.10. A covering of $M^{K_f}(P, \mathcal{X})(\mathbb{C})$: For the rest of the chapter we fix mixed Shimura data $(P, \mathcal{X})$, an open compact subgroup $K_f \subset P(\mathbb{A}_f)$, and a K_f -admissible partial cone decomposition Σ for $(P, \mathcal{X})$. We shall eventually construct the toroidal compactification on $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ associated to Σ . We begin by describing a certain possibly ramified covering of $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ by mixed Shimura varieties associated to all its boundary components

For every rational boundary component $(P_1, \mathcal{X}_1)$ of $(P, \mathcal{X})$ and every $p_f \in P(\mathbb{A}_f)$ let $K_f^1 := P_1(\mathbb{A}_f) \cap p_f \cdot K_f \cdot p_f^{-1}$. Let $\mathcal{X}^+ \subset \mathcal{X}$ be associated to $(P_1, \mathcal{X}_1)$ as in 4.11, and consider the following open subset of $M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C})$:

$$\mathcal{U}(P_1, \mathcal{X}_1, p_f) := P_1(\mathbb{Q}) \backslash \mathcal{X}^+ \times (P_1(\mathbb{A}_f)/K_f^1) \subset P_1(\mathbb{Q}) \backslash \mathcal{X}_1 \times (P_1(\mathbb{A}_f)/K_f^1).$$

Consider the map

$$\beta(P_1, \mathcal{X}_1, p_f) : \mathcal{U}(P_1, \mathcal{X}_1, p_f) \rightarrow M^{K_f}(P, \mathcal{X})(\mathbb{C}), [(x, p_{1,f})] \mapsto [(x, p_{1,f} \cdot p_f)].$$

Let $\mathcal{U}$ be the disjoint union of the $\mathcal{U}(P_1, \mathcal{X}_1, p_f)$ for all $(P_1, \mathcal{X}_1)$ and p_f , and let $\beta : \mathcal{U} \rightarrow M^{K_f}(P, \mathcal{X})(\mathbb{C})$ be the map induced by the $\beta(P_1, \mathcal{X}_1, p_f)$. This is a surjective ramified covering. For all $u, u' \in \mathcal{U}$ write $u \sim u'$ if and only if u and u' are mapped to the same point in $M^{K_f}(P, \mathcal{X})(\mathbb{C})$. Then $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ is isomorphic to $\mathcal{U}/\sim$ as a set. A connected component of $\mathcal{U}$ is of the form $\Gamma_1 \backslash \mathcal{X}^0$ for $\mathcal{X}^0$ a connected component of $\mathcal{X}$ and Γ_1 an arithmetic subgroup of $P_1(\mathbb{Q})$. Its image under β is $\Gamma_1 \backslash \mathcal{X}^0$, where Γ is an arithmetic subgroup of $P(\mathbb{Q})$. This shows that $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ is isomorphic to $\mathcal{U}/\sim$ as a topological space, where $\mathcal{U}/\sim$ carries the quotient topology. Moreover the equivalence relation $\sim$ is induced by analytic automorphisms of $\mathcal{X}$, and locally on $\mathcal{U}$ the map β is a finite possibly ramified covering, so the structure of $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ as a complex space is induced from that of $\mathcal{U}$. Consider the graph $\{(u, u') \in \mathcal{U} \times \mathcal{U} \mid u \sim u'\}$ of $\sim$. Since $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ is Hausdorff, this is a closed subset of $\mathcal{U} \times \mathcal{U}$.

6.11. Explicit description of the equivalence relation: The description of $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ as $\mathcal{U}/\sim$ would not be useful if one could not describe the equivalence relation $\sim$ explicitly. To give such a description we first consider a number of special cases. Let $(P_1, \mathcal{X}_1)$ and $(P'_1, \mathcal{X}'_1)$ be rational boundary components of $(P, \mathcal{X})$, and let $p_f, p'_f \in P(\mathbb{A}_f)$. We are interested in certain maps $\mathcal{U}(P_1, \mathcal{X}_1, p_f) \rightarrow \mathcal{U}(P'_1, \mathcal{X}'_1, p'_f)$.

(a) Suppose that $(P'_1, \mathcal{X}'_1) = (P_1, \mathcal{X}_1)$ and $p'_f = p_f \cdot k_f$ for $k_f \in K_f$. Then $K_f^1 = P_1(\mathbb{A}_f) \cap p_f \cdot K_f \cdot p_f^{-1} = P_1(\mathbb{A}_f) \cap p'_f \cdot K_f \cdot p'^{-1}_f$, hence the identity on $M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C})$ induces an isomorphism

$$[\text{id}] : \mathcal{U}(P_1, \mathcal{X}_1, p_f) \rightarrow \mathcal{U}(P_1, \mathcal{X}_1, p_f \cdot k_f).$$

(b) Suppose that $(P'_1, \mathcal{X}'_1) = (P_1, \mathcal{X}_1)$ and $p'_f = p_{1,f}^{-1} \cdot p_f$ for some $p_{1,f} \in P_1(\mathbb{A}_f)$. Then the map $[\cdot p_{1,f}] : M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C}) \rightarrow M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C})$ induces an isomorphism

$$[\cdot p_{1,f}] : \mathcal{U}(P_1, \mathcal{X}_1, p_f) \rightarrow \mathcal{U}(P_1, \mathcal{X}_1, p_{1,f}^{-1} \cdot p_f).$$

(c) Suppose that $(P'_1, \mathcal{X}'_1) = \text{int}(p)(P_1, \mathcal{X}_1) = (p \cdot P_1 \cdot p^{-1}, p \cdot \mathcal{X}_1)$ and $p'_f = p \cdot p_f$ for some $p \in P(\mathbb{Q})$. Then the map $[\text{int}(p)] : M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C}) \rightarrow M^{K_f^1}(P'_1, \mathcal{X}'_1)(\mathbb{C})$ induces an isomorphism

$$[\text{int}(p)] : \mathcal{U}(P_1, \mathcal{X}_1, p_f) \rightarrow \mathcal{U}(p \cdot P_1 \cdot p^{-1}, p \cdot \mathcal{X}_1, p \cdot p_f).$$

(d) Suppose that $(P_1, \mathcal{X}_1)$ is a rational boundary component of $(P'_1, \mathcal{X}'_1)$, and that $p'_f = p_f$. Then we have a canonical map

$$\beta(P_1, \mathcal{X}_1, P'_1, \mathcal{X}'_1, p_f) : \mathcal{U}(P_1, \mathcal{X}_1, p_f) \rightarrow \mathcal{U}(P'_1, \mathcal{X}'_1, p_f), [(x, p_{1,f})] \mapsto [(x, p_{1,f})].$$

These maps commutes with each other according to the rules:

$$\begin{aligned}
[\text{id}] \circ [\cdot p_{2,f}] &= [\cdot p_{2,f}] \circ [\text{id}], \\
[\text{id}] \circ [\text{int}(p)] &= [\text{int}(p)] \circ [\text{id}], \\
[\text{id}] \circ \beta(P_1, \mathcal{X}_1, P'_1, \mathcal{X}'_1, p_f) &= \beta(P_1, \mathcal{X}_1, P'_1, \mathcal{X}'_1, p_f \cdot k_f) \circ [\text{id}], \\
[\text{int}(p)] \circ [\cdot p_{1,f}] &= [\cdot p \cdot p_{1,f} \cdot p^{-1}] \circ [\text{int}(p)], \\
[\text{int}(p)] \circ \beta(P_1, \mathcal{X}_1, P'_1, \mathcal{X}'_1, p_f) &= \beta(p \cdot P_1 \cdot p^{-1}, p \cdot \mathcal{X}_1, p \cdot P'_1 \cdot p^{-1}, p \cdot \mathcal{X}'_1, p \cdot p_f) \circ [\text{int}(p)], \\
[\cdot p_{1,f}] \circ \beta(P_1, \mathcal{X}_1, P'_1, \mathcal{X}'_1, p_f) &= \beta(P_1, \mathcal{X}_1, P'_1, \mathcal{X}'_1, p_{1,f} \cdot p_f) \circ [\cdot p_{1,f}],
\end{aligned}$$

where in the last case it is necessary to have $p_{1,f} \in P_1(\mathbb{A}_f)$. Also in each of the cases (a) through (d) it is easy to see that the diagram

$$\begin{array}{ccc}
\mathcal{U}(P_1, \mathcal{X}_1, p_f) & \rightarrow & \mathcal{U}(P'_1, \mathcal{X}'_1, p'_f) \\
\beta(P_1, \mathcal{X}_1, p_f) & \searrow & \downarrow \beta(P'_1, \mathcal{X}'_1, p'_f) \text{ cr } M^{K_f}(P, \mathcal{X})(\mathbb{C})
\end{array}$$

is commutative. Thus its graph is a subset of the graph of $\sim$. The following lemma shows how the equivalence relation $\sim$ can be described in terms of these maps.

6.12. Lemma: (a) Let $u \in \mathcal{U}(P_1, \mathcal{X}_1, p_f)$ and $u' \in \mathcal{U}(P'_1, \mathcal{X}'_1, p'_f)$. Then $u \sim u'$ if and only if there exists a rational boundary component $(P_2, \mathcal{X}_2)$ of $(P, \mathcal{X})$, $p \in P(\mathbb{Q})$, $p_{2,f} \in P_2(\mathbb{A}_f)$ and $k_f \in K_f$ such that

- (i) $(P_1, \mathcal{X}_1)$ is a rational boundary component of $(P_2, \mathcal{X}_2)$,
- (ii) $(P'_1, \mathcal{X}'_1)$ is a rational boundary component of $\text{int}(p)(P_2, \mathcal{X}_2)$,
- (iii) $p'_f = p \cdot p_{2,f} \cdot p_f \cdot k_f$, and
- (iv) u and u' have the same image under the following maps:

$$\begin{array}{ccc}
\mathcal{U}(P_1, \mathcal{X}_1, p_f) \ni u & & u' \in \mathcal{U}(P'_1, \mathcal{X}'_1, p'_f) \\
\beta(P_1, \mathcal{X}_1, P_2, \mathcal{X}_2, p_f) \downarrow & & \downarrow \beta(P'_1, \mathcal{X}'_1, p P_2 p^{-1}, p \mathcal{X}_2, p'_f) \\
\mathcal{U}(P_2, \mathcal{X}_2, p_f) & & \mathcal{U}(p \cdot P_2 \cdot p^{-1}, p \cdot \mathcal{X}_2, p'_f) \\
[\cdot p_{2,f}] \uparrow \wr & & \uparrow \wr [\text{int}(p)] \\
\mathcal{U}(P_2, \mathcal{X}_2, p_{2,f} \cdot p_f) & \xrightarrow[\sim]{[\text{id}]} & \mathcal{U}(P_2, \mathcal{X}_2, p_{2,f} \cdot p_f \cdot k_f).
\end{array}$$

(b) The equivalence relation $\sim$ is generated by the graphs of the maps 6.11 (a) to (d).

Proof. (b) follows directly from (a). The “if”-part of (a) is clear from 6.11. So suppose that $u = [(x, p_{1,f})] \in M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C})$ and $u' = [(x', p'_{1,f})] \in M^{K_f^1}(P'_1, \mathcal{X}'_1)(\mathbb{C})$ have the same image in $M^{K_f}(P, \mathcal{X})(\mathbb{C})$. Then there exist $p \in P(\mathbb{Q})$ and $k_f \in K_f$ such that $p \cdot x = x'$ and $p \cdot p_{1,f} \cdot p_f \cdot k_f = p'_{1,f} \cdot p'_f$. Let $(P_2, \mathcal{X}_2)$ be the (unique) improper rational boundary component of $(P, \mathcal{X})$ such that $(P_1, \mathcal{X}_1)$ is a boundary component of $(P_2, \mathcal{X}_2)$. From $x' = p \cdot x \in p \cdot \mathcal{X}_2$ it follows that $(P'_1, \mathcal{X}'_1)$ is a boundary component of $\text{int}(p)(P_2, \mathcal{X}_2)$. Now both $p_{1,f}$ and $p^{-1} \cdot p'_{1,f} \cdot p$ lie in $P_2(\mathbb{A}_f)$, hence

$$\begin{aligned}
p_{2,f} &:= p^{-1} \cdot p'_f \cdot k_f^{-1} \cdot p_f^{-1} \\
&= (p^{-1} \cdot p'_{1,f} \cdot p) \cdot (p^{-1} \cdot p'_{1,f} \cdot p'_f \cdot k_f^{-1} \cdot p_f^{-1}) \\
&= (p^{-1} \cdot p'_{1,f} \cdot p) \cdot p_{1,f}
\end{aligned}$$

lies in $P_2(\mathbb{A}_f)$. One easily verifies that with these choices for $(P_2, \mathcal{X}_2)$, p , $p_{2,f}$, and k_f the points u and u' have the same image in the above diagram. **q.e.d.**

6.13. Compactification of the covering: We want to construct the toroidal compactification of $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ as the quotient of a partial compactification of $\mathcal{U}$ by a suitable extended equivalence relation. First we construct the partial compactification $\overline{\mathcal{U}}$ of $\mathcal{U}$.

For every rational boundary component $(P_1, \mathcal{X}_1)$ of $(P, \mathcal{X})$, and every $p_f \in P(\mathbb{A}_f)$ let $\Sigma_1^0 := ([\cdot p_f]^* \Sigma)|_{(P_1, \mathcal{X}_1)}^0$. By 6.5 this is a K_f^1 -admissible partial cone decomposition for $(P_1, \mathcal{X}_1)$, which is concentrated in the unipotent fibre. Now $M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C})$ is an open subset of $M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1^0)(\mathbb{C})$, and we let $\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$ be the interior of the closure of $\mathcal{U}(P_1, \mathcal{X}_1, p_f)$ in $M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1^0)(\mathbb{C})$. It contains $\mathcal{U}(P_1, \mathcal{X}_1, p_f)$ as an open dense subset. Let us explicitly determine which parts of which strata of $M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1^0)(\mathbb{C})$ are contained in $\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$.

Let $\Gamma_1 \backslash \mathcal{X}_1^0$ be a connected component of $M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C})$. Let $\text{im} : \mathcal{X}_1 \rightarrow U_1(\mathbb{R})(-1)$ be the map “imaginary part” of 4.14, it is equivariant under Γ_1 . Assume first that K_f^1 is neat, then Γ_1 commutes with U_1 , hence $\text{im}|_{\mathcal{X}_1^0}$ factors through a map $\Gamma_1 \backslash \mathcal{X}_1^0 \rightarrow U_1(\mathbb{R})(-1)$. Since by 6.6 the cocharacter group of the relevant torus is an arithmetic subgroup of $U_1(\mathbb{Q})(-1)$, one easily sees that this is a map “ord” as in 5.8. Now by 4.15 (b) the corresponding connected component of $\mathcal{X}$ is $\mathcal{X}^0 = \text{im}^{-1}(C(\mathcal{X}^0, P_1))$, hence the corresponding connected component of $\mathcal{U}(P_1, \mathcal{X}_1, p_f)$ is $\Gamma_1 \backslash \mathcal{X}^0 = \text{ord}^{-1}(C(\mathcal{X}^0, P_1))$. Let $\sigma \in \Sigma_1$, then 5.9 (a) shows that the closure of $\mathcal{U}(P_1, \mathcal{X}_1, p_f)$ meets the σ -stratum precisely in the set $\pi_\sigma(\text{ord}^{-1}(\overline{C(\mathcal{X}^0, P_1)}))$. Hence the intersection of $\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$ with the σ -stratum is just $\pi_\sigma(\text{ord}^{-1}(C(\mathcal{X}^0, P_1))) = \pi_\sigma(\Gamma_1 \backslash \mathcal{X}^0)$. If $\sigma^0 \subset C(\mathcal{X}^0, P_1)$, then $C(\mathcal{X}^0, P_1) + \mathbb{R} \cdot \sigma^0 = U_1(\mathbb{R})(-1)$, hence for such σ $\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$ is a neighborhood of the whole σ -stratum of $M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1^0)(\mathbb{C})$. If K_f^1 is not neat, the same assertion follows by taking quotients.

For later use let us denote by $\partial\mathcal{U}(P_1, \mathcal{X}_1, p_f)$ the union of all σ -strata in $\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$ such that $\sigma^0 \subset C(\mathcal{X}^0, P_1)$. By 5.3 and the above description of the strata it is a closed analytic subset of $\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$, or even of $M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1^0)(\mathbb{C})$. Note that if $(P_1, \mathcal{X}_1)$ is an improper boundary component, then $\partial\mathcal{U}(P_1, \mathcal{X}_1, p_f) = \overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$.

6.14. Extension of the equivalence relation: Let $\overline{\mathcal{U}}$ be the disjoint union of the $\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$ for all $(P_1, \mathcal{X}_1)$ and p_f , this contains $\mathcal{U}$ as an open dense subset. We wish to extend the equivalence relation $\sim$ to $\overline{\mathcal{U}}$ such that the quotient $\overline{\mathcal{U}}/\sim$ is the desired partial toroidal compactification of $M^{K_f}(P, \mathcal{X})(\mathbb{C})$. This extended equivalence relation must satisfy the following conditions:

- (a) Its graph in $\overline{\mathcal{U}} \times \overline{\mathcal{U}}$ is closed (so that the quotient topology on $\overline{\mathcal{U}}/\sim$ is Hausdorff).
- (b) Locally on $\overline{\mathcal{U}}$ it is given by analytic automorphisms, and locally on $\overline{\mathcal{U}}$ the projection $\overline{\mathcal{U}} \rightarrow \overline{\mathcal{U}}/\sim$ is a finite, possibly ramified, covering (so that $\overline{\mathcal{U}}/\sim$ inherits the structure of the complex space).

Moreover we want to be able to explicitly describe the topology on $\overline{\mathcal{U}}/\sim$. Observe that by (a) the graph of the extension must contain the closure of the graph of $\sim$. We shall define an explicit extension, whose graph will turn out to be equal to this closure.

6.15. Extension of the maps of 6.11: To approximate the desired extension of $\sim$ we first consider the maps of 6.11. In 6.11 (a), (b) or (c) the isomorphism $\mathcal{U}(P_1, \mathcal{X}_1, p_f) \rightarrow \mathcal{U}(P'_1, \mathcal{X}'_1, p'_f)$ is induced from an isomorphism $M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C}) \rightarrow M^{K_f'^1}(P'_1, \mathcal{X}'_1)(\mathbb{C})$ as in 3.4. It is easily checked that in each of these cases the hypotheses of 6.7 (a) or (b) hold, so by 6.7 the isomorphism $M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C}) \rightarrow M^{K_f'^1}(P'_1, \mathcal{X}'_1)(\mathbb{C})$ extends uniquely to an isomorphism

$$M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1^0)(\mathbb{C}) \rightarrow M^{K_f'^1}(P'_1, \mathcal{X}'_1, \Sigma_1'^0)(\mathbb{C}).$$

By the definition 6.13 of $\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$ we thus get a unique isomorphism

$$\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f) \rightarrow \overline{\mathcal{U}}(P'_1, \mathcal{X}'_1, p'_f).$$

Now consider the map 6.11 (d). Let $\Sigma_1''^0 := \Sigma_1'^0|_{(P_1, \mathcal{X}_1)}$, this consists of all $\sigma \in \Sigma_1^0$ which are already contained in $\mathcal{C}(P'_1, \mathcal{X}'_1) \times P_1(\mathbb{A}_f) \subset \mathcal{C}(P_1, \mathcal{X}_1) \times P_1(\mathbb{A}_f)$. Thus by 6.7 (a) $M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1''^0)(\mathbb{C})$ is an open subset of $M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1^0)(\mathbb{C})$. Now $M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C}) \hookrightarrow M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1''^0)(\mathbb{C})$ and $M^{K_f'^1}(P'_1, \mathcal{X}'_1)(\mathbb{C}) \hookrightarrow M^{K_f'^1}(P'_1, \mathcal{X}'_1, \Sigma_1'^0)(\mathbb{C})$ are torus embeddings with respect to the same torus. Hence the map $\beta(P_1, \mathcal{X}_1, P'_1, \mathcal{X}'_1, p_f) : \mathcal{U}(P_1, \mathcal{X}_1, p_f) \rightarrow \mathcal{U}(P'_1, \mathcal{X}'_1, p_f)$ of 6.11 (d) extends to a unique holomorphic map

$$\overline{\beta}(P_1, \mathcal{X}_1, P'_1, \mathcal{X}'_1, p_f) : \overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f) \cap M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1''^0)(\mathbb{C}) \rightarrow \overline{\mathcal{U}}(P'_1, \mathcal{X}'_1, p_f).$$

Note that the description of $\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$ in 6.13 implies that the image of this map is a union of connected components of $\overline{\mathcal{U}}(P'_1, \mathcal{X}'_1, p_f)$. Let us also observe the following fact. For every point $u \in \overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$ consider all rational boundary components $(P'_1, \mathcal{X}'_1)$ between $(P, \mathcal{X})$ and $(P_1, \mathcal{X}_1)$ such that this map is defined at u . There exists a unique maximal such $(P'_1, \mathcal{X}'_1)$, for which u is mapped into $\partial\mathcal{U}(P'_1, \mathcal{X}'_1, p_f)$.

We have thus defined an extension for each of the maps of 6.11. Since $\mathcal{U}(P_1, \mathcal{X}_1, p_f)$ is open and dense in $\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$, resp. in $\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f) \cap M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1''^0)(\mathbb{C})$ in the case 6.11 (d), the graphs of these extended maps are contained in the closure of the graph of $\sim$. We shall see below that this closure is equal to the graph of the equivalence relation generated by these extended maps.

6.16. Definition of the extended relation: Let $(P_1, \mathcal{X}_1)$ and $(P'_1, \mathcal{X}'_1)$ be rational boundary components of $(P, \mathcal{X})$ and $p_f, p'_f \in P(\mathbb{A}_f)$. Let $u \in \overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$ and $u' \in \overline{\mathcal{U}}(P'_1, \mathcal{X}'_1, p'_f)$. In analogy to 6.12 define $u \sim u'$ if and only if there exists a rational boundary component $(P_2, \mathcal{X}_2)$ of $(P, \mathcal{X})$, $p \in P(\mathbb{Q})$, $p_{2,f} \in P_2(\mathbb{A}_f)$ and $k_f \in K_f$ such that

- (i) $(P_1, \mathcal{X}_1)$ is a rational boundary component of $(P_2, \mathcal{X}_2)$,
- (ii) $(P'_1, \mathcal{X}'_1)$ is a rational boundary component of $\text{int}(p)(P_2, \mathcal{X}_2)$,
- (iii) $p'_f = p \cdot p_{2,f} \cdot p_f \cdot k_f$, and
- (iv) u and u' have the same image under the following maps:

$$\begin{array}{ccc} M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1''^0)(\mathbb{C}) \cap \overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f) \ni u & & u' \in \overline{\mathcal{U}}(P'_1, \mathcal{X}'_1, p'_f) \cap M^{K_f'^1}(P'_1, \mathcal{X}'_1, \Sigma_1'^0)(\mathbb{C}) \\ \overline{\beta}(P_1, \mathcal{X}_1, P_2, \mathcal{X}_2, p_f) \downarrow & & \downarrow \overline{\beta}(P'_1, \mathcal{X}'_1, pP_2p^{-1}, p\mathcal{X}_2, p'_f) \\ \overline{\mathcal{U}}(P_2, \mathcal{X}_2, p_f) & & \overline{\mathcal{U}}(p \cdot P_2 \cdot p^{-1}, p \cdot \mathcal{X}_2, p'_f) \end{array}$$

$$\begin{array}{ccc}
[\cdot p_{2,f}] \uparrow^{\wr} & & \uparrow^{\wr} [\text{int}(p)] \\
\overline{\mathcal{U}}(P_2, \mathcal{X}_2, p_{2,f} \cdot p_f) & \xrightarrow{\sim [\text{id}]} & \overline{\mathcal{U}}(P_2, \mathcal{X}_2, p_{2,f} \cdot p_f \cdot k_f).
\end{array}$$

Remark. Clearly the graph of this relation is contained in the closure of the graph of the old relation on $\mathcal{U}$.

6.17. Lemma: The relation $\sim$ defined in 6.16 is the equivalence relation on $\overline{\mathcal{U}}$ generated by the graphs of the maps 6.15.

Proof. It suffices to prove that $\sim$ is an equivalence relation. The reflexivity is obvious, and the symmetry follows from the commutativity properties of the maps 6.11 (a) to (c), which also hold for their extensions 6.15. For the transitivity consider $u \in \overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$ and $u' \in \overline{\mathcal{U}}(P'_1, \mathcal{X}'_1, p'_f)$, and $u'' \in \overline{\mathcal{U}}(P''_1, \mathcal{X}''_1, p''_f)$, such that $u \sim u' \sim u''$. Let $(P_2, \mathcal{X}_2)$, p , $p_{2,f}$ and k_f be as above for $u \sim u'$ and $(P'_2, \mathcal{X}'_2)$, p' , $p'_{2,f}$ and k'_f analogously for $u' \sim u''$. Let $\sigma \in \Sigma'_1$ be such that u' is in the σ -stratum of $\overline{\mathcal{U}}(P'_1, \mathcal{X}'_1, p'_f)$, then by assumption σ is contained in $C^*(\mathcal{X}^0, p \cdot P_2 \cdot p^{-1})$ as well as $C^*(\mathcal{X}^0, P'_2)$. Let $(P_3, \mathcal{X}_3)$ be the rational boundary component of $(P, \mathcal{X})$ such that $\sigma^0 \subset C(\mathcal{X}^0, P_3)$, then it follows that both $(p \cdot P_2 \cdot p^{-1}, p \cdot \mathcal{X}_2)$ and $(P'_2, \mathcal{X}'_2)$ are boundary components of $(P_3, \mathcal{X}_3)$. Note the relation

$$\begin{aligned}
\overline{\beta}(P'_1, \mathcal{X}'_1, P_3, \mathcal{X}_3, p'_f) &= \overline{\beta}(P'_2, \mathcal{X}'_2, P_3, \mathcal{X}_3, p'_f) \circ \overline{\beta}(P'_1, \mathcal{X}'_1, P'_2, \mathcal{X}'_2, p'_f) \\
&= \overline{\beta}(p \cdot P_2 \cdot p^{-1}, p \cdot \mathcal{X}_2, P_3, \mathcal{X}_3, p'_f) \circ \overline{\beta}(P'_1, \mathcal{X}'_1, p \cdot P_2 \cdot p^{-1}, p \cdot \mathcal{X}_2, p'_f)
\end{aligned}$$

on $\overline{\mathcal{U}}(P'_1, \mathcal{X}'_1, p'_f) \cap M^{K_f^1}(P'_1, \mathcal{X}'_1, \Sigma_1''^0)(\mathbb{C})$ where $\Sigma_1''^0 := \Sigma_3^0|_{(P'_1, \mathcal{X}'_1)}$ contains σ . In other commutativity relations 6.11, which extend automatically to the maps 6.15, now readily shows that $u \sim u''$ with $(p^{-1} \cdot P_3 \cdot p, p^{-1} \cdot \mathcal{X}_3)$ in place of $(P_2, \mathcal{X}_2)$, $p^{-1} \cdot p'_{2,f} \cdot p \cdot p_{2,f}$ in place of $p_{2,f}$, $k_f \cdot k'_f$ in place of k_f , and $p' \cdot p$ in place of p . **q.e.d.**

6.18. The action of the normalizer: We first study the equivalence relation induced by some of the maps of 6.15 on a fixed $\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$. Let Q be the admissible $\mathbb{Q}$ -parabolic subgroup of P associated to P_1 . The group

$$\text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}_1) \cap (P_1(\mathbb{A}_f) \cdot p_f \cdot K_f \cdot p_f^{-1})$$

acts from the left hand side on $\mathcal{X}_1 \times (P_1(\mathbb{A}_f) \cdot p_f \cdot K_f / K_f)$ by multiplication in each factor. As in 6.3 define

$$\Delta_1 := (\text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}_1) \cap (P_1(\mathbb{A}_f) \cdot p_f \cdot K_f \cdot p_f^{-1})) / P_1(\mathbb{Q}),$$

this is an arithmetic subgroup of $(Q/P_1)(\mathbb{Q})$. Since the map $P_1(\mathbb{A}_f)/K_f^1 \rightarrow (P_1(\mathbb{A}_f) \cdot p_f \cdot K_f / K_f)$, $[p_{1,f}] \mapsto [p_{1,f} \cdot p_f]$ is bijective, we get an action of Δ_1 on

$$\begin{aligned}
P_1(\mathbb{Q}) \backslash \mathcal{X}_1 \times (P_1(\mathbb{A}_f) \cdot p_f \cdot K_f / K_f) &\cong P_1(\mathbb{Q}) \backslash \mathcal{X}_1 \times (P_1(\mathbb{A}_f) / K_f^1) \\
&= M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C}).
\end{aligned}$$

The induced action on $M^{\pi_1(K_f^1)}((P_1, \mathcal{X}_1)/W_1)(\mathbb{C})$ is the same as in 6.3, in particular Δ_1 acts on $M^{\pi_1(K_f^1)}((P_1, \mathcal{X}_1)/W_1)(\mathbb{C})$ through a finite quotient.

Next we want to express the Δ_1 -action on $\mathcal{U}(P_1, \mathcal{X}_1, p_f)$ in terms of the maps 6.11 (a) to (c). For this let $q \in Q(\mathbb{Q})$, $p_{1,f} \in P_1(\mathbb{A}_f)$, and $k_f \in K_f$ be such that $[q] \in \Delta_1$ and $q = p_{1,f} \cdot p_f \cdot k_f \cdot p_f^{-1}$. Denote by $L_{[q]}$ the corresponding automorphism of $M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C})$. One easily verifies that the maps

$$[\text{int}(q)]_{K_f^1 q \cdot K_f^1 \cdot p_{1,f}^{-1}, K_f^1} : M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C}) \rightarrow M^{q \cdot K_f^1 \cdot q^{-1}}(P_1, \mathcal{X}_1)(\mathbb{C})$$

and

$$[\cdot p_{1,f}]_{p_{1,f} \cdot K_f^1 \cdot P_{1,f}^{-1}, K_f^1} : M^{p_{1,f} \cdot K_f^1 \cdot p_{1,f}^{-1}}(P_1, \mathcal{X}_1)(\mathbb{C}) \rightarrow M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C})$$

are well-defined and isomorphisms. Likewise the relations $q \cdot K_f^1 \cdot q^{-1} = p_{1,f} \cdot K_f^1 \cdot p_{1,f}^{-1}$ and

$$L_{[q]} = [\cdot p_{1,f}] \circ [\text{int}(q)]$$

are straightforward. Together with the map $[\text{id}]$ of 6.11 (a) this shows that the restriction of $L_{[q]}$ to $\mathcal{U}(P_1, \mathcal{X}_1, p_f)$ can indeed be expressed in terms of the maps 6.11 (a) to (c). By 6.15 this implies that the Δ_1 -action extends canonically to an action on $\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$.

We just showed that if two points $u, u' \in \overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$ are equivalent under Δ_1 , then they are equivalent under the extensions 6.15 of the maps 6.11 (a) to (c). The converse also holds: Indeed, the latter condition is equivalent to the existence of a diagram as in 6.16 with $(P_2, \mathcal{X}_2) = (P_1, \mathcal{X}_1) = (P'_1, \mathcal{X}'_1)$ and $p'_f = p_f$. Then with $q := p^{-1}$ and $p_{1,f} := p_{2,f}$ we have $u = L_{[q]}(u')$ for $[q] \in \Delta_1$, as desired.

The map from $\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$ to the desired toroidal compactification of $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ now factors through $\Delta_1 \backslash \overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$. The next step is therefore to show that this quotient is a “good” object. For this we need the following important reduction theoretic fact, which is a slight generalization of [AMRT] ch.II cor. p.116.

6.19. Theorem: Let $(P, \mathcal{X})$ be mixed Shimura data, $(P_1, \mathcal{X}_1)$ a rational boundary component, Q the corresponding admissible $\mathbb{Q}$ -parabolic subgroup of P , $\rho : Q \rightarrow \text{GL}(U_1)$ its natural representation on U_1 , and $\mathcal{X}^0$ a connected component of $\mathcal{X}^+$. Let Γ be an arithmetic subgroup of $\rho(\text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^0)) \subset \rho(Q)(\mathbb{Q})$.

- (a) If σ and τ are convex rational polyhedral cones in $C^*(\mathcal{X}^0, P_1)$, then the set $\{\gamma \in \Gamma \mid \gamma\sigma \cap \tau \cap C(\mathcal{X}^0, P_1) \neq \emptyset\}$ is finite.
- (b) There exists a convex rational polyhedral cone σ in $C^*(\mathcal{X}^0, P_1)$ such that $\Gamma \cdot \sigma = C^*(\mathcal{X}^0, P_1)$.

Proof. If P is reductive, then the decomposition [AMRT] ch.III §4.1 of Q implies that $\rho(Q)$ is the group of linear automorphisms of the cone $C(\mathcal{X}^0, P_1)$. In this case (a) the assertion (i) of the corollary [AMRT] ch.II p.116, and by (ii) of [loc. cit.] there exists a convex rational polyhedral cone σ in $C^*(\mathcal{X}^0, P_1)$ such that $\Gamma \cdot \sigma \cap C(\mathcal{X}^0, P_1) = C(\mathcal{X}^0, P_1)$. To get the full assertion of (b) note that there are only finitely many Γ -conjugacy classes of rational boundary components $(P_i, \mathcal{X}_i)$ between $(P, \mathcal{X})$ and $(P_1, \mathcal{X}_1)$. Applying [loc. cit.] (ii) to each of these with $\text{Stab}_\Gamma(C(\mathcal{X}^0, P_1))$ in place of Γ we find finitely many convex rational polyhedral cones σ_i such that $C^*(\mathcal{X}^0, P_1) = \bigcup_i \Gamma \cdot \sigma_i$. Letting $\sigma := \sum_i \sigma_i$ we get assertion (b).

In the general case we may without loss of generality assume that $(P, \mathcal{X})$ is irreducible. Next note that if $\dim(U) > 0$, then by 4.15 (c) we have $C^*(\mathcal{X}^0, P_1) = C^*(\mathcal{X}^0, P_1) + U(\mathbb{R})(-1)$. Hence for both assertions we may replace σ by $\sigma + U(\mathbb{R})(-1)$ and τ by $\tau + U(\mathbb{R})(-1)$. Consider the situation after dividing by U , i.e. $(P_1, \mathcal{X}_1)/U$ as rational boundary component of $(P, \mathcal{X})/U$. Since $U \subset U_1 \subset P_1$, the group Γ maps isomorphically to the corresponding group for $(P, \mathcal{X})/U$, so the assertions are equivalent to those for the image of σ and τ in $C^*(\mathcal{X}^0/U(\mathbb{C}), P_1/U)$. Thus it is enough to treat the case $\dim(U) = 0$. In this case we could mutatis mutandum repeat the proof of [AMRT] ch.II §4.3. But with a trick we can reduce the assertions to the reductive case.

Let us first prove (b). Let $(G, \mathcal{H}) := (P, \mathcal{X})/V$. Let $\bar{U}_1$ and $\bar{P}_1$ be the image of U_1 , resp. P_1 in G , and π the projection $U_1 \rightarrow \bar{U}_1$. Choose a convex rational polyhedral cone $\tau \subset C^*(\mathcal{X}^0, P_1)$ such that

$$\pi(\Gamma \cdot \tau) = C^*(\mathcal{H}^0, \bar{P}_1).$$

Let $\Gamma_{\rho(V)} := \rho(V)(\mathbb{Q}) \cap \Gamma$, this is a lattice in $\rho(V)(\mathbb{Q})$. Thus there exists a compact subset $K \subset \rho(V)(\mathbb{R})$ such that $\Gamma_{\rho(V)} \cdot K = \rho(V)(\mathbb{Q})$. The first assertion of 4.15 (c) implies that $\rho(V)(\mathbb{Q})$ acts transitively on the fibres of the projection $\pi : C^*(\mathcal{X}^0, P_1) \rightarrow C^*(\mathcal{H}^0, \bar{P}_1)$, so it follows

$$\Gamma \cdot K \cdot \tau = C^*(\mathcal{X}^0, P_1).$$

Thus it suffices to show that $K \cdot \tau$ is contained in some convex rational polyhedral cone in $C^*(\mathcal{X}^0, P_1)$. Note that $\rho(V)$ acts trivially on $V \cap U_1$ and $\bar{U}_1 = U_1/(V \cap U_1)$. Thus the action of $\rho(V)$ on U_1 is given by $v \cdot u = u + B(v, \pi(u))$ for some bilinear map $B : \rho(V) \times \bar{U}_1 \rightarrow V \cap U_1$. Choose a finite set of elements $v_i \in \rho(V)(\mathbb{Q})$ whose convex closure contains K . Let $u_j \in U_1(\mathbb{Q})(-1)$ be finitely many elements that generate τ as a convex cone. Then every element of $K \cdot \tau$ is of the form $(\sum_i \alpha_i \cdot v_i) \cdot (\sum_j \beta_j \cdot u_j)$ with $\alpha_i, \beta_j \in \mathbb{R}^{\geq 0}$ such that $\sum_i \alpha_i = 1$. We calculate

$$\begin{aligned} \left(\sum_i \alpha_i \cdot v_i\right) \cdot \left(\sum_j \beta_j \cdot u_j\right) &= \sum_j \beta_j \cdot \left(\sum_i \alpha_i \cdot v_i\right) \cdot u_j \\ &= \sum_j \beta_j \cdot (u_j + B(\sum_i \alpha_i \cdot v_i, \pi(u_j))) \\ &= \sum_j \beta_j \cdot \sum_i \alpha_i \cdot (u_j + B(v_i, \pi(u_j))) \\ &= \sum_i \sum_j \alpha_i \cdot \beta_j \cdot v_i \cdot u_j. \end{aligned}$$

This shows that $K \cdot \tau$ is contained in the convex rational polyhedral cone generated by the elements $v_i \cdot u_j \in U_1(\mathbb{Q})(-1) \cap C^*(\mathcal{X}^0, P_1)$, as desired.

For the proof of (a) we first reduce to a special case. Note that for an embedding $(P, \mathcal{X}) \hookrightarrow (P', \mathcal{X}')$, Γ embeds into the corresponding group for $(P', \mathcal{X}')$, so it suffices to prove the assertion for $(P', \mathcal{X}')$. Choose an embedding

$$(P, \mathcal{X}) \hookrightarrow (G, \mathcal{H}) \times (V_{2,g} \rtimes \mathrm{CSp}_{2g, \mathbb{Q}}, \mathcal{H}'_{2g})$$

as in the proof of 2.26 (b), where $(V_{2,g} \rtimes \mathrm{CSp}_{2g, \mathbb{Q}}, \mathcal{H}'_{2g})$ is the mixed Shimura data defined in 2.25. Since we already know the assertion for $(G, \mathcal{H})$, it remains to treat the case

$(P, \mathcal{X}) = (V_{2,g} \rtimes \mathrm{CSp}_{2g, \mathbb{Q}}, \mathcal{H}'_{2g})$. But this is isomorphic to $(P_{2g}, \mathcal{X}_{2g})/U_{2g}$, so by the equivalence above we are reduced to the case $(P, \mathcal{X}) = (P_{2g}, \mathcal{X}_{2g})$. By 4.25 this is a rational boundary component of $(P', \mathcal{X}') := (\mathrm{CSp}_{2(g+1)}, \mathcal{H}_{2(g+1)})$, it is this fact that we shall exploit. By 4.20 $(P_1, \mathcal{X}_1)$ is also a rational boundary component of $(P', \mathcal{X}')$.

Since $C(\mathcal{X}'^0, P)$ is a half-line in $U(\mathbb{R})(-1)$, it is of the form $\mathbb{R}^{>0} \cdot u_0$ for some $u \in C(\mathcal{X}'^0, P)$. By 4.21 (a) we have $C(\mathcal{X}^0, P_1) = C(\mathcal{X}'^0, P_1) + U(\mathbb{R})(-1)$, since u_0 lies in the closure of $C(\mathcal{X}'^0, P_1)$, it follows that $C(\mathcal{X}^0, P_1) = C(\mathcal{X}'^0, P_1) + \mathbb{R}^{\leq 0} \cdot u_0$. Likewise by 4.22 (b) we have $C^*(\mathcal{X}^0, P_1) = C^*(\mathcal{X}'^0, P_1) + \mathbb{R}^{\leq 0} \cdot u_0$. Thus any convex rational polyhedral cone $\sigma \subset C^*(\mathcal{X}^0, P_1)$ is contained in the convex cone generated by $\pm u_0$ and finitely many $u_\alpha \in C^*(\mathcal{X}'^0, P_1) \cap U(\mathbb{R})(-1)$. Let σ' be the convex rational polyhedral cone generated by u_0 and these u_α , then we have $\sigma' \subset C^*(\mathcal{X}'^0, P_1)$ and $\sigma = \sigma' + \mathbb{R}^{\leq 0} \cdot u_0$. Thus to prove (a) it suffices to consider $\sigma = \sigma' + \mathbb{R}^{\leq 0} \cdot u_0$ and $\tau = \tau' + \mathbb{R}^{\leq 0} \cdot u_0$ for two convex rational polyhedral cones $\sigma', \tau' \subset C^*(\mathcal{X}'^0, P_1)$.

Now let $\gamma \in \Gamma$ such that $\gamma\sigma \cap \tau \cap C(\mathcal{X}^0, P_1) \neq \emptyset$. Then there exists λ, μ and $\nu \in \mathbb{R}^{\leq 0}$ such that

$$\gamma(\sigma' + (-\lambda) \cdot u_0) \cap (\tau' + (-\mu) \cdot u_0) \cap C(\mathcal{X}'^0, P_1) + (-\nu) \cdot u_0 \neq \emptyset.$$

Replacing λ, μ and ν by $\lambda' := \max\{\lambda, \mu, \nu\}$ implies that

$$(\gamma\sigma' \cap \tau' \cap C(\mathcal{X}'^0, P_1)) + (-\lambda') \cdot u_0 \neq \emptyset,$$

hence $\gamma\sigma' \cap \tau' \cap C(\mathcal{X}'^0, P_1) \neq \emptyset$. Let Q' be the normalizer of P_1 in P' , this is the parabolic subgroup of P' associated to $(P_1, \mathcal{X}_1)$ as boundary component of $(P', \mathcal{X}')$, and it contains Q . Thus Γ is contained in the corresponding group for $(P_1, \mathcal{X}_1)$ relative to $(P', \mathcal{X}')$. Thus the assertion follows from the reductive case. **q.e.d.**

6.20. Proposition: Δ_1 acts properly discontinuously on $\overline{U}(P_1, \mathcal{X}_1, p_f)$.

Remark. In fact we shall prove that modulo an arithmetic subgroup of $Z(P)(\mathbb{Q})$ the action is properly discontinuous in the usual sense.

Proof. (Compare [AMRT] ch.III p.272 prop. 2 and p.157 thm. (i). Note that Σ need not be finite.) Let $p_{1,f} \in P_1(\mathbb{A}_f)$, $\mathcal{X}_1^0$ a connected component of $\mathcal{X}_1$ and $\mathcal{X}^0$ the corresponding connected component of $\mathcal{X}$. The stabilizer of $\mathcal{X}_1^0 \times p_{1,f} \cdot p_f \cdot K_f$ in $Q(\mathbb{Q})$ is

$$\Gamma_Q := \mathrm{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^0) \cap (p_{1,f} \cdot p_f) \cdot K_f \cdot (p_{1,f} \cdot p_f)^{-1}.$$

Without loss of generality we may replace K_f by a subgroup of finite index, so we may assume that Γ_Q is neat. Let $\Gamma_1 := P_1(\mathbb{Q}) \cap \Gamma_Q$ and Γ_U the image of $(\{z \in Z(P)(\mathbb{Q}) \mid z|_{\mathcal{X}} = \mathrm{id}\} \times U_1(\mathbb{Q})) \cap \Gamma_Q$ in $U_1(\mathbb{Q})$. By neatness Γ_1 acts trivially on the torus $U_1(\mathbb{R})(-1)/\Gamma_U$, and the corresponding connected component of $M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1^0)(\mathbb{C})$ is the torus embedding $\Gamma_1 \backslash \mathcal{X}_1^0 \hookrightarrow (\Gamma_1 \backslash \mathcal{X}_1^0)_{\Sigma_1^0(\mathcal{X}_1^0, P_1, p_{1,f})}$ with respect to the torus $U_1(\mathbb{C})/\Gamma_U$. The corresponding connected component of $\overline{U}(P_1, \mathcal{X}_1, p_f)$ is the interior of the closure of $\Gamma_1 \backslash \mathcal{X}^0 = \mathrm{ord}^{-1}(C(\mathcal{X}^0, P_1))$, where as in 6.13 the map $\mathrm{ord} : \Gamma_1 \backslash \mathcal{X}_1^0 \rightarrow U_1(\mathbb{R})(-1)$ is induced from the map “imaginary part”. Since $\Gamma_Z := Z(P)(\mathbb{Q}) \cap \Gamma_Q$ acts trivially on $\mathcal{X}^0$, the action of Γ_Q on $(\Gamma_1 \backslash \mathcal{X}_1^0)_{\Sigma_1^0(\mathcal{X}_1^0, P_1, p_{1,f})}$ factors through $\Gamma_Q/\Gamma_Z \cdot \Gamma_1$. We shall show that this is a properly discontinuous action in the usual sense.

Let $\sigma \in \Sigma_1^0(\mathcal{X}_1^0, P_1, p_{1,f})$ and u in the σ -stratum of $\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$. Let $\lambda \in C(\mathcal{X}^0, P_1)$ such that $u \in \pi_\sigma(\text{ord}^{-1}(\lambda))$. There exists a convex rational polyhedral cone $\tau \subset C^*(\mathcal{X}^0, P_1)$ such that τ^0 is a neighborhood of λ . After replacing τ by $\sigma + \tau$ we have $\tau^0 = \tau^0 + \sigma$, hence by 5.9 (a)

$$U := \coprod_{\rho \text{ face of } \sigma} \pi_\rho(\text{ord}^{-1}(\tau^0))$$

is an open neighborhood of u in $\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$. Likewise consider $\sigma' \in \Sigma_1^0(\mathcal{X}_1^0, P_1, p_{1,f})$, u' in the σ' -stratum of $\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$, and let τ' be a convex rational polyhedral cone in $C^*(\mathcal{X}^0, P_1)$ such that

$$U' := \coprod_{\rho' \text{ face of } \sigma'} \pi_{\rho'}(\text{ord}^{-1}(\tau'^0))$$

is an open neighborhood of u' in $\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$. To prove the desired assertion it suffices to show that there exist at most finitely many $[\gamma] \in \Gamma_Q/\Gamma_Z \cdot \Gamma_1$ such that $\gamma U \cap U' \neq \emptyset$.

If $\gamma U \cap U' \neq \emptyset$, there exist faces ρ of σ and ρ' of σ' such that

$$\gamma(\pi_\rho(\text{ord}^{-1}(\tau^0))) \cap \pi_{\rho'}(\text{ord}^{-1}(\tau'^0)) \neq \emptyset.$$

But this implies that $\gamma \cdot \rho = \rho'$, hence by the Γ_Q -equivariance of ord

$$\pi_{\rho'}(\text{ord}^{-1}(\gamma \cdot \tau^0)) \cap \pi_{\rho'}(\text{ord}^{-1}(\tau'^0)) \neq \emptyset.$$

This is equivalent to $\text{ord}^{-1}(\gamma \cdot \tau^0) \cap T_{\rho'} \cdot \text{ord}^{-1}(\tau'^0) \neq \emptyset$, where $T_{\rho'}$ is the subtorus of $U_1(\mathbb{C})/\Gamma_U$ with cocharacter group $\Gamma_U \cap \mathbb{R} \cdot \rho'$. But $T_{\rho'} \cdot \text{ord}^{-1}(\tau'^0) = \text{ord}^{-1}(\tau'^0 + \mathbb{R} \cdot \rho')$, hence

$$(\gamma \cdot \tau^0) \cap (\tau'^0 + \mathbb{R} \cdot \rho') \neq \emptyset.$$

Since ρ' is a convex cone, every element of $\mathbb{R} \cdot \rho'$ can be written as a difference of two elements of ρ' . This implies

$$((\gamma \cdot \tau^0) + \rho') \cap (\tau'^0 + \rho') \neq \emptyset.$$

But ρ' is a face of σ' , and σ' is contained in τ' , so $\tau'^0 + \rho' = \tau'^0$. Likewise it follows that $(\gamma \cdot \tau^0) + \rho' = \gamma \cdot (\tau^0 + \rho) = \gamma \cdot \tau^0$, hence

$$\gamma \cdot \tau^0 \cap \tau'^0 \neq \emptyset.$$

Since τ^0 is contained in $C(\mathcal{X}^0, P_1)$, this implies

$$\gamma \cdot \tau \cap \tau' \cap C(\mathcal{X}^0, P_1) \neq \emptyset.$$

Thus by 6.19 (a) there are finitely many possibilities for $\rho(\gamma)$. But the kernel of $\rho|_\Gamma$ is commensurable with $\Gamma_Z \cdot \Gamma_1$, so there are only finitely many possibilities for $[\gamma] \in \Gamma_Q/\Gamma_Z \cdot \Gamma_1$, as desired. **q.e.d.**

6.21. Proposition: Let π be the projection $(P, \mathcal{X}) \rightarrow (P, \mathcal{X})/W$. Then the composite map $\mathcal{U} \rightarrow M^{K_f}(P, \mathcal{X})(\mathbb{C}) \rightarrow M^{\pi(K_f)}((P, \mathcal{X})/W)(\mathbb{C})$ extends uniquely to a holomorphic map $[\pi]^* : \overline{\mathcal{U}} \rightarrow M^{\pi(K_f)}((P, \mathcal{X})/W)^*(\mathbb{C})$. This map factors through $\overline{\mathcal{U}}/\sim$.

Proof. Since $\mathcal{U}$ is dense in $\overline{\mathcal{U}}$, the extension is unique if it exists. Moreover since $\overline{\mathcal{U}}$ is a normal complex space, if a continuous extension exists, then by Riemann's extension theorem (see [GR] ch.7 §4.2 p.144) this extension is already holomorphic. We first define an explicit extension.

For all $(P_1, \mathcal{X}_1)$ and p_f we define it on $\partial\mathcal{U}(P_1, \mathcal{X}_1, p_f) \subset \overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$ as the composite

$$\begin{aligned} \partial\mathcal{U}(P_1, \mathcal{X}_1, p_f) &\hookrightarrow M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C}) \xrightarrow{\mathbf{1}} \\ &\xrightarrow{\mathbf{1}} \Delta_1 \setminus M^{\pi_1(K_f^1)}((P, \mathcal{X})/W)(\mathbb{C}) \hookrightarrow M^{\pi(K_f)}((P, \mathcal{X})/W)^*(\mathbb{C}). \end{aligned}$$

Every point in $\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f) \setminus \partial\mathcal{U}(P_1, \mathcal{X}_1, p_f)$ is mapped to $\partial\mathcal{U}(P'_1, \mathcal{X}'_1, p_f)$ for a unique rational boundary component $(P'_1, \mathcal{X}'_1)$ between $(P, \mathcal{X})$ and $(P_1, \mathcal{X}_1)$ under the extension 6.15 of the map 6.11 (d). Thus there exists a unique way to extend this map to the whole of $\overline{\mathcal{U}}$ such that it is compatible with these maps. To show that this extended map factors through $\overline{\mathcal{U}}/\sim$ it thus remains to show that it is invariant under the extensions 6.15 of the maps 6.11 (a) to (c). This is an easy calculation, as in 6.11. Also it is easy to check that this map extends the given map $\mathcal{U} \rightarrow M^{K_f}(P, \mathcal{X})(\mathbb{C}) \rightarrow M^{\pi(K_f)}((P, \mathcal{X})/W)(\mathbb{C})$. To prove the assertion it remains to show that it is continuous.

Fix $(P_1, \mathcal{X}_1)$, p_f a point $u \in \overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$, and a neighborhood $\mathcal{V} \subset M^{\pi(K_f)}((P, \mathcal{X})/W)^*(\mathbb{C})$ of the image of u , then we have to show that the inverse image of $\mathcal{V}$ in $\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$ is a neighborhood of u . If $u \notin \partial\mathcal{U}(P_1, \mathcal{X}_1, p_f)$, then the map factors through $\overline{\mathcal{U}}(P'_1, \mathcal{X}'_1, p_f)$ for a boundary component $(P'_1, \mathcal{X}'_1)$ between $(P, \mathcal{X})$ and $(P_1, \mathcal{X}_1)$, and different from $(P_1, \mathcal{X}_1)$, so the assertion follows by induction over $(P_1, \mathcal{X}_1)$. Thus we may suppose $u \in \partial\mathcal{U}(P_1, \mathcal{X}_1, p_f)$. Let $(x_1, p_{1,f}) \in (\mathcal{X}_1/W_1) \times P_1(\mathbb{A}_f)$ be a representative for $\pi_1(u) \in M^{\pi_1(K_f^1)}((P_1, \mathcal{X}_1)/W_1)(\mathbb{C})$, then by the definition 6.2 of the topology on $(\mathcal{X}/W)^*$ we may take $\mathcal{V}$ to be the image of

$$\left(\coprod \psi_2(\text{im}^{-1}(D) \cap \psi_1^{-1}(\mathcal{Y})) \right) \times \{p_{1,f} \cdot p_f\}$$

in $M^{\pi(K_f)}((P, \mathcal{X})/W)^*(\mathbb{C})$, where D is an open core in $C(\mathcal{X}^0, P_1)$, $\mathcal{Y}$ a neighborhood of x_1 in $\mathcal{X}_1/W_1$, and the disjoint union is extended over all rational boundary components $(P_2, \mathcal{X}_2)$ between $(P_1, \mathcal{X}_1)$ and $(P, \mathcal{X})$.

The connected component of $\mathcal{U}(P_1, \mathcal{X}_1, p_f)$ that contains u is the torus embedding of $(\Gamma_1 \setminus \mathcal{X}^0) \times \{p_{1,f}\}$ with respect to the cone decomposition $\Sigma(\mathcal{X}^0, P_1, p_{1,f} \cdot p_f)$. Let $\mathcal{W}$ denote the image of $(\text{im}^{-1}(D) \cap \psi_1^{-1}(\mathcal{Y})) \times \{p_{1,f} \cdot p_f\}$ in $\mathcal{U}(P_1, \mathcal{X}_1, p_f)$, and π_σ the canonical projection to the σ -stratum in the torus embedding, for each $\sigma \in \Sigma(\mathcal{X}^0, P_1, p_{1,f} \cdot p_f)$. Then the inverse image of $\mathcal{V}$ certainly contains the union of all $\pi_\sigma(\mathcal{W})$. By assumption u lies in the τ -stratum for some $\tau \in \Sigma(\mathcal{X}^0, P_1, p_{1,f} \cdot p_f)$ such that $\tau^0 \subset C(\mathcal{X}^0, P_1)$. We also have $\psi_1^{-1}(\mathcal{Y}) = \pi_\tau^{-1}(\mathcal{Y}')$ for a neighborhood $\mathcal{Y}'$ of u in the τ -stratum. Moreover in terms of the torus embedding the map $\text{im} : \mathcal{X}_1^0 \rightarrow U_1(\mathbb{R})(-1)$ is a map ord as in 5.8, and by 6.1 we may without loss of generality assume that D is open and equal to $D + C(\mathcal{X}^0, P_1)$, hence in particular to $D + \tau$. Thus the inverse image of $\mathcal{V}$ contains the set $\coprod \pi_\sigma(\text{ord}^{-1}(D + \tau) \cap \pi_\tau^{-1}(\mathcal{Y}'))$, which by 5.9 (a) is a neighborhood of u . **q.e.d.**

6.22. Proposition Fix $(P_1, \mathcal{X}_1)$, p_f , and a relatively compact subset $\mathcal{Z}_1 \subset \Delta_1 \setminus M^{\pi_1(K_f^1)}((P_1, \mathcal{X}_1)/W_1)(\mathbb{C})$. Consider the morphism $[\pi]^* : \overline{\mathcal{U}} \rightarrow M^{\pi(K_f)}((P, \mathcal{X})/W)^*(\mathbb{C})$ of 6.21 and the projection $[\pi_1] :$

$\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f) \rightarrow \Delta_1 \backslash M^{\pi_1(K_f^1)}((P_1, \mathcal{X}_1)/W_1)(\mathbb{C})$. There exists a neighborhood $\mathcal{V}_1$ of $\mathcal{Z}_1$ in $M^{\pi(K_f)}((P, \mathcal{X})/W)^*(\mathbb{C})$, and a Δ_1 -invariant neighborhood $\mathcal{W}_1$ of $\partial\mathcal{U}(P_1, \mathcal{X}_1, p_f) \cap [\pi_1]^{-1}(\mathcal{Z}_1)$ in $[\pi]^*{}^{-1}(\mathcal{V}_1) \cap \overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$ such that

- (a) Every element of $[\pi]^*{}^{-1}(\mathcal{V}_1) \subset \overline{\mathcal{U}}$ is under $\sim$ equivalent to an element of $\mathcal{W}_1$.
- (b) Any two elements of $\mathcal{W}_1$ are equivalent under $\sim$ if and only they are equivalent under the action of Δ_1 .

Proof. There exist a connected component $(\mathcal{X}_1/W_1)^0$ of $\mathcal{X}_1/W_1$, a relatively compact open subset $\mathcal{Y}_1 \subset (\mathcal{X}_1/W_1)^0$, and a finite subset $\{p_{1,i,f}\} \subset P_1(\mathbb{A}_f)$ such that $\mathcal{Z}_1$ lies in the image of $\mathcal{Y}_1 \times \{p_{1,i,f}\}$ in $\Delta_1 \backslash M^{\pi_1(K_f^1)}((P_1, \mathcal{X}_1)/W_1)(\mathbb{C})$. Consider the subset $\mathcal{E}_1 := \text{im}^{-1}(D) \cap \psi_1^{-1}(\mathcal{Y}_1)$ of $\mathcal{X}$ for some arbitrarily small open core $D \subset C(\mathcal{X}^0, P_1)$. Let $\mathcal{V}_1$ be the image of $(\coprod \psi_2(\mathcal{E}_1)) \times \{p_{1,i,f} \cdot p_f\}$ in $M^{\pi(K_f)}((P, \mathcal{X})/W)^*(\mathbb{C})$, by the definition of the Satake-topology this is a neighborhood of $\mathcal{Z}_1$. Also let $\mathcal{F}_1$ be the image of $\mathcal{E}_1 \times \{p_{1,i,f}\} \subset \mathcal{X}^0 \times P_1(\mathbb{A}_f)$ in $\mathcal{U}(P_1, \mathcal{X}_1, p_f)$ and $\mathcal{W}_1 := \coprod \pi_\sigma(\mathcal{F}_1)$, where the union is extended over all cones $\sigma \in \Sigma(\mathcal{X}^0, P_1, p_{1,i,f} \cdot p_f)$ for some i . By 5.9 (a) this is a neighborhood of $\partial\mathcal{U}(P_1, \mathcal{X}_1, p_f) \cap [\pi_1]^{-1}(\mathcal{Z}_1)$ in $[\pi]^*{}^{-1}(\mathcal{V}_1) \cap \overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$, and it is clearly Δ_1 -invariant.

Consider an element of $\overline{\mathcal{U}}$ in the inverse image of $\mathcal{V}_1$. By the explicit description of the map $[\pi]^*$ in the proof of 6.21 it is equivalent to an element of $\partial\mathcal{U}(P'_1, \mathcal{X}'_1, p'_f)$ for some $(P'_1, \mathcal{X}'_1)$ and p'_f . Let $[(x'_1, p'_{1,f})]$ be its image in $M^{\pi(K_f)}((P, \mathcal{X})/W)^*(\mathbb{C})$, for some $x'_1 \in \mathcal{X}'_1/W'_1$ and $p'_{1,f} \in P'_1(\mathbb{A}_f)$. Then by the definition of the Baily-Borel compactification there exist $p \in P(\mathbb{Q})$ and $k_f \in K_f$ such that $(P_2, \mathcal{X}_2) := (p \cdot P'_1 \cdot p^{-1}, p \cdot \mathcal{X}'_1)$ is between $(P, \mathcal{X})$ and $(P_1, \mathcal{X}_1)$, that $p \cdot x'_1 \in \psi_2(\mathcal{E}_1)$, and that $p \cdot p'_{1,f} \cdot p'_f \cdot k_f = p_{1,i,f} \cdot p_f$ for some $p_{1,i,f}$ in the above finite set. After applying in order the extensions 6.15 of the maps 6.11 (a) for k_f (c) for p , and (b) for $p \cdot (p'_{1,f})^{-1} \cdot p^{-1} \cdot p_{1,i,f}$ we are left with an element of $\partial\mathcal{U}(P_2, \mathcal{X}_2, p_f)$, whose image in $\Delta_2 \backslash M^{\pi_2(K_f^2)}((P_2, \mathcal{X}_2)/W_2)^*(\mathbb{C})$ is represented by $(x_2, p_{1,i,f})$ for some $x_2 \in \psi_2(\mathcal{E}_1)$. It lies in the σ -stratum for some $\sigma \in \Sigma(\mathcal{X}^0, P_2, p_{1,i,f} \cdot p_f) \subset \Sigma(\mathcal{X}^0, P_1, p_{1,i,f} \cdot p_f)$, so under the extension 6.15 of the map 6.11 (d) it is equivalent to an element of $\pi_\sigma(\mathcal{F}_1)$, hence of $\mathcal{W}_1$. This proves (a).

For (b) consider two elements $u, u' \in \mathcal{W}_1$ such that $u \sim u'$. Write $u = \pi_\sigma([(x_1, p_{1,i,f})])$ for some $x \in \mathcal{E}_1$ and for some i , and likewise $u' = \pi_{\sigma'}([(x'_1, p_{1,i',f})])$. Then we have the diagram of 6.16 with the additional relation $p \cdot p_{1,i,f} \cdot p_{2,f}^{-1} \cdot p^{-1} = p_{1,i',f}$. If p stabilizes $(P_1, \mathcal{X}_1)$, then this equation implies $p_{2,f} \in P_1(\mathbb{A}_f)$, so the three lower maps in the diagram of 6.16 all lift to corresponding maps of the $\overline{\mathcal{U}}(P_1, \mathcal{X}_1, \dots)$. Thus by 6.18 u and u' are equivalent under Δ_1 , as desired. It remains to show that p stabilizes $(P_1, \mathcal{X}_1)$, if D was chosen sufficiently small.

For this observe that the equation $p \cdot p_{1,i,f} \cdot p_{2,f}^{-1} \cdot p^{-1} = p_{1,i',f}$ together with that in 6.16 (iii) implies

$$p \in ((p_{1,i',f} \cdot p_f) \cdot K_f \cdot (p_{1,i',f} \cdot p_f)^{-1}) \cdot (p_{1,i',f} \cdot (p_{1,i,f})^{-1}).$$

Here the first factor is an open compact subgroup of $P(\mathbb{A}_f)$, while the second factor runs through a finite set. Since p is an element of $P(\mathbb{Q})$, it lies in a finite union of cosets under an arithmetic subgroup Γ of $P(\mathbb{Q})$. Thus for any fixed $p_0 \in P(\mathbb{Q})$ we have to know that, if D is sufficiently small, then any $p \in \Gamma \cdot p_0$ with $(\coprod \psi_2(\mathcal{E}_1)) \cap p \cdot (\coprod \psi_2(\mathcal{E}_1)) \neq \emptyset$ must stabilize $(P_1, \mathcal{X}_1)$. But this property of the Satake-topology is a direct consequence of [AMRT] ch.III §6.1 thm. 1(iv). q.e.d.

6.23. Corollary: The graph of the equivalence relation $\sim$ on $\overline{\mathcal{U}}$ is closed. In fact it is the closure of the graph of $\sim$ on $\mathcal{U}$.

Proof. The second assertion follows from the first and the remark in 6.16. Consider two points $u, u' \in \overline{\mathcal{U}}$ such that for all neighborhoods $\mathcal{W}$ of u and $\mathcal{W}'$ of u' there exist elements $w \in \mathcal{W}$ and $w' \in \mathcal{W}'$ with $w \sim w'$. Then we have to show that $u \sim u'$. Now by assumption and by 6.21 all neighborhoods of $[\pi]^*(u)$ and of $[\pi]^*(u')$ in $M^{\pi(K_f)}((P, \mathcal{X})/W)^*(\mathbb{C})$ have a common point. Since $M^{\pi(K_f)}((P, \mathcal{X})/W)^*(\mathbb{C})$ is Hausdorff, this implies $[\pi]^*(u) = [\pi]^*(u')$. Choose $\mathcal{V}_1$ and $\mathcal{W}_1$ as in 6.22 for $\mathcal{Z}_1 = \{[\pi]^*(u)\}$. Then by 6.22 (a) both u and u' are equivalent to a point in $\mathcal{W}_1$. If for each of these points we fix the parameters in the diagram of 6.16, then we see that every sufficiently small neighborhood of u , resp. of u' , corresponds to a neighborhood of its (fixed) conjugate in $\mathcal{W}_1$. Thus we are reduced to the case $u, u' \in \mathcal{W}_1$. Then 6.20 implies that u and u' are already conjugate under Δ_1 , hence under $\sim$, as desired. **q.e.d.**

6.24. Definition: Let $(P, \mathcal{X})$ be mixed Shimura data, K_f an open compact subgroup of $P(\mathbb{A}_f)$, and Σ a K_f -admissible partial cone decomposition for $(P, \mathcal{X})$. Let $\overline{\mathcal{U}}$ be as in 6.14, and $\sim$ the equivalence relation on $\overline{\mathcal{U}}$ defined in 6.16. We define the toroidal compactification of $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ with respect to Σ as

$$M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}) := \overline{\mathcal{U}} / \sim.$$

Endowed with the quotient topology, it is a Hausdorff topological space, since by 6.23 the graph of $\sim$ is closed. By 6.22 it is locally isomorphic to certain open subsets $\Delta_1 \setminus \mathcal{W}_1$ of $\Delta_1 \setminus \overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$ for various $(P_1, \mathcal{X}_1)$ and p_f , which by 6.20 is itself locally isomorphic to the quotient of a normal complex space by a finite group that acts analytically. Thus it is a normal complex space. It contains $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ as an open dense subset. If Σ is concentrated in the unipotent fibre, then it is canonically isomorphic to the relative torus embedding defined in 6.6. By 6.21 the projection $[\pi] : M^{K_f}(P, \mathcal{X})(\mathbb{C}) \rightarrow M^{\pi(K_f)}((P, \mathcal{X})/W)(\mathbb{C})$ extends uniquely to a holomorphic map $[\pi]^* : M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}) \rightarrow M^{\pi(K_f)}((P, \mathcal{X})/W)^*(\mathbb{C})$, that is described explicitly in the proof of 6.21.

Actually the term “compactification” is somewhat misused, since in general $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ will not be compact. Nevertheless we keep this term, since the compactness of $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$.

6.25. Proposition: The morphisms of 3.4 extend to arbitrary torus embeddings in the following cases (compare 6.7). More precisely:

(a) Let $(P, \mathcal{X})$ be mixed Shimura data, $p_f \in P(\mathbb{A}_f)$, and $K_f, K'_f \subset P(\mathbb{A}_f)$ open compact subgroups, such that $K'_f \subset p_f \cdot K_f \cdot p_f^{-1}$. Let Σ be a K_f -admissible and Σ' a K'_f -admissible partial cone decomposition for $(P, \mathcal{X})$. Suppose that for every $\sigma' \in \Sigma'$ there exists $\tau \in [\cdot p_f]^* \Sigma$ such that $\sigma' \subset \tau$. Then the map of 3.4 (a) extends uniquely to a holomorphic map

$$[\cdot p_f] = [\cdot p_f]_{K'_f, K_f} : M^{K'_f}(P, \mathcal{X}, \Sigma')(\mathbb{C}) \rightarrow M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}).$$

It is an isomorphism if $K'_f = p_f \cdot K_f \cdot p_f^{-1}$ and $\Sigma' = [\cdot p_f]^* \Sigma$. If $K'_f = K_f$ and $\Sigma' \subset \Sigma$, then it is an open embedding. In that case the complement of the image is a closed analytic subset. If $\Sigma' = \Sigma$, $K'_f = K_f$ and $p_f \in K_f$, then $[\cdot p_f]$ is the identity map. If $\Sigma' = \Sigma$ and K'_f

is a normal subgroup of K_f , then the map $[\cdot 1]_{K'_f, K_f}$ identifies $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ with the quotient of $M^{K'_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ by the finite group K'_f/K_f acting from the right hand side through the morphisms $[\cdot k_f]$ on $M^{K'_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$.

(b) Consider a morphism $\phi : (P_1, \mathcal{X}_1) \rightarrow (P_2, \mathcal{X}_2)$ and open compact subgroups $K_f^1 \subset P_1(\mathbb{A}_f)$ and $K_f^2 \subset P_2(\mathbb{A}_f)$ such that $\phi(K_f^1) \subset K_f^2$. Let Σ_1 be a K_f^1 -admissible partial cone decomposition for $(P_1, \mathcal{X}_1)$ and Σ_2 a K_f^2 -admissible partial cone decomposition for $(P_2, \mathcal{X}_2)$. Suppose that for every $\sigma_1 \in \Sigma_1$ there exists $\sigma_2 \in \Sigma_2$ such that $\phi(\sigma_1) \subset \sigma_2$. Then the map of 3.4 (b) extends to a holomorphic map

$$[\phi] = [\phi]_{K_f^1, K_f^2} : M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1)(\mathbb{C}) \rightarrow M^{K_f^2}(P_2, \mathcal{X}_2, \Sigma_2)(\mathbb{C}).$$

If $\phi : (P, \mathcal{X}) \rightarrow (P, \mathcal{X})$ is an automorphism, $\phi(K_f^1) = K_f^2$ and $\Sigma_1 = \phi^* \Sigma_2$, then $[\phi]$ is an isomorphism.

Outline of proof: To extend the respective maps to the toroidal compactification, one constructs an obvious extension of the corresponding map on the level of $\overline{\mathcal{U}}$. In case (b), one uses the functoriality of boundary components (4.16). Then one shows that this extension is compatible with the equivalence relation of 6.16. This is just a matter of going through the definitions, so the details are left to the reader. Most of the properties of these maps carry over directly to their extensions. The rest is again easily checked on the level of $\overline{\mathcal{U}}$.

Let us indicate one of the more delicate points. It concerns the last statement of (a). There we have $\Sigma' = \Sigma$ and K'_f is a normal subgroup of K_f . Let $\overline{\mathcal{U}}'$ be the analogue of $\overline{\mathcal{U}}$ for K'_f in place of K_f , and consider two points $u_1, u_2 \in \overline{\mathcal{U}}'$ with, say, $u_1 \in \overline{\mathcal{U}}'(P_1, \mathcal{X}_1, p_f)$. Then the crucial step is to verify that the image of u_1 and u_2 in $\overline{\mathcal{U}}$ are equivalent under $\sim$ if and only if there exists $k_f \in K_f$ such that $u_2 \sim [\text{id}](u_1)$, where $[\text{id}]$ is the isomorphism $\overline{\mathcal{U}}'(P_1, \mathcal{X}_1, p_f) \rightarrow \overline{\mathcal{U}}'(P_1, \mathcal{X}_1, p_f \cdot k_f)$ of 6.11 (a), resp. 6.15. But that is clear from the definition of $\sim$. **q.e.d.**

6.26 Proposition: Suppose that K_f is neat and Σ is smooth with respect to K_f . Then $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is smooth.

Proof. Since $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is locally isomorphic to open subsets of $\Delta_1 \backslash \overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$ for various $(P_1, \mathcal{X}_1)$ and p_f , it suffices to show that every such quotient is smooth. By the proof of 6.20 every connected component of this quotient is an open subset of $\Gamma_Q \backslash (\Gamma_1 \backslash \mathcal{X}_1^0)_{\Sigma_1^0(\mathcal{X}_1^0, P_1, p_{1,f})}$, where $\mathcal{X}_1^0$ is a connected component of $\mathcal{X}_1$, $p_{1,f} \in P_1(\mathbb{A}_f)$, $\Gamma_Q = \text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}_1^0) \cap p_{1,f} \cdot p_f \cdot K_f \cdot p_f^{-1}$, $\Gamma_1 = P_1(\mathbb{Q}) \cap \Gamma_Q$, and $\Sigma_1^0 = ([\cdot p_f]^* \Sigma)_{|(P_1, \mathcal{X}_1)}^0$. Since Γ_Q is neat, its image in $(Q/(Z(P) \cdot P_1))(\mathbb{Q})$ is torsion free. Define $\Gamma_{ZP_1} := (Z(P) \cdot P_1)(\mathbb{Q}) \cap \Gamma_Q$, then by 6.20 Γ_Q/Γ_{ZP_1} acts freely on our connected component, so it suffices to show that the quotient $\Gamma_{ZP_1} \backslash (\Gamma_1 \backslash \mathcal{X}_1^0)_{\Sigma_1^0(\mathcal{X}_1^0, P_1, p_{1,f})}$ is smooth.

Now $(\Gamma_1 \backslash \mathcal{X}_1^0)_{\Sigma_1^0(\mathcal{X}_1^0, P_1, p_{1,f})}$ is a relative torus embedding over $\Gamma_1 \backslash \mathcal{X}_1^0/U_1(\mathbb{C})$, so we have the map

$$\Gamma_{ZP_1} \backslash (\Gamma_1 \backslash \mathcal{X}_1^0)_{\Sigma_1^0(\mathcal{X}_1^0, P_1, p_{1,f})} \rightarrow \Gamma_{ZP_1} \backslash \mathcal{X}_1^0/U_1(\mathbb{C}).$$

Let Γ_0 be the subgroup of all elements $\gamma \in \Gamma_{ZP_1}$ which operate trivially on $\mathcal{X}_1^0/U_1(\mathbb{C})$. Clearly it maps to the center $Z(P_1/U_1)(\mathbb{Q})$, but since Γ_{ZP_1} is neat and $Z(P_1/U_1)$ is generated by the image of $Z(P)$ and a torus that is an extension of a $\mathbb{Q}$ -split torus with a torus of compact type, it must be contained in $(Z(P) \cdot U_1)(\mathbb{Q})$. Now 2.11 implies that any element $z \in Z(P)(\mathbb{R})$, which fixes one connected component of $\mathcal{X}$, must operate trivially on $\mathcal{X}$. This shows the equalities

$$\Gamma_0 = (Z(P) \cdot U_1)(\mathbb{Q}) \cap \Gamma_{ZP_1} = (\{z \in Z(P)(\mathbb{Q}) \mid z|_{\mathcal{X}} = \text{id}\} \times U_1(\mathbb{Q})) \cap \Gamma_{ZP_1}.$$

In particular Γ_{ZP_1}/Γ_0 is isomorphic to the image of Γ_{ZP_1} in $((Z(P) \cdot P_1)/(Z(P) \cdot U_1))(\mathbb{Q})$, which is again neat, hence torsion free. On the other hand Γ_{ZP_1} acts properly discontinuously on $\mathcal{X}_1^0$ in the usual sense, so Γ_{ZP_1}/Γ_0 acts freely on $\Gamma_0 \backslash \mathcal{X}_1^0/U_1(\mathbb{C})$. Thus it suffices to prove that $\Gamma_0 \backslash ((\Gamma_0 \cap \Gamma_1) \backslash \mathcal{X}_1^0)_{\Sigma_1^0(\mathcal{X}_1^0, P_1, p_{1,f})}$ is smooth.

If we let Γ_U be the image of Γ_0 under the projection $Z(P) \times U_1 \rightarrow U_1$, we may use 5.4 to write this as $(\Gamma_U \backslash \mathcal{X}_1^0)_{\Sigma_1^0(\mathcal{X}_1^0, P_1, p_{1,f})}$. This is a relative torus embedding over a smooth base with respect to the torus $U_1(\mathbb{C})/\Gamma_U$, so by 5.3 it suffices to show that $\Sigma_1^0(\mathcal{X}_1^0, P_1, p_{1,f})$ is smooth with respect to the lattice $\Gamma_U(-1)$ in $U_1(\mathbb{Q})(-1)$. Applying 6.5 we find that $\Sigma_1^0(\mathcal{X}_1^0, P_1, p_{1,f})$ is the same rational partial polyhedral decomposition of $C^*(\mathcal{X}^0, P_1) \subset C^*(\mathcal{X}_1^0, P_1)$ as $\Sigma(\mathcal{X}^0, P_1, p_{1,f} \cdot p_f)$. Also we get that Γ_U is the image of

$$(\{z \in Z(P)(\mathbb{Q}) \mid z|_{\mathcal{X}} = \text{id}\} \times U_1(\mathbb{Q})) \cap (p_{1,f} \cdot p_f) \cdot K_f \cdot (p_{1,f} \cdot p_f)^{-1}$$

under the projection $Z(P) \times U_1 \rightarrow U_1$. Thus the desired property is precisely that required in the definition 6.4 of the smoothness of Σ . **q.e.d.**

6.27. Proposition: Suppose that Σ is a complete K_f -admissible cone decomposition for $(P, \mathcal{X})$. Then $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is compact.

Proof. Since $M^{\pi(K_f)}((P, \mathcal{X})/W)^*(\mathbb{C})$ is compact, it suffices to show that for all sufficiently small $\mathcal{V}_1$ and $\mathcal{W}_1$ as in 6.22 the image of $\mathcal{W}_1$ is relatively compact in $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$. But this image is isomorphic to $\Delta_1 \backslash \mathcal{W}_1$, so it suffices to show that it is relatively compact in $\Delta_1 \backslash \overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$.

In 6.22 we may replace $\mathcal{W}_1$ by the closure of $\mathcal{W}_1 \cap \mathcal{U}(P_1, \mathcal{X}_1, p_f)$, hence it suffices to show that $\Delta_1 \backslash (\mathcal{W}_1 \cap \mathcal{U}(P_1, \mathcal{X}_1, p_f))$ is relatively compact. By the proof of 6.22 the intersection of $\mathcal{W}_1$ with a connected component of $\mathcal{U}(P_1, \mathcal{X}_1, p_f)$ is the image of $\text{im}^{-1}(D) \cap \psi_1^{-1}(\mathcal{Y}_1)$ in $\Gamma_1 \backslash \mathcal{X}^0$. There is an arithmetic subgroup Γ_Q of $Q(\mathbb{Q})$ such that the image of $\Gamma_1 \backslash \mathcal{X}^0$ in $\Delta_1 \backslash \mathcal{U}(P_1, \mathcal{X}_1, p_f)$ is isomorphic to $\Gamma_Q \backslash \mathcal{X}^0$. After replacing Γ_Q by a smaller subgroup, which does not change the desired assertion, we may assume that $\Gamma_Q = \Gamma_1 \cdot \Gamma'$, where $\Gamma' \subset \Gamma_Q$ centralizes P_1/W_1 . Then Γ' acts trivially on $\mathcal{X}_1^0/W_1$, hence it maps the subset $\psi_1^{-1}(\mathcal{Y}_1) \subset \mathcal{X}^0$ to itself. On the other hand by 6.19 (b) there exists a convex rational polyhedral cone $\tau \subset C^*(\mathcal{X}^0, P_1)$ such that $C^*(\mathcal{X}^0, P_1) = \Gamma' \cdot \tau$. Without loss of generality we may assume that D is a standard core, invariant under Γ' , then it follows that $D = \Gamma' \cdot (\tau \cap D)$. This implies that $\mathcal{W}_1$ is the union of all Γ' -conjugates of the image of $\text{im}^{-1}(\tau \cap D) \cap \psi_1^{-1}(\mathcal{Y}_1)$ in $\Gamma_1 \backslash \mathcal{X}^0$. Thus it suffices to show that this image is relatively compact in $\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$.

For this we may suppose that Γ_1 is neat, then the projections

$$(\Gamma_1 \backslash \mathcal{X}^0)_{\Sigma(\mathcal{X}^0, P_1, p_f)} \xrightarrow{[\pi']} \Gamma_1 \backslash \mathcal{X}_1^0/U_1(\mathbb{C}) \rightarrow \Gamma_1 \backslash (\mathcal{X}_1/W_1)^0,$$

are topologically locally trivial bundles. The fibres of the second projection are compact, so since $\mathcal{Y}_i$ is relatively compact, the image of our set in $\Gamma_1 \backslash \mathcal{X}_1^0 / U_1(\mathbb{C})$ is relatively compact. Thus it suffices to show the relative compactness in every fibre of the first projection. Since the map $\text{im} : \mathcal{X}_1^0 \rightarrow U_1(\mathbb{R})(-1)$ induces a map “ord” as in 5.8, we are reduced to the torus embedding $\Gamma_U \backslash U_1(\mathbb{C})_{\Sigma_1^0(\mathcal{X}_1^0, P_1, p_{1,f})}$, where we have to show that the subset $\text{ord}^{-1}(\tau \cap D)$ is relatively compact in the subset

$$\coprod_{\sigma \in \Sigma(\mathcal{X}^0, P_1, p_f)} \pi_\sigma(\text{ord}^{-1}(C(\mathcal{X}^0, P_1))).$$

Now we use the completeness of Σ . It says on the one hand that $\Sigma(\mathcal{X}^0, P_1, p_f)$ is a complete rational polyhedral decomposition of $C^*(\mathcal{X}^0, P_1)$, hence $\tau \cap C(\mathcal{X}^0, P_1)$ is covered by the $\tau \cap \sigma \cap C(\mathcal{X}^0, P_1)$ for all $\sigma \in \Sigma(\mathcal{X}^0, P_1, p_f)$. On the other hand the finiteness implies that there exist only finitely many such σ for which this set is non-empty. Indeed, it suffices to consider only those σ with $\sigma^0 \subset C(\mathcal{X}^0, P_1)$. By 6.4 (v) the double quotient $P(\mathbb{Q}) \backslash \Sigma / K_f$ is finite, which implies that there are only finitely many Γ' -orbits $[\sigma]$ in $\Sigma(\mathcal{X}^0, P_1, p_f)$ with $\sigma^0 \subset C(\mathcal{X}^0, P_1)$. Moreover for every representative σ , 6.19 (a) says that there exist only finitely many Γ' -conjugates $\gamma\sigma$ such that $\tau \cap \gamma\sigma \cap C(\mathcal{X}^0, P_1)$ is nonempty.

Thus altogether it follows that $\tau \cap C(\mathcal{X}^0, P_1)$ is contained in the union of finitely many $\sigma \in \Sigma(\mathcal{X}^0, P_1, p_f)$. Observe that it is enough to take only those σ with $\mathbb{R} \cdot \sigma = U_1(\mathbb{R})(-1)$, since by completeness every other cone in $\Sigma(\mathcal{X}^0, P_1, p_f)$ is a face of such a cone. So in particular $\tau \cap D$ is contained in the union of finitely many $\sigma \cap D$ for $\sigma \in \Sigma(\mathcal{X}^0, P_1, p_f)$ with $\mathbb{R} \cdot \sigma = U_1(\mathbb{R})(-1)$. Thus it is enough to prove that for every convex rational polyhedral cone $\sigma \subset C^*(\mathcal{X}^0, P_1)$ with $\mathbb{R} \cdot \sigma = U_1(\mathbb{R})(-1)$ the set $\text{ord}^{-1}(\sigma \cap D)$ is relatively compact in the subset

$$\coprod_{\rho \text{ face of } \sigma} \pi_\rho(\text{ord}^{-1}(C(\mathcal{X}^0, P_1)))$$

of the affine torus embedding $(\Gamma_U \backslash U_1(\mathbb{C}))_\sigma$.

By the following lemma 6.28 there exists a compact subset $V \subset C(\mathcal{X}^0, P_1)$ such that $\sigma \cap D \subset V + \sigma$. The assumption $\mathbb{R} \cdot \sigma = U_1(\mathbb{R})(-1)$ means that the σ -stratum is just a point, hence by 5.9 (b)

$$\coprod_{\rho \text{ face of } \sigma} \pi_\rho(\text{ord}^{-1}(V + \sigma))$$

is compact. This set contains $\text{im}^{-1}(\sigma \cap D)$, and is contained in $\coprod_{\rho \text{ face of } \sigma} \pi_\rho(\text{ord}^{-1}(C(\mathcal{X}^0, P_1)))$, as desired. **q.e.d.**

6.28. Lemma: Let $(P, \mathcal{X})$ be mixed Shimura data, $(P_1, \mathcal{X}_1)$ a rational boundary component and $\mathcal{X}^0$ a connected component of $\mathcal{X}$ which maps to $\mathcal{X}_1$. Let $\sigma \subset C^*(\mathcal{X}^0, P_1)$ be a convex rational polyhedral cone, and $D \subset C(\mathcal{X}^0, P_1)$ a core. Then $\sigma \cap D \subset V + \sigma$ for some compact subset $V \subset C(\mathcal{X}^0, P_1)$.

Proof. Without loss of generality we may assume that D is the standard core D_0 of 6.1 and that $\sigma^0 \subset C(\mathcal{X}^0, P_1)$. If $(P_1, \mathcal{X}_1)$ is an improper boundary component of $(P, \mathcal{X})$, then $D_0 = U(\mathbb{R})(-1)$, so the assumption holds with $V = \{0\}$. Otherwise σ is not completely contained in $C(\mathcal{X}^0, P_1)$, since for instance $0 \in \sigma$. Observe that if σ is the union of finitely

many other convex rational polyhedral cones, then it is enough to prove for each of those. Thus after splitting up σ in a suitable way we may assume that there exists a rational boundary component $(P_2, \mathcal{X}_2)$ between $(P_1, \mathcal{X}_1)$ and $(P, \mathcal{X})$, different from $(P_1, \mathcal{X}_1)$, such that $\sigma \subset C^*(\mathcal{X}^0, P_1) \coprod C(\mathcal{X}^0, P_1)$. Let $\rho := \sigma \cap C^*(\mathcal{X}^0, P_2)$, and choose finitely many $u_i \in \sigma \cap C(\mathcal{X}^0, P_1) \cap U_1(\mathbb{Q})(-1)$ which together with ρ generate σ as a convex cone. We may suppose that we have at least one u_i , since otherwise $\sigma = \rho$, so $\sigma \cap D = \sigma \cap C(\mathcal{X}^0, P_1) = \emptyset$ anyway.

Choose a linear form $\ell : U_1(\mathbb{Z})(-1) \rightarrow \mathbb{Z}$ which vanishes on $C^*(\mathcal{X}^0, P_2)$ and is strictly positive on $C(\mathcal{X}^0, P_1)$. Then the definition of D_0 implies $\ell(D_0) \subset \mathbb{R}^{\geq 1}$. After replacing the u_i by positive rational multiples we may assume that $\ell(u_i) = 1$. Let V be the convex closure of these u_i , this is a compact subset of $\{u \in C(\mathcal{X}^0, P_1) \mid \ell(u) = 1\}$. Now $\sigma = \rho + \mathbb{R}^{\geq 0} \cdot V$, so if an element $u + \lambda \cdot v \in \sigma$ for $u \in \rho$, $\lambda \in \mathbb{R}^{\geq 0}$, $v \in V$ lies in D_0 , then $\lambda = \ell(u + \lambda \cdot v) \geq 1$. Thus $\sigma \cap D_0$ is contained in $\rho + \mathbb{R}^{\geq 1} \cdot V = \rho + V + \mathbb{R}^{\geq 0} \cdot V = V + \sigma$, as desired. **q.e.d.**

Chapter 7

Stratification of toroidal compactification

In this chapter we study in detail the natural stratification of the toroidal compactification. In particular we describe the closure of a stratum as the toroidal compactification of a different mixed Shimura variety. If the stratum lies in the torus embedding along the unipotent fibre, then this can be done without extra restrictions. In the general case, however, the closure of a stratum may have self-intersections. To prohibit these we assume a certain condition 7.12 for K_f and the cone decomposition. Also, for simplicity and since we do not need the general case, we restrict attention to the case where Σ is complete, when dealing with an arbitrary stratum. Under these assumptions we shall even identify an open neighborhood of the closure of the stratum with an open neighborhood in a simpler toroidal compactification.

In 7.1–4 we describe the natural stratification of $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$, induced by its definition as a toroidal compactification, in terms of other mixed Shimura varieties. Proposition 7.5 allows to translate certain local questions at the boundary between the two types of compactification, which will be needed in 8.1. The rest of the chapter deals with the closure of a stratum. First we consider the Baily-Borel compactification (7.6). Then we explain (7.7) how a cone decomposition induces one for a boundary stratum, and in 7.8 we prove that the properties “finite” and “complete” are preserved under this operation. In 7.9 we describe the closure of a stratum that already lies in the torus embedding along the unipotent fibre. In 7.11 we explain the main construction in the general case. In 7.12 we state the condition on (K_f, Σ) , and in 7.13 we show it is usually not a big restriction. In 7.15–17 we prove the remaining assertions about the closure of an arbitrary stratum and an open neighborhood of this closure.

7.1. Stratification of a torus embedding along the unipotent fibre: Let $(P, \mathcal{X})$ be irreducible mixed Shimura data, K_f an open compact subgroup of $P(\mathbb{A}_f)$, and Σ a K_f -admissible partial cone decomposition for $(P, \mathcal{X})$ that is concentrated in the unipotent fibre. Then $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ possesses a natural stratification as a torus embedding. We want to describe this stratification in terms of other mixed Shimura varieties.

Since $(P, \mathcal{X})$ is irreducible, it is its own boundary component, so by 6.4 (iv) $\Sigma(\mathcal{X}^0, P, p_f)$

is independent of $p_f \in P(\mathbb{A}_f)$. Thus the data conveyed by Σ consists of a rational partial polyhedral decomposition of $U(\mathbb{R})(-1)$ for every connected component $\mathcal{X}^0$ of $\mathcal{X}$, such that the whole is invariant under the left action of $P(\mathbb{Q})$. Let $[\sigma]$ be a double coset in $P(\mathbb{Q}) \backslash \Sigma / P(\mathbb{A}_f)$. Since $(P, \mathcal{X})$ is irreducible, by 2.14 (a) $P(\mathbb{Q})$ acts through homotheties on $U(\mathbb{R})(-1)$. Thus if we view σ as a cone in $U(\mathbb{R})(-1)$, the subspace $\mathbb{R} \cdot \sigma \subset U(\mathbb{R})(-1)$ does not depend on the choice of the representative $\sigma \in [\sigma]$. Moreover, since σ is a convex rational polyhedral cone, this subspace comes from a unique rational subspace of $U(\mathbb{Q})(-1)$. Thus there exists a unique subgroup $\langle [\sigma] \rangle \subset U$ such that $\mathbb{R} \cdot \sigma = \langle [\sigma] \rangle(\mathbb{R})(-1)$ for every representative $\sigma \in [\sigma]$. Let us also note that $P(\mathbb{Q}) \cap P(\mathbb{R})^0$ maps $\mathcal{X}^0$ and every cone to itself, so the double coset $[\sigma]$ consists of finitely many $P(\mathbb{A}_f)$ -orbits. In other words the set $[\sigma]/P(\mathbb{A}_f)$ is finite.

We now define $P_{[\sigma]} := P/\langle [\sigma] \rangle$ and $U_{[\sigma]} := U/\langle [\sigma] \rangle$, and let $\mathcal{X}_{[\sigma]}$ be the $P_{[\sigma]} \cdot (\mathbb{R}) \cdot U_{[\sigma]}(\mathbb{C})$ -orbit generated by $(\mathcal{X}^0/\langle [\sigma] \rangle(\mathbb{C})) \times \{\sigma \cdot P(\mathbb{A}_f)\}$ in $(\mathcal{X}^0/\langle [\sigma] \rangle(\mathbb{C})) \times ([\sigma]/P(\mathbb{A}_f))$, such that $\sigma \in [\sigma] \cap \Sigma(\mathcal{X}^0, P, 1)$. Clearly this does not depend on the choice of the representative σ . Since $[\sigma]/P(\mathbb{A}_f)$ is a finite set with left action of $P(\mathbb{Q})/(P(\mathbb{Q}) \cap P(\mathbb{R})^0) \cong \pi_0(P_{[\sigma]}(\mathbb{R}) \cdot U_{[\sigma]}(\mathbb{C}))$, the pair $(P_{[\sigma]}, \mathcal{X}_{[\sigma]})$ is mixed Shimura data together with the obvious map $\mathcal{X}_{[\sigma]} \rightarrow \text{Hom}(\mathbb{S}_{\mathbb{C}}, P_{[\sigma], \mathbb{C}})$. The canonical morphism $(P_{[\sigma]}, \mathcal{X}_{[\sigma]}) \rightarrow (P, \mathcal{X})/\langle [\sigma] \rangle$ is an isomorphism if and only if every element of $\text{Stab}_{P(\mathbb{Q})}(\mathcal{X}^0)$ acts by a positive scalar on U . For example, this is so whenever $(P, \mathcal{X})$ is a proper boundary component of some other mixed Shimura data. We denote the canonical projection $P \rightarrow P_{[\sigma]}$ by $\pi_{[\sigma]}$.

7.2. Proposition: Consider $(P, \mathcal{X})$, K_f , and Σ as in 7.1. There is a canonical stratification

$$M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}) \cong \coprod M^{\pi_{[\sigma]}(K_f)}(P_{[\sigma]}, \mathcal{X}_{[\sigma]})(\mathbb{C}),$$

where the disjoint union is extended over all double cosets $[\sigma]$ in $P(\mathbb{Q}) \backslash \Sigma / P(\mathbb{A}_f)$. Each stratum $M^{\pi_{[\sigma]}(K_f)}(P_{[\sigma]}, \mathcal{X}_{[\sigma]})(\mathbb{C})$ is a locally closed analytic subset. Each such stratum is a finite union of strata in the natural stratification of $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ as a torus embedding.

Construction-Proof: Without loss of generality we may assume that K_f is neat. Fix $\mathcal{X}^0$, $x \in \mathcal{X}^0$ and $p_f \in P(\mathbb{A}_f)$. The definition of $\mathcal{X}_{[\sigma]}$ implies that the fibre of $\mathcal{X}_{[\sigma]} \rightarrow \mathcal{X}/\langle [\sigma] \rangle(\mathbb{C})$ over $[(x), p_f]$ is canonically isomorphic to $[\sigma] \cap \Sigma(\mathcal{X}^0, P, p_f)$. With the notations of 6.6 there is a canonical morphism $M^{\pi_{[\sigma]}(K_f)}(P_{[\sigma]}, \mathcal{X}_{[\sigma]})(\mathbb{C}) \rightarrow M^{K'_f}(P', \mathcal{X}')(\mathbb{C})$ and x and p_f determine an isomorphism of the fibre over $[\pi']([(x, p_f)])$ under this map with

$$(\Gamma_U \backslash U(\mathbb{C})/\langle [\sigma] \rangle(\mathbb{C})) \times ([\sigma] \cap \Sigma(\mathcal{X}^0, P, p_f)).$$

By 5.3 this is canonically isomorphic to the union of all strata associated to $[\sigma] \cap \Sigma(\mathcal{X}^0, P, p_f)$ in the torus embedding of $\Gamma(p_f) \backslash U(\mathbb{C})$ with respect to $\Sigma(\mathcal{X}^0, P, p_f)$. This gives the desired map $M^{\pi_{[\sigma]}(K_f)}(P_{[\sigma]}, \mathcal{X}_{[\sigma]})(\mathbb{C}) \rightarrow M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$. The remaining assertions follow from 5.2 and 5.3. **q.e.d.**

Remark. In general the assertion would be false for $(P, \mathcal{X})/\langle [\sigma] \rangle$ in place of $(P_{[\sigma]}, \mathcal{X}_{[\sigma]})$. For example take $(P, \mathcal{X}) = (P_0, h(\mathcal{X}_0))$ as in 2.24 and $\Sigma(\mathcal{X}^0, P, p_f) = \{\sqrt{-1} \cdot \mathbb{R}^{\leq 0}, \{0\}, \sqrt{-1} \cdot \mathbb{R}^{\geq 0}\}$ for every $p_f \in P(\mathbb{A}_f)$. Then there are exactly two double cosets in $P(\mathbb{Q}) \backslash \Sigma / P(\mathbb{A}_f)$, and by the proposition the closed orbit is canonically isomorphic to $M^{\pi(K_f)}(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)(\mathbb{C})$.

This Shimura variety cannot be written as the disjoint union of other Shimura varieties, and replacing $\mathcal{H}_0$ by $h(\mathcal{H}_0)$ in general yields a proper quotient. This situation was one of the reasons to allow for finite coverings of $h(\mathcal{X})$ in the definition 2.1.

7.3. Stratification of an arbitrary toroidal compactification: Now let $(P, \mathcal{X})$ be arbitrary and mixed Shimura data, K_f an open compact subgroup, and Σ a K_f -admissible partial cone decomposition for $(P, \mathcal{X})$. Then $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is stratified by the inverse images of the strata $\Delta_1 \backslash M^{\pi_1(K_f^1)}((P_1, \mathcal{X}_1)/W_1)(\mathbb{C})$ of $M^{\pi(K_f)}((P, \mathcal{X})/W)^*(\mathbb{C})$ as in 6.3. This inverse image is isomorphic to $\Delta_1 \backslash \partial\mathcal{U}(P_1, \mathcal{X}_1, p_f)$. In fact, by the proof of 6.21 $\partial\mathcal{U}(P_1, \mathcal{X}_1, p_f) \subset \bar{U}$ maps to this stratum, and by 6.18 the resulting map factors through $\Delta_1 \backslash \partial\mathcal{U}(P_1, \mathcal{X}_1, p_f)$. By 6.22 this map is a closed embedding over some neighborhood of every relatively compact subgroup of $\Delta_1 \backslash M^{\pi_1(K_f^1)}((P_1, \mathcal{X}_1)/W)(\mathbb{C})$. Thus it is locally closed embedding.

Going one step further, we stratify $\Delta_1 \backslash \partial\mathcal{U}(P_1, \mathcal{X}_1, p_f)$ as in 7.2. Let $\Sigma_1 := ([\cdot p_f]^* \Sigma|_{(P_1, \mathcal{X}_1)})$ and consider a double coset $[\sigma]$ in $P_1(\mathbb{Q}) \backslash \Sigma_1^0 / P_1(\mathbb{A}_f)$ such that $\sigma^0 \subset C(\mathcal{X}^0, P_1)$. Let $M^{\pi_{[\sigma]}(K_f^1)}(P_{[\sigma]}, \mathcal{X}_{[\sigma]})(\mathbb{C})$ be the corresponding stratum of $M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1^0)(\mathbb{C})$ according to 7.2, by 6.13 it is contained in $\partial\mathcal{U}(P_1, \mathcal{X}_1, p_f)$. Since by definition $P_1(\mathbb{Q})$ acts trivially on $P_1(\mathbb{Q}) \backslash \Sigma_1^0 / P_1(\mathbb{A}_f)$, the action of $\text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}_1) \cap (P_1(\mathbb{A}_f) \cdot p_f \cdot K_f \cdot p_f^{-1})$ on Σ_1^0 factors through Δ_1 . Thus the image of $M^{\pi_{[\sigma]}(K_f^1)}(P_{[\sigma]}, \mathcal{X}_{[\sigma]})(\mathbb{C})$ in $\Delta_1 \backslash \partial\mathcal{U}(P_1, \mathcal{X}_1, p_f)$ is isomorphic to

$$\text{Stab}_{\Delta_1}([\sigma]) \backslash M^{\pi_{[\sigma]}(K_f^1)}(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]})(\mathbb{C}).$$

Here note that $\text{Stab}_{\Delta_1}([\sigma])$ acts through a finite quotient on $M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1^0)(\mathbb{C})$, hence on $M^{\pi_{[\sigma]}(K_f^1)}(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]})(\mathbb{C})$, since by 6.19(a) for every arithmetic subgroup $\Gamma_Q \subset Q(\mathbb{Q})$ a subgroup of finite index of $\text{Stab}_{\Gamma_Q}(\sigma)$ lies in $Z(P) \cdot P_1$. As $[\sigma]$ runs through a set of representatives for the Δ_1 -action on the double quotient $P_1(\mathbb{Q}) \backslash \Sigma_1^0 / P_1(\mathbb{A}_f)$, these subsets form a stratification of $\Delta_1 \backslash \partial\mathcal{U}(P_1, \mathcal{X}_1, p_f)$ by locally closed analytic subsets.

If Σ is finite, then there are only finitely many such strata. In fact, by 6.4 (v) Σ is finite if and only if the double quotient $P_1(\mathbb{Q}) \backslash \Sigma / K_f$ is finite. This implies the finiteness of

$$(\text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}_1) \cap (P_1(\mathbb{A}_f) \cdot p_f \cdot K_f \cdot p_f^{-1})) \backslash \Sigma_1^0 / K_f,$$

hence that of $\Delta_1 \backslash (P_1(\mathbb{Q}) \backslash \Sigma_1^0 / P_1(\mathbb{A}_f))$, as desired. In that case we get a finite stratification of $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$.

Assume now that K_f is neat. Then $Q(\mathbb{A}_f) \cap p_f \cdot K_f \cdot p_f^{-1}$ is again neat, so by 0.6 its image in $\text{PGL}(U_1)(\mathbb{A}_f)$ is again neat. Since this image contains that of Δ_1 , which is an arithmetic subgroup, it follows that the image of Δ_1 in $\text{PGL}(U_1)(\mathbb{A}_f)$ is torsion free. Let $[\sigma]$ be a double coset in $P_1(\mathbb{Q}) \backslash \Sigma_1^0 / P_1(\mathbb{A}_f)$ such that $\sigma^0 \subset C(\mathcal{X}^0, P_1)$. From 6.19 (a) it follows that the image of $\text{Stab}_{\Delta_1}([\sigma])$ in $\text{PGL}(U_1)(\mathbb{Q})$ is finite, so by the neatness this image is trivial. This means that $\text{Stab}_{\Delta_1}([\sigma])$ acts trivially on the torus with respect to which $M^{\pi_{[\sigma]}(K_f^1)}(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]})(\mathbb{C})$ is a torus embedding.

7.4. Stratification and functoriality: Consider the situation of 6.25 (a) or (b), and the stratification of 7.3 on both sides. It is easy to see that every stratum is mapped to a

stratum, and this image-stratum can be described explicitly. In fact, consider for example the case 6.25 (b). Let

$$\text{Stab}_{\Delta_{11}}([\sigma_1]) \backslash M^{\pi[\sigma_1]}(K_f^{11})(P_{11, [\sigma_1]}, \mathcal{X}_{11, [\sigma_1]})(\mathbb{C})$$

be a stratum of $M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1)(\mathbb{C})$, associated to a rational boundary component $(P_{11}, \mathcal{X}_{11})$ of $(P_1, \mathcal{X}_1)$, $p_{1,f} \in P_1(\mathbb{A}_f)$, and $\sigma_1 \in \Sigma_1(\mathcal{X}_1^0, P_{11}, p_{1,f})$. Let $(P_{21}, \mathcal{X}_{21})$ be the rational boundary component of $(P_2, \mathcal{X}_2)$ associated to $(P_{11}, \mathcal{X}_{11})$ by functoriality (4.16), $p_{2,f} := \phi(p_{1,f})$, and $\sigma_2 \in \Sigma_2(\mathcal{X}_2^0, P_{21}, p_{2,f})$ the unique cone so that $\phi(\sigma_1) \subset \sigma_2$. Then the image of the above stratum lies in the stratum of $M^{K_f^2}(P_2, \mathcal{X}_2, \Sigma_2)(\mathbb{C})$ associated to $(P_{21}, \mathcal{X}_{21})$, $p_{2,f}$, and σ_2 . For the maps 6.25 (a) the result is analogous.

For later use we prove the following fact.

7.5. Proposition: Recall that with $\mathcal{Z}_1$, $\mathcal{V}_1$ and $\mathcal{W}_1$ as in 6.22, $\Delta_1 \backslash \mathcal{W}_1$ is isomorphic to the inverse image of $\mathcal{V}_1$ in $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$. Given $\mathcal{Z}_1$, the other two sets can be chosen such that there exists a unique holomorphic map $\mathcal{V}_1 \rightarrow \Delta_1 \backslash M^{\pi_1(K_f^1)}((P_1, \mathcal{X}_1)/W_1)(\mathbb{C})$ with which the following diagram commutes:

$$\begin{array}{ccccc} M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}) & \supset & \Delta_1 \backslash \mathcal{W}_1 & \hookrightarrow & \Delta_1 \backslash M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1^0)(\mathbb{C}) \\ \downarrow & & \downarrow & & \downarrow \\ M^{\pi(K_f)}((P, \mathcal{X})/W)^*(\mathbb{C}) & \supset & \mathcal{V}_1 & \longrightarrow & \Delta_1 \backslash M^{\pi_1(K_f^1)}((P_1, \mathcal{X}_1)/W_1)(\mathbb{C}) \end{array}$$

Proof. The fibre in $\Delta_1 \backslash \mathcal{W}_1$ over a point in $M^{\pi(K_f)}((P, \mathcal{X})/W)(\mathbb{C}) \cap \mathcal{V}_1$ is by the extension 6.15 of the map 6.11 (d) contained in a fibre of the map on the right hand side. Thus we already have the desired map on an open dense subset of $\mathcal{V}_1$. If this map extends holomorphically to $\mathcal{V}_1$, then the diagram also commutative for this extension. By Riemann's extension theorem (see [GR] ch.7 §4.2 p.144) it suffices to construct a continuous extension of this map.

As in the proof of 6.22 let $D \subset C(\mathcal{X}^0, P_1)$ be a sufficiently small core, $\mathcal{Y}_1$ a relatively compact subset of $(\mathcal{X}_1/W_1)^0$, $\{p_{1,i,f}\}$ a finite subset of $P_1(\mathbb{A}_f)$, $\mathcal{E}_1 := \text{im}^{-1}(D) \cap \psi_1^{-1}(\mathcal{Y}_1) \subset \mathcal{X}^0$, and $\mathcal{V}_1$ the image of $(\coprod \psi_2(\mathcal{E}_1)) \times \{p_{1,i,f} \cdot p_f\}$ in $M^{\pi(K_f)}((P, \mathcal{X})/W)^*(\mathbb{C})$. Then the proof of 6.22 shows that $\mathcal{V}_1$ is isomorphic to the image of $(\coprod \psi_2(\mathcal{E}_1)) \times \{p_{1,i,f} \cdot p_f\}$ in

$$(\text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^0) \cap P_1(\mathbb{A}_f) \cdot p_f \cdot K_f \cdot p_f^{-1}) \backslash (((\mathcal{X}/W)^*)^0 \times P_1(\mathbb{A}_f) \cdot p_f \cdot K_f / K_f).$$

For every boundary component $(P_2, \mathcal{X}_2)$ between $(P_1, \mathcal{X}_1)$ and $(P, \mathcal{X})$ let

$$\psi_{12} : (\mathcal{X}_2/W_2)^0 \longrightarrow (\mathcal{X}_1/W_1)^0$$

be the unique equivariant map with $\psi_1 = \psi_{12} \circ \psi_2$. These maps patch together to a map $\psi_1^* : \coprod (\mathcal{X}_2/W_2)^0 \rightarrow \mathcal{X}_1$. By the definition of the Satake-topology $\coprod (\mathcal{X}_2/W_2)^0$ is an open subset of $(\mathcal{X}/W)^*$, and ψ_1^* is a continuous map. Thus the induced map

$$\begin{aligned} & (\text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^0) \cap P_1(\mathbb{A}_f) \cdot p_f \cdot K_f \cdot p_f^{-1}) \backslash (\coprod (\mathcal{X}_2/W_2)^0 \times P_1(\mathbb{A}_f) \cdot p_f \cdot K_f / K_f) \longrightarrow \\ & \longrightarrow (\text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^0) \cap P_1(\mathbb{A}_f) \cdot p_f \cdot K_f \cdot p_f^{-1}) \backslash ((\mathcal{X}_1/W_1)^0 \times P_1(\mathbb{A}_f) \cdot p_f \cdot K_f / K_f) \\ & = \Delta_1 \backslash M^{\pi_1(K_f^1)}((P_1, \mathcal{X}_1)/W_1)(\mathbb{C}) \end{aligned}$$

is continuous. Its restriction to $\mathcal{V}_1$ is the desired extension. q.e.d.

7.6. Closure of a stratum in the Baily-Borel compactification: Consider the stratum $\Delta_1 \backslash M^{\pi_1(K_f^1)}((P_1, \mathcal{X}_1)/W_1)(\mathbb{C})$ of $M^{K_f}(P, \mathcal{X})^*(\mathbb{C})$ as in 6.3. The locally closed embedding of this stratum into $M^{K_f}(P, \mathcal{X})^*(\mathbb{C})$ extends uniquely to a holomorphic map

$$\Delta_1 \backslash M^{\pi_1(K_f^1)}((P_1, \mathcal{X}_1)/W_1)^*(\mathbb{C}) \longrightarrow M^{K_f}(P, \mathcal{X})^*(\mathbb{C}).$$

In fact, note that $(\mathcal{X}_1/W_1)^*$ is a closed subset of $\mathcal{X}^*$. As in 6.3 the map

$$(\mathcal{X}_1/W_1)^* \times P_1(\mathbb{A}_f) \rightarrow P(\mathbb{Q}) \backslash \mathcal{X}^* \times P(\mathbb{A}_f)/K_f, (x, p_{1,f}) \mapsto (x, p_{1,f} \cdot p_f)$$

factors through $\Delta_1 \backslash (P_1(\mathbb{Q}) \backslash (\mathcal{X}_1/W_1)^* \times (P_1(\mathbb{A}_f)/K_f^1))$, as desired. By construction this induced map is continuous, and extends the embedding of the stratum. Thus by Riemann's extension theorem (see [GR] ch.7 §4.2 p.144) it is holomorphic. Since $\Delta_1 \backslash M^{\pi_1(K_f^1)}((P_1, \mathcal{X}_1)/W_1)^*(\mathbb{C})$ is compact, its image is the closure of the stratum. By checking what happens to the strata of $\Delta_1 \backslash M^{\pi_1(K_f^1)}((P_1, \mathcal{X}_1)/W_1)^*(\mathbb{C})$ one easily sees that the map is finite. In general, however, we cannot hope that it is injective.

We now want to do the same for the toroidal compactification.

7.7. Induced cone decompositions: Consider irreducible mixed Shimura data $(P, \mathcal{X})$, an open compact subgroup $K_f \subset P(\mathbb{A}_f)$, and K_f -admissible partial cone decomposition Σ . Consider a double coset $[\sigma] \subset P(\mathbb{Q}) \backslash \Sigma / P(\mathbb{A}_f)$ and the associated mixed Shimura data $(P_{[\sigma]}, \mathcal{X}_{[\sigma]})$ defined in 7.1. Let $(P_{1,[\sigma]}, \mathcal{X}_{1,[\sigma]})$ be a rational boundary component of $(P_{[\sigma]}, \mathcal{X}_{[\sigma]})$. By definition every connected component of $\mathcal{X}_{[\sigma]}$ is of the form $(\mathcal{X}_{[\sigma]}^0 = \mathcal{X}^0 / \langle [\sigma] \rangle(\mathbb{C})) \times \{\sigma \cdot P(\mathbb{A}_f)\}$ where $\mathcal{X}^0$ is a connected component of $\mathcal{X}$ and σ a representative of $[\sigma]$ such that $\sigma \in \Sigma(\mathcal{X}^0, P, p_f)$ for some p_f . If $\mathcal{X}_{[\sigma]}^0$ embeds into $\mathcal{X}_{1,[\sigma]}$, let $(P_1, \mathcal{X}_1)$ be the unique rational boundary component of $(P, \mathcal{X})$ such that $P_1 := \pi_{[\sigma]}^{-1}(P_{1,[\sigma]})$ and $\mathcal{X}^0$ embeds into $\mathcal{X}_1$. Clearly it depends only on $(P_{1,[\sigma]}, \mathcal{X}_{1,[\sigma]})$. Let $p_{f,[\sigma]}$ be the image of p_f in $P_{[\sigma]}(\mathbb{A}_f)$.

We can now define a $\pi_{[\sigma]}(K_f)$ -admissible partial cone decomposition $\Sigma_{[\sigma]}$ for $(P_{[\sigma]}, \mathcal{X}_{[\sigma]})$ as follows. For all $\mathcal{X}^0$, p_f , σ , and $(P_{1,[\sigma]}, \mathcal{X}_{1,[\sigma]})$ as above we define

$$\Sigma_{[\sigma]}(\mathcal{X}_{[\sigma]}^0, P_{1,[\sigma]}, p_{1,[\sigma]}) := \{\tau \bmod \langle [\sigma] \rangle \mid \tau \in \Sigma(\mathcal{X}^0, P_1, p_f) \text{ such that } \sigma \text{ is a face of } \tau\}.$$

One easily verifies that this is indeed well-defined and satisfies the conditions 6.4 (i)–(iv).

Denote the double coset in $P_1(\mathbb{Q}) \backslash (\Sigma_{(P_1, \mathcal{X}_1)})^0 / P_1(\mathbb{A}_f)$ generated by σ again by $[\sigma]$. Then applying the construction of 7.1 to $(P_1, \mathcal{X}_1)$ and $[\sigma]$, we get mixed Shimura data which is clearly canonically isomorphic to $(P_{1,[\sigma]}, \mathcal{X}_{1,[\sigma]})$. Moreover with the above construction we have the relation $(\Sigma_{(P_1, \mathcal{X}_1)})_{[\sigma]} = (\Sigma_{[\sigma]})_{(P_{[\sigma]}, \mathcal{X}_{[\sigma]})}$.

7.8. Proposition: Let $(P, \mathcal{X})$ be mixed Shimura data, $K_f \cap P(\mathbb{A}_f)$ an open compact subgroup, and Σ a K_f -admissible partial cone decomposition for $(P, \mathcal{X})$. Let $p_f \in P(\mathbb{A}_f)$, $(P_1, \mathcal{X}_1)$ a rational boundary component of $(P, \mathcal{X})$, and $\Sigma_1 := ([\cdot p_f]^* \Sigma)_{(P_1, \mathcal{X}_1)}$. Let $[\sigma]$ be

a double coset in $P_1(\mathbb{Q}) \backslash \Sigma_1^0 / P_1(\mathbb{A}_f)$ such that $\sigma^0 \subset C(\mathcal{X}^0, P_1, p'_f)$ for some $\mathcal{X}^0$ and p'_f . If Σ is finite, resp. complete, then $\Sigma_{1, [\sigma]}$ possesses the same property.

Proof. Both properties are invariant under $[\cdot p_f]^*$, so we may assume $p_f = 1$. Suppose that Σ is finite. In the notation of 7.7 replace $(P, \mathcal{X})$ by $(P_1, \mathcal{X}_1)$, $(P_1, \mathcal{X}_1)$ by $(P_2, \mathcal{X}_2)$ and so on, and let Q_{12} be the admissible $\mathbb{Q}$ -parabolic subgroup of P_1 associated to $(P_2, \mathcal{X}_2)$. To prove the finiteness of $\Sigma_{1, [\sigma]}$ we then have to show that the quotient $\Gamma_{Q_{12}} \backslash \Sigma_{1, [\sigma]}(\mathcal{X}_{1, [\sigma]}^0, P_{2, [\sigma]}, p_{1, f, [\sigma]})$ is finite, where $\Gamma_{Q_{12}}$ is an arithmetic subgroup of $Q_1(\mathbb{Q})$. Let Q , resp. Q_2 be the admissible $\mathbb{Q}$ -parabolic subgroups of P associated to $(P_1, \mathcal{X}_1)$, resp. $(P_2, \mathcal{X}_2)$, then we have $Q_{12} = P_1 \cap Q_2$. By definition $\Sigma_{1, [\sigma]}(\mathcal{X}_{1, [\sigma]}^0, P_{2, [\sigma]}, p_{1, f, [\sigma]})$ consists of all $\tau \bmod \langle [\sigma] \rangle$ for $\tau \in \Sigma_1(\mathcal{X}_1^0, P_2, p_{1, f})$ such that σ is a face of τ . By definition of Σ_1 we have $\Sigma_1(\mathcal{X}_1^0, P_2, p_{1, f}) = \Sigma(\mathcal{X}^0, P_2, p_{1, f})$, so by the finiteness of Σ these τ lie in finitely many Γ_{Q_2} -orbits for every arithmetic subgroup Γ_{Q_2} of $Q_2(\mathbb{Q})$. We may suppose that $\Gamma_{Q_{12}}$ is contained in Γ_{Q_2} , then it suffices to show that all such τ which lie in a fixed Γ_{Q_2} -orbit, lie in finitely many $\Gamma_{Q_{12}}$ -orbits.

Let us fix such a τ , and consider all $\gamma \in \Gamma_{Q_2}$ such that σ is a face of $\gamma\tau$. Then $\gamma^{-1}\sigma$ is a face of τ , and since τ possesses only finitely many different faces, we are reduced to considering only those $\gamma \in \Gamma_{Q_2}$ with $\gamma\sigma = \sigma$. Every such γ stabilizes $(P_1, \mathcal{X}_1)$, hence lies in an arithmetic subgroup of $Q(\mathbb{Q})$. Since by assumption $\sigma^0 \subset C(\mathcal{X}^0, P_1)$, 6.19 (a) implies that all such γ lie in finitely many coset under $Z(P) \cdot P_1$. Thus we are reduced to γ in an arithmetic subgroup of $(Z(P) \cdot P_1)(\mathbb{Q})$. Since every sufficiently small arithmetic subgroup of $Z(P)(\mathbb{Q})$ acts trivially, we are even reduced to $\gamma \in P_1(\mathbb{Q})$. Thus the τ under consideration lie in finitely many orbits under an arithmetic subgroup of $P_1(\mathbb{Q}) \cap Q_2(\mathbb{Q}) = Q_{12}(\mathbb{Q})$, as desired. This proves the finiteness of $\Sigma_{1, [\sigma]}$.

If Σ is complete, then $C^*(\mathcal{X}^0, P_2)$ is the union of all cones in $\Sigma(\mathcal{X}^0, P_2, p_{1, f})$, and we have to show that $C^*(\mathcal{X}_1^0, P_2)$ is the union of all $\tau + \langle [\sigma] \rangle(\mathbb{R})(-1)$ where $\tau \in \Sigma_1(\mathcal{X}_1^0, P_2, p_{1, f}) = \Sigma(\mathcal{X}^0, P_2, p_{1, f})$ such that σ is a face of τ . By 4.22 (b) we have $C^*(\mathcal{X}_1^0, P_2) = C^*(\mathcal{X}^0, P_2) + U_1(\mathbb{R})(-1)$. Also since $\sigma^0 \subset C(\mathcal{X}^0, P_1)$, we have $U_1(\mathbb{R})(-1) = C(\mathcal{X}^0, P_1) + \langle [\sigma] \rangle(\mathbb{R})(-1) = C^*(\mathcal{X}^0, P_1) + \langle [\sigma] \rangle(\mathbb{R})(-1)$. Together this shows $C^*(\mathcal{X}_1^0, P_2) = C^*(\mathcal{X}^0, P_1) + \langle [\sigma] \rangle(\mathbb{R})(-1)$, so it suffices to show that every element $u \in C^*(\mathcal{X}^0, P_1)$ lies inside $\tau + \langle [\sigma] \rangle(\mathbb{R})(-1)$ for some $\tau \in \Sigma(\mathcal{X}^0, P_2, p_{1, f})$ such that σ is a face of τ .

To prove this fix $u_1 \in \sigma^0$, and consider the convex rational polyhedral cone ρ in $C^*(\mathcal{X}^0, P_2)$ generated by u and u_1 . If ρ^0 does not lie in $C(\mathcal{X}^0, P_2)$, then we may replace $(P_2, \mathcal{X}_2)$ by a rational boundary component between $(P_1, \mathcal{X}_1)$ and $(P_2, \mathcal{X}_2)$, so we may assume $\rho^0 \subset C(\mathcal{X}^0, P_2)$. Fix an arithmetic subgroup Γ_{Q_2} of $Q_2(\mathbb{Q})$, then by 6.19 (a) ρ^0 has non-empty intersection at most with finitely many cones in every Γ_{Q_2} -orbit in $\Sigma(\mathcal{X}^0, P_2, p_{1, f})$. Since Σ is finite, there are only finitely many such orbits, so $\rho^0 \cap \tau \neq \emptyset$ for only finitely many different $\tau \in \Sigma(\mathcal{X}^0, P_2, p_{1, f})$. But the completeness of Σ ρ^0 is covered by the $\rho^0 \cap \tau^0$ for these τ . Thus there exists a $\tau \in \Sigma(\mathcal{X}^0, P_2, p_{1, f})$ which contains the element $\lambda \cdot u + (1 - \lambda) \cdot u_1$ for all sufficiently small $\lambda > 0$. Then clearly $u \in \tau + \langle [\sigma] \rangle(\mathbb{R})(-1)$, as desired. Moreover since τ is closed, it contains u_1 . But u_1 lies in σ^0 as well as in the interior of a unique face of τ . Thus σ is a face of τ , which finishes the proof. **q.e.d.**

7.9. Proposition: Let $(P, \mathcal{X})$ be irreducible mixed Shimura data, K_f an open compact subgroup of $P(\mathbb{A}_f)$, and Σ a K_f -admissible partial cone decomposition. Let $[\sigma]$ be a

double coset in $P(\mathbb{Q}) \backslash \Sigma^0 / P(\mathbb{A}_f)$, and $M^{\pi_{[\sigma]}(K_f)}((P_{[\sigma]}, \mathcal{X}_{[\sigma]})(\mathbb{C}))$ the corresponding stratum of $M^{K_f}(P, \mathcal{X}, \Sigma^0)(\mathbb{C})$ according to 7.2. Then the closure of this stratum in $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is canonically isomorphic to $M^{\pi_{[\sigma]}(K_f)}(P_{[\sigma]}, \mathcal{X}_{[\sigma]}, \Sigma_{[\sigma]})(\mathbb{C})$.

Proof. If Σ is concentrated in the unipotent fibre, then on every connected component in the assertion follows from the construction 6.6 of $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ and the corresponding assertion 5.3 for abstract torus embeddings. For Σ arbitrary consider $\bar{\mathcal{U}}, \bar{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$ etc. as in chapter 6, and denote the analogous object for $(P_{[\sigma]}, \mathcal{X}_{[\sigma]})$ in place of $(P, \mathcal{X})$ by $\bar{\mathcal{U}}_{[\sigma]}, \bar{\mathcal{U}}_{[\sigma]}(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]}, p_{f, [\sigma]})$ etc. To relate $\bar{\mathcal{U}}$ and $\bar{\mathcal{U}}_{[\sigma]}$ we shall construct a certain subset of $\bar{\mathcal{U}}$.

For every $(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]})$ and p_f as in 7.7, we have by 6.13 $\Sigma_1^0 = (([\cdot p_f]^* \Sigma)_{|(P_1, \mathcal{X}_1)})^0$, and $(\Sigma_{[\sigma]})_1^0 = (([\cdot p_{f, [\sigma]}]^* (\Sigma_{[\sigma]}))_{|(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]})})^0$ analogous. The definitions imply $(\Sigma_{[\sigma]})_1^0 = (\Sigma_1^0)_{[\sigma]}$. Since $M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1^0)(\mathbb{C})$ is a torus embedding along the unipotent fibre, we get a closed embedding

$$M^{\pi_{[\sigma]}(K_f^1)}(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]}, (\Sigma_{[\sigma]})_1^0)(\mathbb{C}) \hookrightarrow M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1^0)(\mathbb{C}).$$

The explicit description of $\bar{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$ in 6.13 shows that this induces a closed embedding

$$\bar{\mathcal{U}}_{[\sigma]}(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]}, p_{f, [\sigma]}) \hookrightarrow \bar{\mathcal{U}}(P_1, \mathcal{X}_1, p_f).$$

Denote by $\tilde{\mathcal{U}}_{[\sigma]}$ the union of the images of all these maps. This is a closed subset of $\bar{\mathcal{U}}$, and stable under each of the maps 6.15, hence under the equivalence relation $\sim$. Thus $\tilde{\mathcal{U}}_{[\sigma]} / \sim$ is a closed subset of $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$, and it is this that we shall identify with $M^{\pi_{[\sigma]}(K_f^1)}(P_{[\sigma]}, \mathcal{X}_{[\sigma]}, \Sigma_{[\sigma]})(\mathbb{C})$.

By definition $\bar{\mathcal{U}}_{[\sigma]}$ is the disjoint union

$$\coprod \bar{\mathcal{U}}_{[\sigma]}(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]}, p_{f, [\sigma]}),$$

indexed by all rational boundary components $(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]})$ and all $p_{f, [\sigma]} \in P_{[\sigma]}(\mathbb{A}_f)$. Now $\tilde{\mathcal{U}}_{[\sigma]}$ is the same disjoint union, but indexed by all $(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]})$ and all $p_f \in P(\mathbb{A}_f)$, so there is a natural surjective map $\tilde{\mathcal{U}}_{[\sigma]} \rightarrow \bar{\mathcal{U}}_{[\sigma]}$. Let $\sim_{[\sigma]}$ be the natural equivalence relation on $\bar{\mathcal{U}}_{[\sigma]}$, analogous to $\sim$ on $\bar{\mathcal{U}}$. It remains to show that the given map induces an isomorphism $\tilde{\mathcal{U}}_{[\sigma]} / \sim \cong \tilde{\mathcal{U}}_{[\sigma]} / \sim_{[\sigma]}$.

For this we have to show that any two elements of $\tilde{\mathcal{U}}_{[\sigma]}$ are equivalent under $\sim$ if and only if their images are equivalent under $\sim_{[\sigma]}$. The “only if” part is clear, since it is so for each of the maps 6.15. For the other direction suppose first that their images are equal. Then we have the same $(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]})$ and $p_{f, [\sigma]}$, but two different lifts of $p_{f, [\sigma]}$ to $P(\mathbb{A}_f)$. If these are p_f and $u \cdot p_f$ with $u_f \in \langle [\sigma] \rangle(\mathbb{A}_f)$, then the extension 6.15 of the map 6.11 (b) with $p_{1, f} = u_f$ is an isomorphism $[\cdot u_f] : \bar{\mathcal{U}}(P_1, \mathcal{X}_1, u_f \cdot p_f) \rightarrow \bar{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$. Thus the two points are equivalent under $\sim$ if the following diagram is commutative:

$$\begin{array}{ccc} \bar{\mathcal{U}}(P_1, \mathcal{X}_1, u_f \cdot p_f) & \xrightarrow{[\cdot u_f]} & \bar{\mathcal{U}}(P_1, \mathcal{X}_1, p_f) \\ \uparrow & & \uparrow \\ \bar{\mathcal{U}}_{[\sigma]}(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]}, p_{f, [\sigma]}) & & \end{array}$$

But this is a straightforward functorial property of the stratification of 7.2. Using this, it now remains to prove that whenever two points of $\overline{\mathcal{U}}_{[\sigma]}$ are equivalent under $\sim_{[\sigma]}$, then there exist points in the fibres that are equivalent under $\sim$. But this is clear, since it is so for each of the maps of 6.15. Thus we have proved that $M^{\pi_{[\sigma]}(K_f)}(P_{[\sigma]}, \mathcal{X}_{[\sigma]}, \Sigma_{[\sigma]})(\mathbb{C}) = \overline{\mathcal{U}}_{[\sigma]} / \sim_{[\sigma]} \cong \tilde{\mathcal{U}}_{[\sigma]} / \sim$ is a closed subset of $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$, as desired. **q.e.d.**

7.10. Proposition: Consider an embedding $\phi : (P_1, \mathcal{X}_1) \hookrightarrow (P, \mathcal{X})$ as in 2.13, i.e. where P_1 is a normal subgroup of P , for instance $(P_1, \mathcal{X}_1)$ is an improper rational boundary component of $(P, \mathcal{X})$. Let K_f be an open compact subgroup of $P(\mathbb{A}_f)$. There exist finitely many $p_f^\nu \in P(\mathbb{A}_f)$, such that with $K_f^\nu := P_1(\mathbb{A}_f) \cap p_f^\nu \cdot K_f \cdot (p_f^\nu)^{-1}$ and $\Sigma_\nu := ([\cdot p_f^\nu]^* \Sigma)|_{(P_1, \mathcal{X}_1)}$, the group

$$\Delta_\nu := (\text{Stab}_{P(\mathbb{Q})}(\mathcal{X}_1) \cap (P_1(\mathbb{A}_f) \cdot p_f^\nu \cdot K_f \cdot (p_f^\nu)^{-1})) / P_1(\mathbb{Q}),$$

defined as in 6.18, acts through a finite quotient on $M^{K_f^\nu}(P_1, \mathcal{X}_1, \Sigma_\nu)(\mathbb{C})$, and the maps 6.25 induce an isomorphism

$$\coprod_{\nu} \Delta_\nu \backslash M^{K_f^\nu}(P_1, \mathcal{X}_1, \Sigma_\nu)(\mathbb{C}) \xrightarrow{\coprod [\cdot p_f^\nu] \circ [\phi]} M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}).$$

Moreover if K_f is neat, and P/P_1 is an extension of a $\mathbb{Q}$ -split torus with a torus of compact type, then all $\Delta_\nu = 1$.

Proof. Choose the p_f^ν as a system of representatives of

$$(\text{Stab}_{P(\mathbb{Q})}(\mathcal{X}_1) \cdot P_1(\mathbb{A}_f)) \backslash P(\mathbb{A}_f) / K_f.$$

Denoting the projection $P \rightarrow P/P_1$ by ψ , the group Δ_ν is contained in $(P/P_1)(\mathbb{Q}) \cap \psi(K_f)$, which is an arithmetic subgroup of a torus. By the condition 2.1 (viii) the quotient $P/(P_1 \cdot Z(P))$ is the extension of a $\mathbb{Q}$ -split torus with a torus of compact type, so every sufficiently small arithmetic subgroup of $(P/P_1)(\mathbb{Q})$ is contained in the image of $Z(P)(\mathbb{Q}) \cap Z(P)(\mathbb{R})^0$. But this group acts trivially on $\mathcal{X}_1$, hence trivially on $M^{K_f^\nu}(P_1, \mathcal{X}_1, \Sigma_\nu)(\mathbb{C})$. Thus Δ_ν always acts through a finite quotient. If K_f is neat, then so is $\psi(K_f)$ by 0.6, and if P/P_1 is an extension of a $\mathbb{Q}$ -split torus with a torus of compact type, then Δ_ν must be trivial.

Next if Σ is the trivial cone decomposition, then we are dealing $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ and $M^{K_f^\nu}(P_1, \mathcal{X}_1)(\mathbb{C})$, and the assertion is a special case of 7.3. For arbitrary Σ we still have a generic isomorphism, and since $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is normal, the images of any two distinct individual maps $M^{K_f^\nu}(P_1, \mathcal{X}_1, \Sigma_\nu)(\mathbb{C}) \rightarrow M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ are disjoint. Thus it remains to show that every such map induces an open and closed embedding of $\Delta_\nu \backslash M^{K_f^\nu}(P_1, \mathcal{X}_1, \Sigma_\nu)(\mathbb{C})$. The proof of this is almost the same as that of 7.9, so we leave it to the reader. **q.e.d.**

7.11. The closure of an arbitrary stratum: While in the preceding proposition we studied the closure of a stratum of $M^{K_f}(P, \mathcal{X}, \Sigma^0)(\mathbb{C})$, we now turn to the closure of an arbitrary stratum of $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$. It is not difficult to extend the inclusion

$$\text{Stab}_{\Delta_1}([\sigma]) \backslash M^{\pi_{[\sigma]}(K_f^1)}(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]})(\mathbb{C}) \hookrightarrow M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$$

of 7.3 to a holomorphic map

$$\text{Stab}_{\Delta_1}([\sigma]) \backslash M^{\pi_{[\sigma]}}(K_f^1)((P_{1,[\sigma]}, \mathcal{X}_{1,[\sigma]}, \Sigma_{1,[\sigma]})(\mathbb{C})) \longrightarrow M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}).$$

But in general this map will not be injective. Also we are, for later purposes, interested not only in the closure of the stratum, but also in a canonical description of a small neighborhood of it. For simplicity we shall assume throughout that Σ is complete, since this is the only case that we shall need in the sequel. We begin by constructing a certain map, which we shall later prove to be an open embedding, if a certain condition is met.

So fix $(P, \mathcal{X})$, K_f , Σ , $(P_1, \mathcal{X}_1)$, p_f , K_f^1 , and Σ_1 as in 7.3. Consider $\bar{\mathcal{U}} = \coprod \bar{\mathcal{U}}(P_2, \mathcal{X}_2, p'_f)$ etc. as in chapter 6, and denote the analogous objects for $(P_1, \mathcal{X}_1)$, K_f^1 and Σ_1 in place of $(P, \mathcal{X})$, K_f and Σ by $\bar{\mathcal{U}}_1 = \coprod \bar{\mathcal{U}}_1(P_2, \mathcal{X}_2, p_{1,f})$ etc. Observe that in $\bar{\mathcal{U}}$ the union is extended over all rational boundary components $(P_2, \mathcal{X}_2)$ of $(P, \mathcal{X})$, and all $p'_f \in P(\mathbb{A}_f)$, while in $\bar{\mathcal{U}}_1$ only over those of $(P_1, \mathcal{X}_1)$ and $p_{1,f} \in P_1(\mathbb{A}_f)$. For the latter ones $\bar{\mathcal{U}}_1(P_2, \mathcal{X}_2, p_{1,f})$ is an open subset of $M^{K_f^2}(P_2, \mathcal{X}_2, \Sigma_2^0)(\mathbb{C})$, where

$$K_f^2 = P_2(\mathbb{A}_f) \cap p_{1,f} \cdot K_f^1 \cdot p_{1,f}^{-1} = P_2(\mathbb{A}_f) \cap p_{1,f} \cdot p_f \cdot K_f \cdot (p_{1,f} \cdot p_f)^{-1}$$

and

$$\Sigma_2^0 = (([\cdot p_{1,f}]^* \Sigma_1)_{|(P_2, \mathcal{X}_2)})^0 = (([\cdot p_{1,f} \cdot p_f]^* \Sigma)_{|(P_2, \mathcal{X}_2)})^0.$$

Thus $\bar{\mathcal{U}}(P_2, \mathcal{X}_2, p_{1,f} \cdot p_f)$ is also an open subset of $M^{K_f^2}(P_2, \mathcal{X}_2, \Sigma_2^0)(\mathbb{C})$, and by the explicit description given in 6.13 it is contained in $\bar{\mathcal{U}}_1(P_2, \mathcal{X}_2, p_{1,f})$.

So letting $\bar{\mathcal{U}}_{(1)}$ be the disjoint union of $\bar{\mathcal{U}}(P_2, \mathcal{X}_2, p_{1,f} \cdot p_f)$ for all rational boundary components $(P_2, \mathcal{X}_2)$ of $(P_1, \mathcal{X}_1)$ and all $p_{1,f} \in P_1(\mathbb{A}_f)$, we have canonical embeddings of $\bar{\mathcal{U}}_{(1)}$ into $\bar{\mathcal{U}}$ and into $\bar{\mathcal{U}}_1$.

Let $\sim$ be the equivalence relation on $\bar{\mathcal{U}}$ defined in 6.16, and $\sim_1$ the analogous one on $\bar{\mathcal{U}}_1$. By comparing the definitions of $\sim$ and $\sim_1$ we easily find that $\bar{\mathcal{U}}_{(1)} \hookrightarrow \bar{\mathcal{U}}_1$ is stable under $\sim_1$, that $\bar{\mathcal{U}}_{(1)}/\sim_1 \hookrightarrow \bar{\mathcal{U}}_1/\sim_1 = M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1)(\mathbb{C})$ is stable under the action of Δ_1 on $M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1)(\mathbb{C})$, and that the map $\bar{\mathcal{U}}_{(1)} \rightarrow \bar{\mathcal{U}}$ factors through a map $\Delta_1 \backslash \bar{\mathcal{U}}_{(1)}/\sim_1 \rightarrow \bar{\mathcal{U}}/\sim = M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$. In other words we have extended the maps $M^{K_f}(P, \mathcal{X})(\mathbb{C}) \leftarrow \Delta_1 \backslash \mathcal{U}(P_1, \mathcal{X}_1, p_f) \hookrightarrow \Delta_1 \backslash M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C})$ of 6.10 canonically to holomorphic maps

$$\begin{array}{ccc} \Delta_1 \backslash \bar{\mathcal{U}}_{(1)}/\sim_1 & \xrightarrow{\text{open}} & \Delta_1 \backslash M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1)(\mathbb{C}) \\ \downarrow & & \\ M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}) & & \end{array}$$

Since $\bar{\mathcal{U}}_{(1)}$ contains $\partial \mathcal{U}(P_2, \mathcal{X}_2, p_{1,f} \cdot p_f)$ for all $(P_2, \mathcal{X}_2)$ and $p_{1,f}$, it follows from 7.9 that the closure of $\partial \mathcal{U}(P_1, \mathcal{X}_1, p_f)$ in $M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1)(\mathbb{C})$ is contained in the image of $\bar{\mathcal{U}}_{(1)}/\sim_1$. Fix a double coset $[\sigma] \in P_1(\mathbb{Q}) \backslash \Sigma_1^0 / P_1(\mathbb{A}_f)$ with $\sigma^0 \subset C(\mathcal{X}^0, P_1) \times \{1\}$ for some $\mathcal{X}^0$, then in particular $\bar{\mathcal{U}}_{(1)}/\sim_1$ is an open neighborhood of $M^{\pi_{1,[\sigma]}}(K_f^1)((P_{1,[\sigma]}, \mathcal{X}_{1,[\sigma]}, \Sigma_{1,[\sigma]})(\mathbb{C})) \subset M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1)(\mathbb{C})$. Consider an arbitrarily small open neighborhood $\mathcal{V} \subset \bar{\mathcal{U}}_{(1)}/\sim_1$ of this closure. Since $\text{Stab}_{\Delta_1}([\sigma])$ acts on $M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1)(\mathbb{C})$ through a finite quotient, we

may without loss of generality assume that $\mathcal{V}$ is $\text{Stab}_{\Delta_1}([\sigma])$ -invariant. Consider the maps

$$\begin{array}{ccc} \text{Stab}_{\Delta_1}([\sigma]) \backslash \mathcal{V} & \xrightarrow{\text{open}} & \text{Stab}_{\Delta_1([\sigma])} \backslash M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1)(\mathbb{C}) \\ \downarrow & & \\ M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}) & & \end{array}$$

Since the assumption Σ is complete, 6.27 implies that $M^{\pi_{1, [\sigma]}(K_f^1)}(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]}, \Sigma_{1, [\sigma]})(\mathbb{C})$ is compact. Hence so is its image in $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$, so it is equal to the closure of the corresponding stratum $\text{Stab}_{\Delta_1}([\sigma]) \backslash M^{\pi_{1, [\sigma]}(K_f^1)}(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]})(\mathbb{C})$ of $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$. So far this is the same situation as in 7.6. But in the toroidal compactification things are better. In 7.15 we shall even prove that under certain conditions the vertical map is an open embedding. Then we get in particular a canonical description of the closure of a stratum of $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ as a toroidal embedding of another mixed Shimura variety. Loosely speaking the closure of the stratum has no self-intersections. Moreover we get a canonical isomorphism of every sufficiently small open neighborhood of this closure with a neighborhood of the closure of a stratum in a different mixed Shimura variety, where the situation is much simpler, namely where the stratum is that of a torus embedding along the unipotent fibre.

7.12. A certain condition on cone decompositions: Consider mixed Shimura data $(P, \mathcal{X})$, an open compact subgroup $K_f \subset P(\mathbb{A}_f)$, and a K_f -admissible partial cone decomposition Σ . Consider $p_f \in P(\mathbb{A}_f)$, a connected component $\mathcal{X}^0$ of $\mathcal{X}$, a rational boundary component $(P_1, \mathcal{X}_1)$ of $(P, \mathcal{X})$, and a rational boundary component $(P_2, \mathcal{X}_2)$ of $(P_1, \mathcal{X}_1)$. Let $\tau \in \Sigma(\mathcal{X}^0, P_2, p_f)$ and $\sigma \in \Sigma(\mathcal{X}^0, P_1, p_f) \subset \Sigma(\mathcal{X}^0, P_2, p_f)$ such that σ is a face of τ . Let $p \in P(\mathbb{Q})$, $p_{1, f} \in P(\mathbb{A}_f)$, and $k_f \in K_f$, then with the operations defined in 6.4 $p \cdot p_{1, f} \cdot \sigma \cdot k_f$ is again an element of Σ . Consider the following condition on the pair (K_f, Σ) :

(*) For all p_f , $(P_1, \mathcal{X}_1)$, $(P_2, \mathcal{X}_2)$, τ , σ , p , $p_{1, f}$, and k_f as above, if both σ and $p \cdot p_{1, f} \cdot \sigma \cdot k_f$ are faces of τ , then $p \cdot p_{1, f} \cdot \sigma \cdot k_f = \sigma$.

This condition will turn out to be sufficient for the purpose described in 7.11. If it holds for given (K_f, Σ) , then it clearly holds for any smaller open compact subgroup. It should be noted that this condition is not related to neatness of K_f . Nevertheless the following proposition shows that it is only a mild restriction.

7.13. Proposition: Let $(P, \mathcal{X})$, K_f and Σ be as in 7.12. If Σ is finite, then there exists an open subgroup K'_f of K_f such that the condition of 7.12 holds for (K'_f, Σ) .

Proof. We first show that it is enough to prove that for all fixed τ and σ there exists $K'_f \subset K_f$ such that the condition in 7.12 holds. In fact, by finiteness of Σ , and since every convex rational polyhedral cone has only finitely many different faces, there are, up to the left action of $P(\mathbb{Q})$ and the right action of K_f , only finitely many pairs (σ_i, τ_i) of cones in Σ such that σ_i is a face of τ_i . Suppose that the assertion holds for each of these with $K'_{f, i} \subset K_f$. Choose an open normal subgroup K'_f of K_f that is contained in every $K'_{f, i}$. Then for every $p \in P(\mathbb{Q})$ and $k_f \in K_f$ the conditions also holds for K'_f and $(p \cdot \sigma_i \cdot k_f, p \cdot \tau_i \cdot k_f)$. Thus it holds for K'_f and every (σ, τ) , as desired.

Next fix p_f , $(P_2, \mathcal{X}_2)$ and τ as in 7.12. Let $(\mathbb{R}^{\geq 0} \cdot u_i) \times \{p_f\}$ for $i = 1, \dots, r$ be the different one dimensional faces of τ . Clearly it suffices to prove the assertion for each of these, say for $\sigma = (\mathbb{R}^{\geq 0} \cdot u_i) \times \{p_f\}$. Consider $p \in P(\mathbb{Q})$, $p_{1,f} \in P_1(\mathbb{A}_f)$, and $k_f \in K_f$. By definition of the various operations, $p \cdot p_{1,f} \cdot \sigma \cdot k_f$ is a face of τ if and only if $p \cdot p_{1,f} \cdot p_f \cdot K_f = p_f$, $p \cdot \mathcal{X}^0 = \mathcal{X}^0$, and $\mathbb{R}^{\geq 0} \cdot p \cdot u_i = \mathbb{R}^{\geq 0} \cdot u_i$ for some (unique) $i \in \{1, \dots, r\}$. We have to prove that for K_f sufficiently small these conditions imply $p \cdot p_{1,f} \cdot \sigma \cdot k_f = \sigma$. Since this equality holds if and only if $i = 1$, we have to show that $\mathbb{R}^{\geq 0} \cdot p \cdot u_i \neq \mathbb{R}^{\geq 0} \cdot u_i$ for all $i \neq 1$ and all $p \in \text{Stab}_{P(\mathbb{Q})}(\mathcal{X}^0) \cap p_f \cdot K_f \cdot p_f^{-1} \cdot P_1(\mathbb{A}_f)$, provided that K_f is sufficiently small. Note that since $\mathbb{R}^{\geq 0} \cdot u_1$ and $\mathbb{R}^{\geq 0} \cdot u_i$ are distinct faces of the same convex rational polyhedral cone, we have $\mathbb{R} \cdot u_1 \neq \mathbb{R} \cdot u_i$. Thus we may apply the following lemma 7.14, which proves what we want. **q.e.d.**

7.14. Lemma: Let $(P, \mathcal{X})$ be mixed Shimura data, and $(P_1, \mathcal{X}_1)$, $(P_2, \mathcal{X}_2)$ two rational boundary components of $(P, \mathcal{X})$. Let $0 \neq u_i \in U_i(\mathbb{Q})(-1)$ for $i = 1, 2$ such that $\mathbb{R} \cdot u_1 \neq \mathbb{R} \cdot u_2$. Then there exists an open compact subgroup $K_f \subset P(\mathbb{A}_f)$ such that $\mathbb{R} \cdot p \cdot u_1 \neq \mathbb{R} \cdot u_2$ for all $p \in P(\mathbb{Q}) \cap K_f \cdot P_1(\mathbb{A}_f)$.

Proof. Fix any isomorphism $\lambda : \mathbb{Q} \rightarrow \mathbb{Q}(1)$, and identify U_i with $\text{Lie}(U_i) \subset \text{Lie}(P)$. Then by assumption we have two different lines $\mathbb{Q} \cdot \lambda(u_i)$ in $\text{Lie}(P)(\mathbb{Q})$. We can also view them as two distinct points $[\lambda(u_i)]$ in $(\mathbb{P} \text{Lie}(P))(\mathbb{A}_f) := (\text{Lie}(P)(\mathbb{A}_f) - \{0\})/\mathbb{G}_m(\mathbb{A}_f)$, and it suffices to find K_f such that $[\lambda(u_2)] \notin K_f \cdot P_1(\mathbb{A}_f) \cdot [\lambda(u_1)]$. By the irreducibility of $(P_1, \mathcal{X}_1)$, P_1 acts through scalars on U_1 . Thus it fixes the point $[\lambda(u_1)]$, and $K_f \cdot P_1(\mathbb{A}_f) \cdot [\lambda(u_1)] = K_f \cdot [\lambda(u_1)]$. Choose a neighborhood of $[\lambda(u_1)]$ in $(\mathbb{P} \text{Lie}(P))(\mathbb{A}_f)$ such that a certain $[\lambda(u_2)]$. By the continuity of the $P(\mathbb{A}_f)$ -action we can choose K_f such that $K_f \cdot [\lambda(u_2)]$ is contained in this neighborhood, so we are done. **q.e.d.**

7.15. Proposition: Consider the situation of 7.11. If Σ is complete and the condition of 7.12 holds, then for every sufficiently small $\text{Stab}_{\Delta_1}([\sigma])$ -invariant open neighborhood of $\mathcal{V}$ in $M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1)(\mathbb{C})$ of $M^{\pi_{1, [\sigma]}(K_f^1)}(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]}, \Sigma_{1, [\sigma]})(\mathbb{C})$ the map

$$\text{Stab}_{\Delta_1}([\sigma]) \backslash \mathcal{V} \rightarrow M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$$

of 7.11 is an open embedding.

Proof. Consider the map

$$[\pi_1]^* : M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1)(\mathbb{C}) \rightarrow M^{\pi(K_f^1)}((P_1, \mathcal{X}_1)/W_1)^*(\mathbb{C}).$$

We shall prove that for any two points $x_2, x_3 \in M^{\pi(K_f^1)}((P_1, \mathcal{X}_1)/W_1)^*(\mathbb{C})$ there exist open neighborhoods $\mathcal{V}_i$ of x_i , and open neighborhoods $\mathcal{V}'_i$ of $M^{\pi_{1, [\sigma]}(K_f^1)}(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]}, \Sigma_{1, [\sigma]})(\mathbb{C}) \cap [\pi_1]^{*-1}(\mathcal{V}_i)$, contained in $\overline{\mathcal{U}}_{(1)}/\sim_1$ such that either the image of $\mathcal{V}'_i$ in $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ are disjoint, or the restriction of the map in question to the image of $\mathcal{V}'_2 \cup \mathcal{V}'_3$ in $\text{Stab}_{\Delta_1}([\sigma]) \backslash M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1)(\mathbb{C})$ is an open embedding. By the compactness of $M^{\pi(K_f^1)}((P_1, \mathcal{X}_1)/W_1)^*(\mathbb{C})$ we can then choose finitely many such $\mathcal{V}'_i$, whose union of $\mathcal{V}$ is a $\text{Stab}_{\Delta_1}([\sigma])$ -invariant open neighborhood of $M^{\pi_{1, [\sigma]}(K_f^1)}(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]}, \Sigma_{1, [\sigma]})(\mathbb{C})$ in $\overline{\mathcal{U}}_{(1)}/\sim_1$, such that the above property holds for every

pair of such $\mathcal{V}'_i$. This implies that the restriction to $\text{Stab}_{\Delta_1}([\sigma]) \setminus \mathcal{V}$ of the map in question is injective and locally an open embedding. Thus it is an open embedding, as desired.

To prove the existence of such $\mathcal{V}'_i$, fix rational boundary components $(P_i, \mathcal{X}_i)$ of $(P_1, \mathcal{X}_1)$, and $p_{1,i,f} \in P_1(\mathbb{A}_f)$ for $i = 2, 3$. We may suppose that a fixed connected component $\mathcal{X}_1^0$ of $\mathcal{X}_1$ maps to $\mathcal{X}_i$. Applying 6.22 to this situation we get a Δ_{1i} -invariant open subset $\mathcal{W}_{1i} \subset \partial \mathcal{U}_1(P_i, \mathcal{X}_i, p_{1,i,f})$ such that $\Delta_{1i} \setminus \mathcal{W}_{1i}$ is equal to $[\pi_1]^{*-1}(\mathcal{V}_i)$ as above. Here if Q_i is the admissible $\mathbb{Q}$ -parabolic subgroup of P associated to $(P_i, \mathcal{X}_i)$, then Δ_{1i} is in analogy to 6.18 defined as

$$(\text{Stab}_{(P_1 \cap Q_i)(\mathbb{Q})}(\mathcal{X}_i) \cap P_i(\mathbb{A}_f) \cdot p_{1,i,f} \cdot K_f^1 \cdot p_{1,i,f}^{-1}) / P_i(\mathbb{Q}).$$

By the proof of 6.22 $\mathcal{W}_{1i}$ can be given as follows. There are relatively compact open subset $\mathcal{Y}_i \subset \mathcal{X}_i / W_i$, a finite subset $\{p_{i,\alpha,f}\} \subset P_i(\mathbb{A}_f)$, and an open core $D_{1i} \subset C(\mathcal{X}_1^0, P_i)$, such that $\mathcal{F}_i$ is the image of

$$(\text{im}^{-1}(D_{1i}) \cap \psi_i^{-1}(\mathcal{Y}_i)) \times \{p_{i,\alpha,f}\} \subset \mathcal{X}_1^0 \times P_i(\mathbb{A}_f)$$

in $\mathcal{U}_1(P_i, \mathcal{X}_i, p_{1,i,f})$, and $\mathcal{W}_{1i} := \coprod \pi_\tau(\mathcal{F}_i)$, where the union is extended over all cones $\tau \in \Sigma_1(\mathcal{X}_1^0, P_i, p_{i,\alpha,f} \cdot p_{1,i,f}) = \Sigma(\mathcal{X}^0, P_i, p_{i,\alpha,f} \cdot p_{1,i,f} \cdot p_f)$ for some α . The intersection of $\Delta_{1i} \setminus \mathcal{W}_{1i}$ in $M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1)(\mathbb{C})$ with $M^{\pi_{1,[\sigma]}(K_f^1)}(P_{1,[\sigma]}, \mathcal{X}_{1,[\sigma]}, \Sigma_{1,[\sigma]})(\mathbb{C})$ is the image of the union of all $\pi_\tau(\mathcal{F}_i)$ such that some representative of $[\sigma]$ is a face of τ . We can therefore choose $\mathcal{V}'_i := \Delta_{1i} \setminus \mathcal{W}'_i$ for every small Δ_{1i} -invariant open subset $\mathcal{W}'_i \subset \mathcal{W}_{1i} \cap \overline{\mathcal{U}}(P_i, \mathcal{X}_i, p_{1,i,f} \cdot p_f)$ that contains all these $\pi_\tau(\mathcal{F}_i)$.

Let D_i be an open core in $C(\mathcal{X}^0, P_i)$. If the both D_i and D_{1i} are given as the interior of the standard core with respect to the same lattice, then $D_{1i} = D_i + U_1(\mathbb{R})(-1)$. We shall prove below that $\mathcal{W}'_i$ can be chosen to satisfy the following additional property: its intersection with $\mathcal{U}(P_i, \mathcal{X}_i, p_{1,i,f} \cdot p_f)$ is equal to the image of $(\text{im}^{-1}(D_i \cap \coprod \tau^0) \cap \psi_i^{-1}(\mathcal{Y}_i)) \times \{p_{i,\alpha,f}\}$ in $\mathcal{U}(P_i, \mathcal{X}_i, p_{1,i,f} \cdot p_f)$, where the disjoint union $\coprod \tau^0$ is extended over all $\tau \in \Sigma(\mathcal{X}^0, P_i, p_{i,\alpha,f} \cdot p_{1,i,f} \cdot p_f)$ for all α such that some representative of $[\sigma]$ is a face of τ . Note that such $\mathcal{W}'_i$ is contained in some $\mathcal{W}_i$ as in 6.22, applied to $(P_i, \mathcal{X}_i)$ as a boundary component of $(P, \mathcal{X})$.

If the images of $\mathcal{V}'_2$ and $\mathcal{V}'_3$ in $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ are not disjoint, then the same holds for $\mathcal{W}'_2$ and $\mathcal{W}'_3$, hence also for $\mathcal{W}_2$ and $\mathcal{W}_3$ above. Thus by 6.22, if the D_i are chosen sufficiently small, the two $\partial \mathcal{U}(P_i, \mathcal{X}_i, p_{1,i,f} \cdot p_f)$ must map to the same boundary stratum of $M^{\pi(K_f)}((P, \mathcal{X})/W)^*(\mathbb{C})$. In other words there exist $p \in P(\mathbb{Q})$ and $k_f \in K_f$ such that $\text{int}(p)((P_2, \mathcal{X}_2)) = (P_3, \mathcal{X}_3)$ and $p \cdot p_{1,2,f} \cdot p_f \cdot k_f = p_{1,3,f} \cdot p_f$. Thus by 4.20 (iii) there exists $p \in P(\mathbb{Q})$ such that $\text{int}(p_1)((P_2, \mathcal{X}_2)) = (P_3, \mathcal{X}_3)$, so after applying the extension 6.15 of the map $[\text{int}(p_1)]$ of 6.11 (c) we may assume that $(P_3, \mathcal{X}_3) = (P_2, \mathcal{X}_2)$. Then every such p lies in $Q_2(\mathbb{Q})$.

Now for all such p we also have

$$p \cdot (\text{im}^{-1}(D_2 \cap \coprod \tau^0) \cap \psi_2^{-1}(\mathcal{Y}_2)) \cap (\text{im}^{-1}(D_3 \cap \coprod \tau'^0) \cap \psi_3^{-1}(\mathcal{Y}_3)) \neq \emptyset,$$

hence $(p \cdot \coprod \tau^0 \cap \coprod \tau'^0) \neq \emptyset$. Thus there exist $\tau \in \Sigma(\mathcal{X}^0, P_2, p_{2,\alpha,f} \cdot p_{1,2,f} \cdot p_f)$ and $\tau' \in \Sigma(\mathcal{X}^0, P_3, p_{3,\beta,f} \cdot p_{1,3,f} \cdot p_f)$, such that both τ and τ' have a face in $[\sigma]$, and $p \cdot \tau^0 \cap \tau'^0 \neq \emptyset$. The latter relation implies $p \cdot \tau = \tau'$, which should really be written in the form $p \cdot \tau \cdot k_f = \tau'$ if

we view these cones as subsets of $C^*(\mathcal{X}^0, P_2) \times P(\mathbb{A}_f)$. Let σ , resp. σ' be the representatives of $[\sigma]$ such that σ is a face of τ , and σ' is a face of τ' , then there exist $p_1 \in P_1(\mathbb{Q})$ and $p_{1,f} \in P_1(\mathbb{A}_f)$ such that $\sigma' = p_1 \cdot p_{1,f} \cdot \sigma$. It follows that both σ and $(p^{-1} \cdot p_1) \cdot p_{1,f} \cdot \sigma \cdot k_f$ are faces of τ . Thus the condition in 7.12 implies $(p^{-1} \cdot p_1) \cdot p_{1,f} \cdot \sigma \cdot k_f = \sigma$. As a first consequence, $p^{-1} \cdot p_1$ maps $C(\mathcal{X}^0, P_1)$ to itself, so $p^{-1} \cdot p_1$, hence also p , stabilizes $(P_1, \mathcal{X}_1)$. Thus p lies in $\text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}_1)$. Using this, the relation $p \cdot p_{1,2,f} \cdot p_f \cdot k_f = p_{1,3,f} \cdot p_f$ now implies that p also lies in $P_1(\mathbb{A}_f) \cdot p_f \cdot K_f \cdot p_f^{-1}$. Thus it defines a class $[p]$ in Δ_1 , and this class stabilizes $[\sigma]$. Hence after applying $[\text{int}(p^{-1})]$ to $\mathcal{U}(P_2, \mathcal{X}_2, p_{1,2,f} \cdot p_f)$ we are reduced to the case where $p_{1,3,f} = p_{1,2,f}$.

In this case we may enlarge the $\mathcal{W}'_i$ so that $\mathcal{W}'_3 = \mathcal{W}'_2$. It remains to show that the map $\mathcal{W}'_2 \hookrightarrow \mathcal{W}_2 \rightarrow M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ factors through an open embedding of the image of $\mathcal{W}'_2$ in $\text{Stab}_{\Delta_1}([\sigma]) \backslash M^{K_f}(P_1, \mathcal{X}_1, \Sigma_1)(\mathbb{C})$. Now by 6.22 the map $\mathcal{W}_2 \rightarrow M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ factors through an open embedding of $\Delta_2 \backslash \mathcal{W}_2$, where

$$\Delta_2 := (\text{Stab}_{Q_2(\mathbb{Q})}(\mathcal{X}_2) \cap (P_2(\mathbb{A}_f) \cdot p_{2,f} \cdot p_f \cdot K_f \cdot (p_{2,f} \cdot p_f)^{-1})) / P_2(\mathbb{Q}).$$

The argument above shows that for any $[q] \in \Delta_2$ with $[q] \cdot \mathcal{W}'_2 \cap \mathcal{W}'_2 \neq \emptyset$ we already have $q \in \text{Stab}_{Q_1}(\mathcal{X}_1) \cap (P_1(\mathbb{A}_f) \cdot p_f \cdot K_f \cdot p_f^{-1})$, and the class $[q] \in \Delta_1$ stabilizes $[\sigma]$. Thus the map $\mathcal{W}'_2 \hookrightarrow \mathcal{W}_2 \rightarrow \Delta_2 \backslash \mathcal{W}_2$ factors through an open embedding of the image of $\mathcal{W}'_2$ in $\text{Stab}_{\Delta_1}([\sigma]) \backslash M^{K_f}(P_1, \mathcal{X}_1, \Sigma_1)(\mathbb{C})$, as desired.

It remains to prove the assertion about the special choice of $\mathcal{W}'_i$. This is done in the following lemma. **q.e.d.**

7.16. Lemma: In the notation of the proof of 7.15 there exists a Δ_{1i} -invariant open subset $\mathcal{W}'_i \subset \mathcal{W}_{1i} \cap \overline{\mathcal{U}}(P_i, \mathcal{X}_i, p_{1,i,f} \cdot p_f)$ with the properties:

- (i) It contains the τ -stratum of $\mathcal{W}_{1i}$ for all α and every $\tau \in \Sigma(\mathcal{X}^0, P_i, p_{i,\alpha,f} \cdot p_{1,i,f} \cdot p_f)$, such that some representative of $[\sigma]$ is a face of τ .
- (ii) Its intersection with $\mathcal{U}(P_i, \mathcal{X}_i, p_{1,i,f} \cdot p_f)$ is equal to the image of $(\text{im}^{-1}(D_i \cap \coprod \tau^0) \cap \psi_i^{-1}(\mathcal{Y}_i)) \times \{p_{i,\alpha,f}\}$ in $\mathcal{U}(P_i, \mathcal{X}_i, p_{1,i,f} \cdot p_f)$, where the disjoint union $\coprod \tau^0$ is extended over all $\tau \in \Sigma(\mathcal{X}^0, P_i, p_{i,\alpha,f} \cdot p_{1,i,f} \cdot p_f)$ for all α such that some representative of $[\sigma]$ is a face of τ .

Proof. For every α and every $\rho \in \Sigma(\mathcal{X}^0, P_i, p_{i,\alpha,f} \cdot p_{1,i,f} \cdot p_f)$, such that some representative of $[\sigma]$ is a face of ρ , let $\mathcal{F}_{i,\rho}$ be the image of

$$(\text{im}^{-1}(D_i \cap \coprod \tau^0) \cap \psi_i^{-1}(\mathcal{Y}_i)) \times \{p_{i,\alpha,f}\} \subset \mathcal{X}_1^0 \times P_i(\mathbb{A}_f)$$

in $\mathcal{U}(P_i, \mathcal{X}_i, p_{1,i,f})$, where the disjoint union is extended over all $\tau \in \Sigma(\mathcal{X}^0, P_i, p_{i,\alpha,f} \cdot p_{1,i,f} \cdot p_f)$ such that ρ is a face of τ . Let $\mathcal{W}'_{i,\rho}$ be the union of $\pi_{\rho'}(\mathcal{F}_{i,\rho})$ for all faces ρ' of ρ , and $\mathcal{W}'_i$ the union of all these $\mathcal{W}'_{i,\rho}$. Clearly $\mathcal{W}'_i$ is Δ_{1i} -invariant and satisfies property (ii). To prove the rest it suffices to show that every $\mathcal{W}'_{i,\rho}$ is open and contains the ρ -stratum of $\mathcal{W}_{1i}$.

So fix ρ , and consider the subset $V := D_i \cap \coprod_{\rho \text{ face of } \tau} \tau^0$ above. By definition it satisfies $V = V + \rho$, so to prove the openness of $\mathcal{W}'_{i,\rho}$ it suffices by 5.9 (a) to show that V is open in $U_i(\mathbb{R})(-1)$. Likewise consider the ρ -stratum $\pi_\rho(\mathcal{F}_i)$ of $\mathcal{W}_{1i}$ and the ρ -stratum $\pi_\rho(\mathcal{F}_{i,\rho})$ of $\mathcal{W}'_{i,\rho}$. Since $\mathcal{F}_i$ is the image of $(\text{im}^{-1}(D_{1i} \cap \psi_i^{-1}(\mathcal{Y}_i)) \times \{p_{i,\alpha,f}\})$ in $\mathcal{U}_1(P_i, \mathcal{X}_i, p_{1,i,f})$,

and $\mathcal{F}_{i,\rho}$ that of $(\text{im}^{-1}(D_i \cap \coprod \tau^0) \cap \psi_i^{-1}(\mathcal{Y}_i)) \times \{p_{i,\alpha,f}\}$, they two ρ -strata are equal if and only if $\mathbb{R} \cdot \rho + D_{1i} = \mathbb{R} \cdot \rho + V$.

To prove that V is open in $U_i(\mathbb{R})(-1)$ fix $u \in D_i \cap \tau^0$ such that ρ is a face of τ . Since $u \in D_i \subset C(\mathcal{X}^0, P_i)$, there exists a convex rational polyhedral cone $\pi \subset C^*(\mathcal{X}^0, P_1)$ such that $D_i \cap \pi$ is a neighborhood of u in $U_i(\mathbb{R})(-1)$. As in the proof of 6.27 the completeness of Σ implies that $\pi \cap C(\mathcal{X}^0, P_1)$ is contained in the union of finitely many cones $\tau' \in \Sigma(\mathcal{X}^0, P_i, p_{i,\alpha,f} \cdot p_{1,i,f} \cdot p_f)$. Thus a union of finitely many such τ' is a neighborhood of u . Since each τ' is closed, we may leave out those which do not contain u . On the other hand if $u \in \tau'$, then $\tau^0 \cap \tau' \neq \emptyset$, so τ must be a face of τ' . Hence ρ is a face of τ' , so a union of such τ' is a neighborhood of u . Since D_i is open, this shows that V is a neighborhood of u , as desired.

To prove the equality $\mathbb{R} \cdot \rho + D_{1i} = \mathbb{R} \cdot \rho + V$ note that the inclusion “ $\supset$ ” holds by definition. Thus it remains to show $D_{1i} \subset \mathbb{R} \cdot \rho + V$. Since

$$\begin{aligned} D_{1i} &= D_i + U_1(\mathbb{R})(-1) \\ &= D_i + C^*(\mathcal{X}^0, P_i) + U_1(\mathbb{R})(-1) \\ &= D_i + C^*(\mathcal{X}^0, P_i) + \mathbb{R} \cdot \sigma \\ &= D_i + \mathbb{R} \cdot \sigma \\ &\subset D_i + \mathbb{R} \cdot \rho, \end{aligned}$$

this is equivalent to $D_i \subset \mathbb{R} \cdot \rho + V$. To prove this fix $u \in D_i$ and $u_0 \in \rho^0$, and let π be the convex rational polyhedral cone $\mathbb{R}^{\geq 0} \cdot u + \mathbb{R}^{\geq 0} \cdot u_0$. By the same argument as above $\pi \cap C(\mathcal{X}^0, P_i)$ is covered by finitely many $\tau \in \Sigma(\mathcal{X}^0, P_i, p_{i,\alpha,f} \cdot p_{1,i,f} \cdot p_f)$. Thus there exists one such τ such that $\lambda \cdot u + (1-\lambda) \cdot u_0 \in \tau^0$ for all sufficiently small $\lambda > 0$. This implies that u_0 is contained in the closure of τ^0 , so ρ must be a face of τ . It also shows that $u + \mu \cdot u_0 \in \tau^0$ for every sufficiently big real number μ . Since $D_i = D_i + C^*(\mathcal{X}^0, P_i)$, and $u \in D_i$, we also have $u + \mu \cdot u_0 \in D_i$ for every $\mu > 0$. Thus $u = -\mu \cdot u_0 + (u + \mu \cdot u_0) \in \mathbb{R} \cdot \rho + D_i \cap \tau^0$, as desired. **q.e.d.**

7.17. Corollary: Consider $(P, \mathcal{X})$, K_f , Σ , $(P_1, \mathcal{X}_1)$, p_f , K_f^1 , Σ_1 , and $[\sigma]$ as in 7.3. Assume that Σ is complete, and that (K_f, Σ) satisfies the condition in 7.12. Then:

(a) The closure of the stratum

$$\text{Stab}_{\Delta_1}([\sigma]) \backslash M^{\pi_{1,[\sigma]}(K_f^1)}(P_{1,[\sigma]}, \mathcal{X}_{1,[\sigma]})(\mathbb{C})$$

in $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is canonically isomorphic to

$$\text{Stab}_{\Delta_1}([\sigma]) \backslash M^{\pi_{1,[\sigma]}(K_f^1)}(P_{1,[\sigma]}, \mathcal{X}_{1,[\sigma]}, \Sigma_{1,[\sigma]})(\mathbb{C}),$$

where $\Sigma_{1,[\sigma]}$ is as defined in 7.7.

(b) Every sufficiently small open neighborhood of this closure is canonically isomorphic to $\text{Stab}_{\Delta_1}([\sigma]) \backslash \mathcal{V}$ for some $\text{Stab}_{\Delta_1}([\sigma])$ -invariant open neighborhood $\mathcal{V}$ of $M^{\pi_{1,[\sigma]}(K_f^1)}(P_{1,[\sigma]}, \mathcal{X}_{1,[\sigma]}, \Sigma_{1,[\sigma]})(\mathbb{C})$, in $M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1)(\mathbb{C})$. In particular $M^{K_f}(P, \mathcal{X})(\mathbb{C}) \hookrightarrow M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is a toroidal embedding without self-intersection in the sense of [KKMS] ch.II §1 p.57.

Remark. The assertion also holds without the assumption that Σ is complete.

Proof. Both assertions are direct consequences of 7.11 and 7.15.

q.e.d.

Chapter 8

Construction of ample line bundles

In this chapter we construct ample invertible sheaves on $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ for certain cone decompositions Σ . This is the first step towards the algebraization of $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$; the second step is carried out in the next chapter, where we shall construct the necessary cone decompositions. The main result of this chapter is 8.14, which is a generalization of Tai's theorem in [AMRT] ch.IV §2.1 p.312.

First, we express well-known facts about the canonical sheaves on Shimura varieties in our language. We show (8.1) that the canonical sheaf on $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ with at most logarithmic poles along the boundary is isomorphic to the pullback of a canonically defined invertible sheaf on the Baily-Borel compactification $M^{\pi(K_f)}((P, \mathcal{X})/W)^*(\mathbb{C})$. This sheaf is ample (8.4). Thus, to construct an ample invertible sheaf on $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ it remains to construct one which is ample on every fibre of the projection $[\pi]^* : M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}) \rightarrow M^{\pi(K_f)}((P, \mathcal{X})/W)^*(\mathbb{C})$.

As explained in 8.15, we need a more sophisticated approach for this than in [AMRT] ch.IV §2. The main idea is to interpret some morphisms $M^{K'_f}(P', \mathcal{X}', \Sigma')(\mathbb{C}) \rightarrow M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ of 6.25 as line bundles, and to give sufficient conditions for such line bundles to be relatively ample (or anti-ample) over the Baily-Borel compactification. In 8.5–6 we describe the conditions under which the above morphism represents a line bundle. The ampleness condition has a geometric content like in 5.19, and is better expressed for anti-ample bundles. Thus in 8.13 we give a sufficient condition for the inverse of this line bundle to be relatively ample over $M^{\pi(K_f)}((P, \mathcal{X})/W)^*(\mathbb{C})$.

All in all, this shows that in certain cases an ample line bundle can be described in terms of other mixed Shimura varieties and their internal geometry. This will play a role in chapter 12.

8.1. Proposition: Let $(P, \mathcal{X})$ be mixed Shimura data and K_f a neat open subgroup of $P(\mathbb{A}_f)$. There exists an invertible sheaf $\mathcal{L}_p$ on $M^{\pi(K_f)}((P, \mathcal{X})/W)^*(\mathbb{C})$, up to isomorphism depending only on $(P, \mathcal{X})/U$, such that for every K_f -admissible partial cone decomposition Σ for $(P, \mathcal{X})$ there is an isomorphism

$$([\pi]^*)^* \mathcal{L}_p \cong \omega[\mathrm{dlog}],$$

where $\omega[\text{dlog}]$ is the canonical sheaf on $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ with logarithmic poles along $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}) \setminus M^{K_f}(P, \mathcal{X})(\mathbb{C})$, locally defined as in 5.26.

Proof. First suppose that $(P, \mathcal{X})$ is irreducible, and consider the restriction of $\omega[\text{dlog}]$ to $M^{K_f}(P, \mathcal{X}, \Sigma^0)(\mathbb{C})$. Then in the notations of 6.8 $[\pi'] : M^{K_f}(P, \mathcal{X}, \Sigma^0)(\mathbb{C}) \rightarrow M^{\pi'(K_f)}((P, \mathcal{X})/U)^*(\mathbb{C})$ is a relative torus embedding with respect to the family of tori $M^{K_f^U \rtimes \phi(K_f)}(P_*, \mathcal{X}_*)(\mathbb{C}) \rightarrow M^{\phi(K_f)}(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)(\mathbb{C})$. Fix an invariant differential form on every connected component of $M^{K_f \rtimes \phi(K_f)}(P_*, \mathcal{X}_*)(\mathbb{C})$. Then 5.27 induces an isomorphism between $\omega[\text{dlog}]|_{M^{K_f}(P, \mathcal{X}, \Sigma^0)(\mathbb{C})}$ and the pullback of the canonical sheaf of $M^{\pi'(K_f)}((P, \mathcal{X})/U)(\mathbb{C})$. This isomorphism is canonical up to a scalar factor on each connected component of $M^{K_f}(P, \mathcal{X}, \Sigma^0)(\mathbb{C})$.

Next by 3.14 every fibre of $M^{\pi'(K_f)}((P, \mathcal{X})/U)(\mathbb{C}) \rightarrow M^{\pi(K_f)}((P, \mathcal{X})/W)(\mathbb{C})$ is a compact complex torus. Since this is a smooth family, the relative canonical sheaf is the pullback of a unique invertible sheaf on $M^{\pi(K_f)}((P, \mathcal{X})/W)(\mathbb{C})$, the sheaf of invariant relative top differentials. For the (absolute) canonical sheaf the analogous fact follows. Thus there is a natural invertible sheaf on $M^{\pi(K_f)}((P, \mathcal{X})/W)(\mathbb{C})$, depending only on $(P, \mathcal{X})/U$, whose pullback to $M^{K_f}(P, \mathcal{X}, \Sigma^0)(\mathbb{C})$ is isomorphic to $\omega[\text{dlog}]$. It is important that this isomorphism is canonical up to a scalar factor on each connected component.

Let us now go back to the general case. The preceding remarks, applied to $\mathcal{W}_1 \subset M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1^0)(\mathbb{C})$ as in 6.22, show that the restriction of $\omega[\text{dlog}]$ to $\Delta_1 \setminus \mathcal{W}_1 \subset M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is isomorphic to the pullback of a natural invertible sheaf on $\Delta_1 \setminus M^{\pi_1(K_f^1)}((P_1, \mathcal{X}_1)/W_1)(\mathbb{C})$. Hence by 7.5 it is also isomorphic to the pullback of a natural invertible sheaf on $\mathcal{V}_1 \subset M^{\pi(K_f)}((P, \mathcal{X})/W)^*(\mathbb{C})$. Since these isomorphisms, for a covering of $M^{\pi(K_f)}((P, \mathcal{X})/W)^*(\mathbb{C})$, are canonical up to a scalar on each connected component, they define a global isomorphism between $\omega[\text{dlog}]$ and the pullback of an invertible sheaf $\mathcal{L}_p$ on $M^{\pi(K_f)}((P, \mathcal{X})/W)^*(\mathbb{C})$. Clearly it depends only on $(P, \mathcal{X})/U$, and not on Σ or even the existence of Σ with certain properties, e.g. completeness. **q.e.d.**

8.2. Projectivity of the Baily-Borel compactification: Let $(P, \mathcal{X})$ be pure Shimura data and K_f a neat open subgroup of $P(\mathbb{A}_f)$. Denote by $M^{K_f}(P, \mathcal{X})^+(\mathbb{C})$ the union of $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ with all boundary strata of codimension 1. This is smooth, its complement in $M^{K_f}(P, \mathcal{X})^*(\mathbb{C})$ has codimension ≥ 2 , and the complement of $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ in $M^{K_f}(P, \mathcal{X})^+(\mathbb{C})$ is a smooth divisor. Hence we may consider the sheaf $\omega[\text{dlog}]$ of all top differential forms on $M^{K_f}(P, \mathcal{X})^+(\mathbb{C})$ with at most simple poles along $M^{K_f}(P, \mathcal{X})^+(\mathbb{C}) \setminus M^{K_f}(P, \mathcal{X})(\mathbb{C})$.

For any integer n the space of sections

$$\Gamma(M^{K_f}(P, \mathcal{X})^+(\mathbb{C}), \omega[\text{dlog}]^{\otimes n})$$

is isomorphic to the space of holomorphic automorphic forms of a certain weight. Indeed $\Gamma(M^{K_f}(P, \mathcal{X})(\mathbb{C}), \omega[\text{dlog}]^{\otimes n})$ is isomorphic to a space of holomorphic automorphic forms “without condition at ∞ ” (compare [AMRT] ch.IV §1). By the Koecher principle the condition at ∞ is necessary only for each simple factor of P^{ad} that is isomorphic to $PGL_{2, \mathbb{Q}}$. In that case it is well-known and easy to check that the condition at ∞ is equivalent to the extendability to a section of $\omega[\text{dlog}]^{\otimes n}$.

Fix an integer $n \geq 1$ and consider the projective space

$$\mathbb{P}^N := (\Gamma(M^{K_f}(P, \mathcal{X})^+(\mathbb{C}), \omega[\mathrm{dlog}]^{\otimes n}) \setminus \{0\})/\mathbb{C}^\times.$$

If $\omega[\mathrm{dlog}]^{\otimes n}$ on $M^{K_f}(P, \mathcal{X})^+(\mathbb{C})$ is generated by global sections, then there is a canonical holomorphic map $M^{K_f}(P, \mathcal{X})^+(\mathbb{C}) \rightarrow \mathbb{P}^N$. Now [BB] thm. 10.11 asserts that for some $n \geq 1$, $\omega[\mathrm{dlog}]^{\otimes n}$ is generated by global sections over $M^{K_f}(P, \mathcal{X})^+(\mathbb{C})$, and the canonical map $M^{K_f}(P, \mathcal{X})^+(\mathbb{C}) \rightarrow \mathbb{P}^N$ extends to a closed embedding $M^{K_f}(P, \mathcal{X})^*(\mathbb{C}) \rightarrow \mathbb{P}^N$. Observe that this then follows for every positive multiple of n .

Denote by $\mathcal{O}(1)$ the pullback of $M^{K_f}(P, \mathcal{X})^*(\mathbb{C})$ of the standard ample sheaf on $\mathbb{P}^N$. By the above we have an isomorphism $\mathcal{O}(1)|_{M^{K_f}(P, \mathcal{X})^+(\mathbb{C})} \cong \omega[\mathrm{dlog}]^{\otimes n}$. On the other hand we have by construction $\mathcal{L}_{P|M^{K_f}(P, \mathcal{X})^+(\mathbb{C})} \cong \omega[\mathrm{dlog}]$, since P is reductive. The following lemma now implies $\mathcal{O}(1) \cong \mathcal{L}_P^{\otimes n}$, so $\mathcal{L}_P$ is ample.

8.3. Lemma: Let X be a normal complex space and Z a closed subspace of codimension at least 2. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two invertible sheaves on X , whose restrictions to $X \setminus Z$ are isomorphic. Then they are already isomorphic over X .

Proof. We shall show that any given isomorphism $\mathcal{L}_1|_{X \setminus Z} \cong \mathcal{L}_2|_{X \setminus Z}$ extends to an isomorphism $\mathcal{L}_1 \cong \mathcal{L}_2$. Since this is a local question, we may assume that $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{O}_X$. Then the isomorphism corresponds to a section $f \in \Gamma(X \setminus Z, \mathcal{O}_X^*)$. Now by Riemann's extension theorem (see [GR] ch.7 §4.2 p.144) both f and f^{-1} extend to holomorphic functions on X . Thus f extends to a section in $\Gamma(X, \mathcal{O}_X^*)$, as desired. **q.e.d.**

More generally, we have the following proposition (which will not be needed in the sequel).

8.4. Proposition: For arbitrary mixed Shimura data, the invertible sheaf $\mathcal{L}_P$ defined in 8.1 is ample.

Sketch of proof: Since ampleness is invariant under finite coverings, using 7.10 we may reduce to the case where $(P, \mathcal{X})$ is irreducible. Next, by 8.1 $\mathcal{L}_P$ depends only on $(P, \mathcal{X})/U$, so we may assume that $U = 1$. Put $\mathcal{L}_P^{\mathrm{rel}} := \mathcal{L}_{P \otimes (\mathcal{L}_{P/W})^{\otimes -1}}$. Since by 8.2 $\mathcal{L}_{P/W}$ is ample, it suffices to show that some positive tensor power of $\mathcal{L}_P^{\mathrm{rel}}$ is generated by global sections. If $V = 1$, then we are done. Otherwise let $2 \cdot g = \dim(V) > 0$, then as in the proof of 2.26 we can find a morphism $\phi: (P, \mathcal{X})/V \rightarrow (CS_{P2g, \mathbb{Q}}, \mathcal{H}_{2g})$ and an isomorphism

$$(P, \mathcal{X}) \cong (V_{2g} \rtimes CS_{P2g, \mathbb{Q}}, \mathcal{H}'_{2g}) \times_{(CS_{P2g, \mathbb{Q}}, \mathcal{H}_{2g})} (P, \mathcal{X})/V.$$

Abbreviating $(P', \mathcal{X}') \cong (V_{2g} \rtimes CS_{P2g, \mathbb{Q}}, \mathcal{H}'_{2g})$, it is easy to check from the definition that $\mathcal{L}_P^{\mathrm{rel}} \cong [\phi]^* \mathcal{L}_{P'}^{\mathrm{rel}}$, where $\mathcal{L}_{P'}^{\mathrm{rel}} = \mathcal{L}_{P'} \otimes (\mathcal{L}_{P'/W'})^{\otimes -1}$. Thus it suffices to prove the same assertion for $\mathcal{L}_{P'}^{\mathrm{rel}}$.

By 8.2 this clearly follows from the well-known formula $\mathcal{L}_{P'/W'} \cong (\mathcal{L}_{P'}^{\mathrm{rel}})^{\otimes (g+1)}$. To prove it, note that the restriction of $\mathcal{L}_{P'}^{\mathrm{rel}}$ to $M^{K_f}(CS_{P2g, \mathbb{Q}}, \mathcal{H}_{2g})(\mathbb{C})$ is isomorphic to the pullback of the sheaf of relative top differentials under the zero section, and $\mathcal{L}_{P'/W'}$ is isomorphic to the canonical sheaf. Over each connected of $M^{K_f}(CS_{P2g, \mathbb{Q}}, \mathcal{H}_{2g})(\mathbb{C})$ the formula follows

from an easy calculation with the automorphy factors. If $g \geq 2$, then by lemma 8.3 the formula follows over the whole Baily-Borel compactification, as desired. In the case $g = 1$ one must have a closer look at the boundary to get the same result. Aliter: Let $(P'', \mathcal{X}'')$ be the fibre product of $(P', \mathcal{X}')$ with itself over $(P', \mathcal{X}')/V'$. As above, one easily shows that $(\mathcal{L}_{P'}^{rel})^{\otimes 2} \cong \mathcal{L}_{P''}^{rel}$, and the assertion follows from that for $\mathcal{L}_{P''}^{rel}$. **q.e.d.**

8.5. $\mathbb{C}^\times$ -torsors and cone decompositions: Consider the situation of 3.17, i.e. a $((P_0, \mathcal{X}_0) \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{X}_0))$ -torsor $(P', \mathcal{X}') \rightarrow (P, \mathcal{X})$. Let K'_f be a neat open compact subgroup of $P'(\mathbb{A}_f)$, and K_f its image in $P(\mathbb{A}_f)$. By 3.17 the corresponding mixed Shimura varieties form a $\mathbb{C}^\times$ -torsor. Consider the associated line bundle on $M^{K_f}(P, \mathcal{X})(\mathbb{C})$, and some admissible cone decompositions Σ' resp. Σ . So we want to show that under certain conditions, $M^{K'_f}(P', \mathcal{X}', \Sigma')(\mathbb{C}) \rightarrow M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ has a canonical structure as a line bundle, extending the given line bundle on $M^{K_f}(P, \mathcal{X})(\mathbb{C})$.

Consider a connected component $\mathcal{X}'^0$ of $\mathcal{X}'$, a rational boundary component $(P'_1, \mathcal{X}'_1)$ of $(P', \mathcal{X}')$ such that $\mathcal{X}'^0$ maps to $\mathcal{X}'_1$, and $p'_f \in P'(\mathbb{A}_f)$. They determine corresponding objects $\mathcal{X}^0$, $(P_1, \mathcal{X}_1)$, and p_f for $(P, \mathcal{X})$. Then $\Sigma'(\mathcal{X}'^0, P'_1, p'_f)$ is a rational partial polyhedral decomposition of $C^*(\mathcal{X}'^0, P'_1) \subset U'_1(\mathbb{R})(-1)$, and $\Sigma(\mathcal{X}^0, P_1, p_f)$ one of $C^*(\mathcal{X}^0, P_1) \subset U_1(\mathbb{R})(-1)$. Let $\Gamma_{U'} \subset U'_1(\mathbb{Q})$ be the image of

$$(\{z \in Z(P')(\mathbb{Q}) \mid z|_{\mathcal{X}'} = \text{id}\} \times U'_1(\mathbb{Q})) \cap p'_f \cdot K'_f \cdot p'^{-1}_f$$

under the projection $Z(P') \times U'_1 \rightarrow U'_1$, and define $\Gamma_U \subset U_1(\mathbb{Q})$ likewise. It is easy to see that the short exact sequence $0 \rightarrow U_0 \rightarrow U'_1 \rightarrow U_1 \rightarrow 0$ induces a short exact sequence

$$0 \rightarrow U_0(\mathbb{Q}) \cap \Gamma_{U'} \rightarrow \Gamma_{U'} \rightarrow \Gamma_U \rightarrow 0.$$

Recall the definition of the cone decomposition Σ_0 for $(P_0, \mathcal{X}_0)$ in 6.9: Every connected component $\mathcal{X}_O^0$ of $\mathcal{X}_O$ corresponds to an isomorphism $\lambda : U_O(\mathbb{R})(-1) \rightarrow U_O(\mathbb{R}) = \mathbb{R}$, and for every $p_{O < f} \in P_O(\mathbb{A}_f)$ we have $\Sigma_O(\mathcal{X}_O^0, P_O, p_{O, f}) = \{\{0\}, \lambda^{-1}(\mathbb{R}^{\geq 0})\} \times \{p_{O, f}\}$. Since $\mathcal{X}'^0$ maps to a unique connected component of $\mathcal{X}_O$, we get a canonical convex rational polyhedral cone

$$\sigma_0 := \lambda^{-1}(\mathbb{R}^{\geq 0}) \subset U_O(\mathbb{R})(-1) \subset C^*(\mathcal{X}'^0, P'_1) \subset U'_1(\mathbb{R})(-1).$$

that depends only on $\mathcal{X}'^0$.

Now 5.9 together with the proof of 6.26 tells us which condition to impose on the K'_f -admissible cone decomposition Σ' . Namely:

(*) For every $\mathcal{X}'^0$, $(P'_1, \mathcal{X}'_1)$ and p'_f , and every $\sigma \in \Sigma(\mathcal{X}^0, P_1, p_f)$ there exists a splitting $e_\sigma : U_1 \rightarrow U'_1$, which induces a splitting for the short exact sequence $0 \rightarrow U_O(\mathbb{Q}) \cap \Gamma_{U'} \rightarrow \Gamma_{U'} \rightarrow \Gamma_U \rightarrow 0$, such that

$$\Sigma'(\mathcal{X}'^0, P'_1, p'_f) = \bigcup_{\sigma \in \Sigma(\mathcal{X}^0, P_1, p_f)} \{e_\sigma(\sigma), e_\sigma(\sigma) + \sigma_0\}.$$

8.6. Proposition: If the condition (*) in 8.5 holds, then

$$M^{K'_f}(P', \mathcal{X}', \Sigma')(\mathbb{C}) \rightarrow M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$$

has a canonical structure as a line bundle. Its restriction to $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ is a line bundle associated to the $\mathbb{C}^\times$ -torsor $M^{K'_f}(P', \mathcal{X}')(\mathbb{C}) \rightarrow M^{K_f}(P, \mathcal{X})(\mathbb{C})$ of 3.17.

Proof. By the proof of 6.26 the given map is locally isomorphic to the map

$$(\Gamma_{U'} \backslash \mathcal{X}'_1)_{\Sigma'(\mathcal{X}'^0, P'_1, p'_f)} \longrightarrow (\Gamma_U \backslash \mathcal{X}_1)_{\Sigma(\mathcal{X}^0, P_1, p_f)}$$

with $\Gamma_{U'}$ and Γ_U as in 8.5. It is easily checked from their definition that they lie in a short exact sequence $1 \rightarrow \Gamma_{U_0} \rightarrow \Gamma_{U'} \rightarrow \Gamma_U \rightarrow 1$, where $\Gamma_{U_0} := U_0(\mathbb{Q}) \cap \Gamma_{U'}$. Thus these torus embeddings corresponds to the short exact sequence of tori over $\mathbb{C}$

$$0 \rightarrow U_0(\mathbb{C})/\Gamma_{U_0} \rightarrow U'(\mathbb{C})/\Gamma_{U'} \rightarrow U(\mathbb{C})/\Gamma_U \rightarrow 0,$$

so the assertion follows directly from 5.11. q.e.d.

8.7. Pullback of line bundles: We want to show that the line bundles in 8.6 behave nicely under pullback by the morphisms 6.25. We only consider the case 6.25 (b), the other case being more trivial. So in addition to the data in 8.5, consider a morphism $\phi : (P^*, \mathcal{X}^*) \rightarrow (P, \mathcal{X})$, an open compact subgroup $K_f^* \subset P^*(\mathbb{A}_f)$, and a K_f^* -admissible cone decomposition Σ^* for $(P^*, \mathcal{X}^*)$ such that the morphism $[\phi] : M^{K_f^*}(P^*, \mathcal{X}^*, \Sigma^*)(\mathbb{C}) \rightarrow M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ of 6.25 (b) exists.

Define the fibre product of $(P'^*, \mathcal{X}'^*) := (P', \mathcal{X}') \times_{(P, \mathcal{X})} (P^*, \mathcal{X}^*)$ and $K_f'^* := K_f' \times_{K_f} K_f^*$. Consider a rational boundary component $(P_1^*, \mathcal{X}_1^*)$ of $(P^*, \mathcal{X}^*)$, and let $(P_1, \mathcal{X}_1)$ be the associated rational boundary component of $(P, \mathcal{X})$, and $(P'_1, \mathcal{X}'_1)$ that of $(P', \mathcal{X}')$. Then $(P_1'^*, \mathcal{X}_1'^*) \cong (P'_1, \mathcal{X}'_1) \times_{(P_1, \mathcal{X}_1)} (P_1^*, \mathcal{X}_1^*)$ is the associated rational boundary component of $(P'^*, \mathcal{X}'^*)$, and in particular $U_1'^* \cong U_1^* \times_{U_1} U_1'$. This shows that if $\sigma^* \in \Sigma^*$ and $\sigma' \in \Sigma'$ both map to $\sigma \in \Sigma$, then $\sigma'^* := \sigma' \times_{\sigma} \sigma^*$ is a convex rational polyhedral cone in $U_1'^*(\mathbb{R})(-1)$, for some $(P_1^*, \mathcal{X}_1^*)$. We define Σ'^* as the set of all these σ'^* .

It is now straightforward to prove: $M^{K_f'^*}(P'^*, \mathcal{X}'^*, \Sigma'^*)(\mathbb{C}) \rightarrow M^{K_f^*}(P^*, \mathcal{X}^*, \Sigma^*)(\mathbb{C})$ satisfies the condition (*) of 8.5, and the diagram

$$\begin{array}{ccc} M^{K_f'^*}(P'^*, \mathcal{X}'^*, \Sigma'^*)(\mathbb{C}) & \longrightarrow & M^{K_f'}(P', \mathcal{X}', \Sigma')(\mathbb{C}) \\ \downarrow & & \downarrow \\ M^{K_f^*}(P^*, \mathcal{X}^*, \Sigma^*)(\mathbb{C}) & \longrightarrow & M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}) \end{array}$$

is cartesian. In other words, the left hand side is the pullback of the line bundle on the right hand side.

Later in this chapter we shall need the following two lemmas, which allow us to localize assertions about the line bundle in 8.6.

8.8. Lemma: Consider the situation of 8.6. If K_f' is replaced by $K_f' \cdot K_f^U$ for an arbitrary open compact subgroup K_f^U of $U_O(\mathbb{A}_f)$, then the condition in 8.6 still hold. The line bundle associated to $K_f' \cdot K_f^U$ is isomorphic to a strictly positive tensor power of the line bundle associated to K_f' .

Proof. Consider the situation in the proof of 8.6. By definition $U_O(\mathbb{Q}) \cap \Gamma_{U'}$ is the image in $U_O(\mathbb{Q})$ of

$$(\{z \in Z(P')(\mathbb{Q}) \mid z|_{\mathcal{X}'} = \text{id}\} \times U_0(\mathbb{Q})) \cap (p'_{1,f} \cdot p'_f) \cdot K_f' \cdot (p'_{1,f} \cdot p'_f)^{-1}.$$

This can be written as the intersection with $U_0(\mathbb{Q})$ of the image of

$$(\{z \in Z(P')(\mathbb{Q}) \mid z|_{\mathcal{X}'} = \text{id}\} \times U_0(\mathbb{A}_f)) \cap (p'_{1,f} \cdot p'_f) \cdot K'_f \cdot (p'_{1,f} \cdot p'_f)^{-1}$$

in $U_0(\mathbb{A}_f)$, since K'_f contains an open subgroup of $U_0(\mathbb{A}_f)$. Let $\tilde{K}_f^U$ be the image of $(\{z \in Z(P')(\mathbb{Q}) \mid z|_{\mathcal{X}'} = \text{id}\} \times U_0(\mathbb{A}_f)) \cap K'_f$ in $U_0(\mathbb{A}_f)$, then it follows that $U_0(\mathbb{Q}) \cap \Gamma_{U'} = U_0(\mathbb{Q}) \cap \text{int}(p'_{1,f} \cdot p'_f)(\tilde{K}_f^U)$. If K'_f is replaced by $K'_f \cdot K_f^U$, then clearly $\tilde{K}_f^U$ is replaced by $\tilde{K}_f^U \cdot K_f^U$, and the only effect on the objects considered in 8.5 is that $\Gamma_{U'}$ is replaced by $\Gamma_{U'} + \Gamma'_{U_0}$ with $\Gamma'_{U_0} = U_0(\mathbb{Q}) \cap \text{int}(p'_{1,f} \cdot p'_f)(K_f^U)$. Thus clearly the condition (*) of 8.5 still holds with $K'_f \cdot K_f^U$ in place of K'_f .

More precisely we find that the index

$$\begin{aligned} [\Gamma_{U'} + \Gamma'_{U_0} : \Gamma_{U'}] &= [U_0(\mathbb{Q}) \cap \Gamma_{U'} + \Gamma'_{U_0} : U_0(\mathbb{Q}) \cap \Gamma_{U'}] \\ &= [U_0(\mathbb{Q}) \cap \text{int}(p'_{1,f} \cdot p'_f)(\tilde{K}_f^U \cdot K_f^U) : U_0(\mathbb{Q}) \cap \text{int}(p'_{1,f} \cdot p'_f)(\tilde{K}_f^U)] \\ &= [\tilde{K}_f^U \cdot K_f^U : \tilde{K}_f^U] \end{aligned}$$

is independant of the choice of $(P'_1, \mathcal{X}'_1)$, p'_f , $p'_{1,f}$ and $\mathcal{X}^0$. If this index is n , then locally 5.10 gives a canonical isomorphism of the “new” line bundle with the n^{th} tensor power of the “old” line bundle. These must patch together to a global isomorphism, as desired. **q.e.d.**

8.9. Lemma: Consider the situation of 8.6, and let $(P'_1, \mathcal{X}'_1)$, p'_f be as in 8.5. Define $K_f'^1 := P'_1(\mathbb{A}_f) \cap p'_f \cdot K'_f \cdot p'^{-1}_f$, $\Sigma'_1 := ([p'_f]^* \Sigma')|_{(P'_1, \mathcal{X}'_1)}$, $\mathcal{W}'_1 \subset \bar{\mathcal{U}}'(P'_1, \mathcal{X}'_1, p'_f) \subset M^{K_f'^1}(P'_1, \mathcal{X}'_1, \Sigma_1^0)(\mathbb{C})$, and Δ'_1 in analogy to K_f^1 , Σ_1 , $\mathcal{W}_1 \subset \bar{\mathcal{U}}(P_1, \mathcal{X}_1, p_f) \subset M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1^0)(\mathbb{C})$, and Δ_1 as in 6.10, 6.13, 6.18, and 6.22. By 8.6 the embeddings $\Delta_1 \setminus \mathcal{W}_1 \hookrightarrow M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ and $\Delta'_1 \setminus \mathcal{W}'_1 \hookrightarrow M^{K'_f}(P', \mathcal{X}', \Sigma')(\mathbb{C})$ make $\Delta'_1 \setminus \mathcal{W}'_1 \rightarrow \Delta_1 \setminus \mathcal{W}_1$ a line bundle.

Then by replacing K'_f by $K'_f \cdot K_f^U$ for a sufficiently big open compact subgroup $K_f^U \subset U_0(\mathbb{A}_f)$ we can achieve that

(a) the assumptions of 8.6 are also satisfied for

$$M^{K_f'^1}(P'_1, \mathcal{X}'_1, \Sigma_1^0)(\mathbb{C}) \rightarrow M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1^0)(\mathbb{C}),$$

so $\mathcal{W}'_1 \rightarrow \mathcal{W}_1$ carries a canonical structure as a line bundle, and

(b) this line bundle is isomorphic to the pullback to $\mathcal{W}_1$ of the line bundle $\Delta'_1 \setminus \mathcal{W}'_1 \rightarrow \Delta_1 \setminus \mathcal{W}_1$ above.

Proof. Let $\Gamma_{U'}$ and Γ_U be as in 8.5, but with $p'_{1,f} \cdot p'_f$ in place of p'_f , and define $\Gamma_{U'}^1$ and Γ_U^1 likewise, with $(P', \mathcal{X}')$ in place of $(P'_1, \mathcal{X}'_1)$ and $p'_{1,f}$ in place of p'_f . Explicitly $\Gamma_{U'}$ is the image in $U'_1(\mathbb{Q})$ of

$$(\{z \in Z(P')(\mathbb{Q}) \mid z|_{\mathcal{X}'} = \text{id}\} \times U'_1(\mathbb{Q})) \cap (p'_{1,f} \cdot p'_f) \cdot K'_f \cdot (p'_{1,f} \cdot p'_f)^{-1},$$

while $\Gamma_{U'}^1$ is the image of the possibly smaller subgroup

$$\begin{aligned} (\{z \in Z(P'_1)(\mathbb{Q}) \mid z|_{\mathcal{X}'} = \text{id}\} \times U'_1(\mathbb{Q})) \cap p'_{1,f} \cdot K_f'^1 \cdot (p'_{1,f})^{-1} &= \\ = (\{z \in Z(P'_1)(\mathbb{Q}) \mid z|_{\mathcal{X}'} = \text{id}\} \times U'_1(\mathbb{Q})) \cap (p'_{1,f} \cdot p'_f) \cdot K'_f \cdot (p'_{1,f} \cdot p'_f)^{-1}. \end{aligned}$$

Thus we have the inclusion $\Gamma_{U'}^1 \subset \Gamma_{U'}$ and $\Gamma_U^1 \subset \Gamma_U$. If K'_f is replaced by $K'_f \cdot K_f^U$ for a sufficiently big open compact subgroup $K_f^U \subset U_0(\mathbb{A}_f)$, then $U_0(\mathbb{Q}) \cap \Gamma_{U'}$ is already equal to $U_0(\mathbb{Q}) \cap (p'_{1,f} \cdot p'_f) \cdot K'_f \cdot (p'_{1,f} \cdot p'_f)^{-1}$. Assume that this replacement has been made, then it follows that $\Gamma_{U_0} := U_0(\mathbb{Q}) \cap \Gamma_{U'} = U_0(\mathbb{Q}) \cap \Gamma_{U'}^1$.

Now in (a) everything is clear except that in the condition (*) of 8.5 for all $\sigma \in \Sigma_1(\mathcal{X}_1^0, P_1, p_{1,f}) = \Sigma(\mathcal{X}^0, P_1, p_{1,f} \cdot p_f)$ the splitting e_σ induces a splitting of the corresponding lattices $\Gamma_U^1 \rightarrow \Gamma_{U'}^1$. Denote the projection $U'_1 \rightarrow U_1$ by π_0 , then this is equivalent to saying that $\text{id} - e_{\sigma^0} \circ \pi_0 : U'_1 \rightarrow U_0 \subset U'_1$ induces a splitting $\Gamma_{U'}^1 \rightarrow U_0(\mathbb{Q}) \cap \Gamma_{U'}^1$. But by assumption we have this property for $\Gamma_{U'}$ and Γ_U , that this $\text{id} - e_\sigma \circ \pi_0$ induces a splitting $\Gamma_{U'}^1 \rightarrow \Gamma_{U_0}$. Thus by the equality $\Gamma_{U_0} = U_0(\mathbb{Q}) \cap \Gamma_{U'}^1$ we are done.

For (b) we have to show that the obvious map

$$\overline{U}'(P'_1, \mathcal{X}'_1, p'_f) \longrightarrow \overline{U}(P_1, \mathcal{X}_1, p_f) \times_{(\Delta_1 \backslash \overline{U}(P_1, \mathcal{X}_1, p_f))} (\Delta'_1 \backslash \overline{U}'(P'_1, \mathcal{X}'_1, p'_f))$$

is an isomorphism. By the proof of 6.20 every connected component of $\Delta'_1 \backslash \overline{U}'(P'_1, \mathcal{X}'_1, p'_f)$ is of the form $\Gamma'_Q \backslash ((\Gamma'_1 \backslash \mathcal{X}'^0)_{\Sigma'_1(\mathcal{X}'^0, P'_1, p'_{1,f})})$, where $\Gamma'_Q = \text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0) \cap ((p'_{1,f} \cdot p'_f) \cdot K'_f \cdot (p'_{1,f} \cdot p'_f)^{-1})$ and $\Gamma'_1 = P'_1(\mathbb{Q}) \cap \Gamma'_Q$, and the analogous formula holds for the image of this connected component in $\Delta_1 \backslash \overline{U}(P_1, \mathcal{X}_1, p_f)$. Abbreviate $X' := (\Gamma'_1 \backslash \mathcal{X}'^0)_{\Sigma'_1(\mathcal{X}'^0, P'_1, p'_{1,f})}$ and $X := (\Gamma_1 \backslash \mathcal{X}^0)_{\Sigma_1(\mathcal{X}^0, P_1, p_{1,f})}$, then $X' \rightarrow X$ is a line bundle, and the image connected component is isomorphic to $\Gamma'_Q \backslash X$. Since K'_f is neat, so is Γ'_Q , hence it acts trivially on $U_0(\mathbb{Q})$. This implies that Γ'_Q -action commutes with the action of the torus $\mathbb{C}^\times$ on X' . Moreover the assumption on K'_f above implies

$$\{\gamma \in \Gamma'_Q \mid \gamma|_{\mathcal{X}^0} = \text{id}\} = U_0(\mathbb{Q}) \cap \Gamma'_Q = U_0(\mathbb{Q}) \cap \Gamma'_1,$$

hence

$$X' \cong X \times_{(\Gamma'_Q \backslash X)} (\Gamma'_Q \backslash X'),$$

as desired. **q.e.d.**

8.10. Towards an ampleness criterion: In the situation of 8.5 let us now assume that Σ is complete, so by 6.27 $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is a compact complex space. In particular every fibre of the map $[\pi]^* : M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}) \rightarrow M^{\pi(K_f)}((P, \mathcal{X})/W)^*(\mathbb{C})$ is compact. Our next aim is given a sufficient condition for the inverse of the line bundle of 8.6 to be ample on every such fibre. Theorem 5.19 suggests what that condition could look like. Note that by 7.3 and 7.8 in every fibre we are dealing with the union of torus embeddings of unipotent fibres of certain other mixed Shimura varieties. Thus on the one hand it is plausible to require $|\Sigma'(\mathcal{X}'^0, P'_1, p'_f)|$ to be strictly convex in the sense of 5.16. The other condition in 5.19 means that certain line bundles on the abelian variety part of the unipotent fibre have to be ample. By 3.21 this can be recognized in terms of polarizations of Hodge structures. To express this condition, like the first one, in terms of $|\Sigma'(\mathcal{X}'^0, P'_1, p'_f)|$, we make the following definition.

8.11. Definition: Consider arbitrary mixed Shimura data $(P, \mathcal{X})$, and a connected component $\mathcal{X}^0$ of $\mathcal{X}$. Let $\Psi : V \times V \rightarrow U$ be the pairing induced by the commutator

pairing on the unipotent radical W of P . Let $\sqrt{-1} \in \mathbb{C}$ be a square root of -1 , and define

$$B(\mathcal{X}^0) := \{(2\pi\sqrt{-1})^{-1} \cdot \Psi(v, h_x(\sqrt{-1}v)) \mid 0 \neq v \in V(\mathbb{R}), x \in \mathcal{X}^0\} \subset U(\mathbb{R})(-1).$$

As usual (cf. 1.11) this does not depend on the choice of $\sqrt{-1}$. More generally let $(P_1, \mathcal{X}_1)$ be a rational boundary component of $(P, \mathcal{X})$ such that $\mathcal{X}^0$ corresponds to a connected component $\mathcal{X}_1^0$ of $\mathcal{X}_1$. Then we define $B(\mathcal{X}^0, P_1) := B(\mathcal{X}_1^0)$.

Observe that this definition is almost, but not quite functorial in $(P, \mathcal{X})$. In fact, it is easy to see that it is functorial only with respect to morphisms which are injective on V_1 . By contrast $B(\mathcal{X}^0, P_1) \cup \{0\}$ is always functorial. The definition is made such that we can check on $B(\mathcal{X}^0, P_1)$ whether a pairing on V , that factors through the commutator pairing in W , defines a polarization of Hodge structures. We shall need the following result about $B(\mathcal{X}^0, P_1)$.

8.12. Proposition: $B(\mathcal{X}^0, P_1)$ is contained in the closure of $C(\mathcal{X}^0, P_1)$ in $U_1(\mathbb{R})(-1)$. If P is reductive, then $0 \notin B(\mathcal{X}^0, P_1)$.

Proof. By 4.15 (c) we have $C(\mathcal{X}^0, P_1) = C(\mathcal{X}^0, P_1) + (W \cap U_1)(\mathbb{R})(-1)$, so $C(\mathcal{X}^0, P_1)$ is the inverse image of the corresponding cone for $(P, \mathcal{X})/W$. Thus the assertion follows from that for $(P, \mathcal{X})/W$, and we are reduced to the case that P is reductive. In that case this is, at least up to sign, what is proved in [AMRT] ch.III §4.2 p.231 thm. 2. (There is a slight error in the formulation of that theorem: the proof only shows that every element of $B(\mathcal{X}^0, P_1)$ is nonzero, and is contained in the closure of $C(\mathcal{X}^0, P_1)$. In fact $B(\mathcal{X}^0, P_1)$ is not always a subset of $C(\mathcal{X}^0, P_1)$, even if P^{ad} is $\mathbb{Q}$ -simple.)

We still have to make sure that the sign in [loc. cit.] is the same as above. So we start with $x_1 \in \mathcal{X}_1^0$, the connected component of $\mathcal{X}_1$ associated to $\mathcal{X}^0$. Since the assertion is invariant under conjugation by $U_1(\mathbb{C})$, we may assume that x_1 corresponds to some element $x \in \mathcal{X}^0$. Let ω_x be as in 4.6 (b), fix a choice of $\sqrt{-1}$, and let $J := \text{int}(\omega_x(\sqrt{-1}, 1))|_V$ as in [AMRT] p.230. Since by definition $h_{x_1} = \omega_x \circ h_\infty$ defines pure Hodge structure of weight -1 on V , this operator J acts on V like $\text{int}(\omega_x(h_\infty((\sqrt{-1}))) = \text{int}(h_{x_1}(-\sqrt{-1})) = -1 \cdot \text{int}(h_{x_1}(\sqrt{-1}))$. Thus for any $0 \neq v \in V(\mathbb{R})$ it follows that

$$(2\pi\sqrt{-1})^{-1} \cdot \Psi(v, h_x(\sqrt{-1}v)) = (2\pi\sqrt{-1})^{-1} \cdot (-1) \cdot \Psi(v, Jv) = \sqrt{-1} \cdot (2\pi)^{-1} \cdot \Psi(v, Jv).$$

To identify $C(\mathcal{X}^0, P_1)$ with the correct cone in [AMRT] note that by 4.7 (a) we are still free to choose h_0 in 4.3. If we make the same choice as in [AMRT] p.174 by defining

$$h_0(x + \sqrt{-1}y) := (x + \sqrt{-1}y, \begin{pmatrix} x & y \\ -y & x \end{pmatrix}),$$

then using the definition 4.3 of h_∞ an easy calculation shows that

$$\begin{aligned} h_\infty(x + \sqrt{-1}y) &= (x + \sqrt{-1}y, \begin{pmatrix} x^2+y^2 & \sqrt{-1}(1-x^2-y^2) \\ 0 & 1 \end{pmatrix}) \\ &= \text{int}((1, \begin{pmatrix} 1 & \sqrt{-1} \\ 0 & 1 \end{pmatrix}))((x + \sqrt{-1}y, \begin{pmatrix} x^2+y^2 & 0 \\ 0 & 1 \end{pmatrix})). \end{aligned}$$

Thus the definition 4.14 of the imaginary part implies that $\text{im}(x_1) = \omega_x((1, \begin{pmatrix} 1 & \sqrt{-1} \\ 0 & 1 \end{pmatrix})) \in U_1(\mathbb{R})(-1)$, and by 4.15 (c) $C(\mathcal{X}^0, P_1)$ is the $Q(\mathbb{R})^0$ -orbit in $U_1(\mathbb{R})(-1)$ generated by this

element. On the other hand in [AMRT] p.227 the cone $C \subset U_1(\mathbb{R})$ is defined as the $Q(\mathbb{R})^0$ -orbit generated by $\omega_x((1, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}))$ so we have $\sqrt{-1} \cdot C = C(\mathcal{X}^0, P_1)$. Now as mentioned above the proof of [AMRT] p.231 thm. 2 shows that for all $0 \neq v \in V(\mathbb{R})$ we have $0 \neq \Psi(v, Jv) \in \overline{C}$. Putting everything together we thus get

$$0 \neq (2\pi\sqrt{-1})^{-1} \cdot \Psi(v, h_x(\sqrt{-1})v) = \sqrt{-1} \cdot (2\pi)^{-1} \cdot \Psi(v, Jv) \in \sqrt{-1} \cdot \overline{C} = \overline{C(\mathcal{X}^0, P_1)},$$

as desired.

q.e.d.

8.13. Theorem: In the situation of 8.5 suppose that Σ is complete and condition (*) holds. Denote the homomorphism $P' \rightarrow P$ by π_0 , and for every connected component $\mathcal{X}'^0$ of $\mathcal{X}'$ let $\lambda : U_0(\mathbb{R})(-1) \rightarrow U_0(\mathbb{R}) = \mathbb{R}$ be as in 8.5. Assume that for every $\mathcal{X}'^0$, $(P'_1, \mathcal{X}'_1)$ and p'_f we have:

- (a) $\Sigma'(\mathcal{X}'^0, P'_1, p'_f)$ is strictly convex, and
- (b) for every $\sigma \in \Sigma(\mathcal{X}^0, P_1, p_f)$ with $\mathbb{R} \cdot \sigma = U_1(\mathbb{R})(-1)$ (i.e. of top dimension) the linear form $\lambda \circ (\text{id} - e_\sigma \circ \pi_0)$ is strictly positive on $B(\mathcal{X}'^0, P'_1)$.

Then the inverse of the line bundle

$$M^{K'_f}(P', \mathcal{X}', \Sigma')(\mathbb{C}) \rightarrow M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$$

is ample on every fibre of the map

$$[\pi]^* : M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}) \rightarrow M^{\pi(K_f)}((P, \mathcal{X})/W)^*(\mathbb{C}).$$

Proof. We first assume that $(P, \mathcal{X})$ is irreducible, and prove the assertion over $M^{\pi(K_f)}((P, \mathcal{X})/W)(\mathbb{C})$. Consider the corresponding connected components $\Gamma' \backslash \mathcal{X}'^0 \subset M^{K'_f}(P', \mathcal{X}')(\mathbb{C})$ and $\Gamma \backslash \mathcal{X}^0 \subset M^{K_f}(P, \mathcal{X})(\mathbb{C})$. By 6.6 the torus embeddings along the unipotent fibre $M^{K'_f}(P', \mathcal{X}', \Sigma^0)(\mathbb{C})$ and $M^{K_f}(P, \mathcal{X}, \Sigma^0)(\mathbb{C})$ are on these connected components given as torus embeddings over $\Gamma \backslash \mathcal{X}^0/U(\mathbb{C})$ with respect to tori $T' = U'(\mathbb{C})/\Gamma_{U'}$, resp. $T = U(\mathbb{C})/\Gamma_U$, and the rational partial polyhedral decompositions $\Sigma'(\mathcal{X}'^0, P', p'_f)$, resp. $\Sigma(\mathcal{X}^0, P, p_f)$. Here $\Gamma_{U'}$ and Γ_U are lattices in $U'(\mathbb{Q})$, resp. $U(\mathbb{Q})$, which lie in a short exact sequence $1 \rightarrow \Gamma_{U_0} \rightarrow \Gamma_{U'} \rightarrow \Gamma_U \rightarrow 1$ with $\Gamma_{U_0} := U_0(\mathbb{Q}) \cap \Gamma_{U'}$. As in 3.16 $U_0(\mathbb{C})/\Gamma_{U_0}$ is identified with $\mathbb{C}^\times$ by $U_0(\mathbb{C})/\Gamma_{U_0} = \mathbb{C}/d \cdot \mathbb{Z} \ni [u] \mapsto \exp(\frac{1}{d} \cdot \lambda(u))$ for $d \in \mathbb{Z}^{>0}$. Thus we are given a short exact sequence $1 \rightarrow \mathbb{C}^\times \rightarrow T' \rightarrow T \rightarrow 1$, and the convex rational polyhedral cone σ_0 defined in 8.5 corresponds to the cone $\sigma_0 = \mathbb{R}^{\geq 0} \subset \mathbb{R} = Y_*(\mathbb{G}_m)_{\mathbb{R}}$ considered in 5.11 (hence the same notation). It suffices to verify the condition (b) of 5.19 on every fibre over $\Gamma \backslash \mathcal{X}^0/(U(\mathbb{C}) \cdot W(\mathbb{R}))$.

By assumption $\Sigma(\mathcal{X}^0, P, p_f)$ is a finite and complete rational polyhedral decomposition of $U(\mathbb{R})(-1)$, and by condition (a) $\Sigma'(\mathcal{X}'^0, P'_1, p'_f)$ is strictly convex. Thus the first part of 5.19 (b) is satisfied. For the second part fix $\sigma \in \Sigma(\mathcal{X}^0, P_1, p_f)$ such that $\mathbb{R} \cdot \sigma = U(\mathbb{R})(-1)$, and let e_σ be the associated splitting $U \rightarrow U'$ of 8.5. Then the invertible sheaf $\mathcal{M}_{e_\sigma}$ of 5.19 is the invertible sheaf on $\Gamma \backslash \mathcal{X}^0/(U(\mathbb{C}))$ associated to the $\mathbb{C}^\times$ -torsor $\Gamma' \backslash \mathcal{X}'^0/e_\sigma(U)(\mathbb{C})$, with the $\mathbb{C}^\times$ -torsor structure given through the isomorphism $U_0 \xrightarrow{\sim} U'/e_\sigma(U)$. Since $(P, \mathcal{X})$ is irreducible, by 2.14 (a) $e_\sigma(U)$ is a normal subgroup of P' , so we may consider the mixed Shimura data $(P', \mathcal{X}')/e_\sigma(U)$. To show that 5.19 (b) is verified in every fibre

of $\Gamma \backslash \mathcal{X}^0 / U(\mathbb{C}) \rightarrow \Gamma \backslash \mathcal{X}^0 / (U(\mathbb{C}) \cdot W(\mathbb{R}))$, it thus remains to show that the invers of the $\mathbb{C}^\times$ -torsor

$$M^{\pi_{[\sigma]}(K_f')}(P', \mathcal{X}') / e_\sigma(U)(\mathbb{C}) \rightarrow M^{\pi'(K_f)}((P, \mathcal{X}) / U)(\mathbb{C})$$

is ample on every fibre on $M^{\pi'(K_f)}((P, \mathcal{X}) / U)(\mathbb{C}) \rightarrow M^{\pi(K_f)}((P, \mathcal{X}) / W)(\mathbb{C})$.

To prove this note that the canonical map $U' \rightarrow U' / e_\sigma(U) \cong U_0 \subset U'$ is equal to $\text{id} - e_\sigma \circ \pi_0$. Thus if $\Psi' : V \times V = V' \times V' \rightarrow U'$ is the pairing induced by the commutator in W' , then for $W' / e_\sigma(U)$ we get the pairing $\text{id} - e_\sigma \circ \pi_0 \circ \Psi' : V \times V \rightarrow U_0 \subset U'$. Thus by 3.21 the line bundle in question is ample if the pairing

$$\lambda \circ (\text{id} - e_\sigma \circ \pi_0) \circ \Psi' : V \times V \rightarrow U_0(1) = \mathbb{Q}(1)$$

is a polarization of Hodge structures for every $x \in \mathcal{X}^0$. By the definition of polarization in 1.11 and the definition of $B(\mathcal{X}^0)$ in 8.11 this is just the condition (b) above for $(P'_1, \mathcal{X}'_1) = (P', \mathcal{X}')$. This finishes the special case.

Let us now do the general case. By 7.3 a fibre of $[\pi]^*$ is isomorphic to a fibre of the map $\Delta_1 \backslash \partial \mathcal{U}(P_1, \mathcal{X}_1, p_f) \rightarrow \Delta_1 \backslash M^{\pi_1(K_f^1)}((P_1, \mathcal{X}_1) / W_1)(\mathbb{C})$. Every irreducible component of $\partial \mathcal{U}(P_1, \mathcal{X}_1, p_f)$ is by 7.3 and 7.9 isomorphic to $M^{\pi_{1, [\sigma]}(K_f^1)}(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]}, \Sigma_{1, [\sigma]}^0)(\mathbb{C})$ for some double coset $[\sigma] \in P_1(\mathbb{Q}) \backslash \Sigma_1^0 / P_1(\mathbb{A}_f)$, and the induced map

$$M^{\pi_{1, [\sigma]}(K_f^1)}(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]}, \Sigma_{1, [\sigma]}^0)(\mathbb{C}) \longrightarrow \Delta_1 \backslash \partial \mathcal{U}(P_1, \mathcal{X}_1, p_f).$$

is a finite holomorphic map. Thus it suffices to show the assertion for pullback of our line bundle to $M^{\pi_{1, [\sigma]}(K_f^1)}(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]}, \Sigma_{1, [\sigma]}^0)(\mathbb{C})$. Now we can do the same steps for the mixed Shimura variety $M^{K_f'}(P', \mathcal{X}', \Sigma')(\mathbb{C})$ in place of $M^{K_f}(P, \mathcal{X}, \Sigma)$. By lemma 8.8 we may enlarge K_f' without changing the desired assertion. If we do this the conclusion in lemma 8.9 holds, then it is easy to see that the condition of 8.5 hold also for $(P'_{1, [\sigma']}, \mathcal{X}'_{1, [\sigma']})$ and $\Sigma_{1, [\sigma']}^0$ with $\sigma' = e_\sigma(\sigma)$. It follows that the pullback to $M^{\pi_{1, [\sigma]}(K_f^1)}(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]}, \Sigma_{1, [\sigma]}^0)(\mathbb{C})$ of the line bundle in question is isomorphic to the line bundle

$$M^{\pi'_{1, [\sigma]}(K_f'^1)}(P'_{1, [\sigma']}, \mathcal{X}'_{1, [\sigma']}, \Sigma_{1, [\sigma']}^0)(\mathbb{C}) \longrightarrow M^{\pi_{1, [\sigma]}(K_f^1)}(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]}, \Sigma_{1, [\sigma]}^0)(\mathbb{C}).$$

But this is just the case we have covered in the first part of the proof, so it only remains to show that the conditions (a) and (b) hold for $(P'_{1, [\sigma']}, \mathcal{X}'_{1, [\sigma']})$ and $\Sigma_{1, [\sigma']}^0$.

To prove this let $\Sigma : |\Sigma'(\mathcal{X}^0, P'_1, p'_{1, f} \cdot p'_f)|$ and $\Sigma_{[\sigma]} := |\Sigma'_{1, [\sigma']}(\mathcal{X}^0_{1, [\sigma']}, P'_{1, [\sigma']}, p'_{1, f})|$. We first show that $\Sigma_{[\sigma]} = (\Sigma \bmod \mathbb{R} \cdot \sigma')$. By the definition 7.7 of $\Sigma'_{1, [\sigma']}$, $\Sigma_{[\sigma]}$ is the union of $\tau' \bmod \mathbb{R} \cdot \sigma'$ for all $\tau' \in \Sigma'(\mathcal{X}^0, P'_1, p'_{1, f} \cdot p'_f)$ such that σ' is a face of τ' . Thus the inclusion “ $\subset$ ” is clear. For the reverse inclusion note that by 7.8 $\Sigma_{1, [\sigma]}$ is complete, so $\Sigma_{[\sigma]} + \mathbb{R} \cdot \sigma_0 = C^*(\mathcal{X}^0_{1, [\sigma']}, P'_{1, [\sigma']})$. This means that $C^*(\mathcal{X}^0, P'_1)$ is contained in the union of $e_\tau(\tau) + \mathbb{R} \cdot \sigma' + \mathbb{R} \cdot \sigma_0 = e_\tau(\tau + \mathbb{R} \cdot \sigma_0) + \mathbb{R} \cdot \sigma_0$ for all $\tau \in \Sigma(\mathcal{X}^0, P_1, p_{1, f} \cdot p_f)$ such that σ is a face of τ . It suffices to take only those τ that are of top dimension. For every such τ the convexity condition (a) implies $\Sigma \subset e_\tau(U_1(\mathbb{R})(-1)) + \sigma_0$. Thus it follows that Σ is contained in the union of $e_\tau(\tau + \mathbb{R} \cdot \sigma + \sigma_0) = e_\tau(\tau) + \mathbb{R} \cdot \sigma' + \sigma_0$ for the same τ . But $\sigma' = e_\sigma(\sigma)$ is a face of $e_\tau(\tau) + \sigma_0$, so the reverse inclusion $(\Sigma \bmod \mathbb{R} \cdot \sigma') \subset \Sigma_{[\sigma]}$ follows.

Now the equation $\Sigma_{[\sigma]} = (\Sigma \bmod \mathbb{R} \cdot \sigma')$ and the convexity of Σ implies that $\Sigma_{[\sigma]}$ is convex. To show that $\Sigma'_{1, [\sigma]}(\mathcal{X}'^0_{1, [\sigma]}, P'_{1, [\sigma]}, p'_{1, f})$ is strictly convex, the definition in 5.16 tells us that we have to consider $e_\tau(\tau) \bmod \mathbb{R} \cdot \sigma'$ for every $\tau \in \Sigma(\mathcal{X}^0, P_1, p_{1, f} \cdot p_f)$ such that σ is a face of τ . By the strict convexity of $\Sigma'(\mathcal{X}^0, P'_1, p'_{1, f} \cdot p'_f)$ there exists a linear form ℓ on $U'_1(\mathbb{Q})$ such that $e_\tau(\tau) = \{u \in \Sigma \mid \ell(u) = 0\}$. Since $\sigma' = e_\sigma(\sigma)$ is a face of $e_\tau(\tau)$, this linear form vanishes on $\mathbb{R} \cdot \sigma'$, and hence factors through a linear form ℓ' on $U'_{1, [\sigma']}(\mathbb{Q})$. It follows that $e_\tau(\tau) \bmod \mathbb{R} \cdot \sigma' = \{u \in \Sigma \bmod \mathbb{R} \cdot \sigma' \mid \ell'(u) = 0\} = \{u \in \Sigma_{[\sigma]} \mid \ell'(u) = 0\}$, providing condition (a).

For condition (b) note that the splitting $e_{(\tau \bmod \mathbb{R} \cdot \sigma)} : U_{1, [\sigma]} \rightarrow U'_{1, [\sigma']}$ is just that induced by e_τ , that is more precisely we have the equation

$$\text{id} - e_\tau \circ \pi_0 = (\text{id} - e_{(\tau \bmod \mathbb{R} \cdot \sigma)} \circ \pi_0) \circ \pi_{[\sigma]}.$$

Since moreover

$$B(\mathcal{X}'^0_{1, [\sigma]}, P'_{1, [\sigma]}) = (B(\mathcal{X}'^0, P'_1) \bmod \mathbb{R} \cdot \sigma') = \pi_{[\sigma]}(B(\mathcal{X}'^0, P'_1)),$$

condition (b) follows from that for $(P', \mathcal{X}')$. **q.e.d.**

8.14. Corollary: Let $(P, \mathcal{X})$ be mixed Shimura data, $K_f \subset P(\mathbb{A}_f)$ a neat open compact subgroup, and Σ a complete K_f -admissible cone decomposition for $(P, \mathcal{X})$. Suppose that there exist $(P', \mathcal{X}')$, K'_f , Σ' such that the assumption of 8.13 are verified. Then $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is projective.

More specifically let $\mathcal{L}$ be the invertible sheaf associated to the line bundle $M^{K'_f}(P', \mathcal{X}', \Sigma')(\mathbb{C}) \rightarrow M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ and $\omega[\text{dlog}]$ the canonical sheaf on $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ with logarithmic poles along $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}) \setminus M^{K_f}(P, \mathcal{X})(\mathbb{C})$, locally defined as in 5.26. Then there exists a positive integer n , such that the invertible sheaf

$$\mathcal{L}^{\otimes -1} \otimes (\omega[\text{dlog}])^{\otimes n}$$

is ample.

Proof. Using 8.4 and 8.13 we have the situation of lemma 5.18, with $X := M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$, $Y := M^{K_f}(P, \mathcal{X})^*(\mathbb{C})$, $\mathcal{L}$ replaced by $\mathcal{L}^{\otimes -1}$, and $\mathcal{M} := \mathcal{L}_P$ of 8.4. **q.e.d.**

8.15. Remarks: (a) If P is reductive, then 8.14 essentially implies, and is almost equivalent to, Tai's theorem in [AMRT] ch.IV §2.1 p.312 (Compare also [Nam] thm. 7.23). In fact our construction of an explicit relatively ample sheaf can, as in the proof of [H2] ch.II thm. 7.17, be translated into the construction of an explicit sheaf of ideals, such that $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is the normalization of the blowing up of $M^{K_f}(P, \mathcal{X})^*(\mathbb{C})$ along this sheaf of ideals. Thus the conclusion of 8.14 is equivalent to that in [loc. cit.], and it remains to check that the respective assumptions are more less the same.

We shall only indicate the main point. First note that since P is reductive, after possibly changing the center of P there exists, up to isomorphism, exactly one choice for $(P', \mathcal{X}')$ as in 8.5. Let $(P'_1, \mathcal{X}'_1)$ be a rational boundary component of $(P', \mathcal{X}')$, and

$(P_1, \mathcal{X}_1) := (P'_1, \mathcal{X}'_1)/U_0$. Any splitting $P' \cong U_0 \rtimes P$ induces a splitting $W'_1 \cong U_0 \rtimes W_1$, and this is already a direct product since W_1 acts trivially on U_0 . Since all splittings of P' are conjugate under U_0 , this splitting is canonical! Thus we have a canonical decomposition $U'_1 = U_0 \times U_1$, and $B(\mathcal{X}'^0, P'_1)$ is contained in the second factor. In analogy to 5.20 we can now describe $\Sigma'(\mathcal{X}'^0, P'_1, p'_f)$ in terms of a function $\phi : C^*(\mathcal{X}^0, P_1) \rightarrow \mathbb{R}$, where the splitting associated to any $\sigma \in \Sigma(\mathcal{X}^0, P_1, p_f)$ is, with respect to the canonical decomposition of U'_1 , equal to $(-\lambda^{-1} \circ \phi, \text{id})$ on σ .

It is now easy to see that the condition (a) of 8.13 is equivalent to the strict convexity of ϕ in the sense of condition (2) of [AMRT] ch.IV §2.1 p.310. It can be shown that the condition (1) of [loc. cit.] is equivalent to the assertion that for all σ of top dimension the linear form $\lambda \circ (\text{id} - e_\sigma \circ \pi_0)$ is strictly positive on $\overline{C(\mathcal{X}^0, P_1)} \setminus \{0\}$. By 8.12 it thus implies our condition (b). In general we do not have equivalence, but this is only a very minor difference.

(b) More generally one can describe the line bundle of 8.13 in terms of a function ϕ as in [AMRT] ch.IV §2 if and only if the extension $1 \rightarrow U_0 \rightarrow P' \rightarrow P \rightarrow 1$ splits. In fact, we have described the line bundle locally as an equivariant torus embedding for a short exact sequence of tori $1 \rightarrow \mathbb{G}_m \rightarrow T' \rightarrow T \rightarrow 1$, and a global splitting of this extension amounts to a splitting $P' \cong U_0 \rtimes P$. But 3.21 shows that whenever the abelian variety part of the unipotent fibre is nontrivial, then one can get ample line bundles only with extensions that do not split. Thus we needed a more sophisticated description of relatively ample line bundles than in [AMRT]. A geometric illustration of this will be given in 9.39 (b).

Chapter 9

Algebraization of the toroidal compactification

To finish the algebraization of $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ we construct cone decompositions that satisfy the requirements of the preceding chapter. This construction is an analog of [AMRT] ch.II §§5.3–4. Under a mild condition on $(P, \mathcal{X})$, we begin (9.1–5) with the study of a certain set of splittings that satisfy the condition 8.13 (b). Our cone decompositions will be described in terms of “locally polyhedral” subsets $\Sigma \subset C^*(\mathcal{X}'^0, P'_1)$, defined in 9.8. Their properties (9.9–11) lead to cone decompositions that satisfy the desired conditions (9.12–13), and it remains to construct compatible systems of such subsets Σ . These constructions occupy most of the rest of this chapter.

First study the behavior of Σ with respect to different boundary components (9.14–15). This leads to our first existence theorem 9.16: asserting that some such systems of Σ ’s exists. Next we show how to obtain refinements with specific properties (9.18–20), in particular concerning smoothness and the extendability of the maps 3.4. In 9.21–23 we extend this to arbitrary $(P, \mathcal{X})$. These results show that $M^{K_f}(P, \mathcal{X})(\mathbb{C})$, and certain $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$, possess canonical algebraic structures (9.24–26). In 9.28–33, we extend the existence of an algebraic structure to all $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ for which, loosely speaking, Σ is sufficiently fine or K_f sufficiently small. This implies that there always exists a canonical structure of algebraic space over $\mathbb{C}$; which, however, is not always a scheme (9.34–35).

Having constructed an algebraic structure on most $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$, it follows from the construction that the maps 6.25 are algebraic, and most properties carry over to the algebraic side (9.25). In 9.36–38 we prove the same for the stratifications described in chapter 7, in particular that 7.17 (b) gives an isomorphism of the formal completions. The chapter concludes with an example (9.39).

9.1. Proposition: Let $(P, \mathcal{X})$ be mixed Shimura data, and $(P, \mathcal{X}) \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$ a morphism. As in 8.7 every connected component $\mathcal{X}'^0$ determines an isomorphism $\lambda : U_0(\mathbb{R})(-1) \rightarrow U_0(\mathbb{R}) = \mathbb{R}$ and a convex rational polyhedral cone $\sigma_0 := \lambda^{-1}(\mathbb{R}^{\geq 0}) \subset U_0(\mathbb{R})(-1)$.

(a) Suppose that $(P, \mathcal{X})$ is irreducible. Then there exists a $((P_0, \mathcal{X}_0) \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0))$ -torsor $(P', \mathcal{X}') \rightarrow (P, \mathcal{X})$, and a splitting $\mu : U' \rightarrow U_0$, such that if $\Psi : V \times V \rightarrow U'$ is the pairing

induced by the commutator in W' , then $\lambda \circ \mu \circ \Psi$ defines a polarization of Hodge structures on V_{2g} for every point in $\mathcal{X}'^0$.

(b) Let $(P', \mathcal{X}') \rightarrow (P, \mathcal{X})$ be as in (a). Then there exists pure Shimura data $(\tilde{P}_*, \tilde{\mathcal{X}}_*)$, a rational boundary component $(\tilde{P}, \tilde{\mathcal{X}})$ of $(\tilde{P}_*, \tilde{\mathcal{X}}_*)$, and an embedding $(P', \mathcal{X}') \hookrightarrow (\tilde{P}, \tilde{\mathcal{X}})$, subject to the following condition. For every connected component $\mathcal{X}'^0$ of $\mathcal{X}'$ let $\tilde{\mathcal{X}}^0$ and $\tilde{\mathcal{X}}_*^0$ be the corresponding connected components of $\tilde{\mathcal{X}}$ resp. of $\tilde{\mathcal{X}}_*$. Then the image of σ_0 in $\tilde{U}(\mathbb{R})(-1)$ is contained in $\overline{C(\tilde{\mathcal{X}}_*^0, \tilde{P})}$.

Proof. (a) If $V = 1$, let $(P', \mathcal{X}') := (P_0, \mathcal{X}_0) \times_{(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)} (P, \mathcal{X})$, then any splitting is okay. Otherwise it suffices to take $(P', \mathcal{X}')$ and μ be as in the proof of 2.26 (b).

(b) As in the proof of 2.26 (b) the set of all μ that satisfy (a) above is an open subset of the set of all splittings. This also holds if $V = 1$, since then all splittings are okay. Let $2g := \dim(V)$, then every such splitting defines a morphism $(P', \mathcal{X}') \rightarrow (P_{2g}, \mathcal{X}_{2g})$. As in 2.26 any set of splittings in the above open set, that generates the $\mathbb{Q}$ -vector space $\text{Hom}(U', U_0)$, determines an embedding $(P', \mathcal{X}') \hookrightarrow (P, \mathcal{X})/W \times (P_{2g}, \mathcal{X}_{2g})^r$ for some r . By 4.25 $(P_{2g}, \mathcal{X}_{2g})$ can be interpreted as a rational boundary component of $(CS_{P_{2(g+1)}, \mathbb{Q}}, \mathcal{H}_{2(g+1)})$. So let $(\tilde{P}_*, \tilde{\mathcal{X}}_*) := (P, \mathcal{X})/W \times (CS_{P_{2(g+1)}, \mathbb{Q}}, \mathcal{H}_{2(g+1)})^r$, and let $(\tilde{P}, \tilde{\mathcal{X}})$ be the (unique) irreducible component of $(P, \mathcal{X})/W \times (P_{2g}, \mathcal{X}_{2g})^r$ that contains the image of $(P', \mathcal{X}')$. We are done if we can choose the splittings such that σ_0 maps to $\overline{C(\mathcal{H}_{2(g+1)}^0, P_{2g})}$ under each of the morphisms $(P', \mathcal{X}') \rightarrow (P_{2g}, \mathcal{X}_{2g})$.

If $V = 1$, then this is easy to achieve, since $\overline{C(\mathcal{H}_{2(g+1)}^0, P_{2g})}$ is a half-line in a vector space of dimension 1, and we may replace any splitting μ by $-\mu$. Otherwise by 8.11 the fact that $\lambda \circ \mu \circ \Psi$ defines a polarization of Hodge structures on V means that $\lambda \circ \mu$ is strictly positive on $B(\mathcal{X}'^0, P')$, or equivalently that $\mu(B(\mathcal{X}'^0, P'))$ is contained in $(\sigma_0)^0$. By functoriality (8.11) it follows that $B(\mathcal{X}_{2g}^0) \cap (\sigma_0)^0 \neq \emptyset$, where we denote the image of σ_0 again by σ_0 . But by 8.12 $B(\mathcal{X}_{2g}^0) = B(\mathcal{H}_{2(g+1)}^0, P_{2g}) \subset \overline{C(\mathcal{H}_{2(g+1)}^0, P_{2g})} \setminus \{0\}$, and both σ_0 and $\overline{C(\mathcal{H}_{2(g+1)}^0, P_{2g})}$ are half-lines in a one dimensional vector space, so $\sigma_0 = \overline{C(\mathcal{H}_{2(g+1)}^0, P_{2g})}$ as desired. **q.e.d.**

9.2. A certain set of splittings: Let $(P, \mathcal{X})$ be mixed Shimura data, and $\pi_0 : (P', \mathcal{X}') \rightarrow (P, \mathcal{X})$ a $((P_0, \mathcal{X}_0) \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0))$ -torsor. As in the assertion of 9.1 (a) assume that there exists a (not necessarily P -equivariant) splitting $\mu : U' \rightarrow U_0$, such that if $\Psi : V \times V \rightarrow U'$ is the pairing induced by the commutator in W' , then $\lambda \circ \mu \circ \Psi$ defines a polarization of Hodge structures on V_{2g} for every point in $\mathcal{X}'^0$. Under this assumption, for every irreducible component $(P_{(0)}, \mathcal{X}_{(0)})$ of $(P, \mathcal{X})$, the assertion in 9.1 (b) holds. We fix such data for most of this chapter. We shall eventually construct cone decompositions for the pair $(P', \mathcal{X}') \rightarrow (P, \mathcal{X})$, that satisfy all requirements of 8.13. For this we first concentrate on condition 8.13 (b). The main problem is to produce a sufficient supply of splittings that satisfy this condition.

Consider a rational boundary component $(P'_1, \mathcal{X}'_1)$ of $(P', \mathcal{X}')$, let $(P_1, \mathcal{X}_1) \cong (P'_1, \mathcal{X}'_1)/U_0$ be the corresponding rational boundary component of $(P, \mathcal{X})$, and Q' the associated parabolic subgroup of P' . Let $\mathcal{X}'^0$ be a connected component of $\mathcal{X}'$ that is mapped to $\mathcal{X}'_1$, and let $\mathcal{X}^0$ be the corresponding connected component of $\mathcal{X}$. Let $(P'_{(0)}, \mathcal{X}'_{(0)})$ be the unique irreducible component of $(P', \mathcal{X}')$ with $\mathcal{X}'^0 \subset \mathcal{X}'_{(0)}$.

Consider $(\tilde{P}, \tilde{\mathcal{X}})$, $(\tilde{P}_*, \tilde{\mathcal{X}}_*)$, and an embedding $(P'_{(0)}, \mathcal{X}'_{(0)}) \hookrightarrow (\tilde{P}, \tilde{\mathcal{X}})$ as in 9.1 (b), then by functoriality we get a rational boundary component $(\tilde{P}_1, \tilde{\mathcal{X}}_1)$ of $(\tilde{P}, \tilde{\mathcal{X}})$, and connected components $\tilde{\mathcal{X}}_1^0$, $\tilde{\mathcal{X}}^0$, and $\tilde{\mathcal{X}}_*^0$. Consider a homomorphism $\tilde{\mu} : \tilde{U}_1 \rightarrow U_0$ such that $\lambda \circ \tilde{\mu}$ is strictly positive on $\overline{C(\tilde{\mathcal{X}}_*^0, \tilde{P}_1)} \setminus \{0\}$, and which restricts to the identity on $U_0 \subset U' \hookrightarrow \tilde{U}_1$. Then its restriction to U'_1 is a splitting of the short exact sequence $0 \rightarrow U_0 \rightarrow U'_1 \rightarrow U_1 \rightarrow 0$. The corresponding splitting map $e : U_1 \rightarrow U'_1$ is determined by the equation $\tilde{\mu}|_{U'_1} = \text{id} - e \circ \pi_0$.

Let $\mathcal{E}(\mathcal{X}'^0, P'_1)$ be the set of all splittings $e : U_1 \rightarrow U'_1$ that arise in this way for some choice of $(\tilde{P}, \tilde{\mathcal{X}})$ and $(\tilde{P}_*, \tilde{\mathcal{X}}_*)$. Strictly speaking this is an abuse of notation, since this set depends not only on $\mathcal{X}'^0$ and P'_1 , but also on the torsor structure on $(P'_1, \mathcal{X}'_1) \rightarrow (P, \mathcal{X})$. But since this is fixed, there should be no confusion. Clearly the definition of $\mathcal{E}(\mathcal{X}'^0, P'_1)$ is invariant under $P'(\mathbb{Q})$. In particular the group $\text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0)$ acts on $\mathcal{E}(\mathcal{X}'^0, P'_1)$. We shall denote this operation by $e \mapsto e^q := (u \mapsto q^{-1} \cdot e(q \cdot u))$ for any $q \in \text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0)$. Note that $\text{Stab}_{P'(\mathbb{Q})}(\mathcal{X}'^0)$ acts on U_0 through positive scalars, since we have a morphism $(P', \mathcal{X}') \rightarrow (P_0, \mathcal{X}_0)$. This implies that for every $u' \in U'(\mathbb{R})(-1)$

$$\lambda \circ (\text{id} - e^q \circ \pi_0)(u') = \lambda(q^{-1} \cdot (\text{id} - e \circ \pi_0)(q \cdot u'))$$

is a positive multiple of $\lambda \circ (\text{id} - e \circ \pi_0)(q \cdot u')$.

9.3 Proposition: (a) $\mathcal{E}(\mathcal{X}'^0, P'_1)$ is a nonempty open subset of the space of all splittings $U_1 \rightarrow U'_1$ (open with respect to the archimedean topology on the hyperplane of all splittings in $\text{Hom}(U_1, U'_1)$).

(b) For every $e \in \mathcal{E}(\mathcal{X}'^0, P'_1)$ the linear map $\lambda \circ (\text{id} - e \circ \pi_0)$ is strictly positive on $B(\mathcal{X}'^0, P'_1)$, i.e. satisfies condition 8.13 (b).

Proof. (a) Since by 4.15 (d) $C(\tilde{\mathcal{X}}_*^0, \tilde{P}_1)$ is a nondegenerate selfadjoint cone, the set of all $\lambda \circ \tilde{\mu}$ that are strictly positive on $\overline{C(\tilde{\mathcal{X}}_*^0, \tilde{P}_1)} \setminus \{0\}$ is just the set of rational points in the interior of the dual cone, which is a nonempty subset of the dual space of $\tilde{U}_1(\mathbb{Q})(-1)$. Since by assumption the image of σ_0 lies in $\overline{C(\tilde{\mathcal{X}}_*^0, \tilde{P}_1)}$, and by injectivity is not zero, the restriction of $\tilde{\mu}$ to σ_0 is the multiplication by some positive scalar $\alpha \mathbb{Q}^{>0}$. Thus $\alpha^{-1} \cdot \tilde{\mu}$ satisfies the requirements of 9.2, and runs through a nonempty open subset of splittings $\tilde{U}_1 \rightarrow U_0$. Since the restriction map $\text{Hom}(\tilde{U}_1, U_0) \rightarrow \text{Hom}(U'_1, U_0)$ is surjective, the assertion follows.

(b) By the functoriality 8.11, $B(\mathcal{X}'^0, P'_1)$ is mapped to $B(\tilde{\mathcal{X}}^0, \tilde{P}_1) = B(\tilde{\mathcal{X}}_*^0, \tilde{P}_1)$. By 8.12, the latter is contained in $\overline{C(\tilde{\mathcal{X}}_*^0, \tilde{P}_1)} \setminus \{0\}$. Since by assumption $\lambda \circ \tilde{\mu}$ is strictly positive on $\overline{C(\tilde{\mathcal{X}}_*^0, \tilde{P}_1)} \setminus \{0\}$, it follows that $\lambda \circ \tilde{\mu}|_{U'_1} = \lambda \circ (\text{id} - e \circ \pi_0)$ is strictly positive on $B(\mathcal{X}'^0, P'_1)$, as desired. **q.e.d.**

We shall need the following information about the behavior of $\mathcal{E}(\mathcal{X}'^0, P'_1)$ under restriction to another rational boundary component.

9.4. Proposition: Let $(P'_2, \mathcal{X}'_2)$ be a rational boundary component between $(P'_1, \mathcal{X}'_1)$ and $(P', \mathcal{X}')$.

(a) The restriction map $\text{Hom}(U'_1, U_0) \rightarrow \text{Hom}(U'_2, U_0)$ induces a surjection $\mathcal{E}(\mathcal{X}'^0, P'_1) \rightarrow \mathcal{E}(\mathcal{X}'^0, P'_2)$.

(b) Let $e \in \mathcal{E}(\mathcal{X}'^0, P'_1)$, $A \subset C^*(\mathcal{X}'^0, P'_1) \setminus C^*(\mathcal{X}'^0, P'_2)$ a finite subset, and $\alpha \in \mathbb{Q}$. Then there exists $e' \in \mathcal{E}(\mathcal{X}'^0, P'_1)$ such that $e|_{U_2} = e'|_{U_2}$, $\lambda \circ (\text{id} - e' \circ \pi_0) \geq \lambda \circ (\text{id} - e \circ \pi_0)$ on $C^*(\mathcal{X}'^0, P'_1)$, and $\lambda \circ (\text{id} - e' \circ \pi_0)(u') \geq \alpha$ for every $u' \in A$.

Proof. (a) We first show that every $e \in \mathcal{E}(\mathcal{X}'^0, P'_1)$ restricts to an element of $\mathcal{E}(\mathcal{X}'^0, P'_2)$. Suppose that e comes from a splitting $\tilde{\mu} : \tilde{U}_1 \rightarrow U_0$, then $e|_{U_2}$ comes from the restriction of $\tilde{\mu}$ to $\tilde{U}_2$. since by 4.21 (b) $\overline{C(\tilde{\mathcal{X}}_*^0, \tilde{P}_2)}$ is contained in $\overline{C(\tilde{\mathcal{X}}_*^0, \tilde{P}_1)}$, the map $\tilde{\mu}|_{U_2}$ satisfies the assumptions in 9.2, as desired. For the surjectivity we have to extend a homomorphism $\tilde{\mu}_2 : \tilde{U}_2 \rightarrow U_0$, such that $\lambda \circ \tilde{\mu}_2$ is strictly positive on $\overline{C(\tilde{\mathcal{X}}_*^0, \tilde{P}_2)} \setminus \{0\}$ to a homomorphism $\tilde{\mu} : \tilde{U}_1 \rightarrow U_0$ such that $\lambda \circ \tilde{\mu}$ is strictly positive on $\overline{C(\tilde{\mathcal{X}}_*^0, \tilde{P}_1)} \setminus \{0\}$. This is possible by the following lemma 9.5, applied to $(\tilde{P}_*, \tilde{\mathcal{X}}_*)$ in place of $(P, \mathcal{X})$, $(\tilde{P}_1, \tilde{\mathcal{X}}_1)$ in place of $(P_1, \mathcal{X}_1)$, the analogous rational boundary component in place of $(P_2, \mathcal{X}_2)$, and $\lambda \circ \tilde{\mu}_2$, $\lambda \circ \tilde{\mu}$ in place of μ_2 , resp. μ .

(b) Since $C(\tilde{\mathcal{X}}_2^0/\tilde{W}_2, \tilde{P}_1/(\tilde{P}_1 \cap \tilde{W}_2))$ is a nongenerate selfadjoint cone, there exists a homomorphism $\tilde{\nu} : \tilde{U}_1/\tilde{U}_1 \cap \tilde{W}_2 \rightarrow U_0$ such that $\lambda \circ \tilde{\nu}$ is strictly positive on $\overline{C(\tilde{\mathcal{X}}_2^0/\tilde{W}_2, \tilde{P}_1/(\tilde{P}_1 \cap \tilde{W}_2))} \setminus \{0\}$. Note that by 4.21 (a) $C(\tilde{\mathcal{X}}_2^0/\tilde{W}_2, \tilde{P}_1/(\tilde{P}_1 \cap \tilde{W}_2))$ is the image of $C(\tilde{\mathcal{X}}_*^0, \tilde{P}_1)$, or also of $C(\tilde{\mathcal{X}}^0, \tilde{P}_1)$, under the canonical projection $\tilde{U}_1 \rightarrow \tilde{U}_1/\tilde{U}_1 \cap \tilde{W}_2$. Thus, lifting ν to a homomorphism $\tilde{\nu} : \tilde{U}_1 \rightarrow U_0$, it follows that $\lambda \circ \tilde{\nu}$ is nonnegative on $\overline{C(\tilde{\mathcal{X}}_*^0, \tilde{P}_1)}$, and strictly positive on $C^*(\tilde{\mathcal{X}}^0, \tilde{P}_1) \setminus (\tilde{U}_1 \cap \tilde{W}_2)(\mathbb{R})(-1)$. Now suppose that e comes from a splitting $\tilde{\mu} : \tilde{U}_1 \rightarrow U_0$ as in 9.2. Then for every $\beta \in \mathbb{Q}^{\geq 0}$ the splitting $\tilde{\mu} + \beta \cdot \tilde{\nu}$ also satisfies the requirement of 9.2, and the splitting $e' : U'_1 \rightarrow U_0$ associated to $\tilde{\mu} + \beta \cdot \tilde{\nu}$ has the first two of the three desired properties. By 4.23, the image of A is contained in $C^*(\tilde{\mathcal{X}}^0, \tilde{P}_1) \setminus C^*(\tilde{\mathcal{X}}^0, \tilde{P}_2)$. By 4.22 (c), this set is equal to $C^*(\tilde{\mathcal{X}}^0, \tilde{P}_1) \setminus (\tilde{U}_1 \cap \tilde{W}_2)(\mathbb{R})(-1)$, hence $\lambda \circ \tilde{\nu}$ is strictly positive on the image of A . Thus third property also holds, provided that β is sufficiently large. **q.e.d.**

9.5. Lemma: Let $(P, \mathcal{X})$ be pure Shimura data, $(P_2, \mathcal{X}_2)$ a rational boundary component of $(P, \mathcal{X})$, and $(P_1, \mathcal{X}_1)$ a rational boundary component of $(P_2, \mathcal{X}_2)$. Let $\mathcal{X}^0$ be a connected component of $\mathcal{X}$ that is mapped to $\mathcal{X}_1$. Then for every linear form $\mu_2 : U_2(\mathbb{Q})(-1) \rightarrow \mathbb{Q}$, which is strictly positive on $\overline{C(\mathcal{X}^0, P_2)} \setminus \{0\}$, there exists a linear form $\mu : U_1(\mathbb{Q})(-1) \rightarrow \mathbb{Q}$, which is strictly positive on $\overline{C(\mathcal{X}^0, P_1)} \setminus \{0\}$ such that $\mu_2 = \mu|_{U_2(\mathbb{Q})(-1)}$.

Proof. Since the positivity is an open condition, it is easy to see that it suffices to prove the same assertion with linear forms defined over $\mathbb{R}$. Next note that since 4.21 (b) $\overline{C(\mathcal{X}^0, P_2)}$ is contained in $\overline{C(\mathcal{X}^0, P_1)}$, the restriction of any $\mu : U_1(\mathbb{R})(-1) \rightarrow \mathbb{R}$, that is strictly positive on $\overline{C(\mathcal{X}^0, P_1)} \setminus \{0\}$, is strictly positive on $\overline{C(\mathcal{X}^0, P_2)} \setminus \{0\}$. Thus the desired assertion holds for some μ_2 . Now the set of μ_2 's in question is the interior of the dual cone of $C(\mathcal{X}^0, P_2)$. Let Q_i be the parabolic subgroup of P associated to $(P_i, \mathcal{X}_i)$ for $i = 1, 2$. We are going to show that $(Q_1 \cap Q_2)(\mathbb{R})^0$ acts transitively on $C(\mathcal{X}^0, P_2)$. The same group then also acts transitively on the set of μ_2 's, and since the restriction map $\mu \mapsto \mu|_{U_2(\mathbb{R})(-1)}$ is equivariant, the desired assertion holds for every μ_2 .

Since by 2.14 (a) P_i acts through a nontrivial scalar character on U_i , the image of $P_i(\mathbb{R})^0$ in $\text{GL}(U_i)(\mathbb{R})$ consists of all positive scalars. Clearly $P_1 \subset Q_1 \cap P_2 \subset Q_1 \cap Q_2$, so the image of $(Q_1 \cap Q_2)(\mathbb{R})^0$ in $\text{GL}(U_2)(\mathbb{R})$ contains all positive scalars. On the other hand, by 4.19 (b) we have $Q_2 = (Q_1 \cap Q_2) \cdot P_2$. Since $P_2(\mathbb{R})^0$ acts through positive scalars on $U_2(\mathbb{R})$,

the image of $(Q_1 \cap Q_2)(\mathbb{R})^0$ in $\mathrm{GL}(U_2)(\mathbb{R})$ coincides with that of $Q_2(\mathbb{R})^0$. Now by 4.15 (c) $C(\mathcal{X}^0, P_2)$ is an orbit under $Q_2(\mathbb{R})^0$, so the same follows for the group $(Q_1 \cap Q_2)(\mathbb{R})^0$, as desired. **q.e.d.**

9.6. An almost arithmetic subgroup: Let $\bar{\rho} : Q' \rightarrow \mathrm{PGL}(U'_1)$ be the homomorphism induced by the natural projective representation on U'_1 . We shall be concerned with an arbitrary subgroup

$$\Gamma \subset \mathrm{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0)$$

whose image in $\bar{\rho}(Q')(\mathbb{Q})$ is an arithmetic subgroup. For example if K'_f is an open compact subgroup of $P'(\mathbb{A}_f)$, then

$$\Gamma := \mathrm{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0) \cap K'_f \cdot P'_1(\mathbb{A}_f)$$

is such a group. Indeed, this is a subgroup since Q' normalizes P'_1 , and its image in $\bar{\rho}(Q')(\mathbb{Q})$ is an arithmetic subgroup, since by 2.14 (a) P'_1 acts through scalars on U'_1 . We can also take any arithmetic subgroup of $\mathrm{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0)$. Moreover let $(P'_{(0)}, \mathcal{X}'_{(0)})$ be the unique irreducible component of $(P', \mathcal{X}')$ with $\mathcal{X}'^0 \subset \mathcal{X}'_{(0)}$. Then by the condition 2.1 (viii) every sufficiently small arithmetic subgroup of $P'(\mathbb{Q})$ is contained in $(P'_{(0)}(\mathbb{Q}) \cdot Z(P')(\mathbb{Q}))$. Since $Z(P')$ acts trivially on U'_1 , we can also take any arithmetic subgroup of $\mathrm{Stab}_{(P'_{(0)} \cap Q')(\mathbb{Q})}(\mathcal{X}'^0)$.

Recall (9.2) that $\mathrm{Stab}_{P'(\mathbb{Q})}(\mathcal{X}'^0)$ acts through positive scalars on U_0 . Thus whenever an element $\gamma \in \Gamma$ acts through a scalar on U'_1 , this scalar must be positive, since it is so on U_0 . Thus for all purposes, or objects, which are invariant under positive scalars, we can treat Γ like an arithmetic subgroup of $\rho(Q')(\mathbb{Q})$. Among such objects are in particular all convex rational polyhedral cones. Our aim is to construct Γ -invariant rational partial cone decompositions of $C^*(\mathcal{X}'^0, P'_1)$ and of $C^*(\mathcal{X}^0, P_1)$, that satisfy certain additional conditions.

9.7. Proposition: Let $e \in \mathcal{E}(\mathcal{X}'^0, P'_1)$ and $u' \in C^*(\mathcal{X}'^0, P'_1) \cap U'_1(\mathbb{Q})(-1)$. Then the set

$$\{\gamma \in \Gamma \mid \lambda \circ (\mathrm{id} - e^\gamma \circ \pi_0)(u') \leq 0\}$$

is a finite union of $\mathrm{Stab}_\Gamma(\mathbb{R}^{\geq 0} \cdot u')$ -cosets.

Proof. By the remarks in 9.6 we may without loss of generality assume that Γ is an arithmetic subgroup of $\mathrm{Stab}_{(P'_{(0)} \cap Q')(\mathbb{Q})}(\mathcal{X}'^0)$. Recall that by the remark at the end of 9.2 we may write $\lambda \circ (\mathrm{id} - e \circ \pi_0)(\gamma \cdot u')$ instead of $\lambda \circ (\mathrm{id} - e^\gamma \circ \pi_0)(u')$. Suppose that e comes from a splitting $\tilde{\mu} : \tilde{U}_1 \rightarrow U_0$ as in 9.2, and let $\tilde{u}$ be the image of u' in $C^*(\tilde{\mathcal{X}}^0, \tilde{P}_1) \cap \tilde{U}_1(\mathbb{Q})(-1)$. By the injectivity of $U'_1 \hookrightarrow \tilde{U}_1$ we have to show that the set

$$\{\gamma \cdot \tilde{u} \in \Gamma \text{ such that } \lambda \circ \tilde{\mu}(\gamma \cdot \tilde{u}) \leq 0\}$$

is finite. We shall a fortiori show that for all $\tilde{u} \in C^*(\tilde{\mathcal{X}}^0, \tilde{P}_1) \cap \tilde{U}_1(\mathbb{Q})(-1)$ and $\alpha \in \mathbb{Q}$ the set

$$\{\gamma \cdot \tilde{u} \in \Gamma \text{ such that } \lambda \circ \tilde{\mu}(\gamma \cdot \tilde{u}) \leq \alpha\}$$

is finite.

Since Γ acts on $\tilde{U}_1$ through $\tilde{P}(\mathbb{Q})$, by 2.14 (a) it acts through scalars on $\tilde{U}$. But by 9.2 it acts through positive scalars on U_0 , so since it is an arithmetic subgroup, it acts trivially on $\tilde{U}$. This shows that the assertion is invariant under shifting $\tilde{u}$ by elements of $\tilde{U}(\mathbb{Q})(-1)$, if at the same time α is changed by a constant. Since by 4.22 (b) $C^*(\tilde{\mathcal{X}}^0, \tilde{P}_1) = C^*(\tilde{\mathcal{X}}_*^0, \tilde{P}_1) + \tilde{U}(\mathbb{R})(-1)$, we may therefore assume that $\tilde{u} \in C^*(\tilde{\mathcal{X}}_*^0, \tilde{P}_1) \cap \tilde{U}(\mathbb{Q})(-1)$. Fix a Γ -invariant lattice $\tilde{U}_1(\mathbb{Z})(-1)$ that contains $\tilde{u}$. Since the set $\{\tilde{u} \in C^*(\tilde{\mathcal{X}}_*^0, \tilde{P}_1) \mid \tilde{\mu}(\tilde{u}) \leq \alpha\}$ is compact (cf. [AMRT] ch.II §5.3 p.137), the set $\{\gamma \cdot \tilde{u} \mid \gamma \in \Gamma, \lambda \circ \tilde{\mu}(\gamma \cdot \tilde{u}) \leq \alpha\}$ is a bounded subset of a lattice. It is therefore finite, as desired. **q.e.d.**

9.8. Definition of certain “locally polyhedral” subsets: Consider a nonempty collection $\{e_1, \dots, e_m\}$ of splittings in $\mathcal{E}(\mathcal{X}'^0, P'_1)$. Define

$$\begin{aligned} \Sigma &:= \bigcap_{\gamma \in \Gamma} \bigcap_{i=1}^m \sigma_0 + e_i^\gamma(C^*(\mathcal{X}'^0, P'_1)) \\ &= \{u' \in C^*(\mathcal{X}'^0, P'_1) \mid \forall \gamma \in \Gamma \forall 1 \leq i \leq m \quad \lambda \circ (\text{id} - e_i^\gamma \circ \pi_0)(u') \geq 0\}. \end{aligned}$$

Clearly Σ is a Γ -invariant convex cone, with $\Sigma = \Sigma + \sigma_0$, and relatively closed inside $C^*(\mathcal{X}'^0, P'_1)$ (Caution: in general it is not closed in $U'_1(\mathbb{R})(-1)$). In the following we shall always assume:

Σ contains no nontrivial linear subspace.

Note that every nontrivial subspace of $U'_1(\mathbb{R})(-1)$ that is contained in $C^*(\mathcal{X}'^0, P'_1)$, already lies in $U'_1(\mathbb{R})(-1)$. By definition

$$U'_1(\mathbb{R})(-1) \cap \Sigma \subset \{u' \in U'(\mathbb{R})(-1) \mid \forall 1 \leq i \leq m \quad \lambda \circ (\text{id} - e_i \circ \pi_0)(u') \geq 0\},$$

so the assumption certainly holds if the $\text{id} - e_i \circ \pi_0$ generate the vector space $\text{Hom}(U', U_0)$. Since these are splittings, the differences $e_i - e_j$ factor through homomorphisms $U \rightarrow U_0$, and the assumption is verified if the $e_i - e_j$ generate the $\mathbb{Q}$ -vector space $\text{Hom}(U, U_0)$.

We shall show (in 9.12) that every such Σ gives rise to rational partial polyhedral decompositions satisfying the conditions 8.13 (a) and (b), and the condition (*) in 8.5 except for the integrality of the splittings. The set Σ plays the role that Γ -polyhedral cocores play in [AMRT] ch.II §5.3. The term “locally polyhedral” is justified by the following proposition.

9.9. Proposition: For any convex rational polyhedral cone $\sigma' \subset C^*(\mathcal{X}'^0, P'_1)$, the intersection $\sigma' \cap \Sigma$ is again a convex rational polyhedral cone.

Proof. (Compare [AMRT] ch.II §5.4 p.140f) Every convex rational polyhedral cone is contained in a finite union of simplicial convex rational polyhedral cones of top dimension. Thus we may fix $u'_1, \dots, u'_n \in C^*(\mathcal{X}'^0, P'_1) \cap U'_1(\mathbb{Q})(-1)$ that form a basis of $U'_1(\mathbb{Q})(-1)$, and assume that $\sigma' = \{\sum_{j=1}^n \alpha_j \cdot u'_j \mid \forall 1 \leq j \leq n \quad \alpha_j \in \mathbb{R}^{\geq 0}\}$. Choose a lattice $U'_1(\mathbb{Z})(-1)$ that contains all u'_j . Let $\Gamma' \subset \Gamma$ be the subgroup of all elements that stabilize this lattice, then by 9.6 Γ' is, up to scalars, of finite index in Γ . Since $e_i^\gamma = e_i$ for every scalar γ , the set $\{e_i^\gamma \mid 1 \leq i \leq m, \gamma \in \Gamma\}$ is the union of finitely many Γ' -orbits. Thus there exists a positive integer N so that

$$\lambda \circ (\text{id} - e_i^\gamma \circ \pi_0)(U'_1(\mathbb{Z})(-1)) \subset \frac{1}{N} \cdot \mathbb{Z}$$

for all $1 \leq i \leq m$ and $\gamma \in \Gamma$. Now for all i and j , proposition 9.7 implies that the set $\{\lambda \circ (\text{id} - e_i^\gamma \circ \pi_0)(u'_j) \mid \gamma \in \Gamma\}$ is bounded below. Thus there exists an integer M so that for all $1 \leq i \leq m$, $1 \leq j \leq n$, and $\gamma \in \Gamma$

$$a_{i,j,\gamma} := N \cdot \lambda \circ (\text{id} - e_i^\gamma \circ \pi_0)(u'_j) \in M + \mathbb{Z}^{\geq 0}.$$

By the definitions of σ' and of Σ we get

$$\sigma' \cap \Sigma = \left\{ \sum_{j=1}^n \alpha_j \cdot u'_j \mid \forall 1 \leq j \leq n \alpha_j \in \mathbb{R}^{\geq 0} \text{ and } \forall \gamma \in \Gamma \forall 1 \leq i \leq m \sum_{j=1}^n a_{i,j,\gamma} \cdot \alpha_j \geq 0 \right\}.$$

Let S be the set of all n -tuples $(a_{i,j,\gamma})_{1 \leq j \leq n}$ for $1 \leq i \leq m$ and $\gamma \in \Gamma$, then we can also write

$$\sigma' \cap \Sigma = \left\{ \sum_{j=1}^n \alpha_j \cdot u'_j \mid \forall 1 \leq j \leq n \alpha_j \in \mathbb{R}^{\geq 0} \text{ and } \forall \underline{s} \in S \sum_{j=1}^n s_j \cdot \alpha_j \geq 0 \right\}.$$

Now S is a subset of $(M + \mathbb{Z}^{\geq 0})^n$, so by the lemma [AMRT] ch.II §5.4 p.140 there exists a finite subset $S_0 \subset S$, such that

$$\forall \underline{s} \in S \exists \underline{s}' \in S_0 \text{ so that } \forall 1 \leq j \leq n s_j \geq s'_j.$$

This implies that

$$\sigma' \cap \Sigma = \left\{ \sum_{j=1}^n \alpha_j \cdot u'_j \mid \forall 1 \leq j \leq n \alpha_j \in \mathbb{R}^{\geq 0} \text{ and } \forall \underline{s} \in S_0 \sum_{j=1}^n s_j \cdot \alpha_j \geq 0 \right\},$$

which is a convex rational polyhedral cone, as desired. **q.e.d.**

9.10. Proposition: There exist finitely many $u'_1, \dots, u'_n \in C^*(\mathcal{X}'^0, P'_1) \cap U'_1(\mathbb{Q})(-1)$ such that Σ is the convex closure of the set

$$\bigcup_{\gamma \in \Gamma} \bigcup_{j=1}^n \mathbb{R}^{\geq 0} \cdot \gamma \cdot u'_j.$$

Proof. By 6.19 (b) there exists a convex rational polyhedral cone $\sigma' \subset C^*(\mathcal{X}'^0, P'_1)$ so that $\Gamma \cdot \sigma' = C^*(\mathcal{X}'^0, P'_1)$. Choose the u'_j such that $\sigma' \cap \Sigma$ is the convex closure of $\bigcup_{j=1}^n \mathbb{R}^{\geq 0} \cdot u'_j$, this is possible by 9.9. Then $\Sigma = \Gamma \cdot (\sigma' \cap \Sigma)$ is contained in the convex closure of $\bigcup_{\gamma \in \Gamma} \bigcup_{j=1}^n \mathbb{R}^{\geq 0} \cdot \gamma \cdot u'_j$. But Σ itself is convex, and contains all $\mathbb{R}^{\geq 0} \cdot \gamma \cdot u'_j$, so we have equality. **q.e.d.**

9.11. Proposition: for all $1 \leq i \leq m$ and $\gamma \in \Gamma$, the subset $e_i^\gamma(U_1(\mathbb{R})(-1)) \cap \Sigma$ is a convex rational polyhedral cone.

Proof. By Γ -invariance we may assume $\gamma = 1$. Then the set in question can be written as $\{u' \in \Sigma \mid \lambda \circ (\text{id} - e_i \circ \pi_0)(u') = 0\}$. Let $u'_1, \dots, u'_n$ be as in 9.10, then $\lambda \circ (\text{id} - e_i \circ \pi_0)(\gamma' \cdot u'_j) \geq$

0 for all $1 \leq j \leq n$ and $\gamma' \in \Gamma$. Thus 9.10 implies that $e_i(U_1(\mathbb{R})(-1) \cap \Sigma)$ is the convex closure of the union of all $\mathbb{R}^{\geq 0} \cdot \gamma' \cdot u'_j$ with $\lambda \circ (\text{id} - e_i \circ \pi_0)(\gamma' \cdot u'_j) = 0$. Since

$$\lambda \circ (\text{id} - e_i \circ \pi_0)(\gamma' \cdot u'_j) = \lambda(\gamma' \cdot (\text{id} - e_i^{\gamma'} \circ \pi_0)(u'_j)),$$

this equation is equivalent to $\lambda \circ (\text{id} - e_i^{\gamma'} \circ \pi_0)(u'_j) = 0$. But by 9.7 there are, modulo $\text{Stab}_\Gamma(\mathbb{R}^{\geq 0} \cdot u'_j)$, only finitely pairs (γ', j) with this property. Thus our set is the convex closure of the union of finitely many $\mathbb{R}^{\geq 0} \cdot \gamma' \cdot u'_j$, as desired. **q.e.d.**

9.12. Proposition: Let e_i and Σ be as in 9.8. Let Σ_Σ^+ be the set of all faces of all cones $e_i^\gamma(U_1(\mathbb{R})(-1) \cap \Sigma)$. Let $\Sigma_\Sigma := \{\pi_0(\sigma') \mid \sigma' \in \Sigma_\Sigma^+\}$, and $\Sigma'_\Sigma := \{\sigma', \sigma' + \sigma_0 \mid \sigma' \in \Sigma_\Sigma^+\}$. Then Σ_Σ^+ and Σ'_Σ are rational partial polyhedral decompositions of $C^*(\mathcal{X}^0, P_1)$, and Σ_Σ is a complete rational polyhedral decomposition of $C^*(\mathcal{X}^0, P_1)$. All of them are Γ -invariant and finite modulo Γ . They satisfy the line bundle condition of 5.11 (except for the integrality) i.e. $\Sigma_\Sigma^+ = \{e_\sigma(\sigma) \mid \sigma \in \Sigma_\Sigma\}$ for certain splittings $e_\sigma : U_1 \rightarrow U'_1$. Moreover they satisfy conditions 8.13 (a) and (b).

Proof. We first show that Σ_Σ^+ is a rational partial cone decomposition. By 9.11 Σ_Σ^+ is a nonempty set of convex rational polyhedral cones. Of the conditions 5.1 (i)–(iii) the first holds by definition of Σ_Σ^+ . The last one holds since Σ does not contain a nontrivial linear subspace. For the second one let $\sigma := e_i^\gamma(U_1(\mathbb{R})(-1) \cap \Sigma)$, $\sigma' := e_i^{\gamma'}(U_1(\mathbb{R})(-1) \cap \Sigma)$, τ a face of σ , and τ' a face of σ' . We can write $\tau = \{u' \in \sigma \mid \forall \nu \ell_\nu(u') = 0\}$ with finitely many linear forms $\ell_\nu : U'(\mathbb{Q})(-1) \rightarrow \mathbb{Q}$ that are nonnegative on σ , and likewise $\tau' = \{u' \in \sigma' \mid \forall \nu' \ell'_{\nu'}(u') = 0\}$. Then

$$\tau \cap \tau' = \{u' \in \sigma \cap \sigma' \mid \forall \nu \ell_\nu(u') = 0 \text{ and } \forall \nu' \ell'_{\nu'}(u') = 0\},$$

which by definition is a face of $\sigma \cap \sigma'$. But this cone can be written as $\{u' \in \sigma \mid \lambda \circ (\text{id} - e_i^{\gamma'} \circ \pi_0)(u') = 0\}$, and $\lambda \circ (\text{id} - e_i^{\gamma'} \circ \pi_0)$ is nonnegative on σ , so $\sigma \cap \sigma'$ is a face of σ . thus $\tau \cap \tau'$ is a face of σ , as desired. This proves that Σ_Σ^+ is a rational partial cone decomposition of $C^*(\mathcal{X}^0, P_1)$.

From this it easily follows that Σ_Σ and Σ'_Σ are rational partial cone decompositions. For all γ and i we have

$$\begin{aligned} (e_i^\gamma(U_1(\mathbb{R})(-1) \cap \Sigma) + \sigma_0) &= \\ &= \{u' \in \Sigma \mid \forall \gamma' \in \Gamma \forall 1 \leq i' \leq m \lambda \circ (\text{id} - e_i^{\gamma'} \circ \pi_0)(u') \geq \lambda \circ (\text{id} - e_i^\gamma \circ \pi_0)(u')\} \end{aligned}$$

so 9.7 implies that Σ is the union of all these, i.e. that $|\Sigma'_\Sigma| = \Sigma$. Again by 9.7 we have $\Sigma + U_0(\mathbb{R})(-1) = C^*(\mathcal{X}^0, P_1)$, whence $\pi_0(\Sigma) = C^*(\mathcal{X}^0, P_1)$, and Σ_Σ is a complete rational polyhedral decomposition of $C^*(\mathcal{X}^0, P_1)$. The Γ -invariance and finiteness modulo Γ , as well as the line bundle condition of 5.11 and conditions 8.13 (a) and (b) are clear from the definitions. **q.e.d.**

9.13. Corollary: Let $(P', \mathcal{X}') \rightarrow (P, \mathcal{X})$ be as in 9.2, $K'_f \subset P'(\mathbb{A}_f)$ an open compact subgroup, and $K_f := \pi_0(K'_f)$. Let Σ' be a K'_f -admissible partial cone decomposition

for $(P', \mathcal{X}')$, and Σ a K_f -admissible partial cone decomposition for $(P, \mathcal{X})$. Assume that for every rational boundary component $(P'_1, \mathcal{X}'_1)$ of $(P', \mathcal{X}')$, every connected component $\mathcal{X}^{\prime 0}$ of $\mathcal{X}'$ that maps into $\mathcal{X}'_1$, and every $p'_f \in P'(\mathbb{A}_f)$, we have $\Sigma'(\mathcal{X}^{\prime 0}, P'_1, p'_f) = \Sigma'_\Sigma$ and $\Sigma(\mathcal{X}^0, P_1, \pi_0, (p'_f)) = \Sigma_\Sigma$ for some $\Sigma \subset C^*(\mathcal{X}^{\prime 0}, P'_1)$ as in 9.8. Then after possibly replacing K'_f by $K'_f \cdot K_f^U$ for some open compact subgroup K_f^U of $U_0(\mathbb{A}_f)$, the conditions 8.5 (*) and 8.13 (a) and (b) hold. In particular $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is projective.

Proof. Modulo the left action of $P'(\mathbb{Q})$ and the right action of K'_f there are only finitely many $(\mathcal{X}^{\prime 0}, P'_1, p'_f)$. Since by 9.12 Σ'_Σ is finite modulo Γ , it follows that Σ' is finite. The same for Σ , and since by 9.12 $|\Sigma_\Sigma| = C^*(\mathcal{X}^0, P_1)$, it follows that Σ is complete. Moreover by 9.12 the pair (Σ', Σ) satisfies the conditions 8.13 (a) and (b), plus the line bundle condition (*) of 8.5 except possibly for the integrality of the splitting of the short exact sequence $0 \rightarrow U_0(\mathbb{Q}) \cap \Gamma_{U'} \rightarrow \Gamma_{U'} \rightarrow \Gamma_U \rightarrow 0$. But as in the proof of 8.8, if we replace K'_f by $K'_f \cdot K_f^U$ for a large open compact subgroup K_f^U of $U_0(\mathbb{A}_f)$, then $U_0(\mathbb{Q}) \cap \Gamma_{U'}$ is replaced by a larger lattice. thus for any fixed $\sigma \in \Sigma$ we can choose K_f^U such that the corresponding splitting becomes integral. By the finiteness of Σ , and since this integrality condition is invariant under the operations 6.4 (ii) and (iv), we can even choose K_f^U so that every such splitting becomes integral. If we now replace K'_f by $K'_f \cdot K_f^U$, then the other conditions still hold, and all splittings are integral, as desired. **q.e.d.**

The following fact was implicit in the assumptions of the preceding corollary.

9.14. Proposition: Let Γ be as in 9.6, Σ as in 9.8, and $\Sigma_\Sigma, \Sigma_\Sigma^+, \Sigma'_\Sigma$ as in 9.12. Let $(P'_2, \mathcal{X}'_2)$ be a rational boundary component between $(P'_1, \mathcal{X}'_1)$ and $(P', \mathcal{X}')$, and $\Gamma_2 := \text{Stab}_\Gamma(C^*(\mathcal{X}^{\prime 0}, P'_2))$. Then $\Sigma_2 := C^*(\mathcal{X}^{\prime 0}, P'_2) \cap \Sigma$ can be described as in 9.8, with $(P'_2, \mathcal{X}'_2)$ in place of $(P'_1, \mathcal{X}'_1)$, and Γ_2 in place of Γ . Moreover $\Sigma_{\Sigma_2}^+ = \{\sigma' \in \Sigma_\Sigma^+ \mid \sigma' \subset C^*(\mathcal{X}^{\prime 0}, P'_2)\}$, and likewise for Σ_{Σ_2} and Σ'_{Σ_2} .

Proof. Let us first show that Γ_2 is of the form described in 9.6 with $(P'_2, \mathcal{X}'_2)$ in place of $(P'_1, \mathcal{X}'_1)$. Let Q'_2 be the parabolic subgroup of P' associated to $(P'_2, \mathcal{X}'_2)$, then $\Gamma_2 = Q'_2(\mathbb{Q}) \cap \Gamma \subset \text{Stab}_{Q'_2(\mathbb{Q})}(\mathcal{X}^{\prime 0})$. By 9.6 the image of Γ_2 in $\text{GL}(U'_2)$ is up to scalars commensurable with the image of an arithmetic subgroup of $(Q'_1 \cap Q'_2)(\mathbb{Q})$. But by 4.19 (b) $Q'_2 = (Q' \cap Q'_2) \cdot P'_2$, so since P'_2 acts through scalars on U'_2 , the image of Γ_2 in $\text{GL}(U'_2)$ is up to scalars commensurable with the image of an arithmetic subgroup of $(Q'_2)(\mathbb{Q})$, as desired.

According to 6.19 (b) choose a convex rational polyhedral cone $\sigma \subset C^*(\mathcal{X}^{\prime 0}, P'_2)$ such that $\Gamma_2 \cdot \sigma = C^*(\mathcal{X}^{\prime 0}, P'_2)$, such a cone must contain $U'(\mathbb{R})(-1)$. By 9.9 $\sigma \cap \Sigma$ is again a convex rational polyhedral cone. Since by definition

$$\sigma \cap \Sigma = \{u' \in \sigma \mid \forall \gamma \in \Gamma \forall 1 \leq i \leq m \lambda \circ (\text{id} - e_i^\gamma \circ \pi_0)(u') \geq 0\},$$

there exists a finite set S of pairs (γ, i) such that

$$\sigma \cap \Sigma = \{u' \in \sigma \mid \forall (\gamma, i) \in S \lambda \circ (\text{id} - e_i^\gamma \circ \pi_0)(u') \geq 0\}.$$

Write $\{e'_1, \dots, e'_n\} := \{e_i^\gamma|_{U'_2} \mid (\gamma, i) \in S\}$. We have

$$\begin{aligned} \sigma \cap \Sigma &= \{u' \in \sigma \mid \forall 1 \leq j \leq n \lambda \circ (\text{id} - e'_j \circ \pi_0)(u') \geq 0\} \\ &= \{u' \in \sigma \mid \forall 1 \leq j \leq n \forall \gamma \in \Gamma_2 \lambda \circ (\text{id} - e_j'^\gamma \circ \pi_0)(u') \geq 0\}, \end{aligned}$$

hence

$$\Sigma_2 = \Gamma_2 \cdot (\sigma \cap \Sigma) = \{u' \in C^*(\mathcal{X}'^0, P'_2) \mid \forall 1 \leq j \leq n \ \forall \gamma \in \Gamma_2 \ \lambda \circ (\text{id} - e_j'^\gamma \circ \pi_0)(u') \geq 0\},$$

as in 9.8. The assertions about Σ_{Σ_2} , $\Sigma_{\Sigma_2}^+$, and Σ'_{Σ_2} are obvious. **q.e.d.**

The following construction is a kind of inverse to 9.14, and will be needed in the construction of cone decompositions as in 9.13.

9.15. Proposition: Let Γ be as in 9.6, and $\Delta \subset C^*(\mathcal{X}'^0, P'_1) \setminus C(\mathcal{X}'^0, P'_1)$ a subset, such that for every rational boundary component $(P'_2, \mathcal{X}'_2)$ between $(P'_1, \mathcal{X}'_1)$ and $(P', \mathcal{X}')$, different from $(P'_1, \mathcal{X}'_1)$, the intersection $C^*(\mathcal{X}'^0, P'_1) \cap \Delta$ is of the form described in 9.8 with $(P'_2, \mathcal{X}'_2)$ in place of $(P'_1, \mathcal{X}'_1)$ and some group of Γ_2 . Then there exists a Γ -invariant subset $\Sigma \subset C^*(\mathcal{X}'^0, P'_1)$ as in 9.8, such that $\Sigma \setminus C(\mathcal{X}'^0, P'_1) = \Delta$.

Proof. In the case $(P'_1, \mathcal{X}'_1) = (P', \mathcal{X}')$ the assertion is just the existence of some Σ as in 9.8. But by 9.3 (a) there exist splittings e_i satisfying the condition in 9.8, so this is clear. Now assume that $(P'_1, \mathcal{X}'_1)$ is a proper rational boundary component of $(P', \mathcal{X}')$. There are only finitely many Γ -conjugacy classes of rational boundary components between $(P'_1, \mathcal{X}'_1)$ and $(P', \mathcal{X}')$. Let $(P'_\nu, \mathcal{X}'_\nu)$ be representatives for these conjugacy classes. Let $\Gamma_\nu := \text{Stab}_\Gamma(C^*(\mathcal{X}'^0, P'_\nu))$, by the proof of 9.14 this group is a valid choice as in 9.6 for $(P'_\nu, \mathcal{X}'_\nu)$ in place of $(P'_1, \mathcal{X}'_1)$. Thus by assumption there exist finitely many $e_{\nu,i} \in \mathcal{E}(\mathcal{X}'^0, P'_\nu)$ such that

$$(C^*(\mathcal{X}'^0, P'_\nu) \cap \Delta = \{u' \in C^*(\mathcal{X}'^0, P'_\nu) \mid \forall \gamma \in \Gamma_\nu \ \forall i \ \lambda \circ (\text{id} - e_{\nu,i}^\gamma \circ \pi_0)(u') \geq 0\}.$$

Also by 9.10 there exist finitely many $u'_{\nu,j} \in C^*(\mathcal{X}'^0, P'_\nu) \cap U'_\nu(\mathbb{Q})(-1)$ such that $\Delta \cap C^*(\mathcal{X}'^0, P'_\nu)$ is the convex closure of the set

$$\bigcup_{\gamma \in \Gamma_\nu} \bigcup_j \mathbb{R}^{\geq 0} \cdot \gamma \cdot u'_{\nu,j}.$$

Let us fix ν and i for a moment. By 9.4 (a) there exists $\tilde{e}_{\nu,i} \in \mathcal{E}(\mathcal{X}'^0, P'_1)$ such that $e_{\nu,i} = \tilde{e}_{\nu,i}|_{U'_\nu}$. We would be happy if we had

$$\lambda \circ (\text{id} - \tilde{e}_{\nu,i}^\gamma \circ \pi_0)(u'_{\mu,j}) \geq 0$$

for all μ, j , and all $\gamma \in \Gamma$. Since we have only finitely many $u'_{\mu,j}$, by 9.7 this inequality fails at most if $\gamma \cdot u'_{\mu,j}$ is contained in one of finitely many half-lines. Since $u'_{\mu,j} \in \Delta$ these points can not lie in $C^*(\mathcal{X}'^0, P'_\nu)$. Thus by 9.4 (b) there exists $\tilde{\tilde{e}}_{\nu,i} \in \mathcal{E}(\mathcal{X}'^0, P'_1)$ such that $e_{\nu,i} = \tilde{\tilde{e}}_{\nu,i}|_{U'_\nu}$, that $\lambda \circ (\text{id} - \tilde{\tilde{e}}_{\nu,i} \circ \pi_0) \geq \lambda \circ (\text{id} - \tilde{e}_{\nu,i} \circ \pi_0)$ on $C^*(\mathcal{X}'^0, P'_1)$, and that $\lambda \circ (\text{id} - \tilde{\tilde{e}}_{\nu,i} \circ \pi_0)$ is nonnegative on each of these finitely many half-lines. Hence the desired inequalities hold for $\tilde{\tilde{e}}_{\nu,i}$ in place of $\tilde{e}_{\nu,i}$.

Now we define Σ as in 9.8 using the splittings $\tilde{\tilde{e}}_{\nu,i}$. To prove $\Sigma \setminus C(\mathcal{X}'^0, P'_1) = \Delta$, by Γ -invariance it suffices to prove $C^*(\mathcal{X}'^0, P'_\nu) \cap \Sigma = C^*(\mathcal{X}'^0, P'_\nu) \cap \Delta$ for every ν . But by definition

$$\begin{aligned} C^*(\mathcal{X}'^0, P'_\nu) \cap \Sigma &= \{u' \in C^*(\mathcal{X}'^0, P'_\nu) \mid \forall \gamma \in \Gamma \ \forall \mu \ \forall i \ \lambda \circ (\text{id} - e_{\nu,i}^\gamma \circ \pi_0)(u') \geq 0\} \\ &\subset \{u' \in C^*(\mathcal{X}'^0, P'_\nu) \mid \forall \gamma \in \Gamma_\nu \ \forall i \ \lambda \circ (\text{id} - e_{\nu,i}^\gamma \circ \pi_0)(u') \geq 0\} \\ &= C^*(\mathcal{X}'^0, P'_\nu) \cap \Delta, \end{aligned}$$

so one inclusion is clear. For the other inclusion it suffices to prove that all $u'_{\mu,j} \in \Sigma$, and this follows from the choice of the $\tilde{e}_{\nu,i}$. Finally by assumption $(P', \mathcal{X}')$ is one of the $(P'_\nu, \mathcal{X}'_\nu)$, so the equality $U'(\mathbb{R})(-1) \cap \Sigma = C^*(\mathcal{X}'^0, P') \cap \Sigma = C^*(\mathcal{X}'^0, P') \cap \Delta$ implies in particular that Σ , like Δ , contains no nontrivial linear subspace. **q.e.d.**

9.16. Proposition: Let $(P', \mathcal{X}') \rightarrow (P, \mathcal{X})$ be as in 9.2, $K'_f \subset P'(\mathbb{A}_f)$ an open compact subgroup, and $K_f := \pi_0(K'_f)$. Then there exist admissible cone decompositions Σ' and Σ as in 9.13.

Proof. By induction we shall construct admissible cone decompositions $\Sigma'_1 \subsetneq \Sigma'_2 \subsetneq \dots$ and $\Sigma_1 \subsetneq \Sigma_2 \subsetneq \dots$ with the following property: For every rational boundary component $(P'_1, \mathcal{X}'_1)$ of $(P', \mathcal{X}')$, every connected component $\mathcal{X}'^0$ of $\mathcal{X}'$, and every $p'_f \in P'(\mathbb{A}_f)$ either $|\Sigma'_i(\mathcal{X}'^0, P'_1, p'_f)| \subset C^*(\mathcal{X}'^0, P'_1) \setminus C(\mathcal{X}'^0, P'_1)$ and $|\Sigma_i(\mathcal{X}^0, P_1, \pi_0(p'_f))| \subset C^*(\mathcal{X}^0, P_1) \setminus C(\mathcal{X}^0, P_1)$, or there exists Σ as in 9.8 with $\Sigma'_i(\mathcal{X}'^0, P'_1, p'_f) = \Sigma'_\Sigma$ and $\Sigma_i(\mathcal{X}^0, P_1, \pi_0(p'_f)) = \Sigma_\Sigma$. Since modulo the left action of $P'(\mathbb{Q})$ and the right action of K'_f there are only finitely many triples $(\mathcal{X}'^0, P'_1, p'_f)$, this sequence stops at some place, at which Σ'_i and Σ_i have the desired properties.

Suppose that either $i = 0$, or Σ'_i and Σ_i are already constructed. We want to construct Σ'_{i+1} and Σ_{i+1} . Choose $(\mathcal{X}'^0, P'_1, p'_f)$ such that $\Delta := |\Sigma'_i(\mathcal{X}'^0, P'_1, p'_f)|$ has the property required in 9.15. Let

$$\Gamma := \text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0) \cap p'_f \cdot K'_f \cdot p'^{-1}_f \cdot P'_1(\mathbb{A}_f),$$

then by the K'_f -admissibility of Σ'_i the set Δ is invariant under Γ . If we let Σ be as in 9.15, then by 9.14 the cone decomposition $\Sigma'_{i+1}(\mathcal{X}'^0, P'_1, p'_f) := \Sigma'_\Sigma$ contains $\Sigma'_i(\mathcal{X}'^0, P'_1, p'_f)$, and $\Sigma_{i+1}(\mathcal{X}^0, P_1, \pi_0(p'_f)) := \Sigma_\Sigma$ contains $\Sigma_i(\mathcal{X}^0, P_1, \pi_0(p'_f))$. By the following lemma, $\Sigma'_{i+1}(\mathcal{X}'^0, P'_1, p'_f) \cup \Sigma'_i$ and $\Sigma_{i+1}(\mathcal{X}^0, P_1, \pi_0(p'_f)) \cup \Sigma_i$ extend to a K'_f -, resp. K_f -admissible cone decompositions. By construction they have the desired properties. **q.e.d.**

9.17. Lemma: Let $(P, \mathcal{X})$ be mixed Shimura data, $K_f \subset P(\mathbb{A}_f)$ an open compact subgroup, and Σ a K_f -admissible partial cone decomposition for $(P, \mathcal{X})$. Let $(P_1, \mathcal{X}_1)$ be a rational boundary component of $(P, \mathcal{X})$, $\mathcal{X}^0$ be a connected component of $\mathcal{X}$ that maps to $\mathcal{X}_1$, and $p_f \in P(\mathbb{A}_f)$. Let Q be a parabolic subgroup of P associated to $(P_1, \mathcal{X}_1)$, and

$$\Gamma_1 := \text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^0) \cap p_f \cdot K_f \cdot p_f^{-1} \cdot P_1(\mathbb{A}_f).$$

Let Σ_0 be a rational partial polyhedral decomposition of $C^*(\mathcal{X}^0, P_1)$, containing $\Sigma(\mathcal{X}^0, P_1, p_f)$, such that $\sigma^0 \subset C(\mathcal{X}^0, P_1)$ for every $\sigma \in \Sigma_0 \setminus \Sigma(\mathcal{X}^0, P_1, p_f)$. Let $\mathcal{T}(\mathcal{X}^0, P_1, p_f) := \Sigma(\mathcal{X}^0, P_1, p_f) \cup \Sigma_0$. Then $\Sigma \cup \mathcal{T}(\mathcal{X}^0, P_1, p_f)$ extends to a K_f -admissible partial cone decomposition for $(P, \mathcal{X})$ if and only if Σ_0 is Γ_1 -invariant.

Proof. Since the conditions 6.4 (ii)-(iv) already hold for Σ , it is easily checked that they hold for $\mathcal{T}$ if and only if for all $p, p' \in P(\mathbb{Q})$, $p_{1,f}, p'_{1,f} \in P_1(\mathbb{A}_f)$, $k_f, k'_f \in K_f$, we have $p \cdot p_{1,f} \cdot \Sigma_0 \cdot k_f = p' \cdot p'_{1,f} \cdot \Sigma_0 \cdot k'_f$, if $p \cdot \mathcal{X}^0 = p' \cdot \mathcal{X}^0$, $p \cdot p_{1,f} \cdot p_f \cdot k_f = p' \cdot p'_{1,f} \cdot p_f \cdot k'_f$, and $\text{int}(p)(P_1) = \text{int}(p')(P_1)$. Here Σ_0 is identified with the corresponding decomposition

of $C^*(\mathcal{X}^0, P_1) \times \{p_f\}$. In terms of the original decomposition of $C^*(\mathcal{X}^0, P_1)$ this means that $p \cdot \Sigma_0 = p' \cdot \Sigma_0$ for all $p, p' \in P(\mathbb{Q})$ such that $p'^{-1} \cdot p \cdot \mathcal{X}^0 = \mathcal{X}^0$, $p'^{-1} \cdot p \in P(\mathbb{A}_f) \cdot p_f \cdot K_f \cdot p_f^{-1} \cdot P_f(\mathbb{A}_f)$ and $\text{int}(p'^{-1} \cdot p)(P_1) = P_1$. Writing $q = p'^{-1} \cdot p$ this is equivalent to $q \cdot \Sigma_0 = \Sigma_0$ for all $q \in \text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}^0) \cap P_1(\mathbb{A}_f) \cdot p_f \cdot K_f \cdot p_f^{-1} \cdot P_1(\mathbb{A}_f)$. But this set is just Γ_1 , since $Q(\mathbb{Q})$ normalizes $P_1(\mathbb{A}_f)$. **q.e.d.**

The following analog of 5.22 permits to construct refinements of cone decompositions Σ as in 9.13.

9.18. Propositions: Let Γ be as in 9.6, and Σ as in 9.8. Consider a Γ -invariant function $f : C^*(\mathcal{X}^0, P_1) \rightarrow U_0(\mathbb{R})(-1)$, such that the restriction of $\lambda \circ f$ to every $\sigma \in \Sigma_\Sigma$ is a piecewise linear convex rational function, with respect to the rational structure given by $U_1(\mathbb{Q})(-1)$. For every $\epsilon \in \mathbb{Q}^{>0}$ define

$$\begin{aligned} T &:= \{u' \in C^*(\mathcal{X}'^0, P'_1) \mid u' + \epsilon \cdot f \circ \pi_0(u') \in \Sigma\} \\ &= \{u' + u_0 \mid u' \in \Sigma, u_0 \in U_0(\mathbb{R})(-1) \text{ with } \lambda(u_0 + \epsilon \cdot f \circ \pi_0(u')) \geq 0\}. \end{aligned}$$

If ϵ is sufficiently small, then T is again as in 9.8, and

$$\Sigma_T = \bigcup_{\sigma \in \Sigma_\Sigma} \Sigma(\lambda \circ f|_\sigma).$$

Proof. Choose representatives σ_i for the finitely many Γ -conjugacy classes in Σ_Σ with $\mathbb{R} \cdot \sigma_i = U_1(\mathbb{R})(-1)$, and let $e_i \in \mathcal{E}(\mathcal{X}'^0, P'_1)$ be the associated splittings. Let $\tau_{i,\mu}$ be the cones in $\Sigma(\lambda \circ f|_{\sigma_i})$ with $\mathbb{R} \cdot \tau_{i,\mu} = U_1(\mathbb{R})(-1)$ and $\ell_{i,\mu} : U_1 \rightarrow U_0$ the unique linear maps such that $f = \ell_{i,\mu}$ on $\tau_{i,\mu}$. Then by definition

$$T \cap \pi_0^{-1}(\tau_{i,\mu}) = \{u' \in \pi_0^{-1}(\tau_{i,\mu}) \mid \lambda \circ (\text{id}(e_i - \epsilon \cdot \ell_{i,\mu}) \circ \pi_0)(u') \geq 0\}.$$

If ϵ is sufficiently small, then by 9.3 (a) $e_i - \epsilon \cdot \ell_{i,\mu} \in \mathcal{E}(\mathcal{X}'^0, P'_1)$. To finish we need that $\lambda \circ (\text{id} - (e_i - \epsilon \cdot \ell_{i,\mu}) \circ \pi_0)$ is strictly positive on $T \setminus \pi_0^{-1}(\tau_{i,\mu})$. In fact, it then follows that T is as in 9.8 with respect to all $e_i - \epsilon \cdot \ell_{i,\mu}$, and all $\tau_{i,\mu}$ are in Σ_T . This implies that all $\Sigma(\lambda \circ f|_{\sigma_i})$ are contained in Σ_T , and since both Σ_T and $\bigcup_{\sigma \in \Sigma_\Sigma} \Sigma(\lambda \circ f|_\sigma)$ are complete rational polyhedral decompositions of $C^*(\mathcal{X}^0, P_1)$, the desired equality follows.

Let $\mathbb{R}^{\geq 0} \cdot u_{j,\nu}$ be all one dimensional cones in $\Sigma(\lambda \circ f|_{\sigma_j})$. Since

$$T \cap \pi_0^{-1}(\mathbb{R}^{\geq 0} \cdot u_{j,\nu}) = \sigma_0 + \mathbb{R}^{\geq 0} \cdot (e_j - \epsilon \cdot f)(u_{j,\nu}),$$

we have to show that for all $\gamma \in \Gamma$

$$\lambda \circ (\text{id} - (e_i - \epsilon \cdot \ell_{i,\mu}) \circ \pi_0)(\gamma \cdot (e_j - \epsilon \cdot f)(u_{j,\nu})) > 0,$$

unless $\gamma \cdot u_{j,\nu} \in \tau_{i,\mu}$. Fix i, j, μ, ν , and $\epsilon_0 > 0$ such that $e_{\tau_i} - \epsilon \cdot \ell_{i,\nu} \in \mathcal{E}(\mathcal{X}'^0, P'_1)$ for all $\epsilon_0 > \epsilon \in \mathbb{Q}^{>0}$ then by 9.7 each of the four inequalities

$$\lambda \circ (\text{id} - (e_i - \epsilon_1 \cdot \ell_{i,\mu}) \circ \pi_0)(\gamma \cdot (e_j - \epsilon_2 \cdot f)(u_{j,\nu})) > 0$$

with $\epsilon_1, \epsilon_2 \in \{0, \epsilon_0\}$ fails at most if $\gamma \cdot u_{j,\nu}$ lies in one of finitely many half-lines. This shows that there exists a finite number of $\gamma_k \in \Gamma$, such that for every $\epsilon_0 > \epsilon \in \mathbb{Q}^{>0}$ the inequality

above fails at most if $\gamma \cdot u_{j,\nu} \in \mathbb{R}^{\geq 0} \cdot \gamma_k \cdot u_{j,\nu}$ for some k . Thus we are reduced to showing the inequality for some fixed $\gamma \cdot u_{j,\nu} \notin \tau_{i,\mu}$. If $\gamma \cdot u_{j,\nu} \notin \sigma_i$, then

$$\lambda \circ (\text{id} - e_i \circ \pi_0)(\gamma \cdot (e_j(u_{j,\nu}))) > 0$$

by the definition of Σ_Σ , so the inequality certainly holds if ϵ is sufficiently small. Otherwise $\gamma \cdot u_{j,\nu} \in \sigma_i$ implies that $\mathbb{R}^{\geq 0} \cdot \gamma \cdot u_{j,\nu} \in \Sigma(\lambda \circ f|_{\sigma_i})$, so we may assume that $j = i$ and $\gamma = 1$. Then we have

$$\lambda \circ (\text{id} - (e_i - \epsilon \cdot \ell_{i,\mu}) \circ \pi_0) \circ (e_i - \epsilon \cdot f)(u_{j,\nu}) = \epsilon \cdot \lambda \circ (\ell_{i,\nu} - f)(u_{i,\nu}),$$

so the assertion is equivalent to $\lambda \circ (\ell_{i,\mu} - f)(u_{i,\nu}) > 0$. But by the definition of $\tau_{i,\mu}$ and $\ell_{i,\mu}$ this inequality holds on all of $\sigma_i - \tau_{i,\mu}$, as desired. **q.e.d.9.19. Proposition:**

Let $(P', \mathcal{X}') \rightarrow (P, \mathcal{X})$, K'_f , K_f , Σ' and Σ be as in 9.13. Let $\tilde{\Sigma}$ be a refinement of Σ . Then there exist $\mathcal{T}'$ and $\mathcal{T}$ satisfying the same conditions of 9.13, such that $\mathcal{T}$ is a refinement of $\tilde{\Sigma}$.

Proof. The essential points of the proof are 5.21 and 9.18. To be able to use 9.18 we need a function $f : \mathcal{C}(P, \mathcal{X}) \times P(\mathbb{A}_f) \rightarrow U_0(\mathbb{R})(-1)$ with the following properties:

- (i) The restriction of $\lambda \circ f$ to every $\sigma \in \Sigma$ is a piecewise linear convex rational function.
- (ii) For every $\sigma \in \Sigma$, $\Sigma(\lambda \circ f|_\sigma)$ is a refinement of $\{\tau \in \tilde{\Sigma} \mid \tau \subset \sigma\}$.
- (iii) f is invariant under the three actions 6.4 (ii)–(iv).

Suppose that such a function is given. For every $(\mathcal{X}'^0, P'_1, p'_f)$ let $\Sigma := |\Sigma'(\mathcal{X}'^0, P'_1, p'_f)|$, define T as in 9.18, and let $\mathcal{T}'(\mathcal{X}'^0, P'_1, p'_f) := \Sigma'_T$ and $\mathcal{T}'(\mathcal{X}'^0, P_1, \pi_0(p'_f)) := \Sigma_T$. By 9.18 we can make ϵ so small that any finite number of $\mathcal{T}'(\mathcal{X}'^0, P'_1, p'_f)$ and $\mathcal{T}(\mathcal{X}'^0, P_1, \pi_0(p'_f))$ satisfy all our requirements. Since by assumption this construction of $\mathcal{T}'$ and $\mathcal{T}$ is invariant under the actions in 6.4 (ii)–(iv), and there are only finitely many triples $(\mathcal{X}'^0, P'_1, p'_f)$ modulo this action, there exists ϵ so that this works for all such triples, as desired.

We construct the desired function by induction over the classes of cones in Σ modulo the three operations 6.4 (ii)–(iv). If it has not yet been defined on all of these, choose $(\mathcal{X}'^0, P'_1, p'_f)$ and $\sigma \in \Sigma(\mathcal{X}'^0, P_1, \pi_0(p'_f))$ with $\sigma^0 \subset C(\mathcal{X}'^0, P_1)$, such that f is already defined on every proper face of σ , but not on σ itself. By 5.21 there exists a piecewise linear convex rational function $g : \sigma \rightarrow \mathbb{R}$, which coincides with $\lambda \circ f$ on every proper face of σ , such that $\Sigma(g)$ is a refinement of $\{\tau \in \tilde{\Sigma} \mid \tau \subset \sigma\}$.

Let us check what the invariance conditions say about the desired extension of f . Let $\Gamma_1 := \text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0) \cap P'_1(\mathbb{A}_f) \cdot p'_f \cdot K'_f \cdot p'^{-1}_f$. It is easy to see that any extension of f to σ can be extended invariantly to the whole conjugacy class of σ if and only if $f|_\sigma$ is invariant under the action of $\text{Stab}_{\Gamma_1}(\sigma)$. Let a_γ be the scalar through which an element $\gamma \in \Gamma_1$ acts on U_0 , by 9.2 this scalar is positive. Since $\lambda \circ f|_\sigma$ is to be a piecewise linear convex rational function, we have $\lambda \circ f^\gamma(u) = \lambda(a_\gamma^{-1} \cdot f(\gamma \cdot u)) = \lambda \circ f(a_\gamma^{-1} \cdot \gamma \cdot u)$ for all $u \in \sigma$, so we want $\lambda \circ f|_\sigma$ to be invariant under the twisted action $(\gamma, u) \mapsto a_\gamma^{-1} \cdot \gamma \cdot u$ of $\text{Stab}_{\Gamma_1}(\sigma)$. Since P'_1 acts through scalars on U'_1 , it acts trivially under this twisted action. But from 6.19 (a) it follows that $\text{Stab}_{\Gamma_1}(\sigma)$ is finite modulo $P'_1(\mathbb{Q}) \cap \Gamma_1$, so all we need is that $\lambda \circ f|_\sigma$ is invariant under a finite group of automorphisms of $U_1(\mathbb{Q})(-1)$ that fixes σ . Now the

function $g : \sigma \rightarrow \mathbb{R}$ above is not necessarily invariant. But by assumption it coincides with $\lambda \circ f$ on $\sigma \setminus \sigma^0$, so at least on this subset it is invariant. Thus if we replace g by the average of all its conjugates under that finite group, then it is invariant while still satisfying the other conditions. If we now define f to be equal to $\lambda^{-1} \circ g$ on σ , then we can extend it to the whole conjugacy class of σ , as desired. **q.e.d.**

9.20. Proposition: In 9.19 we can achieve in addition that Σ is smooth with respect to K_f , and that the condition 7.12 (*) is satisfied. Then in particular $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is smooth, projective, and the complement $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}) \setminus M^{K_f}(P, \mathcal{X})(\mathbb{C})$ is a union of smooth divisors with only normal crossings.

Proof. (For the smoothness compare [N] thm. 7.20, thm. 7.26.) If in the proof of 9.19 we construct f using the strenghtness 5.23 of 5.21, then the resulting cone decomposition is smooth. To prove that in addition the condition 7.12 (*) can be satisfied, we may therefore assume that $\tilde{\Sigma} = \Sigma$, and is already smooth. We want to apply the same procedure as in the proof of 9.19. Using 5.25 we can choose the new function f such that the associated refinement $\mathcal{T}$ of Σ is the barycentric subdivision of Σ with respect to the lattices given in 6.4. By 5.24 $\mathcal{T}$ is again smooth, and it remains to show that it satisfies the condition 7.12(*).

For this choose $\tau \in \mathcal{T}(\mathcal{X}^0, P_2, p_f)$ as in 7.12, let τ' be a face of τ that lies in $\mathcal{T}(\mathcal{X}^0, P_1, p_f)$, and suppose that $p \cdot p_{1,f} \cdot \tau' \cdot k_f$ is also a face of τ . We have to show that $p \cdot p_{1,f} \cdot \tau' \cdot k_f = \tau'$. Choose $\sigma \in \Sigma(\mathcal{X}^0, P_2, p_f)$ so that $\tau^0 \subset \sigma^0$, and $\sigma' \in \Sigma(\mathcal{X}^0, P_1, p_f)$ so that $\tau'^0 \subset \sigma'^0$. Then by assumption $(p \cdot p_{1,f} \cdot \sigma' \cdot k_f)^0 \supset (p \cdot p_{1,f} \cdot \tau' \cdot k_f)^0 \subset \tau \subset \sigma$, which implies $(p \cdot p_{1,f} \cdot \sigma' \cdot k_f)^0 \cap \sigma \neq \emptyset$, so $p \cdot p_{1,f} \cdot \sigma' \cdot k_f$ is a face of σ . Now we have an isomorphism $p \cdot p_{1,f} \cdot \sigma' \cdot k_f \cong \sigma'$, and by assumption both $p \cdot p_{1,f} \cdot \tau' \cdot k_f$ and τ' are faces of τ , which occurs in the barycentric subdivision of σ . As explained in 5.24 it follows that $p \cdot p_{1,f} \cdot \tau' \cdot k_f = \tau'$, as desired. **q.e.d.**

9.21. Theorem: Let $(P, \mathcal{X})$ be arbitrary mixed Shimura data, and $K_f \subset P(\mathbb{A}_f)$ a neat open compact subgroup. There exists a K_f -admissible complete cone decomposition Σ for $(P, \mathcal{X})$ such that $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is smooth, projective, and the complement $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}) \setminus M^{K_f}(P, \mathcal{X})(\mathbb{C})$ is a union of smooth divisors with only normal crossings. Moreover if $\mathcal{T}$ is any K_f -admissible complete cone decomposition for $(P, \mathcal{X})$, then Σ can be chosen to be a refinement of $\mathcal{T}$. Some ample line bundle on $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ can be described by algebraic constructions involving $\omega[\text{dlog}]$ and line bundles as in 8.13.

Proof. We shall reduce the assertion to 9.20. The problem is to relate $(P, \mathcal{X})$ with mixed Shimura data as in 9.2. First let $(\tilde{P}, \tilde{\mathcal{X}}) := (P, \mathcal{X}) \times (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$. Consider the open compact subgroup $\hat{\mathbb{Z}}^\times$ of $\mathbb{G}_m(\mathbb{A}_f)$, then

$$M^{(\hat{\mathbb{Z}})^\times}(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)(\mathbb{C}) = \mathbb{Q}^\times \setminus \mathcal{H}_0 \times (\mathbb{G}_m(\mathbb{A}_f)/(\hat{\mathbb{Z}})^\times) \cong \{\pm 1\} \setminus \mathcal{H}_0$$

is just a single point. Let $\tilde{K}_f := K_f \times (\hat{\mathbb{Z}})^\times$, then there is an obvious one-to-one relation between all K_f -admissible partial cone decompositions Σ for $(P, \mathcal{X})$ and all $\tilde{K}_f$ -admissible partial cone decompositions $\tilde{\Sigma}$ for $(\tilde{P}, \tilde{\mathcal{X}})$, such that the projection $(\tilde{P}, \tilde{\mathcal{X}}) \rightarrow (P, \mathcal{X})$ induces isomorphisms

$$\begin{aligned} M^{\tilde{K}_f}(\tilde{P}, \tilde{\mathcal{X}}, \tilde{\Sigma})(\mathbb{C}) &\cong M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}) \times M^{(\hat{\mathbb{Z}})^\times}(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)(\mathbb{C}) \\ &\xrightarrow{\sim} M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}). \end{aligned}$$

This reduces the assertion to that for $(\tilde{P}, \tilde{\mathcal{X}})$ and $\tilde{K}_f$.

Next let $(\tilde{P}_{(0)}, \tilde{\mathcal{X}}_{(0)}) \hookrightarrow (\tilde{P}, \tilde{\mathcal{X}})$ be an irreducible component. Since the other projection defines a morphism $(\tilde{P}_{(0)}, \tilde{\mathcal{X}}_{(0)}) \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$, we can find a $((P_0, \mathcal{X}_0) \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0))$ -torsor $(\tilde{P}'_{(0)}, \tilde{\mathcal{X}}'_{(0)}) \rightarrow (\tilde{P}_{(0)}, \tilde{\mathcal{X}}_{(0)})$ as in 9.1 (a). Let $P_* := \tilde{P}_{(0)} \cdot Z(\tilde{P})^0 \subset \tilde{P}$, and $(P_*, \mathcal{X}_*) \hookrightarrow (\tilde{P}, \tilde{\mathcal{X}})$ be the corresponding embedding as in 2.13. Since the extension class of the torsor $(\tilde{P}'_{(0)}, \tilde{\mathcal{X}}'_{(0)}) \rightarrow (\tilde{P}_{(0)}, \tilde{\mathcal{X}}_{(0)})$ is fixed by $Z(\tilde{P})$, we can extend it to a torsor $(P'_*, \mathcal{X}'_*) \rightarrow (P_*, \mathcal{X}_*)$. This torsor satisfies the requirements of 9.2. Now by 7.10 we have an isomorphism

$$\coprod_{\nu} \Delta_{\nu} \backslash M^{K_f^{\nu}}(P_*, \mathcal{X}_*, \Sigma_{\nu})(\mathbb{C}) \xrightarrow{\coprod [\cdot p_f^{\nu}] \circ [\phi]} M^{\tilde{K}_f}(\tilde{P}, \tilde{\mathcal{X}}, \Sigma)(\mathbb{C})$$

with finitely many $p_f^{\nu} \in \tilde{P}(\mathbb{A}_f)$, $K_f^{\nu} := P_*(\mathbb{A}_f) \cap p_f^{\nu} \cdot K_f \cdot (p_f^{\nu})^{-1}$, $\Sigma_{\nu} := ([\cdot p_f^{\nu}]^* \tilde{\Sigma})|_{(P_*, \mathcal{X}_*)}$, and

$$\Delta_{\nu} := (\text{Stab}_{\tilde{P}(\mathbb{Q})}(\mathcal{X}_*) \cap (P_*(\mathbb{A}_f) \cdot p_f^{\nu} \cdot \tilde{K}_f \cdot (p_f^{\nu})^{-1})) / P_*(\mathbb{Q}).$$

We claim that in our situation we have $\Delta_{\nu} = 1$. In fact, since $\tilde{P} = P \times \mathbb{G}_{m, \mathbb{Q}}$, we have $\{1\} \times \mathbb{G}_{m, \mathbb{Q}} \subset Z(\tilde{P})^0 \subset P_*$. Thus the image of $P_*(\mathbb{A}_f) \cdot p_f^{\nu} \cdot \tilde{K}_f \cdot (p_f^{\nu})^{-1}$ in $(P/P_*)(\mathbb{A}_f)$ is equal to that of $K_f \subset \tilde{K}_f = K_f \times (\hat{\mathbb{Z}})^{\times}$, which by 0.6 is neat. As in the proof of 7.10 it follows that Δ_{ν} is contained in a neat arithmetic subgroup of $(P/P_*)(\mathbb{Q})$, which by 2.1 (viii) must be trivial.

Now for any fixed collection of p_f^{ν} , as before there is an obvious one-to-one relation between $\tilde{K}_f$ -admissible cone decompositions $\tilde{\Sigma}$ for $(\tilde{P}, \tilde{\mathcal{X}})$, and collections of K_f^{ν} -admissible partial cone decomposition Σ_{ν} for $(P_*, \mathcal{X}_*)$. Thus the assertion is reduced to $(P_*, \mathcal{X}_*)$. But in that case it follows from 9.20. **q.e.d.**

9.22. Functoriality of cone decompositions: Consider a map $[\phi] : M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C}) \rightarrow M^{K_f^1}(P_2, \mathcal{X}_2)(\mathbb{C})$ of 3.4 (b). Let Σ_i be a K_f^i -admissible cone decomposition of every $(P_i, \mathcal{X}_i)$. Then we define another K_f^1 -admissible partial cone decomposition $\mathcal{T}_1$ for $(P_1, \mathcal{X}_1)$ as follows. Every rational boundary component $(P_{11}, \mathcal{X}_{11})$ of $(P_1, \mathcal{X}_1)$ determines a rational boundary component $(P_{21}, \mathcal{X}_{21})$ of $(P_2, \mathcal{X}_2)$, and every connected component $\mathcal{X}_1^0$ of $\mathcal{X}_1$ maps to a unique connected component $\mathcal{X}_2^0$ of $\mathcal{X}_2$. The homomorphism $\phi : U_{11} \rightarrow U_{21}$ then maps $C^*(\mathcal{X}_1^0, P_{11})$ to $C^*(\mathcal{X}_2^0, P_{21})$. For all such $(P_1, \mathcal{X}_1)$ and $\mathcal{X}_1^0$, and every $p_f \in P_1(\mathbb{A}_f)$, we define

$$\mathcal{T}_1(\mathcal{X}_1^0, P_{11}, p_{1,f}) := \{\sigma_1 \cap \phi^{-1}(\sigma_2) \mid \sigma_1 \in \Sigma_1(\mathcal{X}_1^0, P_{11}, p_{1,f}), \sigma_2 \in \Sigma_2(\mathcal{X}_2^0, P_{21}, \phi(p_f))\}.$$

It is clear from the definition that $\mathcal{T}_1$ is a K_f^1 -admissible partial cone decomposition for $(P_1, \mathcal{X}_1)$. If Σ_1 and Σ_2 are finite or complete, then $\mathcal{T}_1$ has the same property. Moreover $\mathcal{T}_1 = \Sigma_1$ if and only if the condition in 6.25 (b) holds for the pair (Σ_1, Σ_2) . In general by 6.25 (b) we have the following maps

$$M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1)(\mathbb{C}) \leftarrow M^{K_f^1}(P_1, \mathcal{X}_1, \mathcal{T}_1)(\mathbb{C}) \longrightarrow M^{K_f^2}(P_2, \mathcal{X}_2, \Sigma_2)(\mathbb{C}).$$

A particular case is the identity map on $(P_1, \mathcal{X}_1)$, where $\mathcal{T}_1$ is the coarsest common refinement of Σ_1 and Σ_2 . If $(P_2, \mathcal{X}_2) = (P_1, \mathcal{X}_1)$ and $p_f \in P_1(\mathbb{A}_f)$, then applying this to Σ_1 and

$[\cdot p_f]^* \Sigma_2$ yields the analogous statements for the morphism $[\cdot p_f]$ of 3.4 (a). The same arguments work for any finite number of morphisms of 3.4 (a) and (b).

9.23. Corollary: In 9.21 suppose that in addition a finite number of maps $M^{K_f}(P, \mathcal{X})(\mathbb{C}) \rightarrow M^{K_f^i}(P_i, \mathcal{X}_i)(\mathbb{C})$ of 3.4 (a) and (b) are given, and for every i a complete $K - f^i$ -admissible cone decomposition Σ_i for $(P_i, \mathcal{X}_i)$. Then one can choose Σ such that all the maps $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}) \rightarrow M^{K_f^i}(P_i, \mathcal{X}_i, \Sigma_i)(\mathbb{C})$ (of 6.25 (a) and (b)) are defined.

Proof. By 9.22 there exists a complete K_f -admissible cone decomposition $\mathcal{T}$ for $(P, \mathcal{X})$ such that the conditions in 6.25 (a) and (b) are verified for all the morphisms $M^{K_f}(P, \mathcal{X}, \mathcal{T})(\mathbb{C}) \rightarrow M^{K_f^i}(P_i, \mathcal{X}_i, \Sigma_i)(\mathbb{C})$. If in 9.21 Σ is a refinement of $\mathcal{T}$, then the same holds for Σ in place of $\mathcal{T}$, as desired. **q.e.d.**

9.24. Proposition: For every mixed Shimura data $(P, \mathcal{X})$, and every open compact subgroup $K_f \subset P(\mathbb{A}_f)$, the mixed Shimura variety $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ possesses a canonical structure of a normal quasiprojective algebraic variety over $\mathbb{C}$, such that all the maps of 3.4 become algebraic morphisms. Whenever Σ is a complete K_f -admissible cone decomposition for $(P, \mathcal{X})$ such that $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is projective, then the open embedding $M^{K_f}(P, \mathcal{X})(\mathbb{C}) \hookrightarrow M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is algebraic.

Proof. Let Σ be a complete K_f -admissible cone decomposition for $(P, \mathcal{X})$ such that $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is projective. Since by 6.25 (a) the complement of $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ in $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is a closed analytic set, by Chow's theorem (see [S] §19 prop.13) it is a Zariski-closed subset. Thus its complement $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ inherits the structure of a quasiprojective variety over $\mathbb{C}$.

To prove the functoriality of this algebraic structure consider other mixed Shimura data $(P^*, \mathcal{X}^*)$, an open compact subgroup $K_f^* \subset P^*(\mathbb{A}_f)$, and a complete K_f^* -admissible cone decomposition for $(P^*, \mathcal{X}^*)$, such that $M^{K_f^*}(P^*, \mathcal{X}^*, \Sigma^*)(\mathbb{C})$ is projective. Consider a map $M^{K_f}(P, \mathcal{X})(\mathbb{C}) \rightarrow M^{K_f^*}(P^*, \mathcal{X}^*)(\mathbb{C})$ of 3.4 (a) or (b). Choose a neat open normal subgroup $K_f^+ \subset K_f$, then by 9.23 there exists a refinement $\mathcal{T}$ of Σ , such that the map $M^{K_f^+}(P, \mathcal{X}, \mathcal{T})(\mathbb{C}) \rightarrow M^{K_f^*}(P^*, \mathcal{X}^*, \Sigma^*)(\mathbb{C})$ of 6.25 (a) or (b) is defined. Then the maps

$$M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}) \xleftarrow{[\text{id}]} M^{K_f^+}(P, \mathcal{X}, \mathcal{T})(\mathbb{C}) \rightarrow M^{K_f^*}(P^*, \mathcal{X}^*, \Sigma^*)(\mathbb{C})$$

are holomorphic maps between projective complex spaces, hence algebraic. In particular the maps

$$M^{K_f}(P, \mathcal{X})(\mathbb{C}) \xleftarrow{[\text{id}]} M^{K_f^+}(P, \mathcal{X})(\mathbb{C}) \rightarrow M^{K_f^*}(P^*, \mathcal{X}^*)(\mathbb{C})$$

are algebraic with respect to the induced algebraic structures. Recall that by 6.25 (a) $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ is as complex space the quotient of $M^{K_f^+}(P, \mathcal{X})(\mathbb{C})$ by the finite group of K_f/K_f^+ . Since the map on the right hand side factors through $M^{K_f}(P, \mathcal{X})(\mathbb{C})$, the map $M^{K_f}(P, \mathcal{X})(\mathbb{C}) \rightarrow M^{K_f^*}(P^*, \mathcal{X}^*)(\mathbb{C})$ is also algebraic, as desired. **q.e.d.**

A special case of the functoriality is that the algebraic structure on $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ is independant of the choice of Σ . Thus the assertions are already proved whenever there exists a complete K_f -admissible cone decomposition for $(P, \mathcal{X})$ such that $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is

projective. By 9.21 this is the case if K_f is neat. For arbitrary K_f choose a neat open normal subgroup $K_f^+ \subset K_f$. By the functoriality for $M^{K_f^+}(P, \mathcal{X})(\mathbb{C})$ the action of K_f/K_f^+ respects the canonical algebraic structure on $M^{K_f^+}(P, \mathcal{X})(\mathbb{C})$. Since this is a quasiprojective variety, and by [SGA1] exp.VIII cor. 7.7 the quotient of every quasiprojective algebraic variety by a finite group exists and is again quasiprojective, we get a canonical quasiprojective algebraic structure on $M^{K_f}(P, \mathcal{X})(\mathbb{C})$. The functoriality for this algebraic structure follows by the same arguments as above, in particular it does not depend on the choice of K_f^+ . **q.e.d.**

9.25. Definition: For every mixed Shimura data $(P, \mathcal{X})$ and every open compact subgroup $K_f \subset P(\mathbb{A}_f)$ let

$$M_{\mathbb{C}}^{K_f}(P, \mathcal{X})$$

be the normal algebraic variety over $\mathbb{C}$ associated by 9.24 to the normal complex space $M^{K_f}(P, \mathcal{X})(\mathbb{C})$. Let Σ be a K_f -admissible partial cone decomposition for $(P, \mathcal{X})$. If there exists an algebraic structure on $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ extending the above algebraic structure on $M^{K_f}(P, \mathcal{X})(\mathbb{C})$, we let

$$M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma)$$

be the associated normal algebraic variety over $\mathbb{C}$. Furthermore if P is reductive, then we denote the normal projective variety corresponding to the Baily-Borel compactification $M^{K_f}(P, \mathcal{X})^*(\mathbb{C})$ by

$$M_{\mathbb{C}}^{K_f}(P, \mathcal{X})^*.$$

In all these cases the variety is, strictly speaking, defined by a universal property, so it really consists of a complex variety X together with an isomorphism $X(\mathbb{C}) \cong M^{K_f}(P, \mathcal{X})(\mathbb{C})$, resp. $X(\mathbb{C}) \cong M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$, resp. $X(\mathbb{C}) \cong M^{K_f}(P, \mathcal{X})^*(\mathbb{C})$. We have to prove that it is unique up to canonical isomorphism. For $M_{\mathbb{C}}^{K_f}(P, \mathcal{X})$ this follows from 9.24, for $M_{\mathbb{C}}^{K_f}(P, \mathcal{X})^*$ by projectivity. For $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma)$ recall that $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ is dense in $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$, hence Zariski-dense with respect to any algebraic structure. It is easily seen that an analytic map between two normal complex varieties is an algebraic morphism if it is an algebraic morphism on a Zariski-open dense subset. Thus $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma)$ is well-defined whenever such an algebraic structure exists.

By 9.24 we already know that all the maps 3.4 (a) and (b) correspond to unique algebraic morphisms between different $M_{\mathbb{C}}^{K_f}(P, \mathcal{X})^*$. By the argument above, the same holds for the maps 6.25 (a) and (b), if $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma)$ etc. exists. For the Baily-Borel compactification the same follows by projectivity. Moreover for arbitrary P , by the argument above the map $[\pi]^*$ defined in 6.24 corresponds to a morphism $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma) \rightarrow M_{\mathbb{C}}^{\pi(K_f)}((P, \mathcal{X})/W)^*$ whenever $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma)$ exists. In all these cases we shall use the same symbols to denote the corresponding algebraic morphisms.

The assertion of 6.25 (functoriality, open embeddings, isomorphisms, quotients by finite groups), 6.26 (smoothness), 8.6 (line bundle structure), 8.13–14 (ampleness) are equally valid for these algebraic varieties, since these are properties that translate directly between the algebraic and the analytic category, once all the objects and maps are algebraic.

9.26. Remark: From the proof of 9.24 it follows that the system of algebraic structures on all $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ is uniquely determined by the properties expressed in 9.24. One may ask whether these algebraic structures can be characterized uniquely without reference to toroidal compactifications. As it turns out, they are already characterized uniquely by the functoriality with respect to all the maps 3.4.

Let us briefly indicate why. If P is reductive and possesses no almost direct factor to type $SL_{2, \mathbb{Q}}$, then the boundary of $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ in the Baily-Borel compactification has codimension ≥ 2 , so it possesses at most one algebraic structure. For $P = GL_{2, \mathbb{Q}}$ the uniqueness follows via an embedding $(GL_{2, \mathbb{Q}}, \mathcal{H}_2) \hookrightarrow (CS_{P_4, \mathbb{Q}}, \mathcal{H}_4)$, as in the proof of 8.4. Thus we have uniqueness whenever P is reductive. Next, if $U = 1$, then $M^{K_f}(P, \mathcal{X})(\mathbb{C}) \rightarrow M^{\pi(K_f)}((P, \mathcal{X})/W)(\mathbb{C})$ is proper, so again the uniqueness follows by functoriality. Finally observe that every torus possesses a unique algebraic structure. Thus for arbitrary P , and sufficiently small K_f , we may use the torsor structure on $M^{K_f}(P, \mathcal{X})(\mathbb{C}) \rightarrow M^{\pi'(K_f)}((P, \mathcal{X})/U)(\mathbb{C})$ to get the result.

9.27. Remarks about the existence of $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma)$: (a) By the definition in 9.25, $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma)$ exists whenever $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is projective, in particular in the situations of 8.13, 9.13, or 9.21. If $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma)$ exists, then the same follows for every K_f -admissible partial cone decomposition contained in Σ . If Σ is the union of any (finite or infinite) collection Σ_i , such that $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma_i)$ exists, then by gluing it follows that $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma)$ exists. These three facts, together with 9.16–21, shows that $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma)$ exists for a large class of cone decompositions.

(b) In the situation of 8.13 (e.g. 9.13) not only $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma)$, but also the “line bundle” $M_{\mathbb{C}}^{K'_f}(P', \mathcal{X}', \Sigma')$ exists as an algebraic variety and is quasiprojective. In fact, by ampleness one can construct a canonical ample line bundle on the associated $\mathbb{P}^1(\mathbb{C})$ -bundle obtained by adding the ∞ -section. This, too, can be expressed in the formalism of chapter 8.

(c) The knowledge so far about the existence of $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma)$ is probably sufficient for most purposes. Nevertheless it seems somewhat unsatisfactory to stop here and not try to find out more about the algebraicity of our toroidal compactifications. Therefore we include more results in this direction, although the proof of 9.29 is rather technical. See also 9.35 below.

First we show (using 9.20) that $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma)$ exists whenever Σ is sufficiently “fine”.

9.28. Proposition: Let $(P', \mathcal{X}') \rightarrow (P, \mathcal{X})$, K'_f , K_f , Σ' , and Σ be as in 9.13. Assume that the condition 7.12 (*) is satisfied for Σ . Let $\mathcal{T}$ be a K_f -admissible partial cone decomposition for $(P, \mathcal{X})$ such that every $\tau \in \mathcal{T}$ is contained in some $\sigma \in \Sigma$. Then $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \mathcal{T})$ exists. In fact there exists a collection of $\overline{\mathcal{T}}'_i$ and $\overline{\mathcal{T}}_i$ as in 9.13, and $\mathcal{T}_i \subset \overline{\mathcal{T}}_i$, such that $\mathcal{T}$ is the union of all $\mathcal{T}_i$.

Proof. It suffices to prove that for every convex rational polyhedral cone τ contained in some $\sigma \in \Sigma$ there exist $\overline{\mathcal{T}}'$ and $\overline{\mathcal{T}}$ as in 9.13 such that $\tau \in \overline{\mathcal{T}}$. Using the same procedure as in the proof of 9.19 it suffices to construct the function f such that $\tau \in \Sigma(\lambda \circ f|_{\sigma})$. To do this choose a piecewise linear convex rational function $g : \sigma \rightarrow \mathbb{R}$ such that $\tau \in \Sigma(g)$. To show that such a function exists, take any linear rational function $\ell : \sigma \rightarrow \mathbb{R}$ that is strictly

positive on $\sigma \setminus \{0\}$. Then let g be the smallest nonnegative convex function $\sigma \rightarrow \mathbb{R}^{\geq 0}$ which is $\geq \ell$ on τ . It is clear that $g = \ell$ on τ , and $g > \ell$ on $\sigma \setminus \tau$, hence $\tau \in \Sigma(g)$, as desired.

Now define $f|_{\sigma} := \lambda^{-1} \circ g : \sigma \rightarrow U_0(\mathbb{R})(-1)$. The condition 7.12 (*) implies that no two different faces of σ are equivalent under the actions 6.4 (ii)–(iv). This in turn implies that whenever these actions induce an automorphism of a face ρ of σ , then this automorphism maps every face of ρ to itself. But by 6.19 (a) and 9.6 some power of this automorphism acts like a scalar on ρ . This shows that the automorphisms acts by the same scalar on every one dimensional face of ρ , hence it acts like a scalar on ρ . Thus there is no obstruction to extending $f|_{\sigma}$ invariantly to all conjugates of σ under the actions 6.4 (ii)–(iv). Finally by the proof of 9.19 (with $\tilde{\Sigma} = \Sigma$) it can be extended to all of $\mathcal{C}(P, \mathcal{X}) \times P(\mathbb{A}_f) \rightarrow U_0(\mathbb{R})(-1)$ in the desired way. **q.e.d.**

9.29. Proposition: Let $(P', \mathcal{X}') \rightarrow (P, \mathcal{X})$ be as in 9.2, $(P_1, \mathcal{X}_1)$ a rational boundary component of $(P, \mathcal{X})$, and $\mathcal{X}^0$ a connected component of $\mathcal{X}$ that maps into $\mathcal{X}_1$. Let $p_f \in P(\mathbb{A}_f)$, and σ a convex rational polyhedral cone in $C(\mathcal{X}^0, P_1)$ that does not contain a nontrivial linear subspace. Then there exists an open compact subgroup $K'_f \subset P'(\mathbb{A}_f)$ and Σ and Σ' as in 9.13, such that $\sigma \in \Sigma(\mathcal{X}^0, P_1, p_f)$.

Remark. Incidentally, this shows that the cones in a complete admissible decomposition can be arbitrarily large.

Proof. Using the operation 6.5 (a) we may assume that $p_f = 1$. Without loss of generality we may assume that $\mathbb{R} \cdot \sigma = U_1(\mathbb{R})(-1)$, since otherwise we may (repeatedly) replace σ by $\sigma + \mathbb{R}^{\geq 0} \cdot u'$ for arbitrary $u' \in C^*(\mathcal{X}^0, P_1) \cap U_1(\mathbb{Q})(-1) \setminus \mathbb{R} \cdot \sigma$. Let $(P'_1, \mathcal{X}'_1)$ be the rational boundary component of $(P', \mathcal{X}')$ with $(P_1, \mathcal{X}_1) = (P'_1, \mathcal{X}'_1)/U_0$, and $\mathcal{X}'^0$ the connected component of $\mathcal{X}'$ corresponding to $\mathcal{X}^0$. We want to apply the same inductive procedure as in the proof of 9.16. Suppose that for some K'_f we have constructed a K'_f -admissible partial cone decomposition Σ' for $(P', \mathcal{X}')$, such that for some $\Sigma \subset C^*(\mathcal{X}'^0, P'_1)$ as in 9.8 $\Sigma'(\mathcal{X}'^0, P'_1, 1) = \Sigma'_{\Sigma}$, and $\sigma \in \Sigma_{\Sigma}$. Then as in the proof of 9.16 it can be extended to the desired Σ' . We therefore have to construct $\Sigma \subset C^*(\mathcal{X}'^0, P'_1)$ as in 9.8 such that $\sigma \in \Sigma_{\Sigma}$ and $\Sigma'(\mathcal{X}'^0, P'_1, 1) = \Sigma'_{\Sigma}$ extends to a K'_f -admissible partial cone decomposition. While the first property is not difficult to achieve, the second one makes the proof complicated.

Fix an arbitrary $e_1 \in \mathcal{E}(\mathcal{X}'^0, P'_1)$. By 9.3 (a) we may fix $e_2, \dots, e_m \in \mathcal{E}(\mathcal{X}'^0, P'_1)$ such that the $\lambda \circ (e_1 - e_i) \in \text{Hom}(U_1(\mathbb{R})(-1), \mathbb{R})$ span the dual cone $\check{\sigma}$, in other words

$$\sigma = \{u \in C^*(\mathcal{X}^0, P_1) \mid \forall 2 \leq i \leq m \quad \lambda \circ (e_1 - e_i)(u) \geq 0\}.$$

Let $u_1, \dots, u_n \in C^*(\mathcal{X}^0, P_1) \cap U_1(\mathbb{Q})(-1)$ such that $\sigma = \sum_{j=1}^n \mathbb{R}^{\geq 0} \cdot u_j$. Suppose that K'_f has been chosen, and let

$$\Gamma_1 := \text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0) \cap K'_f \cdot P'_1(\mathbb{A}_f).$$

Define Σ_0 as in 9.8 using this group and the sections $e_1, \dots, e_m$. Since σ does not contain a nontrivial linear subspace, its dual cone generates the dual of $U_1(\mathbb{R})(-1)$, so by the choice of the e_i the assumption in 9.8 holds. Suppose that all $e_i(u_j)$ lie in Σ_0 . Then $e_1(\sigma) \subset \Sigma_0$, and the choice of $e_2, \dots, e_m$ implies that $\lambda \circ (\text{id} - e_1 \circ \pi)$ is strictly positive on $\Sigma_0 \setminus e_1(\sigma)$. Thus $\sigma \in \Sigma_{\Sigma_0}$, which is one of the desired properties. But Σ_0 is not yet the right subset,

since Σ'_{Σ_0} does not necessarily extend to a K'_f -admissible partial cone decomposition. To remedy this we shall construct the “right” Σ as a Γ_1 -invariant subset of Σ_0 that contains all the $e_1(u_j)$. By the same argument it then follows that $\sigma \in \Sigma_\Sigma$, and it remains to choose Σ such that Σ'_Σ extends.

Let us check what this extendability condition amounts to. By lemma 9.32 below Σ'_Σ extends to a K'_f -admissible partial cone decomposition if and only if for every rational boundary component $(P'_2, \mathcal{X}'_2)$ between $(P'_1, \mathcal{X}'_1)$ and $(P', \mathcal{X}')$, and every $q' \in \text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0) \cap K'_f \cdot P'_2(\mathbb{A}_f)$, we have

$$\{q' \cdot \tau \mid \tau \in \Sigma'_\Sigma, \tau \subset C^*(\mathcal{X}'^0, P'_2)\} = \{\tau \in \Sigma'_\Sigma \mid \tau \subset q' \cdot C^*(\mathcal{X}'^0, P'_2)\}.$$

By the definition of Σ'_Σ this can be translated into the following equality in terms of Σ :

$$q' \cdot (\Sigma \cap C^*(\mathcal{X}'^0, P'_2)) = \Sigma \cap q' \cdot C^*(\mathcal{X}'^0, P'_2).$$

Together with the condition $e_1(u_j) \in \Sigma$ this has the following consequence. Every u_j lies in $C(\mathcal{X}^0, P_j)$ for a unique rational boundary component $(P'_j, \mathcal{X}'_j)$ between $(P'_1, \mathcal{X}'_1)$ and $(P', \mathcal{X}')$. Thus $e_1(u_j) \in \Sigma$ implies $q' \cdot e_1(u_j) \in \Sigma$ for all j and $q' \in \text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0) \cap K'_f \cdot P'_j(\mathbb{A}_f)$.

We shall construct Σ by induction over the $(P'_2, \mathcal{X}'_2)$ for which the above equality holds. Consider an increasing sequence $\emptyset = \mathfrak{A}_0 \subsetneq \mathfrak{A}_1 \subsetneq \dots$ of Γ_1 -invariant collections of rational boundary components between $(P'_1, \mathcal{X}'_1)$ and $(P', \mathcal{X}')$, such that whenever $(P'_2, \mathcal{X}'_2) \in \mathfrak{A}_\nu$, then so is every rational boundary component between $(P'_2, \mathcal{X}'_2)$ and $(P', \mathcal{X}')$. We shall construct these $\mathfrak{A}_\nu$ and a sequence of Γ_1 -invariant subsets $\Sigma_0 \supset \Sigma_1 \supset \dots$ as in 9.8, such that the above equality holds for all Σ_ν and $(P'_2, \mathcal{X}'_2) \in \mathfrak{A}_\nu$. Furthermore we require that $q' \cdot e_1(u_j) \in \Sigma_\nu$ for all ν, j , and $q' \in \text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0) \cap K'_f \cdot P'_j(\mathbb{A}_f)$. Since modulo Γ_1 there are only finitely many rational boundary components between $(P'_1, \mathcal{X}'_1)$ and $(P', \mathcal{X}')$, this sequence stops at some point, where $\Sigma := \Sigma_\nu$ has all the desired properties.

Of course this works only if K'_f is chosen in advance. So we first show that it can be chosen so that the conditions hold for Σ_0 . Since $\mathfrak{A}_0 = \emptyset$, only the condition $q' \cdot e_1(u_j) \in \Sigma_0$ matters. By the definition of Σ_0 this is equivalent to

$$\lambda \circ (\text{id} - e_i^\gamma \circ \pi_0)(q' \cdot e_1(u_j)) \geq 0$$

for all $i, j, \gamma \in \Gamma_1$, and $q' \in \text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0) \cap K'_f \cdot P'_j(\mathbb{A}_f)$. By the remark in 9.2 this is equivalent to $\lambda \circ (\text{id} - e_i \circ \pi_0)(\gamma \cdot q' \cdot e_1(u_j)) \geq 0$, so by lemma 9.30 below we are reduced to $\gamma = 1$. Since there are only finitely many i and j , the desired K'_f exists by lemma 9.31 below. We fix such K'_f once and for all.

Next assume that Σ_ν has been constructed. Fix a rational boundary component $(P'_2, \mathcal{X}'_2)$ between $(P'_1, \mathcal{X}'_1)$ and $(P', \mathcal{X}')$, such that $(P'_2, \mathcal{X}'_2) \notin \mathfrak{A}_\nu$, but $\mathfrak{A}_\nu$ contains every boundary component between $(P'_2, \mathcal{X}'_2)$ and $(P', \mathcal{X}')$ that is different from $(P'_2, \mathcal{X}'_2)$. Let $\mathfrak{A}_{\nu+1}$ be the union of $\mathfrak{A}_\nu$ with the set of all Γ_1 -conjugates of $(P'_2, \mathcal{X}'_2)$. Let Q'_2 be the parabolic subgroup of P' associated to $(P'_2, \mathcal{X}'_2)$, and define $\Gamma_2 := \text{Stab}_{(Q' \cap Q'_2)(\mathbb{Q})}(\mathcal{X}'^0) \cap K'_f \cdot P'_2(\mathbb{A}_f)$, this group contains $\text{Stab}_{\Gamma_1}(C^*(\mathcal{X}'^0, P'_2))$ as a subgroup of finite index. Thus the set

$$\Sigma' := \bigcap_{\gamma \in \Gamma_2} \gamma \cdot (\Sigma_\nu \cap C^*(\mathcal{X}'^0, P'_2))$$

is of the form of 9.8 for $(P'_2, \mathcal{X}'_2)$ in place of $(P'_1, \mathcal{X}'_1)$. We shall construct a Γ_1 -invariant subset $\Sigma_{\nu+1} \subset \Sigma_\nu$ as in 9.8, such that

- (a) $q' \cdot e_1(u_j) \in \Sigma_{\nu+1}$ for all j and all $q' \in \text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0) \cap K'_f \cdot P'_j(\mathbb{A}_f)$,
- (b) $\Sigma_{\nu+1} \cap C^*(\mathcal{X}'^0, P'_3) = \Sigma_\nu \cap C^*(\mathcal{X}'^0, P'_3)$ for every $(P'_3 \mathcal{X}'_3) \in \mathfrak{A}_\nu$, and
- (c) $\Sigma_{\nu+1} \cap q' \cdot C^*(\mathcal{X}'^0, P'_2) = q' \cdot \Sigma'$ for every $q' \in \text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0) \cap K'_f \cdot P'_2(\mathbb{A}_f)$.

Then by (c) and by $\Gamma - 1$ -invariance the extendability condition holds for all Γ_1 -conjugates of $(P'_2, \mathcal{X}'_2)$, and by (b) and the assumption about Σ_ν it holds for all $(P'_3 \mathcal{X}'_3) \in \mathfrak{A}_\nu$, so it holds for all $(P'_3 \mathcal{X}'_3) \in \mathfrak{A}_{\nu+1}$, as desired.

Using the fact that $\Sigma_{\nu+1}$ is Γ_1 -invariant we can reduce this infinity of conditions to a finite number. First by lemma 9.30 below it suffices to require (c) only for a finite subset $\mathfrak{R} \subset \text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0) \cap K'_f \cdot P'_2(\mathbb{A}_f)$. For the same reason it suffices to require (a) only for a finite subset $\mathfrak{R}_j \subset \text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0) \cap K'_f \cdot P'_j(\mathbb{A}_f)$. For (b) we are reduced to a finite number of $(P'_3 \mathcal{X}'_3) \in \mathfrak{A}_\nu$. By 9.14 every $\Sigma_\nu \cap C^*(\mathcal{X}'^0, P'_3)$ is of the form in 9.8 with respect to the group $\text{Stab}_{\Gamma_1}(C^*(\mathcal{X}'^0, P'_3))$, so by 9.10 this set is the convex closure of the union of all $\text{Stab}_{\Gamma_1}(C^*(\mathcal{X}'^0, P'_3))$ -orbits of finitely many rational half-lines. Therefore by $\text{Stab}_{\Gamma_1}(C^*(\mathcal{X}'^0, P'_3))$ -invariance the condition (b) is equivalent to $\mathcal{U} \subset \Sigma_{\nu+1}$ for some finite subset

$$\mathcal{U} \subset \sum_{\nu} \cap U'_1(\mathbb{Q})(-1) \cap \bigcup_{(P'_3 \mathcal{X}'_3) \in \mathfrak{A}_\nu} C^*(\mathcal{X}'^0, P'_3).$$

Now for every $q' \in \mathfrak{R}$ let $\Gamma_{2,q'} := \text{Stab}_{\Gamma_1}(C^*(\mathcal{X}'^0, q' \cdot P'_2 \cdot q'^{-1}))$, and choose finitely many $\bar{e}_{q',i} \in \mathcal{E}(\mathcal{X}'^0, q' \cdot P'_2 \cdot q'^{-1})$ so that $q' \cdot \Sigma'$ is of the form in 9.8 with this group and these splittings. Lift every $\bar{e}_{q',i}$ to an element $\tilde{e}_{q',i} \in \mathcal{E}(\mathcal{X}'^0, P'_1)$, and define $\Sigma_{\nu+1}$ as in 9.8 with the group Γ_1 , and as splittings these $\tilde{e}_{q',i}$ and those that define Σ_ν . It remains to show that the conditions (a)–(c) hold for a suitable choice of the liftings $\tilde{e}_{q',i}$. Let us rewrite these conditions in terms of the $\tilde{e}_{q',i}$. For (c) note that for every $q' \in \mathfrak{R}$ the inclusion $\Sigma_{\nu+1} \cap q' \cdot C^*(\mathcal{X}'^0, P'_2) \subset q' \cdot \Sigma'$ holds by construction. For the other inclusion we can as above write $q' \cdot \Sigma'$ as the convex closure of the union of all $\text{Stab}_{\Gamma_1}(q' \cdot C^*(\mathcal{X}'^0, P'_2))$ -orbits of finitely many rational half-lines, so it suffices to show that $\mathcal{V} \subset \Sigma_{\nu+1}$ for some finite subset

$$\mathcal{V} \subset \bigcup_{q' \in \mathfrak{R}} q' \cdot \Sigma' \cap U'_1(\mathbb{Q})(-1).$$

Altogether it follows that the conditions (a)–(c) hold if and only if $\Sigma_{\nu+1}$ contains the finite subset

$$\mathcal{W} := \{q' \cdot e_1(u_j) \mid 1 \leq j \leq n, q' \in \mathfrak{R}_j\} \cup \mathcal{U} \cup \mathcal{V} \subset \Sigma_\nu \cap U'_1(\mathbb{Q})(-1).$$

By the definition of $\Sigma_{\nu+1}$ this means that for all $q' \in \mathfrak{R}$, i , $u' \in \mathcal{W}$ and $\gamma \in \Gamma_1$ we need

$$\lambda \circ (\text{id} - \tilde{e}_{q',i} \circ \pi_0)(\gamma \cdot u') \geq 0.$$

By the argument in the proof of 9.15 one can choose the liftings $\tilde{e}_{q',i}$ such that this inequality fails at most if $\gamma \cdot u' \in q' \cdot C^*(\mathcal{X}'^0, P'_2)$. Thus it remains to prove $\Gamma_1 \cdot \mathcal{W} \cap q' \cdot C^*(\mathcal{X}'^0, P'_2) \subset q' \cdot \Sigma'$ for every $q' \in \mathfrak{R}$.

Translated back into the original formulation of these conditions we have to prove for all $q' \in \text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0) \cap K'_f \cdot P'_2(\mathbb{A}_f)$:

(a') For all j and $q'' \in \text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0) \cap K'_f \cdot P'_j(\mathbb{A}_f)$:

$$q'' \cdot e_1(u_j) \in q' \cdot C^*(\mathcal{X}'^0, P'_2) \Rightarrow q'' \cdot e_1(u_j) \in q' \cdot \Sigma'.$$

(b') $\Sigma_\nu \cap C^*(\mathcal{X}'^0, P'_3) \cap q' \cdot C^*(\mathcal{X}'^0, P'_2) \subset q' \cdot \Sigma'$ for every $(P'_3 \mathcal{X}'_3) \in \mathfrak{A}_\nu$.

(c') $q'' \cdot \Sigma' \cap q' \cdot C^*(\mathcal{X}'^0, P'_2) \subset q' \cdot \Sigma'$ for all $q'' \in \text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0) \cap K'_f \cdot P'_2(\mathbb{A}_f)$.

Since $q' \cdot \gamma_2 \in \text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0) \cap K'_f \cdot P'_2(\mathbb{A}_f)$ for every $\gamma_2 \in \Gamma_2$, by the definition of Σ' we may replace the inclusions $\subset q' \cdot \Sigma'$ by the inclusions $\subset q' \cdot \Sigma_\nu$. Now in (a') the assumption implies that $q'' \cdot C(\mathcal{X}'^0, P'_j) \cap q' \cdot C^*(\mathcal{X}'^0, P'_2) \neq \emptyset$, so $\text{int}(q')((P'_2, \mathcal{X}'_2))$ is a rational boundary component of $\text{int}(q'')((P'_j, \mathcal{X}'_j))$, and in particular $\text{int}(q''^{-1} \cdot q')((P'_2, \mathcal{X}'_2)) \subset (P'_j, \mathcal{X}'_j)$. Let $p'_{2,f} \in P'_2(\mathbb{A}_f)$ such that $q' \in K'_f \cdot p'_{2,f}$, then it follows that

$$\begin{aligned} q'^{-1} &= p'_{2,f} \cdot q'^{-1} \cdot q'' \cdot \text{int}(q''^{-1} \cdot q')(p'^{-1}_{2,f}) \\ &\in K'_f \cdot K'_f \cdot P'_j(\mathbb{A}_f) \cdot \text{int}(q''^{-1} \cdot q')((P'_2, \mathcal{X}'_2)) = K'_f \cdot P'_j(\mathbb{A}_f), \end{aligned}$$

whence $q'^{-1} \cdot q'' \in \text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0) \cap K'_f \cdot P'_j(\mathbb{A}_f)$. Thus without loss of generality we may assume $q' = 1$, so we are reduced to proving $q' \cdot e_1(u_j) \in \Sigma_\nu$ for all j , and $q' \in \text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0) \cap K'_f \cdot P'_j(\mathbb{A}_f)$. But this is one of the assumptions about Σ_ν .

In (b') it suffices to consider only those $(P'_3 \mathcal{X}'_3) \in \mathfrak{A}_\nu$ for which $\text{int}(q')((P'_2, \mathcal{X}'_2))$ is a rational boundary component of $(P'_3 \mathcal{X}'_3)$. As above it follows that $q'^{-1} \in \text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0) \cap K'_f \cdot P'_3(\mathbb{A}_f)$, and the inclusion

$$\begin{aligned} \Sigma_\nu \cap C^*(\mathcal{X}'^0, P'_3) &\subset q' \cdot \Sigma_\nu \\ \Leftrightarrow q'^{-1} \cdot (\Sigma_\nu \cap C^*(\mathcal{X}'^0, P'_3)) &\subset \Sigma_\nu \end{aligned}$$

follows from the invariance condition for $(P'_3 \mathcal{X}'_3)$ and Σ_ν . Finally for (c') the left hand side is always contained in $q'' \cdot C^*(\mathcal{X}'^0, P'_3)$ for a rational boundary component $(P'_3 \mathcal{X}'_3)$ between $(P'_2 \mathcal{X}'_2)$ and $(P' \mathcal{X}')$, such that $\text{int}(q')((P'_2, \mathcal{X}'_2))$ is a rational boundary component of $\text{int}(q'')((P'_3 \mathcal{X}'_3))$. Thus as above we are reduced to prove the inclusion

$$q'' \cdot (\Sigma' \cap C^*(\mathcal{X}'^0, P'_3)) \subset \Sigma_\nu$$

for all $q'' \in \text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0) \cap K'_f \cdot P'_3(\mathbb{A}_f)$ such that $q'' \cdot C^*(\mathcal{X}'^0, P'_3) \subset C^*(\mathcal{X}'^0, P'_2)$. In the case $(P'_3 \mathcal{X}'_3) = (P'_2 \mathcal{X}'_2)$ we must have $q'' \in \Gamma_2$, so this follows from the definition of Σ' . Otherwise we have $(P'_3 \mathcal{X}'_3) \in \mathfrak{A}_\nu$, so this follows from the invariance condition for $(P'_3 \mathcal{X}'_3)$ and Σ_ν . Thus the conditions (a')–(c') are satisfied. **q.e.d.**

9.30. Lemma: Let $(P, \mathcal{X})$ be mixed Shimura data, $(P_2, \mathcal{X}_2)$ a rational boundary component of $(P, \mathcal{X})$, and $(P_1, \mathcal{X}_1)$ a rational boundary component of $(P_2, \mathcal{X}_2)$. Let $\mathcal{X}^0$ be a connected component of $\mathcal{X}$, Q the parabolic subgroup of P associated to $(P_1, \mathcal{X}_1)$, and K_f an open compact subgroup of $P(\mathbb{A}_f)$. Let

$$\Gamma_1 := \text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^0) \cap K_f \cdot P_1(\mathbb{A}_f).$$

There exists a finite subset $\mathfrak{R} \subset \text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^0) \cap K_f \cdot P_2(\mathbb{A}_f)$ such that

$$\text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^0) \cap K_f \cdot P_2(\mathbb{A}_f) = \Gamma_1 \cdot \mathfrak{R} \cdot \text{Stab}_{(P_2 \cap Q)(\mathbb{Q})}(\mathcal{X}^0).$$

Proof. For abbreviation write $A := \text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^0) \cap K_f \cdot P_2(\mathbb{A}_f)$. Then we have $\Gamma_1 \cdot A = A$. Indeed, let $q \in A$ and $q' \in \Gamma_1$, then obviously $q' \cdot q \in \text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^0)$, and

$$q' \cdot q \in K_f \cdot P_1(\mathbb{A}_f) \cdot q \subset K_f \cdot (K_f \cdot P_2(\mathbb{A}_f)) \cdot (q^{-1} \cdot P_1(\mathbb{A}_f) \cdot q) = K_f \cdot P_2(\mathbb{A}_f),$$

since q normalizes $P_1(\mathbb{A}_f)$. Clearly $\text{Stab}_{(P_2 \cap Q)(\mathbb{Q})}(\mathcal{X}^0)$ acts on the right hand side on A , and we have to show that the double quotient $\Gamma_1 \backslash A / \text{Stab}_{(P_2 \cap Q)(\mathbb{Q})}(\mathcal{X}^0)$ is finite.

Since $P_2 \cap Q$ is a parabolic subgroup of P_2 , for every open compact subgroup $K_f^{P_2} \subset P_2(\mathbb{A}_f)$ the double quotient $K_f^{P_2} \backslash P_2(\mathbb{A}_f) / (P_2 \cap Q)(\mathbb{Q})$ is finite. In particular there exists a finite subset $\mathfrak{R} \subset P_2(\mathbb{A}_f)$ such that

$$P_2(\mathbb{A}_f) = (P_2(\mathbb{A}_f) \cap K_f) \cdot \mathfrak{R} \cdot \text{Stab}_{(P_2 \cap Q)(\mathbb{Q})}(\mathcal{X}^0).$$

This implies

$$\begin{aligned} A &= \text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^0) \cap K_f \cdot (P_2(\mathbb{A}_f) \cap K_f) \cdot \mathfrak{R} \cdot \text{Stab}_{(P_2 \cap Q)(\mathbb{Q})}(\mathcal{X}^0) \\ &= (\text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^0) \cap K_f \cdot \mathfrak{R}) \cdot \text{Stab}_{(P_2 \cap Q)(\mathbb{Q})}(\mathcal{X}^0), \end{aligned}$$

and that

$$(\text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^0) \cap K_f) \backslash (\text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^0) \cap K_f \cdot \mathfrak{R})$$

is finite. Since $\text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^0) \cap K_f$ is contained in Γ_1 , we are done. **q.e.d.**

9.31. Lemma: Consider the situation of 9.2. Let $(P'_2, \mathcal{X}'_2)$ be a rational boundary component between $(P'_1, \mathcal{X}'_1)$ and $(P', \mathcal{X}')$. Let $e \in \mathcal{E}(\mathcal{X}'^0, P'_1)$ and $u' \in C(\mathcal{X}'^0, P'_2) \cap U'_2(\mathbb{Q})(-1)$ such that $\lambda \circ (\text{id} - e \circ \pi_0)(u') \geq 0$. There exists an open compact subgroup $K'_f \subset P'(\mathbb{A}_f)$ such that for all $q' \in \text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0) \cap K'_f \cdot P'_2(\mathbb{A}_f)$

$$\lambda \circ (\text{id} - e \circ \pi_0)(q' \cdot u') \geq 0.$$

Proof. If $P'_2, \mathcal{X}'_2 = (P', \mathcal{X}')$, then the inequalities always hold, since in this case $\text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0) \cap K'_f \cdot P'_2(\mathbb{A}_f) = \text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}'^0)$ acts by positive scalars on U' . Otherwise fix an arbitrary K'_f . We first show that there are finitely many half-lines $\mathbb{R}^{\geq 0} \cdot u'_i$ such that the inequality fails at most if $\mathbb{R}^{\geq 0} \cdot q' \cdot u' = \mathbb{R}^{\geq 0} \cdot u'_i$ for some i . To prove this let

$$\Gamma_1 := \text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0) \cap K'_f \cdot P'_1(\mathbb{A}_f),$$

and a fix subset $\mathfrak{R} = \{q'_1, \dots, q'_n\} \subset \text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^0)$ as in 9.30. Then for any $q' = \gamma \cdot q'_j \cdot p_2 \in \text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0) \cap K'_f \cdot P'_2(\mathbb{A}_f)$ with $p_2 \in \text{Stab}_{P'_2(\mathbb{Q})}(\mathcal{X}'^0)$ and $\gamma \in \Gamma_1$, the element $q' \cdot u'$ is a positive multiple of $\gamma \cdot q'_j \cdot u'$. By 9.7 there are at most finitely many half-lines $\mathbb{R}^{\geq 0} \cdot \gamma \cdot q'_j \cdot u'$, for which the inequality fails, as desired.

Next suppose that $\mathbb{R} \cdot u'_i = \mathbb{R} \cdot u'$ for some i . Then u'_i is a negative multiple of u' , since by assumption the inequality fails for u'_i . Since both u' and u'_i lie in the convex set $C(\mathcal{X}'^0, P'_2)$, this contains 0, in contradiction to the assumption $(P'_2, \mathcal{X}'_2) \neq (P', \mathcal{X}')$. Thus $\mathbb{R} \cdot u'_i \neq \mathbb{R} \cdot u'$ for all i . Now consider an open compact subgroup $K''_f \subset K'_f$. If the inequality fails for some $q' \in \text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0) \cap K''_f \cdot P'_2(\mathbb{A}_f)$, then $\mathbb{R}^{\geq 0} \cdot q' \cdot u' = \mathbb{R}^{\geq 0} \cdot u'_i$ for some i .

But for every i , by 7.14 this cannot happen if K_f'' is sufficiently small. Since there are only finitely many i , we are done. **q.e.d.**

9.32. Lemma: Let $(P, \mathcal{X})$ be mixed Shimura data, $(P_1, \mathcal{X}_1)$ a rational boundary component of $(P, \mathcal{X})$, and $\mathcal{X}^0$ a connected component of $\mathcal{X}$ that maps to $\mathcal{X}_1$. Let Q be the parabolic subgroup of P associated to $(P_1, \mathcal{X}_1)$. Furthermore let $K_f \subset P(\mathbb{A}_f)$ be an open compact subgroup, and Σ_0 a rational partial polyhedral decomposition of $C^*(\mathcal{X}^0, P_1)$. There exists a K_f -admissible partial cone decomposition Σ for $(P, \mathcal{X})$ with $\Sigma(\mathcal{X}^0, P_1, 1) = \Sigma_0$ if and only if the following condition holds:

For every rational boundary component $(P_2, \mathcal{X}_2)$ between $(P_1, \mathcal{X}_1)$ and $(P, \mathcal{X})$, and every $q \in \text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^0) \cap K_f \cdot P_2(\mathbb{A}_f)$

$$\{q \cdot \sigma \mid \sigma \in \Sigma_0, \sigma \subset C^*(\mathcal{X}^0, P_2)\} = \{\sigma \in \Sigma_0 \mid \sigma \subset q \cdot C^*(\mathcal{X}^0, P_2)\}.$$

Proof. If Σ is to exist, then for every $(P_2, \mathcal{X}_2)$ above we must have $\Sigma(\mathcal{X}^0, P_2, 1) := \{\sigma \in \Sigma_0 \mid \sigma \subset C^*(\mathcal{X}^0, P_2)\}$. By the conditions 6.4 (ii)–(iv) we must also have

$$\Sigma(p \cdot \mathcal{X}^0, p \cdot P_2 \cdot p^{-1}, p_{2,f} \cdot k_f) = \{p \cdot \sigma \mid \sigma \in \Sigma_0, \sigma \subset C^*(\mathcal{X}^0, P_2)\}$$

for all $p \in P(\mathbb{Q})$, $p_{2,f} \in P_2(\mathbb{A}_f)$, and $k_f \in K_f$. Thus a necessary condition is that the right hand side does not change if p , $p_{2,f}$, k_f are replaced by other elements such that $p \cdot \mathcal{X}^0$, $p \cdot P_2 \cdot p^{-1}$, $p \cdot p_{2,f} \cdot k_f$ do not change. It is quite obvious that this condition is also sufficient. Now this condition holds if and only if for all $(P_2, \mathcal{X}_2)$ and $p \in \text{Stab}_{P(\mathbb{Q})}(\mathcal{X}^0) \cap K_f \cdot P_2(\mathbb{A}_f)$, if $(P_1, \mathcal{X}_1)$ is a boundary component of $\text{int}(p)((P_2, \mathcal{X}_2))$, then

$$\{p \cdot \sigma \mid \sigma \in \Sigma_0, \sigma \subset C^*(\mathcal{X}^0, P_2)\} = \{\sigma \in \Sigma_0 \mid \sigma \subset p \cdot C^*(\mathcal{X}^0, P_2)\}.$$

For every such p , both $(P_1, \mathcal{X}_1)$ and $\text{int}(p^{-1})((P_1, \mathcal{X}_1))$ are rational boundary components of $(P_2, \mathcal{X}_2)$, so there exists $p_2 \in \text{Stab}_{P_2(\mathbb{Q})}(\mathcal{X}^0)$ such that $\text{int}(p^{-1})((P_1, \mathcal{X}_1)) = \text{int}(p_2^{-1})((P_1, \mathcal{X}_1))$. In other words $q := p \cdot p_2^{-1}$ normalizes $(P_1, \mathcal{X}_1)$, hence lies in $Q(\mathbb{Q})$. But $p_2 \cdot C^*(\mathcal{X}^0, P_2) = C^*(\mathcal{X}^0, P_2)$, and acts by scalars on $U_2(\mathbb{R})(-1)$. If $(P_2, \mathcal{X}_2)$ is a proper boundary component, then this scalar is positive, and in the above equation we may replace p by q , as in the desired assertion. Otherwise we anyway have $Q = P$, and may even take $p_2 = 1$. **q.e.d.**

9.33. Corollary (of 9.29): Consider mixed Shimura data $(P, \mathcal{X})$, an open compact subgroup $K_f \subset P(\mathbb{A}_f)$, and a finite K_f -admissible cone decomposition Σ for $(P, \mathcal{X})$. Then $M_{\mathbb{C}}^{K_f^+}(P, \mathcal{X}, \Sigma)$ exists for every sufficiently small open compact subgroup $K_f^+ \subset K_f$. In fact $M_{\mathbb{C}}^{K_f^+}(P, \mathcal{X}, \Sigma)$ is covered by quasiprojective $M_{\mathbb{C}}^{K_f^+}(P, \mathcal{X}, \Sigma_i)$, for which an ample line bundle can be described as in 8.14.

Remark. Except for the last sentence this can be proved with less pain than taken 9.29, if one disregards the invariance condition 6.4 (iv).

Proof. Since we may choose K_f^+ neat, as in the proof of 9.21 we can reduce to the case where $(P, \mathcal{X})$ is as in 9.2. Let σ_i be (finitely many) representatives in Σ for the two actions 6.4 (ii)–(iii) of $P(\mathbb{Q})$ and K_f . By 9.29 for every i there exists an open compact subgroup

$K_f^i \subset P(\mathbb{A}_f)$, and a K_f^i -admissible partial cone decomposition Σ_i for $(P, \mathcal{X})$, that contains σ , and such that $M_{\mathbb{C}}^{K_f^i}(P, \mathcal{X}, \Sigma_i)$ exists. Since we have only finitely many i , we can find an open compact subgroup K_f^* , contained in the intersection of all K_f^i , that is a normal subgroup of K_f . Then every $M_{\mathbb{C}}^{K_f^*}(P, \mathcal{X}, \Sigma_i)$ exists, and so does $M_{\mathbb{C}}^{K_f^*}(P, \mathcal{X}, [\cdot k_f]^* \Sigma_i)$ for every $k_f \in K_f$ modulo K_f^* . Now we may replace every Σ_i by the smallest K_f^* -admissible cone decomposition that contains σ_i . Then Σ_i is contained in Σ , and by construction Σ is the union of all $[\cdot k_f]^* \Sigma_i$. Thus by gluing $M_{\mathbb{C}}^{K_f^*}(P, \mathcal{X}, \Sigma_i)$ exists, as desired. **q.e.d.**

9.34. Corollary: (a) “ $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma)$ ” always exists as an algebraic space.

(b) Suppose that $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma)$ exists. If Σ is finite, resp. complete, then $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma)$ is of finite type, resp. proper, over $\mathbb{C}$.

Proof. (a) In the category of algebraic spaces every quotient of a variety by a finite group exists (see [K] introduction). Thus by 9.33, “ $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma)$ ” exists as the quotient of $M_{\mathbb{C}}^{K_f^+}(P, \mathcal{X}, \Sigma)$ by K_f/K_f^+ , whenever Σ is finite. By gluing along open subspaces of finite type, the same follows for arbitrary Σ .

(b) Both assertions are invariant under quotients by finite groups. Thus the first assertion follows from 9.34, and the second follows from this and the compactness result 6.27. **q.e.d.**

9.35. Remarks: (a) From now on we implicitly assume the existence of $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma)$ as a scheme, whenever we write these symbols. Since by 9.28 and 9.33, it exists whenever Σ is sufficiently fine or K_f sufficiently small, this will not be a real restriction in the applications we have in mind. In any case, mutatis mutandum our assertions will also be valid in the category of algebraic spaces, without this restriction.

(b) Counterexample: $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma)$ does not always exist. It is a nice exercise to realize the example [H2] app.B ex 3.4.2 in the toroidal compactification for the mixed Shimura data $(P, \mathcal{X}) = (P_{2g}, \mathcal{X}_{2g})$ of 2.25 with $g = 1$, or for $(P, \mathcal{X}) = (\mathcal{S}_{P_4, \mathbb{Q}}, \mathcal{H}_4)$ of 2.7.

Finally, we consider the stratification of chapter 7.

9.36. Proposition: The stratifications in 7.2, 7.3, and 7.10 are algebraic, and the assertion of 7.17 (a) holds algebraically.

Proof. By [S] §19 prop. 15 the map in 7.17 (a) is algebraic, since both sides are compact. This shows the second assertion. In particular this yields an algebraic morphism.

$$\mathrm{Stab}_{\Delta_1}([\sigma]) \backslash M_{\mathbb{C}}^{\pi_{[\sigma]}(K_f^1)}(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]}) \longrightarrow M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma),$$

which proves the other assertions whenever Σ is complete. By 9.33 the general case follows from this, since the assertions are invariant under replacing the open compact subgroup by a smaller one. **q.e.d.**

9.37. Proposition: The isomorphism of 7.17(b) induces a canonical isomorphism between the formal completion of $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma)$ and of $\text{Stab}_{\Delta_1}([\sigma]) \backslash M_{\mathbb{C}}^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1)$ along

$$\text{Stab}_{\Delta_1}([\sigma]) \backslash M_{\mathbb{C}}^{\pi_{[\sigma]}(K_f^1)}(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]}, \Sigma_{1, [\sigma]}).$$

Proof. This is a special case of the following lemma. **q.e.d.**

9.38. Lemma: Let X be a proper algebraic variety over $\mathbb{C}$, $Y \subset X$ a closed subvariety, and $\mathcal{X}_Y$ the formal completion of X along Y . Let X_1 be another algebraic variety over $\mathbb{C}$, $Y_1 \subset X_1$ a closed subvariety, and $\mathcal{X}_{1, Y_1}$ the formal completion of X_1 along Y_1 . Suppose that we are given open analytic neighborhoods $Y(\mathbb{C}) \subset U \subset X(\mathbb{C})$ and $Y_1(\mathbb{C}) \subset U_1 \subset X_1(\mathbb{C})$, and an analytic isomorphism $U \cong U_1$ that restricts to an isomorphism $Y(\mathbb{C}) \cong Y_1(\mathbb{C})$:

$$\begin{array}{ccccc} Y(\mathbb{C}) & \subset & U & \subset & X(\mathbb{C}) \\ \downarrow \wr & & \downarrow \wr & & \\ Y_1(\mathbb{C}) & \subset & U_1 & \subset & X_1(\mathbb{C}) \end{array}$$

Then this isomorphism corresponds to a canonical isomorphism $\mathcal{X}_Y \cong \mathcal{X}_{1, Y_1}$:

$$\begin{array}{ccc} Y & \subset & \mathcal{X}_Y \\ \downarrow \wr & & \downarrow \wr \\ Y_1 & \subset & \mathcal{X}_{1, Y_1} \end{array}$$

Proof. The isomorphism $U \cong U_1$ induces an isomorphism between the nonreduced complex spaces $(Y(\mathbb{C}), \mathcal{O}_X^{an}/(\mathfrak{I}_Y^{an})^n)$ and $(Y_1(\mathbb{C}), \mathcal{O}_{X_1}^{an}/(\mathfrak{I}_{Y_1}^{an})^n)$. Since X is proper over $\mathbb{C}$, it follows that $Y(\mathbb{C})$, and hence $Y_1(\mathbb{C})$ is compact. Thus by [S] §19 prop. 15 the analytic isomorphism corresponds to a unique algebraic isomorphism between the infinitesimal neighborhoods $(Y(\mathcal{O}_X)/(\mathfrak{I}_Y)^n)$ and $(Y_1(\mathcal{O}_{X_1})/(\mathfrak{I}_{Y_1})^n)$. By taking inverse limits the assertion follows. (Compare also [H1] ch.VI ex. 2.9) **q.e.d.**

9.39. Examples: (a) As pointed out in 9.8, our sets Σ play the role of the Γ -polyhedral cocores in [AMRT] ch.II §5.3. In the case where P is reductive, the precise correspondence is the following. As explained in 8.15 (a), we have a canonical splitting of U'_1 , whence a canonical decomposition $C^*(\mathcal{X}^0, P'_1) \cong \mathbb{R} \cdot \sigma_0 \times C^*(\mathcal{X}^0, P_1)$. Fix some $u_0 \in (\sigma_0)^0 \cap U'(\mathbb{Q})(-1)$. The correspondence between Γ -polyhedral cocores $K \subset C^*(\mathcal{X}^0, P_1)$, and our sets Σ , can then be given by

$$\begin{aligned} K &= \{u \in C^*(\mathcal{X}^0, P_1) \mid (-u_0, u) \in \Sigma\}, \text{ and} \\ \Sigma &= \text{convex closure of } \sigma_0 \cup \{-u_0\} \times K. \end{aligned}$$

(b) We include an example not subsumed by (a), which at the same time illustrates the remark 8.15 (b). Let $(P', \mathcal{X}')$ be the mixed Shimura data $(P_{2g}, \mathcal{X}_{2g})$ defined in 2.25

for $g := 1$, and $(P, \mathcal{X}) := (P', \mathcal{X}')/U'$. Let $K'_f \subset P'(\mathbb{A}_f)$ be an open compact subgroup whose image in $(P'/W')(\mathbb{A}_f) \cong GL_2(\mathbb{A}_f)$ is conjugate to $GL_2(\hat{\mathbb{Z}})$. Then modulo the actions of K'_f and of $P'(\mathbb{Q})$, there is precisely one conjugacy class of proper rational boundary components. Describing it as in 4.27, we have $U'_1 \cong (\mathbb{G}_{a, \mathbb{Q}})^{\oplus 3}$,

$$\lambda(C^*(\mathcal{X}'^0, P'_1) \cong \{(x, y, z) \in \mathbb{R}^3 \mid z > 0 \text{ or } y = z = 0\},$$

and $\Gamma_1 := \text{Stab}_{Q'(\mathbb{Q})}(\mathcal{X}'^0) \cap (K'_f \cdot P'_1(\mathbb{A}_f))$ acts on this through positive scalars and the unipotent transformations

$$(x, y, z) \mapsto (x + 2ny + n^2z, y + nz, z)$$

for all $n \in \mathbb{Z}$.

A K'_f -admissible cone decomposition as in 9.13 now corresponds to a subset $\Sigma \subset C^*(\mathcal{X}'^0, P'_1)$ as in 9.8 with respect to the above unipotent transformations. In terms of the above identification, 9.1 shows that $\sigma_0 = \mathbb{R}^{\geq 0} \cdot (1, 0, 0)$. Clearly $\Sigma = \sigma_0 \cup \mathbb{R}^{> 0} \cdot \{(x, y, 1) \in \Sigma\}$ for such Σ , so we can describe it in terms of its intersection with the plane $z = 1$. This must be the convex closure of $\Gamma_1 \cdot \{(x_i + \mathbb{R}^{\geq 0}, y_i, 1) \mid 1 \leq i \leq n\}$ for some $x_i, y_i \in \mathbb{Q}$. In the special case $n = 1$ and $y_1 = 0$, we get

A little calculation shows that the associated splittings lie in $\mathcal{E}(\mathcal{X}'^0, P'_1)$ if and only if $x_1 < 0$. In this case the assumptions of 9.13 are verified.

Coming back to 8.15 (b), we now see a geometric reason for the necessity of quadratic terms in the action of Γ_1 : the strict convexity condition. At the same time, it prohibits the existence of a splitting of the unipotent extension $(P', \mathcal{X}') \rightarrow (P, \mathcal{X})$.

Chapter 10

Moduli schemes of abelian varieties

In this chapter we describe an interpretation of some mixed Shimura varieties as moduli schemes, together with the universal family, of abelian varieties or of similar algebraic objects. Given this interpretation, we show that some of the maps 3.4 and 7.17 can be expressed in modular terms: so that, in particular, they are defined over $\mathbb{Q}$.

We first describe a well-known, suitable rigidified, moduli functor of polarized abelian varieties with level structure. It is representable by a smooth scheme $\underline{\mathcal{M}}_d$ over $\mathbb{Q}$ (10.1–6). Certain mixed Shimura varieties associated to the mixed Shimura data $(P_{2g}, \mathcal{X}_{2g})$ defined in 2.25 naturally carry the structure required by this functor (10.7–8). This yields isomorphisms between these mixed Shimura varieties on one side, and $\underline{\mathcal{M}}_{d,\mathbb{C}}$, together with its universal family, on the other side (10.9–10). We show that some of the maps of 3.4 (a) can be described in terms of modular data (10.11–14).

In 10.15–16 we do the same for the mixed Shimura data $(P_0, \mathcal{X}_0)$ defined in 2.24. This is more or less a special case of $(P_{2g}, \mathcal{X}_{2g})$, where the moduli functor degenerates to one for roots of unity.

The rest of the chapter deals with one special case of the above, but takes into account the toroidal compactification. We describe the Tate-curve (10.17–18), and the moduli scheme for generalized elliptic curves with d -structure (10.19–20), in terms of toroidal compactifications of mixed Shimura varieties. The main result 10.22 is a modular interpretation for the isomorphism 7.17/9.37.

10.1. Abelian schemes: Let S be a scheme over $\mathbb{Q}$, and $A \rightarrow S$ an abelian scheme (see [M3] ch.6). We denote the group operation on A by $(a, b) \mapsto a + b$. For every positive integer d let $[d] : A \rightarrow A$ be the morphism $a \mapsto d \cdot a = a + \dots + a$ (d times). The subgroup scheme $\ker([d]) \subset A$ is finite and étale over S , since d is invertible over S . Let $X \rightarrow A$ be a $\mathbb{G}_m$ -torsor. For every abelian scheme $T \rightarrow S$ let A_T be the abelian scheme $A \times_S T$ over T , and X_T the $\mathbb{G}_m$ -torsor $X \times_S T$ over A_T . For every section $a \in A(T)$ let $T_a : A_T \rightarrow A_T$ be the morphism $b \mapsto b + a$. Consider the functor

$$T \mapsto H(X)(T) := \{a \in A(T) \mid T_a^* X_T \cong X_T\}.$$

The condition $T_a^* X_T \cong X_T$ is equivalent to the existence of an isomorphism $f : X_T \rightarrow X_T$

of schemes over T such that the two diagrams

$$\begin{array}{ccc} X_T & \xrightarrow{f} & X_T \\ \downarrow & & \downarrow \\ A_T & \xrightarrow{T_a} & A_T \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{G}_m \times X_T & \xrightarrow{\text{id} \times f} & \mathbb{G}_m \times X_T \\ \downarrow & & \downarrow \\ X_T & \xrightarrow{f} & X_T \end{array}$$

commute, where the vertical arrows in the diagram on the right hand side represent the action of $\mathbb{G}_m$. This functor is represented by a closed subgroup scheme $H(X) \subset A$, which is smooth over S . If X is relatively ample with respect to $A \rightarrow S$, then $H(X)$ is finite and étale over S .

10.2. The canonical pairing: Let $a, b \in H(X)(T)$ be sections and f resp. g corresponding isomorphisms $X_T \rightarrow X_T$. Then $f \circ g \circ f^{-1} \circ g^{-1}$ is an automorphism of the $\mathbb{G}_m$ -torsor X_T over A_T , so there exists a unique section $\alpha \in \mathbb{G}_m(T)$ such that $f \circ g \circ f^{-1} \circ g^{-1}$ is equal to the multiplication by α . Since f and g commute with the $\mathbb{G}_m$ -action, α depends only on a and b . thus the map

$$H(X)(T) \times H(X)(T) \rightarrow \mathbb{G}_m(T), (a, b) \mapsto e(X)(a, b) := \alpha$$

is well-defined. This morphism of functors is represented by a unique morphism of schemes $e(X) : H(X) \times_S H(X) \rightarrow \mathbb{G}_{m,S}$. It is easily seen that this is an alternating bilinear form. If X is relatively ample with respect to $A \rightarrow S$, then this is a perfect duality (see [M2]§1 thm. 1).

Let $A = \tilde{A}/\Gamma$ be an abelian variety over $\mathbb{C}$, where $\tilde{A}$ is the universal covering of A in the topological sense, and $E : \Gamma \times \Gamma \rightarrow 2\pi\sqrt{-1} \cdot \mathbb{Z}$ the alternating form that corresponds to the Chern class of X (compare 3.18–19). Let $\Gamma^\perp := \{\gamma \in \Gamma \otimes \mathbb{T} \mid \forall \gamma' \in \Gamma \ E(\gamma, \gamma') \in 2\pi\sqrt{-1} \cdot \mathbb{Z}\}$, then by [M1] §9 prop. p.84 (iii) we have $H(X) = \Gamma^\perp/\Gamma$. The canonical pairing $e(X)$ is according to [M1] §24 p.236 given by

$$(\Gamma^\perp/\Gamma) \times (\Gamma^\perp/\Gamma) \rightarrow \mathbb{C}^\times, ([\gamma], [\gamma']) \mapsto \exp(E(\gamma, \gamma')).$$

10.3. Level structure: Fix integers $g \geq 1$ and $d \geq 1$, and free $\mathbb{Z}/d\mathbb{Z}$ -modules $V[d]$ of rank $2g$ and $U[d]$ of rank 1. Fix also a perfect alternating paring $\Psi[d] : V[d] \times V[d] \rightarrow U[d]$. We consider abelian schemes $A \rightarrow S$ of constant relative dimension g , together with a $\mathbb{G}_m$ -torsor $X \rightarrow A$ such that $H(X) = A[d]$ for the above d . Consider the constant finite group schemes $V[d] \times S$ and $U[d] \times S$ over S . A symplectic d -structure on $X \rightarrow A \rightarrow S$ consists of an isomorphism $\lambda : V[d] \times S \rightarrow A[d]$ and a monomorphism $\mu : U[d] \times S \hookrightarrow \mathbb{G}_{m,S}$, such that the following diagram commutes:

$$\begin{array}{ccc} (V[d] \times S) \times_S (V[d] \times S) & \xrightarrow{\Psi[d]} & U[d] \times S \\ \lambda \times \lambda \downarrow & & \downarrow \mu \\ A[d] \times_S A[d] & \xrightarrow{e(X)} & \mathbb{G}_{m,S} \end{array}$$

Locally in the étale topology on S there always exists a symplectic d -structure.

10.4. Normalization of $\mathbb{G}_m$ -torsor: Let $[-1] : A \rightarrow A$ be the inverse $a \mapsto -a$, and $i : A[2] \hookrightarrow A$ the inclusion. The equation $[-1] \circ i = i$ yields a canonical $i^*[-1]^*X \cong i^*X$. The $\mathbb{G}_m$ -torsor X is called totally symmetric if there exists an isomorphism $[-1]^*X \cong X$ that agrees with this canonical isomorphism on $A[2]$. In other words X is totally symmetric if there exists an isomorphism of schemes $f : X \rightarrow X$ such that the three diagrams

$$\begin{array}{ccccc} X & \xrightarrow{f} & X & \mathbb{G}_m \times X & \xrightarrow{\text{id} \times f} & \mathbb{G}_m \times X & & X \times_A A[2] \\ \downarrow & & \downarrow & \downarrow & & \downarrow & & \begin{array}{ccc} & \swarrow^{pr_1} & \searrow^{pr_1} \\ X & \xrightarrow{f} & X \end{array} \\ A & \xrightarrow{[-1]} & A & X & \xrightarrow{f} & X & & \end{array}$$

are commutative. If S is the spectrum of an algebraically closed field, the remark in [M2] §2 p.307 says that every totally symmetric $\mathbb{G}_m$ -torsor is uniquely determined by its Chern class.

Let $e : S \hookrightarrow A$ be the zero section. A normalized $\mathbb{G}_m$ -torsor is a $\mathbb{G}_m$ -torsor $X \rightarrow A$ together with an isomorphism $e^*X \cong \mathbb{G}_{m,S}$, i.e. with a trivialization along the zero section. The trivialization is uniquely determined by its restriction to the “identity”-section in $\mathbb{G}_{m,S}$. Thus it is equivalent to a morphism of schemes $\tau : S \rightarrow X$ such that the diagram

$$\begin{array}{ccc} & & X \\ & \nearrow \tau & \downarrow \\ S & \xrightarrow{e} & A \end{array}$$

commutes. In the following we shall be interested in normalized totally symmetric $\mathbb{G}_m$ -torsors (X, τ) on $A \rightarrow S$. By the above remark such an object uniquely determined up to isomorphism by the relative Chern class of X .

10.4. A moduli problem: Let $g, d, V[d], U[d]$ and $\Psi[d]$ be as in 10.3. We suppose that d is even and at least 4. For every scheme S over $\mathbf{Spec}(\mathbb{Q})$ let

$$\mathcal{M}_d(S) := \left\{ \begin{array}{l} \text{isomorphism classes of quintuples } (A, X, \tau, \lambda, \mu), \text{ where:} \\ A \rightarrow S \text{ is an abelian scheme of relative dimension } g, \\ (X, \tau) \text{ is a relatively ample normalized totally symmetric} \\ \mathbb{G}_m\text{-torsor on } A \rightarrow S \text{ such that } H(X) = A[d], \text{ and} \\ (\lambda, \mu) \text{ a symplectic } d\text{-structure on } X \rightarrow A \rightarrow S. \end{array} \right\}$$

In the category of complex spaces we define $\mathcal{M}_d(S)$ analogously. The following theorem is well-known.

10.6. Theorem: The functor $\mathcal{M}_d$ on the category of schemes is representable by a smooth quasiprojective scheme $\mathcal{M}_d$ over $\mathbb{Q}$.

Proof. The functor $A_{g,d^g,d}$ of [M3] thm. 7.9 is representable by a quasiprojective scheme over $\mathbf{Spec}(\mathbb{Z})$. Our functor $\mathcal{M}_d$ is isomorphic to an open and closed subfunctor of $A_{g,d^g,d} \times \mathbf{Spec}(\mathbb{Q})$, hence $\mathcal{M}_d$ is representable by a quasiprojective scheme $\underline{\mathcal{M}}_d$ over $\mathbb{Q}$. Since $d \geq 4$ an object $(A, X, \tau, \lambda, \mu)$ possesses no nontrivial automorphism, hence $\underline{\mathcal{M}}_d$ is smooth. **q.e.d.**

10.7. Construction of the moduli scheme as Shimura variety: Consider the mixed Shimura data $(P, \mathcal{X}) = (P_{2g}, \mathcal{X}_{2g})$ defined in 2.25. We shall describe a natural structure as in 10.5 on certain mixed Shimura varieties associated to $(P_{2g}, \mathcal{X}_{2g})$.

As explained in 2.15, we may assume that $W = U \times V$ with the group operation $(u, v) \cdot (u', v') := (u + u' + \frac{1}{2} \cdot \Psi(v, v'), v + v')$ and the $G = CS_{P_{2g}, \mathbb{Q}}$ -action $g((u, v)) = (g(u), g(v))$. Choose $\mathbb{Z}$ -structures on U and V such that Ψ induces a unimodular pairing $V(\mathbb{Z}) \times V(\mathbb{Z}) \rightarrow U(\mathbb{Z})$. Since G acts faithfully on V this also gives a $\mathbb{Z}$ -structure on G . For every even positive integer d we define the open compact subgroups

$$K_f = K_f(d) := \{g \in G(\hat{\mathbb{Z}}) \mid g \equiv 1 \pmod{d}\},$$

$$K_f^W = K_f^W(d) := (d \cdot U(\hat{\mathbb{Z}})) \times (d \cdot V(\hat{\mathbb{Z}})).$$

Since d is even, the definition of the group operation on W implies that K_f^W is indeed a subgroup of $W(\mathbb{A}_f)$. Clearly it is normalized by K_f . Thus $K_f^P := K_f^P(d) := K_f^W \rtimes K_f$ is an open compact subgroup of $P(\mathbb{A}_f)$. From now on we assume $d \geq 4$. Let $K_f^U := U(\mathbb{A}_f) \cap K_f^W$ and $K_f^V := K_f^W / K_f^U$, and write

$$\begin{aligned} M &= M(d) := M_{\mathbb{C}}^{K_f}(CS_{P_{2g}, \mathbb{Q}}, \mathcal{H}_{2g}), \\ M_V &= M_V(d) := M_{\mathbb{C}}^{K_f^V \rtimes K_f}(V \rtimes CS_{P_{2g}, \mathbb{Q}}, \mathcal{H}_{2g}), \text{ and} \\ M_W &= M_W(d) := M_{\mathbb{C}}^{K_f^P}(P_{2g}, \mathcal{X}_{2g}), \end{aligned}$$

(see 9.25). The projections

$$(P_{2g}, \mathcal{X}_{2g}) \longrightarrow (V_{2g} \rtimes CS_{P_{2g}, \mathbb{Q}}, \mathcal{H}'_{2g}) \longrightarrow (CS_{P_{2g}, \mathbb{Q}}, \mathcal{H}_{2g}).$$

induces holomorphic maps $M_W \rightarrow M_V \rightarrow M$.

By 3.14, $M_V \rightarrow M$ is a family of abelian varieties of relative dimension g . Remember from 2.25 the structure on $(P_{2g}, \mathcal{X}_{2g}) \rightarrow (V_{2g} \rtimes CS_{P_{2g}, \mathbb{Q}}, \mathcal{H}'_{2g})$ as a $((P_0, \mathcal{X}_0) \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0))$ -torsor. The image of K_f in $\mathbb{G}_m(\mathbb{A}_f)$ is

$$K_f^* = K_f^*(d) := \{t \in \mathbb{G}_m(\mathbb{A}_f) \mid t \equiv 1 \pmod{d}\},$$

so by 3.12 (b), $M_W \rightarrow M_V$ is a torsor under the group scheme $M_{\mathbb{C}}^{K_f^U \rtimes K_f^*}(P_0, \mathcal{X}_0) \rightarrow M_{\mathbb{C}}^{K_f^*}(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$. As in 3.17, under the isomorphism

$$M_{\mathbb{C}}^{K_f^U \rtimes K_f^*}(P_0, \mathcal{X}_0) \xrightarrow{\sim} \mathbb{G}_m \times M_{\mathbb{C}}^{K_f^*}(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$$

defined in 3.16, this turns $M_W \rightarrow M_V$ into a $\mathbb{G}_m$ -torsor. By 3.21 its inverse is relatively ample with respect to $M_V \rightarrow M$. From now on we replace this $\mathbb{G}_m$ -action by its twist under the automorphism $\mathbb{G}_m \rightarrow \mathbb{G}_m, z \mapsto z^{-1}$. This corresponds to replacing the $\mathbb{G}_m$ -torsor $M_W \rightarrow M_V$ by its inverse, but as a scheme the “new” M_W is equal to the “old” one. With this new structure, it is relatively ample.

Let $e : (CS_{P_{2g}, \mathbb{Q}}, \mathcal{H}_{2g}) \hookrightarrow (V_{2g} \rtimes CS_{P_{2g}, \mathbb{Q}}, \mathcal{H}'_{2g})$ and $\tau : (CS_{P_{2g}, \mathbb{Q}}, \mathcal{H}_{2g}) \hookrightarrow (P_{2g}, \mathcal{X}_{2g})$ be the morphism that corresponds to the splitting $CS_{P_{2g}, \mathbb{Q}}, \mathcal{H}_{2g} \hookrightarrow V_{2g} \rtimes CS_{P_{2g}, \mathbb{Q}}$ and

$CS_{P_{2g}, \mathbb{Q}}, \mathcal{H}_{2g} \hookrightarrow P_{2g}$. They induce morphisms $[e]$ and $[\tau]$ such that the following diagram commutes.

$$\begin{array}{ccccc}
 & & M & & \\
 & & \downarrow [\tau] & & \searrow \text{id} \\
 & & \swarrow & \downarrow [e] & \searrow \\
 M_W & \xrightarrow{[\pi']} & M_V & \xrightarrow{[\pi'']} & M
 \end{array}$$

Since $[e]$ is the zero section, this defines a normalization of the $\mathbb{G}_m$ -torsor $M_W \rightarrow M_V$.

Define $V[d] := V(\mathbb{Z})/d \cdot V(\mathbb{Z}) \cong V(\hat{\mathbb{Z}})/K_f^V$, and $U[d] := U(\mathbb{Z})/d \cdot U(\mathbb{Z}) \cong U(\hat{\mathbb{Z}})/K_f^U$. Since $\Psi : V(\mathbb{Z}) \times V(\mathbb{Z}) \rightarrow U(\mathbb{Z})$ is unimodular, it induces a perfect alternating pairing $\Psi[d] : V[d] \times V[d] \rightarrow U[d]$. The group $V(\hat{\mathbb{Z}})$ normalizes $K_f^V \rtimes K_f$ in $V(\mathbb{A}_f) \rtimes G(\mathbb{A}_f)$. Thus 3.15 gives an isomorphism $\lambda : V[d] \times M \rightarrow M_V[d]$ of group schemes over M . Likewise $U(\hat{\mathbb{Z}})$ normalizes $K_f^U \rtimes K_f^*$ in $P_0(\mathbb{A}_f)$, so 3.15 yields a monomorphism $U[d] \times M_{\mathbb{C}}^{K_f^*}(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0) \hookrightarrow M_{\mathbb{C}}^{K_f^U \rtimes K_f^*}(P_0, \mathcal{X}_0)$. Under the above identifications we get a monomorphism $\mu : U[d] \times M \hookrightarrow \mathbb{G}_m \times M$.

10.8. Proposition: The quintuple $(M_W, M_V, [\tau], \lambda, \mu)$ is an element of $\underline{\mathcal{M}}_d(M)$, and hence corresponds to a unique morphism $M \rightarrow \underline{\mathcal{M}}_{d, \mathbb{C}}$.

Proof. We have to show:

- (a) $M_W \rightarrow M_V$ is totally symmetric.
- (b) $H(M_W) = M_V[d]$.
- (c) The pair (λ, μ) is a symplectic d -structure on $M_W \rightarrow M_V \rightarrow M$.

It suffices to verify these assertions fibrewise and in the category of complex spaces. Fix a point $[(y, g_f)] \in M(\mathbb{C}) = M^{K_f}(CS_{P_{2g}, \mathbb{Q}}, \mathcal{H}_{2g})(\mathbb{C})$. Since $CS_{P_{2g}}(\mathbb{A}_f)G(\mathbb{A}_f) = G(\mathbb{Q}) \cdot G(\hat{\mathbb{Z}})$, we may assume that $(y, g_f) \in \mathcal{H}_{2g} \times G(\hat{\mathbb{Z}})$. Let $\Gamma_V := d \cdot V(\mathbb{Z})$ and $\Gamma_W := (d \cdot U(\mathbb{Z})) \times (d \cdot V(\mathbb{Z}))$, then the fibres of M_W and M_V over $[(y, g_f)]$ are isomorphic to $X := \Gamma_W \backslash W(\mathbb{R}) \cdot U(\mathbb{C})$, respectively $A := \Gamma_V \backslash V(\mathbb{R})$.

For (a) we consider the homomorphism $W \rightarrow W$, $(u, v) \mapsto (u, -v)$. It maps K_f^W to itself and commutes with the action of G , so it induces an automorphism ι of the Shimura data $(P_{2g}, \mathcal{X}_{2g})$, and hence an automorphism $[\iota]$ of M_W . It operates in X by the same formula $(u, v) \mapsto (u, -v)$, and it commutes with the action of $U(\mathbb{C})$ on X and therefore with its structure as $\mathbb{C}^\times$ -torsor. Every point in $A[2]$ is of the form $[v]$ für $v \in \frac{d}{2} \cdot V(\mathbb{Z})$. For such v we have $(0, 2v) \in \Gamma_W$, hence for any $u \in U(\mathbb{C})$

$$(0, 2v) \cdot (u, -v) = (u + \frac{1}{2} \cdot \Psi(2v, -v), 2v - v) = (u, v),$$

since $\Psi(v, v) = 0$. This shows $[\iota]([(u, v)]) = [(u, -v)] = [(u, v)]$, so $[\iota]$ is equal to the identity over $A[2]$. By 10.4 this proves (a).

Since we have inverted the $\mathbb{G}_m$ -torsor $M_W \rightarrow M_V$, 3.19 implies that the Chern class of $X \rightarrow A$ now corresponds to the alternating pairing $+\Psi : \Gamma_V \times \Gamma_V \rightarrow \Gamma_U := U(\mathbb{Q}) \cap \Gamma_W$. Via the isomorphism 3.16, we get the equation $E = \frac{1}{d} \cdot \lambda_{\psi(y)} \circ \Psi$. It is verified at once that $(\Gamma_V)^\perp = V(\mathbb{Z})$, whence $K(X) = V(\mathbb{Z})/\Gamma_V = A[d]$, which implies (b).

By 10.2 this also shows that $e(X) : A[d] \times A[d] \rightarrow \mathbb{C}^\times$ is given by

$$([v], [v']) \mapsto \exp\left(\frac{1}{d} \cdot \lambda_{\psi(y)} \circ \Psi(v, v')\right).$$

By 3.15 λ is given by

$$\begin{aligned} V[d] &\cong V(\hat{\mathbb{Z}})/d \cdot V(\hat{\mathbb{Z}}) \rightarrow U(\hat{\mathbb{Z}})/d \cdot V(\hat{\mathbb{Z}}) \cong V(\mathbb{Z})/d \cdot V(\mathbb{Z}) = A[d], \\ [v_f] &\mapsto [g_f(v_f)], \end{aligned}$$

and by 3.15 and 3.16, μ is given by

$$\begin{aligned} U[d] &= U(\hat{\mathbb{Z}})/d \cdot U(\hat{\mathbb{Z}}) \rightarrow U(\hat{\mathbb{Z}})/d \cdot U(\hat{\mathbb{Z}}) = U(\mathbb{Z})/d \cdot U(\mathbb{Z}) \rightarrow \mathbb{C}^\times, \\ [u_f] &\mapsto [-g_f(u_f)] = [u] \mapsto \exp\left(-\frac{1}{d} \cdot \lambda_{\psi(y)}(u)\right). \end{aligned}$$

The commutativity of the diagram 10.3 is therefore equivalent to the identity

$$\Psi(-g_f(v_f), -g_f(v'_f)) \equiv \Psi(v_f, v'_f) \bmod d \cdot U(\hat{\mathbb{Z}}).$$

This congruence follows from the bilinearity and the G -equivariance of the pairing Ψ . This proves (c). **q.e.d.**

10.9. Proposition: The morphism $M \rightarrow \underline{\mathcal{M}}_{d,\mathbb{C}}$ of 10.8 is an isomorphism.

Proof. Both are smooth varieties over $\mathbb{C}$, so it suffices to prove the bijectivity on their $\mathbb{C}$ -valued points. Consider $(A, X, \tau, \lambda, \mu) \in \mathcal{M}_d(\mathbb{C})$. Write $A = \tilde{A}/\Gamma$, where $\tilde{A}$ is the universal covering of A , and let $E : \Gamma \times \Gamma \rightarrow 2\pi\sqrt{-1} \cdot \mathbb{Z}$ be the pairing associated to X . Let λ_0 be an isomorphism $\mathbb{Z} \rightarrow 2\pi\sqrt{-1} \cdot \mathbb{Z}$. Then $\Gamma^\perp = \frac{1}{d} \cdot \Gamma$ is in $\Gamma \otimes \mathbb{Q}$, and the pairing $\Gamma^\perp \times \Gamma^\perp \rightarrow \mathbb{Z}$ induced by $d^2 \cdot \lambda_0^{-1} \circ E$ is unimodular. Thus there exists an isomorphism $\phi : \Gamma \rightarrow \Gamma_V$ such that this pairing corresponds to the pairing $\Psi : V(\mathbb{Z}) \times V(\mathbb{Z}) \rightarrow U(\mathbb{Z}) = \mathbb{Z}$. The complex structure on $A \cong V(\mathbb{R})/\Gamma_V$ then corresponds to a unique action of $\mathbb{S}$ on $V(\mathbb{R})$. By [M1]§2 cor. p.18 we have $E(\sqrt{-1} \cdot v, \sqrt{-1} \cdot v') = E(v, v')$ for all $v, v' \in V(\mathbb{R})$, hence this action factors through a unique homomorphism $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$. Since X is ample, $-\lambda_0 \circ \Psi$ is a polarization of the Hodge structure on V defined by h , hence $h \in \mathcal{H}_{2g}$. The given isomorphism $\lambda : V[d] \rightarrow A[d] = V[d]$ is symplectic, so there exists $g_f \in G(\hat{\mathbb{Z}})$ such that λ is induced by conjugation with g_f . We have already chosen an isomorphism between A and the fibre of $M_V \rightarrow M$ over the point $[(h, g_f)]$. The Chern classes of the two normalized totally symmetric $\mathbb{C}^\times$ -torsors correspond to each other, so by 10.4 they are isomorphic. Since μ is uniquely determined by λ and the commutativity of the diagram 10.3, the d -structures also correspond. Thus $(A, X, \tau, \lambda, \mu)$ is in the image of the morphism $M \rightarrow \underline{\mathcal{M}}_{d,\mathbb{C}}$, and the surjectivity is proved.

Every other isomorphism $\Gamma \rightarrow \Gamma_V$, such that E corresponds to the pairing $\Psi : V(\mathbb{Z}) \times V(\mathbb{Z}) \rightarrow U(\mathbb{Z}) = \mathbb{Z}$, must be the form $\text{int}(g) \circ \phi$ for some $g \in G(\mathbb{Z})$. The homomorphism h must then be replaced by $\text{int}(g) \circ h$ and g_f by $g \cdot g_f \bmod K_f$. But $[(\text{int}(g) \circ h, g \cdot g_f)] = [(h, g_f)]$ in M , this proves the injectivity. **q.e.d.**

10.10. Corollary: Let $X_d \rightarrow A_d \rightarrow \underline{\mathcal{M}}_d$ be the universal family, these are smooth quasiprojective schemes over $\mathbb{Q}$. The data in 10.8 determines isomorphisms $M_W(d) \rightarrow X_{d,\mathbb{C}}$, $M_V(d) \rightarrow A_{d,\mathbb{C}}$ and $M(d) \rightarrow \underline{\mathcal{M}}_{d,\mathbb{C}}$, compatible with all described structures.

Proof. This follows from 10.9 and the universal property of $\mathcal{M}_d$.

q.e.d.

We now study the relation of the modular interpretation of M_W with some of the maps of 3.4 (a).

10.11. Proposition: Let $d \geq 4$ be an even integer. Any element $w_f \in K_f^W(2)$ normalizes $K_f^P(d)$, and the automorphism of the scheme $X_{d,\mathbb{C}}$ corresponding to $[\cdot w_f] : M_W \rightarrow M_W$ can be described uniquely in terms of modular data.

Proof. Let $(u_f, v_f) = w_f$, and consider $w'_f = (u'_f, v'_f) \in K_f^W(d)$. We have $[w_f, w'_f] = \Psi(v_f, v'_f) \in U(\mathbb{A}_f)$, and since $v_f \in 2 \cdot V(\hat{\mathbb{Z}})$ and $v'_f \in d \cdot V(\hat{\mathbb{Z}})$, it follows that $\Psi(v_f, v'_f) \in 2d \cdot U(\hat{\mathbb{Z}}) \subset K_f^U(d)$. Thus w_f normalizes $K_f^W(d)$. Next let $g_f \in K_f(d)$, then

$$\begin{aligned} [w_f, g_f] &= w_f \cdot g_f(w_f)^{-1} = (u_f, v_f) \cdot (g_f(u_f), g_f(v_f))^{-1} \\ &= (u_f, v_f) \cdot (-g_f(u_f), -g_f(v_f)) \\ &= (u_f - g_f(u_f) - \frac{1}{2} \cdot \Psi(v_f, g_f(v_f)), v_f - g_f(v_f)). \end{aligned}$$

Since $g_f \equiv 1 \pmod{d}$ and $v_f \in 2 \cdot V(\hat{\mathbb{Z}})$, we have $v_f - g_f(v_f) \in 2d \cdot V(\hat{\mathbb{Z}})$ and $u_f - g_f(u_f) \in 2d \cdot U(\hat{\mathbb{Z}})$. Since Ψ is alternating, we have

$$\frac{1}{2} \cdot \Psi(v_f, g_f(v_f)) = \frac{1}{2} \cdot \Psi(v_f, g_f(v_f) - v_f) \in \frac{1}{2} \cdot 2 \cdot 2d \cdot U(\hat{\mathbb{Z}}) \subset d \cdot U(\hat{\mathbb{Z}}).$$

Thus $[w_f, g_f] \in K_f^W(d)$, which proves that w_f normalizes $K_f^P(d)$.

For the second assertion we write $w_f = (u_f, 0) \cdot (0, 2v_f)$ with $u_f \in 2 \cdot U(\hat{\mathbb{Z}})$ and $v_f \in V(\hat{\mathbb{Z}})$. If $v_f = 0$, then $[\cdot w_f]$ operates on the $\mathbb{C}^\times$ -torsor $M_W \rightarrow M_V$ by multiplication by $\mu(-[u_f]) \in \mathbb{C}^\times$, so this map is given in modular terms. From now on we assume $u_f = 0$.

As in the proof of 10.8 consider a fibre $X := \Gamma_W \backslash W(\mathbb{R}) \cdot U(\mathbb{C})$ of $M_W \rightarrow M$ over $[(y, g_f)]$, where $g_f \in G(\hat{\mathbb{Z}})$. By 3.15 $[\cdot w_f]$ operates on X by $[w'] \mapsto [w^{-1} \cdot w']$ for any $w \in W(\mathbb{Q})$ such that $g_f^{-1}(w) \equiv w_f \pmod{K_f^W}$. Let $v \in V(\mathbb{Z})$ such that $g_f^{-1}(v) \equiv v_f \pmod{d \cdot V(\hat{\mathbb{Z}})}$, then we can for instance choose $w = (0, 2v)$. The isomorphism $[\cdot w_f]$ then induces the translation $[v'] \mapsto [v' - 2v]$ on $A := \Gamma_V \backslash V(\mathbb{R})$, and since U is in the center of W , it commutes with the action of $\mathbb{C}^\times$. Now $[v] \in H(X)$, and every holomorphic isomorphism of $\mathbb{C}^\times$ -torsors $X \rightarrow X$ over the translation $T_{[-v]}$ is of the form $f : [w'] \mapsto [(u, v)^{-1} \cdot w']$ for some $u \in U(\mathbb{C})$. The isomorphism $[i] : X \rightarrow X$, $[(u', v')] \mapsto [(u', -v')]$ in the proof of 10.8 is the unique isomorphism $X \rightarrow X$ over the inversion map on A , which induces the identity map over the zero element of A . For arbitrary $u \in U(\mathbb{C})$ we have

$$[i]((u, v)) \cdot (u, v)^{-1} = (u, -v) \cdot (-u, -v) = (0, -2v) = (0, 2v)^{-1}.$$

Therefore $[i] \circ f^{-1} \circ [i] \circ f = [\cdot w_f]$ on X , and this isomorphism does not depend on the choice of u . This shows that $[\cdot w_f]$ is uniquely determined by the modular data (compare [M2] §2 p.319 remark 3).

q.e.d.

10.12. Lemma: Let $g_f \in G(\mathbb{A}_f)$. Then for every sufficiently divisible even integer $d \geq 4$ there exists an even integer $d' \geq 4$ such that

$$K_f^P(d') \subset g_f \cdot K_f^P(d) \cdot g_f^{-1} \subset K_f^P(2).$$

Proof. This follows immediately from the fact that the $K_f^P(d)$ are cofinal in the system of all open compact subgroups of $P(\mathbb{A}_f)$. **q.e.d.**

10.13. Proposition: Let g_f , d and d' as in 10.12, and define $d'' := d \cdot d'$. The morphism $X_{d'', \mathbb{C}} \rightarrow X_{d, \mathbb{C}}$ corresponding to the map $[\cdot g_f] : M_W(d'') \rightarrow M_W(d)$ can be described uniquely in terms of modular data.

Proof. By the inclusions $K_f^P(d'') \subset K_f^P(d') \subset g_f \cdot K_f^P(d) \cdot g_f^{-1}$ the map $[\cdot g_f] : M_W(d'') \rightarrow M_W(d)$ is defined. The fibre over a point $[(y, g'_f)] \in M(d'')(\mathbb{C})$ is mapped to the fibre over $[(y, g'_f \cdot g_f)] \in M(d)(\mathbb{C})$. Let $\Gamma_W'' := W(\mathbb{Q}) \cap g'_f(K_f^W(d''))$ and $\Gamma_W := W(\mathbb{Q}) \cap (g'_f \cdot g_f)(K_f^W(d))$, then with the identification of 3.13, $[\cdot g_f]$ is on these fibres given by

$$X'' := \Gamma_W'' \backslash W(\mathbb{R}) \cdot U(\mathbb{C}) \longrightarrow X := \Gamma_W \backslash W(\mathbb{R}) \cdot U(\mathbb{C}), \quad [w] \mapsto [w].$$

Consider the isomorphism

$$g_f(K_f^W(d))/K_f^W(d'') \xrightarrow{g'_f(\cdot)} (g'_f \cdot g_f)(K_f^W(d))/g'_f(K_f^W(d'')) \cong \Gamma_W/\Gamma_W''.$$

Let $[w_f] \in g_f(K_f^W(d))/K_f^W(d'')$, and $[w]$ its image in Γ_W/Γ_W'' . Since by assumption $g_f(K_f^W(d)) \subset K_f^W(2)$, it follows, as in the proof of 10.11, that $[\cdot w_f]$ is on X'' given by $([w'] \mapsto [w^{-1} \cdot w'])$. Hence $[\cdot g_f]$ induces an isomorphism $X''/g_f(K_f^W(d)) \rightarrow X$. By 10.11 this quotient is uniquely determined by the modular data on X'' . It remains to show that the other structures on X are determined by those X'' .

According to 3.16 we have the identification

$$(U(\mathbb{Q}) \cap \Gamma_W'') \backslash U(\mathbb{C}) \rightarrow \mathbb{C}^\times, \quad [u] \mapsto \exp\left(\frac{1}{d''} \cdot \lambda_{\psi(y)}(u)\right),$$

and since we have inverted the $\mathbb{C}^\times$ -torsor $M_W \rightarrow M_V$, this group operates on X'' by $[w] \mapsto [u^{-1} \cdot w]$. The analogous formula holds for X , and the identification lie in a commutative diagram

$$\begin{array}{ccc} (U(\mathbb{Q}) \cap \Gamma_W'') \backslash U(\mathbb{C}) & \xrightarrow{\sim} & \mathbb{C}^\times \\ [u] \mapsto [u] \downarrow & & \downarrow z \mapsto z^{d'} \\ (U(\mathbb{Q}) \cap \Gamma_W) \backslash U(\mathbb{C}) & \xrightarrow{\sim} & \mathbb{C}^\times, \end{array}$$

which is made such that the projection $X'' \rightarrow X$ is equivariant under the two actions. Hence the $\mathbb{C}^\times$ -action on X comes from that on X'' . Since $g_f \in G(\mathbb{A}_f)$, the map $[\cdot g_f]$ commutes with the section $[\tau]$, so the normalization of the $\mathbb{C}^\times$ -torsor X'' induces that of X . Let Γ_V and Γ_V'' be the images of Γ_W and Γ_W'' in $V(\mathbb{Q})$, and let $A := \Gamma_V \backslash V(\mathbb{R})$ and $A'' := \Gamma_V'' \backslash V(\mathbb{R})$. It remains to show that the symplectic d -structure on A comes from the symplectic d'' -structure on A'' .

The symplectic d'' -structure on A'' is uniquely determined by the isomorphism $\lambda : V[d''] \rightarrow A''[d'']$. Under the canonical identification

$$A''[d''] = \left(\frac{1}{d''} \cdot \Gamma_V''\right) / \Gamma_V'' \cong g'_f(V(\hat{\mathbb{Z}})) / g'_f(d'' \cdot V(\hat{\mathbb{Z}}))$$

this isomorphism is by 3.15 of the form

$$V[d''] = V(\hat{\mathbb{Z}})/d'' \cdot V(\hat{\mathbb{Z}}) \rightarrow g'_f(V(\hat{\mathbb{Z}}))/g'_f(d'' \cdot V(\hat{\mathbb{Z}})), [v_f] \mapsto [-g'_f(v_f)].$$

The analogous formula hold for A . Since $d'' = d \cdot d'$, and by assumption $d' \cdot V(\hat{\mathbb{Z}}) \subset g_f(d \cdot V(\hat{\mathbb{Z}}))$, the subgroup $\frac{1}{d} \cdot \Gamma_V''$ is already contained in Γ_V . Thus the isogeny $A'' \rightarrow A$ factors through the endomorphism $[d] : A'' \rightarrow A''$. So let us consider the new isogeny

$$A'' = \Gamma_V'' \backslash V(\mathbb{R}) \rightarrow A = \Gamma_V \backslash V(\mathbb{R}), [v] \mapsto [\frac{1}{d} \cdot v].$$

The inverse image of $A[d]$ under this map is $\Gamma_V/\Gamma_V'' = (g'_f \cdot g_f)(d \cdot V(\hat{\mathbb{Z}}))/g'_f(d'' \cdot V(\hat{\mathbb{Z}}))$. Since by assumption $g_f(d \cdot V(\hat{\mathbb{Z}})) \subset 2 \cdot V(\hat{\mathbb{Z}})$, we have $(g'_f \cdot g_f)(d \cdot V(\hat{\mathbb{Z}})) \subset g'_f(V(\hat{\mathbb{Z}}))$, so this inverse image lies in $A''[d'']$. For the d'' -structure on A'' and the d -structure on A we therefore have the commutative diagram

$$\begin{array}{ccc} [d \cdot g_f(v_f)] \in V[d''] & \longrightarrow & A''[d''] \hookrightarrow A'' \ni [v] \\ \uparrow & & \downarrow \\ v[f] \in V(\hat{\mathbb{Z}}) & \xrightarrow{\text{id}} & V[d] \longrightarrow A[d] \hookrightarrow A \ni [\frac{1}{d} \cdot v]. \end{array}$$

This implies the assertion. **q.e.d.**

10.14. Corollary: The automorphism of $X_{d,\mathbb{C}}$ in 10.11, and the morphism $X_{d'',\mathbb{C}} \rightarrow X_{d,\mathbb{C}}$ in 10.13, both descend to morphisms of schemes over $\mathbb{Q}$.

Proof. By 10.11, 10.13, and the universal property of $X_d \rightarrow \underline{\mathcal{M}}_d$. **q.e.d.**

Remark. $P_{2g}(\mathbb{A}_f)$ is generated by $K_f^W(2)$ together with $CS_{P_{2g}}(\mathbb{A}_f)$. Thus, for our purposes, the corollary holds for sufficiently many maps.

We now want to obtain the same results for the mixed Shimura data $(P_0, \mathcal{X}_0)$ defined in 2.24. This is, mutatis mutandum, a special case of the above.

10.15. A moduli scheme for roots of unity: For any positive integer d , let $\mu_{d,\mathbb{Q}} \subset \mathbb{G}_{m,\mathbb{Q}}$ be the kernel of the homomorphism $t \rightarrow t^d$. Let $\underline{M}_d^0 \subset \mathbb{G}_{m,\mathbb{Q}}$ be the reduced closed subscheme $\mu_{d,\mathbb{Q}} - \bigcup_{d'|d, d' < d} \mu_{d',\mathbb{Q}}$ of all primitive d^{th} roots of unity. For any scheme S over $\mathbb{Q}$, it is equivalent to give an S -valued point on $\underline{M}_d^0$, or to give an isomorphism $(\mathbb{Z}/d \cdot \mathbb{Z}) \times S \cong \mu_{d,S}$. In other words, $\underline{M}_d^0$ represents the functor

$$\underline{\text{Isom}}((\mathbb{Z}/d \cdot \mathbb{Z}) \times \mathbf{Spec}(\mathbb{Q}), \mu_{d,\mathbb{Q}}).$$

Consider the mixed Shimura data $(P_0, \mathcal{X}_0)$ defined in 2.24. Let

$$\begin{aligned} K_f^U(d) &:= d \cdot \hat{\mathbb{Z}} \subset \mathbb{A}_f = U_0(\mathbb{A}_f) \text{ and} \\ K_f(d) &:= \{t_f \in \mathbb{G}_m(\hat{\mathbb{Z}}) \mid t_f \equiv 1 \pmod{d}\}, \end{aligned}$$

then $K_f^P(d) := K_f^U(d) \rtimes K_f(d)$ is an open compact subgroup of $P_0(\mathbb{A}_f)$. Every $K_f^P(d)$ is a normal subgroup of $K_f^P(1)$. Put

$$\begin{aligned} M^0(d) &:= M_{\mathbb{C}}^{K_f(d)}(\mathbb{G}_{m,\mathbb{Q}}, \mathcal{H}_0) \text{ and} \\ M_U^0(d) &:= M_{\mathbb{C}}^{K_f^P(d)}(P_0, \mathcal{X}_0). \end{aligned}$$

The canonical isomorphism $M_U^0(d)(\mathbb{C}) \cong \mathbb{C}^\times \times M^0(d)(\mathbb{C})$ defined in 3.6 gives an isomorphism $M_U^0(d) \xrightarrow{\sim} \mathbb{G}_m \times M^0(d)$. Since $K_f^U(1)/K_f^U(d) = \hat{\mathbb{Z}}/d \cdot \hat{\mathbb{Z}} \cong \mathbb{Z}/d \cdot \mathbb{Z}$, 3.15 defines a monomorphism $(\mathbb{Z}/d \cdot \mathbb{Z}) \times M^0(d) \hookrightarrow M_U^0(d) \xrightarrow{\sim} \mathbb{G}_m \times M^0(d)$, whence an isomorphism

$$(\mathbb{Z}/d \cdot \mathbb{Z}) \times M^0(d) \xrightarrow{\sim} \mu_d \times M^0(d).$$

By the universal property of $\underline{M}_d^0$, this defines a morphism $M^0(d) \rightarrow \underline{M}_{d,\mathbb{C}}^0$. Defining $\underline{X}_d^0 := \mathbb{G}_{m,\mathbb{Q}} \times \underline{M}_d^0$, we in turn get a morphism $M_U^0(d) \rightarrow \underline{X}_{d,\mathbb{C}}^0$.

10.16. Proposition: (a) The morphisms $M^0(d) \rightarrow \underline{M}_{d,\mathbb{C}}^0$ and $M_U^0(d) \rightarrow \underline{X}_{d,\mathbb{C}}^0$ are isomorphisms.

(b) Let $p_f \in K_f^U(1) \cup \mathbb{G}_m(\mathbb{A}_f)$, and d', d such that the map $[\cdot p_f] : M_U^0(d') \rightarrow M_U^0(d)$ of 3.4 (a) is defined. Then with the identifications in (a) this map corresponds to a morphism $\underline{X}_{d'}^0 \rightarrow \underline{X}_d^0$ over $\mathbb{Q}$.

Proof. (a) Let $e : (\mathbb{G}_{m,\mathbb{Q}}, \mathcal{H}_0) \hookrightarrow (P_0, \mathcal{X}_0)$ be the given section, then the isomorphism $M_U^0(d)(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}^\times \times M^0(d)(\mathbb{C})$ of 3.16 is given by

$$[(u \cdot e(y), e(t_f))] \mapsto (\exp(\frac{1}{d} \cdot \lambda_y(u)), [(y, t_f)])$$

for $u \in U(\mathbb{C})$ and $t \in \mathcal{H}_0$, and $t_f \in \mathbb{G}_m(\hat{\mathbb{Z}})$. Using 3.15 it follows that the isomorphism $(\mathbb{Z}/d \cdot \mathbb{Z}) \times M^0(d) \xrightarrow{\sim} \mu_d \times M^0(d)$ is given by

$$([u], [(y, t_f)]) \mapsto (\exp(\frac{1}{d} \cdot \lambda_y(u))^{t_f}, [(y, t_f)])$$

for $[u] \in \mathbb{Z}/d \cdot \mathbb{Z}$, $y \in \mathcal{H}_0$, and $t_f \in \mathbb{G}_m(\hat{\mathbb{Z}})$. Clearly every isomorphism $\mathbb{Z}/d \cdot \mathbb{Z} \cong \mu_d(\mathbb{C})$ occurs in precisely on fibre over $M^0(d)(\mathbb{C})$, so the map $M^0(d)(\mathbb{C}) \rightarrow \underline{M}_d^0(\mathbb{C})$ is a bijection. Since both are normal reduced schemes, $M^0(d) \rightarrow \underline{M}_{d,\mathbb{C}}^0$ must be an isomorphism. This implies (a).

(b) If $p_f = u_f \in \hat{\mathbb{Z}} = K_f^U(1)$, then by the definition 3.15 the map $[\cdot u_f] : M_U^0(d) \rightarrow M_U^0(d)$ corresponds to the automorphism $(t, \zeta) \mapsto (t \cdot \zeta^{u_f}, \zeta)$ of $\mathbb{G}_{m,\mathbb{Q}} \times \underline{M}_d^0$. Next if $p_f = e(t_f)$ with $t_f \in \mathbb{G}_m(\hat{\mathbb{Z}})$, then the formulas in (a) show that the map $[\cdot e(t_f)] : M_U^0(d) \rightarrow M_U^0(d)$ corresponds to the automorphism $(t, \zeta) \mapsto (t, \zeta^{t_f})$ of $\mathbb{G}_{m,\mathbb{Q}} \times \underline{M}_d^0$. It remains to consider the case $p_f = e(t)$ with $t \in \mathbb{Q}^{>0} \subset \mathbb{G}_m(\mathbb{Q})$. Since $e(t) \cdot K_f^P(d) \cdot e(t)^{-1} = K_f^P(d)(t \cdot d)$, d' must be a multiple of $t \cdot d$. On $\mathbb{C}^\times \times M^0(d)(\mathbb{C})$ the map $[\cdot e(t)] : M_U^0(d') \rightarrow M_U^0(d)$ is given by

$$\begin{aligned} (z, [(y, t_f)]) &\hat{=} [(\lambda_y^{-1}(d' \cdot \log(z)) \cdot e(y), e(t_f))] \\ &\mapsto [(\lambda_y^{-1}(d' \cdot \log(z)) \cdot e(y), e(t_f) \cdot e(t))] \\ &= [(e(t)^{-1} \cdot \lambda_y^{-1}(d' \cdot \log(z)) \cdot e(y), e(t_f))] \\ &= [(t^{-1}(\lambda_y^{-1}(d' \cdot \log(z))) \cdot e(y), e(t_f))] \\ &\hat{=} (\exp(\frac{1}{d} \cdot \lambda_y(\frac{1}{t} \cdot \lambda_y^{-1}(d' \cdot \log(z)))), [(y, t_f)]) \\ &= (z^{d''}, [(y, t_f)]), \end{aligned}$$

where $d'' = \frac{d'}{d \cdot t} \in \mathbb{Z}$. Thus it corresponds to the morphism

$$\mathbb{G}_{m,\mathbb{Q}} \times \underline{M}_{d'}^0 \rightarrow \mathbb{G}_{m,\mathbb{Q}} \times \underline{M}_d^0, (t, \zeta) \mapsto (t^{d''}, \zeta^{d'/d}).$$

In each of the three cases the morphism $\underline{X}_{d'}^0 \rightarrow \underline{X}_d^0$ is defined over $\mathbb{Q}$, as desired. **q.e.d.**

For the rest of this chapter we shall do something similar for the compactification of a mixed Shimura variety. In a very special case, we shall give a modular interpretation for the isomorphism 7.17. The following construction is well-known.

10.17. The Tate curve as a torus embedding: Consider the projection $pr_1 : \mathbb{G}_{m,\mathbb{Q}}^2 \rightarrow \mathbb{G}_{m,\mathbb{Q}}$, we may consider this as the (relative) commutative group scheme obtained from $\mathbb{G}_{m,\mathbb{Q}} \rightarrow \mathbf{Spec}(\mathbb{Q})$ by base change $(\) \times_{\mathbb{Q}} \mathbb{G}_{m,\mathbb{Q}}$. The torus embedding associated to $\sigma := \mathbb{R}^{\geq 0} \subset \mathbb{R} = Y_*(\mathbb{G}_{m,\mathbb{Q}})_{\mathbb{R}}$ is just $\mathbb{G}_{m,\mathbb{Q}} \hookrightarrow \mathbb{A}_{\mathbb{Q}}^1$. Let $\mathcal{T}$ be the partial cone decomposition of $\mathbb{R}^2 = Y_*(\mathbb{G}_{m,\mathbb{Q}}^2)_{\mathbb{R}}$ that consists of the cones $\mathbb{R}^{\geq 0} \cdot (1, n) + \mathbb{R}^{\geq 0} \cdot (1, n+1)$ for all $n \in \mathbb{Z}$, and their faces; as illustrated in the diagram:

Let $\mathcal{T}^r \subset \mathcal{T}$ be the subset of all cones of dimension ≤ 1 . Write $Z := (\mathbb{G}_{m,\mathbb{Q}}^2)_{\mathcal{T}}$ and $Z^r := (\mathbb{G}_{m,\mathbb{Q}}^2)_{\mathcal{T}^r}$.

By the criterion 5.4 for the functoriality of torus embeddings, the projection $\mathbb{G}_{m,\mathbb{Q}}^2 \rightarrow \mathbb{G}_{m,\mathbb{Q}}$ extends to a morphism $Z \rightarrow \mathbb{A}_{\mathbb{Q}}^1$. The inverse image of $\mathbb{G}_{m,\mathbb{Q}} \subset \mathbb{A}_{\mathbb{Q}}^1$ is just $\mathbb{G}_{m,\mathbb{Q}}^2$, but the fibre over 0 is a infinite sequence of projective lines, with the point 0 of each component glued to the point ∞ of the next one. It is easily checked that the morphism $Z \rightarrow \mathbb{A}_{\mathbb{Q}}^1$ is flat. The open subscheme, on which it is smooth, is just Z^r . The structure of commutative group scheme on $pr_1 : \mathbb{G}_{m,\mathbb{Q}}^2 \rightarrow \mathbb{G}_{m,\mathbb{Q}}$ is given by the identity section, the inversion, and the group operation. These are

$$\begin{array}{lll} \mathbb{G}_{m,\mathbb{Q}} \rightarrow \mathbb{G}_{m,\mathbb{Q}}, & s & \mapsto (s, 1), \\ \mathbb{G}_{m,\mathbb{Q}}^2 \rightarrow \mathbb{G}_{m,\mathbb{Q}}, & (s, t) & \mapsto (s, t^{-1}), \\ \mathbb{G}_{m,\mathbb{Q}}^2 \times_{\mathbb{G}_{m,\mathbb{Q}}} \mathbb{G}_{m,\mathbb{Q}}^2 \cong \mathbb{G}_{m,\mathbb{Q}}^3 \rightarrow \mathbb{G}_{m,\mathbb{Q}}^2, & (s, t_1, t_2) & \mapsto (s, t_1 \cdot t_2) \end{array}$$

respectively. We claim that they extend to morphisms

$$\mathbb{A}_{\mathbb{Q}}^1 \rightarrow Z^r, Z^r \rightarrow Z^r, \text{ and } Z^r \times_{\mathbb{A}_{\mathbb{Q}}^1} Z^r \rightarrow Z^r \text{ and } Z^r \times_{\mathbb{A}_{\mathbb{Q}}^1} Z \rightarrow Z$$

respectively. In fact, for the first two morphisms this is a direct application of 5.4. It is easily checked that the embedding

$$\mathbb{G}_{m,\mathbb{Q}}^3 \cong \mathbb{G}_{m,\mathbb{Q}}^2 \times_{\mathbb{G}_{m,\mathbb{Q}}} \mathbb{G}_{m,\mathbb{Q}}^2 \hookrightarrow Z^r \times_{\mathbb{A}_{\mathbb{Q}}^1} Z$$

is the torus embedding with respect to the partial cone decomposition of $\mathbb{R}^3$, consisting of the cones

$$\mathbb{R}^{\geq 0} \cdot (1, m, n) + \mathbb{R}^{\geq 0} \cdot (1, m, n+1)$$

for all $m, n \in \mathbb{Z}$, and their faces. Since such a cone maps to the cone

$$\mathbb{R}^{\geq 0} \cdot (1, m, n) + \mathbb{R}^{\geq 0} \cdot (1, m + n + 1) \in \mathcal{T},$$

by functoriality we get a morphism $Z^r \times_{\mathbb{A}_{\mathbb{Q}}^1} Z \rightarrow Z$. The same argument applies to $Z^r \times_{\mathbb{A}_{\mathbb{Q}}^1} Z^r \rightarrow Z^r$. Since $\mathbb{G}_{m, \mathbb{Q}}^2$ is dense in Z^r , the group axioms for $pr_1 : \mathbb{G}_{m, \mathbb{Q}}^2 \rightarrow \mathbb{G}_{m, \mathbb{Q}}$ extend, so $Z^r \rightarrow \mathbb{A}_{\mathbb{Q}}^1$ is a commutative group scheme. Likewise, the morphism $Z^r \times_{\mathbb{A}_{\mathbb{Q}}^1} Z \rightarrow Z$ defines a group action. ($Z^r \rightarrow \mathbb{A}_{\mathbb{Q}}^1$ is the Néron-model of $\mathbb{G}_m$, but we will not need this fact.)

Consider the action of $a \in \mathbb{Z}$ on $\mathbb{G}_{m, \mathbb{Q}}^2$ through $(s, t) \mapsto (s, s^a \cdot t)$. Clearly $\mathcal{T}$ is invariant under these substitutions, and $\mathbb{Z}$ cyclically and transitively permutes the irreducible components of the special fibre. The identity component of the special fibre is again $\mathbb{G}_{m, \mathbb{Q}}$, so we get a canonical group isomorphism between the special fibre of $Z^r \rightarrow \mathbb{A}_{\mathbb{Q}}^1$ and $\mathbb{Z} \times \mathbb{G}_{m, \mathbb{Q}}$.

Let $\mathcal{Z}$ be the formal completion of the special fibre in Z . For any positive integer d we can form the quotient $\mathcal{Z}/d \cdot \mathbb{Z}$. This is formally proper and flat of relative dimension 1 over the completion of $\mathbb{A}_{\mathbb{Q}}^1$ in $\{0\}$, so the formal scheme is algebraizable, yielding a proper flat scheme $E \rightarrow \mathbf{Spec} \mathbb{Q}[[s]]$. Letting $E^r \subset E$ be the open subscheme which is smooth over $\mathbf{Spec} \mathbb{Q}[[s]]$, we obtain the structure of a commutative group scheme on $E^r \rightarrow \mathbf{Spec} \mathbb{Q}[[s]]$, and the action $E^r \times_{\mathbf{Spec} \mathbb{Q}[[s]]} E \rightarrow E$. This is a generalized elliptic curve in the sense of [DR] II.1.12. It is easy to show that $E \rightarrow \mathbf{Spec} \mathbb{Q}[[s]]$ is the universal deformation of the closed fibre as a generalized elliptic curve (compare [DR] III.1.2).

The kernel of the multiplication by d on $\mathcal{Z}/d \cdot \mathbb{Z}$ is the formal scheme associated to $(\mathbb{Z}/d \cdot \mathbb{Z}) \times \mu_d$, so $E^r[d]$ is canonically isomorphic to

$$(\mathbb{Z}/d \cdot \mathbb{Z}) \times \mu_d \times \mathbf{Spec} \mathbb{Q}[[s]] \rightarrow \mathbf{Spec} \mathbb{Q}[[s]].$$

A d -structure on $E \rightarrow \mathbf{Spec} \mathbb{Q}[[s]]$, in the sense of [DR] IV.2.3, is an isomorphism $(\mathbb{Z}/d \cdot \mathbb{Z})^2 \cong E^r[d]$. Thus giving a d -structure is in our case equivalent to giving an isomorphism $\mathbb{Z}/d \cdot \mathbb{Z} \cong \mu_d$. In particular, if $\underline{M}_d^0$ is as in 10.15, there is a canonical d -structure on $E \times \underline{M}_d^0 \rightarrow \mathbf{Spec} \mathbb{Q}[[s]] \times \underline{M}_d^0$. Again, this is the universal deformation of the closed fibre as a generalized elliptic curve with d -structure.

10.18. The Tate curve in terms of mixed Shimura varieties: Consider the mixed Shimura data $(P_0, \mathcal{X}_0)$ defined as in 2.24. Let $(\tilde{P}_0, \tilde{\mathcal{X}}_0)$ be the fibre product of $(P_0, \mathcal{X}_0)$ with itself over $(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$; we have canonically $\tilde{P}_0 \cong (U_0 \times U_0) \rtimes \mathbb{G}_{m, \mathbb{Q}}$. Consider the open compact subgroups

$$\begin{aligned} K_f^{U_0} &:= d \cdot \hat{\mathbb{Z}} \subset \mathbb{A}_f = U_0(\mathbb{A}_f), \\ K_f^{\mathbb{G}_m} &:= \{t \in \hat{\mathbb{Z}}^\times \mid t_f \equiv 1 \pmod{d}\}, \\ K_f^{P_0} &:= K_f^{U_0} \rtimes K_f^{\mathbb{G}_m} \subset P_0(\mathbb{A}_f), \text{ and} \\ K_f^{\tilde{P}_0} &:= (K_f^{U_0} \times K_f^{U_0}) \rtimes K_f^{\mathbb{G}_m} \subset \tilde{P}_0(\mathbb{A}_f). \end{aligned}$$

With the identification of 10.15 we have canonical isomorphisms

$$\begin{aligned} M_U^0 &:= M_C^{K_f^{P_0}}(P_0, \mathcal{X}_0) \cong \mathbb{G}_m \times \underline{M}_{d, \mathbb{C}}^0, \\ \tilde{M}_U^0 &:= M_C^{K_f^{\tilde{P}_0}}(\tilde{P}_0, \tilde{\mathcal{X}}_0) \cong \mathbb{G}_m^2 \times \underline{M}_{d, \mathbb{C}}^0. \end{aligned}$$

By 6.8–9, the partial cone decompositions $\mathcal{T}^r \subset \mathcal{T}$ of 10.17 correspond to unique $K_f^{\tilde{P}_0}$ -admissible partial cone decompositions $\Sigma_0^r \subset \Sigma_0$ for $(\tilde{P}_0, \tilde{\mathcal{X}}_0)$. Thus if $\bar{\Sigma}_0$ denotes the cone decomposition for $(P_0, \mathcal{X}_0)$ considered in 6.9, we get canonical isomorphisms

$$\begin{array}{ccc} \overline{\tilde{M}}_U^{0,r} & := M_{\mathbb{C}}^{K_f^{\tilde{P}_0}}(\tilde{P}_0, \tilde{\mathcal{X}}_0, \Sigma_0^r) \cong & Z^r \times \underline{M}_{d,\mathbb{C}}^0 \\ \cap & & \cap \\ \overline{\tilde{M}}_U^0 & := M_{\mathbb{C}}^{K_f^{\tilde{P}_0}}(\tilde{P}_0, \tilde{\mathcal{X}}_0, \Sigma_0) \cong & Z \times \underline{M}_{d,\mathbb{C}}^0 \\ \downarrow & & \downarrow \\ \overline{M}_U^0 & := M_{\mathbb{C}}^{K_f^{P_0}}(P_0, \mathcal{X}_0, \bar{\Sigma}_0) \cong & \mathbb{A}^1 \times \underline{M}_{d,\mathbb{C}}^0. \end{array}$$

Recall (2.21, 3.12 (a)) that the group structure on $\tilde{M}_U^0 \rightarrow M_U^0$ is given in terms of maps 3.4 (b). We can directly translate 10.17: the condition of 6.25 (b) holds for each of the necessary maps. The action of $\mathbb{Z}$ on $Z \times \underline{M}_{d,\mathbb{C}}$ corresponds to the action on $\overline{\tilde{M}}_U^0$ induced by the maps $[\phi_a]$, where ϕ_a is the automorphism of $(\tilde{P}_0, \tilde{\mathcal{X}}_0)$ given by

$$\tilde{P}_0 = (U_0 \times U_0) \rtimes \mathbb{G}_{m,\mathbb{Q}} \ni (u_1, u_2, t) \mapsto (u_1, a \cdot u_1 + u_2, t).$$

By 10.15 the canonical isomorphism $\mathbb{Z}/d \cdot \mathbb{Z} \cong \mu_d$ over $\underline{M}_{d,\mathbb{C}}$ can be expressed in terms of maps 3.4 (a). Thus 10.17 shows that, over the formal completion of $\overline{M}_U^0$ along the boundary $\overline{M}_U^0 \setminus M_U^0$, we have a canonical generalized elliptic curve with d -structure, and this structure can be defined purely in terms of maps 6.25.

We now “glue” the objects just constructed into the boundary of other objects.

10.19. A certain toroidal compactification: Consider the Shimura data $(\mathrm{GL}_{2,\mathbb{Q}}, \mathcal{H}_2)$ defined in 2.7, and its unipotent extension $(P, \mathcal{X}) := (V_2 \times \mathrm{GL}_{2,\mathbb{Q}}, \mathcal{H}'_2)$, defined in 2.25. Here V_2 is just the standard 2-dimensional representation of $\mathrm{GL}_{2,\mathbb{Q}}$. For every positive integer d we let

$$\begin{aligned} K_f &= K_f(d) := \{g \in \mathrm{GL}_2(\hat{\mathbb{Z}}) \mid g \equiv 1 \pmod{d}\}, \\ K_f^V &= K_f^V(d) := d \cdot \hat{\mathbb{Z}}^2, \text{ and} \\ K_f^P &= K_f^P(d) := K_f^V \rtimes K_f, \end{aligned}$$

and define

$$\begin{aligned} M &:= M_{\mathbb{C}}^{K_f}(\mathrm{GL}_{2,\mathbb{Q}}, \mathcal{H}_2), \\ M_V &:= M_{\mathbb{C}}^{K_f^P}(V_2 \rtimes \mathrm{GL}_{2,\mathbb{Q}}, \mathcal{H}'_2). \end{aligned}$$

As explained in 4.15, we identify the mixed Shimura data $(P_0, \mathcal{X}_0)$ with a rational boundary component of $(\mathrm{GL}_{2,\mathbb{Q}}, \mathcal{H}_2)$. Every proper rational boundary component is $\mathrm{GL}_2(\mathbb{Q})$ -conjugate to this one. The rational boundary component is $(V_2 \rtimes \mathrm{GL}_{2,\mathbb{Q}}, \mathcal{H}'_2)$ associated to $(P_0, \mathcal{X}_0)$ is isomorphic to $(\tilde{P}_0, \tilde{\mathcal{X}}_0)$. Embedding all our groups into $\mathrm{GL}_{3,\mathbb{Q}}$ we have

$$P \cong \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{P}_0 \cong \begin{pmatrix} * & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q \cong \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix},$$

where Q is the normalizer of $\tilde{P}_0$. We identify the unipotent radical of $\tilde{P}_0$ with $\mathbb{G}_{a,\mathbb{Q}}^2$ by

$$(x, y) \longleftrightarrow \begin{pmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The cone decomposition of 10.18 correspond canonically to $K_f^P(1)$ -admissible cone decompositions for $(P, \mathcal{X})$. In fact, since $P(\mathbb{A}_f) = Q(\mathbb{Q}) \cdot K_f^P(1)$, there are precisely two conjugacy classes of (P_1, p_f) modulo the actions 6.4 (ii) and (iii). One of these corresponds to $P_1 = P$, for which there is nothing to specify. Since the other is represented by $(\tilde{P}_0, 1)$, any cone decomposition Σ for $(P, \mathcal{X})$ is determined by $\Sigma|_{(\tilde{P}_0, \tilde{\mathcal{X}}_0)}$. By 4.26, the cone $C^*(\mathcal{X}^0, \tilde{P}_0)$ corresponds to $\{0\} \cup \mathbb{R}^{>0} \times \mathbb{R}$ under the above identification. Moreover, $\Gamma := \text{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^0) \cap \tilde{P}_0(\mathbb{A}_f) \cdot K_f^P(1)$ acts on this through

$$(x, y) \mapsto b \cdot (x, \pm y + ax) \text{ for } a \in \mathbb{Z} \text{ and } b \in \mathbb{Q}^{>0}.$$

Thus there are unique $K_f^P(1)$ -admissible partial cone decompositions $\Sigma^r \subset \Sigma$ for $(P, \mathcal{X})$ with $\Sigma_{|(\tilde{P}_0, \tilde{\mathcal{X}}_0)}^r = \Sigma_0^r$ and $\Sigma_{|(\tilde{P}_0, \tilde{\mathcal{X}}_0)} = \Sigma_0$, and the latter is complete. There exists a unique complete $K_f(1)$ -admissible cone decomposition for $(\text{GL}_{2,\mathbb{Q}}\mathcal{H}_2)$, which we denote by $\bar{\Sigma}$. Of course, all these cone decompositions remain admissible for $K_f^P = K_f^0(d)$, resp. $K_f = K_f(d)$. We define

$$\begin{aligned} \overline{M}_V &:= M_{\mathbb{C}}^{K_f^P}(V_2 \rtimes \text{GL}_{2,\mathbb{Q}}, \mathcal{H}'_2, \Sigma), \\ \overline{M}_V^r &:= M_{\mathbb{C}}^{K_f^P}(V_2 \rtimes \text{GL}_{2,\mathbb{Q}}, \mathcal{H}'_2, \Sigma^r), \text{ and} \\ \overline{M} &:= M_{\mathbb{C}}^{K_f}(\text{GL}_{2,\mathbb{Q}}, \mathcal{H}_2, \bar{\Sigma}) = M_{\mathbb{C}}^{K_f}(\text{GL}_{2,\mathbb{Q}}, \mathcal{H}_2)^*. \end{aligned}$$

(By 9.39 (b) we know that $\overline{M}_V$ is a projective variety for every d , but we shall not need this fact.)

10.20. The moduli scheme of generalized elliptic curves in terms of mixed Shimura varieties: From now on we assume that $d \geq 3$. Then the fibres of $M_V \rightarrow M$ are elliptic curves. Consider the boundary stratum $M_{\mathbb{C}}^{K_f^{\mathbb{G}_m}}(\mathbb{G}_{m,\mathbb{Q}}, \mathcal{H}_0)$ of $\overline{M}$, associated to $(P_0, \mathcal{X}_0)$ and $p_f = 1$ (see 6.3). This is canonically identified with the unique proper boundary stratum of $\overline{M}_U^0$, and 9.37 yields an isomorphism between its formal neighborhoods in $\overline{M}$ and $\overline{M}_U^0$ respectively. Likewise, for the rational boundary component $(\tilde{P}_0, \tilde{\mathcal{X}}_0)$ of $(P, \mathcal{X})$, and $p_f = 1$, the group Δ_1 of 7.3 acts on $\overline{M}_U^0$ just like the group $d \cdot \mathbb{Z}$ in 10.18. Thus, by 7.17, the inverse image of $M_{\mathbb{C}}^{K_f^{\mathbb{G}_m}}(\mathbb{G}_{m,\mathbb{Q}}, \mathcal{H}_0)$ of $\overline{M}_V \rightarrow \overline{M}$ is isomorphic to $(\overline{M}_U^0 \setminus \tilde{M}_U^0)/d \cdot \mathbb{Z}$, and 9.37 yields an isomorphism between its formal neighborhood in $\overline{M}_V$ and the quotient by $d \cdot \mathbb{Z}$ of the formal neighborhood of $\overline{M}_U^0 \setminus \tilde{M}_U^0$ in $\overline{M}_U^0$.

Recall (10.7–8) that the structure of $M_V \rightarrow M$ as an elliptic curve with d -structure can be defined purely in terms of the maps 3.4. As in 10.18, we find that this structure extends and makes $\overline{M}_V \rightarrow \overline{M}$ into a generalized elliptic curve with d -structure.

Now let $\overline{\mathcal{M}}_d$ be the moduli scheme of generalized elliptic curves over $\mathbb{Q}$ with d -structure. By [DR] IV.2.9, it is smooth, projective, of dimension 1 over $\mathbb{Q}$. Consider the moduli scheme $\mathcal{M}_d$ in 10.6 for $g = 1$. One easily checks that, for $g = 1$, the data $(A, X, \tau, \lambda, \mu)$ in 10.5 is determined up to unique isomorphism by (A, λ) . Thus the forgetful functor induces an open embedding $\mathcal{M} \hookrightarrow \overline{\mathcal{M}}_d$, and the image is Zariski-dense. By the universal property of $\overline{\mathcal{M}}_d$, the structures defined above induce a morphism $\overline{M} \rightarrow \overline{\mathcal{M}}_{d, \mathbb{C}}$. This morphism extends the isomorphism $M \xrightarrow{\sim} \overline{\mathcal{M}}_{d, \mathbb{C}}$ of 10.9. It is easy to see that $\overline{M} \rightarrow \overline{\mathcal{M}}_{d, \mathbb{C}}$ is an isomorphism (for instance because $\overline{M}$ is also proper and smooth). If $E_d \rightarrow \overline{\mathcal{M}}_d$ is the universal family, then this also induces an isomorphism $\overline{M}_V \cong E_{d, \mathbb{C}}$.

10.21 Summary of 10.17–20: To tie everything up, consider the generalized elliptic curve with d -structure, constructed in 10.17. By the universal property of $\overline{\mathcal{M}}_d$, it defines a morphism $\mathbf{Spec} \mathbb{Q}[[s]] \times \mathcal{M}_d^0 \rightarrow \overline{\mathcal{M}}_d$. Since the former is a universal deformation, this gives an isomorphism of the completion of $\overline{\mathcal{M}}_d$ along $\mathcal{M}_d^0 \hookrightarrow \overline{\mathcal{M}}_d$.

In summary, the different objects and morphisms we have constructed form the following diagram.

Here $\mathcal{FC}()$ denotes formal completion along a closed subscheme that was specified above. All diagonal arrows are isomorphisms. These, and the arrows in the foreground, are all defined in terms of modular data, and the latter descend to $\mathbb{Q}$. The arrows in the background have been defined in terms of maps 6.25. By the universal property of the moduli schemes involved, the whole diagram is commutative! We have thus proved the following proposition.

10.22 Proposition: With the identification $\overline{M} \cong \overline{\mathcal{M}}_{d, \mathbb{C}}$ of 10.20, and $\overline{M}_U^0 \cong \mathbb{A}^1 \times \mathcal{M}_{d, \mathbb{C}}^0$

of 10.18, the isomorphism 9.37 of the formal completions of $\overline{M}$ and $\overline{M}_U^0$ corresponds to an isomorphism of the formal completions of $\overline{\mathcal{M}}_d$ and $\mathbb{A}^1 \times \underline{M}_{d,\mathbb{C}}^0$, that is defined over $\mathbb{Q}$.

Chapter 11

Canonical models

We introduce the concept of canonical models, and prove the existence and uniqueness of canonical models for all mixed Shimura varieties. The existence is proved by reduction to the case of usual Shimura varieties, where it is known by [Mi1] thm. 7.2. A central role is played by the modular interpretation of certain mixed Shimura varieties, and by the main theorem of Shimura and Taniyama about abelian varieties with complex multiplication. In the whole chapter, we closely follow [D1].

After defining canonical models (11.1–5), we study their functorial behaviour. In 11.9 we show that if a canonical model exists for some mixed Shimura data, then this induces a canonical model on any embedded mixed Shimura variety. This implies functoriality (11.10), and in particular uniqueness (11.11) of canonical models. We prove the existence of canonical models for those mixed Shimura varieties, whose modular interpretation was studied in the previous chapter (11.14–16). The main theorem (11.18) on the existence of canonical models in general is then reduced to [Mi1]thm. 7.2.

11.1. The reflex field: Let $(P, \mathcal{X})$ be mixed Shimura data. For every $x \in \mathcal{X}$ consider the homomorphism $h_x \circ \mu : \mathbb{G}_{m, \mathbb{C}} \hookrightarrow \mathbb{S}_{\mathbb{C}} \rightarrow P_{\mathbb{C}}$, where $\mu : \mathbb{G}_{m, \mathbb{C}} \hookrightarrow \mathbb{S}_{\mathbb{C}}$ is the homomorphism defined in 1.3. Since the different h_x are all conjugate under $P(\mathbb{C})$, we get a conjugacy of cocharacters of $P_{\mathbb{C}}$ that depends only on P and $\mathcal{X}$. The automorphism group of $\mathbb{C}$ acts continuously on the (discrete) set of all such conjugacy classes, so the given conjugacy class is defined over some number field $E \subset \mathbb{C}$.

Definition. The reflex field of $(P, \mathcal{X})$, denoted by $E(P, \mathcal{X})$, is the field of definition of the conjugacy class of $h_x \circ \mu$ for any $x \in \mathcal{X}$.

- 11.2. Properties:**
- (a) It is important that the reflex field is given as a subfield of $\mathbb{C}$.
 - (b) For any morphism of mixed Shimura data $(P_1, \mathcal{X}_1) \rightarrow (P_2, \mathcal{X}_2)$ we have $E(P_1, \mathcal{X}_1) \supset E(P_2, \mathcal{X}_2)$. This follows at once from the fact that the map from the set of conjugacy classes of cocharacters of $P_{1, \mathbb{C}}$ to the set of conjugacy classes of cocharacters of $P_{2, \mathbb{C}}$ is invariant under $\text{Aut}(\mathbb{C})$.
 - (c) The set of all conjugacy classes of cocharacters of $P_{\mathbb{C}}$ depends only on P/W . Thus $E(P, \mathcal{X}) = E((P, \mathcal{X})/W) = E((P, h(\mathcal{X}))/W)$. In particular, $E(P_1, \mathcal{X}_1) = E(P_2, \mathcal{X}_2)$ for any unipotent extension $(P_1, \mathcal{X}_1) \rightarrow (P_2, \mathcal{X}_2)$.

(d) If the central torus of G splits over a totally real field or a CM -field, then the reflex field is totally real or a CM -field. This follows from (c) and [D1] 3.8.

11.3. The class of field isomorphism: Let E be a number field. Class field theory gives a canonical isomorphism

$$\mathrm{Gal}(E^{ab}/E) \cong \pi_0(\mathbb{G}_m(\mathbb{A}_E)/\mathbb{G}_m(E)).$$

In order to avoid a sign ambiguity we state it explicitly. Let v be a nonarchimedean prime of E , π_v a local parameter in E_v , and E' an abelian extension of E that is unramified at v . Then the composite map

$$\pi_0(\mathbb{G}_m(\mathbb{A}_E)/\mathbb{G}_m(E)) \xrightarrow{\sim} \mathrm{Gal}(E^{ab}/E) \rtimes \mathrm{Gal}(E'/E)$$

maps the idele $(\dots, 1, \pi_v^{-1}, 1, \dots)$ to the Frobenius at v . This convention determines the class field isomorphism uniquely.

In the case $E = \mathbb{Q}$ the inclusion $\hat{\mathbb{Z}}^\times \hookrightarrow \mathbb{G}_m(\mathbb{A}_f) \hookrightarrow \mathbb{G}_m(\mathbb{A})$ induce an isomorphism

$$\hat{\mathbb{Z}}^\times \cong \mathbb{X}_f^\times / (\mathbb{Q}^\times)^+ \cong (\mathbb{R}^\times)^0 \backslash \mathbb{A}^\times / \mathbb{Q}^\times \cong \pi_0(\mathbb{G}_m(\mathbb{A})/\mathbb{G}_m(\mathbb{Q})).$$

Its composite with the above class field isomorphism is precisely the cyclotomic character

$$\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \xrightarrow{\sim} \hat{\mathbb{Z}}^\times, \quad \sigma \mapsto a_\sigma,$$

which is given by the condition: For any root of unity $\zeta \in \mathbb{Q}^{ab}$ we have $\sigma(\zeta) = \zeta^{a_\sigma}$.

11.4. The reciprocity law for a torus: Consider Shimura data $(T, \mathcal{Y})$, where T is a torus. Let $E = E(T, \mathcal{Y})$ be its reflex field. By definition E is the field of definition of the cocharacters $h_y \circ \mu : \mathbb{G}_{m, \mathbb{C}} \rightarrow T_{\mathbb{C}}$ for an arbitrary $y \in \mathcal{Y}$. Let $\mu_h : \mathbb{G}_{m, E} \rightarrow T_E$ be the corresponding homomorphism over E , and $\mathfrak{R}_{E/\mathbb{Q}}(\mu_h) : \mathfrak{R}_{E/\mathbb{Q}}\mathbb{G}_{m, E} \rightarrow \mathfrak{R}_{E/\mathbb{Q}}T_E$ its Weil restriction. The composite map

$$\mathfrak{R}_{E/\mathbb{Q}}\mathbb{G}_{m, E} \xrightarrow{\mathfrak{R}_{E/\mathbb{Q}}(\mu_h)} \mathfrak{R}_{E/\mathbb{Q}}T_E \xrightarrow{\mathrm{Norm}_{E/\mathbb{Q}}} T$$

induces a continuous homomorphism

$$\pi_0(\mathbb{G}_m(\mathbb{A}_E)/\mathbb{G}_m(E)) \longrightarrow \pi_0(T(\mathbb{A})/T(\mathbb{Q})).$$

Together with the class field isomorphism of 11.3

$$\mathrm{Gal}(\bar{E}/E) \rtimes \mathrm{Gal}(E^{ab}/E) \xrightarrow{\sim} \pi_0(\mathbb{G}_m(\mathbb{A}_E)/\mathbb{G}_m(E))$$

we get a homomorphism

$$\psi : \mathrm{Gal}(\bar{E}/E) \longrightarrow \pi_0(T(\mathbb{A})/T(\mathbb{Q})),$$

which is canonically associated to the Shimura data $(T, \mathcal{Y})$. This homomorphism ψ is reciprocity law for $(T, \mathcal{Y})$.

Consider the natural action $T(\mathbb{A})$ from the left hand side on $\mathcal{Y} \times T(\mathbb{A}_f)$:

$$\begin{aligned} T(\mathbb{A}) = T(\mathbb{R}) \times T(\mathbb{A}_f) \ni (t_\infty, t_f) : \mathcal{Y} \times T(\mathbb{A}_f) &\rightarrow \mathcal{Y} \times T(\mathbb{A}_f), \\ (y, t'_f) &\mapsto (t_\infty \cdot y, t_f \cdot t'_f). \end{aligned}$$

Let $K_f \subset (\mathbb{A}_f)$ be an open compact subgroup. Since T is abelian, this action induces a transitive action of $T(\mathbb{A})/T(\mathbb{Q})$ on $M^{K_f}(T, \mathcal{Y})(\mathbb{C}) = T(\mathbb{Q}) \backslash \mathcal{Y} \times (T(\mathbb{A}_f)/K_f)$. This is a finite discrete set, so this action factors over $\pi_0(T(\mathbb{A})/T(\mathbb{Q}))$. We define an action of $\text{Gal}(\bar{E}/E)$ from the left hand side on $M^{K_f}(T, \mathcal{Y})(\mathbb{C})$ by the formula

$$\sigma(x) := \psi(\sigma)(x).$$

This action defines descent data for $M_{\mathbb{C}}^{K_f}(T, \mathcal{Y})$, yielding a model $M^{K_f}(T, \mathcal{Y})$ over E , i.e. a scheme $M^{K_f}(T, \mathcal{Y})$ over E together with an isomorphism $M_{\mathbb{C}}^{K_f}(T, \mathcal{Y}) \cong M^{K_f}(T, \mathcal{Y}) \times_E \mathbb{C}$, unique up to canonical isomorphism. This is the canonical model of $M^{K_f}(T, \mathcal{Y})(\mathbb{C})$.

Let $t_f \in T(\mathbb{A}_f)$ and $K'_f \subset K_f$ open compact subgroups of $T(\mathbb{A}_f)$. Then the map $[\cdot t_f] : M_{\mathbb{C}}^{K'_f}(T, \mathcal{Y}) \rightarrow M_{\mathbb{C}}^{K_f}(T, \mathcal{Y})$ is defined. Since T is abelian, this map is equivariant under the above action of $T(\mathbb{A})$, hence also under the action of $\text{Gal}(\bar{E}/E)$. Therefore it corresponds to a morphism of the canonical models $M^{K'_f}(T, \mathcal{Y}) \rightarrow M^{K_f}(T, \mathcal{Y})$, defined over E , which we again denote by $[\cdot t_f]$.

11.5. Definition: Let $(P, \mathcal{X})$ be arbitrary mixed Shimura data. A canonical model for $(P, \mathcal{X})$ consists of a scheme $M^{K_f}(P, \mathcal{X})$ over $E(P, \mathcal{X})$ for all open compact subgroups $K_f \subset P(\mathbb{A}_f)$, together with an isomorphism $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}) \cong M^{K_f}(P, \mathcal{X}) \times_{E(P, \mathcal{X})} \mathbb{C}$, such that :

(a) Every morphism $[\cdot p_f]_{K'_f, K_f} : M_{\mathbb{C}}^{K'_f}(P, \mathcal{X}) \rightarrow M_{\mathbb{C}}^{K_f}(P, \mathcal{X})$ descends to a morphism defined over $E(P, \mathcal{X})$

$$[\cdot p_f]_{K'_f, K_f} : M^{K'_f}(P, \mathcal{X}) \rightarrow M^{K_f}(P, \mathcal{X}).$$

(b) If $\phi : (T, \mathcal{Y}) \hookrightarrow (P, \mathcal{X})$ is an embedding, where T is a torus, and $K_f^T \subset T(\mathbb{A}_f)$ and $K_f \subset P(\mathbb{A}_f)$ open compact subgroups such that $\phi(K_f^T) \subset K_f$, then the morphism $[\phi]_{K_f^T, K_f} : M_{\mathbb{C}}^{K_f^T}(T, \mathcal{Y}) \rightarrow M_{\mathbb{C}}^{K_f}(P, \mathcal{X})$ descends to $E(T, \mathcal{Y})$:

$$[\phi]_{K_f^T, K_f} : M^{K_f^T}(T, \mathcal{Y}) \rightarrow M^{K_f}(P, \mathcal{X}) \times_{E(P, \mathcal{X})} E(T, \mathcal{Y}).$$

In order to study the functorial behavior of canonical models, we first need some technical facts.

11.6. Lemma: Let $(P, \mathcal{X})$ be mixed Shimura data. For every finite extension E' of $E(P, \mathcal{X})$ in $\mathbb{C}$ there exists an embedding $(T, \mathcal{Y}) \hookrightarrow (P, \mathcal{X})$, with T a torus, such that $E(T, \mathcal{Y}) \cap E' = E(P, \mathcal{X})$.

Proof. For P reductive and $h : \mathcal{X} \rightarrow \text{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$ injective this is the assertion of [D1] thm. 5.1. Thus for $P = G$ reductive we already know it for $(G, h(\mathcal{X}))$. Let

$(\phi, \psi) : (T, \mathcal{Y}) \hookrightarrow (G, h(\mathcal{X}))$ be an embedding, then $\mathcal{Y}$ consists of one element only. Choose $x \in \mathcal{X}$ such that $h(x) \in \psi(\mathcal{Y})$. Let $\mathcal{Y}'$ be the $T(\mathbb{R})$ -orbit in $\mathcal{X}$ that is generated by x , and let $\psi' : \mathcal{Y}' \hookrightarrow \mathcal{X}$ be the inclusion. Then $(\phi, \psi') : (T, \mathcal{Y}') \hookrightarrow (G, \mathcal{X})$ is an embedding of Shimura data. Since by 11.2 $E(G, \mathcal{H}) = E(G, h(\mathcal{H}))$, the assertion follows for $(G, \mathcal{X})$. Finally let $P = W \rtimes G$ be arbitrary and $(P, \mathcal{X})/W = (G, \mathcal{H})$. Since by 11.2 $E(P, \mathcal{X}) = E(G, \mathcal{H})$, and there exists a section $(G, \mathcal{H}) \hookrightarrow (P, \mathcal{X})$, the assertion follows from that for $(G, \mathcal{H})$. **q.e.d.**

11.7. Lemma: Let $(\phi, \psi) : (P', \mathcal{X}') \rightarrow (P, \mathcal{X})$ be a morphism of mixed Shimura data, and $K_f \subset P(\mathbb{A}_f)$ an open compact subgroup. Then the union of the images of every map $[\cdot p_f] \circ [\phi] : M^{K'_f}(P', \mathcal{X}')(\mathbb{C}) \rightarrow M^{K_f}(P, \mathcal{X})(\mathbb{C})$ for all K'_f and p_f for which it is defined, is Zariski-dense in $M^{K_f}(P, \mathcal{X})$.

Proof. (Compare [D1] 5.2) The set in question is

$$P(\mathbb{Q}) \backslash (P(\mathbb{Q}) \cdot \psi(\mathcal{X}')) \times (P(\mathbb{A}_f)/K_f) \subset P(\mathbb{Q}) \backslash \mathcal{X} \times (P(\mathbb{A}_f)/K_f),$$

and it suffices to show that it is not contained in any proper closed complex analytic subset. Since $P(\mathbb{Q})$ is dense in $P(\mathbb{R})$, the closure of $P(\mathbb{Q}) \cdot \psi(\mathcal{X}')$ in $\mathcal{X}$ is equal to $P(\mathbb{R}) \cdot \psi(\mathcal{X}')$. Thus it suffices to show that no $P(\mathbb{R})$ -orbit in $\mathcal{X}$ is contained in a proper closed complex analytic subset of $\mathcal{X}$. Consider the projection $\mathcal{X} \rightarrow \mathcal{Z} := U(\mathbb{C}) \backslash \mathcal{X}$. Since $P(\mathbb{R})$ acts transitively on $\mathcal{Z}$, it suffices to prove the assertion in every fibre. But $\mathcal{X} \rightarrow \mathcal{Z}$ is a holomorphic $U(\mathbb{C})$ -torsor, and $U(\mathbb{R})$ is not contained in any proper closed complex analytic subset of $U(\mathbb{C})$. **q.e.d.**

11.8. Lemma: Let E be a subfield of $\mathbb{C}$, X a scheme over E , and $Y_{\mathbb{C}}$ a reduced closed subscheme of $X \times_E \mathbb{C}$. Suppose that we are given a collection of overfields $E \subset E_i \subset \mathbb{C}$ whose intersection is E . Suppose that for every i we are given schemes $Z_{i,\alpha}$ over E_i with morphisms $\phi_{i,\alpha} : Z_{i,\alpha} \rightarrow X \times_E E_i$ over E_i , such that $Y_{\mathbb{C}}$ is the Zariski-closure of

$$\bigcup_{\alpha} \phi_{i,\alpha}(Z_{i,\alpha} \times E_{\alpha} \mathbb{C}).$$

Then $Y_{\mathbb{C}} = Y \times_E \mathbb{C}$ for a unique reduced closed subscheme $Y \subset X$ defined over E .

Proof. (Compare [D1] 5.3) The data defines descent data for $Y_{\mathbb{C}}$ as closed subschemes of $X_{\mathbb{C}}$. By [SGA1]éxp.VIII cor. 1.9 this descent data is effective. **q.e.d.**

A central role is played by the following proposition.

11.9. Proposition: Let $(P_1, \mathcal{X}_1) \hookrightarrow (P_2, \mathcal{X}_2)$ be an embedding of mixed Shimura data. If a canonical model exists for $(P_2, \mathcal{X}_2)$, then there exists a canonical model for $(P_1, \mathcal{X}_1)$, such that every morphism $[\phi] : M_{\mathbb{C}}^{K_1^f}(P_1, \mathcal{X}_1) \rightarrow M_{\mathbb{C}}^{K_2^f}(P_2, \mathcal{X}_2)$ descends to a morphism

$$M^{K_1^f}(P_1, \mathcal{X}_1) \rightarrow M^{K_2^f}(P_2, \mathcal{X}_2) \times_{E(P_2, \mathcal{X}_2)} E(P_1, \mathcal{X}_1)$$

defined over $E(P_1, \mathcal{X}_1)$.

Proof. (Compare [D1] 5.7) By 11.2 (b), $E(P_2, \mathcal{X}_2)$ is contained in $E(P_1, \mathcal{X}_1)$, so the assertion makes sense. Let us write $E := E(P_1, \mathcal{X}_1)$. Let $K_f^1 \subset P_1(\mathbb{A}_f)$ be an open compact

subgroup, and $K_f^2 \subset P_2(\mathbb{A}_f)$ an open compact subgroup according to 3.8, such that $[\phi] : M_{\mathbb{C}}^{K_f^1}(P_1, \mathcal{X}_1) \rightarrow M_{\mathbb{C}}^{K_f^2}(P_2, \mathcal{X}_2)$ is a closed embedding. Let $X := M_{\mathbb{C}}^{K_f^2}(P_2, \mathcal{X}_2) \times_{E(P_2, \mathcal{X}_2)} E$, and $Y_{\mathbb{C}} \subset X_{\mathbb{C}} := X \times_E \mathbb{C}$ the image of $[\phi]$, this is a closed subscheme. For any embedding $e : (T, \mathcal{Y}) \hookrightarrow (P_1, \mathcal{X}_1)$ with T a torus, and all $p_f \in P_1(\mathbb{A}_f)$, choose an open compact subgroup $K_f^T \subset T(\mathbb{A}_f)$ such that $[\cdot p_f] \circ [e] : M_{\mathbb{C}}^{K_f^T}(T, \mathcal{Y}) \rightarrow M_{\mathbb{C}}^{K_f^1}(P_1, \mathcal{X}_1)$ is defined. Then $[\phi] \circ [\cdot p_f] \circ [e] : M_{\mathbb{C}}^{K_f^T}(T, \mathcal{Y}) \rightarrow M_{\mathbb{C}}^{K_f^2}(P_2, \mathcal{X}_2)$ is also defined and by 3.5 equal to $[\cdot \phi(p_f)] \circ [\phi \circ e]$. By assumption this map descends to a morphism $M_{\mathbb{C}}^{K_f^T}(T, \mathcal{Y}) \rightarrow X \times_E E(T, \mathcal{Y})$ defined over $E(T, \mathcal{Y})$. Over $\mathbb{C}$ it factors through $Y_{\mathbb{C}}$, and by 11.6 and 11.7 the conditions of 11.8 are satisfied for the collection of all these morphisms. 11.8 now implies that $Y_{\mathbb{C}}$ descends to a unique reduced closed subscheme $Y \subset X$ defined over E .

Let $K_f^{2'} \supset K_f^2$ be another open compact subgroup of $P_2(\mathbb{A}_f)$. By 3.5 we have $[\phi]_{K_f^1, K_f^{2'}} = [\text{id}]_{K_f^2, K_f^{2'}} \circ [\phi]_{K_f^1, K_f^2}$, and by assumption $[\text{id}]_{K_f^2, K_f^{2'}}$ descends to a morphism $M_{\mathbb{C}}^{K_f^2}(P_2, \mathcal{X}_2) \rightarrow M_{\mathbb{C}}^{K_f^{2'}}(P_2, \mathcal{X}_2)$ defined over $E(P_2, \mathcal{X}_2)$. Therefore $[\phi]_{K_f^1, K_f^{2'}} : M_{\mathbb{C}}^{K_f^1}(P_1, \mathcal{X}_1) \rightarrow M_{\mathbb{C}}^{K_f^{2'}}(P_2, \mathcal{X}_2)$ descends to the morphism over E

$$[\text{id}]_{K_f^2, K_f^{2'}} \circ [\phi]_{K_f^1, K_f^2} : Y \rightarrow M_{\mathbb{C}}^{K_f^{2'}}(P_2, \mathcal{X}_2).$$

If this is also a closed embedding, and we define $Y' \subset M_{\mathbb{C}}^{K_f^{2'}}(P_2, \mathcal{X}_2) \times_{E(P_2, \mathcal{X}_2)} E$ in the same way as $Y \subset X$, then we get a canonical isomorphism $Y \rightarrow Y'$ defined over E . Applying this to the intersection of two K_f^2 shows that, up to canonical isomorphism, Y is independent of the choice of K_f^2 . Thus we may define $M_{\mathbb{C}}^{K_f^1}(P_1, \mathcal{X}_1) := Y$. The condition 11.5 (b) being clear by construction, it remains to prove 11.5 (a).

So let $p_f \in P_1(\mathbb{A}_f)$, and let K_f^1 and $K_f^{1'}$ be open compact subgroups of $P_1(\mathbb{A}_f)$ such that $[\cdot p_f] : M_{\mathbb{C}}^{K_f^{1'}}(P_1, \mathcal{X}_1) \rightarrow M_{\mathbb{C}}^{K_f^1}(P_1, \mathcal{X}_1)$ is defined. Let $K_f^2 \subset P_2(\mathbb{A}_f)$ be as above, and $K_f^{2'} \subset P_2(\mathbb{A}_f)$ likewise for $K_f^{1'}$. Since $K_f^{1'} \subset p_f \cdot K_f^1 \cdot p_f^{-1} \hookrightarrow \phi(p_f) \cdot K_f^2 \cdot \phi(p_f)^{-1}$ we may replace $K_f^{2'}$ by $K_f^{2'} \cap \phi(p_f) \cdot K_f^2 \cdot \phi(p_f)^{-1}$. Then the map $[\cdot \phi(p_f)] : M_{\mathbb{C}}^{K_f^{2'}}(P_2, \mathcal{X}_2) \rightarrow M_{\mathbb{C}}^{K_f^2}(P_2, \mathcal{X}_2)$ is also defined. By assumption it descends to a morphism of the canonical models, defined over $E(P_2, \mathcal{X}_2)$. Since by 3.5 (c) we have $[\phi] \circ [\cdot p_f] = [\cdot \phi(p_f)] \circ [\phi]$, the morphism $[\cdot p_f]$ descends to a morphism $M_{\mathbb{C}}^{K_f^{1'}}(P_1, \mathcal{X}_1) \rightarrow M_{\mathbb{C}}^{K_f^1}(P_1, \mathcal{X}_1)$ defined over E , as desired. **q.e.d.**

Now we can prove functoriality for canonical models.

11.10. Proposition: Let $\phi : (P_1, \mathcal{X}_1) \rightarrow (P_2, \mathcal{X}_2)$ be a morphism of mixed Shimura data, and $K_f^1 \subset P_1(\mathbb{A}_f)$ and $K_f^2 \subset P_2(\mathbb{A}_f)$ open compact subgroups such that $\phi(K_f^1) \subset K_f^2$. If canonical subsets exist for $(P_1, \mathcal{X}_1)$ and $(P_2, \mathcal{X}_2)$, then the morphism $[\phi]_{K_f^1, K_f^2} : M_{\mathbb{C}}^{K_f^1}(P_1, \mathcal{X}_1) \rightarrow M_{\mathbb{C}}^{K_f^2}(P_2, \mathcal{X}_2)$ descends to a morphism defined over $E(P_1, \mathcal{X}_1)$:

$$[\phi]_{K_f^1, K_f^2} : M_{\mathbb{C}}^{K_f^1}(P_1, \mathcal{X}_1) \rightarrow M_{\mathbb{C}}^{K_f^2}(P_2, \mathcal{X}_2) \times_{E(P_2, \mathcal{X}_2)} E(P_1, \mathcal{X}_1).$$

are defined over $E(P_1 \times P_2, \mathcal{X}_1 \times \mathcal{X}_2) = E(P_1, \mathcal{X}_1)$. Applying 11.9 to the morphism e yields a “new” canonical model for $(P_1, \mathcal{X}_1)$, such that all morphisms

$$[e] : M_{\mathbb{C}}^{K_f^1}(P_1, \mathcal{X}_1)^{(new)} \hookrightarrow M_{\mathbb{C}}^{K_f^1 \times K_f^2}(P_1 \times P_2, \mathcal{X}_1, \mathcal{X}_2)$$

are defined over $E(P_1, \mathcal{X}_1)$. Its composite with the projection to the first factor is a morphism defined over $E(P_1, \mathcal{X}_1)$:

$$[pr_1] \circ [e] : M^{K_f^1}(P_1, \mathcal{X}_1)^{(new)} \rightarrow M^{K_f^1}(P_1, \mathcal{X}_1).$$

Since this is an isomorphism after base change to $\mathbb{C}$, it is an isomorphism itself. From $[pr_2] \circ [e] = [\phi] \circ [pr_1] \circ [e]$ it follows that $[\phi]$ descends to a morphism $M^{K_f^1}(P_1, \mathcal{X}_1) \rightarrow M^{K_f^2}(P_2, \mathcal{X}_2) \times_{E(P_2, \mathcal{X}_2)} E(P_1, \mathcal{X}_1)$ defined over $E(P_1, \mathcal{X}_1)$, as desired. **q.e.d.**

11.11. Corollary: If a canonical model exists for Shimura data $(P, \mathcal{X})$, then it is uniquely determined up to isomorphism.

Proof. (Compare [D1] 5.5) Apply 11.10 to $(P_1, \mathcal{X}_1) = (P_2, \mathcal{X}_2) = (P, \mathcal{X})$. **q.e.d.**

11.12 Proposition: Let $\phi : (P_1, \mathcal{X}_1) \hookrightarrow (P, \mathcal{X})$ be an improper rational boundary component. If $(P_1, \mathcal{X}_1)$ possesses a canonical model, then so does $(P, \mathcal{X})$.

Proof. Since P_1 is normal in P , $E := E(P, \mathcal{X}) = E(P_1, \mathcal{X}_1)$. Let $K_f \subset P(\mathbb{A}_f)$ be an open compact subgroup. By 7.10 there exist finitely many $p_f^\nu \in P(\mathbb{A}_f)$, such that with $K_f^\nu := P_1(\mathbb{A}_f) \cap p_f^\nu \cdot K_f \cdot (p_f^\nu)^{-1}$ and

$$\Delta_\nu := (\text{Stab}_{P(\mathbb{Q})}(\mathcal{X}_1) \cap (P_1(\mathbb{A}_f) \cdot p_f^\nu \cdot K_f \cdot (p_f^\nu)^{-1})) / P_1(\mathbb{Q}),$$

the maps 3.4 induce an isomorphism

$$\coprod_{\nu} \Delta_\nu \backslash M_{\mathbb{C}}^{K_f^1}(P_1, \mathcal{X}_1) \xrightarrow{\coprod [\cdot p_f^\nu] \circ [\phi]} M_{\mathbb{C}}^{K_f}(P, \mathcal{X}).$$

Since Δ_ν acts through maps 3.4, by 11.5 and 11.10 it acts on the canonical model $M^{K_f^\nu}(P_1, \mathcal{X}_1)$. We may therefore define $M^{K_f}(P, \mathcal{X})$ as $\coprod_{\nu} \Delta_\nu \backslash M^{K_f^\nu}(P_1, \mathcal{X}_1)$ under this isomorphism.

To verify 11.5 (a) let K'_f be another open compact subgroup of $P(\mathbb{A}_f)$, and $p_f \in P(\mathbb{A}_f)$ such that $K'_f \subset p_f \cdot K_f \cdot p_f^{-1}$. We have to show that for every $p'_f \in P(\mathbb{A}_f)$ with $K_f^1 := P_1(\mathbb{A}_f) \cap p'_f \cdot K'_f \cdot p'^{-1}_f$ the morphism $[\cdot p_f] \circ [\cdot p'_f] \circ [\phi]$ descends to a morphism $M^{K_f^1}(P_1, \mathcal{X}_1) \rightarrow M^{K_f}(P, \mathcal{X})$ defined over E . There exists a unique ν so that p'_f and $p'_f \cdot p_f$ represent the same class in $(\text{Stab}_{P(\mathbb{Q})}(\mathcal{X}_1) \cdot P_1(\mathbb{A}_f)) \backslash P(\mathbb{A}_f) / K_f$. Writing $p'_f \cdot p_f = p \cdot p_{1,f} \cdot p'^{\nu}_f \cdot k_f$ accordingly, the properties 3.5 (a) through (e) imply

$$\begin{aligned} [\cdot p_f] \circ [\cdot p'_f] \circ [\phi] &= [\cdot p'_f \cdot p_f] \circ [\phi] = [\cdot k_f] \circ [\cdot p'^{\nu}_f] \circ [\cdot p_{1,f}] \circ [\cdot p] \circ [\phi] \\ &= [\cdot p'^{\nu}_f] \circ [\phi] \circ [\cdot p_{1,f}] \circ [\text{int}(p^{-1})|_{(P_1, \mathcal{X}_1)}]. \end{aligned}$$

By 11.5 and 11.10 the morphisms $[\cdot p_{1,f}]$ and $[\text{int}(p^{-1})|_{(P_1, \mathcal{X}_1)}]$ descend to E , and $[\cdot p'^{\nu}_f] \circ [\phi]$ does so by definition. This proves 11.5 (a); in particular the model $M^{K_f}(P, \mathcal{X})$ is independent of the choice of the p_f^ν .

To prove 11.5 (b) let $\psi : (T, \mathcal{Y}) \hookrightarrow (P, \mathcal{X})$ be an embedding with T a torus. By the same arguments as above we may reduce to the case where $(T, \mathcal{Y})$ is irreducible. Then $\psi(T)$ is contained in P_1 . Fix $p \in P(\mathbb{Q})$ such that $p \cdot \psi(\mathcal{Y})$ has a non-empty intersection with $\mathcal{X}_1$.

Since $\psi(T(\mathbb{R})) \subset P_1(\mathbb{R})$, it then follows that $p \cdot \psi(\mathcal{Y}) \subset \mathcal{X}_1$, whence $\text{int}(p) \circ \psi = \phi \circ \psi_1$ for a morphism $\psi_1 : (T, \mathcal{Y}) \hookrightarrow (P_1, \mathcal{X}_1)$. By 3.5 it follows that

$$[\psi] = [\text{int}(p)^{-1}] \circ [\phi] \circ [\psi_1] = [\cdot p] \circ [\phi] \circ [\psi_1].$$

Here $[\cdot p] \circ [\phi]$ descends to E by the above definition of $M^{K_f}(P, \mathcal{X})$, and $[\psi_1]$ descends to $E(T, \mathcal{Y})$ by the assumption on $(P_1, \mathcal{X}_1)$. **q.e.d.**

In order to apply the results of the previous chapter, we need the following technical lemma.

11.13. Lemma: Let $(P, \mathcal{X})$ be mixed Shimura data, $(G, \mathcal{H}) := (P, \mathcal{X})/W$, and $e : (G, \mathcal{H}) \hookrightarrow (P, \mathcal{X})$ a splitting. Let $\mathcal{P} \subset P(\mathbb{A}_f)$ be a subset that generates $P(\mathbb{A}_f)$ as an abstract semigroup. Let $\mathcal{K}$ be a cofinal system of open compact subgroups of $P(\mathbb{A}_f)$. Suppose that for every $K_f \in \mathcal{K}$ we are given a model $M^{K_f}(P, \mathcal{X})$ of $M_{\mathbb{C}}^{K_f}(P, \mathcal{X})$ over $E(P, \mathcal{X})$. Assume that:

(a) For all $p_f \in \mathcal{P}$, $K_f \in \mathcal{K}$, and all sufficiently small $K'_f \in \mathcal{K}$ (here the bound depends on p_f and K_f), the morphism $[\cdot p_f]_{K'_f, K_f}$ descends to a morphism $M^{K'_f}(P, \mathcal{X}) \rightarrow M^{K_f}(P, \mathcal{X})$ defined over $E(P, \mathcal{X})$.

(b) For every embedding $\phi : (T, \mathcal{Y}) \hookrightarrow (G, \mathcal{H})$ with T a torus, every $K_f \in \mathcal{K}$, and every open compact subgroup $K_f^T \subset T(\mathbb{A}_f)$ such that $e \circ \phi(K_f^T) \subset K_f$, the morphism $[e \circ \phi]_{K_f^T, K_f}$ descends to a morphism $M^{K_f^T}(T, \mathcal{Y}) \rightarrow M^{K_f}(P, \mathcal{X}) \times_{E(P, \mathcal{X})} E(T, \mathcal{Y})$ defined over $E(T, \mathcal{Y})$.

Then there exists a canonical model for $(P, \mathcal{X})$.

Proof. If condition (a) holds for $p_f, p'_f \in P(\mathbb{A}_f)$, then it clearly holds for $p_f \cdot p'_f$ as well. Since by assumption $P(\mathbb{A}_f) = \mathcal{P} \cup \mathcal{P} \cdot \mathcal{P} \cup \mathcal{P} \cdot \mathcal{P} \cdot \mathcal{P} \cup \dots$, (a) holds for all $p_f \in P(\mathbb{A}_f)$, and from now on we may assume $\mathcal{P} = P(\mathbb{A}_f)$.

Write $E := E(P, \mathcal{X})$. Let $K_f \subset P(\mathbb{A}_f)$ be an arbitrary open compact subgroup, and fix $K'_f \in \mathcal{K}$ with $K'_f \subset K_f$. We shall show that there exists a model $M^{K_f}(P, \mathcal{X})$ over E of $M_{\mathbb{C}}^{K_f}(P, \mathcal{X})$, such that the morphism $[\text{id}] : M_{\mathbb{C}}^{K'_f}(P, \mathcal{X}) \rightarrow M_{\mathbb{C}}^{K_f}(P, \mathcal{X})$ descends to a morphism $M^{K'_f}(P, \mathcal{X}) \rightarrow M^{K_f}(P, \mathcal{X})$. Let $K''_f := \bigcap_{k_f \in K_f} k_f \cdot K'_f \cdot k_f^{-1}$, and k'_f a set of representatives of K_f/K''_f . Fix some $K'''_f \in \mathcal{K}$ with $K'''_f \subset K''_f$. Let Y be the image of the morphism

$$M^{K'''_f}(P, \mathcal{X}) \xrightarrow{\pi[\cdot k'_f]} \prod_{\nu} M_{\mathbb{C}}^{K'_f}(P, \mathcal{X}),$$

with the structure of a reduced closed subscheme. By construction, over $\mathbb{C}$ this map factors through a closed embedding

$$M_{\mathbb{C}}^{K'''_f}(P, \mathcal{X}) \cong Y_{\mathbb{C}} \hookrightarrow \prod_{\nu} M_{\mathbb{C}}^{K'_f}(P, \mathcal{X}),$$

in other words Y is a model of $M_{\mathbb{C}}^{K'''_f}(P, \mathcal{X})$ over E . This closed embedding is equivariant under the following action of K_f/K''_f : on $M_{\mathbb{C}}^{K'''_f}(P, \mathcal{X})$ through the automorphism $[\cdot k_f]$, and on $\prod_{\nu} M_{\mathbb{C}}^{K'_f}(P, \mathcal{X})$ through the obvious permutations. This shows that this action descends

to an action on Y . Our schemes being quasiprojective, we may form the quotient, which is the desired scheme $M^{K_f}(P, \mathcal{X})$.

We now prove 11.5 for the models just constructed. For 11.5 (a) consider p_f, K_f, K'_f so that $K'_f \subset p_f \cdot K_f \cdot p_f^{-1}$. Let $\tilde{K}_f, \tilde{K}'_f \in \mathcal{K}$ so that $\tilde{K}_f \subset K_f, \tilde{K}'_f \subset K'_f$, and the maps $[\text{id}]_{\tilde{K}_f, K_f}$ and $[\text{id}]_{\tilde{K}'_f, K'_f}$ descend to morphisms $M^{\tilde{K}_f}(P, \mathcal{X}) \rightarrow M^{K_f}(P, \mathcal{X})$ and $M^{\tilde{K}'_f}(P, \mathcal{X}) \rightarrow M^{K'_f}(P, \mathcal{X})$. Let $\tilde{K}''_f \in \mathcal{K}$ so that $\tilde{K}''_f \subset \tilde{K}'_f \cap p_f \cdot \tilde{K}_f \cdot p_f^{-1}$. Then we have the commutative diagram

$$\begin{array}{ccccc} M^{\tilde{K}''_f}(P, \mathcal{X}) & \xrightarrow{[\text{id}]} & M^{\tilde{K}'_f}(P, \mathcal{X}) & \xrightarrow{[\text{id}]} & M^{K'_f}(P, \mathcal{X}) \\ & \searrow [p_f] & & \downarrow [p_f] & \\ & & M^{\tilde{K}_f}(P, \mathcal{X}) & \xrightarrow{[\text{id}]} & M^{K_f}(P, \mathcal{X}), \end{array}$$

where the solid arrows are morphisms over E , but the dotted arrow is a priori defined only over $\mathbb{C}$. Since the upper line consists of finite flat coverings, the dotted arrow also descends to E , as desired. This also proves that the model $M^{K_f}(P, \mathcal{X})$ is independent of the choice of K'_f .

For 11.5 (b) let $\phi : (T, \mathcal{Y}) \hookrightarrow (P, \mathcal{X})$ be an embedding with T a torus. By conjugacy of all splittings of P , there exists $p \in P(\mathbb{Q})$ such that $\text{int}(p) \circ \phi$ factors through G . Thus $\phi = \text{int}(p^{-1}) \circ e \circ \phi'$ for some embedding $\phi' : (T, \mathcal{Y}) \hookrightarrow (G, \mathcal{H})$. Having proved 11.5 (a), we may assume $p = 1$. Let $K_f \subset P(\mathbb{A}_f)$ and $K_f^T \subset T(\mathbb{A}_f)$ be open compact subgroups such that $\phi(K_f^T) \subset K_f$. Let $\tilde{K}_f \in \mathcal{K}$ so that $\tilde{K}_f \subset K_f$ and the map $[\text{id}]_{\tilde{K}_f, K_f}$ descends to a morphism $M^{\tilde{K}_f}(P, \mathcal{X}) \rightarrow M^{K_f}(P, \mathcal{X})$. With $\tilde{K}_f^T := K_f^T \cap \phi^{-1}(\tilde{K}_f)$ we get a commutative diagram

$$\begin{array}{ccc} M^{\tilde{K}_f^T}(T, \mathcal{Y}) & \xrightarrow{[\text{id}]} & M^{K_f^T}(T, \mathcal{Y}) \\ [\phi] \downarrow & & \downarrow [\phi] \\ M^{\tilde{K}_f}(P, \mathcal{X}) \times_E E(T, \mathcal{Y}) & \xrightarrow{[\text{id}]} & M^{K_f}(P, \mathcal{X}) \times_E E(T, \mathcal{Y}), \end{array}$$

and the assertion follows as above. **q.e.d.**

11.14. Proposition: There exists a canonical model for the mixed Shimura data $(P_0, \mathcal{X}_0)$ defined in 2.24. If $K_f = K_f^U \rtimes K_f^*$ according to the decomposition $P_0 = U_0 \rtimes \mathbb{G}_{m, \mathbb{Q}}$, descends to an isomorphism over $\mathbb{Q}$

$$M^{K_f}(P_0, \mathcal{X}_0) \cong \mathbb{G}_{m, \mathbb{Q}} \times M^{K_f^*}(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}).$$

Proof. The reflex field of $(P_0, \mathcal{X}_0)$ is $\mathbb{Q}$. We use the notations of 10.15. Clearly the set $\mathcal{P} := K_f^U(1) \cup \mathbb{G}_m(\mathbb{A}_f)$ generates $P_0(\mathbb{A}_f)$ as a semi group, and the collection $\mathcal{K} = \{K_f^P(d) \mid d \geq 1\}$ of open compact subgroups is cofinal. Letting $M^{K_f^P(d)}(P_0, \mathcal{X}_0) := \underline{X}_d^0$ for every $d \geq 1$, condition 11.13 (a) directly follows from 10.16 (b). Since by construction the zero section descends to a morphism $\underline{M}_d^0 \rightarrow \underline{X}_d^0$ over $\mathbb{Q}$, condition 11.13 (b) is equivalent to saying that the identification in 10.16 (a) turns $\underline{M}_d^0$ into the canonical model of $M_{\mathbb{C}}^{K_f^P(d)}(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H})$. This is so by the explicit description of the reciprocity law for $\mathbb{G}_{m, \mathbb{Q}}$ in 11.3, and the definition of the identification in 10.15. The first assertion now follows from 11.13.

The second statement holds by construction if $K_f = K_f^P(d)$ for some d . In general some conjugate of K_f by an element of $\mathbb{G}_m(\mathbb{Q}) \subset P_0(\mathbb{Q})$ is of the form $K_f^U(d) \rtimes K_f^*$ for some $K_f(d) \subset K_f^* \subset K_f(1)$, and by the equivariance of the isomorphism 3.16 it is enough to prove the assertion for the latter group. But $M^{K_f^U(d) \rtimes K_f^*}(P_0, \mathcal{X}_0)$ is isomorphic to the quotient of $\underline{X}_d$ by the right action of $K_f^*/K_f(d)$. As in the proof of 10.16 (b) one finds that through the isomorphism $\underline{X}_d^0 \cong \mathbb{G}_m \times \underline{M}_d^0$ this group acts only on the second factor. Thus $\underline{X}_d^0/K_f^* \cong \mathbb{G}_m \times (\underline{M}_d^0/K_f^*) \cong \mathbb{G}_m \times M^{K_f^*}(\mathbb{G}_m, \mathbb{Q}, \mathcal{H})$, and this is just the isomorphism defined in 3.16. **q.e.d.**

11.15. Proposition: There exists a canonical model for the mixed Shimura data $(P_0, h(\mathcal{X}_0))$ defined in 2.24.

Proof. Let $\iota : (P_0, \mathcal{X}_0) \rightarrow (P_0, \mathcal{X}_0)$ be the automorphism that is the identity on P_0 and on $h(\mathcal{X}_0)$, but interchanges the two connected components of $\mathcal{X}_0$. For every open compact subgroup $K_f^P \subset P_0(\mathbb{A}_f)$ the automorphism $[\iota]$ of $M_{\mathbb{C}}^{K_f^P}(P_0, \mathcal{X}_0)$ is defined, and by 11.10 and 11.14 it descends to an automorphism of its canonical model over $\mathbb{Q}$. It defines an action of a group of order 2, and over $\mathbb{C}$ the quotient is canonically isomorphic to $M_{\mathbb{C}}^{K_f^P}(P_0, h(\mathcal{X}_0))$. We define $M^{K_f^P}(P_0, h(\mathcal{X}_0))$ as the quotient of $M^{K_f^P}(P_0, \mathcal{X}_0)$ by this action; this is possible since the scheme is affine.

We show that this defines a canonical model for $(P_0, h(\mathcal{X}_0))$. The property 11.5 (a) follows directly from the corresponding property for $(P_0, \mathcal{X}_0)$, and the universal property of the quotient by a finite group. Let $(\phi, \psi) : (T, \mathcal{Y}) \hookrightarrow (P_0, h(\mathcal{X}_0))$ be an embedding with T a torus. As in the proof of 11.6, we can construct an embedding $(\phi, \psi') : (T, \mathcal{Y}') \hookrightarrow (P_0, \mathcal{X}_0)$ such that the composite $(T, \mathcal{Y}') \hookrightarrow (P_0, \mathcal{X}_0) \rightarrow (P_0, h(\mathcal{X}_0))$ factors through $(T, \mathcal{Y})$. The condition 11.5 (b) now follows as in the proof of 11.13. **q.e.d.**

11.16. Theorem: For every $g \geq 1$, there exists a canonical model for the mixed Shimura data $(P_{2g}, \mathcal{X}_{2g})$ defined as in 2.25. For the open compact subgroup in 10.7, this canonical model is isomorphic to the universal family X_d over the moduli $\underline{\mathcal{M}}_d$ of 10.6.

Proof. The reflex field of $(P_{2g}, \mathcal{X}_{2g})$ is $\mathbb{Q}$. We use the notations of chapter 10, in particular 10.7. Clearly the set $\mathcal{P} := K_f^W(2) \cup CS_{P_{2g}}(\mathbb{A}_f)$ generates $P_{2g}(\mathbb{A}_f)$ as a semigroup, and the collection $\mathcal{K} = \{K_f^P(d) \mid 2|d \geq 4\}$ of open compact subgroups is cofinal. Let $M^{K_f^P(d)}(P_{2g}, \mathcal{X}_{2g}) := X_d$ for all even $d \geq 4$, where X_d is as in 10.10. By 10.13, condition 11.13 (a) holds for $p_f = 1$. This, together with 10.11, proves the same for all $p_f \in K_f^W(2)$. Fix $p_f \in CS_{P_{2g}}(\mathbb{A}_f)$ and an even integer $d \geq 4$. Let $\tilde{d}$ be a multiple of d , and $2|d' \geq 4$, so that $K_f^P(d') \subset p_f \cdot K_f^P(\tilde{d}) \cdot p_f^{-1} \subset K_f^P(2)$ as in 10.12. With $d'' := d' \cdot \tilde{d}$, 10.13 implies that both $[\cdot p_f]_{K_f^P(d''), K_f^P(\tilde{d})}$ and $[\text{id}]_{K_f^P(\tilde{d}), K_f^P(d)}$ descends to morphisms over $\mathbb{Q}$, so 11.13 (a) holds for all $p_f \in \mathcal{P}$.

It remains to verify 11.13 (b). Since, in the notations of 10.7, the section $M(d) \rightarrow M_W(d)$ corresponds to the morphism $\tau : \underline{\mathcal{M}}_d \rightarrow X_d$, defined over $\mathbb{Q}$, this is equivalent to saying that the identification in 10.9 turns $\underline{\mathcal{M}}_d$ into the canonical model of $M_{\mathbb{C}}^{K_f^P(d)}(CS_{P_{2g}, \mathbb{Q}}, \mathcal{H}_{2g})$. This is a consequence of the main theorem of Shimura and Taniyama about abelian varieties with complex multiplications (see [D1] thm. 4.19, thm. 4.21). Note that the contrast

to [D1] we have a different sign convention at two places, namely on the one hand in the definition of the operation of the torus $\mathbb{S}$ in a Hodge structure (cf. [D1] 1.3), on the other hand in the definition of the reciprocity law for a torus (cf. [D1] 3.9). These two sign changes cancel in translating [loc. cit.] thm. 4.19. **q.e.d.**

11.17. Theorem: Let $(G, \mathcal{H})$ be Shimura data with G reductive and $h : \mathcal{H} \rightarrow \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, G_{\mathbb{C}})$ injective. Then there exists a canonical model for $(G, \mathcal{H})$.

Proof. see [Mi1] thm. 7.2.

11.18. Theorem: There exists a canonical model for every mixed Shimura data.

Proof. By 11.12 it suffices to consider irreducible $(P, \mathcal{X})$. Suppose first that $V = 1$. Choose an embedding

$$(P, \mathcal{X}) \hookrightarrow (T, \mathcal{Y}) \times (G, \mathcal{H}) \times \prod_{i=1}^n (P_0, h(\mathcal{X}_0))$$

as in 2.26 (a). By 11.4, 11.17, and 11.15 each of the factors on the right hand side possesses a canonical model. Thus the assertion follows from 11.9.

Next suppose $2g := \dim(V) > 0$. According to 2.26 (b) choose a morphism $\phi : (P, \mathcal{X}) \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H})$, a $(P_0, \mathcal{X}_0) \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H})$ -torsor $(P_1, \mathcal{X}_1) \rightarrow (P, \mathcal{X})$, and an embedding

$$(P_1, \mathcal{X}_1) \hookrightarrow (T, \mathcal{Y}) \times (G, \mathcal{H}) \times \prod_{i=1}^n (P_{2g}, \mathcal{X}_{2g}).$$

By 11.4, 11.17 and 11.16 each factor on the right hand side possesses a canonical model. Thus, by 11.19, $(P_1, \mathcal{X}_1)$ also possesses a canonical model. It remains to push this down to $(P, \mathcal{X})$.

Let $K_f^1 \subset P_1(\mathbb{A}_f)$ be a neat open compact subgroup, and K_f its image in $P(\mathbb{A}_f)$. We abbreviate $M_1 := M^{K_f^1}(P_1, \mathcal{X}_1)$, $M_{\mathbb{C}} := M_{\mathbb{C}}^{K_f}(P, \mathcal{X})$, and $\tilde{M} := M^{\pi(K_f)}((P, \mathcal{X})/W)$; the latter exists by the first part of the present proof. The $\mathbb{G}_m$ -action on $M_{1, \mathbb{C}}$ is given in terms of maps 3.4, so by 11.10 and 11.14 it induces a $\mathbb{G}_{m, E}$ -action on M_1 , where $E = E(P, \mathcal{X})$. Since $M_{1, \mathbb{C}} \rightarrow M_{\mathbb{C}}$ is a $\mathbb{G}_m$ -torsor, it is Zariski-locally trivial, so the descent data for $M_{1, \mathbb{C}}$ relative to $E \subset \mathbb{C}$ induces descent data for $M_{\mathbb{C}}$. Since by 11.10 the composite morphism $M_{1, \mathbb{C}} \rightarrow M_{\mathbb{C}} \rightarrow \tilde{M}_{\mathbb{C}}$ descends to E , we may view this as descent data with respect to $\tilde{M}_{\mathbb{C}} \rightarrow \tilde{M}$. If $L_{\mathbb{C}}$ denotes the line bundle on $M_{\mathbb{C}}$ that corresponds to the inverse of the $\mathbb{G}_m$ -torsor $M_{1, \mathbb{C}} \rightarrow M_{\mathbb{C}}$, then we have descent data for the pair $(M_{\mathbb{C}}, L_{\mathbb{C}})$. Recall that by 3.21, $L_{\mathbb{C}}$ is relatively ample on $M_{\mathbb{C}} \rightarrow \tilde{M}_{\mathbb{C}}$. Thus by [SGA1] exp.VIII cor, 7.8 this descent data is effective.

For all neat open compact subgroups $K_f \subset P(\mathbb{A}_f)$ we have thus defined a model of $M_{\mathbb{C}}^{K_f}(P, \mathcal{X})$ over E . Letting $\mathcal{K}$ be the set of these K_f , and $\mathcal{P} := P(\mathbb{A}_f)$, it remains to verify the conditions 11.13. These follow easily from the construction, by the same arguments as in the proof of 11.13. (Condition 11.13 (a) is an application of the descent of morphisms [SGA1] exp.VIII thm. 5.2) **q.e.d.**

Chapter 12

Canonical model of the compactification

In this chapter we formulate and prove the main result of this thesis: the extension of the canonical model to both the Baily-Borel compactification and the toroidal compactifications, together with an explicit description of the model induced on the boundary. Before stating these theorems (12.3–5), we prove an assertion relating the reflex field of mixed Shimura data with that of boundary component (12.1–2). For the structure of the proofs see 12.6–7; suffice it to say here that we are reaping the benefits of the consistent use of the adelic formalism, in particular in chapter 6. The proofs occupy 12.8–17. In 12.18–20, we apply the results to q -expansion of modular forms. We conclude with two examples (12.21–22).

12.1. Proposition: Let $(P_1, \mathcal{X}_1)$ be a rational boundary component of some mixed Shimura data $(P, \mathcal{X})$. Then

$$E(P_1, \mathcal{X}_1) = E(P, \mathcal{X})$$

Proof. Let $x \in \mathcal{X}$, and $x_1 \in \mathcal{X}_1$ the corresponding point. We claim: $h_x \circ \mu$ is conjugate to $h_{x_1} \circ \mu$ under $P(\mathbb{C})$. To prove this by the definition 4.11 of h_{x_1} , it suffices to show that the two cocharacters $h_0 \circ \mu$ and $h_\infty \circ \mu$ of $H_{0,\mathbb{C}}$ are conjugate under $H_0(\mathbb{C})$, where H_0, h_0, h_∞ are as in 4.3. But it is easily checked that both of these are conjugate to the map

$$\mathbb{G}_m(\mathbb{C}) \rightarrow H_0(\mathbb{C}), \quad z \mapsto (z, \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}).$$

Let $Q \subset P$ be the parabolic subgroup associated to $(P_1, \mathcal{X}_1)$. By the following lemma 12.2, we have inclusions

$$Y^\natural(P_1) \hookrightarrow Y^\natural(Q) \supset Y^\natural(Q)^+ \xrightarrow{\sim} Y^\natural(P).$$

By the claim above, and by 4.4, our conjugacy class $[h_x \circ \mu] = [h_{x_1} \circ \mu]$ lies in each of these subsets. By the definition of the reflex field, the assertion follows. **q.e.d.**

12.2. Lemma: For any linear algebraic group G denote by $Y^\natural(G)$ the (discrete) set of G -conjugacy classes of cocharacters of G .

- (a) If U is the unipotent radical of G , then $Y^{\natural}(G) \cong Y^{\natural}(G/U)$.
- (b) If H is a connected normal subgroup of G , then $Y^{\natural}(H) \hookrightarrow Y^{\natural}(G)$.
- (c) If G is reductive, $P \subset G$ a parabolic subgroup, and U the unipotent radical of P , then the subset

$$Y^{\natural}(P)^+ := \{\lambda \in Y^{\natural}(P) \mid \lambda \text{ has nonnegative weights on } U\}$$

maps isomorphically to $Y^{\natural}(G)$.

Proof. Assertion (a) follows from the conjugacy of all maximal tori. Assertion (b) holds, because G and H have the same image in $\text{Aut}(H)$. For (c), fix a Borel subgroup B inside P . The fact that every weight is conjugate to a unique dominant weight, means that every element of $Y^{\natural}(G)$ possesses a unique representative in $Y^{\natural}(B)^+$. Applying this fact to $B \subset P$, the assertion follows. **q.e.d.**

We are now in the position to formulate our main results.

12.3. Main Theorem for the Baily-Borel compactification: Consider pure Shimura data $(P, \mathcal{X})$ and an open compact subgroup $K_f \subset P(\mathbb{A}_f)$.

- (a) The canonical model $M^{K_f}(P, \mathcal{X})$ extends uniquely to a scheme $M^{K_f}(P, \mathcal{X})^*$ over $E(P, \mathcal{X})$, with an isomorphism

$$M_{\mathbb{C}}^{K_f}(P, \mathcal{X})^* = M^{K_f}(P, \mathcal{X})^* \times_{E(P, \mathcal{X})} \mathbb{C}.$$

- (b) All morphisms 3.4 correspond to morphisms of these schemes, defined over $E(P, \mathcal{X})$, in the case 3.4 (a), and over $E(P_1, \mathcal{X}_1)$ in the case 3.4 (b) (compare 6.2, 9.25). The functoriality of these morphisms is preserved, as well as the assertions of 8.2 and 8.4 (ampleness).
- (c) The stratification in 6.3, and the maps in 7.6, are defined over the common reflex field $E(P, \mathcal{X})$.

12.4. Main Theorem for the toroidal compactification: Let $(P, \mathcal{X})$ be mixed Shimura data, $K_f \subset P(\mathbb{A}_f)$ an open compact subgroup, and Σ a K_f -admissible partial cone decomposition for $(P, \mathcal{X})$. We assume that $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma)$ exists and can be covered by quasiprojective $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma_i)$ for $\Sigma_i \subset \Sigma$, such that, on each $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma_i)$, some ample line bundle can be described as in terms of $\omega[\text{dlog}]$ and line bundles as in 8.13 (e.g. in the situation of 9.21, 9.28, or 9.33).

- (a) The canonical model $M^{K_f}(P, \mathcal{X})$ extends uniquely to a scheme $M^{K_f}(P, \mathcal{X}, \Sigma)$ over $E(P, \mathcal{X})$, with an isomorphism

$$M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma) = M^{K_f}(P, \mathcal{X}, \Sigma) \times_{E(P, \mathcal{X})} \mathbb{C}.$$

- (b) All morphisms 6.25 (a) $\cong$ 11.5 (a) correspond to morphisms of these schemes, defined over $E(P, \mathcal{X})$. The same holds for the morphism $[\pi]^*$ in 6.24. All morphisms 6.25 (b) $\cong$ 11.10 correspond to morphisms of these schemes, defined over $E(P_1, \mathcal{X}_1)$. The assertions of 6.25 (functoriality, open embeddings, isomorphisms, quotients by finite groups), 6.26 (smoothness), 8.6 (line bundle structure), and 8.13–14 (ampleness) are equally valid for these schemes.

(c) The stratifications in 7.2, 7.3, and 7.10 (see 9.36) are defined over the common reflex field $E(P, \mathcal{X})$, and the assertion of 7.17 (a) holds over $E(P, \mathcal{X})$. The isomorphism of 7.17 (b) (see 9.37) induces a canonical isomorphism between the formal completion of $M^{K_f}(P, \mathcal{X}, \Sigma)$ and of $\text{Stab}_{\Delta_1}([\sigma]) \backslash M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1)$ along

$$\text{Stab}_{\Delta_1}([\sigma]) \backslash M^{\pi[\sigma](K_f^1)}(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]}, \Sigma_{1, [\sigma]}).$$

12.5. Corollary: Without the assumption on $(P, \mathcal{X})$, K_f , and Σ , “ $M^{K_f}(P, \mathcal{X}, \Sigma)$ ” always exists as an algebraic space. The remaining assertions of 12.4 carry over literally.

Proof. As in 9.34 (a), using 9.33 the assertion reduces to that in the theorem. **q.e.d.**

12.6. Beginning of the proof of 12.3 and 12.4: The central points in both theorems (which imply the others) are 12.3 (a), 12.4 (a): the existence of an extension of the canonical models; and the last assertion of 12.4 (c): the isomorphism of formal schemes at the boundary. The remaining assertions say that a certain morphism descends to the reflex field. For each such morphism, by the effectivity of descent of morphisms ([SGA1] exp.VIII thm. 5.2), it suffices to show that there is canonical descent data. Since all our schemes are normal, this is certainly so if the morphism descends on some open dense subscheme. Taking into account 9.25, 11.5 (a) and 11.10, this proves that part (b) of either theorem follows from part (a). In particular, this proves the uniqueness in (a). Likewise, in 12.4 (c) it suffices to prove the assertion for the formal neighborhood of the stratum $\text{Stab}_{\Delta_1}([\sigma]) \backslash M^{\pi[\sigma](K_f^1)}(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]})$ instead of its closure. This assertion, together with 12.3 (a), also implies 12.3 (c). Indeed, fix any complete K_f -admissible cone decomposition Σ for $(P, \mathcal{X})$, so that 12.4 holds for $M^{K_f}(P, \mathcal{X}, \Sigma)$. The map of 6.3 lies in the commutative diagram

$$\begin{array}{ccc} \text{Stab}_{\Delta_1}([\sigma]) \backslash M_{\mathbb{C}}^{\pi[\sigma](K_f^1)}(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]}) & \hookrightarrow & M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma) \\ \downarrow & & \downarrow \\ \Delta_1 \backslash M_{\mathbb{C}}^{K_f^1}((P_1, \mathcal{X}_1)/W_1) & \hookrightarrow & M_{\mathbb{C}}^{K_f}(P, \mathcal{X})^* \end{array}$$

for some cone σ . By part (b) of both theorems, the two vertical maps descend to $E(P, \mathcal{X})$. By assumption, the upper inclusion descends to E , so the same follows for the lower one, as desired.

By similar arguments, we may assume that K_f is arbitrarily small. In fact, either theorem reduces to the quasiprojective case, in which quotients by finite groups exist. Thus part (a) of either theorem reduces to all sufficiently small open compact subgroups, and the rest follows from the universal property of quotients by finite groups.

12.7. Strategy of proof: Of the statements that remain to be proved, the existence of $M^{K_f}(P, \mathcal{X}, \Sigma)$ and the isomorphism of formal neighborhoods over the reflex field are intimately connected. In fact, we shall construct $M^{K_f}(P, \mathcal{X}, \Sigma)$ in such a way that 12.4 (c) holds automatically. The existence of $M^{K_f}(P, \mathcal{X})^*$ for reductive P will be proved on the way. The intuitive idea is the following.

In a way, we can view $M^{K_f}(P, \mathcal{X}, \Sigma)$ as being glued together from formal neighborhoods of torus embeddings along the unipotent fibre of other mixed Shimura varieties, much as in the construction of $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ in chapter 6. Of course, one cannot glue formal schemes in the way that would be needed here, so we have to argue more indirectly. In any case, we must show that the “gluing isomorphism” given by 9.37 descends to the reflex field. To prove this we use an analog of “special points”: namely certain embedded mixed Shimura varieties, for which the assertions can be shown directly. The descent is then effected by a density argument similar to 11.6–10.

This way of proving 12.4 (c) seems to assume 12.4 (a), but it proves this assertion as well. In fact, when we have not yet shown the existence of $M^{K_f}(P, \mathcal{X}, \Sigma)$, we can at least interpret the argument as proving that the descent data for $M_{\mathbb{C}}^{K_f}(P, \mathcal{X})$ with respect to $E(P, \mathcal{X}) \subset \mathbb{C}$ extends to descent data for $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma)$, and it remains to show that this descent is effective. Here our assumption about the ample line bundle comes in. Since it can be expressed purely in terms of the toroidal compactification of other mixed Shimura data, we get compatible descent data for this ample line bundle as well. In such a situation, descent is always effective, and we are done.

In detail, there are the following steps. First, we prove 12.4 in the case of torus embeddings along the unipotent fibre (12.8). Next, using modular interpretation we do the same for $(P, \mathcal{X}) = (\mathrm{GL}_{2, \mathbb{Q}} \mathcal{H}_2)$ (see 12.9). Taken together, these special cases imply 12.4 for certain $(P, \mathcal{X})$, and cone decompositions which are “supported” only on certain rational boundary components (see 12.10–11). In particular, if P is reductive, this holds for all boundary components of codimension 1 in the Baily-Borel compactification; and this is sufficient to prove 12.3 (a) (see 12.12). In 12.13, we prove that the embedded mixed Shimura varieties of the above special type satisfy the density condition alluded to before. Some descent questions are then dealt with the general context (12.14–16), before (in 12.17) we put everything together to finish the proof.

12.8. The case of a torus embedding along the unipotent fibre: We first prove 12.4 in the situation of 6.8. Recall that $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}) \rightarrow M_{\mathbb{C}}^{\pi'(K_f)}((P, \mathcal{X})/U)$ is a torsor under a (relative) torus, namely either $M_{\mathbb{C}}^{K_f^*}(P_*, \mathcal{X}_*) \rightarrow M_{\mathbb{C}}^{\phi(K_f)}(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$, or the same with $\mathcal{X}_*$, $\mathcal{H}_0$ replaced by $h(\mathcal{X}_*)$, $h(\mathcal{H}_0)$ respectively. Moreover there exists a unique K_f^* -admissible cone decomposition Σ_* for $(P_*, \mathcal{X}_*)$, resp. for $(P_*, h(\mathcal{X}_*))$, such that $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}) \hookrightarrow M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma)$ is the torus embedding associated to the torus embedding $M_{\mathbb{C}}^{K_f^*}(P_*, \mathcal{X}_*) \hookrightarrow M_{\mathbb{C}}^{K_f^*}(P_*, \mathcal{X}_*, \Sigma_*)$ (resp. with $h(\mathcal{H}_*)$ in place of $\mathcal{X}_*$). By 11.2, the common reflex field of $(P_*, \mathcal{X}_*)$, $(P_*, h(\mathcal{X}_*))$, $(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$, and $(\mathbb{G}_{m, \mathbb{Q}}, h(\mathcal{H}_0))$ is $\mathbb{Q}$, which is of course contained in $E(P, \mathcal{X}) = E(P', \mathcal{X}')$. Thus, by 11.10, the torsor structure descends to the canonical model $M^{K_f}(P, \mathcal{X})$. The assertion 12.4 (a) for $M^{K_f}(P, \mathcal{X}, \Sigma)$ now follows if we know it for $M^{K_f^*}(P_*, \mathcal{X}_*, \Sigma_*)$, resp. for $M^{K_f^*}(P_*, h(\mathcal{X}_*), \Sigma_*)$. The first sentence of (c) is then true by the definition of the stratification in terms of the canonical projection (see 5.2). The last assertion of (c) is a tautology.

It remains to prove 12.4 (a) for $(P_*, \mathcal{X}_*)$ and $(P_*, h(\mathcal{X}_*))$. We begin with $(P_*, \mathcal{X}_*)$.

Letting $\Gamma_U := U(\mathbb{Q}) \cap K_f^U$, by 6.9 the map 3.16 induces canonical isomorphisms

$$\begin{array}{ccc} M_{\mathbb{C}}^{K_f^*}(P_*, \mathcal{X}_*) & \xrightarrow{\sim} & \mathbb{G}_{m, \mathbb{C}} \otimes \Gamma_U \times M_{\mathbb{C}}^{\phi(K_f)}(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0) \\ \cap & & \cap \quad \parallel \\ M_{\mathbb{C}}^{K_f^*}(P_*, \mathcal{X}_*, \Sigma_*) & \xrightarrow{\sim} & (\mathbb{G}_{m, \mathbb{C}} \otimes \Gamma_U)_{\Sigma_U} \times M_{\mathbb{C}}^{\phi(K_f)}(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0). \end{array}$$

By 11.14, the first line descends to an isomorphism

$$M^{K_f^*}(P_*, \mathcal{X}_*) \xrightarrow{\sim} \mathbb{G}_{m, \mathbb{Q}} \otimes \Gamma_U \times M^{\phi(K_f)}(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$$

for the canonical models. Thus we may define $M^{K_f^*}(P_*, \mathcal{X}_*, \Sigma_*)$ as $(\mathbb{G}_{m, \mathbb{Q}} \otimes \Gamma_U)_{\Sigma_U} \times M^{\phi(K_f)}(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$.

Finally we show how this implies the assertion for $(P_*, h(\mathcal{X}_*))$. As in the proof of 11.15, let ι be the automorphism of $(P_*, \mathcal{X}_*)$ that is the identity on P_* and on $h(\mathcal{X}_*)$, but interchanges the two connected components of $\mathcal{X}_*$. By assumption, this induces an involution $[\iota]$ on the canonical model $M^{K_f^*}(P_*, \mathcal{X}_*, \Sigma_*)$. We define $M^{K_f^*}(P_*, h(\mathcal{X}_*), \Sigma_*)$ as the quotient of $M^{K_f^*}(P_*, \mathcal{X}_*, \Sigma_*)$ by $[\iota]$; this quotient exists, since $M^{K_f^*}(P_*, \mathcal{X}_*, \Sigma_*)$ is covered by affine $M^{K_f^*}(P_*, \mathcal{X}_*, \Sigma_i)$, each of which is stable under $[\iota]$. Since $M^{K_f^*}(P_*, h(\mathcal{X}_*))$ is the quotient of $M^{K_f^*}(P_*, \mathcal{X}_*)$, and $M_{\mathbb{C}}^{K_f^*}(P_*, h(\mathcal{X}_*), \Sigma_*)$ the quotient of $M_{\mathbb{C}}^{K_f^*}(P_*, \mathcal{X}_*, \Sigma_*)$, our definition gives the desired extension of the canonical model.

12.9. The case $P = GL_{2, \mathbb{Q}}$: Next we prove 12.4 for $(P, \mathcal{X}) = (GL_{2, \mathbb{Q}}, \mathcal{H}_2)$, as defined in 2.7. In this case every K_f -admissible cone decomposition is contained in a unique complete cone decomposition Σ , with $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma) \cong M_{\mathbb{C}}^{K_f}(P, \mathcal{X})^*$. It suffices to consider this Σ . By projectivity it suffices to prove the assertions for a cofinal system of open compact subgroups. So we may assume that $K_f = K_f(d)$ for $d \geq 3$, as in 10.19. Let $\overline{\mathcal{M}}_d$ be the moduli scheme of generalized elliptic curves over $\mathbb{Q}$ with d -structure, and $\mathcal{M}_d \subset \overline{\mathcal{M}}_d$ the open subscheme parametrizing (honest) elliptic curves. By 10.20 and 11.16, we have compatible isomorphisms

$$M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma) \cong \overline{\mathcal{M}}_{d, \mathbb{C}} \text{ and } M^{K_f}(P, \mathcal{X}) \cong \mathcal{M}_d.$$

To prove 12.4 (a), we just set $M^{K_f}(P, \mathcal{X}, \Sigma) := \overline{\mathcal{M}}_d$.

In 12.4 (c), the last statement implies the earlier ones. Since in our case all the pairs $(p_f(P_1, \mathcal{X}_1))$ consisting an element of $P(\mathbb{A}_f)$ and a proper rational boundary component of $(P, \mathcal{X})$, are conjugate under the left action of $GL_2(\mathbb{Q})$ and the right action of $GL_2(\hat{\mathbb{Z}})$ (which normalizes K_f), it suffices to prove that assertion in any one instance. This instance is provided by 10.22, taking into account 11.14 and 12.8.

Now we can prove the assertion for a certain type of boundary components.

12.10. The case of boundary components “of codimension 1” for reductive P : Consider pure Shimura data $(P, \mathcal{X})$ and surjective homomorphism $P \twoheadrightarrow \mathrm{PGL}_{2, \mathbb{Q}}$. The inverse image of any rational Borel subgroup of $\mathrm{PGL}_{2, \mathbb{Q}}$ is an admissible $\mathbb{Q}$ -parabolic subgroup of P . We shall prove 12.4 in the case that Σ “is supported” only on those rational

boundary components associated to such parabolic subgroups. If $(P_1, \mathcal{X}_1)$ is such a rational boundary component, then by assumption U_1 has dimension 1, so as in 12.9, we have to prove the assertion for just one Σ , the maximal one with the above property. In other words, we want to prove the assertion for $M_{\mathbb{C}}^{K_f}(P, \mathcal{X})^+$, as defined in 8.2.

Let us first assume that the surjection $P \twoheadrightarrow \mathrm{PGL}_{2, \mathbb{Q}}$ lifts to a morphism $(P, \mathcal{X}) \rightarrow (\mathrm{GL}_{2, \mathbb{Q}}, \mathcal{H}_2)$. Then $\mathrm{PGL}_{2, \mathbb{Q}}$ lifts to an almost direct factor of P^{der} , isomorphic to $\mathrm{SL}_{2, \mathbb{Q}}$. Let $(P', \mathcal{X}') := (P, \mathcal{X})/\mathrm{SL}_{2, \mathbb{Q}}$ (see 2.9), then since $\mathrm{SL}_2(\mathbb{R})$ is connected, we get an embedding $(P, \mathcal{X}) \hookrightarrow (P', \mathcal{X}') \times (\mathrm{GL}_{2, \mathbb{Q}}, \mathcal{H}_2)$. Since $E(\mathrm{GL}_{2, \mathbb{Q}}, \mathcal{H}_2) = \mathbb{Q}$, we clearly have $E(P, \mathcal{X}) = E(P', \mathcal{X}')$, which is also reflex field of this product. Using 12.9, the assertion of 12.4 follows for the product, with the product of trivial cone decomposition for $(P', \mathcal{X}')$, and the unique maximal one for $(\mathrm{GL}_{2, \mathbb{Q}}, \mathcal{H}_2)$. By descent of closed subschemes ([SGA1]éxp.VIII cor. 1.9), resp. of morphisms, the assertions also follows for $(P, \mathcal{X})$.

In general case, let $\tilde{P}$ be the difference kernel of the two maps $P \times \mathrm{GL}_{2, \mathbb{Q}} \rightarrow \mathrm{PGL}_{2, \mathbb{Q}}$. The inclusion $\tilde{P} \hookrightarrow P \times \mathrm{GL}_{2, \mathbb{Q}}$ corresponds to an embedding $(\tilde{P}, \tilde{\mathcal{X}}) \hookrightarrow (P, \mathcal{X}) \times (\mathrm{GL}_{2, \mathbb{Q}}, \mathcal{H}_2)$ for some Shimura data $(\tilde{P}, \tilde{\mathcal{X}})$. The kernel of the projection $\tilde{P} \twoheadrightarrow P$ is isomorphic to $\mathbb{G}_{m, \mathbb{Q}}$, and $(P, \mathcal{X}) \cong (\tilde{P}, \tilde{\mathcal{X}})/\mathbb{G}_{m, \mathbb{Q}}$. Clearly $E(\tilde{P}, \tilde{\mathcal{X}}) \rightarrow E(P, \mathcal{X})$. By assumption, the desired assertions hold for $(\tilde{P}, \tilde{\mathcal{X}})$. Let $\tilde{\Sigma}, \Sigma$ be the respective partial cone decompositions satisfying the condition above. For all open compact subgroups $\tilde{K}_f$ mapping to K_f , the morphism $M^{\tilde{K}_f}(\tilde{P}, \tilde{\mathcal{X}}, \tilde{\Sigma})(\mathbb{C}) \rightarrow M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is finite; and even surjective, since $H^1(\mathbb{A}, \mathbb{G}_m) = 0$. Thus we may define $M^{K_f}(P, \mathcal{X}, \Sigma)$ as a finite quotient of $M^{\tilde{K}_f}(\tilde{P}, \tilde{\mathcal{X}}, \tilde{\Sigma})$ which exists by quasi-projectivity. This proves 12.4 (a), and the compatibilities (c) follow by descent of morphisms.

As the last special case, we prove 12.4 for certain unipotent extensions of the Shimura data of 12.10.

12.11. Generalization of 12.10: Let $(P, \mathcal{X})$ and Σ be as in 12.10, only this time assume that $(P, \mathcal{X})$ is irreducible. Let $(P', \mathcal{X}')$ be a unipotent extension of $(P, \mathcal{X})$ whose unipotent radical W' is pure of weight - 2, i.e. $W' = U'$. Let Σ' be a finite admissible partial cone decomposition “supported” only on the rational boundary components $(P'_1, \mathcal{X}'_1)$ associated to those $(P_1, \mathcal{X}_1)$ in 12.10. As the last special case, we prove 12.4 for all such Σ' that are sufficiently fine, in the sense explained below, and a cofinal system of $K'_f \subset P'(\mathbb{A}_f)$.

With $(P_*, \mathcal{X}_*)$ defined as in 12.8, $(P', \mathcal{X}')$ is (non-canonically) isomorphic to the fibre product $(P_*, h(\mathcal{X}_*)) \times_{(P_0, h(\mathcal{X}_0))} (P, \mathcal{X})$ (see 2.20). Let $K_f^* = K_f^U \rtimes K_f^T \subset P_*(\mathbb{A}_f)$ be neat, $K_f \subset P(\mathbb{A}_f)$ mapping to K_f^T , and let $K'_f \subset P'(\mathbb{A}_f)$ be their fibre product. By 3.11,

$$M_{\mathbb{C}}^{K'_f}(P', \mathcal{X}') \cong M_{\mathbb{C}}^{K_f^*}(P_*, h(\mathcal{X}_*)) \times_{M_{\mathbb{C}}^{K_f^T}(\mathbb{G}_{m, \mathbb{Q}}, h(\mathcal{H}_0))} M_{\mathbb{C}}^{K_f}(P, \mathcal{X}).$$

Fix any K_f^* -admissible complete cone decomposition Σ_* for $(P_*, h(\mathcal{X}_*))$. By pullback, this Σ_* , together with the partial cone decomposition Σ considered in 12.10 induce a finite K'_f -admissible partial cone decomposition Σ'_0 for $(P', \mathcal{X}')$. Clearly,

$$M_{\mathbb{C}}^{K'_f}(P', \mathcal{X}', \Sigma'_0) \cong M_{\mathbb{C}}^{K_f^*}(P_*, h(\mathcal{X}_*), \Sigma_*) \times_{M_{\mathbb{C}}^{K_f^T}(\mathbb{G}_{m, \mathbb{Q}}, h(\mathcal{H}_0))} M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma);$$

indeed, for any $(P'_1, \mathcal{X}'_1)$ as above we have a corresponding isomorphism

$$M_{\mathbb{C}}^{K_f^{1'}}(P'_1, \mathcal{X}'_1, \Sigma'_{01}) \cong M_{\mathbb{C}}^{K_f^*}(P_*, h(\mathcal{X}_*), \Sigma_*) \times_{M_{\mathbb{C}}^{K_f^T}(\mathbb{G}_{m, \mathbb{Q}}, h(\mathcal{H}_0))} M_{\mathbb{C}}^{K_f^1}(P_1, \mathcal{X}_1).$$

By 12.8 and 12.10 these fibre products descend to the canonical models over $E(P', \mathcal{X}') = E(P, \mathcal{X})$, so we may define

$$M^{K_f'}(P', \mathcal{X}', \Sigma'_0) := M^{K_f^*}(P_*, h(\mathcal{X}_*), \Sigma_*) \times_{M_{\mathbb{C}}^{K_f^T}(\mathbb{G}_{m, \mathbb{Q}}, h(\mathcal{H}_0))} M^{K_f}(P, \mathcal{X}, \Sigma).$$

This proves all of 12.4 for the given Σ'_0 .

More generally, let Σ' be any finite partial refinement of Σ'_0 . Note that, since Σ'_0 has the property that $|\Sigma'_0(\mathcal{X}'^0, P'_1, p'_f)| = C^*(\mathcal{X}'^0, P'_1)$ for all p'_f and all $(P'_1, \mathcal{X}'_1)$ as above, this includes any sufficiently fine decomposition “supported” only on these $(P'_1, \mathcal{X}'_1)$. Then, both

$$M_{\mathbb{C}}^{K_f'}(P', \mathcal{X}', \Sigma') \rightarrow M_{\mathbb{C}}^{K_f'}(P', \mathcal{X}', \Sigma'_0)$$

and the various

$$M_{\mathbb{C}}^{K_f^{1'}}(P'_1, \mathcal{X}'_1, \Sigma'_1) \rightarrow M_{\mathbb{C}}^{K_f^{1'}}(P'_1, \mathcal{X}'_1, \Sigma'_{01})$$

are allowable modifications in the sense of [KKMS] ch.II §2 p.87 def. 3. Since they are formally isomorphic, they correspond to the same “f.r.p.p. decomposition” (see [loc. cit.] p.86 def.2). By the equivalence between allowable modifications and f.r.p.p. decompositions [loc. cit.] p.90 thm. 6*), one of them descends to $E(P, \mathcal{X})$ if and only the other does so. But by 12.8 the second does descend, hence so does the first, and the compatibility is automatic.

12.12. Proof of 12.3 (a): Consider pure mixed Shimura data $(P, \mathcal{X})$ and neat $K_f \subset P(\mathbb{A}_f)$. By 12.10, the canonical model extends to a model $M^{K_f}(P, \mathcal{X})^+$ of $M_{\mathbb{C}}^{K_f}(P, \mathcal{X})^+$, over $E := E(P, \mathcal{X})$. Define $M^{K_f}(P, \mathcal{X})^*$ as the closure of $M^{K_f}(P, \mathcal{X})^+$ inside

$$\mathbb{P}_E^N \cong \mathbb{P}\Gamma(M^{K_f}(P, \mathcal{X})^+, \omega[\mathrm{dlog}]^{\otimes n})$$

for any sufficiently large n . By 8.2, this is a model $M_{\mathbb{C}}^{K_f}(P, \mathcal{X})^*$, as desired.

The following lemma implies that every boundary component can be deformed, along a given collection of tangent directions, into embedded mixed Shimura data of the special type considered in 12.11.

12.13. Embedding Lemma: Let $(P_1, \mathcal{X}_1)$ be a proper rational boundary component of some mixed Shimura data $(P, \mathcal{X})$. Let $U \subset U'_1 \subset U_1$ with $\dim(U'_1/U) = 1$, and such that, for some ($\Leftrightarrow$ for all) connected components $\mathcal{X}^0$ of $\mathcal{X}^+$ (see 4.11),

$$C(\mathcal{X}^0, P_1) \cap U'_1(\mathbb{R})(-1) \neq \emptyset.$$

Then there exists irreducible mixed Shimura data $(P', \mathcal{X}')$, an embedding $(P', \mathcal{X}') \hookrightarrow (P, \mathcal{X})$, and a rational boundary component $(P'_1, \mathcal{X}'_1)$ of $(P', \mathcal{X}')$, such that $(P_1, \mathcal{X}_1)$ is the rational boundary component of $(P, \mathcal{X})$ associated to $(P'_1, \mathcal{X}'_1)$ by functoriality (4.16), and

- (a) P' contains U , and P'/U is reductive,
- (b) $P'_1 \cap W_1 = U'_1$ and $P'_1/U'_1 \xrightarrow{\sim} P_1/W_1$, and
- (c) $(P'/U)^{ad} \cong (P'_1/U'_1)^{ad} \times \mathrm{PGL}_{2,\mathbb{Q}}$.

Proof. Without loss of generality we may assume $U = 1$. Since $C(\mathcal{X}^0, P_1)$ is open, there exists an element $u_1 \in C(\mathcal{X}^0, P_1) \cap U'_1(\mathbb{Q})(-1)$. Take a point $x \in \mathcal{X}^0$ that maps to a point $x_1 \in \mathcal{X}_1$ whose imaginary part is u_1 . By definition (4.14), $(\mathrm{int}(u_1^{-1}) \circ h_{x_1} : \mathbb{S} \rightarrow P_{1,\mathbb{R}})$ is defined over $\mathbb{R}$. Fix a splitting $P_1 = W_1 \rtimes G_1$, defined over $\mathbb{Q}$. Then there exists a unique element $u'_1 \in U_1(\mathbb{R})$ so that $\mathrm{int}(u'_1) \circ \mathrm{int}(u_1^{-1}) \circ h_{x_1}$ factors through $G_{1,\mathbb{R}}$. After replacing x by $u'_1 \cdot x$, and x_1 by $u'_1 \cdot x_1$, which has the same imaginary part, $\mathrm{int}(u_1^{-1}) \circ h_{x_1}$ factors through $G_{1,\mathbb{R}}$.

Next, let ω_x be as in 4.6. Since $U = 1$, is defined over $\mathbb{R}$. Observe that, in the notations of 4.3, there exists a unique nonzero element $u_0 \in U_0(\mathbb{R})(-1)$ such that $\mathrm{int}(u_0^{-1}) \circ h_\infty$ is defined over $\mathbb{R}$. The identity $h_{x_1} = \omega_x \circ h_\infty$ implies $u_1 = \omega_x(u_0)$. Since $\dim(U'_1) = 1$, it follows that $\omega_x(U_0) = U'_{1,\mathbb{R}}$. Now, by 4.9 (b),

$$\mathrm{int}(u_1)(G_{1,\mathbb{C}}) \cdot \omega_x(H_0)_{\mathbb{C}}$$

is a subgroup of $P_{\mathbb{C}}$. Since $u_1 \in \omega_x(H_0)(\mathbb{C})$, this is equal to $G_{1,\mathbb{C}} \cdot \omega_x(H_0)_{\mathbb{C}}$. As in the proof of 4.9 (a), this group has a parabolic subgroup

$$G_{1,\mathbb{C}} \cdot U'_{1,\mathbb{C}} \cdot (\omega_x \circ h_0 \circ w(\mathbb{G}_{m,\mathbb{C}})),$$

which is contained in $(G_1 \cdot U'_1 \cdot Z(P))_{\mathbb{C}}$. Thus $(G_1 \cdot U'_1 \cdot Z(P))_{\mathbb{C}}$ is a parabolic subgroup of the group

$$P'_{\mathbb{C}} := G_{1,\mathbb{C}} \cdot \omega_x(H_0)_{\mathbb{C}} \cdot Z(P)_{\mathbb{C}}.$$

In particular, some parabolic subgroup of $P'_{\mathbb{C}}$ is defined over $\mathbb{Q}$. For the same reason as in the proof of 4.9 (a), $P'_{\mathbb{C}} = P' \times_{\mathbb{Q}} \mathbb{C}$ for a unique subgroup $P' \subset P$.

Now we are practically done. Defining $\mathcal{X}'$ to be the $P'(\mathbb{R})$ -orbit in $\mathcal{X}$ generated by x , we get an embedding $(P', \mathcal{X}') \hookrightarrow (P, \mathcal{X})$. There is an obvious rational boundary component $(U'_1 \rtimes G_1, \mathcal{X}'_1)$ of $(P', \mathcal{X}')$. The assertions (a)–(c) hold by construction. If the $(P', \mathcal{X}')$ so constructed is not irreducible, just replace it by an irreducible component. **q.e.d.**

Next we want to prove effectivity of descent in a certain general situation where the descent data involves formal completions.

12.14. An analog of Čech-resolution: Let $\overline{X}$ be a locally noetherian scheme, and Y a closed subscheme of $\overline{X}$. Let $\hat{X}$ denote the formal completion of $\overline{X}$ along Y . Denote the inclusion $X := \overline{X} \setminus Y \hookrightarrow \overline{X}$ by j , the canonical morphism $\hat{X} \rightarrow \overline{X}$ by k . Since $\overline{X}$ is locally noetherian, [B-AC] ch.III §4 thm. 3 implies that $k_*\mathcal{O}_{\hat{X}}$ is flat over $\mathcal{O}_X$. Since $j \coprod k : X \coprod \hat{X} \rightarrow \overline{X}$ is surjective, it is a faithfully flat morphism of formal schemes.

Claim. There is a canonical exact sequence

$$0 \longrightarrow \mathcal{O}_{\overline{X}} \longrightarrow j_*\mathcal{O}_X \oplus k_*\mathcal{O}_{\hat{X}} \longrightarrow j_*\mathcal{O}_X \otimes_{\mathcal{O}_{\overline{X}}} k_*\mathcal{O}_{\hat{X}},$$

with the morphisms given as $s \mapsto (s, s)$ and $(s, t) \mapsto s \otimes 1 - 1 \otimes t$ respectively. (If $j \coprod k$ were an open covering, this would be the Čech-resolution.)

Proof. The assertion is local in $\overline{X}$, so we may assume $\overline{X}$ to be affine, say $\overline{X} = \mathbf{Spec}(A)$. If $I \subset A$ is the ideal of Y , then $\hat{X}$ is the formal spectrum of $\hat{A}$, the I -adic completion of A . Fix an affine covering $\mathbf{Spec}(\tilde{A}) \rightarrow X$, then by the sheaf property we have an exact sequence

$$0 \longrightarrow A^0 := \Gamma(X, \mathcal{O}_X) \longrightarrow \tilde{A} \longrightarrow \tilde{A} \otimes_A \tilde{A},$$

where the homomorphism on the right is given as $\tilde{a} \mapsto \tilde{a} \otimes 1 - 1 \otimes \tilde{a}$. Since $\tilde{A}$ is flat over A , $\tilde{A} \oplus \tilde{A}$ is faithfully flat, and we get an exact sequence

$$0 \longrightarrow A \longrightarrow \tilde{A} \oplus \hat{A} \longrightarrow (\tilde{A} \oplus \hat{A}) \otimes_A (\tilde{A} \oplus \hat{A}).$$

We shall prove the this sequence stays exact if the rightmost term is replaced by its quotient $\tilde{A} \otimes_A (\tilde{A} \oplus \hat{A})$. Contemplating the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & \tilde{A} \oplus \hat{A} & \longrightarrow & (\tilde{A} \otimes_A \tilde{A}) \oplus (\tilde{A} \otimes_A \hat{A}) \\ & & \parallel & & \uparrow & & \uparrow \\ 0 & \longrightarrow & A & \longrightarrow & A^0 \oplus \hat{A} & \longrightarrow & 0 \oplus A^0 \otimes_A \hat{A} \end{array}$$

we find that this will imply the exactness of the second line, as desired.

To prove the exactness of the first line K be the kernel of the map $A \rightarrow \tilde{A}$. Since $\text{supp}(K) \subset Y$ and A is noetherian, it is annihilated by some power of I , whence $K \otimes_A \hat{A} \cong K$. Tensoring our sequence with the short exact sequence $0 \rightarrow K \rightarrow A \rightarrow A/K \rightarrow 0$, a little diagram chasing shows that it suffices to prove the assertion for A/K in place of A . In other words we may assume that the map $A \rightarrow \tilde{A}$ is injective. In this case, by the flatness of $\hat{A}$, the natural map $\hat{A} \otimes_A \hat{A} \rightarrow \hat{A} \otimes_A \tilde{A} \otimes_A \hat{A}$ is also injective. Now let $(\tilde{a}, \hat{a}) \in \tilde{A} \oplus \hat{A}$ so that $\tilde{a} \otimes 1 = 1 \otimes \hat{a}$ in $\tilde{A} \otimes_A \hat{A}$. In $\tilde{A} \otimes_A \hat{A} \otimes_A \hat{A}$ we can calculate $1 \otimes \hat{a} \otimes 1 = \tilde{a} \otimes 1 \otimes 1 = 1 \otimes 1 \otimes \hat{a}$, whence by injectivity $\hat{a} \otimes 1 = 1 \otimes \hat{a}$ in $\hat{A} \otimes_A \hat{A}$. This fact implies that our sequence stays exact when we drop the terms $\hat{A} \otimes_A (\tilde{A} \oplus \hat{A})$, as required. **q.e.d.**

12.15. Functoriality: Let us consider $j : X \rightarrow \overline{X} \supset Y$ and $k : \hat{X} \rightarrow \overline{X}$ as in 12.14, and another such situation with $j' : X' \rightarrow \overline{X}' \supset Y'$ and $k' : \hat{X}' \rightarrow \overline{X}'$. Any morphism $\bar{\phi} : \overline{X}' \rightarrow \overline{X}$ with $\bar{\phi}^{-1}(Y) = Y'$ induces compatible morphisms $\phi : X' \rightarrow X$ and $\hat{\phi} : \hat{X}' \rightarrow \hat{X}$. In particular we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\overline{X}} & \hookrightarrow & j_* \mathcal{O}_X \oplus k_* \mathcal{O}_{\hat{X}} \\ \downarrow & & \downarrow \\ \bar{\phi}_* \mathcal{O}_{\overline{X}'} & \hookrightarrow \bar{\phi}_* j'_* \mathcal{O}_{X'} \oplus \bar{\phi}_* k'_* \mathcal{O}_{\hat{X}'} \cong & j_* \phi_* \mathcal{O}_{X'} \oplus k_* \hat{\phi}_* \mathcal{O}_{\hat{X}'} \end{array}$$

Claim. If $\hat{\phi}$ is scheme theoretically dominant (cf. 5.6), this diagram is cartesian.

Proof. Contemplating the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_{\overline{X}} & \longrightarrow & j_* \mathcal{O}_X \oplus k_* \mathcal{O}_{\hat{X}} & \longrightarrow & j_* \mathcal{O}_X \otimes_{\mathcal{O}_{\overline{X}}} k_* \mathcal{O}_{\hat{X}} \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \bar{\phi}_* \mathcal{O}_{\overline{X}'} & \longrightarrow & \bar{\phi}_* j'_* \mathcal{O}_{X'} \oplus \bar{\phi}_* k'_* \mathcal{O}_{\hat{X}'} & \longrightarrow & \bar{\phi}_* (j'_* \mathcal{O}_{X'} \otimes_{\mathcal{O}_{\overline{X}}} k'_* \mathcal{O}_{\hat{X}'}) \end{array}$$

we see it suffices to show that the vertical arrow on the right is injective. Again the assertion is local on $\overline{X}$, so let $A \supset I$, $\tilde{A}$, A^0 , and $\hat{A}$ be as in 12.14. Let $\mathbf{Spec}(A') \rightarrow \overline{X}'$ be an affine

covering of $\overline{X}'$, and define $\tilde{A}'$, $\hat{A}'$ likewise. Since $A^0 \otimes_A \hat{A} \hookrightarrow \tilde{A} \otimes_A \hat{A}$, it suffices to prove the injectivity of the natural homomorphism

$$\tilde{A} \otimes_A \hat{A} \longrightarrow \tilde{A} \otimes_A \tilde{A}' \otimes_{A'} \hat{A}'.$$

By the assumption on $\hat{\phi}$, we have $\hat{A} \hookrightarrow \hat{A}'$. Thus, by tensoring with the A -flat $\tilde{A}$, we get $\tilde{A} \otimes_A \hat{A} \hookrightarrow \tilde{A} \otimes_A \hat{A}' \cong (\tilde{A} \otimes_A A') \otimes_{A'} \hat{A}'$. On the other hand, $\tilde{A} \otimes_A A'$ injects into $\tilde{A} \otimes_A \tilde{A}'$ since

$$\begin{aligned} \mathbf{Spec}(\tilde{A} \otimes_A \tilde{A}') &\cong \mathbf{Spec}(\tilde{A}) \times_X \mathbf{Spec}(\tilde{A}') \\ &\longrightarrow \mathbf{Spec}(\tilde{A}) \times_X X' \cong \mathbf{Spec}(\tilde{A}) \times_{\overline{X}} \overline{X}' \cong \mathbf{Spec}(\tilde{A} \otimes_A A') \end{aligned}$$

is faithfully flat. Since $\hat{A}'$ is flat over A' , the assertion follows. **q.e.d.**

12.16. Descent Lemma: Let $T \rightarrow S$ be a faithfully flat, quasicompact morphism of locally noetherian schemes. Let $X_T \subset \overline{X}_T \supset Y_T$ and $\hat{X}_T$ be as in 12.14, this time as schemes over T . Let $\overline{L}_T$ be an invertible sheaf on $\overline{X}_T$, $L_T := \overline{L}_T|_{X_T}$, and let $\hat{L}_T$ be the associated invertible sheaf on $\hat{X}_T$. Assume that we are given models (X, L) of (X_T, L_T) , and $(\mathcal{X}, \mathcal{L})$ of $(\hat{X}_T, \hat{L}_T)$, over S . Consider another situation as in 12.14, this time with schemes $X' \subset \overline{X}' \supset Y'$, $\hat{X}'$ over S , and with compatible invertible sheaves $\overline{L}'$, L' , $\hat{L}'$. Finally, suppose that we are given a morphism $\overline{\phi} : \overline{X}'_T \rightarrow \overline{X}_T$ over T such that $\overline{\phi}^{-1}(Y_T) = Y'_T$, and an isomorphism $\overline{\psi} : \overline{\phi}^* \overline{L}_T \xrightarrow{\sim} \overline{L}'_T$. They induce isomorphisms $\phi : X'_T \rightarrow X_T$ and $\hat{\phi} : \hat{X}'_T \rightarrow \hat{X}_T \cong \mathcal{X}_T$, and isomorphisms $\psi : \phi^*(L_T) \xrightarrow{\sim} L'_T$ and $\hat{\psi} : \hat{\phi}^*(\hat{L}_T) \xrightarrow{\sim} \hat{L}'_T$. Let us assume that

- (i) (ϕ, ψ) descends to S .
- (ii) $(\hat{\phi}, \hat{\psi})$ descends to S .
- (iii) The morphism $\hat{\phi} : \hat{X}'_T \rightarrow \mathcal{X}_T$ is scheme-theoretically dominant.
- (iv) $\overline{L}_T$ is relatively ample with respect to $\overline{X}_T \rightarrow T$.

Claim. There exists a unique model $(\overline{X}, \overline{L})$ for $(\overline{X}_T, \overline{L}_T)$ over S , compatible with all the morphisms above.

Proof. For any nonnegative integer n , we have a commutative diagram for $\overline{L}_T^{\otimes n}$, analogous to that in 12.15:

$$\begin{array}{ccc} \overline{L}_T^{\otimes n} & \hookrightarrow & j_* L_T^{\otimes n} \oplus k_* \hat{L}_T^{\otimes n} \\ \downarrow \phi_* & & \downarrow \\ \phi_* \overline{L}_T^{\otimes n} & \hookrightarrow & \phi_* j_* L_T^{\otimes n} \oplus \phi_* k_* \hat{L}_T^{\otimes n} \cong j_* L_T^{\otimes n} \oplus k_* \hat{\phi}_* \hat{L}_T^{\otimes n} \end{array}$$

Since $\overline{L}_T$ is locally free, 12.15 implies that this diagram is also cartesian. Denote the structure morphisms by $\overline{f} : \overline{X}_T \rightarrow T$, $f : X_T \rightarrow T$, and $\hat{f} : \hat{X}_T \rightarrow T$ respectively, and analogously for $\overline{X}'$, etc. Applying $\overline{f}_*$ to the above diagram, we obtain the diagram

$$\begin{array}{ccc} \overline{f}_*(\overline{L}_T^{\otimes n}) & \hookrightarrow & f_{T,*}(L_T^{\otimes n}) \oplus \hat{f}_{T,*}(\hat{L}_T^{\otimes n}) \\ \downarrow \overline{f}'_* & & \downarrow \\ \overline{f}'_*(\overline{L}_T^{\otimes n}) & \hookrightarrow & f'_{T,*}(L_T^{\otimes n}) \oplus \hat{f}'_{T,*}(\hat{L}_T^{\otimes n}) \end{array}$$

which is, by the left exactness of $\bar{f}_*$, again cartesian. In particular, $\bar{f}_*(\bar{L}_T^{\otimes n})$ is uniquely determined as a subsheaf of $f_{T,*}(L_T^{\otimes n}) \oplus \hat{f}_{T,*}(\hat{L}_T^{\otimes n})$ by the bottom part and the right hand part of this diagram. Now by our data, both these parts come from sheaves and morphisms over S . Invoking [SGA1] éxp.VIII cor. 1.8, this shows that $\bar{f}_*(\bar{L}_T^{\otimes n})$ comes from a canonical quasi-coherent sheaf $\mathfrak{A}_n$ on S .

Put $\mathfrak{A} := \bigoplus_{n \geq 0} \mathfrak{A}_n$, then its pullback to T is canonically isomorphic to the graded algebra $\bigoplus_{n \geq 0} \bar{f}_*(\bar{L}_T^{\otimes n})$. By descent of homomorphisms of sheaves (see [SGA1] éxp.VIII cor. 1.2), the algebra structure descends to one on $\mathfrak{A}$. Put $P := \mathbf{Proj}(\mathfrak{A})$, then by the relative ampleness of $\bar{L}_T$, there is a canonical open embedding $\bar{X}_T \hookrightarrow P_T$. The image of this embedding is the union of the images of X_T and of Y_T , each of which is stable by the descent data on P_T . Thus, by [SGA1] éxp.VIII cor. 1.9, the complement of $\bar{X}_T$ in P_T descends to a closed subscheme of P , hence $\bar{X}_T$ descends to an open subscheme of P , as desired. The compatibility with the various morphisms holds by construction. **q.e.d.**

12.17. End of the proof of 12.4: It remains to prove 12.4 (a), and that in 12.4 (c) the isomorphism of formal neighborhoods of $\mathrm{Stab}_{\Delta_1}([\sigma])M^{\pi([\sigma])(K_f^1)}(P_{1,[\sigma]}, \mathcal{X}_{1,[\sigma]})$ descends to $E := E(P, \mathcal{X})$. Let $(P, \mathcal{X})$, K_f , Σ be as in 12.4. It suffices to prove the assertion for every Σ_i , so we may assume $\Sigma = \Sigma_i$ and Σ is finite. Then, by assumption and by 8.13, there is a line bundle $M_{\mathbb{C}}^{K'_f}(P', \mathcal{X}', \Sigma') \rightarrow M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma)$ so that, if $\mathcal{M}$ denotes the associated invertible sheaf, $\mathcal{M}^{\otimes -1}$ is relatively ample with respect to $M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma) \rightarrow M_{\mathbb{C}}^{\pi(K_f)}((P, \mathcal{X})/W)^*$. By 7.13 we may further assume that the condition 7.12 (*) is satisfied for Σ .

We proceed by induction over the (finite) number of double classes in $P(\mathbb{Q}) \backslash \Sigma / K_f$. Fix $p_f \in P(\mathbb{A}_f)$, and a rational boundary component $(P_1, \mathcal{X}_1)$ of $(P, \mathcal{X})$. Let $K_f^1 := P_1(\mathbb{A}_f) \cap p_f \cdot K_f \cdot p_f^{-1}$ and $\Sigma_1^0 := (([\cdot p_f]^* \Sigma)_{|(P_1, \mathcal{X}_1)})^0$, and fix a double coset $[\sigma] \in P_1(\mathbb{Q}) \backslash \Sigma_1^0 / K_f^1$ such that $\sigma^0 \subset C(\mathcal{X}^0, P_1)$ for some $\mathcal{X}^0$. By gluing, it suffices to prove the assertions in the case that Σ is the smallest K_f -admissible partial cone decomposition that contains σ . Let $\Sigma_0 \subset \Sigma$ be the unique K_f -admissible partial cone decomposition that contains all proper faces of σ , but not σ itself. Let $\Sigma'_0 \subset \Sigma'$ be analogous, then we may assume that both $M^{K_f}(P, \mathcal{X}, \Sigma_0)$ and $M^{K'_f}(P', \mathcal{X}', \Sigma'_0)$ exist. We have to prove the same for $M^{K_f}(P, \mathcal{X}, \Sigma)$ and $M^{K'_f}(P', \mathcal{X}', \Sigma')$, and that the isomorphism 9.37 descends to E . This will follow by applying 12.16. Without loss of generality, we may assume $p_f = 1$. By 12.8, we may assume that $(P_1, \mathcal{X}_1)$ is a proper rational boundary component of $(P, \mathcal{X})$.

In the setup of 12.16, put $T := \mathbf{Spec}(\mathbb{C})$ and $S := \mathbf{Spec}(E)$. As models over S we define $X := M^{K_f}(P, \mathcal{X}, \Sigma_0)$, and let $\mathcal{X}$ be the formal completion of $\mathrm{Stab}_{\Delta_1}([\sigma]) \backslash M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1^0)$ along the stratum $Y := \mathrm{Stab}_{\Delta_1}([\sigma]) \backslash M^{\pi([\sigma])(K_f^1)}(P_{1,[\sigma]}, \mathcal{X}_{1,[\sigma]})$. We let $\bar{X}_T := M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma)$, with the isomorphism $\hat{X}_T \cong \mathcal{X} \times_S T$ given by 9.37. For the line bundle, fix any ample line bundle $\mathcal{N}$ on $M^{\pi(K_f)}((P, \mathcal{X})/W)^*$. Then, for some positive integer n , $\bar{L}_T := \mathcal{M}^{\otimes -1} \otimes \mathcal{N}_T^{\otimes n}$ is ample. By assumption, its restriction to X_T has an obvious model over S , and the same holds for its pullback to $\mathcal{X}_T$; with $M^{K'_f}(P', \mathcal{X}', \Sigma'_0) \rightarrow M^{K_f}(P, \mathcal{X}, \Sigma_0)$, respectively

$$\mathrm{Stab}_{\Delta_1}([\sigma]) \backslash M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1^0) \rightarrow \mathrm{Stab}_{\Delta_1}([\sigma]) \backslash M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1^0)$$

in place of $M^{K'_f}(P', \mathcal{X}', \Sigma') \rightarrow M^{K_f}(P, \mathcal{X}, \Sigma)$.

To define $\overline{X}'$, we apply 12.13 to $(P, \mathcal{X})$ with all $U \subset U'_1 \subset U_1$ such that $\sigma^0 \cap U'_1(\mathbb{R})(-1) \neq \emptyset$. Denote by $i_\alpha : (P_\alpha, \mathcal{X}_\alpha) \hookrightarrow (P, \mathcal{X})$ all embedded mixed Shimura varieties thus obtained. Denote by $w_{1,f}^\beta$ all elements of $W_1(\mathbb{A}_f)$. For all α and β , let $K_f^{\alpha\beta}$ be some open compact subgroup inside $P_\alpha(\mathbb{A}_f) \cap w_{1,f}^\beta \cdot K_f \cdot (w_{1,f}^\beta)^{-1}$, and let $\Sigma_{\alpha\beta}^*$ denote the smallest $K_f^{\alpha\beta}$ -admissible partial cone decomposition for $(P_\alpha, \mathcal{X}_\alpha)$ that contains $U_\alpha(\mathbb{R})(-1) \cap \sigma \cdot (w_{1,f}^\beta)^{-1}$. By 12.11, we can choose $K_f^{\alpha\beta}$ and a refinement $\Sigma_{\alpha\beta}$ of $\Sigma_{\alpha\beta}^*$ such that 12.4 holds for $(P_\alpha, \mathcal{X}_\alpha)$, $K_f^{\alpha\beta}$ and $\Sigma_{\alpha\beta}$. Observe that, using 12.1, the characterization in 12.13 implies $E(P_\alpha, \mathcal{X}_\alpha) = E(P_1, \mathcal{X}_1) = E$. The inverse image of X_T under the map

$$[\cdot w_{1,f}^\beta] \circ [i_\alpha] : M_{\mathbb{C}}^{K_f^{\alpha\beta}}(P_\alpha, \mathcal{X}_\alpha, \Sigma_{\alpha\beta}) \rightarrow M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma) = \overline{X}_T$$

is just $M_{\mathbb{C}}^{K_f^{\alpha\beta}}(P_\alpha, \mathcal{X}_\alpha, \Sigma_{\alpha\beta 0})$ for some partial cone decomposition $\Sigma_{\alpha\beta 0} \subset \Sigma_{\alpha\beta}$. We let $\overline{X}'$ be the disjoint union of all $M_{\mathbb{C}}^{K_f^{\alpha\beta}}(P_\alpha, \mathcal{X}_\alpha, \Sigma_{\alpha\beta})$, with $\overline{\phi} : \overline{X}'_T \rightarrow \overline{X}_T$ the disjoint union of the above morphisms, and X' the disjoint union of all $M_{\mathbb{C}}^{K_f^{\alpha\beta}}(P_\alpha, \mathcal{X}_\alpha, \Sigma_{\alpha\beta 0})$.

By 8.7, the pullback of the line bundle $M_{\mathbb{C}}^{K'_f}(P', \mathcal{X}', \Sigma') \rightarrow M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma)$ is the line bundle $M_{\mathbb{C}}^{K_{\alpha\beta}^{\alpha\beta'}}(P'_\alpha, \mathcal{X}'_\alpha, \Sigma'_\alpha) \rightarrow M_{\mathbb{C}}^{K_f^{\alpha\beta}}(P_\alpha, \mathcal{X}_\alpha, \Sigma_{\alpha\beta})$ for some $(P'_\alpha, \mathcal{X}'_\alpha)$, Σ'_α and $K_{\alpha\beta}^{\alpha\beta'}$, for some α . Thus by 12.11 it has an obvious model over S . By the same argument as in 12.6, the composite morphism $[\pi]^* \circ [\cdot w_{1,f}^\beta] \circ [i_\alpha]$ descends to a morphism $M_{\mathbb{C}}^{K_f^{\alpha\beta}}(P_\alpha, \mathcal{X}_\alpha, \Sigma_{\alpha\beta}) \rightarrow M^{\pi(K_f)}((P, \mathcal{X})/W)^*$. Using the pullback of $\mathcal{N}$ under this morphism, we get an obvious model of $\overline{\phi}^*(\overline{L}_T)$ over S .

Now all objects needed in 12.16 have been defined. The conditions (i), (ii), and (iv) hold by construction. To get the desired assertion, it remains to show that the morphism $\hat{\phi} : \hat{X}'_T \rightarrow \mathcal{X}_T$ is scheme-theoretically dominant. Recall how this morphism is defined: for all α and β , it comes by formal completion, and by taking the quotient by $\text{Stab}_{\Delta_1}([\sigma])$, from a morphism

$$[\cdot w_{1,f}^\beta] \circ [i_\alpha] : M_{\mathbb{C}}^{K_{\alpha\beta 1}^{\alpha\beta 1}}(P_{\alpha 1}, \mathcal{X}_{\alpha 1}, \Sigma_{\alpha\beta 1}) \rightarrow M_{\mathbb{C}}^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1^0),$$

where $(P_{\alpha 1}, \mathcal{X}_{\alpha 1})$ is a rational boundary component of $(P_\alpha, \mathcal{X}_\alpha)$. It suffices to prove the same assertion without taking the quotient by $\text{Stab}_{\Delta_1}([\sigma])$.

By 6.8–9, since $(P_1, \mathcal{X}_1)$ is a proper rational boundary component, $M_{\mathbb{C}}^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1^0)$ is a relative torus embedding over $M_{\mathbb{C}}^{\pi'(K_f^1)}((P_1, \mathcal{X}_1)/U_1)$ with respect to the split torus $\mathbb{G}_{m,\mathbb{C}} \otimes (U_1(\mathbb{Q}) \cap K_f^1)$. A neighborhood of our stratum is given by the affine relative torus embedding with respect to the cone $\lambda(\sigma) \subset U_1(\mathbb{R})$. Let us introduce new notation. Let T be the torus over $\mathbb{C}$ with cocharacter group $\mathbb{R} \cdot \lambda(\sigma) \cap U_1(\mathbb{Q}) \cap K_f^1$. Then

$$X := M_{\mathbb{C}}^{K_f^1}(P_1, \mathcal{X}_1) \longrightarrow S := M_{\mathbb{C}}^{\pi_{[\sigma]}(K_f^1)}(P_{1,[\sigma]}, \mathcal{X}_{1,[\sigma]})$$

is a T -torsor, and letting $\hat{X}_{\lambda(\sigma)}$ be as in 5.7, our formal scheme $\mathcal{X}_T$ becomes a finite quotient of $\hat{X}_{\lambda(\sigma)}$.

Next,

$$M_{\mathbb{C}}^{K_f^{\alpha\beta 1}}(P_{\alpha 1}, \mathcal{X}_{\alpha 1}, \Sigma_{\alpha\beta 1}^0) \rightarrow M_{\mathbb{C}}^{\pi(K_f^{\alpha\beta 1})}((P_{\alpha 1}, \mathcal{X}_{\alpha 1})/W_{\alpha 1})$$

is a relative torus embedding with respect to the torus with cocharacter group $U_{\alpha 1}(\mathbb{Q}) \cap K_f^{\alpha\beta 1}$. We are only interested in a neighborhood of those strata that map to the closed stratum of $\hat{X}_{\lambda(\sigma)}$; this is a torus embedding with respect to the subtorus $T_{\alpha\beta}$ with cocharacter group $\mathbb{R} \cdot \lambda(\sigma) \cap U_{\alpha 1}(\mathbb{Q}) \cap K_f^{\alpha\beta 1}$. The morphism

$$[\cdot w_{1,f}^{\beta}] \circ [i_{\alpha}] : M_{\mathbb{C}}^{K_f^{\alpha\beta 1}}(P_{\alpha 1}, \mathcal{X}_{\alpha 1}) \rightarrow M_{\mathbb{C}}^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1^0)$$

is still defined if we replace $K_f^{\alpha\beta 1}$ by $K_f^{\alpha\beta 1} \cdot (\mathbb{R} \cdot \lambda(\sigma) \cap U_{\alpha 1}(\mathbb{Q}) \cap K_f^1)$, which corresponds to dividing by a finite subgroup of $T_{\alpha\beta}$. Then $T_{\alpha\beta}$ becomes canonically identified with a subtorus of T . Our neighborhood is the torus embedding with respect to some partial cone decompositions of $\mathbb{R} \cdot \lambda(\sigma) \cap U_{\alpha 1}(\mathbb{R})$ with support $\lambda(\sigma) \cap U_{\alpha 1}(\mathbb{R})$. Let $\tau_{\alpha\beta}^{\gamma}$ denote all those cones in this decomposition for which $(\tau_{\alpha\beta}^{\gamma})^0 \subset \lambda(\sigma)^0$. Since W_1 centralizes U_1 , we find that neither $T_{\alpha\beta} \subset T$, nor the set of these $\tau_{\alpha\beta}^{\gamma}$ depends on $w_{1,f}^{\beta}$. Therefore we may write T_{α} and τ_{α}^{γ} . Fixing α and γ , we now let $T_{\alpha}^{\gamma} \subset T_{\alpha}$ be the subtorus with cocharacter group $\mathbb{R} \cdot \tau_{\alpha}^{\gamma} \cap U_{\alpha 1}(\mathbb{Q}) \cap K_f^1$. We let $X_{\alpha}^{\gamma} \rightarrow S_{\alpha}^{\gamma}$ be the disjoint union of the T_{α}^{γ} -torsors

$$M_{\mathbb{C}}^{K_f^{\alpha\beta 1}}(P_{\alpha 1}, \mathcal{X}_{\alpha 1}) \rightarrow M_{\mathbb{C}}^{\pi_{[\tau_{\alpha}^{\gamma}]}(K_f^{\alpha\beta 1})}(P_{\alpha 1, [\tau_{\alpha}^{\gamma}]}, \mathcal{X}_{\alpha 1, [\tau_{\alpha}^{\gamma}]})$$

for all β . The maps $[\cdot w_{1,f}^{\beta}] \circ [i_{\alpha}]$ induce a T_{α}^{γ} -equivariant morphism $(\Psi_{\alpha}^{\gamma}, \psi_{\alpha}^{\gamma}) : (X_{\alpha}^{\gamma}, S_{\alpha}^{\gamma}) \rightarrow (X, S)$. Since the induced morphism of the formal completions $\hat{\Psi}_{\alpha}^{\gamma} : \hat{X}_{\alpha, \tau_{\alpha}^{\gamma}}^{\gamma} \rightarrow \hat{X}_{\lambda(\sigma)}$ factors through the morphism which we have to prove to be scheme-theoretically dominant, it suffices to prove that the disjoint union of all $\hat{\Psi}_{\alpha}^{\gamma}$ is scheme-theoretically dominant.

This will follow by applying 5.7. In fact, conditions (i) and (ii) of 5.7 hold by construction. To finish, it therefore remains to prove that every $\psi_{\alpha}^{\gamma} : S_{\alpha}^{\gamma} \rightarrow S$ is scheme-theoretically dominant. Since S is a normal scheme, it suffices to show that its image is Zariski-dense. Recall that by 12.13 (b), $P_{\alpha 1}$ has the same reduced part as P_1 . Thus $P_1(\mathbb{A}_f) = P_{\alpha 1}(\mathbb{A}_f) \cdot W_1(\mathbb{A}_f)$, which by the definition of ψ_{α}^{γ} , implies that its image is the same as the union of the images of the maps

$$[\cdot p_{1, [\sigma], f}] \circ [i_{\alpha}] : M_{\mathbb{C}}^{K_f^{\alpha 1*}}(P_{\alpha 1, [\tau_{\alpha}^{\gamma}]}, \mathcal{X}_{\alpha 1, [\tau_{\alpha}^{\gamma}]}) \longrightarrow M_{\mathbb{C}}^{\pi_{[\sigma]}(K_f^1)}(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]})$$

for all $p_{1, [\sigma], f} \in P_{1, [\sigma]}(\mathbb{A}_f)$, and some $K_f^{\alpha 1*}$ for which these morphisms are defined. By 11.7 this is Zariski-dense, as desired. This finishes the proof of 12.3 and of 12.4. **q.e.d.**

As an application of 12.4, we study q -expansions in our setting.

12.18. Completions and coherent sheaves: Consider irreducible mixed Shimura data $(P, \mathcal{X})$, a neat open compact subgroup $K_f \subset P(\mathbb{A}_f)$, a K_f -admissible cone decomposition Σ for $(P, \mathcal{X})$. Assume that Σ is concentrated in the unipotent fibre, and that there exists a morphism $(P, \mathcal{X}) \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$. Then by 6.8–9 (and, of course, 12.4) $M^{K_f}(P, \mathcal{X}, \Sigma)$ is a

relative torus embedding with respect to a split torus $\mathbb{G}_m \otimes \Gamma_U$ and a rational partial polyhedral cone decomposition Σ_U of $U(\mathbb{R})$. Fix a coset $[\sigma] \in \Sigma/P(\mathbb{A}_f)$, and assume that Σ is the smallest K_f -admissible cone decomposition containing σ . Then Σ_U consists of all faces of the single cone $\lambda(\sigma)$, for λ associated to the connected component $\mathcal{X}^0$ with $\sigma \in \Sigma(P, \mathcal{X}^0, p_f)$. Letting $T := \mathbb{G}_m \otimes (\Gamma_U \cap \mathbb{R} \cdot \sigma)$, we have $\mathbb{R} \cdot \sigma = Y_*(T)_{\mathbb{R}}$, and $M^{K_f}(P, \mathcal{X}, \Sigma)$ is also a torus embedding with respect to T . The closed stratum is just $M^{\pi_{[\sigma]}}(K_f)(P_{[\sigma]}, \mathcal{X}_{[\sigma]})$ in the notation of 7.1. Let us denote the canonical projection $M^{K_f}(P, \mathcal{X}, \Sigma) \rightarrow M^{\pi_{[\sigma]}}(K_f)(P_{[\sigma]}, \mathcal{X}_{[\sigma]})$ by π_{σ} .

Every character $\chi \in X^*(T) = \text{Hom}(\Gamma_U \cap \mathbb{R} \cdot \lambda(\sigma), \mathbb{Z})$ induces a morphism $U(\mathbb{Q}) \cap \mathbb{R} \cdot \lambda(\sigma)/\mathbb{Q} \cdot \ker(\chi) \cong \mathbb{Q} = U_0(\mathbb{Q})$, and so defines the structure of a $((P_0, \mathcal{X}_0) \rightarrow (\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0))$ -torsor on $(P, \mathcal{X})/\langle \ker(\chi) \rangle \rightarrow (P_{[\sigma]}, \mathcal{X}_{[\sigma]})$. This defines a $\mathbb{G}_m$ -torsor structure on

$$M^{(K_f \bmod \langle \ker(\chi) \rangle)}((P, \mathcal{X})/\langle \ker(\chi) \rangle) \longrightarrow M^{\pi_{[\sigma]}}(K_f)(P_{[\sigma]}, \mathcal{X}_{[\sigma]}).$$

Let us denote the associated invertible sheaf by $\mathcal{L}_{\chi}$. By 5.13, the direct image under π_0 of the structure sheaf of $M^{K_f}(P, \mathcal{X}, \Sigma)$ is the direct sum of all those $\mathcal{L}_{\chi}$, for which χ is nonnegative on $\lambda(\sigma)$. Denote by $\hat{M}$ the formal completion of $M^{K_f}(P, \mathcal{X}, \Sigma)$ along the closed stratum, and by $\hat{\pi}_{\sigma} : \hat{M} \rightarrow M^{\pi_{[\sigma]}}(K_f)(P_{[\sigma]}, \mathcal{X}_{[\sigma]})$ the retraction induced by π_0 . By 5.13, we have a canonical isomorphism

$$\hat{\pi}_{\sigma,*} \mathcal{O}_{\hat{M}} \cong \prod_{\chi} \mathcal{L}_{\chi},$$

the sum extended over the same $\chi \in X^*(T)$.

More generally, let $\mathcal{F}$ be a locally free coherent sheaf on $M^{K_f}(P, \mathcal{X}, \Sigma)$, and $\hat{\mathcal{F}}$ the associated coherent sheaf on $\hat{M}$. Assume that we are given a coherent sheaf $\mathcal{G}$ on $M^{\pi_{[\sigma]}}(K_f)(P_{[\sigma]}, \mathcal{X}_{[\sigma]})$ and an isomorphism $\mathcal{F} \cong \pi_{\sigma}^* \mathcal{G}$, whence an isomorphism $\hat{\mathcal{F}} \cong \hat{\pi}_{\sigma,*} \mathcal{G}$. Then we have canonical isomorphisms

$$\pi_{\sigma,*} \mathcal{F} \cong \bigoplus_{\chi} \mathcal{L}_{\chi} \otimes \mathcal{G} \text{ and } \hat{\pi}_{\sigma,*} \hat{\mathcal{F}} \cong \prod_{\chi} \mathcal{L}_{\chi} \otimes \mathcal{G}.$$

In particular, for any $E = E(P, \mathcal{X})$ -algebra R the given data determines an isomorphism

$$\Gamma(\hat{M} \times_E \mathbf{Spec}(R), \hat{\mathcal{F}}) \cong \prod_{\chi} \Gamma(M^{\pi_{[\sigma]}}(K_f)(P_{[\sigma]}, \mathcal{X}_{[\sigma]}) \times_E \mathbf{Spec}(R), \mathcal{L}_{\chi} \otimes \mathcal{G}).$$

12.19. Definition of q -expansions: Let $M^{K_f}(P, \mathcal{X}, \Sigma)$ be a toroidal compactification; for simplicity we assume that K_f is neat. Fix a stratum $\text{Stab}_{\Delta_1}([\sigma]) \setminus M^{\pi_{[\sigma]}}(K_f^1)(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]})$ of $M^{K_f}(P, \mathcal{X}, \Sigma) \setminus M^{K_f}(P, \mathcal{X})$. Let $\hat{M}_1$ denote the formal completion of $M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1^0)$ along $M^{\pi_{[\sigma]}}(K_f^1)(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]})$, and $\hat{\pi}_1 : \hat{M}_1 \rightarrow M^{\pi_{[\sigma]}}(K_f^1)(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]})$ the retraction, as in 12.18. Let $\mathcal{F}$ be a locally free coherent sheaf on $M^{K_f}(P, \mathcal{X}, \Sigma)$, and $\hat{\mathcal{F}}_1$ its pullback to $\hat{M}_1$ under the morphism 12.4. Assume that we are given an isomorphism $\hat{\mathcal{F}}_1 \cong \hat{\pi}_1^* \mathcal{G}_1$, for some locally free sheaf $\mathcal{G}_1$ on $M^{\pi_{[\sigma]}}(K_f^1)(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]})$. Let $E = E(P, \mathcal{X})$, and R an E -algebra. By pullback of sections and by 12.18, this data determines a map

$$\begin{aligned} \Gamma(M^{K_f}(P, \mathcal{X}, \Sigma) \times_E \mathbf{Spec}(R), \mathcal{F}) \\ \longrightarrow \prod_{\chi} \Gamma(M^{\pi_{[\sigma]}}(K_f^1)(P_{1, [\sigma]}, \mathcal{X}_{1, [\sigma]}) \times_E \mathbf{Spec}(R), \mathcal{L}_{\chi} \otimes \mathcal{G}_1), \end{aligned}$$

the product being extended over all χ as in 12.18. This map can be interpreted as associating to each section of $\mathcal{F}$ its “ q -expansion coefficients”.

For certain $\mathcal{F}$ this can be made more explicit. Assume first that $\mathcal{F}$ is defined in terms of the internal geometry of our toroidal compactifications: for instance in terms of sheaves of differentials possibly of logarithmic poles (compare 8.1), for the given $(P, \mathcal{X})$ or for other mixed Shimura varieties, and/or in terms of line bundles in 8.6. Then the same definition gives rise to a coherent sheaf $\mathcal{F}_1$ on $M_f^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1^0)$, and 12.4 (c) yields a canonical isomorphism between $\hat{\mathcal{F}}_1$ and the coherent sheaf on $\hat{M}_1$ associated to $\mathcal{F}_1$. Next, by 8.1 a canonical $\mathcal{G}_1$ exists in the case $\mathcal{F} = \omega[\mathrm{dlog}]$. By functoriality, the same follows for any other sheaf defined in terms of $\omega[\mathrm{dlog}]$ for the given $(P, \mathcal{X})$ or for other mixed Shimura varieties. In particular, for $\mathcal{F} = \omega[\mathrm{dlog}]^{\otimes n}$ and Σ complete, the above definition of q -expansion applies to the space

$$\Gamma(M^{K_f}(P, \mathcal{X}, \Sigma) \times_E \mathbf{Spec}(R), \omega[\mathrm{dlog}]^{\otimes n})$$

of all R -valued automorphic forms f of a certain weight. The “ q -expansion” of f is a (in general infinite) collection of R -valued automorphic forms $f_{1,\alpha}$ on $M^{\pi[\sigma]}(K_f^1)(P_{1,[\sigma]}, \mathcal{X}_{1,[\sigma]})$ of certain other weights. These automorphic forms are generalizations of Jacobi modular forms; the condition at infinity being played by extendability to some toroidal compactification.

12.20. A q -expansion principle for $E(P, \mathcal{X})$ -rationality: In the situation of 12.19, let us now consider a finite number of strata $\mathrm{Stab}_{\Delta_i}([\sigma_i]) \backslash M^{\pi[\sigma_i]}(K_f^i)(P_{i,[\sigma_i]}, \mathcal{X}_{i,[\sigma_i]})$, and write $\hat{M}_i, \hat{\mathcal{F}}_i, \hat{\pi}_i$ accordingly. We assume that the map

$$\coprod_i \mathrm{Stab}_{\Delta_i}([\sigma_i]) \backslash M^{\pi[\sigma_i]}(K_f^i)(P_{i,[\sigma_i]}, \mathcal{X}_{i,[\sigma_i]})(\mathbb{C}) \longrightarrow M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$$

induces a surjection on the sets of connected components. Then, since $M^{K_f}(P, \mathcal{X}, \Sigma)$ is a normal scheme, our assumption implies that pullback of sections induces injections

$$\begin{array}{ccc} \Gamma(M^{K_f}(P, \mathcal{X}, \Sigma), \mathcal{F}) & \hookrightarrow & \bigoplus_i \Gamma(\hat{M}_i, \hat{\mathcal{F}}_i) \\ \downarrow & & \downarrow \\ \Gamma(M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma), \mathcal{F}) & \hookrightarrow & \bigoplus_i \Gamma(\hat{M}_{i,\mathbb{C}}, \hat{\mathcal{F}}_i) \end{array}$$

Clearly this diagram is cartesian, i.e. $\Gamma(M^{K_f}(P, \mathcal{X}, \Sigma), \mathcal{F})$ consists of those sections in $\Gamma(M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma), \mathcal{F})$ whose pullback to every $\hat{M}_{i,\mathbb{C}}$ is already defined over E . If for every i we are given a product decomposition $\hat{\pi}_{i,*}(\hat{\mathcal{F}}_i) \cong \prod_{\alpha} (\mathcal{G}_i \otimes \mathcal{L}_{i,\alpha})$, then this means that a section in $\Gamma(M_{\mathbb{C}}^{K_f}(P, \mathcal{X}, \Sigma), \mathcal{F})$ is defined over E if and only if every “coefficient in the q -expansion” is defined over E , i.e. lies in

$$\Gamma(M^{\pi[\sigma_i]}(K_f^i)(P_{i,[\sigma_i]}, \mathcal{X}_{i,[\sigma_i]}), \mathcal{G}_1 \otimes CL_{1,\alpha}).$$

Sometimes the above surjectivity condition already holds for just one stratum. This is so for instance if $(P, \mathcal{X})$ is irreducible, the semisimple part, $(P/W)^{\mathrm{der}}$, of P is simply connected,

and there exists an algebraic “local cross-section” $(P/W)/(P/W)^{der} \hookrightarrow P_1$. Indeed, under these circumstances we have $P(\mathbb{A}) = P_1(\mathbb{A}) \cdot P^{der}(\mathbb{A})$, and the surjectivity follows from 3.9.

Remark. The same result has been obtained by M. Harris for arbitrary “arithmetic vector bundles” on pure Shimura varieties, in terms of the Baily-Borel compactification (see [Ha1],[Ha2]).

12.21. Example: Hilbert-modular varieties: Fix a totally real number field F . Let $P := \mathfrak{R}_{F/\mathbb{Q}} \mathrm{GL}_{2,F}$, and $\mathcal{X} \cong h(\mathcal{X})$ be the $P(\mathbb{R})$ -orbit generated by

$$h : \mathbb{S}(\mathbb{R}) \cong \mathcal{C}^\times \longrightarrow P(\mathbb{R}) \cong (\mathrm{GL}_2(\mathbb{R}))^{[F/\mathbb{Q}]},$$

$$x + \sqrt{-1}y \mapsto \left(\begin{pmatrix} x & y \\ -y & x \end{pmatrix}, \dots, \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \right).$$

This orbit does not depend on the choice of $\sqrt{-1}$ (of course, using such a choice it can be identified with $(\mathbb{C} \setminus \mathbb{R})^{[F/\mathbb{Q}]}$). The reflex field of $(P, \mathcal{X})$ is $\mathbb{Q}$, not F ! There is precisely one conjugacy class of proper rational boundary components of $(P, \mathcal{X})$, e.g. with representative

$$P_1 := \begin{pmatrix} \mathbb{G}_{m,\mathbb{Q}} & \mathfrak{R}_{F/\mathbb{Q}} \mathbb{G}_{a,F} \\ 0 & 1 \end{pmatrix}.$$

Here every h_{x_1} for $x_1 \in \mathcal{X}_1$ is of the form $\mathcal{C}^\times \ni z \mapsto \begin{pmatrix} z & \bar{z} \\ 0 & 1 \end{pmatrix}$, and $(P_1, \mathcal{X}_1)$ is isomorphic to the $[F/\mathbb{Q}]$ -fold fibre product of $(P_0, \mathcal{X}_0)$ of 2.24 with itself over $(\mathbb{G}_{m,\mathbb{Q}}, \mathcal{H}_0)$. In particular, $(P_1, \mathcal{X}_1)/W_1 \cong (\mathbb{G}_{m,\mathbb{Q}}, \mathcal{H}_0)$.

It follows from 12.3 that the Baily-Borel compactification $M^{K_f}(P, \mathcal{X})^*$, the boundary is a finite disjoint union of certain $M^{K_f^0}(\mathbb{G}_{m,\mathbb{Q}}, \mathcal{H}_0)$. In particular, the field of definition of every geometric point of the boundary, i.e. of $M^{K_f}(P, \mathcal{X})^*(\mathbb{C}) \setminus M^{K_f}(P, \mathcal{X})(\mathbb{C})$, is an abelian extension of $\mathbb{Q}$. Of course, this field of definition may be larger than the field of definition of the associated connected component of $M^{K_f}(P, \mathcal{X})^*(\mathbb{C})$. In the toroidal compactification, it is well-known how the arithmetic of F comes into play. This is described in M. Rapoport’s thesis [Rap], in a strictly more general setting (i.e. not restricted to characteristic zero).

12.22. Example: The group $CU(n, 1)$: Let $E \subset \mathbb{C}$ be an imaginary quadratic field (with a fixed embedding into $\mathbb{C}$), V an E -vector space of dimension $n + 1 \geq 3$, and $H : V \times V \rightarrow E$ a nondegenerate hermitian form that becomes of type $(n, 1)$ over $\mathbb{C}$. We let P be the group of unitary similitudes of V with respect to H , i.e.

$$P := \{g \in \mathfrak{R}_{E/\mathbb{Q}} \mathrm{GL}_E(V) \mid H(gv, gv') = t \cdot H(v, v') \text{ for some } t \in \mathfrak{R}_{E/\mathbb{Q}} \mathbb{G}_{m,E}\}.$$

Every homomorphism $h : \mathbb{S} \rightarrow \mathfrak{R}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_{\mathbb{C}}(V \otimes_E \mathbb{C})$ such that $V \otimes_E \mathbb{C}$ becomes of Hodge type $\{(-1, 0), (0, -1)\}$ induces another structure of $\mathbb{C}$ -vector space on $V \otimes_E \mathbb{C}$ which commutes with the old $\mathbb{C}$ -structure. Thus it defines a decomposition $V \otimes_E \mathbb{C} = V^+ \oplus V^-$, where V^+ (resp. V^-) is the subspace where the two $\mathbb{C}$ -structures coincide (resp. differ by complex conjugation). If we further require that V^+ and V^- are orthogonal with respect to H and that $H|_{V^-}$ is negative definite (whence $\dim(V^-) = 1$, $\dim(V^+) = n$, and $H|_{V^+}$ is positive definite), then h must factor through $P_{\mathbb{R}}$. The set $\mathcal{X}$ of all such h constitutes a $P(\mathbb{R})$ -orbit, and $(P, \mathcal{X})$ is pure Shimura data. Since $n \geq 2$, $P(\mathbb{R})$ and hence $\mathcal{X}$ are connected. The reflex field of $(P, \mathcal{X})$ is E .

Every maximal proper $\mathbb{Q}$ -parabolic subgroup $Q \subset P$ is the normalizer of a nontrivial isotropic subspace $V' \subset V$. In the given case, we must have $\dim(V') = 1$, and Q stabilizes $(V')^\perp$ which is of codimension 1. If $(P_1, \mathcal{X}_1)$ is the rational boundary component of $(P, \mathcal{X})$, then for every $x_1 \in \mathcal{X}_1$ and $z \in \mathbb{S}(\mathbb{R}) = \mathcal{C}^\times$, $h_{x_1}(z)$ acts on $V' \otimes_E \mathbb{C}$ through multiplication by $z\bar{z}$, on $((V')^\perp/V') \otimes_E \mathbb{C}$ through multiplication by z , and trivially on $(V/(V')^\perp) \otimes_E \mathbb{C}$. Since the semisimple part of Q is totally anisotropic over $\mathbb{R}$, P_1 is solvable, and, in fact, $P_1/W_1 \cong \mathfrak{R}_{E/\mathbb{Q}}\mathbb{G}_{m,E}$. Since $\mathcal{X}$ is connected, we have $|\mathcal{X}_1| = 1$, and the associated homomorphism is the isomorphism

$$\mathbb{S} = \mathfrak{R}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}} \cong (\mathfrak{R}_{E/\mathbb{Q}}\mathbb{G}_{m,E}) \times_E \mathbb{C}.$$

Moreover $\dim(U_1) = 1$, and $\dim(V_1) = 2(n-1)$.

By 12.3, the boundary $M^{K_f}(P, \mathcal{X})^* \setminus M^{K_f}(P, \mathcal{X})$ of the Baily-Borel compactification is a finite disjoint union of certain $M^{\pi(K_f^1)}((P_1, \mathcal{X}_1)/W_1)$. In particular, the field of definition of every geometric point of the boundary is an abelian extension of E . Since $\dim(U_1) = 1$, there is a unique complete decomposition Σ for $(P, \mathcal{X})$. By 12.4, the boundary of the associated toroidal compactification is a finite disjoint union of certain $M^{\pi'(K_f^1)}((P_1, \mathcal{X}_1)/U_1)$. The family $M^{\pi'(K_f^1)}((P_1, \mathcal{X}_1)/U_1) \rightarrow M^{\pi(K_f^1)}((P_1, \mathcal{X}_1)/W_1)$ is a torsor under a family of abelian varieties of dimension $n-1$, isogenous to the $(n-1)$ -fold product with itself of an elliptic curve with complex multiplication by E . By 12.4 (c) the normal bundle of $M^{\pi'(K_f^1)}((P_1, \mathcal{X}_1)/U_1)$ in $M^{K_f}(P, \mathcal{X}, \Sigma)$ is isomorphic to the line bundle $M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1) \rightarrow M^{\pi'(K_f^1)}((P_1, \mathcal{X}_1)/U_1)$. The coefficients of the q -expansion of a modular form

$$f \in \Gamma(M^{K_f}(P, \mathcal{X}, \Sigma), \omega[\mathrm{dlog}]^{\otimes n})$$

are sections of all nonpositive powers of the invertible sheaf associated to this line bundle. (Since the stratum is blown down in the Baily-Borel compactification, the normal bundle is anti-ample: which explains in another way why we have the nonpositive powers of this invertible sheaf.)

Since V is of Hodge type $\{(-1, 0), (0, -1)\}$, we can also consider the mixed Shimura data $V \rtimes (P, \mathcal{X})$. The resulting family of abelian varieties inherits multiplication by E , which leads to a modular interpretation of $M^{K_f}(P, \mathcal{X})$. A suitable extension of this family in terms of toroidal compactification can, as in 10.17–10.22, be described as degeneration of such abelian varieties. All these structures can, of course, be described in terms of morphisms in 12.4.

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List of frequently used symbols

$\mathbb{A}, \mathbb{A}_f, \mathbb{A}_K, \mathbb{A}_{K,f}$	adele ring	0.2
Ad_G, ad_G	adjoint operation	0.3
$A[d]$	group of d -division points	10.1
$B(\mathcal{X}^0), B(\mathcal{X}^0, P_1)$	set of commutators	8.11
$\beta(P_1, \mathcal{X}_1, P'_1, \mathcal{X}'_1, p_f), \bar{\beta}(P_1, \mathcal{X}_1, P'_1, \mathcal{X}'_1, p_f)$	certain maps in $\mathcal{U}, \bar{\mathcal{U}}$	6.11, 6.15
$\mathbb{C}$	set of complex numbers	0.2
$\mathcal{C}(P, \mathcal{X})$	conical complex	4.24
$C(\mathcal{X}^0, P_1)$	open cone associated to boundary component	4.15
$C^*(\mathcal{X}^0, P_1)$	union of all boundary components of $C(\mathcal{X}^0, P_1)$	4.22
$(CS_{P_{2g}}, \mathcal{H}_{2g})$	Shimura data for the symplectic group	2.7
Δ_1	normalizer of a boundary stratum in the Baily-Borel compactification	6.3, 6.18, 7.3
e_σ	splitting associated to a cone σ	5.11, 8.5
$\exp L$	subgroup associated to sub-Lie algebra	0.3
$E(P, \mathcal{X})$	reflex field	11.1
$e(X)$	canonical pairing $H(X) \times H(X) \rightarrow \mathbb{G}_m$	10.2
$\mathcal{E}(\mathcal{X}^0, P'_1)$	set of splittings	9.2
$F^P M_{\mathbb{C}}$	Hodge filtration	1.1
G	semisimple part P/W	2.1
G^{ad}, G^{der}	adjoint, derived group	0.3
$(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0), (\mathbb{G}_{m, \mathbb{Q}}, h(\mathcal{H}_0))$	standard Shimura data	2.8
h	homomorphism $\mathbb{S}_{\mathbb{C}} \rightarrow P_{\mathbb{C}}$	1.4
h	equivariant map $\mathcal{X} \rightarrow \text{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$	2.1
H_0, h_0, h_∞	reference group and homomorphisms	4.3
$H(X)$	group of translations normalizing X	10.1
$\mathcal{H}_W$	conjugacy class of $h : \mathbb{S}_{\mathbb{C}} \rightarrow P_{\mathbb{C}}$	1.6
im	imaginary part	4.14
int_G	interior automorphism	0.3
$\text{int}(p)$	interior automorphism of $(P, \mathcal{X})$	3.5
$K_f(d), K_f^P(d), K_f^U(d), K_f^W(d)$	special open compact subgroups	10.7, 10.15
$\mathcal{L}_P$	canonical invertible sheaf on $M^{K_f}(P, \mathcal{X})^*(\mathbb{C})$	8.1
$\mathcal{L}_{\mathcal{X}}$	invertible sheaf occuring in q -expansion	12.18
λ	isomorphism $\mathbb{Z} \rightarrow \mathbb{Z}(1)$	2.8, 3.16, 8.5
$M^{p,q}$	Hodge decomposition	1.1
$M(n)$	Tate twist	1.11
$M^{K_f}(P, \mathcal{X})(\mathbb{C}), M_{\mathbb{C}}^{K_f}(P, \mathcal{X}), M^{K_f}(P, \mathcal{X})$	(mixed) Shimura variety	3.1, 9.25, 11.5
$M^{K_f}(P, \mathcal{X})^*(\mathbb{C}), M_{\mathbb{C}}^{K_f}(P, \mathcal{X})^*, M^{K_f}(P, \mathcal{X})^*$	Baily-Borel compactification	6.2, 9.25, 12.3
$M^{K_f}(P, \mathcal{X})(\mathbb{C}), M_{\mathbb{C}}^{K_f}(P, \mathcal{X}), M^{K_f}(P, \mathcal{X})$		

	toroidal compactification	6.24, 9.25, 9.34, 12.4–5
$M^{K_f}(P, \mathcal{X})^+(\mathbb{C})$	union of $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ with all boundary strata of codimension 1	8.2
$M(d), M_V(d), M_W(d), M^0(d), M_U^0(d)$	special mixed Shimura varieties	10.7, 10.15
$\mathcal{M}_d$	moduli scheme of abelian varieties	10.6, 11.16
$\underline{M}_d^0, \underline{X}_d^0$	“moduli scheme” of roots of unity	10.15
μ	canonical cocharacter of $\mathbb{S}$	1.3
ord_T	homomorphism $T(\mathbb{C}) \rightarrow Y_*(T)_{\mathbb{R}}$	5.8
$\omega_{X/S}^T$	sheaf of invariant differentials	5.26
$\omega[\text{dlog}], \omega_{X_{\Sigma}}[\text{dlog}], \omega_{X_{\Sigma}}/S[\text{dlog}]$	sheaf of differentials with at most logarithmic poles along the boundary	5.26, 8.1
$(P, \mathcal{X}), (P, \mathcal{X}, h)$	(mixed) Shimura data	2.1
$(P, \mathcal{X})/P_0$	quotient mixed Shimura data	2.9
$(P_1, \mathcal{X}_1) \times_{(P, \mathcal{X})} (P_2, \mathcal{X}_2)$	fibre product	2.20
$(P_0, \mathcal{X}_0)$	unipotent extension of $(\mathbb{G}_{m, \mathbb{Q}}, \mathcal{H}_0)$	2.24
$(P_{2g}, \mathcal{X}_{2g})$	unipotent extension of $(CS_{P_{2g}}, \mathcal{H}_{2g})$	2.25
$(P_{[\sigma]}, \mathcal{X}_{[\sigma]})$	mixed Shimura data associated to $[\sigma]$	7.1
π	the positive real number	0.2
π, π'	projections $P \rightarrow G = P/W, P \rightarrow P/U$	2.1
$\pi_0(X)$	set of connected components	0.1
π_{σ}	canonical projection $T_{\sigma} \rightarrow \tilde{T}_{\sigma}$	5.2
$\pi_{[\sigma]}$	projection $(P, \mathcal{X}) \rightarrow (P_{[\sigma]}, \mathcal{X}_{[\sigma]})$	7.1
$[\pi]^*$	projection to Baily-Borel compactification	6.24, 9.25, 12.4
Q	admissible $\mathbb{Q}$ -parabolic subgroup	4.5
$\mathfrak{R}_{L/K}X$	Weil restriction	0.2
$\mathbb{R}$	set of real numbers	0.2
$\mathbb{S}, \mathbb{S}^1$	Deligne torus	1.3
$\Sigma, \Sigma(\mathcal{X}^0, P_1, p_f)$	admissible (partial) cone decomposition	6.4
Σ^0	subset of all cones along the unipotent fibre	6.5
$\Sigma_{ (P_1, \mathcal{X}_1)}$	restriction to a boundary component	6.5
$\Sigma_{[\sigma]}$	induced cone decomposition	7.7
$\Sigma(f)$	cone decomposition associated to a piecewise linear convex rational function	5.20
$\Sigma_{\Sigma}^+, \Sigma'_{\Sigma}, \Sigma_{\Sigma}$	cone decompositions associated to Σ	9.12
$ \Sigma $	support of a cone decomposition	5.1
σ^0	interior of a cone	5.1
σ_0	standard cone	5.10, 8.5, 9.1
$\check{\sigma}$	dual cone	5.1
$[\sigma]$	double coset of cones	7.1
Σ	locally polyhedral subset	9.8
T_{Σ}	torus embedding	5.3
T_{σ}	affine torus embedding	5.2

$\tilde{T}_\sigma$	orbit in a torus embedding	5.2
U	weight -2 subgroup of P	2.1
$\mathcal{U}, \mathcal{U}(P_1, \mathcal{X}_1, p_f)$	covering of $M^{K_f}(P, \mathcal{X})(\mathbb{C})$	6.10
$\bar{\mathcal{U}}, \bar{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$	covering of $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$	6.13
V	weight -1 subquotient of P	2.1
$V[d], U[d]$	standard $\mathbb{Z}/d\mathbb{Z}$ -modules	10.3, 10.7
w	“weight” of $\mathbb{S}$	1.3
W	unipotent radical of P	2.1
$W_n M$	weight filtration	1.1
$X_d \rightarrow A_d \rightarrow \underline{\mathcal{M}}_d$	universal family of abelian varieties	10.10, 11.16
X_Σ	relative torus embedding	5.5
$X^*(T)$	character group of a torus T	5.2
$\mathcal{X}^+$	open subset of $\mathcal{X}$ that maps to $\mathcal{X}_1$	4.11
$\mathcal{X}^*$	union of hermitain symmetric domain with all its rational boundary components	6.2
$Y_*(T)$	cocharacter group of torus T	5.2
$\mathbb{Z}(n)$	Tate Hodge structure	1.11
$\sim$	equivalence relation on $\mathcal{U}$ or $\bar{\mathcal{U}}$	6.10, 6.16
$\sqrt{-1}$	the complex number	0.2
$[,]$	commutator	0.3
$[\cdot p_f], [\phi]$	morphisms of mixed Shimura varieties	3.4, 6.2, 6.25, 9.25, 12.3, 12.4
$[\cdot p_f]^* \Sigma, \phi^* \Sigma$	pullback of admissible cone decomposition	6.5

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