Projet de Master

Multilinear Theory of Commutative Group Schemes

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0 Introduction

In this paper we study the multilinear morphisms between group schemes and associated constructions. We will also do some explicit calculations and give examples that show that this theory behaves in a way that one would naturally expect. When we are studying homomorphisms of commutative group schemes, we are naturally led to look at multilinear morphisms between them, because on the one hand they are obvious generalizations of homomorphisms and on the other hand they make it possible to have a group scheme version of multilinear algebra. Although some of the results of multilinear algebra are no longer valid in this new setting, there are many similarities between these two theories, as we will see. Let $G_1, \ldots, G_n$ and H be commutative group schemes over a base scheme S. A multilinear morphism $\varphi : G_1 \times \cdots \times G_n \to H$ is a morphism of schemes over S that is linear in each G_i. The group of all such multilinear morphisms is denoted by $\mathrm{Mult}(G_1 \times \cdots \times G_n, H)$. Natural examples of multilinear morphisms are the Weil pairings on the torsion groups of an abelian variety, and the pairing $G \times G^* \to \mathbb{G}_m$ between a finite and flat commutative group scheme G and its Cartier dual G^*.

In the first section we study groups of multilinear morphisms and related concepts. We define the so-called inner Hom of two commutative group schemes G and H, denoted by $\underline{\mathrm{Hom}}(G, H)$, as being the group scheme representing the functor

$$\mathfrak{Sch}_S \longrightarrow \mathfrak{Ab}, \qquad T \mapsto \mathrm{Hom}(G_T, H_T)$$

from the category of schemes over S to the category of abelian groups. Theorem 3.10 in [6] states that this group scheme exists whenever the group G is finite and flat over S and is affine (or of finite type) if H is affine (or of finite type). We show that this construction commutes with the base change, i.e., $\underline{\mathrm{Hom}}(G, H)_T \cong \underline{\mathrm{Hom}}(G_T, H_T)$ for any S-scheme T and that the functors $\underline{\mathrm{Hom}}(-, H)$ and $\underline{\mathrm{Hom}}(G, -)$ from the category of commutative group schemes over a base field to itself are left exact, which is not very surprising, since these functors are constructed from left exact functors $\mathrm{Hom}(-, H_T)$ and $\mathrm{Hom}(G_T, -)$ by varying T.
It turns out that the group scheme $\underline{\mathrm{Hom}}(G, H)$ need not be flat (or finite) even if both G and H are flat (or finite). We show this by giving one example in each case.

We can generalize the definition of inner Hom as follows. Define $\underline{\mathrm{Mult}}(G_1 \times \cdots \times G_n, H)$ to be the group scheme representing the functor

$$\mathfrak{Sch}_S \longrightarrow \mathfrak{Ab}, \qquad T \mapsto \mathrm{Mult}(G_{1,T} \times \cdots \times G_{n,T}, H_T).$$

The conditions under which this group exists are identical to those for $\underline{\mathrm{Hom}}(G, H)$, i.e., flatness and finiteness of G_i. It is affine or of finite type if H has these properties.

Then we study the group of multilinear morphisms $\mathrm{Mult}(G_1 \times \cdots \times G_n, H)$. Consider the case where $G := G_1 = \cdots = G_n$ and write G^n for the product

of n copies of G. The group $\mathrm{Mult}(G^n, H)$ has two distinguished subgroups, namely, the group $\mathrm{Sym}(G^n, H)$ of symmetric multilinear morphisms and the group $\mathrm{Alt}(G^n, H)$ of alternating multilinear morphisms. The first one is the group of multilinear morphisms that are invariant under the obvious action of the symmetric group S_n on G^n and the second one the group of multilinear morphisms that vanish when two factors are equal.
In the same way that we construct $\underline{\mathrm{Mult}}(G_1 \times \cdots \times G_n, H)$ from $\mathrm{Mult}(G_1 \times \cdots \times G_n, H)$, we can "schematize" the groups $\mathrm{Sym}(G^n, H)$ and $\mathrm{Alt}(G^n, H)$ and obtain $\underline{\mathrm{Sym}}(G^n, H)$ and $\underline{\mathrm{Alt}}(G^n, H)$, in order to take into account the behavior of these groups over different base schemes.

In section 1, we establish the following propositions, which show that our definitions lead to a coherent theory.

Proposition 1.12. *Let $G_1, \ldots, G_r, H_1, \ldots, H_s, F$ be commutative group schemes over a base scheme S. We have a natural isomorphism*

$$\mathrm{Mult}(G_1 \times \cdots \times G_r, \underline{\mathrm{Mult}}(H_1 \times \cdots \times H_s, F)) \cong \mathrm{Mult}(G_1 \times \cdots \times G_r \times H_1 \times \cdots \times H_s, F)$$

functorial in all arguments.

In particular we have $\mathrm{Mult}(F \times G, H) \cong \mathrm{Hom}(F, \underline{\mathrm{Hom}}(G, H))$. We could therefore take this isomorphism for the definition of $\underline{\mathrm{Hom}}(G, H)$, i.e., $\underline{\mathrm{Hom}}(G, H)$ is the unique group scheme such that we have a natural isomorphism

$$\mathrm{Mult}(- \times G, H) \cong \mathrm{Hom}(-, \underline{\mathrm{Hom}}(G, H)).$$

It also shows how naturally multilinear morphisms arise when one is looking at the homomorphisms between group schemes. The next important result is a generalization of Proposition 1.12:

Proposition 1.14. *Let $G_1, \ldots, G_r, H_1, \ldots, H_s, F$ be commutative group schemes over a base scheme S. We have a natural isomorphism*

$$\underline{\mathrm{Mult}}(G_1 \times \cdots \times G_r, \underline{\mathrm{Mult}}(H_1 \times \cdots \times H_s, F)) \cong \underline{\mathrm{Mult}}(G_1 \times \cdots \times G_r \times H_1 \times \cdots \times H_s, F)$$

functorial in all arguments.

Then, we give some concrete examples and we show the following isomorphisms, where the base scheme is $\mathbf{Spec}\, k$ with k a field of characteristic p and α_{p^n} denotes the kernel of the nth Frobenius of the additive group $\mathbb{G}_a$ over k, i.e., $\alpha_{p^n}(R) = \{\, a \in R \mid a^{p^n} = 0 \,\}$ for any k-algebra R:

- $\underline{\mathrm{Mult}}(\alpha_p^n, \mathbb{G}_m) \cong \mathbb{G}_a \quad \forall\, n \geq 1.$
- $\underline{\mathrm{Mult}}(\alpha_{p^{n_1}} \times \cdots \times \alpha_{p^{n_r}}, \mathbb{G}_a) \cong \mathbb{G}_a^{n_1 \cdots n_r}.$
- $\underline{\mathrm{Hom}}(\alpha_{p^n}, \alpha_{p^m}) \cong \mathbb{G}_a^n \quad \text{if } m \geq n.$
- $\underline{\mathrm{Hom}}(\alpha_{p^n}, \alpha_{p^m}) \cong \alpha_{p^m}^{n-m} \times \mathbb{G}_a^m \quad \text{if } m < n.$

- $\underline{\mathrm{Sym}}(\alpha_{p^n}^r, \mathbb{G}_a) \cong \mathbb{G}_a^{\binom{n+r-1}{r-1}}$.
- $\underline{\mathrm{Alt}}(\alpha_{p^n}^r, \mathbb{G}_a) \cong \mathbb{G}_a^{\binom{n}{r}}$ if $p > 2$ (where $\binom{n}{r} = 0$ if $r > n$).

In section 2, we make the "dual" constructions of the first section. A multilinear morphism from $G_1 \times \cdots \times G_n$ to a commutative group scheme is not a homomorphism of group schemes and therefore is not a morphism in the category of group schemes, but we would like to work inside this category. Thus, we should somehow look at these multilinear morphisms inside this category, that is, we should replace $G_1 \times \cdots \times G_n$ by a commutative group scheme such that for any commutative group scheme H and any multilinear morphism from the product $G_1 \times \cdots \times G_n$ to H, there is a unique homomorphism from this new commutative group scheme to H, that satisfies a certain universal property. This is possible thanks to the tensor product of $G_1, \ldots, G_n$. Let the tensor product of commutative group schemes $G_1, \ldots, G_n$ over S be a commutative group scheme $G_1 \otimes \cdots \otimes G_n$ together with a "universal" multilinear morphism $\varphi : G_1 \times \cdots \times G_n \to G_1 \otimes \cdots \otimes G_n$ that yields an isomorphism

$$\mathrm{Hom}(G_1 \otimes \cdots \otimes G_n, H) \cong \mathrm{Mult}(G_1 \times \cdots \times G_n, H), \qquad \psi \mapsto \psi \circ \varphi$$

for all commutative group schemes H over S. This universal property determines the tensor product up to unique isomorphism, if it exists. Theorem 4.3 in [6] says that the tensor product exists and is pro-finite if the base scheme S is the spectrum of a field and the G_i are finite over S, and with the notations of the first section it is isomorphic to the inverse limit $\varprojlim G_\alpha^*$ where G_α runs through all finite subgroup schemes of $\underline{\mathrm{Mult}}(G_1 \times \cdots \times G_n, \mathbb{G}_m)$. By abuse of notation, we can write this inverse limit as $\underline{\mathrm{Mult}}(G_1 \times \cdots \times G_n, \mathbb{G}_m)^*$. This shows that all information about the tensor product $G_1 \otimes \cdots \otimes G_n$ and hence about multilinear morphisms from $G_1 \times \cdots \times G_n$ to commutative group schemes can be read off from the group of multilinear morphisms $\mathrm{Mult}(G_{1,T} \times \cdots \times G_{n,T}, \mathbb{G}_{m,T})$ for all extensions $T \to S$ of the base scheme.
Despite our expectations, the construction of the tensor product does not commute with the base change, that is, we don't have in general $(G \otimes H)_T \cong G_T \otimes H_T$ for an S-scheme T. This makes the calculations more difficult.

In a similar fashion, we define the symmetric power $S^n G$, respectively the alternating power $\Lambda^n G$, of a commutative group scheme G over S to be the unique commutative group schemes that characterize, in the same way as the tensor product, the group $\mathrm{Sym}(G^n, H)$, respectively $\mathrm{Alt}(G^n, H)$ for all commutative group schemes H over S. Again, if S is the spectrum of a field and G is finite over S, these group schemes exist, are pro-finite and constructed as quotients of the n-fold tensor product $G^{\otimes n}$, similar to the same constructions for modules over commutative rings.

Then we do some explicit calculations and show the following isomorphisms for $n > 1$, where $\mathbb{G}_a^* := \varprojlim_i G_i^*$ and G_i runs through all finite subgroup schemes of $\mathbb{G}_a$:

- $S^n\alpha_p \cong \alpha_p^{\otimes n} \cong \mathbb{G}_a^*$
- $\Lambda^n\alpha_p \cong \alpha_p^{\otimes n}$ if $p = 2$.
- $\Lambda^n\alpha_p = 0$ if $p > 2$.

And more generally:

- $\underline{\mathrm{Sym}}(\alpha_p^n, H) = \underline{\mathrm{Mult}}(\alpha_p^n, H)$.
- $\underline{\mathrm{Alt}}(\alpha_p^n, H) = \underline{\mathrm{Mult}}(\alpha_p^n, H)$ if $p = 2$.
- $\underline{\mathrm{Alt}}(\alpha_p^n, H) = 0$ if $p > 2$.

For the remainder of section 2 we work on alternating multilinear morphisms and the alternating powers. Our main results are:

Theorem 2.16. *Assume that $0 \to G' \xrightarrow{\iota} G \xrightarrow{\pi} G'' \to 0$ is a short exact sequence of commutative group schemes. Let m be a non negative integer and write $m = m' + m''$ with non negative integers m' and m''. Consider the diagram*

$$\begin{tikzcd} \mathrm{Alt}(G^m, H) \arrow[r, "\rho"] & \mathrm{Alt}(G'^{m'} \times G^{m''}, H) \\ & \mathrm{Alt}(G'^{m'} \times G''^{m''}, H) \arrow[u, hook, "\pi^*"'] \end{tikzcd}$$

where ρ is the restriction map.

(a) *If $\Lambda^{m''+1}G'' = 0$, then ρ is injective.*

(b) *If $\Lambda^{m'+1}G' = 0$, then ρ factors through π^*.*

(c) *If both conditions hold, then there is a natural epimorphism*

$$\zeta : \Lambda^{m'}G' \otimes \Lambda^{m''}G'' \twoheadrightarrow \Lambda^m G.$$

(d) *If furthermore the sequence is split, then the epimorphism ζ is an isomorphism.*

We know that this theorem is true for modules of finite length over local rings. The condition $\Lambda^m M = 0$ for a such module M is guaranteed if m is greater than the length of M. One would desire that the same thing holds for commutative group schemes. Restricting to local-local commutative group schemes over a base field of odd characteristic p, we show:

Proposition 2.17. *Let G be a local-local commutative group scheme of order p^n with p an odd prime number. We have:*

(a) *$\Lambda^m G = 0$ for all $m > n$.*

(b) *$\Lambda^n G$ is a quotient of $\alpha_p^{\otimes n}$.*

We see that for a local-local commutative group scheme of order p^n for an odd prime number p, the exponent n plays somehow the role of the length of modules of finite length. Another example that shows this analogy is:

Corollary 2.18. *Let G and H be local-local commutative group schemes of order p^n and p^m respectively, with p an odd prime number. Then we have a natural isomorphism*

$$\Lambda^{n+m}(G \oplus H) \cong \Lambda^n G \otimes \Lambda^m H.$$

Finally, we have the following important result:

Proposition 2.26. *Let the base scheme be* $\mathbf{Spec}\, k$ *for a perfect field k of odd characteristic p, and n a positive integer. Then there is an isomorphism*

$$\Lambda^n \alpha_{p^n} \cong \alpha_p.$$

The results proved in this paper may hold in a more general context (see Remark 2.23), however, we do not intend to state them with minimal hypotheses.

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Conventions. We suppose some familiarity with the elementary theory of schemes and group schemes. Throughout the paper, all schemes are assumed to be separated and quasi-compact. We usually consider schemes over a fixed base scheme S, and in this case morphisms and fiber products are taken over S, unless otherwise noted. The pullback of a scheme X over S via any morphism $T \to S$ is denoted by X_T. When there is no ambiguity, we write $\mathbb{G}_a, \mathbb{G}_m$ or α_{p^n} instead of $\mathbb{G}_{a,S}, \mathbb{G}_{m,S}$ or $\alpha_{p^n,S}$.

1 $\underline{\mathrm{Hom}}(-,-)$ and multilinear morphisms

Definition 1.1. *Let G and H be commutative group schemes over a base scheme S. Define a contravariant functor $\underline{\mathrm{Hom}}(G,H)$ from the category of S-schemes to the category of abelian groups as follows:*

$$T \mapsto \underline{\mathrm{Hom}}(G,H)(T) := \mathrm{Hom}_T(G_T, H_T).$$

If this functor is representable by a group scheme over S, that group scheme is also denoted by $\underline{\mathrm{Hom}}(G,H)$ and is called the inner Hom from G to H.

Remark 1.2. According to Theorem 3.10 in [6], if G is finite and flat over S, then $\underline{\mathrm{Hom}}(G,H)$ is representable and if in addition H is affine, resp. of finite type over S, then $\underline{\mathrm{Hom}}(G,H)$ has the same property. So, in order to assure the existence of $\underline{\mathrm{Hom}}(G,H)$, in the sequel, every time we write $\underline{\mathrm{Hom}}(G,H)$, we assume that G is finite and flat over the base scheme, without explicitly mentioning it.

Proposition 1.3. *Let H be an affine commutative group scheme over a field k. Then the functors $\underline{\mathrm{Hom}}(-,H)$ and $\underline{\mathrm{Hom}}(H,-)$ from the category of affine commutative group schemes over k to itself are left exact.*

Proof. Suppose that $0 \to N \xrightarrow{i} G \xrightarrow{\pi} Q \to 0$ is a short exact sequence of affine group schemes over a field k and denote by A, B the Hopf algebras representing G, Q and by I_B the augmentation ideal of B. Then the Hopf algebra representing N is $A/(I_B \cdot A)$. Let R be a k-algebra. Since it is flat over k, we have an injection $B \otimes_k R \hookrightarrow A \otimes_k R$ and therefore $G_R \xrightarrow{\pi_R} Q_R$ is a quotient morphism. We have also that $(I_B \cdot A) \otimes_k R = (I_B \otimes_k R) \cdot (A \otimes_k R)$ and so by flatness we have $(A/(I_B \cdot A)) \otimes_k R \cong A \otimes_k R/((I_B \cdot A) \otimes_k R) = A \otimes_k R/(I_B \otimes_k R)(A \otimes_k R)$. It implies that N_R is the kernel of $G_R \xrightarrow{\pi_R} Q_R$. Consequently the short sequence $0 \to N_R \xrightarrow{i_R} G_R \xrightarrow{\pi_R} Q_R \to 0$ is exact. Now, fix an affine commutative group scheme H. We show that the sequence

$$0 \to \underline{\mathrm{Hom}}(Q,H) \xrightarrow{\pi^*} \underline{\mathrm{Hom}}(G,H) \xrightarrow{i^*} \underline{\mathrm{Hom}}(N,H)$$

is exact. It is equivalent to the exactness of the sequence

$$0 \to \underline{\mathrm{Hom}}(Q,H)(R) \xrightarrow{\pi^*} \underline{\mathrm{Hom}}(G,H)(R) \xrightarrow{i^*} \underline{\mathrm{Hom}}(N,H)(R)$$

for every k-algebra R, i.e., the exactness of the sequence

$$0 \to \mathrm{Hom}_R(Q_R,H_R) \xrightarrow{\pi^*} \mathrm{Hom}_R(G_R,H_R) \xrightarrow{i^*} \mathrm{Hom}_R(N_R,H_R).$$

Assume we have shown that for any homomorphism $\varphi : G_R \to H_R$ such that $\varphi \circ i_R = 0$, then there exists a unique homomorphism $\psi : Q_R \to H_R$ with $\varphi = \psi \circ \pi_R$, i.e., the following diagram is commutative

$$\begin{tikzcd}
0 \arrow[r] & N_R \arrow[r, "i_R"] \arrow[dr, "0"'] & G_R \arrow[r, "\pi_R"] \arrow[d, "\varphi"] & Q_R \arrow[r] \arrow[dl, "\exists!\ \psi"] & 0 \\
& & H_R. & &
\end{tikzcd}$$

Then the exactness is clear; indeed, pick a morphism $f : Q_R \to H_R$ with $f \circ \pi_R = 0$ then putting $\varphi := 0$ the zero morphism, there are two morphisms $Q_R \to H_R$, namely f and the zero morphism, whose composition with π_R are φ and from the assumption they should be equal. This shows the injectivity of

$$\mathrm{Hom}_R(Q_R,H_R) \xrightarrow{\pi_R^*} \mathrm{Hom}_R(G_R,H_R).$$

Clearly we have $\mathrm{Im}\,\pi_R^* \subset \mathrm{Ker}\, i_R^*$. Let $g : G_R \to H_R$ be an element of $\mathrm{Ker}\, i_R^*$, i.e., $g \circ i_R^* = 0$, then according to the assumption there is a $\psi : Q_R \to H_R$ with

$\pi_R \circ \psi = g$, or in other words $g = \pi_R^*(\psi)$ and thus $\operatorname{Ker} i_R^* \subset \operatorname{Im} \pi_R^*$.

It is thus sufficient to show that the assumption holds. But this is obvious, since as we proved above, the morphism $G_R \xrightarrow{\pi_R} Q_R$ is the cokernel of the injection $N_R \hookrightarrow G_R$ in the category of affine commutative group schemes. Similarly, the fact that $N_R \xrightarrow{i_R} G_R$ is the kernel of the quotient morphism $G_R \xrightarrow{\pi_R} Q_r$ implies that given any homomorphism $\varphi : H_R \to G_R$ with trivial composition $\pi_R \circ \varphi$ there is a unique homomorphism $\psi : H_R \to N_R$ such that the following diagram is commutative

$$\begin{tikzcd}
0 \arrow[r] & N_R \arrow[r, "i_R"] & G_R \arrow[r, "\pi_R"] & Q_R \arrow[r] & 0 \\
& & H_R. \arrow[ul, "\exists!\ \psi"] \arrow[u, "\varphi"] \arrow[ur, "0"'] & &
\end{tikzcd}$$

And this implies as above the exactness of the following short sequence

$$0 \to \operatorname{Hom}_R(H_R, N_R) \xrightarrow{i_R^*} \operatorname{Hom}_R(H_R, G_R) \xrightarrow{\pi_r^*} \operatorname{Hom}_R(H_R, Q_R)$$

for every k-algebra R, and consequently the following sequence of group schemes is exact

$$0 \to \underline{\operatorname{Hom}}(H, N) \xrightarrow{i^*} \underline{\operatorname{Hom}}(H, G) \xrightarrow{\pi^*} \underline{\operatorname{Hom}}(H, Q).$$

□

A natural question that one may ask is to know to what extent the inner Hom, $\underline{\operatorname{Hom}}(G, H)$ shares the properties of G and H. Examples of such properties are finiteness or flatness. It is quite easy to see that $\underline{\operatorname{Hom}}(\alpha_p, \alpha_p) \cong \mathbb{G}_a$ (and we will give a detailed proof later), so we observe that despite the finiteness of α_p, the group scheme $\underline{\operatorname{Hom}}(\alpha_p, \alpha_p)$ is not finite and thus, this property is not preserved by the construction of inner Hom. In the following example, we show that in fact, the flatness also has this "defect" and is not preserved by this construction.

Example 1.4. Here we give an example of finite flat commutative group schemes over a Λ-algebra R such that the group scheme $\underline{\operatorname{Hom}}(G, H)$ is not flat. We refer the reader to the paper [5] for a discussion of group schemes of prime order, their classification and the definition of Λ. We know that the field $\mathbb{F}_p$ is canonically a Λ-algebra and therefore any $\mathbb{F}_p$-algebra is canonically a Λ-algebra. Put $R := \mathbb{F}_p[x]$, the polynomial ring in one variable over the field $\mathbb{F}_p$. Then any elements a, $b \in R$ satisfying $ab = 0$ define a group scheme $G_{a,b} := \mathbf{Spec}\, R[y]/(y^p - ay)$ together with the comultiplication

$$y \mapsto y \otimes 1 + 1 \otimes y + b(1-p)^{-1} \cdot \sum_{i=1}^{p-1} \frac{y^i}{w_i} \otimes \frac{y^{p-i}}{w_{p-i}}$$

and according to Proposition 3.11 in [6], if c, $d \in R$ are such that $cd = 0$, we have

$$\underline{\operatorname{Hom}}(G_{a,b}, G_{c,d}) \cong \mathbf{Spec}\, R[y]/(ay^p - cy, dy^p - by)$$

with the comultiplication

$$y \mapsto y \otimes 1 + 1 \otimes y + ad(1-p)^{-1} \cdot \sum_{i=1}^{p-1} \frac{y^i}{w_i} \otimes \frac{y^{p-i}}{w_{p-i}}.$$

The group scheme $G_{0,x}$, represented by the Hopf algebra $R[y]/(y^p)$, is flat over R, because this Hopf algebra is a torsion-free module over the principal ideal domain R and so is flat over R. But the group scheme $\underline{\mathrm{Hom}}(G_{0,x}, G_{0,x})$ is represented by the Hopf algebra $R[y]/(xy^p - xy)$ which has the torsion element $y^p - y$ (which is annihilated by x) and therefore is not flat over R. It follows then that $\underline{\mathrm{Hom}}(G_{0,x}, G_{0,x})$ is not flat over R.

Recall that if $G_1, \ldots, G_r, H$ are commutative group schemes over a base S, then $\mathrm{Mult}(G_1 \times \cdots \times G_r, H)$ is the group of all multilinear morphisms from $G_1 \times \cdots \times G_r$ to H, i.e. morphisms that are linear in each factor G_i or equivalently morphisms which have the property that for any S-scheme T the induced morphism $G_1(T) \times \cdots \times G_r(T) \to H(T)$ is multilinear. We can then generalize the definition of inner Hom as follows:

Definition 1.5. *Let $G_1, G_2, \ldots, G_r, H$ be commutative group schemes over a base scheme S. Define a contravariant functor from the category of S-schemes to the category of abelian groups as follows:*

$$T \mapsto \underline{\mathrm{Mult}}(G_1 \times G_2 \times \cdots \times G_r, H)(T) := \mathrm{Mult}_T(G_{1,T} \times G_{2,T} \times \cdots \times G_{r,T}, H_T).$$

If this functor is representable by a group scheme over S, we will also denote that group scheme by $\underline{\mathrm{Mult}}(G_1 \times G_2 \times \cdots \times G_r, H)$.

For any positive integer r we denote by G^r the product of r copies of G, and for any $1 \leq i, j \leq r$ we let $\Delta^r_{ij} \subset G^r$ or $\Delta^r_{ij} G$ (if we want to make explicit the group scheme G) denote the closed subscheme defined by equating the ith and j^{th} components.

Definition 1.6. *Let G and H be as above.*

(i) *A multilinear morphism $G^r \to H$ is called* symmetric *if it is invariant under permutation of the factors. The group of all such symmetric multilinear morphisms is denoted* $\mathrm{Sym}(G^r, H)$.

(ii) *A multilinear morphism $G^r \to H$ is called* alternating *if its restriction to Δ^r_{ij} is trivial for all $1 \leq i, j \leq r$. The group of all such alternating multilinear morphisms is denoted* $\mathrm{Alt}(G^r, H)$.

Remark 1.7.

1) Let $\varphi : G^r \to H$ be a multilinear morphism. Then one can see easily that φ is symmetric if and only if the induced morphism $\varphi(T) : G^r(T) = G(T)^r \to H(T)$ is symmetric for all S-schemes T.

2) We have a natural action of the symmetric group S_r on G^r. This action induces an action on the group $\mathrm{Mult}(G_1 \times \cdots \times G_r, H)$ and the subgroup $\mathrm{Sym}(G^r, H)$ is precisely the subgroup of fixed points, i.e. $\mathrm{Sym}(G^r, H) = \mathrm{Mult}(G^r, H)^{S_r}$.

3) Similarly to 1), if $\psi : G^r \to H$ is a multilinear morphism, then ψ is alternating if and only if $\psi(T) : G^R(T) = G(T)^r \to H(T)$ is alternating for all S-schemes T.

4) The usual calculation shows that any alternating morphism is antisymmetric, i.e. a permutation of the factors multiplies the morphism by the sign of the permutation.

We can make definitions similar to Definition 1.5 for the group of symmetric and alternating multilinear morphisms:

Definition 1.8. *Let G, H be commutative group schemes over S. Then denote by $\underline{\mathrm{Sym}}(G^r, H)$ and $\underline{\mathrm{Alt}}(G^r, H)$ respectively the contravariant functors*

$$T \mapsto \underline{\mathrm{Sym}}(G^r, H)(T) := \mathrm{Sym}_T(G^r_T, H_T)$$

and

$$T \mapsto \underline{\mathrm{Alt}}(G^r, H)(T) := \mathrm{Alt}_T(G^r_T, H_T)$$

respectively. If $\underline{\mathrm{Sym}}(G^r, H)$ resp. $\underline{\mathrm{Alt}}(G^r, H)$, is representable by a commutative group scheme, we will also denote this group scheme by $\underline{\mathrm{Sym}}(G^r, H)$ resp. $\underline{\mathrm{Alt}}(G^r, H)$.

We are now going to prove a general proposition on multilinear morphisms which will be used throughout the paper, but we first establish two lemmas:

Lemma 1.9. *If $\underline{\mathrm{Hom}}(G, H)$ is representable, there is a natural isomorphism*

$$\mathrm{Mult}(G_1 \times \cdots \times G_r, \underline{\mathrm{Hom}}(G, H)) \cong \mathrm{Mult}(G_1 \times \cdots \times G_r \times G, H),$$

functorial in all arguments.

Proof. By the definition of $\underline{\mathrm{Hom}}(G, H)$, giving a morphism of schemes $\varphi : G_1 \times \cdots \times G_r \to \underline{\mathrm{Hom}}(G, H)$ is equivalent to giving a morphism of schemes $\widetilde{\varphi} : G_1 \times \cdots \times G_r \times G \to H$ which is linear in G. Since the group structure of $\underline{\mathrm{Hom}}(G, H)$ is induced by that of H, one sees easily that φ is linear in G_i if and only if $\widetilde{\varphi}$ is linear in G_i. This completes the proof. □

Now, we give an "underline" version of this lemma in order to show our general result of this type:

Lemma 1.10. *If $\underline{\mathrm{Mult}}(G_1 \times \cdots \times G_{r-1}, \underline{\mathrm{Hom}}(G, H))$ and $\underline{\mathrm{Hom}}(G, H)$ are representable, then $\underline{\mathrm{Mult}}(G_1 \times \cdots \times G_r, H)$ is representable and there is a natural isomorphism*

$$\underline{\mathrm{Mult}}(G_1 \times \cdots \times G_r, \underline{\mathrm{Hom}}(G, H)) \cong \underline{\mathrm{Mult}}(G_1 \times \cdots \times G_r \times G, H)$$

functorial in all arguments.

Proof. If we establish the isomorphism, the representability will follow directly from it, because if two functors are naturally isomorphic and one is representable, the other is representable too. We show thus only the isomorphism. We show at first that for any commutative group schemes G and H over S and any S-scheme T, we have $\underline{\mathrm{Hom}}(G_T, H_T) \cong \underline{\mathrm{Hom}}(G, H)_T$. Indeed, if X is any T-scheme, then $\underline{\mathrm{Hom}}(G_T, H_T)(X) = \mathrm{Hom}_X((G_T)_X, (H_T)_X) = \mathrm{Hom}_X(G_X, H_X) = \underline{\mathrm{Hom}}(G, H)(X) = \underline{\mathrm{Hom}}(G, H)_T(X)$. Now, we have

$$\underline{\mathrm{Mult}}(G_1 \times G_2 \times \cdots \times G_r, H)(T) = \mathrm{Mult}(G_{1,T} \times G_{2,T} \times \cdots \times G_{r,T}, H_T)$$

and by Lemma 1.9 this is isomorphic to

$$\mathrm{Mult}(G_{1,T} \times G_{2,T} \times \cdots \times G_{r-1,T}, \underline{\mathrm{Hom}}(G_{r,T}, H_T)).$$

By the above discussion, it is isomorphic to

$$\mathrm{Mult}(G_{1,T} \times G_{2,T} \times \cdots \times G_{r-1,T}, \underline{\mathrm{Hom}}(G_r, H)_T) =$$

$$\underline{\mathrm{Mult}}(G_1 \times G_2 \times \cdots \times G_{r-1}, \underline{\mathrm{Hom}}(G_r, H))(T).$$

This achieves the proof. □

Remark 1.11.

1) Assume that $G_1, \ldots, G_r$ are finite and flat over S. We can show by induction on r that $\underline{\mathrm{Mult}}(G_1 \times \cdots \times G_r, H)$ is representable by a commutative group scheme. If furthermore H is affine or resp. of finite type, then $\underline{\mathrm{Mult}}(G_1 \times \cdots \times G_r, H)$ has the same property. Indeed, if $r = 1$ then this is exactly Theorem 3.10 in [6]. So let $r > 1$ and suppose that the statement is true for $r - 1$. By the induction hypothesis, $\underline{\mathrm{Mult}}(G_1 \times \cdots \times G_{r-1}, \underline{\mathrm{Hom}}(G_r, H))$ is representable and has the same properties (affineness or being of finite type) of $\underline{\mathrm{Hom}}(G, H)$ which has itself the same properties as H according to Theorem 3.10 in [6]. From Lemma 1.10, it follows that

$$\underline{\mathrm{Mult}}(G_1 \times \cdots \times G_{r-1}, \underline{\mathrm{Hom}}(G, H)) \cong \underline{\mathrm{Mult}}(G_1 \times \cdots \times G_r, H).$$

Hence, the right hand side is representable and has the same properties as H.

2) Let G be finite and flat over S. By definition 1.8, it is clear that the functors $\underline{\mathrm{Sym}}(G^r, H)$ and $\underline{\mathrm{Alt}}(G^r, H)$ are subfunctors of the representable functor $\underline{\mathrm{Mult}}(G^r, H)$. Since the conditions defining these subfunctors are closed conditions (given by equations), they are represented by closed subgroup schemes.

3) We will thus make the assumption that every time we use $\underline{\mathrm{Mult}}(G_1 \times \cdots \times G_1, H), \underline{\mathrm{Sym}}(G^r, H)$ or $\underline{\mathrm{Alt}}(G^r, H)$, the group schemes $G_1, \ldots, G_r$ and G are finite and flat over S and we will no longer worry about the representability of these functors.

Here is the desired proposition:

Proposition 1.12. *Let $G_1, \ldots, G_r, H_1, \ldots, H_s, F$ be commutative group schemes over a base scheme S. We have a natural isomorphism*

$$\mathrm{Mult}(G_1 \times \cdots \times G_r, \underline{\mathrm{Mult}}(H_1 \times \cdots \times H_s, F)) \cong \mathrm{Mult}(G_1 \times \cdots \times G_r \times H_1 \times \cdots \times H_s, F)$$

functorial in all arguments.

Proof. We prove this proposition by induction on s. If $s = 1$, then it is exactly the Lemma 1.9. So assume that $s > 1$ and that the proposition is true for $s - 1$. We have a series of isomorphisms:

$$\mathrm{Mult}(G_1 \times \cdots \times G_r \times H_1 \times \cdots \times H_s, F) \overset{1.9}{\cong}$$

$$\mathrm{Mult}(G_1 \times \cdots \times G_r \times H_1 \times \cdots \times H_{s-1}, \underline{\mathrm{Hom}}(H_s, F)) \underset{\text{hyp.}}{\overset{\text{ind.}}{\cong}}$$

$$\mathrm{Mult}(G_1 \times \cdots \times G_r, \underline{\mathrm{Mult}}(H_1 \times \cdots \times H_{s-1}, \underline{\mathrm{Hom}}(H_s, F))) \overset{1.10}{\cong}$$

$$\mathrm{Mult}(G_1 \times \cdots \times G_r, \underline{\mathrm{Mult}}(H_1 \times \cdots \times H_s, F)).$$

□

Remark 1.13. Let $G_1 \ldots, G_r, H_1 \ldots, H_s, G, H$ and F be commutative group schemes over a base scheme S. There is a natural action of the symmetric group S_n on H^n that induces an action on the group scheme $\underline{\mathrm{Mult}}(H^n, F)$ which itself induces an action on the group

$$\mathrm{Mult}(G_1 \times \cdots \times G_r, \underline{\mathrm{Mult}}(H^n, F)).$$

We also have a natural action of this group on the group

$$\mathrm{Mult}(G_1 \times \cdots \times G_1 \times H^n, H).$$

One checks that the isomorphism in the proposition is invariant under the action of S_n. Similarly, we have an action of the symmetric group S_m on

$$\mathrm{Mult}(G^m, \underline{\mathrm{Mult}}(H_1 \times \cdots \times H_s, F)) \text{ and } \mathrm{Mult}(G^m \times H_1 \times \cdots \times H_s, F)$$

induced by its action on G^m. Again, one can easily verify that the isomorphism in the proposition is invariant under this action of S_m.

In the same way that Lemma 1.10 follows from Lemma 1.9 the following proposition can be deduced from Proposition 1.12; we will thus omit the proof:

Proposition 1.14. *Let $G_1, \ldots, G_r, H_1, \ldots, H_s, F$ be commutative group schemes over a base scheme S. We have a natural isomorphism*

$$\underline{\mathrm{Mult}}(G_1 \times \cdots \times G_r, \underline{\mathrm{Mult}}(H_1 \times \cdots \times H_s, F)) \cong \underline{\mathrm{Mult}}(G_1 \times \cdots \times G_r \times H_1 \times \cdots \times H_s, F)$$

functorial in all arguments. □

Fix a base scheme $S = \mathbf{Spec}\, k$ for a field k and let $G_1, G_2, \ldots, G_r$ be finite commutative group schemes and let H be a commutative group scheme. Then by Proposition 1.12, we have an isomorphism that is functorial in H:

$$\mathrm{Mult}(H \times G_1 \times G_2 \times \cdots \times G_r, \mathbb{G}_m) \cong \mathrm{Hom}(H, \underline{\mathrm{Mult}}(G_1 \times \cdots \times G_r, \mathbb{G}_m)).$$

Let us write $\widetilde{G}$ for $\underline{\mathrm{Mult}}(G_1 \times \cdots \times G_r, \mathbb{G}_m)$. Then this means that we have a natural isomorphism

$$\mathrm{Mult}(- \times G_1 \times \cdots \times G_r, \mathbb{G}_m) \xrightarrow{\tau} \mathrm{Hom}(-, \widetilde{G}) = \widetilde{G}(-)$$

or in other words, the group scheme $\widetilde{G}$ represents the functor

$$\mathrm{Mult}(- \times G_1 \times \cdots \times G_r, \mathbb{G}_m).$$

Assume that we have a multilinear morphism

$$\varphi : H \times G_1 \times \cdots G_r \to \mathbb{G}_m,$$

then by functoriality, we have a commutative diagram:

$$\begin{tikzcd}
\mathrm{Mult}(\widetilde{G} \times G_1 \times \cdots\ G_r, \mathbb{G}_m) \arrow[r, "\tau_{\widetilde{G}}", "\cong"'] \arrow[d, "(\tau_H(\varphi)\times 1\times\cdots\times 1)^*"'] & \mathrm{Hom}(\widetilde{G}, \widetilde{G}) \arrow[d, "\tau_H(\varphi)"'] \\
\mathrm{Mult}(H \times G_1 \times \cdots \times G_r, \mathbb{G}_m) \arrow[r, "\tau_H", "\cong"'] & \mathrm{Hom}(H, \widetilde{G})
\end{tikzcd}$$

where $\tau_H(\varphi)^*(f) = f \circ \tau_H(\varphi)$ and

$$(\tau_H(\varphi) \times 1 \times \cdots \times 1)^*(g) = (\tau_H(\varphi) \times 1 \times \cdots \times 1) \circ g.$$

Proposition 1.15. *Assume that $\psi_{\widetilde{G}} \in \mathrm{Mult}(\widetilde{G} \times G_1 \times \cdots \times G_r, \mathbb{G}_m)$ is such that $\tau_{\widetilde{G}}(\psi_{\widetilde{G}}) = \mathrm{Id}_{\widetilde{G}}$. Then the pair $(\psi_{\widetilde{G}}, \widetilde{G})$ satisfies the following universal property: Given any multilinear morphism $\varphi : H \times G_1 \times \cdots \times G_r \to \mathbb{G}_m$ there is a unique homomorphism $\widetilde{\varphi} : H \to \widetilde{G}$ such that the following diagram commutes*

$$\begin{tikzcd}
H \times G_1 \times \cdots \times G_r \arrow[rr, "\varphi"] \arrow[dr, "\widetilde{\varphi}\times 1\times\cdots\times 1"'] & & \mathbb{G}_m \\
& \widetilde{G} \times G_1 \times \cdots \times G_r. \arrow[ur, "\psi_{\widetilde{G}}"'] &
\end{tikzcd}$$

Proof. We show that the unique morphism $\widetilde{\varphi}$ is $\tau_H(\varphi)$. Since $\tau_{\widetilde{G}}(\psi_{\widetilde{G}}) = \mathrm{Id}_{\widetilde{G}}$, we have by the commutative diagram before the proposition

$$\tau_H((\tau_H(\varphi) \times 1 \times \cdots \times 1) \circ \psi_{\widetilde{G}}) = \tau_H((\tau_H(\varphi) \times 1 \times \cdots \times 1)^*(\psi_{\widetilde{G}})) =$$

$$\tau_H(\varphi)^*(\tau_{\widetilde{G}}(\psi_{\widetilde{G}})) = \tau_H(\varphi)^*(\mathrm{Id}_{\widetilde{G}}) = \tau_H(\varphi).$$

But, τ_H is injective and therefore $\varphi = \tau_H(\varphi) \times 1 \times \cdots \times 1 \circ \psi_{\widetilde{G}}$. This shows the existence of $\widetilde{\varphi}$.

Now, if we have a morphism $f : H \to \widetilde{G}$ with the property that $\varphi = f \times 1 \times \cdots \times 1 \circ \psi_{\widetilde{G}}$, then by surjectivity of τ_H, there exists a multilinear morphism $\varphi' : H \times G_1 \times \cdots \times G_r \to \mathbb{G}_m$ such that $\tau_H(\varphi') = f$, and from what we have shown above

$$\varphi' = \tau_H(\varphi') \times 1 \times \cdots \times 1 \circ \psi_{\widetilde{G}} = f \times 1 \times \cdots \times 1 \circ \psi_{\widetilde{G}} = \varphi$$

Thus $\tau_H(\varphi) = \tau_H(\varphi') = f$. This proves the uniqueness of $\widetilde{\varphi}$. □

Definition 1.16. *We call the group scheme* $\underline{\mathrm{Mult}}(G_1 \times \cdots \times G_r, \mathbb{G}_m)$*, resp. the multilinear morphism* $\psi : \underline{\mathrm{Mult}}(G_1 \times \cdots \times G_r, \mathbb{G}_m) \times G_1 \times \cdots \times G_r \to \mathbb{G}_m$ *defined in Proposition 1.15,* the universal group scheme resp. the universal multilinear morphism associated to $G_1, \ldots, G_r$.

Lemma 1.17. *Let G, H be commutative group schemes over a base scheme S and let Γ be a finite group acting on G. Then we have a natural isomorphism*

$$\mathrm{Hom}(H, G)^\Gamma \cong \mathrm{Hom}(H, G^\Gamma),$$

where G^Γ is the subgroup scheme of fixed points, in other words, $G^\Gamma(T) = G(T)^\Gamma$ for any S-scheme T, where $G(T)^\Gamma$ is the subgroup of fixed points of the abelian Γ-group $G(T)$ and the action of Γ on $\mathrm{Hom}(H, G)$ is induced by its action on G. More precisely, the image of the inclusion $\mathrm{Hom}(H, G^\Gamma) \hookrightarrow \mathrm{Hom}(H, G)$ is the group of fixed points $\mathrm{Hom}(H, G)^\Gamma$.

Proof. Let $\varphi : H \to G^\Gamma$ and $\gamma \in \Gamma$ be given. The image of φ under the inclusion in the lemma is the composition $H \xrightarrow{\varphi} G^\Gamma \hookrightarrow G$ and under the action of γ on $\mathrm{Hom}(H, G)$ it maps to the morphism $H \xrightarrow{\varphi} G^\Gamma \hookrightarrow G \xrightarrow{\gamma\cdot} G$. But by definition of G^Γ, we have that the composition $G^\Gamma \hookrightarrow G \xrightarrow{\gamma\cdot} G$ is the same as the composition $G^\Gamma \hookrightarrow G$ and therefore γ fixes the image of φ and hence it is an element of $\mathrm{Hom}(H, G)^\Gamma$. We have thus an inclusion $\mathrm{Hom}(H, G^\Gamma) \subset \mathrm{Hom}(H, G)^\Gamma$, where we have identified $\mathrm{Hom}(H, G^\Gamma)$ with its image.
Now, assume that we have a morphism $\psi : H \to G$ which lies inside the group of fixed points. This means that the composition $\gamma \circ \psi$ for any $\gamma\cdot : G \to G$ is equal to ψ and therefore ψ must factor through G^Γ. This gives the inclusion $\mathrm{Hom}(H, G)^\Gamma \subset \mathrm{Hom}(H, G^\Gamma)$ and the lemma is proved. □

We are now going to apply this lemma to the particular case, where the acting group is the symmetric group S_n which acts on the group scheme $\underline{\mathrm{Mult}}(G^n, F)$ with G and F two commutative group schemes. The lemma states that we have an isomorphism

$$\mathrm{Hom}(H, \underline{\mathrm{Mult}}(G^n, F)^{S_n}) \cong \mathrm{Hom}(H, \underline{\mathrm{Mult}}(G^n, F))^{S_n}$$

By Definitions 1.6 and 1.8 and Remark 1.7, $\underline{\mathrm{Sym}}(G^n, F)$ is exactly the group of fixed points $\underline{\mathrm{Mult}}(G^n, F)^{S_n}$, and therefore we can rewrite the last isomorphism as

$$\mathrm{Hom}(H, \underline{\mathrm{Sym}}(G^n, F)) \cong \mathrm{Hom}(H, \underline{\mathrm{Mult}}(G^n, F))^{S_n} \qquad (\bigstar)$$

We now apply Proposition 1.12 and Remark 1.13: taking the fixed points of both sides of the isomorphism in Proposition 1.12, we will again get an isomorphism. We can thus apply it to our situation, and obtain the isomorphism:

$$\mathrm{Hom}(H, \underline{\mathrm{Mult}}(G^n, F))^{S_n} \cong \mathrm{Mult}(H \times G^n, F)^{S_n}.$$

Combining this with (★), we have the following proposition:

Proposition 1.18. *With the above notations, there is a natural isomorphism*

$$\mathrm{Hom}(H, \underline{\mathrm{Sym}}(G^n, F)) \cong \mathrm{Mult}(H \times G^n, F)^{S_n}$$

functorial in all arguments. □

Remark 1.19.

1) We recall that the action of S_n on the right hand side consists of permuting the factors of G^n and consequently, the group $\mathrm{Mult}(H \times G^n, F)^{S_n}$ contains the multilinear morphisms from $H \times G^n$ to F that are symmetric in G^n.

2) Note that the functoriality of this isomorphism in H implies that the group scheme $\underline{\mathrm{Sym}}(G^n, F)$ represents the functor $\mathrm{Mult}(- \times G^n, F)^{S_n}$.

3) It is clear that if we change $H \times G^n$ to $G^n \times H$ the proposition remains valid; we have thus another natural and functorial isomorphism

$$\mathrm{Hom}(H, \underline{\mathrm{Sym}}(G^n, F)) \cong \mathrm{Mult}(G^n \times H, F)^{S_n}.$$

Similar arguments prove the following proposition:

Proposition 1.20. *Let $G, H_1, \ldots, H_r$ and F be commutative group schemes. We have a natural isomorphism*

$$\mathrm{Mult}(H_1 \times \cdots \times H_r \times G^n, F)^{S_n} \cong \mathrm{Mult}(H_1 \times \cdots \times H_r, \underline{\mathrm{Sym}}(G^n, F)).$$

□

We can show, with slight modifications of arguments, similar results concerning the group of alternating multilinear morphisms and in particular the following proposition:

Proposition 1.21. *With above notations we have a natural isomorphism*

$$\mathrm{Alt}(H_1 \times \cdots \times H_r \times G^n, F) \cong \mathrm{Mult}(H_1 \times \cdots \times H_r, \underline{\mathrm{Alt}}(G^n, F))$$

where the first group is the group of multilinear morphisms that are alternating in G^n. □

Remark 1.22. We could show Propositions 1.20 and 1.21 using the isomorphism given in Proposition 1.12. Indeed, under that isomorphism the image of an element of $\mathrm{Mult}(H_1 \times \cdots \times H_r \times G^n, F)^{S_n}$ lies inside the subgroup $\mathrm{Mult}(H_1 \times \cdots \times H_r, \underline{\mathrm{Sym}}(G^n, F))$ of $\mathrm{Mult}(H_1 \times \cdots \times H_r, \underline{\mathrm{Mult}}(G^n, F))$ and vice versa. This is what we explained in Remark 1.13. The same argument works in the alternating case.

Let G, H and F be as above and assume that we are in a situation where any multilinear morphism from G^n to F is symmetric. Then we have in particular

$$\mathrm{Mult}(G^n, \underline{\mathrm{Hom}}(H, F)) = \mathrm{Mult}(G^n, \underline{\mathrm{Hom}}(H, F))^{S_n} \qquad (\blacktriangle)$$

According to Proposition 1.12 we have an isomorphism

$$\mathrm{Mult}(G^n \times H, F) \cong \mathrm{Mult}(G^n, \underline{\mathrm{Hom}}(H, F)),$$

and referring again to Remark 1.13, we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Mult}(G^n \times H, F)^{S_n} \arrow[r, "\cong"] \arrow[d, hook, "i"'] & \mathrm{Mult}(G^n, \underline{\mathrm{Hom}}(H, F))^{S_n} \arrow[d, hook, "j"] \\
\mathrm{Mult}(G^n \times H, F) \arrow[r, "\cong"] & \mathrm{Mult}(G^n, \underline{\mathrm{Hom}}(H, F)).
\end{tikzcd}$$

We deduce from $(\blacktriangle)$ that j is an equality and therefore i is an equality as well. Another use of Proposition 1.12 and Proposition 1.18 gives the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Mult}(G^n \times H, F)^{S_n} \arrow[r, "\cong", "1.18"'] \arrow[d, hook, "i"', "\shortparallel"] & \mathrm{Hom}(H, \underline{\mathrm{Sym}}(G^n, F)) \\
\mathrm{Mult}(G^n \times H, F) \arrow[r, "\cong", "1.12"'] & \mathrm{Hom}(H, \underline{\mathrm{Mult}}(G^n, F)). \arrow[u, hook, "u"']
\end{tikzcd}$$

As we have seen, i is an equality, which implies that u is also an equality. Since this is true for any commutative group scheme H, it follows that $\underline{\mathrm{Sym}}(G^n, H) = \underline{\mathrm{Mult}}(G^n, F)$.
Suppose that the base scheme S is defined over $\mathbb{F}_p$ with $p > 2$ and that T is any S-scheme. Take an alternating multilinear morphism $\psi : G_T^n \to F_T$, i.e. an element of $\underline{\mathrm{Alt}}(G^n, F)(T)$. Since $\underline{\mathrm{Sym}}(G^n, H) = \underline{\mathrm{Mult}}(G^n, F)$ and so $\underline{\mathrm{Sym}}(G^n, H)(T) = \underline{\mathrm{Mult}}(G^n, F)(T)$, this ψ should be symmetric. As it is also alternating and the characteristic is odd, we deduce that ψ is the zero morphism. Recapitulating, we have:

Proposition 1.23. *Suppose that G and F are commutative group schemes over a base scheme S. Suppose also that any multilinear morphism from G^n to any commutative group scheme is symmetric. Then if $\underline{\mathrm{Mult}}(G^n, F)$ exists, we have*

- $\underline{\mathrm{Sym}}(G^n, F) = \underline{\mathrm{Mult}}(G^n, F)$ *and*
- $\underline{\mathrm{Alt}}(G^n, F) = 0$ *if S is defined over $\mathbb{F}_p$ with $p > 2$.* $\square$

Example 1.24. Let $n \geq 2$, then there is an isomorphism

$$\underline{\mathrm{Mult}}(\underbrace{\alpha_p \times \cdots \times \alpha_p}_{n \text{ times}}, \mathbb{G}_m) \cong \mathbb{G}_a.$$

Proof. We first show that $\underline{\mathrm{Hom}}(\alpha_p, \mathbb{G}_a) \cong \mathbb{G}_a$. Indeed, let R be a k-algebra and $\varphi : k[X] \otimes_k R = R[X] \to k[X]/(X^p) \otimes_k R = R[X]/(X^p)$ be an R-Hopf algebra homomorphism, write x for the image of X in $R[X]/(X^p)$, and let $\varphi(X) = a_0 + a_1 x + \ldots + a_{p-1}x^{p-1} \in R[X]/(X^p)$. Then being a Hopf algebra homomorphism amounts to saying that

$$\sum_{i=0}^{p-1} a_i(1 \otimes x^i + x^i \otimes 1) = \sum_{i=0}^{p-1} a_i(1 \otimes x + x \otimes 1)^i.$$

Since $\{x^j \otimes x^l\}_{j,l=0}^{p-1}$ form an R-basis of $R[X]/(X^p) \otimes_k R[X]/(X^p)$, a_i should be zero for $i \neq 1$. We have therefore $\varphi(X) = a_1 x$ for an element $a_1 \in R$. Consequently, R-Hopf algebra homomorphisms from $R[X]$ to $R[X]/(X^p)$ are of the form $X \mapsto r \cdot x$, and any such morphism is an R-Hopf algebra homomorphism. Moreover, the sum of two such homomorphisms $X \mapsto r \cdot x$ and $X \mapsto s \cdot x$ is $X \mapsto (r+s) \cdot x$. The parameter r thus defines an isomorphism $\underline{\mathrm{Hom}}(\alpha_p, \mathbb{G}_a) \cong \mathbb{G}_a$.

Secondly, since $(rx)^p = 0$ in $R[X]/(X^p)$, we find that any homomorphism $\alpha_{p,R} \to \mathbb{G}_{a,R}$ factors through $\alpha_{p,R} \subset \mathbb{G}_{a,R}$. Therefore

$$\underline{\mathrm{Hom}}(\alpha_p, \alpha_p) = \underline{\mathrm{Hom}}(\alpha_p, \mathbb{G}_a) \cong \mathbb{G}_a.$$

We also have canonical isomorphisms, $\alpha_p^* \cong \alpha_p$ and $\underline{\mathrm{Hom}}(\alpha_p, \mathbb{G}_m) \cong \alpha_p^*$. Finally, putting all this together and using Lemma 1.10, we obtain

$$\underline{\mathrm{Mult}}(\alpha_p^n, \mathbb{G}_m) \cong \underline{\mathrm{Mult}}(\alpha_p^{n-1}, \underline{\mathrm{Hom}}(\alpha_p, \mathbb{G}_m)) \cong \underline{\mathrm{Mult}}(\alpha_p^{n-1}, \alpha_p^*) \cong$$

$$\underline{\mathrm{Mult}}(\alpha_p^{n-1}, \alpha_p) \cong \underline{\mathrm{Mult}}(\alpha_p^{n-1}, \underline{\mathrm{Hom}}(\alpha_p, \alpha_p)) \cong \underline{\mathrm{Mult}}(\alpha_p^{n-1}, \mathbb{G}_a) \cong$$

$$\underline{\mathrm{Mult}}(\alpha_p^{n-2}, \underline{\mathrm{Hom}}(\alpha_p, \mathbb{G}_a)) \cong \underline{\mathrm{Mult}}(\alpha_p^{n-2}, \mathbb{G}_a) \cong \cdots \cong \underline{\mathrm{Hom}}(\alpha_p, \mathbb{G}_a) \cong \mathbb{G}_a.$$

□

Example 1.25. Let n, $m \geq 1$ be natural numbers, we have $\underline{\mathrm{Hom}}(\alpha_{p^n}, \alpha_{p^m}) \cong$

- $\mathbb{G}_a^n = \overbrace{\mathbb{G}_a \times \cdots \times \mathbb{G}_a}^{n \text{ times}}$ if $m \geq n$
- $\alpha_{p^m}^{n-m} \times \mathbb{G}_a^m = \underbrace{\alpha_{p^m} \times \cdots \times \alpha_{p^m}}_{n-m \text{ times}} \times \underbrace{\mathbb{G}_a \times \cdots \times \mathbb{G}_a}_{m \text{ times}}$ if $m < n$.

Proof. Let R be a k-algebra and $\varphi : \alpha_{p^n,R} \to \alpha_{p^m,R}$ be a homomorphism. Then φ corresponds to a Hopf algebra homomorphism $R[X]/(X^{p^m}) \to R[X]/(X^{p^n})$ which we denote again by φ. Let x be the class of X in $R[X]/(X^{p^m})$ and $R[X]/(X^{p^n})$. Write $\varphi(x) = a_0 + a_1 x + \cdots + a_{p^n-1}x^{p^n-1}$. This element of $R[X]/(X^{p^n})$ should fulfill two conditions, namely $\varphi(x)^{p^m} = 0$ and

$$\Delta(\varphi(x)) = (\varphi \otimes \varphi)(\Delta(x)) = 1 \otimes \varphi(x) + \varphi(x) \otimes 1.$$

We first exploit the second condition, which gives

$$\Delta(\sum_{i=0}^{p^n-1} a_i x^i) = 1 \otimes (\sum_{i=0}^{p^n-1} a_i x^i) + (\sum_{i=0}^{p^n-1} a_i x^i) \otimes 1.$$

We thus have

$$\sum_{i=0}^{p^n-1} a_i \Delta(x)^i = \sum_{i=0}^{p^n-1} a_i (1 \otimes x + x \otimes 1)^i = \sum_{i=0}^{p^n-1} a_i (1 \otimes x^i + x^i \otimes 1).$$

Since the elements $x^i \otimes x^j$ are linearly independent, we must have $a_i(1 \otimes x + x \otimes 1)^i = a_i(1 \otimes x^i + x^i \otimes 1)$ for all i, i.e., we have

$$a_i(\sum_{j=0}^{i} \binom{i}{j} x^j \otimes x^{i-j} - 1 \otimes x^i + x^i \otimes 1) = a_i \sum_{j=1}^{i-1} \binom{i}{j} x^j \otimes x^{i-j} = 0.$$

If i is not a power of p, then by Lucas theorem there is a j with $\binom{i}{j}$ not divisible by p and therefore, from the linear independence of $x^j \otimes x^{i-j}$ we deduce that $a_i = 0$ for these i's, and we can write

$$\varphi(x) = a_1 x + a_p x^p + a_{p^2} x^{p^2} + \cdots + a_{p^{n-1}} x^{p^{n-1}}.$$

Consider now the first condition. If $m \geq n$, this condition is automatically satisfied and any n-tuple $(a_1, a_p, \ldots, a_{p^{n-1}})$ gives rise to a unique homomorphism $\alpha_{p^n} \to \alpha_{p^m}$, and one sees easily that the component-wise addition of these n-tuples corresponds to the addition of homomorphisms $\alpha_{p^n} \to \alpha_{p^m}$. This gives the isomorphism

$$\underline{\mathrm{Hom}}(\alpha_{p^n}, \alpha_{p^m}) \cong \mathbb{G}_a^n.$$

If $m < n$, the first condition implies that

$$(a_1 x + a_p x^p + a_{p^2} x^{p^2} + \cdots + a_{p^{n-1}} x^{p^{n-1}})^{p^m} = 0,$$

i.e.,

$$a_1^{p^m} x^{p^m} + a_p^{p^m} x^{p^{m+1}} + a_{p^2}^{p^m} x^{p^{m+2}} + \cdots + a_{p^{n-1}}^{p^m} x^{p^{m+n-1}} = 0.$$

For indices i with $m + i \geq n$ we have $x^{p^{m+i}} = 0$, therefore we must have

$$a_1^{p^m} x^{p^m} + a_p^{p^m} x^{p^{m+1}} + a_{p^2}^{p^m} x^{p^{m+2}} + \cdots + a_{p^{n-m-1}}^{p^m} x^{p^{n-1}} = 0$$

which implies that $a_{p^i}^{p^m} = 0$ for all $0 \leq i \leq m - n - 1$ and there is no condition on other a_i's. Consequently, the n-tuples $(a_1, a_p, \ldots, a_{p^{n-1}})$ belong to the group $\alpha_{p^m}^{n-m}(R) \times \mathbb{G}_a^n(R)$. Again, the component-wise addition of these tuples corresponds to the addition in $\mathrm{Hom}(\alpha_{p^n}, \alpha_{p^m})$, and therefore we have an isomorphism

$$\underline{\mathrm{Hom}}(\alpha_{p^n}, \alpha_{p^m}) \cong \alpha_{p^m}^{n-m} \times \mathbb{G}_a^m.$$

□

Remark 1.26.

1) In both cases, any n-tuple $(a_1, \dots, a_n)$ with $a_i \in \mathbb{G}_a(R)$ (in the second case, the first m entries are in fact in $\alpha_{p^m}^{n-m}(R)$) defines a Hopf algebra homomorphism $\varphi : R[X]/(X^{p^m}) \to R[X]/(X^{p^n})$ with

$$\varphi(x) = a_1 x + a_p x^p + a_{p^2} x^{p^2} + \cdots + a_{p^{n-1}} x^{p^{n-1}}.$$

It then follows that the homomorphism $\alpha_{p^n,R} \to \alpha_{p^m,R}$ corresponding to φ, sends $s \in \alpha_{p^n}(S)$ to

$$a_1 s + a_p s^p + a_{p^2} s^{p^2} + \cdots + a_{p^{n-1}} s^{p^{n-1}} \in \alpha_{p^m}(S)$$

for any R-algebra S.

2) The same arguments as in the first case of the example, show that for any positive integer n, there is an isomorphism $\underline{\mathrm{Hom}}(\alpha_{p^n}, \mathbb{G}_a) \cong \mathbb{G}_a^n$. And as in the first part of the remark, this isomorphism sends any n-tuple $(a_1 \dots, a_n) \in \mathbb{G}_a(R)^n$ to the homomorphism $\alpha_{p^n,R} \to \mathbb{G}_{a,R}$ that sends an element $s \in \alpha_{p^n}(S)$ to the element

$$a_1 s + a_p s^p + a_{p^2} s^{p^2} + \cdots + a_{p^{n-1}} s^{p^{n-1}} \in \mathbb{G}_a(S)$$

for any R-algebra S.

Example 1.27. We can go further and show that for any positive integers $n_1, n_2, \dots, n_r$ we have an isomorphism

$$\underline{\mathrm{Mult}}(\alpha_{p^{n_1}} \times \alpha_{p^{n_2}} \times \cdots \times \alpha_{p^{n_r}}, \mathbb{G}_a) \cong \mathbb{G}_a^{n_1 n_2 \cdots n_r}$$

given by the following formula: An element $\overrightarrow{a}_r := (a_{i_1,i_2,\dots,i_r}) \in \mathbb{G}_a(R)^{n_1 n_2 \cdots n_r}$ corresponds to the homomorphism $\varphi_{\overrightarrow{a}_r} : \alpha_{p^{n_1},R} \times \alpha_{p^{n_2},R} \times \cdots \times \alpha_{p^{n_r},R} \to \mathbb{G}_{a,R}$ which sends an r-tuple

$$(s_1, s_2 \dots, s_r) \in \alpha_{p^{n_1}}(S) \times \alpha_{p^{n_2}}(S) \times \cdots \times \alpha_{p^{n_r}}(S)$$

to the element

$$\sum_{i_1,i_2,\dots,i_r} a_{i_1,i_2,\dots,i_r} s_1^{p^{i_1}} s_2^{p^{i_2}} \cdots s_r^{p^{i_r}} \in \mathbb{G}_a(S), \quad i_j \in \{0, 1, \dots, n_j - 1\} \ \forall 1 \le j \le r$$

for any R-algebra S.

Proof. We recall that for commutative group schemes G, H_1 and H_2 we have

$$\mathrm{Hom}(G, H_1 \times H_2) \cong \mathrm{Hom}(G, H_1) \times \mathrm{Hom}(G, H_2)$$

and therefore, we also have the underlined version

$$\underline{\mathrm{Hom}}(G, H_1 \times H_2) \cong \underline{\mathrm{Hom}}(G, H_1) \times \underline{\mathrm{Hom}}(G, H_2).$$

We show the statement by induction on r. If $r = 1$, then this is exactly Remark 1.26 point 2). Suppose that $r > 1$ and the statement is true for $r - 1$. Let us

fix a k-algebra R, an R-algebra S and an S-algebra T for the rest of the proof. We have

$$\underline{\mathrm{Mult}}(\alpha_{p^{n_1}} \times \alpha_{p^{n_2}} \times \cdots \times \alpha_{p^{n_r}}, \mathbb{G}_a) \cong \underline{\mathrm{Hom}}(\alpha_{p^{n_1}}, \underline{\mathrm{Mult}}(\alpha_{p^{n_2}} \times \cdots \times \alpha_{p^{n_r}}, \mathbb{G}_a))$$

by Proposition 1.14. By the induction hypothesis, we have

$$\underline{\mathrm{Mult}}(\alpha_{p^{n_2}} \times \cdots \times \alpha_{p^{n_r}}, \mathbb{G}_a) \cong \mathbb{G}_a^{n_2 \cdots n_r}$$

and under this isomorphism an element $\overrightarrow{a}_{r-1} = (a_{i_2,\ldots,i_r}) \in \mathbb{G}_a(R)^{n_2\cdots n_r}$ is sent to the homomorphism $\varphi_{\overrightarrow{a}_{r-1}} : \alpha_{p^{n_2},R} \times \cdots \times \alpha_{p^{n_r},R} \to \mathbb{G}_{a,R}$ defined above. Combining this isomorphism with the last one, we obtain:

$$\underline{\mathrm{Mult}}(\alpha_{p^{n_1}} \times \alpha_{p^{n_2}} \times \cdots \times \alpha_{p^{n_r}}, \mathbb{G}_a) \cong \underline{\mathrm{Hom}}(\alpha_{p^{n_1}}, \mathbb{G}_a^{n_2\cdots n_r}) \cong \underline{\mathrm{Hom}}(\alpha_{p^{n_1}}, \mathbb{G}_a)^{n_2\cdots n_r}.$$

By Remark 1.26 2), $\underline{\mathrm{Hom}}(\alpha_{p^{n_1}}, \mathbb{G}_a)^{n_2\cdots n_r} \cong \mathbb{G}_a^{n_1 n_2 \ldots n_r}$. Now we consider the image of an element $\overrightarrow{a}_r = (a_{i_1,i_2,\ldots,i_r}) \in \mathbb{G}_a(R)^{n_1 n_2 \cdots n_r}$ under these isomorphisms. The isomorphism $(\mathbb{G}_a^{n_1})^{n_2\cdots n_r} \cong \mathbb{G}_a^{n_1\cdots n_r}$ (it is in fact a rearranging of entries) maps this element to $(A_{i_2,\ldots,i_r})$ where each

$$A_{i_2,\ldots,i_r} = (a_{0,i_2,\ldots,i_r}, a_{1,i_2,\ldots,i_r}, \ldots, a_{n_1-1,i_2,\ldots,i_r})$$

is a vector in $\mathbb{G}_a(R)^{n_1}$. Under the isomorphism $\underline{\mathrm{Hom}}(\alpha_{p^{n_1}}, \mathbb{G}_a) \cong \mathbb{G}_a^{n_1}$, each vector $A_{i_2,\ldots,i_r}$ is sent to the homomorphism $\varphi_{i_2,\ldots,i_r} : \alpha_{p^{n_1},R} \to \mathbb{G}_{a,R}$ defined by the vector $A_{i_2,\ldots,i_r}$, as in the statement of Remark 1.26 2), i.e.,

$$\varphi_{i_2,\ldots,i_r}(s) = a_{0,i_2,\ldots,i_r} + a_{1,i_2,\ldots,i_r} s^p + \cdots + a_{n_1-1,i_2,\ldots,i_r} s^{p^{n_1-1}}$$

for any $s \in \alpha_{p^{n_1}}(S)$. So we have an element $(\varphi_{i_2,\ldots,i_r}) \in \mathrm{Hom}(\alpha_{p^{n_1},R}, \mathbb{G}_{a,R})^{n_2\cdots n_r}$. Under the isomorphism $\mathrm{Hom}(\alpha_{p^{n_1},R}, \mathbb{G}_{a,R}^{n_2\cdots n_r}) \cong \mathrm{Hom}(\alpha_{p^{n_1},R}, \mathbb{G}_{a,R})^{n_2\cdots n_r}$, this element goes to

$$\varphi : \alpha_{p^{n_1},R} \to \mathbb{G}_{a,R}^{n_2\cdots n_r}, \quad s \mapsto (\varphi_{i_2,\ldots,i_r}(s)) \in \mathbb{G}_a(S)^{n_2\cdots n_r}$$

where s is an element of $\alpha_{p^{n_1}}(S)$. The element $(\varphi_{i_2,\ldots,i_r}(s))$ corresponds by the induction hypothesis to the multilinear morphism $\varphi' : \alpha_{p^{n_2},S} \times \cdots \times \alpha_{p^{n_r},S} \to \mathbb{G}_{a,S}$ which sends an r-tuple $(t_2, \ldots, t_r) \in \alpha_{p^{n_2}}(T) \times \cdots \times \alpha_{p^{n_r}}(T)$ to the element $\sum_{i_2,\ldots,i_r} \varphi_{i_2,\ldots,i_r}(s) t_2^{p^{i_2}} \cdots t_r^{p^{i_r}} =$

$$\sum_{i_2,\ldots,i_r} \sum_{i_1=0}^{n_1-1} a_{i_1,i_2\ldots,i_r} s^{p^{i_1}} t_2^{p^{i_2}} \cdots t_r^{p^{i_r}} = \sum_{i_1,\ldots,i_r} a_{i_1,i_2\ldots,i_r} s^{p^{n_1}} t_2^{p^{i_2}} \cdots t_r^{p^{i_r}}.$$

It follows from the isomorphism

$$\underline{\mathrm{Mult}}(\alpha_{p^{n_1}} \times \alpha_{p^{n_2}} \times \cdots \times \alpha_{p^{n_r}}, \mathbb{G}_a) \cong \underline{\mathrm{Hom}}(\alpha_{p^{n_1}}, \underline{\mathrm{Mult}}(\alpha_{p^{n_2}} \times \cdots \times \alpha_{p^{n_r}}, \mathbb{G}_a))$$

that φ' is sent to $\varphi_{\overrightarrow{a}_r} : \alpha_{p^{n_1},R} \times \alpha_{p^{n_2},R} \times \cdots \times \alpha_{p^{n_r},R} \to \mathbb{G}_{a,R}$ as in the statement of the example. The proof is thus achieved. □

Remark 1.28.

1) It is easy to check that the corresponding Hopf algebra homomorphism defined by $\varphi_{\overrightarrow{a}_r} : \alpha_{p^{n_1},R} \times \alpha_{p^{n_2},R} \times \cdots \times \alpha_{p^{n_r},R} \to \mathbb{G}_{a,R}$ is the R-homomorphism

$$\varphi^\sharp : R[X] \to R[Y_1]/(Y_1^{p^{n_1}}) \otimes \cdots \otimes R[Y_r]/(Y_r^{p^{n_r}})$$

that sends X to $\sum_{i_1,\ldots,i_r} a_{i_1\ldots,i_r} y_1^{p^{i_1}} \otimes \cdots \otimes y_r^{p^{i_r}}$ where y_j is the is the image of Y_j in $R[Y_j]/(Y_j^{p^{n_j}})$.

2) With the same methods as in the proof of Example 1.27 and the second part of Example 1.25, one can determine the group scheme

$$\underline{\mathrm{Mult}}(\alpha_{p^{n_1}} \times \alpha_{p^{n_2}} \times \cdots \times \alpha_{p^{n_r}}, \alpha_{p^l}).$$

It would obviously depend on l, r and n_i's. The formula for the general case (arbitrary l, r and different n_i's) is rather complicated and we would not give it here, but for $l = 1, r = n$ and $n_1 = n_2 \cdots = n_n = n$ we have

$$\underline{\mathrm{Mult}}(\alpha_{p^n}^n, \alpha_p) \cong \alpha_p^{(n-1)^n} \times \mathbb{G}_a^{(n-1)^n}.$$

We can use this example in order to calculate other interesting groups of multilinear morphisms.

Example 1.29. Let k be a field of characteristic p. We have isomorphisms:

- $\underline{\mathrm{Sym}}(\alpha_{p^n}^r, \mathbb{G}_a) \cong \mathbb{G}_a^{\binom{n+r-1}{r-1}}$
- $\underline{\mathrm{Alt}}(\alpha_{p^n}^r, \mathbb{G}_a) \cong \mathbb{G}_a^{\binom{n}{r}}$ if $p > 2$,

with the convention that $\binom{a}{b} = 0$ whenever $b > a$.

Proof. Let R be a k-algebra and $\varphi : \alpha_{p^n,R}^r \to \mathbb{G}_{a,R}$ be an element of $\mathrm{Mult}(\alpha_{p^n,R}^r, \mathbb{G}_{a,R})$, which is isomorphic to $\mathbb{G}_a(R)^{n^r}$ by Example 1.27. So there is an element $(a_{i_1,\ldots,i_r}) \in \mathbb{G}_a(R)^{n^r}$ that corresponds in the way explained in that example to φ. By the the first part of Remark 1.28, the R-Hopf algebra homomorphism corresponding to φ is the homomorphism

$$\varphi^\sharp : R[X] \to R[Y]/(Y^{p^n}) \otimes \cdots \otimes R[Y]/(Y^{p^n})$$

that sends X to $\sum_{i_1,\ldots,i_r} a_{i_1\ldots,i_r} y^{p^{i_1}} \otimes \cdots \otimes y^{p^{i_r}}$. The action of the symmetric group S_r on $\alpha_{p^n}^r$ and therefore on its representing Hopf algebra

$$\underbrace{R[Y]/(Y^{p^n}) \otimes \cdots \otimes R[Y]/(Y^{p^n})}_{r \text{ times}}$$

permutes $y^{p^{i_j}}$'s, i.e., if $\sigma \in S_r$, then $\sigma(y^{p^{i_1}} \otimes \cdots \otimes y^{p^{i_r}}) = y^{p^{i_{\sigma 1}}} \otimes \cdots \otimes y^{p^{i_{\sigma r}}}$. We have thus,

$$\sigma(\sum_{i_1,\ldots,i_r} a_{i_1\ldots,i_r} y^{p^{i_1}} \otimes \cdots \otimes y^{p^{i_r}}) = \sum_{i_1,\ldots,i_r} a_{i_{\sigma 1}\ldots,i_{\sigma r}} y^{p^{i_1}} \otimes \cdots \otimes y^{p^{i_r}}$$

- φ is symmetric in $\alpha_{p^n}^r$ if and only if $\varphi^\sharp$ is symmetric in the sense that it is invariant under composition with any permutation, i.e., we must have $\varphi^\sharp \circ \sigma = \varphi^\sharp$, or in other words, $\sigma(\sum_{i_1,\dots,i_r} a_{i_1\dots,i_r} y^{p^{i_1}} \otimes \cdots \otimes y^{p^{i_r}}) = \sum_{i_1,\dots,i_r} a_{i_1\dots,i_r} y^{p^{i_1}} \otimes \cdots \otimes y^{p^{i_r}}$, for all permutations $\sigma \in S_r$. But the elements $y^{p^{i_1}} \otimes \cdots \otimes y^{p^{i_r}}$ are linearly independent over R. It follows then that $a_{i_1\dots,i_r} = a_{i_{\sigma 1}\dots,i_{\sigma r}}$, for all permutations $\sigma \in S_r$. This is the only condition on $a_{i_1,\dots,i_r}$ for the homomorphism φ to be symmetric. So the number of different classes of $a_{i_1,\dots,i_r}$'s under the action of S_r is equal to the number of sequences of indices $i_1 \leq i_2 \leq \cdots \leq i_r$ with $0 \leq i_j \leq n-1$, because with the action of S_r we can reorder the indices in this way. This number is $\binom{n+r-1}{r-1}$. Hence, we have $\mathrm{Sym}(\alpha_{p^n,R}^r, \mathbb{G}_{a,R}) \cong \mathbb{G}_a(R)^{\binom{n+r-1}{r-1}}$, which implies that $\underline{\mathrm{Sym}}(\alpha_{p^n}^r, \mathbb{G}_a) \cong \mathbb{G}_a^{\binom{n+r-1}{r-1}}$.

- φ is alternating if and only if it is antisymmetric (since the characteristic is odd). Then it is antisymmetric if and only if $\varphi^\sharp$ is antisymmetric. Arguing in the same way as above, $\varphi^\sharp$ is antisymmetric if and only if $a_{i_{\sigma 1}\dots,i_{\sigma r}} = \mathrm{sgn}(\sigma) a_{i_1\dots,i_r}$. In particular, every time two indices i_{j_i} and i_{j_2} are equal $a_{i_1,\dots,i_r}$ vanishes. Here one uses again the fact that p is odd, indeed on one hand interchanging y's in j_1^{th} and j_2^{th} factor of $(R[Y]/(Y^{p^n}))^{\otimes r}$ doesn't change the sign (y appears with the same power in these factors) and on the other hand it changes the sign (it is antisymmetric) and since p is odd the coefficient $a_{i_1,\dots,i_r}$ should be zero. Therefore the number of possible nonzero $a_{i_1,\dots,i_r}$'s, i.e., those with no restriction, is equal to the number of sequences of indices $i_1 < \cdots < i_r$ with $0 \leq i_j \leq n-1$. If r is greater than n then this number is zero, otherwise this number is $\binom{n}{r}$, so with the convention mentioned above it is always $\binom{n}{r}$, and we have consequently $\mathrm{Alt}(\alpha_{p^n,R}^r, \mathbb{G}_{a,R}) \cong \mathbb{G}_a(R)^{\binom{n}{r}}$. It follows at once that $\underline{\mathrm{Alt}}(\alpha_{p^n}^r, \mathbb{G}_a) \cong \mathbb{G}_a^{\binom{n}{r}}$.

□

2 Tensor product and related constructions

Definition 2.1. *Let $G_1 \dots, G_r, G$ be commutative group schemes over a base scheme S. A multilinear morphism $\varphi : G_1 \times \cdots \times G_r \to G$, or by abuse of terminology, the group scheme G, is called a* tensor product of $G_1, \dots, G_r$ *if, for all commutative group schemes H over S, the induced map*

$$\mathrm{Hom}(G, H) \to \mathrm{Mult}(G_1 \times \cdots \times G_r, H), \quad \psi \mapsto \psi \circ \varphi,$$

is an isomorphism. If such G and φ exist, we write $G_1 \otimes \cdots \otimes G_r$ for G.

Remark 2.2.

1) The defining universal property of the tensor product, makes it unique up to unique isomorphism to the extent that it exists, and if this is so, the tensor product is functorial and right exact in all arguments.

2) According to Theorem 4.3 in [6], if $S = \mathbf{Spec}\, k$ for a field k, and $G_1, \dots, G_r$ are finite over S, then $G_1 \otimes \cdots \otimes G_r$ exists and is pro-finite over S, i.e., it is an inverse limit of finite group schemes over S. It is in fact the inverse limit $\varprojlim G_\alpha^*$ where G_α runs over all finite subgroup schemes of $\underline{\mathrm{Mult}}(G_1 \times \cdots \times G_r, \mathbb{G}_m)$. Again, every time we use the tensor product of group schemes, we will assume the hypotheses in this theorem so that this tensor product exists.

3) One would expect that the construction of tensor product commutes with the base change, i.e., $(G_1 \otimes \cdots \otimes G_r)_T \cong G_{1,T} \otimes \cdots \otimes G_{r,T}$. But this is not true as the example $\alpha_{p,k} \otimes \alpha_{p,k}$ shows. Indeed, we have for any field L that $\alpha_{p,L} \otimes \alpha_{p,L} \cong \varprojlim G_i^*$ where G_i runs over all finite subgroup schemes of $\mathbb{G}_{a,L}$ and if k'/k is a transcendental field extension of characteristic p, then there are finite subgroup schemes of $\mathbb{G}_{a,k'}$ that do not lie in a finite subgroup scheme defined over k, so the inverse limit over k' is taken over a much larger system than over k. But for finite field extensions this problem does not occur.

Definition 2.3. *Let G be a commutative group scheme over a base scheme S.*

(i) *A symmetric multilinear morphism $\varphi : G^r \to G'$, or by abuse of terminology, the group scheme G', is called an rth* symmetric power *of G, if for all commutative group schemes H over S, the induced map*

$$\mathrm{Hom}(G', H) \to \mathrm{Sym}(G^r, H), \quad \psi \mapsto \psi \circ \varphi,$$

is an isomorphism. If such G' and φ exist, we write $S^r G$ for G'.

(ii) *An alternating multilinear morphism $\varphi : G^r \to G'$, or by abuse of terminology, the group scheme G', is called an rth* alternating power *of G, if for all commutative group schemes H over S, the induced map*

$$\mathrm{Hom}(G', H) \to \mathrm{Alt}(G^r, H), \quad \psi \mapsto \psi \circ \varphi,$$

is an isomorphism. If such G' and φ exist, we write $\Lambda^r G$ for G'.

Remark 2.4. *Again, if $S^r G$ resp. $\Lambda^r G$ exists, it with the multilinear morphism $G^r \to S^r G$, resp. $G^r \to \Lambda^r G$, is unique up to unique isomorphism.*

Proposition 2.5. *Let G be a commutative group scheme. If $\Lambda^n G = 0$ then we have $\Lambda^m G = 0$ for all $m \geq n$.*

Proof. We show that $\Lambda^{n+1} G = 0$; the result follows immediately by induction. Let H be a commutative group scheme. By the definition, we have an isomorphism $\mathrm{Hom}(\Lambda^{n+1} G, H) \cong \mathrm{Alt}(G^{n+1}, H)$. Under the isomorphism

$$\mathrm{Mult}(G^{n+1}, H) \cong \mathrm{Mult}(G^n, \underline{\mathrm{Hom}}(G, H))$$

given in Lemma 1.9, the image of the subgroup $\mathrm{Alt}(G^{n+1}, H)$ lies in the subgroup $\mathrm{Alt}(G^n, \underline{\mathrm{Hom}}(G, H))$ of $\mathrm{Mult}(G^n, \underline{\mathrm{Hom}}(G, H))$. Again by definition, we have

$$\mathrm{Mult}(G^n, \underline{\mathrm{Hom}}(G, H)) \cong \mathrm{Hom}(\Lambda^n G, \underline{\mathrm{Hom}}(G, H)).$$

The latter group is trivial by hypothesis. Therefore, we have $\mathrm{Hom}(\Lambda^{n+1} G, H) = 0$ for all commutative group schemes H, which implies that $\Lambda^{n+1} G = 0$. $\square$

In order to show the existence of symmetric and alternating powers of a commutative group scheme G under good conditions, we have to make a digression on quotients of inverse limits and the notion of largest quotient:

Quotients of inverse limits

Let $G = \varprojlim G_\alpha$ be a filtered inverse limit of affine commutative group schemes and $G \twoheadrightarrow H$ a quotient morphism. Let A_α, A resp. B be the Hopf algebras representing the group schemes G_α, G resp. H. We have $B \subset A$ and $A = \bigcup_\alpha A_\alpha$ where the union is filtered; and consequently $B = \bigcup_\alpha B_\alpha$ where $B_\alpha = B \cap A_\alpha$ and this union is filtered too. Using the fact that $(A_\alpha \otimes_k A_\alpha) \cap (B \otimes_k B) = (A_\alpha \cap B) \otimes_k (A_\alpha \cap B) = B_\alpha \otimes_k B_\alpha$, we see that B_α is in fact a Hopf algebra. It follows then that $H = \varprojlim H_\alpha$ where $H_\alpha = \mathbf{Spec}\, B_\alpha$ is the group scheme associated to B_α.

Largest quotient

Let Γ be a finite group acting on an abelian group G. Then the *largest quotient* of G where Γ acts trivially is a quotient $\pi : G \twoheadrightarrow \widetilde{G}$ with the following universal property: given an abelian group H with a trivial Γ-action and a Γ-equivariant homomorphism $\varphi : G \to H$ there exists a unique homomorphism $\widetilde{\varphi} : \widetilde{G} \to H$ which makes the following diagram commute

$$\begin{tikzcd} G \arrow[rr, "\varphi"] \arrow[dr, two heads, "\pi"'] & & H \\ & \widetilde{G}. \arrow[ur, dashed, "\exists!\ \widetilde{\varphi}"'] & \end{tikzcd}$$

It is easy to see that the largest quotient is unique up to unique isomorphism and if we write it as quotient of G by a subgroup then it is unique. One can verify easily that explicitly the largest quotient is the cokernel of the homomorphism

$$(\prod_{\gamma\in\Gamma} G) \to G, \quad (g_\gamma) \mapsto \sum_{\gamma\in\Gamma} (\gamma \cdot g_\gamma - g_\gamma).$$

Now let G be a commutative group scheme with a Γ-action. We can define in the same fashion the largest quotient of G where Γ acts trivially. Using the fact that this Γ-action induces an action on every abelian group $G(X)$ for all schemes X, one sees easily that the cokernel of the morphism

$$(\prod_{\gamma\in\Gamma} G) \to G, \quad (g_\gamma) \mapsto \sum_{\gamma\in\Gamma} (\gamma \cdot g_\gamma - g_\gamma)$$

in the category of finite commutative group schemes over the field k is indeed the largest quotient of G in this category under the action of Γ.

Now we are ready to show the following theorem:

Theorem 2.6. *If $S = \mathbf{Spec}\, k$ for a field k, and G is finite over S, then $S^r G$ and $\Lambda^r G$ exist and are pro-finite over S.*

Proof. Under the stated assumptions, the tensor product $G^{\otimes r}$ of r factors of G exists and is pro-finite over S by Theorem 4.3 in [6]. By its universal property the tensor product inherits an action of the symmetric group S_r. It is now clear that the largest quotient of $G^{\otimes r}$ where S_r acts trivially is a symmetric power $S^r G$.

Now let G' be the inverse limit of all finite quotients H_α of $G^{\otimes r}$ with the property that the composite morphism $\Delta^r_{ij} \hookrightarrow G^r \to G^{\otimes r} \twoheadrightarrow H_\alpha$ is trivial for all i, j. We want to show that G' is an alternating power $\Lambda^r G$. Given a morphism $\varphi : G^{\otimes r} \to K$ with trivial composition $\Delta^r_{ij} \hookrightarrow G^r \to G^{\otimes r} \xrightarrow{\varphi} K$ for all i, j, let K' be the image of φ, then since $K' \hookrightarrow K$ is a monomorphism, the composite $\Delta^r_{ij} \hookrightarrow G^r \to G^{\otimes r} \twoheadrightarrow K'$ is zero. According to what we have shown about quotients of inverse limits and since $G^{\otimes r}$ is pro-finite, its quotient K' is pro-finite too. We can thus write $K' = \varprojlim K_\beta$ for finite K_β's. We have therefore a unique morphism $G' = \varprojlim H_\alpha \to K'$ (since $G^{\otimes r} \twoheadrightarrow K' \twoheadrightarrow K_\beta$ is a finite quotient of $G^{\otimes r}$ with trivial composition with $\Delta^r_{ij} \hookrightarrow G^r \to G^{\otimes r}$, K_β appears in the filtered system of the inverse limit $\varprojlim H_\alpha$). It follows that $G' = \Lambda^r G$. □

Proposition 2.7. *Let G_1, G_2 and F be commutative group schemes and r_1, r_2 two positive integers. We have a natural isomorphism*

$$\mathrm{Alt}(G_1^{r_1} \times G_2^{r_2}, F) \cong \mathrm{Hom}(\Lambda^{r_1} G_1 \otimes \Lambda^{r_2} G_2, F).$$

Proof. Using Proposition 1.21 we have a natural isomorphism

$$\mathrm{Alt}(G_1^{r_1} \times G_2^{r_2}, F) \cong \mathrm{Alt}(G_1^{r_1}, \underline{\mathrm{Alt}}(G_2^{r_2}, F))$$

and by definition of $\Lambda^{r_1} G_1$ this is isomorphic to $\mathrm{Hom}(\Lambda^{r_1} G_1, \underline{\mathrm{Alt}}(G_2^{r_2}, F))$ which is again by Proposition 1.21 isomorphic to $\mathrm{Alt}(\Lambda^{r_1} G_1 \times G_2^{r_2}, F) \cong$

$$\mathrm{Alt}(G_2^{r_2}, \underline{\mathrm{Hom}}(\Lambda^{r_1} G_1, F)) \cong \mathrm{Hom}(\Lambda^{r_2} G_2, \underline{\mathrm{Hom}}(\Lambda^{r_1} G_1, F))$$

$$\cong \mathrm{Mult}(\Lambda^{r_1} G_1 \times \Lambda^{r_2} G_2, F) \cong \mathrm{Hom}(\Lambda^{r_1} G_1 \otimes \Lambda^{r_2} G_2, F).$$

□

Remark 2.8.

1) Let $\lambda_{1,2} : G_1^{r_1} \times G_2^{r_2} \to \Lambda^{r_1} G_1 \otimes \Lambda^{r_2} G_2$ be the multilinear morphism in $\mathrm{Alt}(G_1^{r_1} \times G_2^{r_2}, \Lambda^{r_1} G_1 \otimes \Lambda^{r_2} G_2)$ that maps to the identity of $\Lambda^{r_1} G_1 \otimes \Lambda^{r_2} G_2$ by the isomorphism given in the Proposition 2.7. Then, one can easily see that the group scheme $\Lambda^{r_1} G_1 \otimes \Lambda^{r_2} G_2$ has the following universal property: Given any multilinear morphism $\varphi : G_1^{r_1} \times G_2^{r_2} \to F$ which is alternating in

$G_1^{r_1}$ and $G_2^{r_2}$, there exists a unique homomorphism $\widetilde{\varphi} : \Lambda^{r_1}G_1 \otimes \Lambda^{r_2}G_2 \to F$ making the following diagram commute:

$$\begin{tikzcd} G_1^{r_1} \times G_2^{r_2} \arrow[rr, "\varphi"] \arrow[dr, "\lambda_{1,2}"'] & & F \\ & \Lambda^{r_1}G_1 \otimes \Lambda^{r_2}G_2. \arrow[ur, "\exists!\widetilde{\varphi}"'] & \end{tikzcd}$$

2) It is clear that we can generalize the Proposition 2.7, i.e., if $G_1, \ldots, G_n$ are commutative group schemes and $r_1, \ldots, r_n$ are positive integers, then there is a multilinear morphism

$$\lambda_{1,\ldots,n} : G_1^{r_1} \times \cdots \times G_n^{r_n} \to \Lambda^{r_1}G_1 \otimes \cdots \otimes \Lambda^{r_n}G_n$$

alternating in each $G_i^{r_i}$ such that the homomorphism

$$\mathrm{Hom}(\Lambda^{r_1}G_1 \otimes \cdots \otimes \Lambda^{r_n}G_n, F) \to \mathrm{Alt}(G_1^{r_1} \times \cdots \times G_n^{r_n}, F)$$

$$\varphi \mapsto \varphi \circ \lambda_{1,\ldots,n}$$

is an isomorphism.

We know from the construction of the tensor product, which is given in the proof of Theorem 4.3 in [6], that $G_1 \otimes \cdots \otimes G_r \cong \varprojlim G_\alpha^*$ where G_α runs through all finite subgroups of $\underline{\mathrm{Mult}}(G_1 \times \cdots \times G_r, \mathbb{G}_m)$ and we know that if this group scheme is isomorphic to another group scheme H, then the corresponding inverse limits of it and H are isomorphic too and we deduce that the tensor product of G_1, $G_2, \ldots,$ G_r is uniquely determined up to unique isomorphism by $(\psi, \underline{\mathrm{Mult}}(G_1 \times \cdots \times G_r, \mathbb{G}_m))$, where ψ is the universal multilinear morphism associated to $G_1 \ldots, G_r$.
Now suppose that $G_1 = G_2 = \cdots = G_r$ and the universal multilinear morphism $\psi : \underline{\mathrm{Mult}}(G^r, \mathbb{G}_m) \times G^r \to \mathbb{G}_m$ is symmetric (resp. alternating) in G^r. Then any multilinear morphism $\varphi : H \times G^r \to \mathbb{G}_m$ is symmetric (resp. alternating) which follows from the commutativity of the diagram

$$\begin{tikzcd} H \times G^r \arrow[rr, "\varphi"] \arrow[dr, "\widetilde{\varphi} \times \mathrm{Id}_{G^r}"'] & & \mathbb{G}_m \\ & \widetilde{G} \times G^r \arrow[ur, "\psi_{\widetilde{G}}"'] & \end{tikzcd}$$

with the notations of Proposition 1.15.
If H is a finite commutative group scheme and $\varphi : G^r \to H$ is multilinear, then φ is symmetric (resp. alternating) if and only if the corresponding multilinear morphism $H^* \times G^r \to \mathbb{G}_m$ given by Cartier duality and Lemma 1.9 is symmetric (resp. alternating) but as we have seen, in our situation any such multilinear morphism is symmetric (resp. alternating) and thus

$$\mathrm{Hom}(G^{\otimes r}, H) \cong \mathrm{Mult}(G^r, H) \cong \mathrm{Sym}(G^r, H) \cong \mathrm{Hom}(S^r G, H).$$

If H is of finite type, then any multilinear morphism $\varphi : G^r \to H$ factors through a finite subgroup H' of H, i.e., we have a commutative diagram:

$$\begin{tikzcd} G^r \arrow[rr, "\varphi"] \arrow[dr, "\varphi'"'] & & H \\ & H' \arrow[ur, hook] & \end{tikzcd}$$

As we proved above, φ' is symmetric (resp. alternating), hence φ is symmetric (resp. alternating) as well and we have again

$$\mathrm{Hom}(G^{\otimes r}, H) \cong \mathrm{Mult}(G^r, H) \cong \mathrm{Sym}(G^r, H) \cong \mathrm{Hom}(S^r G, H).$$

Now if we are in the general case, i.e., H is any commutative group scheme, then we can write it as an inverse limit of commutative group schemes of finite type, say $H = \varprojlim H_\alpha$. We have

$$\mathrm{Hom}(G^{\otimes r}, H) \cong \mathrm{Hom}(G^{\otimes r}, \varprojlim H_\alpha) \cong \varprojlim \mathrm{Hom}(G^{\otimes r}, H_\alpha) \cong$$

$$\varprojlim \mathrm{Hom}(S^r G, H_\alpha) \cong \mathrm{Hom}(S^r G, \varprojlim H_\alpha) \cong \mathrm{Hom}(S^r G, H).$$

We have thus in any case that $\mathrm{Hom}(G^{\otimes r}, H) \cong \mathrm{Hom}(S^r G, H)$ and it implies that $G^{\otimes r} \cong S^r G$. The same arguments hold in the alternating case and we have in this case that $G^{\otimes r} \cong \Lambda^r G$.

Example 2.9. Here we give a concrete example and calculate the tensor product $\alpha_p \otimes \alpha_p$, the symmetric power $S^2\alpha_p$ and the alternating power $\Lambda^2\alpha_p$ over $S = \mathbf{Spec}\, k$ for a field k of characteristic $p > 0$. In order to do this, we try to find the universal group scheme associated to α_p, α_p (see Definition 1.16) that we denote by $\widetilde{\alpha_p}^2$ in this example and the universal multilinear morphism $\psi_{\widetilde{\alpha_p}^2} : \widetilde{\alpha_p}^2 \times \alpha_p \times \alpha_p \to \mathbb{G}_m$. The universal group $\widetilde{\alpha_p}^2$ is the group $\underline{\mathrm{Mult}}(\alpha_p^2, \mathbb{G}_m) \overset{1.10}{\cong} \underline{\mathrm{Hom}}(\alpha_p, \underline{\mathrm{Hom}}(\alpha_p, \mathbb{G}_m))$, and we have isomorphisms

$$\underline{\mathrm{Hom}}(\alpha_p, \mathbb{G}_m) \cong \alpha_p^*, \ \alpha_p^* \cong \alpha_p \ \text{ and } \ \mathbb{G}_a \cong \underline{\mathrm{Hom}}(\alpha_p, \alpha_p). \qquad (\blacktriangle)$$

It follows then that

$$\widetilde{\alpha_p}^2 \cong \underline{\mathrm{Hom}}(\alpha_p, \underline{\mathrm{Hom}}(\alpha_p, \mathbb{G}_m)) \cong \underline{\mathrm{Hom}}(\alpha_p, \alpha_p^*) \cong \underline{\mathrm{Hom}}(\alpha_p, \alpha_p) \cong \mathbb{G}_a. \qquad (\bigstar)$$

Using these isomorphisms and those stated at the beginning of this section we can write:

$$\mathrm{Mult}(\mathbb{G}_a \times \alpha_p \times \alpha_p, \mathbb{G}_m) \cong \mathrm{Mult}(\mathbb{G}_a \times \alpha_p, \underline{\mathrm{Hom}}(\alpha_p, \mathbb{G}_m))$$

$$\cong \mathrm{Hom}(\mathbb{G}_a, \underline{\mathrm{Hom}}(\alpha_p, \underline{\mathrm{Hom}}(\alpha_p, \mathbb{G}_m))) \cong \mathrm{Hom}(\mathbb{G}_a, \widetilde{\alpha_p}^2).$$

If we identify $\widetilde{\alpha_p}^2$ with $\mathbb{G}_a$ via the isomorphism $(\bigstar)$, then in order to find the universal multilinear morphism $\varphi : \mathbb{G}_a \times \alpha_p \times \alpha_p \to \mathbb{G}_m$ we have to chase through these isomorphism and find the element of $\mathrm{Mult}(\mathbb{G}_a \times \alpha_p \times \alpha_p, \mathbb{G}_m)$

corresponding to the inverse of the isomorphism $(\bigstar)$ in $\mathrm{Hom}(\mathbb{G}_a, \widetilde{\alpha_p}^2)$.

Before doing this, we explain the isomorphisms $(\blacktriangle)$. The isomorphism $\mathbb{G}_a \cong \underline{\mathrm{Hom}}(\alpha_p, \alpha_p)$ is given for any k-algebra R, by the morphism $r \mapsto \lambda_r$ where, $\lambda_r : \alpha_{p,R} \to \alpha_{p,R}$ is defined by

$$\lambda_r(S) : \alpha_p(S) \to \alpha_p(S) \quad s \mapsto r \cdot s.$$

The isomorphism $\underline{\mathrm{Hom}}(\alpha_p, \mathbb{G}_m) \cong \alpha_p^*$ is a general fact about finite commutative group schemes and we explain it in the case where $G = \mathbf{Spec}\, A$ is a finite affine commutative group scheme over k. Given $f \in G^*(R)$ for a k-algebra R, by definition, f is a k-algebra homomorphism from A^*, the dual of A, to the k-algebra R. So f defines an R-algebra homomorphism from $A^* \otimes_k R \cong (A \otimes_k R)^*$ to R which we denote again by f, which is in particular R-linear and by duality $((A \otimes_k R)^*)^* \cong A \otimes_k R$. It follows that there is an element $a \in A \otimes_k R$ such that for any $g \in (A \otimes_k R)^*$, we have $f(g) = g(a)$. The homomorphism f being an R-algebra homomorphism is equivalent to a being a *group-like element*, i.e., an element such that $\Delta(a) = a \otimes a$ and $\varepsilon(a) = 1$. This shows that the elements of $G^*(R)$ are in bijection with group-like elements of $A \otimes_k R$. But any such element defines in a unique way a homomorphism $\theta_{R,a} : G_R \to \mathbb{G}_{m,R}$ as follows: given an R-algebra S and an element $\psi \in G(S)$, i.e., a k-algebra homomorphism $A \to S$, then $\theta_{R,a}(\psi) \in \mathbb{G}_m(S)$ is the composite

$$k[x, x^{-1}] \xrightarrow{i_a} A \otimes_k R \xrightarrow{\psi \otimes 1} S \otimes_k R \xrightarrow{m} S$$

where $i_a(x) = a$ and $m(s \otimes r) = r \cdot s$. Hence we have the isomorphism $G^* \cong \underline{\mathrm{Hom}}(G, \mathbb{G}_m)$.

Now we explain the isomorphism $\alpha_p \cong \alpha_p^*$. We first give the isomorphism of Hopf algebras, then the isomorphism between the group schemes with α_P^* regarded as $\underline{\mathrm{Hom}}(\alpha_p, \mathbb{G}_m)$ as explained above. The Hopf algebras of α_p and α_p^* are respectively $k[Y]/(Y^p)$ and $(k[Y]/(Y^p))^*$. The elements $1, y, y^2, \ldots, y^{p-1}$, the images of $1, Y, Y^2, \ldots, Y^{p-1}$ in $k[Y]/(Y^p)$, form a k-basis of $k[Y]/(Y^p)$ and denote by $\xi_0, \xi_1, \xi_2, \ldots, \xi_{p-1}$ the dual basis of $(k[Y]/(Y^p))^*$. A direct calculation shows that the morphism sending ξ_i to $\frac{1}{i!} \cdot y^i$ gives a Hopf algebra isomorphism between $(k[Y]/(Y^p))^*$ and $k[Y]/(Y^p)$. This isomorphism defines for any k-algebra R an isomorphism of abelian groups $\alpha_p(R) \to \alpha_p^*(R)$ as follows: an element $r \in \alpha_p(R)$ defines a k-algebra homomorphism $k[Y]/(Y^p) \to R$ sending y to r; we have thus a k-algebra homomorphism $(k[Y]/(Y^p))^* \to R$ sending ξ_i to $\frac{1}{i!} r^i$ and it gives canonically an R-algebra homomorphism $\gamma_r : (k[Y]/(Y^p) \otimes_k R)^* \cong (k[Y]/(Y^p))^* \otimes_k R \to R$, which sends $\xi_i \otimes 1$ to $\frac{1}{i!} r^i$. Suppose that γ_r corresponds to the group-like element $\sum_{i=0}^{p-1} y^i \otimes u_i \in k[Y]/(Y^p) \otimes_k R$. So by definition, we have for any element g of $(k[Y]/(Y^p) \otimes_k R)^*$ that $\gamma_r(g) = g(\sum_{i=0}^{p-1} y^i \otimes u_i)$. In particular taking $g = \xi_j \otimes 1$ and we obtain:

$$\frac{1}{j!} r^j = \gamma_r(\xi_j \otimes 1) = (\xi_j \otimes 1)(\sum_{i=0}^{p-1} y^i \otimes u_i) = \sum_{i=0}^{p-1} \xi_j(y^i) u_i = u_j.$$

We deduce that the element $r \in \alpha_p(R)$ corresponds to the group-like element $U_r := \sum_{i=0}^{p-1} \frac{1}{i!} \cdot y^i \otimes r^i$, which itself corresponds to the morphism $\theta_{R,r} : \alpha_{p,R} \to \mathbb{G}_{m,R}$ defined for any R-algebra S by $\theta_{R,r}(S) : \alpha_p(S) \to \mathbb{G}_m(S)$ sending an element $s \in \alpha_p(S)$ to the composite

$$k[x, x^{-1}] \xrightarrow{i_{U_r}} k[Y]/(Y^p) \otimes_k R \xrightarrow{\psi_s \otimes 1} S \otimes_k R \xrightarrow{m} S$$

where $\psi_s(y) = s$. The image of x via this composite is $\sum_{i=0}^{p-1} \frac{1}{i!}(rs)^i$. Thus, regarding $\mathbb{G}_m(S)$ as a subset of S, i.e., the group of invertible elements, this k-algebra homomorphism is the element $\sum_{i=0}^{p-1} \frac{1}{i!}(rs)^i$.

Now, we can proceed to find the desired multilinear morphism $\mathbb{G}_a \times \alpha_p \times \alpha_p \to \mathbb{G}_m$. From the above arguments, it is clear that the isomorphism $\varphi : \mathbb{G}_a \cong \widetilde{\alpha_p}^2$ is given for any k-algebra R, by the morphism

$$\varphi_R : \mathbb{G}_a(R) \to \mathrm{Hom}_R(\alpha_{p,R}, \alpha^*_{p,R}), \quad r \mapsto \varphi_{R,r}$$

where $\varphi_{R,r} : \alpha_{p,R} \to \alpha^*_{p,R}$ is defined as follows: if S is an R-algebra, then $\varphi_{R,r}(S) : \alpha_p(S) \to \alpha^*_p(S)$ sends an element $s \in \alpha_p(S)$ to the group-like element $\sum_{i=0}^{p-1} \frac{1}{i!}(rs)^i$ or in other words, to the element $\varphi_{R,r,s}$ in $\mathrm{Hom}_S(\alpha_{p,S}, \mathbb{G}_{m,S})$ which sends an element $t \in \alpha_p(T)$ for an S-algebra T to the element $\sum_{i=0}^{p-1} \frac{1}{i!}(rst)^i$. Under the isomorphism

$$\mathrm{Hom}(\mathbb{G}_a, \underline{\mathrm{Hom}}(\alpha_p, \underline{\mathrm{Hom}}(\alpha_p, \mathbb{G}_m))) \cong \mathrm{Mult}(\mathbb{G}_a \times \alpha_p, \underline{\mathrm{Hom}}(\alpha_p, \mathbb{G}_m))$$

φ is mapped to the multilinear morphism $\widetilde{\varphi} : \mathbb{G}_a \times \alpha_p \to \underline{\mathrm{Hom}}(\alpha_p, \mathbb{G}_m)$ that sends the element $(r, s) \in \mathbb{G}_a(R) \times \alpha_p(R)$ to $\varphi_{R,r,s} \in \mathrm{Hom}_R(\alpha_{p,R}, \mathbb{G}_{m,R})$ for any k-algebra R. And under the isomorphism

$$\mathrm{Mult}(\mathbb{G}_a \times \alpha_p, \underline{\mathrm{Hom}}(\alpha_p, \mathbb{G}_m)) \cong \mathrm{Mult}(\mathbb{G}_a \times \alpha_p \times \alpha_p, \mathbb{G}_m)$$

$\widetilde{\varphi}$ is sent to the multilinear morphism $\widehat{\varphi} : \mathbb{G}_a \times \alpha_p \times \alpha_p \to \mathbb{G}_m$ which maps the triple $(r, s, t) \in \mathbb{G}_a(R) \times \alpha_p(R) \times \alpha_p(R)$ to the element $\sum_{i=0}^{p-1} \frac{1}{i!}(rst)^i \in \mathbb{G}_m(R)$ for any k-algebra R. The morphism $\widehat{\varphi}$ is our universal multilinear morphism. It is clearly symmetric in the second and third arguments and it follows from preceding discussion that we have $S^2\alpha_p \cong \alpha_p \otimes \alpha_p$. Therefore, any multilinear morphism $\alpha_p \times \alpha_p \to H$ to any commutative group scheme H is symmetric and if the characteristic p is 2, then it is automatically alternating and we have that $\Lambda^2\alpha_p \cong S^2\alpha_p \cong \alpha_p \otimes \alpha_p$. If the characteristic is not 2 and if this multilinear morphism is alternating, then it is trivial and it follows that the alternating group $\Lambda^2\alpha_p$ is trivial.

Example 2.10. Using Example 1.24, we see that the universal group $\underline{\mathrm{Mult}}(\alpha_p^n, \mathbb{G}_m)$ associated to the n-fold tensor product $\alpha_p \otimes \cdots \otimes \alpha_p$ with $n \geq 2$ is isomorphic to $\mathbb{G}_a$. Then similar calculations show that the universal multilinear morphism

$$\psi : \underline{\mathrm{Mult}}(\alpha_p^n, \mathbb{G}_m) \times \alpha_p^n \to \mathbb{G}_m$$

is given for any k-algebra R, by the morphism

$$(s, r_1, \dots, r_n) \mapsto \sum_{i=0}^{p-1} \frac{(s \cdot r_1 \cdots r_n)^i}{i!}.$$

This morphism is clearly symmetric in α_p^n and we have therefore

- $S^n \alpha_p \cong \alpha_p^{\otimes n}$
- $\Lambda^n \alpha_p = 0$ if $p \neq 2$ and
- $\Lambda^n \alpha_p \cong \alpha_p^{\otimes n}$ if $p = 2$.

From the construction of tensor products, we know that $\alpha_p \otimes \cdots \otimes \alpha_p \cong \varprojlim G_\gamma^*$, where G_γ runs through all finite subgroups of $\underline{\mathrm{Mult}}(\alpha_p^n, \mathbb{G}_m)$. But the latter group is by Example 1.24 isomorphic to the additive group $\mathbb{G}_a$. Thus, tensor products $\alpha_p^{\otimes n}$ for any $n \geq 2$ are isomorphic which implies that the symmetric powers $S^n \alpha_p$ and the alternating powers $\Lambda^n \alpha_p$ are also independent from n for $n \geq 2$.
This result together with Proposition 1.23 imply that for any commutative group scheme F we have

- $\underline{\mathrm{Sym}}(\alpha_p^n, F) = \underline{\mathrm{Mult}}(\alpha_p^n, F)$ and
- $\underline{\mathrm{Alt}}(\alpha_p^n, F) = 0$ if $p > 2$ and
- $\underline{\mathrm{Alt}}(\alpha_p^n, F) = \underline{\mathrm{Mult}}(\alpha_p^n, F)$ if $p = 2$

For the rest of this section, let H be an arbitrary commutative group scheme.

Notation. Let G', G'' be subgroup schemes of G and F a commutative group scheme. By $\mathrm{Alt}(G'^r \times G''^s \times F^t, H)$ we mean the group of multilinear morphisms that are alternating in G'^r, G''^s and $(G' \cap G'')^{r+s}$ and when we say that a multilinear morphism $G'^r \times G''^s \times F^t \to H$ is alternating, we mean that it belongs to the group $\mathrm{Alt}(G'^r \times G''^s \times F^t, H)$. Likewise, we define the group $\mathrm{Alt}(G_1^{r_1} \times \cdots \times G_n^{r_n} \times F_1^{s_1} \times \cdots \times F_m^{s_m}, H)$ with G_i subgroup schemes of G and F_j's arbitrary commutative group schemes.

Lemma 2.11. *Let $\pi : G \twoheadrightarrow G''$ be an epimorphism and let $\varphi : G''^r \to H$ be a multilinear morphism such that the composition $\varphi \circ \pi^r : G^r \to H$ is alternating. Then φ is alternating as well.*

Proof. The morphism π induces a morphism $\Delta\pi : \Delta G \to \Delta G''$ between diagonals and since the morphism π is epimorphic, the morphism $\Delta\pi$ is epimorphic too.
Similarly, we have an induced epimorphism between $\Delta_{ij}^r G \subset G^r$ and $\Delta_{ij}^r G'' \subset G''^r$ for all $1 \leq i < j \leq r$, which we denote by $\Delta_{ij}^r \pi$. In order to show that φ is alternating, we must show that for any $1 \leq i < j \leq r$ the composition

$\Delta^r_{ij}G'' \hookrightarrow G''^r \xrightarrow{\varphi} H$ is trivial. But we have a commutative diagram

$$\begin{tikzcd}
\Delta^r_{ij}G \arrow[r, "\Delta\pi"] \arrow[d, hook, "\iota"'] & \Delta^r_{ij}G'' \arrow[d, hook, "\iota''"] & \\
G^r \arrow[r, "\pi^r"'] & G''^r \arrow[r, "\varphi"'] & H.
\end{tikzcd}$$

Since the composite $\varphi \circ \pi^r$ is alternating, the composition $\varphi \circ \pi^r \circ \iota$ is trivial, and so is the composition $\varphi \circ \iota'' \circ \Delta\pi$. The morphism $\Delta\pi$ is epimorphic and it follows that $\varphi \circ \iota''$ is trivial. $\square$

Remark 2.12. Let G' be a subgroup scheme of G and $\pi : G \twoheadrightarrow G''$ an epimorphism. It can be shown in the same fashion that if the composition of a multilinear morphism $G''^r \times G^s \times G'^t \to H$ with the epimorphism $\pi^r \times \mathrm{Id}_{G'^s} \times \mathrm{Id}_{G'^t} : G^r \times G^s \times G'^t \to G''^r \times G^s \times G'^t$ is alternating, then this multilinear morphism is also alternating.

Lemma 2.13. *Let $G_1 \ldots, G_r$ be commutative group schemes and $\psi : G_1 \times \cdots \times G_r \to H$ a multilinear morphism. Assume that for some $1 \leq i \leq r$ we have a short exact sequence $0 \to G'_i \xrightarrow{\iota} G_i \xrightarrow{\pi} G''_i \to 0$. If the restriction $\psi|_{G_1 \times \cdots \times G'_i \times \cdots \times G_r}$ is zero, then there is a unique multilinear morphism $\psi' : G_1 \times \cdots \times G''_i \times \cdots \times G_r \to H$ such that $\psi = \psi' \circ (\mathrm{Id}_{G_1} \times \cdots \times \pi \times \cdots \times \mathrm{Id}_{G_r})$ with π at the ith place.*

Proof. By functoriality of the isomorphism in Proposition 1.12 we have a commutative diagram:

$$\begin{tikzcd}
\mathrm{Mult}(G_1 \times \cdots \times G_r, H) \arrow[r, "\cong"] \arrow[d, "\overline{\pi}"] & \mathrm{Hom}(G''_i, \underline{\mathrm{Mult}}(G_1 \times \cdots \times \check{G}_i \times \cdots \times G_r, H)) \arrow[d, "\pi^*"] \\
\mathrm{Mult}(G_1 \times \cdots \times G_r, H) \arrow[r, "\cong"] \arrow[d, "\overline{\iota}"] & \mathrm{Hom}(G_i, \underline{\mathrm{Mult}}(G_1 \times \cdots \times \check{G}_i \times \cdots \times G_r, H)) \arrow[d, "\iota^*"] \\
\mathrm{Mult}(G_1 \times \cdots \times G_r, H) \arrow[r, "\cong"] & \mathrm{Hom}(G'_i, \underline{\mathrm{Mult}}(G_1 \times \cdots \times \check{G}_i \times \cdots \times G_r, H))
\end{tikzcd}$$

where the indicated maps are the obvious ones and $\check{G}_i$ means that this factor is omitted. The right column is exact and π^* is injective, because the sequence $0 \to G'_i \xrightarrow{\iota} G_i \xrightarrow{\pi} G''_i \to 0$ is exact and the functor $\mathrm{Hom}(-, F)$ is left exact for any commutative group scheme F. Therefore, the left column is exact too and $\overline{\pi}$ is injective. The morphism ψ is an element of $\mathrm{Mult}(G_1 \times \cdots \times G_r, H)$ which goes to zero under the map $\overline{\iota}$ (restriction map). By exactness, there is a unique multilinear morphism $\psi' \in \mathrm{Mult}(G_1 \times \cdots \times G''_i \times \cdots \times G_r, H)$ which is mapped to ψ under $\overline{\pi}$. This proves the lemma. $\square$

Lemma 2.14. *Let G be a commutative group scheme, and let $0 \to G' \xrightarrow{\iota} G \xrightarrow{\pi} G'' \to 0$ be a short exact sequence. Then the restriction map*

$$\mathrm{Alt}(G^m, H) \to \mathrm{Mult}(G' \times G^{m-1}, H)$$

is injective, whenever $\Lambda^m G'' = 0$.

Proof. Let $\varphi : G^m \to H$ be an alternating morphism and assume that the restriction $\varphi|_{G'\times G^{m-1}}$ is zero. We will show that there is a multilinear morphism $\varphi' : G''^m \to H$ such that $\varphi = \varphi' \circ \pi^m$. The result will then follow, since by Lemma 2.13 φ' is also alternating, it is so inside the group $\mathrm{Alt}(G''^m, H) \cong \mathrm{Hom}(\Lambda^m G, H)$, which is trivial by the hypothesis. It follows that φ' and consequently φ are zero.
Note that since φ is alternating and the restriction $\varphi|_{G'\times G^{m-1}}$ is zero, the restrictions $\varphi|_{G^i\times G'\times G^{m-i-1}}$ are zero for any $1 \le i \le m-1$.
Put $\varphi_0 = \varphi$. We show by induction on $0 \le i \le m$ that there is a multilinear morphism $\varphi_i : G''^i \times G^{m-i} \to H$ such that $\varphi = \varphi_i \circ (\pi^i \times \mathrm{Id}_{G^{m-i}})$. This is clear for $i = 0$, so let $i > 0$ and assume that we have φ_{i-1} with the stated property. Consider the following commutative diagram

$$\begin{tikzcd}
G^{i-1} \times G' \times G^{m-i} \arrow[r, hook, "\rho"] \arrow[d, "\widehat{\pi}"'] & G^{i-1} \times G \times G^{m-i} \arrow[d, "\widetilde{\pi}"] \arrow[dr, "\varphi"] & \\
G''^{i-1} \times G' \times G^{m-i} \arrow[r, hook, "\rho'"] & G''^{i-1} \times G \times G^{m-i} \arrow[r, "\varphi_{i-1}"] & H
\end{tikzcd}$$

where $\widehat{\pi} = \pi^{i-1} \times \mathrm{Id}_{G'} \times \mathrm{Id}_{G^{m-i}}$, $\widetilde{\pi} = \pi^{i-1} \times \mathrm{Id}_G \times \mathrm{Id}_{G^{m-i}}$ and ρ, ρ' are the inclusion morphisms. We have by hypothesis, $0 = \varphi \circ \rho = \varphi_{i-1} \circ \widetilde{\pi} \circ \rho$ which implies that $\varphi_{i-1} \circ \rho' \circ \widehat{\pi} = 0$. The morphism $\widehat{\pi}$ is epimorphic and so $\varphi_{i-1} \circ \rho'$, the restriction of φ_{i-1}, is zero. We can therefore apply Lemma 2.14, so there is a multilinear morphism $\varphi_i : G''^i \times G^{m-i} \to H$ such that $\varphi_i = \varphi_{i-1} \circ (\mathrm{Id}_{G''^{i-1}} \times \pi \times \mathrm{Id}_{G^{m-i}})$. We have thus $\varphi = \varphi_i \circ (\pi^i \times \mathrm{Id}_{G^{m-i}})$.
Now put $i = m$, the statement says that there is a multilinear morphism $\varphi_m : G''^m \to H$ with $\varphi = \varphi_m \circ \pi^m$. This φ_m is the required φ'. □

Remark 2.15.

1) In Lemma 2.14, obviously the other restriction maps, i.e., restrictions to $G^r \times G' \times G^{n-r-1}$ for $1 \le r \le n-1$ are injective too.

2) It is clear that the image of the restriction map in Lemma 2.14 lies inside the group $\mathrm{Alt}(G' \times G^{m-1}, H)$. We have thus the injection

$$\mathrm{Alt}(G^m, H) \hookrightarrow \mathrm{Alt}(G' \times G^{m-1}, H).$$

Theorem 2.16. *Assume that $0 \to G' \xrightarrow{\iota} G \xrightarrow{\pi} G'' \to 0$ is a short exact sequence of commutative group schemes. Let m be a nonnegative integer and write $m = m' + m''$ with non negative integers m' and m''. Consider the diagram*

$$\begin{tikzcd}
\mathrm{Alt}(G^m, H) \arrow[r, "\rho"] & \mathrm{Alt}(G'^{m'} \times G^{m''}, H) \\
& \mathrm{Alt}(G'^{m'} \times G''^{m''}, H) \arrow[u, hook, "\pi^*"']
\end{tikzcd}$$

where ρ is the restriction map.

(a) *If $\Lambda^{m''+1} G'' = 0$, then ρ is injective.*

(b) *If* $\Lambda^{m'+1}G' = 0$*, then* ρ *factors through* π^**.*

(c) *If both conditions hold, then there is a natural epimorphism*

$$\zeta : \Lambda^{m'}G' \otimes \Lambda^{m''}G'' \twoheadrightarrow \Lambda^m G.$$

(d) *If furthermore the sequence is split, then the epimorphism* ζ *is an isomorphism.*

Proof. If $m = 0$ then $m' = 0 = m''$ and all statements are trivially true, so assume $m > 0$. We prove each point of the proposition separately.

(a) Fix m. We show by induction on $0 \leq m' \leq m$ that the restriction map gives an injective map

$$\mathrm{Alt}(G^m, H) \hookrightarrow \mathrm{Alt}(G^{m''}, \underline{\mathrm{Mult}}(G'^{m'}, H)).$$

If $m' = 0$ then $m'' = m$, and ρ is the identity map, so there is nothing to show. So assume that $0 < m' \leq m$ and that the statement is true for $m' - 1$ and $m'' + 1$ in place of m' and m''. Then $\Lambda^{m''+1}G'' = 0$ implies $\Lambda^{m''+2}G'' = 0$ by Proposition 2.5; so by the induction hypothesis we have an injection

$$\mathrm{Alt}(G^m, H) \hookrightarrow \mathrm{Alt}(G^{m''+1}, \underline{\mathrm{Mult}}(G'^{m'-1}, H)).$$

Since by hypothesis we have $\Lambda^{m''+1}G'' = 0$ we can use Lemma 2.14, and we have thus an injection

$$\mathrm{Alt}(G^{m''+1}, \underline{\mathrm{Mult}}(G'^{m'-1}, H)) \hookrightarrow \mathrm{Alt}(G^{m''} \times G', \underline{\mathrm{Mult}}(G'^{m'-1}, H)).$$

The latter group is inside the group

$$\mathrm{Alt}(G^{m''}, \underline{\mathrm{Hom}}(G', \underline{\mathrm{Mult}}(G'^{m'-1}, H))).$$

By Proposition 1.14, $\underline{\mathrm{Hom}}(G', \underline{\mathrm{Mult}}(G'^{m'-1}, H)) \cong \underline{\mathrm{Mult}}(G'^{m'}, H)$. Putting these together, we conclude that there is an injection

$$\mathrm{Alt}(G^m, H) \hookrightarrow \mathrm{Alt}(G^{m''}, \underline{\mathrm{Mult}}(G'^{m'}, H)).$$

Following through the above isomorphisms and inclusions, one verifies that this injection is induced by the restriction map.

Under the isomorphism

$$\mathrm{Mult}(G^{m''}, \underline{\mathrm{Mult}}(G'^{m'}, H)) \cong \mathrm{Mult}(G^{m''} \times G'^{m'}, H)$$

given by Proposition 1.12, the image of $\mathrm{Alt}(G^m, H)$ by ι lies inside the group $\mathrm{Alt}(G^{m''} \times G'^{m'}, H)$ and we can easily see that the injection

$$\mathrm{Alt}(G^m, H) \hookrightarrow \mathrm{Alt}(G^{m''} \times G'^{m'}, H)$$

thus obtained is given by the restriction map.

(b) Choose an alternating multilinear morphism $\varphi : G^m \to H$ and write φ_0 for the restriction $\varphi|_{G'^{m'} \times G^{m''}}$. For any $0 \leq j \leq m'' - 1$ the restriction of φ_0 to the subgroup scheme $G'^{m'} \times G^j \times G' \times G^{m''-j-1}$ belongs to the group

$$\mathrm{Alt}(G'^{m'} \times G^j \times G' \times G^{m''-j-1}, H) \hookrightarrow \mathrm{Alt}(G'^{m'+1}, \underline{\mathrm{Mult}}(G^{m''-1}, H)).$$

The latter group is isomorphic to $\mathrm{Hom}(\Lambda^{m'+1}G', \underline{\mathrm{Mult}}(G^{m''-1}, H))$, which is zero by assumption. Therefore, the restriction $\varphi_0|_{G'^{m'} \times G^j \times G' \times G^{m''-j-1}}$ is zero.

Now we show by induction on $0 \leq i \leq m''$, that there exists a multilinear morphism $\varphi_i : G''^i \times G^{m''-i} \times G'^{m'} \to H$ such that the composition

$$G^i \times G^{m''-i} \times G'^{m'} \xrightarrow{\overline{\pi}} G''^i \times G^{m''-i} \times G'^{m'} \xrightarrow{\varphi_i} H$$

is φ_0, where $\overline{\pi} = \pi^i \times \mathrm{Id}_{G^{m''-i}} \times \mathrm{Id}_{G'^{m'}}$. If $i = 0$ then we have nothing to show, so let $i < m''$ and assume that we have constructed φ_i with the desired property and we construct φ_{i+1}. Consider the following commutative diagram:

$$\begin{tikzcd}
G^i \times G' \times G^{m''-i-1} \times G'^{m'} \arrow[r, hook, "\rho"] \arrow[d, two heads, "\widehat{\pi}"'] & G^i \times G^{m''-i} \times G'^{m'} \arrow[d, two heads, "\overline{\pi}"] \arrow[dr, "\varphi_0"] & \\
G''^i \times G' \times G^{m''-i-1} \times G'^{m'} \arrow[r, hook, "\rho'"] & G''^i \times G^{m''-i} \times G'^{m'} \arrow[r, "\varphi_i"] & H.
\end{tikzcd}$$

As we have said above, the restriction of φ_0, $\varphi_0 \circ \rho$, is zero. By the induction hypothesis, we have $\varphi_0 = \varphi_i \circ \overline{\pi}$ and therefore, $0 = \varphi_0 \circ \rho = \varphi_i \circ \overline{\pi} \circ \rho = \varphi_i \circ \rho' \circ \widehat{\pi}$. The morphism $\widehat{\pi}$ being epimorphic, we conclude that the restriction of φ_i, i.e., $\varphi_i \circ \rho'$ is zero. This allows us to use Lemma 2.13 in order to find a multilinear morphism $\varphi_{i+1} : G''^{i+1} \times G^{m''-i-1} \times G'^{m'} \to H$ such that $\varphi_i = \varphi_{i+1} \circ (\mathrm{Id}_{G''^i} \times \pi \times \mathrm{Id}_{G^{m''-i-1}} \times \mathrm{Id}_{G'^{m'}})$. It follows at once that $\varphi_0 = \varphi_{i+1} \circ (\pi^{i+1} \times \mathrm{Id}_{G^{m-i-2}} \times \mathrm{Id}_{G'})$.

Put $i = m''$, then the statement says that there is a multilinear morphism $\varphi_{m''} : G''^{m''} \times G'^{m'} \to H$ such that $\varphi_0 = \varphi_{m''} \circ (\pi^{m''} \times \mathrm{Id}_{G'^{m'}})$. Since φ_0 is alternating, by Remark 2.12, $\varphi_{m''}$ is also alternating.

(c) If both conditions hold, then by (a), ρ is injective and therefore the homomorphism $\mathrm{Alt}(G^m, H) \to \mathrm{Alt}(G'^{m'} \times G''^{m''}, H)$ defined in (b) is injective as well. So we obtain

$$\mathrm{Hom}(\Lambda^m G, H) \cong \mathrm{Alt}(G^m, H) \hookrightarrow \mathrm{Alt}(G'^{m'} \times G''^{m''}, H) \overset{2.7}{\cong}$$
$$\mathrm{Hom}(\Lambda^{m'} G' \otimes \Lambda^{m''} G'', H)$$

which is natural, in other words we have a natural injection of functors

$$\tau : \mathrm{Hom}(\Lambda^m G, -) \hookrightarrow \mathrm{Hom}(\Lambda^{m'} G' \otimes \Lambda^{m''} G'', -).$$

It is a known fact that any natural transformation between such functors is induced by a unique morphism $\zeta : \Lambda^{m'} G' \otimes \Lambda^{m''} G'' \to \Lambda^m G$, in fact,

this morphism is the image of the identity morphism of $\Lambda^m G$ under this transformation. This means that for any commutative group scheme H, $\tau_H : \mathrm{Hom}(\Lambda^m G, H) \to \mathrm{Hom}(\Lambda^{m'} G' \otimes \Lambda^{m''} G'', H)$ sends a morphism $f : \Lambda^m G \to H$ to the morphism $f \circ \zeta$. The injectivity of τ implies that ζ is epimorphic.

(d) Let $s : G'' \to G$ be a section of π, i.e., $\pi \circ s = \mathrm{Id}_{G''}$ and $r : G \to G'$ the corresponding retraction of ι, that is, $r \circ \iota = \mathrm{Id}$ and that the short sequence

$$0 \to G'' \xrightarrow{s} G \xrightarrow{r} G' \to 0$$

is exact. Then we show that the map $\mu : \mathrm{Alt}(G^m, H) \to \mathrm{Alt}(G'^{m'} \times G''^{m''}, H)$ whose composition with π^* is ρ (given by (b)) is induced by the inclusion $j := \iota^{m'} \times s^{m''} : G'^{m'} \times G''^{m''} \hookrightarrow G^m$. Indeed, given a morphism $f \in \mathrm{Alt}(G^m, H)$, we have $\rho(f) = \pi^*(\mu(f))$, or in other words, $\mu(f) \circ (\mathrm{Id}_{G'}^{m'} \times \pi^{m''}) = f \circ (\iota^{m'} \times \mathrm{Id}_G^{m''})$. Hence the following diagram is commutative

$$\begin{tikzcd}
G'^{m'} \times G''^{m''} \arrow[d, hook, "\widetilde{s}"'] \arrow[dr, hook, "j"] \arrow[drr, "\mu(f)"] \arrow[dd, bend right=60, "\mathrm{Id}"'] & & \\
G'^{m'} \times G^{m''} \arrow[r, hook, "i"] \arrow[d, two heads, "\widetilde{\pi}"'] & G^m \arrow[r, "f"] & H \\
G'^{m'} \times G''^{m''} \arrow[urr, "\mu(f)"'] & &
\end{tikzcd}$$

where $\widetilde{s}, \widetilde{\pi}$ and i are respectively the morphisms $\mathrm{Id}_{G'}^{m'} \times s^{m''}, \mathrm{Id}_{G'}^{m'} \times \pi^{m''}$ and the inclusion $\iota^{m'} \times \mathrm{Id}_G^{m''}$. Consequently, $\mu(f) = f \circ j$. This shows that μ is induced by j as we claimed.

Now define a morphism $\omega : G^m \to G'^{m'} \times G''^{m''}$ as follows: for any k-algebra R ω sends an element $(g_1, \ldots, g_m) \in G(R)^m$ to

$$\sum_\sigma \mathrm{sgn}(\sigma, \tau)(r(g_{\sigma(1)}), \ldots, r(g_{\sigma(m')}), \pi(g_{\tau(1)}), \ldots, \pi(g_{\tau(m'')}))$$

where the sum runs over all length m' subsequences $\sigma = (\sigma(1), \ldots, \sigma(m'))$ of $(1, 2, \ldots, m)$ with complementary subsequences $\tau = (\tau(1), \ldots, \tau(m''))$ and $\mathrm{sgn}(\sigma, \tau)$ is the signature of (σ, τ) as a permutation of m elements. This morphism induces a homomorphism

$$\omega^* : \mathrm{Alt}(G'^{m'} \times G''^{m''}, H) \to \mathrm{Mult}(G^m, H)$$

and it is straightforward to see that in fact the image lies inside the subgroup $\mathrm{Alt}(G^m, H)$. We also denote by ω^* the homomorphism $\mathrm{Alt}(G'^{m'} \times G''^{m''}, H) \to \mathrm{Alt}(G^m, H)$ obtained by restricting the codomain of ω^*. Since the composites $r \circ s$ and $\pi \circ \iota$ are trivial and $r \circ \iota$ and $\pi \circ s$ are the identity morphisms, we see that the composition $\omega \circ j$ is the identity morphism of $G'^{m'} \times G''^{m''}$. Therefore the composite $\mu \circ \omega^*$ is the identity homomorphism. Consequently, the homomorphism $\mu : \mathrm{Alt}(G^m, H) \to$

$\mathrm{Alt}(G'^{m'} \times G''^{m''}, H)$ is an epimorphism. We know from (c) that it is a monomorphism, and hence it is an isomorphism. We obtain thus

$$\mathrm{Hom}(\Lambda^m G, H) \cong \mathrm{Alt}(G^m, H) \xrightarrow{\cong}$$
$$\mathrm{Alt}(G'^{m'} \times G''^{m''}, H) \cong \mathrm{Hom}(\Lambda^{m'} G' \otimes \Lambda^{m''} G'', H).$$

As we know, this homomorphism is induced by the morphism

$$\zeta : \Lambda^{m'} G' \otimes \Lambda^{m''} G'' \to \Lambda^m G.$$

Since it is an isomorphism, the morphism ζ must be an isomorphism as well.

□

Proposition 2.17. *Let G be a local-local commutative group scheme of order p^n with p an odd prime number. We have:*

(a) $\Lambda^m G = 0$ *for all* $m > n$.

(b) $\Lambda^n G$ *is a quotient of* $\alpha_p^{\otimes n}$.

Proof. We know that any subgroup of a local-local commutative group scheme is again local-local. We can thus prove the proposition by induction on n. If $n = 1$, then G is necessarily isomorphic to α_p, hence the equality $\Lambda^m \alpha_p = 0$ follows from Example 2.10 and we have obviously $\Lambda^1 \alpha_p = \alpha_p$, which is a quotient of itself. So assume that $n > 1$ and that the two statements are true for positive integers less that n. Take a proper subgroup scheme G' of G and let G'' be the quotient of G by G', that is, we have a short exact sequence $0 \to G' \to G \to G'' \to 0$. We know that the order of commutative group schemes is multiplicative, i.e., $|G| = |G'| \cdot |G''|$. So if $|G'| = p^{n'}$ and $|G''| = p^{n''}$, we have $n = n' + n''$. Take $m \geq n$, we can write $m = m' + m''$ where $m'' = n''$ and $m' = m - n''$ and we have $m' \geq n'$. Since G' is a proper subgroup scheme of G, we have $n' < n$ and so $n'' < n$. Therefore, by the induction hypothesis we have $\Lambda^{m'+1} G' = 0 = \Lambda^{m''+1} G''$. We can thus apply the third point of Theorem 2.16, and we have an epimorphism

$$\zeta : \Lambda^{m'} G' \otimes \Lambda^{m''} G'' \twoheadrightarrow \Lambda^m G.$$

If $m > n$, then $m' > n'$ and we have $\Lambda^{m'} G' = 0$ by the induction hypothesis and so the tensor product $\Lambda^{m'} G' \otimes \Lambda^{m''} G''$ vanishes. Since ζ is epimorphic, we conclude that $\Lambda^m G = 0$.
If $m = n$, then $m' = n'$. By the induction hypothesis, we have epimorphisms $\xi' : \alpha_p^{\otimes n'} \twoheadrightarrow \Lambda^{n'} G'$ and $\xi'' : \alpha_p^{\otimes n''} \twoheadrightarrow \Lambda^{n''} G''$. As we said in Remark 2.2, the tensor product is right exact, and we have thus an epimorphism

$$\alpha_p^{\otimes n} \cong \alpha_p^{\otimes n'} \otimes \alpha_p^{\otimes n''} \xrightarrow{\xi' \otimes \xi''} \Lambda^{n'} G' \otimes \Lambda^{n''} G''.$$

Composing this epimorphism with ζ we obtain the desired epimorphism

$$\alpha_p^{\otimes n} \twoheadrightarrow \Lambda^n G.$$

□

Corollary 2.18. *Let G and H be local-local commutative group schemes of order p^n and p^m respectively, with p an odd prime number. Then we have a natural isomorphism*

$$\Lambda^{n+m}(G \oplus H) \cong \Lambda^n G \otimes \Lambda^m H.$$

Proof. By Proposition 2.17, we know that $\Lambda^{n+1}G = 0 = \Lambda^{m+1}H$. The result follows at once from the last point of Theorem 2.16. □

Lemma 2.19. *Let G be an affine commutative group scheme and H, F two finite subgroup schemes. Then there is a finite subgroup scheme of G that contains both H and F.*

Proof. Consider the homomorphism

$$\mu : F \oplus H \to G, \quad (f, h) \mapsto \mu(f, h) := f + h.$$

The image of this homomorphism contains both F and H and its order is less than or equal to the order of $F \oplus H$ which is finite. It is thus a finite subgroup scheme of G. □

Lemma 2.20. *Let I be a filtered system and for any $i \in I$, $0 \to N_i \xrightarrow{f_i} G_i \xrightarrow{g_i} Q_i \to 0$ be a short exact sequence of affine commutative group schemes. Then the short sequence*

$$0 \longrightarrow \varprojlim_{i \in I} N_i \xrightarrow{\varprojlim f_i} \varprojlim_{i \in I} G_i \xrightarrow{\varprojlim g_i} \varprojlim_{i \in I} Q_i \longrightarrow 0$$

is exact, in other words, taking filtered inverse limits is an exact functor.

Proof. To simplify the notation, we denote by N, G and Q the group schemes $\varprojlim N_i, \varprojlim G_i$ and $\varprojlim Q_i$. Let B_i, A_i and C_i denote respectively the Hopf algebras associated to the group schemes N_i, G_i and Q_i. Let also B, A and C denote respectively the Hopf algebras of the group schemes N, G and Q, i.e., $B = \bigcup_i B_i, A = \bigcup_i A_i$ and $C = \bigcup_i C_i$. Finally, let $f_i : N_i \to G_i$ and $g_i : G_i \to Q_i$ be respectively the morphism associated to the morphisms $f_i' : A_i \to B_i$ and $g_i' : C_i \to A_i$.
The morphism $f := \varprojlim f_i : N \to G$ is associated to the morphism $\bigcup_i f_i' : \bigcup_i A_i \to \bigcup_i B_i$. Since f_i' is surjective for all $i \in I$, their union $\bigcup_i f_i'$ is surjective too and consequently $\varprojlim f_i$ is a monomorphism. Similarly, since each $g_i' : C_i \to A_i$ is injective, the union $\bigcup_i g'i : \bigcup_i C_i \to \bigcup_i A_i$ is injective and so the morphism $g := \varprojlim g_i$ is an epimorphism. It remains to show that Q is the quotient of $f : N \to G$.
Let Q', with associated Hopf algebra C', be the quotient of f, i.e., we have a short exact sequence

$$0 \to N \xrightarrow{f} G \longrightarrow Q' \to 0.$$

We know that C' equals

$$\{x \in A \mid \Delta x \equiv x \otimes 1 \bmod A \otimes J\},$$

the subspace of the regular representation where N acts trivially.
We have for all $i \in I$ a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, two heads] & G \arrow[r] \arrow[d, two heads] & Q' \arrow[r] & 0 \\
0 \arrow[r] & N_i \arrow[r] & G_i \arrow[r] & Q_i \arrow[r] & 0.
\end{tikzcd}$$

We want to show that $Q' = \varprojlim Q_i$. Since the composite $N \to G \to G_i \to Q_i$ is trivial, there exists a unique morphism $Q \to Q_i$ which makes the right square in the diagram commute and since the composite $G \to G_i \to Q_i$ is an epimorphism the induced morphism $Q' \to Q_i$ is an epimorphism too. We can thus complete the diagram as follows

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, two heads] & G \arrow[r] \arrow[d, two heads] & Q' \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & N_i \arrow[r] & G_i \arrow[r] & Q_i \arrow[r] & 0.
\end{tikzcd}$$

Writing $B_i = A_i/J_i$ and expressing the above commutative diagram in terms of Hopf algebras we obtain for all $i \in I$ the following commutative diagram

$$\begin{tikzcd}
C_i \arrow[r, hook] \arrow[d, hook] & A_i \arrow[r, two heads] \arrow[d, hook] & A_i/J_i \arrow[d, hook] \\
C' \arrow[r, hook] & A \arrow[r, two heads] & A/J.
\end{tikzcd}$$

It follows then that $J_i \subset J$ and $J_i = J \cap A_i$. We can also deduce from this that $J_i = A_i \cap J_j$ whenever $A_i \subset A_j$. The inclusions $C_i \subset C'$ give an inclusion $\bigcup_i C_i \subset C'$ and we should prove that this inclusion is in fact an equality. Note that the union $\bigcup_i C_i$ is filtered, for if given two indices i and j there is an index l such that $A_i, A_j \subset A_l$ and we have

$$C_l = \{x \in A_l \mid \Delta x \equiv x \otimes 1 \bmod A_l \otimes J_l\}$$

contains both C_i and C_j. Now let $x \in C'$, so $\Delta x - x \otimes 1 \in A \otimes J$. Since $\bigcup_i A_i = A$, x is in some A_i and so $\Delta x - x \otimes 1 \in A_i \otimes A_i$ which implies that $\Delta x - x \otimes 1 \in A_i \otimes A_i \cap A \otimes J = A_i \otimes (A_i \cap J) = A_i \otimes J_i$ and therefore x is in C_i. It follows at once that $Q' = \varprojlim Q_i$.

□

Lemma 2.21. *Let $\mathbb{G}_a^*$ denote the group scheme $\varprojlim_{G_i \subset \mathbb{G}_a} G_i^*$ where the limit is over all finite subgroup schemes of $\mathbb{G}_a$. Then we have a short exact sequence*

$$\mathbb{G}_a^{*(p)} \xrightarrow{V} \mathbb{G}_a^* \to \alpha_p \to 0$$

where V is the Verschiebung.

Proof. A straightforward calculation shows that

$$\alpha_p^{\otimes n^{(p)}} \cong (\varprojlim_{G_i \subset \mathbb{G}_a} G_i^*)^{(p)} \cong \varprojlim_{G_i \subset \mathbb{G}_a} (G_i^{*^{(p)}})$$

and that the Verschiebung $V : (\varprojlim_{G_i \subset \mathbb{G}_a} G_i^*)^{(p)} \to \varprojlim_{G_i \subset \mathbb{G}_a} G_i^*$ is the inverse limit of the Verschiebungen $V_i : G_i^{*^{(p)}} \to G_i^*$. According to Lemma 2.20, the cokernel of the inverse limit of the V_i is the inverse limit of the cokernel of the V_i, i.e., $\operatorname{Coker} \varprojlim_i V_i = \varprojlim_i \operatorname{Coker} V_i$.
Now we have $G_i^{*(p)} \cong G_i^{(p)^*}$ and $V_i : G_i^{*(p)} \to G_i^*$ is the dual of the Frobenius $F_i : G_i \to G_i^{(p)}$. By duality, we have $\operatorname{Coker} F_i^* \cong (\operatorname{Ker} F_i)^*$. Putting these facts together, we obtain $\operatorname{Coker} V \cong \varprojlim (\operatorname{Ker} F_i)^*$. From Lemma 2.19 we deduce that the finite subgroup schemes of $\mathbb{G}_a$ that contain α_p form a cofinal system and we can thus suppose that every G_i contains α_p. It follows that the kernel of the Frobenius $F_i : G_i \to G_i^{(p)}$ is equal to α_p. Hence

$$\operatorname{Coker} V \cong \varprojlim (\operatorname{Ker} F_i)^* \cong \varprojlim \alpha_p^* \cong \alpha_p^* \cong \alpha_p.$$

□

Lemma 2.22. *Let $G_1, \dots, G_n$ and H be finite commutative group schemes and $\varphi : G_1 \times \cdots \times G_n \to H$ a multilinear morphism. Then we have a commutative diagram*

$$\begin{tikzcd}
G_1^{(p)} \times G_2^{(p)} \times \cdots \times G_n^{(p)} \arrow[r, "\varphi^{(p)}"] & H^{(p)} \arrow[dd, "V_H"] \\
G_1^{(p)} \times G_2 \times \cdots \times G_n \arrow[u, "\widetilde{F}"] \arrow[d, "\widetilde{V}"'] & \\
G_1 \times G_2 \times \cdots \times G_n \arrow[r, "\varphi"'] & H
\end{tikzcd}$$

where $\widetilde{F} = \mathrm{Id}_{G_1^{(p)}} \times F_{G_2} \times \cdots \times F_{G_n}$ and $F_{G_i} : G_i \to G_i^{(p)}$ is the Frobenius of G_i and $\widetilde{V} = V_{G_1} \times \mathrm{Id}_{G_2} \times \cdots \times \mathrm{Id}_{G_n}$, and V_{G_1} and V_H are the Verschiebungen of G_1 and H.

Proof. Consider the following diagram

$$\begin{tikzcd}
\mathrm{Mult}(G_1 \times G_2 \times \cdots \times G_n, H) \arrow[r, "\theta_1", "\cong"'] \arrow[d, "(-)\circ\widetilde{V}"'] & \mathrm{Mult}(G_2 \times \cdots \times G_n \times H^*, G_1^*) \arrow[d, "F_{G_1^*}\circ(-)"] \\
\mathrm{Mult}(G_1^{(p)} \times G_2 \times \cdots \times G_n, H) \arrow[r, "\theta_2", "\cong"'] & \mathrm{Mult}(G_2 \times \cdots \times G_n \times H^*, G_1^{(p)^*}) \\
\mathrm{Mult}(G_1^{(p)} \times G_2 \times \cdots \times G_n, H^{(p)}) \arrow[r, "\theta_3", "\cong"'] \arrow[u, "V_H\circ(-)"] & \mathrm{Mult}(G_2 \times \cdots \times G_n \times H^{(p)^*}, G_1^{(p)^*}) \arrow[u, "F_H^*"'] \\
\mathrm{Mult}(G_1^{(p)} \times \cdots \times G_n^{(p)}, H^{(p)}) \arrow[r, "\theta_4", "\cong"'] \arrow[u, "(-)\circ\widetilde{F}"] & \mathrm{Mult}(G_2^{(p)} \times \cdots \times G_n^{(p)} \times H^{(p)^*}, G_1^{(p)^*}) \arrow[u, "F^*"']
\end{tikzcd}$$

where the horizontal homomorphisms are the isomorphisms given by Lemma 1.9 (note that $(-)^* = \underline{\mathrm{Hom}}(-, \mathbb{G}_m)$) and F_H^* and F^* are respectively the homomorphisms $(-) \circ (\mathrm{Id}_{G_2} \times \cdots \times \mathrm{Id}_{G_n} \times F_{H^*})$ and $(-) \circ (F_{G_1} \times \cdots \times F_{G_n} \times \mathrm{Id}_{H^{(p)*}})$. Using the facts that the isomorphism in Lemma 1.9 is functorial and under the identification $(-)^{(p)^*} \cong (-)^{*^{(p)}}$, the dual of the Verschiebung of a commutative group scheme is the Frobenius of the dual group scheme, we deduce that this diagram is commutative.
The commutativity of the upper square implies that

$$F_{G_1^*} \circ \theta_1(\varphi) = \theta_2(\varphi \circ \widetilde{V}) \qquad (\bigstar)$$

The commutativity of the two bottom squares implies that

$$\theta_4(\varphi^{(p)}) \circ (F_{G_2} \times \cdots \times F_{G_n} \times \mathrm{Id}_{H^{(p)*}}) \circ (\mathrm{Id}_{G_2} \times \cdots \times \mathrm{Id}_{G_n} \times F_{H^*}) = \theta_2(V_H \circ \varphi^{(p)} \circ \widetilde{F}).$$

The composition $(F_{G_2} \times \cdots \times F_{G_n} \times \mathrm{Id}_{H^{(p)*}}) \circ (\mathrm{Id}_{G_2} \times \cdots \times \mathrm{Id}_{G_n} \times F_{H^*})$ equals $(F_{G_2} \times \cdots \times F_{G_n} \times F_{H^*})$ and one can easily check that the isomorphism θ_1 given in Lemma 1.9 is compatible with the pullback of the Frobenius, i.e., $\theta_4(\varphi^{(p)}) = \theta_1(\varphi)^{(p)}$. We have thus

$$\theta_1(\varphi)^{(p)} \circ (F_{G_2} \times \cdots \times F_{G_n} \times F_{H^*}) = \theta_2(V_H \circ \varphi^{(p)} \circ \widetilde{F}) \qquad (\blacktriangle)$$

Writing $\overrightarrow{F}$ for $(F_{G_2} \times \cdots \times F_{G_n} \times F_{H^*})$, we know that there is a commutative diagram

$$\begin{tikzcd}
G_2 \times \cdots \times G_n \times H^* \arrow[r, "\theta_1(\varphi)"] \arrow[d, "\overrightarrow{F}"'] & G_1^* \arrow[d, "F_{G_1^*}"] \\
G_2^{(p)} \times \cdots \times G_n^{(p)} \times H^{*^{(p)}} \arrow[r, "\theta_1(\varphi)^{(p)}"'] & G_1^{*^{(p)}}.
\end{tikzcd}$$

This together with $(\bigstar)$ and $(\blacktriangle)$ imply that $\theta_2(\varphi \circ \widetilde{V}) = \theta_2(V_H \circ \varphi^{(p)} \circ \widetilde{F})$. But θ_2 is injective and therefore $\varphi \circ \widetilde{V} = V_H \circ \varphi^{(p)} \circ \widetilde{F}$.

□

Remark 2.23. This lemma is true more generally, i.e., with G_i and H arbitrary commutative group schemes and not necessarily finite. But the proof is more complicated and in the sequel, we will only need the weaker version.

Let G be a commutative group scheme over a field k of characteristic p and $\kappa : G^n \to \Lambda^n G$ the universal alternating morphism defining $\Lambda^n G$. Then taking the pullback of κ and using the isomorphism $(G^n)^{(p)} \cong (G^{(p)})^n$, we obtain an alternating morphism $\kappa^{(p)} : G^{(p)} \to (\Lambda^n G)^{(p)}$. Therefore, there is a unique homomorphism $\eta : \Lambda^n(G^{(p)}) \to (\Lambda^n G)^{(p)}$ such that $\eta \circ \kappa' = \kappa^{(p)}$, where $\kappa' : (G^{(p)})^n \to \Lambda^n(G^{(p)})$ is the universal alternating morphism of $\Lambda^n(G^{(p)})$.

Lemma 2.24. *Let the base field k be perfect of odd characteristic p and G a commutative group scheme over k. Then the homomorphism*

$$\eta : \Lambda^n(G^{(p)}) \to (\Lambda^n G)^{(p)}$$

is a natural isomorphism and therefore $(\Lambda^n G)^{(p)}$ together with the alternating morphism $\kappa^{(p)} : (G^{(p)})^n \to (\Lambda^n G)^{(p)}$ is an alternating n^{th} power of $G^{(p)}$.

Proof. Note that since the field k is perfect, the functor $(-)^{(p)}$ from the category of affine commutative group schemes over k to itself is an equivalence of categories. Using the above notation, we have thus a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(\Lambda^n G, H) \arrow[r, "(-)\circ\kappa", "\cong"'] \arrow[d, "(-)^{(p)}"', "\cong"] & \mathrm{Alt}(G^n, H) \arrow[d, "(-)^{(p)}"', "\cong"] \\
\mathrm{Hom}((\Lambda^n G)^{(p)}, H^{(p)}) \arrow[r, "(-)\circ\kappa^{(p)}"] \arrow[dr, "(-)\circ\eta"'] & \mathrm{Alt}((G^{(p)})^n, H^{(p)}) \\
& \mathrm{Hom}(\Lambda^n(G^{(p)}), H^{(p)}). \arrow[u, "(-)\circ\kappa'"', "\cong"]
\end{tikzcd}$$

The above square is commutative because of the functoriality of $(-)^{(p)}$. It implies that the homomorphism

$$(-)\circ\kappa^{(p)} : \mathrm{Hom}((\Lambda^n G)^{(p)}, H^{(p)}) \to \mathrm{Alt}((G^{(p)})^n, H^{(p)})$$

is an isomorphism and so the homomorphism $(-)\circ\eta$ is also an isomorphism. Since the functor $(-)^{(p)}$ is an equivalence of categories, we can write any commutative group scheme as $H^{(p)}$ for some commutative group scheme H. Consequently η is an isomorphism. $\square$

Lemma 2.25. *Let the base field k be a perfect field of odd characteristic p and n a positive integer. Then the Verschiebung*

$$V : (\Lambda^n \alpha_{p^n})^{(p)} \to \Lambda^n \alpha_{p^n}$$

is trivial.

Proof. If we show that every element φ of $\mathrm{Alt}(\alpha_{p^n}^n, H)$ is annihilated by the Verschiebung V_H of H, i.e., the composite

$$(\alpha_{p^n}^n)^{(p)} \xrightarrow{\varphi^{(p)}} H^{(p)} \xrightarrow{V_H} H$$

is zero, then for every element ψ of $\mathrm{Hom}(\Lambda^n \alpha_{p^n}, H)$ we will have $\psi \circ V = 0$ and hence $V = 0$ (by putting $H = \Lambda^n \alpha_{p^n}$ and ψ the identity homomorphism). Indeed, let $\psi : \Lambda^n \alpha_{p^n} \to H$ be a homomorphism and put $\psi' := \psi \circ \kappa$. Consider the following commutative diagram

$$\begin{tikzcd}
(\alpha_{p^n}^{(p)})^n \arrow[r, "\kappa^{(p)}"] \arrow[dr, "\psi'^{(p)}"'] & (\Lambda^n \alpha_{p^n})^{(p)} \arrow[r, "V"] \arrow[d, "\psi^{(p)}"] & \Lambda^n \alpha_{p^n} \arrow[d, "\psi"] \\
& H^{(p)} \arrow[r, "V_H"'] & H.
\end{tikzcd}$$

By hypothesis, $V_H \circ \psi'^{(p)} = 0$ and therefore $\psi \circ V \circ \kappa^{(p)} = 0$. But according to Lemma 2.24

$$(-)\circ\kappa^{(p)} : \mathrm{Hom}((\Lambda^n G)^{(p)}, H) \to \mathrm{Alt}((G^{(p)})^n, H)$$

is an isomorphism, which implies that $\psi \circ V = 0$.

So we should show that for every H, every element φ of $\mathrm{Alt}(\alpha_{p^n}^n, H)$ is annihilated by the Verschiebung V_H. We show this in 3 steps.

Step 1) We show the statement for H finite. According to Lemma 2.22, we have the following commutative diagram

$$\begin{tikzcd}
\alpha_{p^n}^{(p)} \times \alpha_{p^n}^{(p)} \times \cdots \times \alpha_{p^n}^{(p)} \arrow[r, "\varphi^{(p)}"] & H^{(p)} \arrow[d, "V_H"] \\
\alpha_{p^n}^{(p)} \times \alpha_{p^n} \times \cdots \times \alpha_{p^n} \arrow[u, "\widetilde{F}"] \arrow[d, "\widetilde{V}"'] & \\
\alpha_{p^n} \times \alpha_{p^n} \times \cdots \times \alpha_{p^n} \arrow[r, "\varphi"'] & H
\end{tikzcd}$$

where $\widetilde{V} = V_{\alpha_{p^n}} \times \mathrm{Id}_{\alpha_{p^n}} \times \cdots \times \mathrm{Id}_{\alpha_{p^n}}$ and $\widetilde{F} = \mathrm{Id}_{\alpha_{p^n}^{(p)}} \times F_{\alpha_{p^n}} \times \cdots \times F_{\alpha_{p^n}}$. But the Verschiebung is trivial on α_{p^n}, so the composite $\varphi \circ \widetilde{V}$ is trivial, because φ is multilinear, and hence $V_H \circ \varphi^{(p)} \circ \widetilde{F} = 0$. We want to show that this implies $V_H \circ \varphi^{(p)}$ is zero.

We know that we can write the Frobenius $F : \alpha_{p^n} \to \alpha_{p^n}^{(p)}$ as the composite

$$\alpha_{p^n} \xrightarrow{q} \alpha_{p^{n-1}} \overset{i}{\hookrightarrow} \alpha_{p^n} \underset{\cong}{\xrightarrow{\theta}} \alpha_{p^n}^{(p)}$$

where the epimorphism and the monomorphism are the natural ones and the isomorphism $\theta : \alpha_{p^n} \cong \alpha_{p^n}^{(p)}$ is given by the Hopf algebra isomorphism

$$k[X]/(X^{p^n}) \otimes_{k,\sigma} k \to k[X]/(X^{p^n}), \quad \overline{x} \otimes a \mapsto a \cdot \overline{x}, \quad \forall a \in k.$$

We can thus write $\widetilde{F}$ as the composition

$$\alpha_{p^n}^{(p)} \times \alpha_{p^n} \times \cdots \times \alpha_{p^n} \xrightarrow{\widetilde{q}} \alpha_{p^n} \times \alpha_{p^{n-1}} \times \cdots \times \alpha_{p^{n-1}} \xrightarrow{\widetilde{i}} \alpha_{p^n}^n \xrightarrow{\theta^n} (\alpha_{p^n}^{(p)})^n$$

where $\widetilde{q}$ is $\theta^{-1} \times q \times \cdots \times q$ and $\widetilde{i}$ is the restriction map $\mathrm{Id}_{\alpha_{p^n}} \times i \times \cdots \times i$. Since $\widetilde{q}$ is epimorphic and the composition $V_H \circ \varphi^{(p)} \circ \widetilde{F}$ is zero, we have that $V_H \circ \varphi^{(p)} \circ \theta^n \circ \widetilde{i} = 0$. Since φ is alternating, $\varphi^{(p)}$ is alternating too. Therefore the morphism $V_H \circ \varphi^{(p)} \circ \theta^n$ is alternating. It has a trivial restriction to $\alpha_{p^n} \times \alpha_{p^{n-1}} \times \cdots \times \alpha_{p^{n-1}}$ and we have a short exact sequence $0 \to \alpha_{p^{n-1}} \to \alpha_{p^n} \to \alpha_p \to 0$. We can thus apply Theorem 2.16 (a) and conclude that the morphism $V_H \circ \varphi^{(p)} \circ \theta^n$ is zero as well. Since θ^n is an isomorphism, the morphism $V_H \circ \varphi^{(p)}$ is zero.

Step 2) We show the statement with H of finite type. According to Proposition 2.3 in [6], the morphism φ factors through a finite subgroup scheme H'

of H, i.e., the following diagram is commutative

$$\begin{tikzcd} \alpha_{p^n}^n \arrow[rr, "\varphi"] \arrow[dr, "\varphi'"] & & H \\ & H'. \arrow[ur, hook] & \end{tikzcd}$$

We have thus a commutative diagram

$$\begin{tikzcd} (\alpha_{p^n}^{(p)})^n \arrow[r, "\varphi^{(p)}"] \arrow[dr, "\varphi'^{(p)}"'] & H^{(p)} \arrow[r, "V_H"] & H \\ & H'^{(p)} \arrow[u, hook] \arrow[r, "V_{H'}"'] & H'. \arrow[u, hook] \end{tikzcd}$$

By step 1, we have $V_{H'} \circ \varphi'^{(p)} = 0$. Hence $V_H \circ \varphi^{(p)} = 0$.

Step 3) Now we show the statement for general H. We know that we can write $H = \varprojlim H_i$ with commutative schemes H_i of finite type. Let $\lambda_i : H \to H_i$ be the canonical homomorphisms of the inverse limit and put $\varphi_i := \lambda_i \circ \varphi$. For ever i we have a commutative diagram

$$\begin{tikzcd} (\alpha_{p^n}^{(p)})^n \arrow[r, "\varphi^{(p)}"] \arrow[dr, "\varphi_i^{(p)}"'] & H^{(p)} \arrow[r, "V_H"] \arrow[d, "\lambda_i^{(p)}"] & H \arrow[d, "\lambda_i"] \\ & H_i \arrow[r, "V_{H_i}"'] & H_i. \end{tikzcd}$$

By Step 2, the composition $V_{H_i} \circ \varphi_i^{(p)}$ is trivial and thus we have for all i that the composition $\lambda_i \circ V_H \circ \varphi^{(p)} = 0$. Since $H = \varprojlim H_i$, we conclude that $V_H \circ \varphi^{(p)} = 0$.

□

Proposition 2.26. *Let the base scheme be* $\mathbf{Spec}\, k$ *for a perfect field of odd characteristic* p *and* n *a positive integer. Then there is an isomorphism*

$$\Lambda^n \alpha_{p^n} \cong \alpha_p.$$

Proof. If $n = 1$ then this is a tautology, so assume $n > 1$. We know by Example 1.29 that $\underline{\mathrm{Alt}}(\alpha_{p^n}^n, \mathbb{G}_a) \cong \mathbb{G}_a$ and therefore, $\mathrm{Hom}(\Lambda^n \alpha_{p^n}, \mathbb{G}_a) \cong \mathbb{G}_a(k) = k$. This shows that the group scheme $\Lambda^n \alpha_{p^n}$ is not trivial. Assume that we have an epimorphism $\pi : \alpha_p \twoheadrightarrow \Lambda^n \alpha_{p^n}$. This implies that π is in fact an isomorphism, because its kernel could not be the whole group scheme α_p (since otherwise the image, $\Lambda^n \alpha_{p^n}$, would be trivial) and since α_p is simple, the kernel should be zero. Consequently, π is a monomorphism too and hence an isomorphism. It is thus sufficient to show that such an epimorphism exists.

We know from Proposition 2.17 that there is an epimorphism $\theta : \alpha_p^{\otimes n} \twoheadrightarrow \Lambda^n \alpha_{p^n}$. Consider the following commutative diagram

$$\begin{tikzcd}
\alpha_p^{\otimes n^{(p)}} \arrow[r, two heads, "\theta^{(p)}"] \arrow[d, "V"'] & \Lambda^n \alpha_{p^n}^{(p)} \arrow[d, "V'"] \\
\alpha_p^{\otimes n} \arrow[r, two heads, "\theta"] \arrow[d, two heads] & \Lambda^n \alpha_{p^n} \arrow[d] \\
C \arrow[r, two heads, "\bar{\theta}"] & C'
\end{tikzcd}$$

where V and V' are Verschiebung and C and C' are the cokernels of V and V'. By Lemma 2.21, C is isomorphic to α_p. We know from Lemma 2.25 that the image of V' is zero and hence its cokernel C' is isomorphic to $\Lambda^n \alpha_{p^n}$. Since $C \xrightarrow{\bar{\theta}} C'$ is epimorphic, we get the desired epimorphism $\alpha_p \twoheadrightarrow \Lambda^n \alpha_{p^n}$. $\square$

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