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Algebraic Monodromy Groups of A-Motives

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Geb ich mich mit dem Allgemeinen versöhnlich
Oder lebe ich lieber ungewöhnlich?

Sportfreunde Stiller

Lesson 2: Theorems are not miracles, but incestuous relationships
between overdetermined inbred mathematical objects.

Doron Zeilberger

Acknowledgments

Since Richard Pink lectured on Algebraic Geometry during the last years of my student days at ETH Zürich, it is more than fair to say that he has been my greatest mathematical influence, and most important guide to the vast body of mathematical knowledge known as Arithmetic Geometry. Among other things, his interest in the positive characteristic theory, his unfailing eye for the main idea paired with his knack for technical details, and his insistence on precision in both mathematical content and linguistic implementation have been rather influential. I would like to thank him for the past years under his wings.

Equally important for this thesis was Akio Tamagawas article [Tam95], which was shown to me by Matthias Traulsen. It is abundantly clear that many of the ideas that will be developed in the following originate in his work. By writing his thesis in Japanese, and switching subject without publishing a substantial account of his results in English language, he bestowed upon me the pleasure of reworking the interesting subject matter. I wish to thank him for encouragement, dating back already to the time after my diploma thesis, and e-mail correspondance with respect to technical problems. Incidentally, I have never met him in person.

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This thesis is dedicated to my late grandmother, Agnes “Omi” Feller.

Nicolas Stalder, Zürich, August 2007.

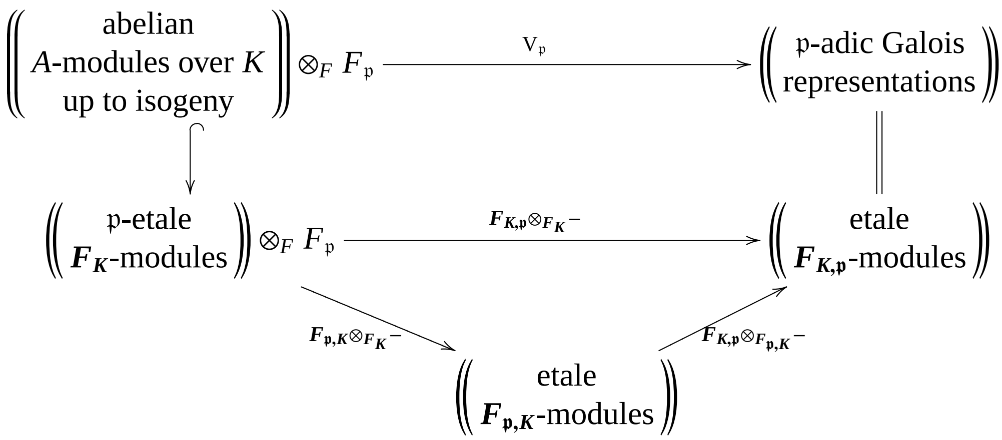
Contents

English Abstract

Let F be a global field of positive characteristic, fix a place ∞ and let A denote the ring of all elements of F integral outside ∞. Let K be a field of finite type over its prime field, equipped with a unital ring homomorphism $\iota : A \to K$.

In this thesis we study the Galois representations associated to A-modules over K, in particular with respect to the question of their semisimplicity. An A-module is a smooth commutative group scheme G of finite type over K, equipped with an action $\phi : A \to \mathrm{End}_K(G)$ which is (in a sense to be explained later) compatible with the action of A on $\mathrm{Lie}\, G$ given by ι. The Galois representations we are speaking of are the Tate modules $\mathrm{V}_{\mathfrak{p}}(G, \phi)$ which arise by collecting the $\mathfrak{p}$-power torsion points of $G(K^{\mathrm{sep}})$ with respect to ϕ.

For this, one introduces the notion of isogeny between A-modules. In Chapters II and III we collect from and complement the literature. One obtains a category of abelian A-modules up to isogeny and its classification in terms of abelian A-motives up to isogeny. The latter objects are special cases of the more general notion of *restricted modules*, and may or may not be *etale* at a given place $\mathfrak{p}$ of F. Choosing a place $\mathfrak{p} \notin \{\ker \iota, \infty\}$, we have the following diagram of categories and functors, which translates our problems into problems of semilinear algebra:



The Semisimplicity Conjecture states that $\mathrm{V}_\mathfrak{p}$ respects and reflects semisimplicity of objects, whereas the Tate Conjecture states that $\mathrm{V}_\mathfrak{p}$ is fully faithful. They both follow from the respective statements for the functor $\boldsymbol{F}_{K,\mathfrak{p}} \otimes_{\boldsymbol{F}_K} -$ in the middle row, which in turn follow from the respective statements for the functors in the lower left and lower right corner of the diagram.

In Chapter IV we generalise the functor $\boldsymbol{F}_{\mathfrak{p},K} \otimes_{\boldsymbol{F}_K} -$ to more general scalar extensions F'/F, and prove the required results using fairly straightforward extensions of results on scalar extension of modules over algebras as in [Bou81].

In Chapter V we prove the required results for $\boldsymbol{F}_{K,\mathfrak{p}} \otimes_{\boldsymbol{F}_{\mathfrak{p},K}} -$ by constructing a left quasi-inverse functor $\mathrm{Q}_\mathfrak{p}$. It has the additional property of characterising the essential image of $\boldsymbol{F}_{K,\mathfrak{p}} \otimes_{\boldsymbol{F}_{\mathfrak{p},K}} -$. This is done using ideas of Akio Tamagawa, and is cast in language formally analogous to the Fontaine theory of p-adic Galois representations.

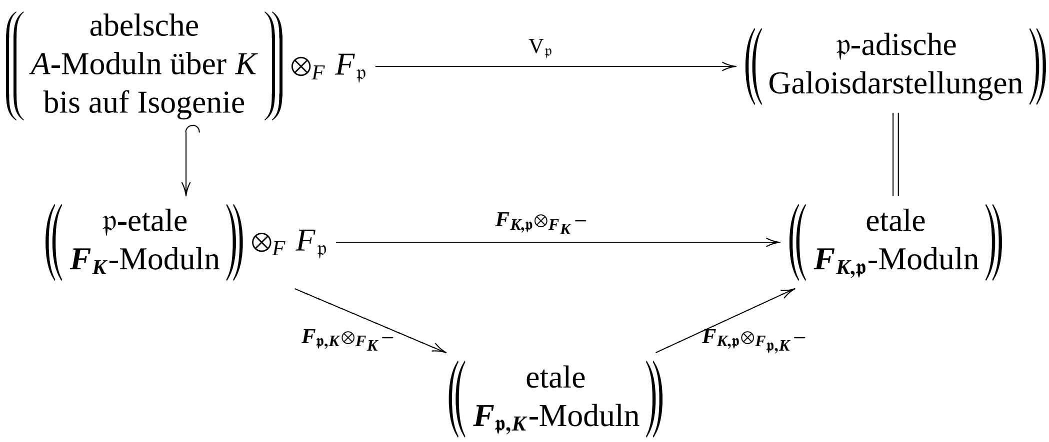
Finally, in Chapters VII and VIII some complements on Tannakian categories and a general result from representation theory allow us to prove that certain algebraic monodromy groups (which coincide with the Zariski closure of the image of the absolute Galois group of K) are reductive.

Deutsche Zusammenfassung

Sei F ein globaler Körper positiver Charakteristik, mit fixierter Stelle ∞, und sei A der Ring der Elemente von F, welcher ausserhalb ∞ ganz sind. Sei K ein Körper endlichen Typs über seinem Primkörper, ausgerüstet mit einem unitalen Ringhomomorphismus $\iota : A \to K$.

In der vorliegenden Arbeit untersuchen wir die Galoisdarstellungen, welche A-Moduln über K assoziiert sind, insbesonders in Hinsicht auf die Frage ihrer Halbeinfachkeit. Ein A-Modul ist ein glattes kommutatives Gruppenschema G von endlichem Typ über K, ausgerüstet mit einer Operation $\phi : A \to \mathrm{End}_K(G)$ welche (in einem später zu erklärenden Sinn) mit der Operation von A auf $\mathrm{Lie}\, G$ durch ι kompatibel ist. Die genannten Galoisdarstellungen sind die Tatemoduln $\mathrm{V}_{\mathfrak{p}}(G, \phi)$, welche durch das Zusammenfassen der $\mathfrak{p}$-Potenz Torsionspunkten von $G(K^{\mathrm{sep}})$ bezüglich ϕ entstehen.

Dazu führt man den Begriff der Isogenie zwischen A-Moduln ein. In den Kapiteln II und III sammeln wir Ergebnisse aus und ergänzen wir die bestehende Literatur. Man erhält eine Kategorie abelscher A-Moduln bis auf Isogenie, und eine Klassifikation durch abelsche A-Motive bis auf Isogenie. Letztere sind spezielle Fälle des allgemeineren Begriffs *restringierter Moduln*, und können an einer gegebenen Stelle $\mathfrak{p}$ von F entweder *etale* sein, oder eben nicht. F”ur eine Stelle $\mathfrak{p} \notin \{\ker \iota, \infty\}$ erhalten wir folgendes kommutative Diagramm:



Es übersetzt unsere Probleme in Probleme der semilinearen Algebra. Die Halbeinfachkeitsvermutung besagt dass $\mathrm{V}_{\mathfrak{p}}$ die Halbeinfachkeit sowohl erhält als auch reflektiert, wohingegen die Tatevermutung besagt, dass $\mathrm{V}_{\mathfrak{p}}$ volltreu ist. Beide Vermutungen folgen aus den entsprechenden Aussagen für den Funktor $\boldsymbol{F}_{\boldsymbol{K},\mathfrak{p}} \otimes_{\boldsymbol{F}_{\boldsymbol{K}}} -$ in der mittleren Reihe, welche wiederum aus den entsprechenden Aussagen für die Funktoren in der unteren linken und unteren rechten Ecke des Diagramms folgen.

In Kapitel IV verallgemeinern wir den Funktor $\boldsymbol{F}_{\mathfrak{p},\boldsymbol{K}} \otimes_{\boldsymbol{F}_{\boldsymbol{K}}} -$ zu allgemeineren Skalarerweiterungen F'/F, und beweisen die benötigten Resultate mittels relativ einfachen Erweiterungen der Resultate über Skalarerweiterung von Moduln über Algebren wie in [Bou81].

In Kapitel V beweisen wir die benötigten Resultate für $\boldsymbol{F}_{\boldsymbol{K},\mathfrak{p}} \otimes_{\boldsymbol{F}_{\mathfrak{p},\boldsymbol{K}}} -$ indem wir einen linksinversen Funktor $\mathrm{Q}_{\mathfrak{p}}$ konstruieren. Er hat die zusätzliche Eigenschaft, das essentielle Bild von $\boldsymbol{F}_{\boldsymbol{K},\mathfrak{p}} \otimes_{\boldsymbol{F}_{\mathfrak{p},\boldsymbol{K}}} -$ zu charakterisieren. Dabei verwenden wir Ideen von Akio Tamagawa, und eine Sprache, welche formal analog zur Fontainetheorie p-adischer Galoisdarstellungen ist.

Schliesslich erlauben uns in den Kapiteln VII und VIII einige Ergänzungen zur Theorie der Tannakakategorien und ein allgemeines Result aus der Darstellungstheorie, zu zeigen dass gewisse algebraische Monodromiegruppen (sie stimmen mit dem Zariskiabschluss des Bildes der absoluten Galoisgruppe von K überein) reduktiv sind.

Conventions

All rings are unital, as are all ring homomorphisms.
All categories are additive, as are all functors.

Caveat Emptor

We assume throughout that two universes $\mathscr{U} \subset \mathscr{V}$ have been chosen, sapienti sat!

Introduction

In order to put into context the theory of t-motives in general, and the results of this thesis in particular, we start by giving a bird's eye view of Alexandre Grothendieck's theory of motives.

$\mathbb{Q}$-motives and their monodromy groups

The idea behind motives was and is to "linearise" the geometric category of smooth projective algebraic varieties over a given base field. For this, a rather dazzling array of cohomology theories had already been developed and employed, ranging from singular and de Rham over etale and ℓ-adic to crystalline cohomology and more! In all cases, such a cohomology theory is given by a functor

$$V : \left(\!\!\left(\begin{array}{c}\text{smooth projective}\\ \text{algebraic varieties over}\\ \text{a given field } K\end{array}\right)\!\!\right) \to \left(\!\!\left(\begin{array}{c}\text{finite-dimensional vector spaces}\\ \text{over a given field } F_0 \text{ of characteristic } 0,\\ \text{possibly with additional}\\ \text{algebraic structure}\end{array}\right)\!\!\right).$$

The question then naturally arose of how many "substantially different" cohomology theories exist, that is, are there relations between them, or does there even exists a "universal" such cohomology theory

$$M : \left(\!\!\left(\begin{array}{c}\text{smooth projective}\\ \text{algebraic varieties over}\\ \text{a given field } K\end{array}\right)\!\!\right) \to \left(\!\!\left(\text{"motives"}\right)\!\!\right),$$

with "motives" some $\mathbb{Q}$-linear abelian category, universal in the sense that every other (classical) cohomology theory V "factors" as $V = V_{\mathrm{mot}} \circ M$ for some $\mathbb{Q}$-linear exact functor V_{mot} from "motives" to the target of V. In fact, Grothendieck proposed a construction[1] of such a category $\mathbb{Q}\text{-Mot}_K$, which is by now accepted

[1] Pure motives for numerical equivalence with coefficients in $\mathbb{Q}$, cf. "The Standard Conjectures" in [JKS94].

as being the “correct” one. In many instances, i.e., for the classical cohomology theories, a factorisation through (or extension to) $\mathbb{Q}$-Mot$_K$ has been proven.

But this “universality” is only one side of the story. The other has to do with *tensor products*. It is the closely connected theory of Tannakian categories, devised – again – by Grothendieck with the aim of reducing the study of motives to the representation theory of reductive groups. How?

Given a linear algebraic group G over a field F, one can reconstruct G from the category $\mathrm{Rep}_F(G)$ of its finite-dimensional representations over F with the help of the “forgetful” functor $\mathrm{Rep}_F(G) \to \mathrm{Vec}_F$, where Vec_F denotes the category of finite-dimensional F-vector spaces. Conversely, one may ask oneself which F-linear abelian categories arise as $\mathrm{Rep}_F(G)$ for some group G. Such categories have several distinguishing properties:

- **Finiteness:** Every object has a composition series of finite length and its endomorphism ring is a finite-dimensional F-algebra.
- **Tensor products:** To every pair of objects there is associated their “tensor product”, in a functorial, associative and commutative fashion.
- **Rigidity:** Every object has a “dual”, and is isomorphic to its bidual.

Axiomatising these properties in a suitable way, one arrives at the notion of a pre-Tannakian category over F (that is, a finite rigid abelian tensor category over F, cf. Definition 26.1). Such a category $\mathscr{T}$ is called Tannakian over F if there exists a field extension $F' \supset F$ and an exact faithful F-linear functor $\omega : \mathscr{T} \to \mathrm{Vec}_{F'}$ compatible with tensor products, a *fibre functor*. If there exists a fibre functor with $F' = F$, then one calls $\mathscr{T}$ a *neutral* Tannakian category over F.

The (algebraic) monodromy group of a Tannakian category depends on the choice of fibre functor ω, and is given by the automorphisms of ω as a tensor functor (Definitions 1.4 and 26.2). The monodromy group of an object X of a Tannakian category $\mathscr{T}$ is the monodromy group of the subcategory of $\mathscr{T}$ “generated” by X (Definitions 1.6 and 26.2).

Grothendieck and Neantro Saavedra Rivano succeeded in showing (we quote this in Theorem 26.4) that any neutral Tannakian category is equivalent to $\mathrm{Rep}_F(G)$, where G is the monodromy group of $\mathscr{T}$. Additionally, there exists a dictionary between the $\mathscr{T}$’s and the G’s, which states for instance that if F is of characteristic zero, then $\mathscr{T}$ is semisimple (all of its objects are isomorphic to direct sums of simple objects) if and only if the monodromy groups of all of its objects are reductive.

The target categories of all classical cohomology theories are Tannakian categories, namely the categories of finite-dimensional vector spaces over $\mathbb{Q}$ for singular cohomology, Hodge structures for de Rham cohomology, Galois representations for ℓ-adic cohomology and Dieudonné modules for crystalline cohomology. The category $\mathbb{Q}$-Mot$_K$ proposed by Grothendieck does in fact have a built-in "tensor product" suitable for these purposes, derived essentially from the direct product of varieties. Uwe Jannsen has proven that $\mathbb{Q}$-Mot$_K$ is semisimple abelian [Jan92], and it is conjectured that $\mathbb{Q}$-Mot$_K$ is in fact a Tannakian category.

An avatar of the theory of motives is the classical procedure of associating to a smooth projective algebraic *curve* its Jacobian. This is an abelian variety, and may hence be considered to be a "linearisation" of the curve. (At least over $\mathbb{C}$, an abelian variety is determined by a full $\mathbb{Z}$-lattice in a $\mathbb{C}$-vector space, and homomorphisms among abelian varieties extend to $\mathbb{C}$-linear homomorphisms of the associated $\mathbb{C}$-vector spaces). The $\mathbb{Z}$-linear category of abelian varieties becomes a $\mathbb{Q}$-linear semisimple (Poincaré's reducibility theorem!) abelian category if one inverts isogenies, that is, if one formally adjoins inverses to the endomorphisms given by "multiplication by n" for $n \geq 1$. The closure of this category of abelian varieties up to isogeny under duality with respect to tensor products in $\mathbb{Q}$-Mot$_K$ is a Tannakian category [Jan92].

Returning to the theme of "universality", one may ask, given a cohomology theory

$$\mathbb{Q}\text{-Mot}_K \xrightarrow{V_{\text{mot}}} \mathscr{T} \longrightarrow \text{Vec}_{F_0}$$

with values in a neutral Tannakian category $\mathscr{T}$ over F_0, whether qualitative properties of a motive M are mirrored in its associated cohomology $V_{\text{mot}}(M)$. Three examples of possible questions for a given M in $\mathbb{Q}$-Mot$_K$:

- **Endomorphism algebras:** Whereas End(M) is a finite-dimensional $\mathbb{Q}$-algebra, End $(V_{\text{mot}}(M))$ is a finite-dimensional F_0-algebra. So it is natural to ask whether the natural homomorphism

  $$F_0 \otimes_{\mathbb{Q}} \text{End}(M) \to \text{End}\,(V_{\text{mot}}(M))$$

  is an isomorphism. This is known as the "Tate Conjecture".

- **Semisimiplicity:** Does the fact that M is semisimple imply that $V_{\text{mot}}(M)$ is semisimple? This is sometimes subsumed under the "Tate Conjecture", but other authors refer to it as the "Grothendieck-Serre Conjecture" or simply as the "Semisimplicity Conjecture", as we will.

- **Monodromy groups:** Tannakian duality assigns to M two (a priori different) monodromy groups, that of M and that of $V_{\text{mot}}(M)$, since both $\mathbb{Q}$-Mot$_K$ and $\mathscr{T}$ are Tannakian. Do these coincide? For practical purposes, this would mean that one could calculate the monodromy group of the motive of a variety inside $\mathscr{T}$, without reference to $\mathbb{Q}$-Mot$_K$.

These are open questions in general. If K is finitely generated as a field over its prime field, and one considers the ℓ-adic cohomology (with $\ell \neq \mathrm{char}(K)$) of an abelian variety over K, then the combined efforts of John Tate ([Tat66], for K finite), Shigefumi Mori and Yuri Zarhin ([Mor78, Zar76] independently, for $\mathrm{char}(K) > 2$) and Gerd Faltings ([Fal83], for $\mathrm{char}(K) = 0$) have shown that the answer to the first two questions is positive. We will see later that this implies a positive answer to the third question.

A-motives and their monodromy groups

We turn to an introduction to the subject matter proper of this thesis. The astute reader will not fail to have noticed that, independent of the choice of K and in particular of its characteristic, the category $\mathbb{Q}$-Mot_K is always $\mathbb{Q}$-linear, so in particular it is only of use when one considers cohomology theories with values in Tannakian categories $\mathscr{T}$ over fields F_0 of characteristic zero. For instance, with ℓ-adic cohomology one obtains representations of the absolute Galois group of K over the field $\mathbb{Q}_\ell$ of ℓ-adic numbers.

With a slightly different background, Vladimir Drinfeld [Dri74] (for dimension 1, with a view towards the "Langlands Correspondance") and Greg Anderson [And86] (for higher dimensions, with a view towards tensor products) introduced the concept of abelian A-modules. What is this?

Put simply, the idea is to replace $\mathbb{Q}$ by a global field F of positive characteristic, and then mimic the theory of abelian varieties. More precisely, letting k denote the finite field of constants of F, one chooses a place ∞ of F, lets A denote the subring of F consisting of those elements that are integral outside ∞ (this is the replacement of $\mathbb{Z}$), chooses a base field K containing k and chooses a k-linear homomorphism $\iota : A \to K$. This homomorphism is new to the theory, since there exists a unique unital homomorphism $\mathbb{Z} \to K$ for every base field K.

An *A-module over K* then consists of a vector group G over K, that is, a group scheme over K which is isomorphic to a finite product of copies of the additive group $\mathbb{G}_a$ over the algebraic closure of K (cf. Definition 8.1), and an action of A on G, that is, a k-linear ring homomorphism

$$\phi : A \longrightarrow \mathrm{End}_K(G),$$

which must fulfill an additional condition relating the induced action of A on $\mathrm{Lie}(G)$ with the characteristic homomorphism ι (cf. Definition 8.2).

These A-modules are in duality with and classified by A-motives, which we do not define here, but they are are elements of a concrete abelian category of modules over a certain non-commutative ring (Definition 10.1). In particular, this allows one to define directly the tensor product of two A-motives! A further technical condition (Definition 10.5) defines the full subcategories of *abelian* A-modules and A-motives, and it turns out that the categories of abelian A-modules and abelian A-motives are anti-equivalent. This lets one work directly with the technically simpler category of A-motives.

Inverting isogenies, which in this case means formally adjoining inverses to the endomorphisms given by "multiplication by a" for all $0 \neq a \in A$, one obtains an F-linear abelian category of A-motives up to isogeny (Definition 12.1). After adding formal duals with respect to the tensor product for each object, one obtains the pre-Tannakian category of all *A-isomotives* (Definition 12.6), which we will denote in this introduction by $F\text{-Mot}_K$.

Despite the formal analogy between $F\text{-Mot}_K$ and $\mathbb{Q}\text{-Mot}_K$, there are major differences, of which we mention the following. The category $F\text{-Mot}_K$ is "simpler" in the sense that is given by definition as a subcategory of a concrete category of modules over a ring, whereas progress with studying $\mathbb{Q}\text{-Mot}_K$ is blocked, partially due to the lack of such a concrete interpretation. On the other hand, $F\text{-Mot}_K$ is "less simple" since there is no analogue of Poincaré's reducibility theorem, and it turns out that there do exist elements of $F\text{-Mot}_K$ which are *not* semisimple (Example 13.5). Also, objects of $F\text{-Mot}_K$ need not be "pure", nor even "mixed" in the sense that there always exist filtrations by "pure" objects. In this sense, $F\text{-Mot}_K$ is more general even than the hypothetical category of mixed motives over $\mathbb{Q}$.

Mimicking the definitions of classical cohomology theories, various authors have defined and studied cohomology theories for $F\text{-Mot}_K$ (e.g., [Pap05] and [Tae07] for "Betti cohomology"). In this thesis, we are interested in the analogue of ℓ-adic cohomology, that is, of the Tate modules of abelian varieties. For abelian A-modules one may copy the definition of the Tate module of an abelian variety verbatim, mutatis mutandi, and for a place $\mathfrak{p} \neq \ker(\iota), \infty$ of F, one obtains a faithful F-linear exact functor compatible with tensor products

$$\mathrm{V}_{\mathfrak{p}} : F\text{-Mot}_K \longrightarrow \mathrm{Rep}_{F_{\mathfrak{p}}}(\Gamma_K),$$

where $F_{\mathfrak{p}}$ is the completion of F at $\mathfrak{p}$, and $\mathrm{Rep}_{F_{\mathfrak{p}}}(\Gamma_K)$ is the category of continuous finite-dimensional representations of Γ_K, the absolute Galois group of K, over $F_{\mathfrak{p}}$.

Now $\mathrm{Rep}_{F_{\mathfrak{p}}}(\Gamma_K)$ is a neutral Tannakian category over $F_{\mathfrak{p}}$, so it follows that $F\text{-Mot}_K$ is a Tannakian category over F, and one may ask the questions about endomorphism rings, semisimplicity and monodromy groups in our situation for fields K finitely generated over their prime field, as before for $\mathbb{Q}\text{-Mot}_K$ and in particular abelian varieties.

- **Endomorphism algebras:** The analogue of the "Tate Conjecture" turns out to hold true for finitely generated base fields K. This has been known for a while, and been proven independently by Y. Taguchi [Tag96] and A.

Tamagawa [Tam94]. In this thesis, we reprove it (Proposition 19.2), using ideas of A. Tamagawa [Tam95, Tam04].

- **Semisimiplicity:** Concerning the "Semisimplicity Conjecture", we stress again that there exist non-semisimple objects in F-Mot$_K$, and in fact the Tate Conjecture implies that the Tate module of such an object is not semisimple (Lemma 3.2). On the other hand, a one-dimensional abelian A-module – traditionally called a Drinfeld module – is semisimple precisely because it is one-dimensional. Y. Taguchi has proven that the Tate modules of Drinfeld modules are semisimple [Tag91, Tag93].

  In Theorem 20.1, we prove in full generality that semisimple objects of F-Mot$_K$ have semisimple Tate modules for finitely generated base fields K, again using ideas of A. Tamagawa [Tam95, Tam04].

- **Monodromy groups:** The Tate Conjecture and Semisimplicity Conjecture together have two consequences for the relevant monodromy groups. We note first that the algebraic monodromy group of a continuous representation V of Γ_K over $F_\mathfrak{p}$ is the Zariski closure of the image of Γ_K in $\mathrm{GL}(V)(F_\mathfrak{p})$ (Theorem 27.3).

  So, given M in F-Mot$_K$, to show that the monodromy groups of M and $\mathrm{V}_\mathfrak{p}(M)$ coincide means showing that the image of Γ_K in $\mathrm{Aut}_{F_\mathfrak{p}}(\mathrm{V}_\mathfrak{p}\, M)$ may be identified naturally with a Zariski-dense subgroup of the algebraic monodromy group of M. We prove this in Theorem 28.1(a).

- The question of whether the algebraic monodromy group of a semisimple object of F-Mot$_K$ is reductive is more subtle, since in positive characteristic an algebraic group with a faithful semisimple representation need not be reductive, due to the phenomenon of inseparability and contrary to what is the case in characteristic zero. However, we do prove that if M is semisimple and, additionally, one assumes that the endomorphism algebra of M is separable (Definition 23.16), then identity component of the algebraic monodromy group of M is a reductive group, this is Theorem 28.1(b).

- **Scalar extension of abelian categories:** For the proof of these consequences for monodromy groups, we introduce the notion of *scalar extension* for abelian categories linear over fields and satisfying a certain finiteness condition (Definition 1.8). Its construction is inspired by [Del87] and [Mil92, Appendix A]. We develop its basic properties, find its universal property (Theorem 24.1) and discuss compatibilities with tensor products. The main results for our applications are Theorems 26.6 and 26.9.

How to prove all this

We start by formalising the properties we hope our functor $\mathrm{V}_\mathfrak{p}$ to have: Given any field extension $F' \supset F$, an F-linear abelian category $\mathscr{A}$ and an F'-linear abelian category $\mathscr{A}'$, consider an F-linear exact functor

$$V : \mathscr{A} \longrightarrow \mathscr{A}'.$$

If the induced homomorphism $F' \otimes_F \mathrm{Hom}_{\mathscr{A}}(X, Y) \to \mathrm{Hom}_{\mathscr{A}'}(VX, VY)$ is an isomorphism for all objects X, Y of $\mathscr{A}$, we say that V is *F'/F-fully faithful*.

If V maps semisimple objects of $\mathscr{A}$ to semisimple objects of $\mathscr{A}'$, we say that V is *semisimple*.

Both properties are "transitive": Assume that $F'' \supset F'$ is another field extension, $\mathscr{A}''$ is an F''-linear abelian category and $V' : \mathscr{A} \to \mathscr{A}''$ is an F'-linear exact functor. If V is F'/F-fully faithful and V' is F''/F'-fully faithful, then $V' \circ V$ is F''/F-fully faithful. And if V and V' are both semisimple then so is $V' \circ V$. This allows to "factor the proof" of both of these properties in a given situation. We are interested in the case $F'' = F' = F_\mathfrak{p}$.

Recall that abelian A-modules up to isogeny are classified by their associated A-isomotives (Theorem 10.8), which are modules over a certain noncommutative ring. The continuous representations of Γ_K over $F_\mathfrak{p}$ are also classified by associated modules over a certain noncommutative ring (Proposition 7.3). This is a major difference and simplification to the situation for representations of global Galois groups in characteristic zero.

It turns out that, under these identifications, $\mathrm{V}_\mathfrak{p}$ translates to a functor of a rather simple form, associating to an M in $F\text{-}\mathrm{Mot}_K$ the tensor product $R_\mathfrak{p} \otimes_R M$, where R and $R_\mathfrak{p}$ are certain rings (Proposition 14.4 and the following remarks).

Moreover, there exists an explicit $F_\mathfrak{p}$-linear category which fulfills the purpose of factoring the above translation of $\mathrm{V}_\mathfrak{p}$ into a composite

$$R_\mathfrak{p} \otimes_R (-) = (R_\mathfrak{p} \otimes_{R'_\mathfrak{p}} (-)) \circ (R'_\mathfrak{p} \otimes_R (-))$$

for a certain intermediate ring $R_\mathfrak{p} \supset R'_\mathfrak{p} \supset R$. Philosophically speaking, this corresponds to passing from the F-linear abelian category $F\text{-}\mathrm{Mot}_K$ to an $F_\mathfrak{p}$-linear abelian category in a "minimal" way.

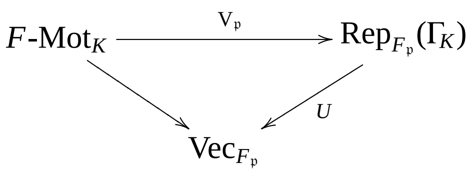
In Chapter IV, we prove that the first factor of this decomposition of $\mathrm{V}_\mathfrak{p}$ is $F_\mathfrak{p}/F$-fully faithful (Proposition 15.2) and semisimple (Theorem 16.4) by direct computations, reminiscent of and inspired by what one does for the scalar extensions of algebras as in [Bou81].

In Chapter V, using and generalising clever yet not formally and fully published ideas of Tamagawa [Tam95, Tam04], we prove that the second factor of this decomposition of $\mathrm{V}_{\mathfrak{p}}$ admits a right-adjoint which is left-quasiinverse (Theorem 17.18). Formally, this implies that the second factor is $F_{\mathfrak{p}}/F_{\mathfrak{p}}$-fully faithful, that is, fully faithful, and even maps simple objects to simple objects, which in turn clearly implies that this factor is semisimple.

In Chapter VI, we deduce from these results of Chapters IV and V that $\mathrm{V}_{\mathfrak{p}}$ is $F_{\mathfrak{p}}/F$-fully faithful and semisimple, which means that both the Tate Conjecture and the Semisimplicity Conjecture are true for

$$\mathrm{V}_{\mathfrak{p}} : F\text{-Mot}_K \to \mathrm{Rep}_{F_{\mathfrak{p}}}(\Gamma_K).$$

In order to discuss the consequences for algebraic monodromy groups, consider the following commutative diagram, where U denotes the forgetful functor:



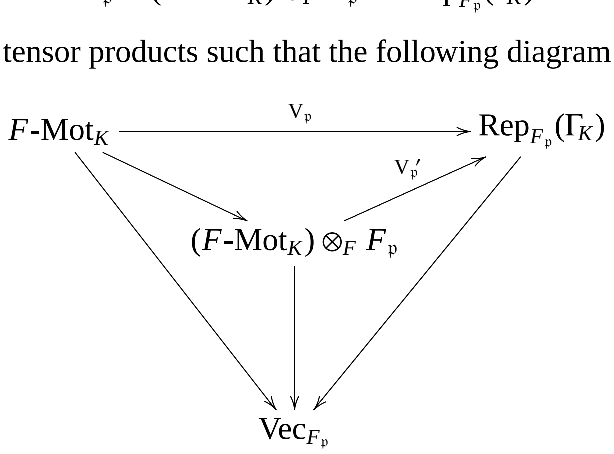
To compare the monodromy groups of F-Mot$_K$ and $\mathrm{Rep}_{F_{\mathfrak{p}}}(\Gamma_K)$ is, by definition, to compare the automorphisms of $U \circ \mathrm{V}_{\mathfrak{p}}$ and U.

For this, we have found it useful to consider for a given field extension $F' \supset F$ the general question of associating to an F-linear abelian category $\mathscr{A}$ a "universal" F'-linear abelian category $\mathscr{A} \otimes_F F'$, its "scalar extension" from F to F'. In Chapter VII we address this question for F-linear abelian categories satisfying a certain finiteness condition (F-finiteness, Definition 1.8) enjoyed by Tannakian categories.

We develop the universal property of $\mathscr{A} \otimes_F F'$ (Theorem 24.1) and discuss the influence of tensor products in $\mathscr{A}$. The outcome in our situation is that we obtain a Tannakian category $(F\text{-Mot}_K) \otimes_F F_{\mathfrak{p}}$ and an $F_{\mathfrak{p}}$-linear exact functor

$$\mathrm{V}_{\mathfrak{p}}' : (F\text{-Mot}_K) \otimes_F F_{\mathfrak{p}} \to \mathrm{Rep}_{F_{\mathfrak{p}}}(\Gamma_K)$$

compatible with tensor products such that the following diagram commutes:



Thus we may compare first the monodromy groups of F-Mot$_K$ and $(F\text{-Mot}_K) \otimes_F F_\mathfrak{p}$, and then the monodromy groups of $(F\text{-Mot}_K) \otimes_F F_\mathfrak{p}$ and $\mathrm{Rep}_{F_\mathfrak{p}}(\Gamma_K)$.

For the first comparison, the "universality" of passing from F-Mot$_K$ to its scalar extension $(F\text{-Mot}_K) \otimes_F F_\mathfrak{p}$ implies that the monodromy groups of F-Mot$_K$ and $(F\text{-Mot}_K) \otimes_F F_\mathfrak{p}$ coincide.

For the second comparison, the fact that $\mathrm{V}_\mathfrak{p}$ fulfills the Tate and Semisimplicity Conjecture implies first that $\mathrm{V}_\mathfrak{p}'$ is fully faithful and maps simple objects to simple objects (Theorem 25.6), which in turn implies that $(F\text{-Mot}_K) \otimes_F F_\mathfrak{p}$ and its essential image in $\mathrm{Rep}_{F_\mathfrak{p}}(\Gamma_K)$ are equivalent (Theorem 26.9), so that the monodromy group of $(F\text{-Mot}_K) \otimes_F F_\mathfrak{p}$ coincides with that of its essential image in $\mathrm{Rep}_{F_\mathfrak{p}}(\Gamma_K)$.

In combination, these comparisons imply that for every object M of F-Mot$_K$ the monodromy group of M coincides with the monodromy group of $\mathrm{V}_\mathfrak{p}(M)$.

The claim about the reductivity of the monodromy groups of semisimple objects with separable endomorphism rings then follows, using ingredients from the representation theory of unipotent groups and further generalities on scalar extensions of abelian categories linear over a field, applied to $\mathrm{Rep}_{F_\mathfrak{p}} G$, where G is the monodromy group in question, and the field extension $\overline{F_\mathfrak{p}} \supset F_\mathfrak{p}$, where $\overline{F_\mathfrak{p}}$ is an algebraic closure of $F_\mathfrak{p}$.

Index of Notation

Rings

k	a fixed finite field
F	a global field of positive characteristic, with constant field k (Exception: In Chapter IV, F may be any field containing k)
$\mathfrak{p}$	a place of F
$F_\mathfrak{p}$	the completion of F at $\mathfrak{p}$
K	a field containing k
K^{sep}	a fixed separable closure of K
F_K	the total ring of quotients of $F \otimes_k K$
$F_{\mathfrak{p},K}$	the total ring of quotients of $F_\mathfrak{p} \otimes_k K$
$F_{K,\mathfrak{p}}$	the "completion" of F_K at $\mathfrak{p}$ (cf. Example 6.11(b))

Groups

Γ_K	the absolute Galois group $\text{Gal}(K^{\text{sep}}/K)$ of K

Categories

$A\text{-Mot}_K$	the category of A-motives over K (Definition 10.1)
$\text{Rep}_{F_\mathfrak{p}}(\Gamma_K)$	the category of all finite-dimensional continuous representations of Γ_K over $F_\mathfrak{p}$

Functors

$\text{V}_\mathfrak{p}$	the rational Tate module functor

Symbols

The symbol $\longrightarrow$ denotes either a homomorphism of objects or a functor.
The symbol $\Longrightarrow$ denotes a a homomorphism of functors (natural transformation).
The symbol $\iff$ is an abbreviation for "if and only if".
The symbol $\therefore$ denotes the end of a proof.

Chapter I

Preliminaries

1 Properties of categories and functors

We refer to [Wei94] for basic category theoretic notions and terminology. In the following, all categories and functors are assumed to be additive.

Let R be a commutative ring.

Definition 1.1. A category is *R-linear* if all Hom-groups are endowed with structures of R-modules such that composition is R-bilinear. A functor between R-linear categories is *R-linear* if it commutes with the respective R-module structures on Hom-groups.

Let $R \longrightarrow R'$ be a homomorphism of commutative rings.

Definition 1.2. Let $\mathscr{C}$ be an R-linear category, $\mathscr{C}'$ be an R'-linear category. An R-linear functor $V : \mathscr{C} \to \mathscr{C}'$ is called *R'/R-fully faithful* if for every pair X, Y of objects of $\mathscr{C}$ the R'-linear homomorphism

$$R' \otimes_R \mathrm{Hom}_{\mathscr{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathscr{C}'}(VX, VY)$$

induced by V is an isomorphism of R'-modules.

Definition 1.3. Let $\mathscr{C}$ be an R-linear category. The *additive scalar extension* of $\mathscr{C}$ from R to R' is the category $R' \odot_R \mathscr{C}$ which has the same objects as $\mathscr{C}$ and for which

$$\mathrm{Hom}_{R'\odot_R\mathscr{C}}(X, Y) := R' \otimes_R \mathrm{Hom}_{\mathscr{C}}(X, Y) \qquad \text{for all } X, Y \text{ in } \mathscr{C}.$$

Clearly, $R' \odot_R \mathscr{C}$ is an R'-linear category, and we have a natural R-linear R'/R-fully faithful functor $\mathscr{C} \longrightarrow R \odot_R \mathscr{C}$. Moreover, it has the following characterising universal property: For every R'-linear category $\mathscr{C}'$ and every R-linear functor

$\mathscr{C} \longrightarrow \mathscr{C}'$ there is a unique R'-linear functor $R' \odot_R \mathscr{C} \longrightarrow \mathscr{C}'$ extending the given functor, in the sense that it factors as

$$\mathscr{C} \longrightarrow R' \odot_R \mathscr{C} \longrightarrow \mathscr{C}'.$$

Note that if $\mathscr{C}$ is abelian, then $R' \odot_R \mathscr{C}$ is usually not abelian. We will come back to this question in Definition 23.11.

Definition 1.4. In these and the following definitions of this section, for more precise definitions we refer to [Del82] and [Del90].

(a) A *tensor category* is a category $\mathscr{T}$ equipped with a bilinear functor

$$\otimes : \mathscr{T} \times \mathscr{T} \longrightarrow \mathscr{T}$$

and sufficiently many (associativity, commutativity and unity) constraints such that the tensor product of an unordered finite set of objects is well-defined. In particular, there exists a unit object $\mathbf{1}$. One tends to suppress mention of the constraints.

(b) An *abelian tensor category* is an tensor category which is abelian and whose tensor product is right exact.

(c) A *tensor category over R* is a tensor category $(\mathscr{T}, \otimes)$ equipped with a ring isomorphism $R \longrightarrow \mathrm{End}(\mathbf{1})$. Using this isomorphism and the constraints, $\mathscr{T}$ becomes R-linear and $\otimes$ R-bilinear [Del82, Remark after Definition 1.15].

(d) A *tensor functor* is a functor $\mathscr{T} \xrightarrow{\omega} \mathscr{T}'$ between two tensor categories $\mathscr{T}$ and $\mathscr{T}'$ equipped with tensor constraints, that is, functorial isomorphisms $\omega(X) \otimes \omega(Y) \longrightarrow \omega(X \otimes Y)$ compatible with with the associativity, commutativity and unity constraints of $\mathscr{T}$ and $\mathscr{T}'$.

(e) A *morphism of tensor functors* $\omega, \omega' : \mathscr{T} \to \mathscr{T}'$ is a natural transformation $\eta : \omega \Longrightarrow \omega'$ commuting with the respective tensor constraints. We let $\mathrm{Hom}^{\otimes}(\omega, \omega')$ denote the set of morphisms of tensor functors $\omega \Rightarrow \omega'$, and let $\mathrm{Aut}^{\otimes}(\omega)$ denote the set auf tensor automorphisms of ω.

If $\mathscr{T}$ is a tensor category (over R), then the opposite category $\mathscr{T}^{\mathrm{op}}$ inherits a structure of tensor category (over R) by setting $X^{\mathrm{op}} \otimes Y^{\mathrm{op}} := (X \otimes Y)^{\mathrm{op}}$ for $X, Y \in \mathscr{T}$.

If a tensor functor $\mathscr{T} \to \mathscr{T}'$ is an equivalence of categories, there exists a tensor functor $\mathscr{T}' \to \mathscr{T}$ such that the both possible compositions are isomorphic as tensor functors to the respective identity functors [Del82, Proposition 1.11].

Definition 1.5. (a) An object X of a tensor category is *dualisable* if there exists an object $X^\vee$ (a *dual* of X) and homomorphisms $\delta_X : \mathbf{1} \to X \otimes X^\vee$ and $\mathrm{ev}_X : X^\vee \otimes X \to \mathbf{1}$ such that the composite homomorphisms $X \to X \otimes X^\vee \otimes X \to X$ and $X^\vee \to X^\vee \otimes X \otimes X^\vee \to X^\vee$ are equal to the respective identities. If X is dualisable, then so is $X^\vee$ and one has a canonical isomorphism $X \cong X^{\vee\vee}$. If both X and Y are dualisable, then $X \otimes Y$ is dualisable and one has a canonical isomorphism $(X \otimes Y)^\vee \cong X^\vee \otimes Y^\vee$.

(b) If X is dualisable, and Y is any other object, we set

$$\mathbf{Hom}(X, Y) := X^\vee \otimes Y,$$

and call this object the *inner Hom* of X and Y. The existence of isomorphisms $\mathrm{Hom}(Z \otimes X, Y) \to \mathrm{Hom}(Z, \mathbf{Hom}(X, Y))$, natural in Z, follows. In particular, one has $\mathrm{Hom}(X, Y) = \mathrm{Hom}(\mathbf{1}, \mathbf{Hom}(X, Y))$.

(c) A tensor category is *rigid* if every object is dualisable.

If $\mathscr{T}$ is a rigid tensor category, then dualisation extends [Del82, Remark after Definition 1.7] to a tensor equivalence of categories $\mathscr{T}^{\mathrm{op}} \longrightarrow \mathscr{T}$, mapping an object $X^{\mathrm{op}} \in \mathscr{T}^{\mathrm{op}}$ to $X^\vee$, and the opposite of a homomorphism $X \xrightarrow{f} Y$ in $\mathscr{T}$ to the unique map $f^\vee : Y^\vee \to X^\vee$ satisfying

$$\mathrm{ev}_Y \circ (\mathrm{id}_{Y^\vee} \otimes f) = \mathrm{ev}_X \circ (f^\vee \otimes \mathrm{id}_X) : Y^\vee \otimes X \longrightarrow \mathbf{1}.$$

We remark that if $\mathscr{T}$ is an abelian rigid tensor category, then its tensor product is exact in both variables [Del82, Proposition 1.16]. Furthermore, every morphism of tensor functors between two given rigid tensor categories is an isomorphism [Del82, Proposition 1.13].

Definition 1.6. Let $\mathscr{T}$ be an abelian rigid tensor category. For every object $X \in \mathscr{T}$, we let $⟪X⟫_\otimes$ denote the smallest full subcategory of $\mathscr{T}$ containing X and closed under subquotients, tensor products, and duals. If $\mathscr{T} = ⟪X⟫_\otimes$ for some object X of $\mathscr{T}$, we say that $\mathscr{T}$ is *finitely generated* as a rigid abelian tensor category.

Definition 1.7. An abelian category $\mathscr{A}$ is *finite* if every object has a composition series of finite length.

Definition 1.8. Let F be a field. An F-linear abelian category $\mathscr{A}$ is *F-finite* if it is finite and for every pair X, Y of objects of $\mathscr{A}$ the F-vector space $\mathrm{Hom}_{\mathscr{A}}(X, Y)$ is finite-dimensional.

2 Semisimplicity of objects

Let $\mathscr{A}$ be an abelian category. An object of $\mathscr{A}$ is *simple* if it is non-zero, and has no non-trivial subquotients other than itself. It is *semisimple* if it is (isomorphic to) a direct sum of simple objects. In general, of course, an abelian category has non-semisimple objects. We let $\mathscr{A}^{\mathrm{ss}}$ denote the full abelian subcategory of $\mathscr{A}$ consisting of the semisimple objects of $\mathscr{A}$.

An object $X \in \mathscr{A}$ is *finite* if it has a composition series of finite length, i.e., there is a finite exhaustive filtration

$$0 = X^0 \subset X^1 \subset \cdots \subset X^\ell = X$$

of X, such that every successive subquotient X_{i+1}/X_i is simple. The length $\lg(X) := \ell$ of such a series is well-defined, and called the *length* of the object X.

For the remainder of this section, we will assume that $\mathscr{A}$ is *finite*, meaning that all of its objects are finite.

Definition 2.1. Let X be an object of $\mathscr{A}$. The *socle* $\mathrm{soc}(X)$ of X is the sum of its simple subobjects, i.e., its largest semisimple subobject. We define the (ascending) *socle filtration* of X as follows: We set $\mathrm{soc}^0(X) := 0$, $\mathrm{soc}^1(X) := \mathrm{soc}(X)$. For $i \geq 1$ we consider the homomorphism

$$X \xrightarrow{\pi_i} X/\mathrm{soc}^i(X)$$

and set $\mathrm{soc}^{i+1}(X) := \pi_i^{-1}(\mathrm{soc}(X/\mathrm{soc}^i(X)))$. The *socle length* of X is the smallest integer $\mathrm{slg}(X)$ such that $\mathrm{soc}^{\mathrm{slg}(X)}(X) = X$.

Somewhat dually, the *radical* $\mathrm{rad}(X)$ of X is the intersection of the kernels of homomorphisms from X to a simple object, i.e., the kernel of the homomorphism to its largest semisimple quotient object.

So, by definition, X is semisimple if and only if $X = \mathrm{soc}(X)$. Similarly, X is semisimple if and only if $\mathrm{rad}(X) = 0$.

Proposition 2.2. *(a) The assignments* soc *and* rad, *and the socle filtration are functorial.*

(b) The functor soc *is right adjoint to the inclusion of categories $\mathscr{A}^{\mathrm{ss}} \subset \mathscr{A}$. In particular, it is left exact.*

Proof. (a): Let us show that given a homomorphism $X \xrightarrow{f} Y$ of objects of $\mathscr{A}$, then $f(\mathrm{soc}\, X) \subset \mathrm{soc}(Y)$. By definition, $\mathrm{soc}(X)$ is the sum of the simple subobjects of X, hence we may restrict to such a simple subobject $S \subset X$. Then $f(S)$ is either zero or isomorphic to S, and is in any case contained in a simple subobject of Y, hence $f(S) \subset \operatorname{soc}(Y)$. Hence soc is a functor. It follows by induction that the socle filtration is functorial. The proof that rad is a functor is dual.

(b): We must show that for every semisimple object X and every object Y the homomorphism

$$\operatorname{Hom}(X, Y) \to \operatorname{Hom}(X, \operatorname{soc} Y)$$

is a bijection. For $f \in \operatorname{Hom}(X, Y)$ the object $f(X)$ is semisimple as a quotient object of X, so we have $f(X) \subset \operatorname{soc}(Y)$ and the homomorphism is welldefined. It follows that it is a bijection, since we may extend any element of $\operatorname{Hom}(X, \operatorname{soc}(Y))$ by post-composition with the inclusion $\operatorname{soc}(Y) \subset Y$. $\therefore$

Definition 2.3. The *semisimplification* X^{ss} of an object $X \in \mathscr{A}$ is the object underlying the graded object associated to the socle filtration of X, i.e.,

$$X^{\mathrm{ss}} := \bigoplus_{i \geq 0} \operatorname{soc}^{i+1}(X) / \operatorname{soc}^{i}(X).$$

By Proposition 2.2(a), this extends to a functor $(-)^{\mathrm{ss}} : \mathscr{A} \to \mathscr{A}^{\mathrm{ss}}$ of semisimplification.

3 Semisimplicity of functors

Let F'/F be a field extension. We consider an F-linear abelian category $\mathscr{A}$, an F'-linear abelian category $\mathscr{B}$, and an F-linear additive functor

$$\mathscr{A} \xrightarrow{V} \mathscr{B}.$$

Definition 3.1. V is *semisimple* if it maps semisimple objects in $\mathscr{A}$ to semisimple objects in $\mathscr{B}$.

For the rest of this section, we assume that V is exact and F'/F-fully faithful (cf. Definition 1.2). This implies that V maps non-zero objects to non-zero objects.

Lemma 3.2. *Assume that V is exact and F'/F-fully faithful. Let $\alpha : 0 \to A' \to A \to A'' \to 0$ be a short exact sequence in $\mathscr{A}$. Then α splits if and only if $V(\alpha)$ splits.*

Proof. Clearly, if α splits, then so does $V(\alpha)$.

Conversely, let us assume that $V(\alpha)$ splits. It suffices to show that $\mathrm{id}_{A''}$ is in the image of the natural homomorphism $\operatorname{Hom}_{\mathscr{A}}(A'', A) \to \operatorname{Hom}_{\mathscr{A}}(A'', A'')$. This image coincides with the intersection of $\operatorname{Hom}_{\mathscr{A}}(A'', A'')$ and the image of the natural homomorphism $\operatorname{Hom}_{\mathscr{A}}(A'', A) \otimes_F F' \to \operatorname{Hom}_{\mathscr{A}}(A'', A'') \otimes_F F'$. Moreover, by F'/F-full faithfulness, we may identify this latter image with the image of the natural homomorphism $\mathrm{Hom}_{\mathscr{B}}(V(A''), V(A)) \to \mathrm{Hom}_{\mathscr{B}}(V(A''), V(A''))$. By assumption, $\mathrm{id}_{V(A'')} = V(\mathrm{id}_{A''})$ is an element of this image, and under our natural identifications it is also clearly an element of $\mathrm{Hom}_{\mathscr{A}}(A'', A'')$, therefore we are done. ∴

Remark 3.3. One might paraphrase the "if" direction of Lemma 3.2 by saying that the homomorphism

$$V:\ \mathrm{Ext}^1(A'', A') \longrightarrow \mathrm{Ext}^1(VA'', VA')$$

induced by V on the Yoneda groups of extension classes is *injective*.

For the rest of this section, assume that both $\mathscr{A}$ and $\mathscr{B}$ are *finite*, in the sense that all objects have finite length.

Theorem 3.4. *Assume that $\mathscr{A}$ and $\mathscr{B}$ are finite, and that V is exact and F'/F-fully faithful. The following properties of V are equivalent:*

(a) For every semisimple object A of $\mathscr{A}$, the object $V(A)$ is semisimple.

(b) For every object A of $\mathscr{A}$, we have that A is semisimple if and only if $V(A)$ is semisimple.

(c) For every object A of $\mathscr{A}$, we have $V(\mathrm{soc}\,A) = \mathrm{soc}(VA)$.

If $F' = F$, the above properties are also equivalent to each of the following:

(d) For every simple object A of $\mathscr{A}$, the object $V(A)$ is simple.

(e) For every object A of $\mathscr{A}$, we have that A is simple if and only if $V(A)$ is simple.

Proof. The implication (a) $\Longrightarrow$ (b) follows from Lemma 3.2, whereas the implication (b) $\Longrightarrow$ (a) is clear.

The implication (c) $\Longrightarrow$ (a) follows directly: If A is semisimple, then $A = \mathrm{soc}(A)$, so by (c) we have $V(A) = V(\mathrm{soc}\,A) = \mathrm{soc}(VA)$, which implies that $V(A)$ is semisimple.

The hard work is in the implication (b) $\Longrightarrow$ (c). I thank my advisor Richard Pink for his help with this proof. If A is semisimple, then $V(A)$ is also semisimple by (b), so we have $\mathrm{soc}(VA) = V(A) = V(\mathrm{soc}\,A)$. We may apply this to the semisimple object $\mathrm{soc}(A)$, which gives $V(\mathrm{soc}\,A) = \mathrm{soc}(V(\mathrm{soc}\,A)) \subset \mathrm{soc}(VA)$, so we have $V(\mathrm{soc}\,A) \subset \mathrm{soc}(VA)$ in the general case.

It remains to show that $\mathrm{soc}(VA) \subset V(\mathrm{soc}\,A)$ for non-semisimple A. Consider a nonsplit short exact sequence

$$\alpha: \quad 0 \to A' \to A \to A'' \to 0$$

in $\mathscr{A}$, where A'' is simple. We claim that in such a situation we have $\operatorname{soc}(VA) \subset \operatorname{soc}(VA')$. Let us, for a moment, take this claim for granted. By induction over the length of A, we may assume that $\operatorname{soc}(VA') \subset V(\operatorname{soc} A')$. Combining this with the claim, we obtain

$$\operatorname{soc}(VA) \subset \operatorname{soc}(VA') \subset V(\operatorname{soc} A') \subset V(\operatorname{soc} A),$$

and we are done.

Let us prove the claim. If it is false, then there exists a simple subobject $B' \subset \operatorname{soc}(VA)$ not contained in $\operatorname{soc}(VA')$. It is then also not contained in $V(A')$, therefore the natural map $\psi : B' \to V(A'')$ is a monomorphism. Since A'' is simple, by (b) the object $V(A'')$ is semisimple, so ψ has a retraction ϕ. We shall show that this implies that our original short exact sequence α splits – a contradiction.

Set $E'' := \operatorname{End}_{\mathscr{A}}(A'')$. Since A'' is simple, this is a skew field. The natural homomorphism

$$\operatorname{Hom}_{\mathscr{A}}(A'', A) \to \operatorname{End}_{\mathscr{A}}(A'') = E'' \tag{3.5}$$

is E''-linear, if we equip both sides with the right E''-module structure given by pre-composition. Therefore, its image is either 0 or E''. In the latter case, $\operatorname{id}_{A''}$ is in the image, the short exact sequence α splits, and we obtain our desired contradiction.

Now the image of the homomorphism (3.5) is zero if and only if the image of its scalar extension to F'

$$\operatorname{Hom}_{\mathscr{B}}(VA'', VA) = \operatorname{Hom}_{\mathscr{A}}(A'', A) \otimes_F F' \longrightarrow \operatorname{End}_{\mathscr{A}}(A'') \otimes_F F' = \operatorname{End}_{\mathscr{B}}(VA'')$$

is zero. But the element $V(A'') \xrightarrow{\phi} B' \subset V(A)$ of the left hand side maps to the projection of $V(A'')$ onto its direct factor B', which is a nonzero element of the right hand side $\operatorname{End}_{\mathscr{B}}(VA'')$. So we have proven our claim, and thereby the implication (b) $\Longrightarrow$ (c).

Let us now assume that $F' = F$. Since V is exact and fully faithful, if an object A is non-simple, then so is $V(A)$, so (d) and (e) are equivalent. By additivity of V, property (d) implies property (a). Conversely, given property (a) and a simple object A of $\mathscr{A}$, we know that $V(A)$ is semisimple. However, since V is fully faithful, $\operatorname{End}_{\mathscr{B}}(VA) = \operatorname{End}_{\mathscr{A}}(A)$ is a skew field, so $V(A)$ is simple. ∴

Proposition 3.6. *Assume that $\mathscr{A}$ and $\mathscr{B}$ are finite, $F' = F$ and that V is exact and fully faithful. If the essential image of V is closed under subquotients in $\mathscr{B}$, then V is semisimple.*

Proof. It is enough to show that if A is a simple object of $\mathscr{A}$, then VA is simple, by definition of semisimplicity of functors or using Theorem 3.4. Since V is faithful and exact, VA is not zero.

Assume that VA is not simple, then there exists an exact a nonzero object $B' \subset VA$ such that B'/VA is not zero. Since the image of V is closed under subquotients, there exists a (non-zero!) object A' in $\mathscr{A}$ such that $B' \cong VA'$. Since V is full, there exists a homomorphism $A' \to A$ inducing the inclusion $VA' \cong B' \subset VA$. This is a monomorphism, since otherwise its non-zero kernel is mapped to zero, which cannot happen since V is faithful and exact. On the other hand, it is not an epimorphism, since if it were, then the homomorphism $A' \to A$ would be an isomorphism, which is not the case since the induced homomorphism $VA' \to VA$ is not an isomorphism. Therefore, we have found a non-trivial subobject A' of the simple object A, a contradiction. $\therefore$

4 Semilinear algebra

We shall call *bold ring* a pair $\boldsymbol{R} = (R, \sigma)$ consisting of a commutative ring R and an injective flat ring endomorphism σ of R. The *scalar ring* of $\boldsymbol{R}$ is the subring $R^\sigma := \{r \in R : \sigma(r) = r\}$ of σ-invariants of R. We fix such a bold ring $\boldsymbol{R}$ throughout this section.

A homomorphism of bold rings is a ring homomorphism of the underlying rings that commutes with the respective ring endomorphisms.

Here are some constructions with bold rings: Given three bold rings $\boldsymbol{R}_0 = (R_0, \sigma_0)$, $\boldsymbol{R}_1 = (R_1, \sigma_1)$ and $\boldsymbol{R}_2 = (R_2, \sigma_2)$ together with homomorphisms of bold rings $f_i : \boldsymbol{R}_0 \to \boldsymbol{R}_i$ for $i = 1, 2$, then $\boldsymbol{R}_1 \otimes_{\boldsymbol{R}_0} \boldsymbol{R}_2 := (R_1 \otimes_{R_0} R_2, \sigma_1 \otimes \sigma_2)$ is a bold ring.

If $\boldsymbol{R}_2 = (R_2, \sigma_2)$ is a bold ring, R_0 is a subring of the ring of scalars of $\boldsymbol{R}_2$, and R_1 is a commutative R_0-algebra, then $R_1 \otimes_{R_0} \boldsymbol{R}_2 := (R_1 \otimes_{R_0} R_2, \mathrm{id} \otimes \sigma_2)$ is a bold ring.

Definition 4.1. An $\boldsymbol{R}$-*module* is a pair $\boldsymbol{M} = (M, \tau)$ consisting of an R-module M and a σ-linear homomorphism $\tau : M \longrightarrow M$, that is, an additive homomorphism such that

$$\tau(r \cdot m) = \sigma(r) \cdot \tau(m) \quad \forall\, r \in R,\ m \in M.$$

A *homomorphism of* $\boldsymbol{R}$-*modules* is an R-linear homomorphism of the underlying R-modules that commutes with the respective σ-linear endomorphisms. We denote the abelian category of $\boldsymbol{R}$-modules as $\boldsymbol{R}$-Mod.

An $\boldsymbol{R}$-module $\boldsymbol{M} = (M, \tau)$ is *finitely generated* if M is finitely generated as an R-module. If R is Noetherian, we let $\boldsymbol{R}$-mod denote the full abelian subcategory of finitely generated $\boldsymbol{R}$-modules of $\boldsymbol{R}$-Mod.

If $\boldsymbol{M}$ is an $\boldsymbol{R}$-module, and $n \geq 0$, then (M, τ^n) is a (R, σ^n)-module.

If $\boldsymbol{M}$ is an $\boldsymbol{R}$-module, its *invariant submodule* is the R^σ-submodule

$$M^\tau := \{m \in M : \ \tau(m) = m\}$$

of M. Clearly, $(-)^\tau$ extends to a left-exact covariant functor from $\boldsymbol{R}$-modules to R^σ-modules.

Note that if σ is not surjective, then the image of τ for a given $\boldsymbol{R}$-module $\boldsymbol{M}$ is in general not an R-submodule of M. However, for every $\boldsymbol{R}$-module $\boldsymbol{M}$ we let

$$\mathrm{Lie}^*(\boldsymbol{M}) := M/R \cdot \tau(M)$$

denote the R-module quotient of M by the R-submodule of M generated by the image of τ. Clearly, Lie^* extends to a right-exact covariant functor from $\boldsymbol{R}$-modules to R-modules.

Given two $\boldsymbol{R}$-modules $\boldsymbol{M} = (M, \tau_M)$ and $\boldsymbol{N} = (N, \tau_N)$, their *tensor product*

$$\boldsymbol{M} \otimes_{\boldsymbol{R}} \boldsymbol{N} := (M \otimes_R N, \tau)$$

is the R-module $M \otimes_R N$, equipped with the diagonal σ-linear endomorphism mapping $m \otimes n \in \otimes_R N$ to $\tau(m \otimes n) := \tau_M(m) \otimes \tau_N(n)$.

The *unit* object of $\boldsymbol{R}$-Mod is $\mathbf{1} = (R, \sigma)$.

Proposition 4.2. *The category* $\boldsymbol{R}$-Mod *is an abelian tensor category over* R^σ*. If* R *is Noetherian, then so is* $\boldsymbol{R}$-mod*.*

Proof. This follows from the well-known fact that the category of R-modules is an abelian tensor category over R. ∴

Let $\boldsymbol{R} \xrightarrow{f} \boldsymbol{R}' = (R', \sigma')$ be a homomorphism of bold rings. To any $\boldsymbol{R}$-module $\boldsymbol{M}$ we may associate the $\boldsymbol{R}'$-module $f_*\boldsymbol{M} := \boldsymbol{R}' \otimes_{\boldsymbol{R}} \boldsymbol{M} := (R' \otimes_R M, \sigma' \otimes \tau)$, the *base extension* of $\boldsymbol{M}$ along f. On the other hand, any $\boldsymbol{R}'$-module $\boldsymbol{M}'$ gives rise to an $\boldsymbol{R}$-module $f^*\boldsymbol{M}'$ via *restriction* along f. By abuse of notation, we sometimes denote this $\boldsymbol{R}$-module as $\boldsymbol{M}'$ as well. Both base extension and restriction extend to covariant functors in $\boldsymbol{M}$. If $\boldsymbol{R}' \xrightarrow{g} \boldsymbol{R}''$ is another homomorphism of bold rings, then we have $(gf)_* = g_* f_*$ and $(gf)^* = f^* g^*$.

Remark 4.3. Other points of view towards $\boldsymbol{R}$-modules are sometimes useful:

(a) **Noncommutative algebra:** To $\boldsymbol{R}$ we can associate the (in general) noncommutative ring $R\{\tau\}$ freely generated by R and τ, subject to the noncommutation rule $\tau \cdot r = \sigma(r) \cdot \tau$ for $r \in R$ (cf. [Gos96] or [Tha04]). Then $\boldsymbol{R}$-Mod is equivalent to the category of left $R\{\tau\}$-modules. If R is a field, then $R\{\tau\}$ is a left principal ideal ring, in particular $R\{\tau\}$ is Noetherian.

A subset $S \subset M$ is said to $\boldsymbol{R}$-*generate* $\boldsymbol{M}$ if it generates M as an $R\{\tau\}$-module. If this can be accomplished with a finite subset, we say that $\boldsymbol{M}$ is *finitely* $\boldsymbol{R}$-*generated*. Note that this property is weaker than the property of being finitely generated.

(b) **Linearisation:** Given a bold ring $\boldsymbol{R} = (R, \sigma)$ and an R-module M, let $\sigma_* M := R \otimes_{\sigma,R} M$ denote the base extension of M along σ. Then we have a natural σ-linear homomorphism $1 \otimes \mathrm{id} : M \longrightarrow \sigma_* M$.

If $\tau_{\mathrm{lin}} : \sigma_* M \longrightarrow M$ is R-linear, then $\tau := \tau_{\mathrm{lin}} \circ (1 \otimes \mathrm{id}) : M \longrightarrow M$ is σ-linear, so $\boldsymbol{M} = (M, \tau)$ is an $\boldsymbol{R}$-module. Conversely, if $\boldsymbol{M} = (M, \tau)$ is an $\boldsymbol{R}$-module, then $\tau_{\mathrm{lin}} := \mathrm{id} \otimes \tau : \sigma_* M \longrightarrow M$ is R-linear.

All in all, the datum of an $\boldsymbol{R}$-module $\boldsymbol{M} = (M, \tau)$ is equivalent to the datum consisting of the R-module M together with the *linearisation* τ_{lin} of τ.

Note that we have $\mathrm{Lie}^*(\boldsymbol{M}) = \mathrm{coker}(\tau_{\mathrm{lin}})$.

We let $\tau^n_{\mathrm{lin}} : (\sigma^n)_* M \to M$ denote the linearisation of τ^n.

(c) **Matrices:** Let $\boldsymbol{M} = (M, \tau)$ be an $\boldsymbol{R}$-module such that M is free of finite rank n over R. Identifying M and R^n by choice of a basis, τ corresponds to the homomorphism $R^n \longrightarrow R^n$, $(r_i) \mapsto \Delta \cdot (\sigma(r_i))$ for some matrix $\Delta \in \mathrm{Mat}_{n \times n}(R)$. Moreover, the natural corresponding choice of basis of $\sigma_* M$ lets τ_{lin} correspond to the homomorphism $R^n \longrightarrow R^n$, $(r_i) \mapsto \Delta \cdot (r_i)$.

5 Global bold rings and their modules

In this thesis we will have to deal with a great abundance of bold rings. Several structural results, which are false for general bold rings and their modules, but true for the bold rings we use, reappear in various places. Hence we try to distil these common properties by defining “global” bold rings and “nondegenerate” modules.

Definition 5.1. A *global bold ring* (resp. *local bold ring*, resp. *bold field*) is a bold ring $\boldsymbol{R} = (R, \sigma)$ with the following properties:

(a) R^σ is a Dedekind domain (resp. local Dedekind domain, resp. field).

(b) R decomposes as $R = R_1 \times \cdots \times R_s$ for R-subalgebras $R_i \subset R$ such that

- (b_1) The factors R_i are Dedekind domains (resp. semilocal Dedekind domains, resp. fields).
- (b_2) σ permutes these factors transitively, in the sense that $\sigma(R_i) \subset R_{i-1}$ for $i \in \mathbb{Z}/s$.

Note that we have the following inclusions:

$$\{\text{bold fields}\} \subset \{\text{local bold rings}\} \subset \{\text{global bold rings}\} \subset \{\text{bold rings}\}.$$

Our motivation for calling a global bold ring "global" is that we seek to study a module $\boldsymbol{M}$ over a global bold ring $\boldsymbol{R}$ by means of its "localisations" $\boldsymbol{R}' \otimes_{\boldsymbol{R}} \boldsymbol{M}$ for local bold rings (or even bold fields) $\boldsymbol{R}' \supset \boldsymbol{R}$.

The reader might wish to consult the next section which contains the examples of global bold rings we are interested in, in order to get a feeling for what a global bold ring might be.

It would be easier to deal with global bold rings for which R is connected, but the applications we have in mind call for the generality given in Definition 5.1.

For the rest of this section, assume that $\boldsymbol{R} = (R, \sigma)$ is a global bold ring. We will be considering only finitely generated $\boldsymbol{R}$-modules, since this allows us to use the structure theory of finitely generated modules over Dedekind domains (cf. [Jac90, Section 10.6]). And this we will do freely.

There is a unique extension of σ to a ring endomorphism of $\mathrm{Frac}(R)$, the total ring of fractions of R. We let $\mathrm{Frac}(\boldsymbol{R})$ denote this bold ring, it is a bold field. We have several basic definitions to make.

Given a finitely generated R-module M the decomposition $R = R_1 \times \cdots \times R_s$ gives a decomposition $M = M_1 \times \cdots \times M_s$, where $M_i = R_i \otimes_R M$. We set

$$\mathrm{Tor}(M) := \mathrm{Tor}(M_1) \times \cdots \times \mathrm{Tor}(M_s),$$

where $\mathrm{Tor}(M_i) := \{m \in M_i \mid \exists\, 0 \neq r \in R_i : \ r \cdot m = 0\}$, the usual notion of torsion for modules over Dedekind domains. One says that M is *torsion* if $M = \mathrm{Tor}(M)$, and *torsion-free* if $\mathrm{Tor}(M) = 0$.

Definition 5.2. An $\boldsymbol{R}$-module $\boldsymbol{M} = (M, \tau)$ is *non-degenerate* if it is finitely generated and both kernel and cokernel of τ_{lin} are torsion R-modules.

Proposition 5.3. *The full subcategory of non-degenerate $\boldsymbol{R}$-modules (in $\boldsymbol{R}$-*Mod*) is an abelian tensor category over R^σ.*

Proof. By Proposition 4.2 the category of finitely generated $\boldsymbol{R}$-modules is an abelian tensor category, since R is Noetherian. The unit $\mathbf{1} = (R, \sigma)$ is non-degenerate. One checks that the category in question is closed under subquotients and tensor products. ∴

There are two particular (extremal) types of non-degenerate $\boldsymbol{R}$-modules:

Definition 5.4. Consider a non-degenerate $\boldsymbol{R}$-module $\boldsymbol{M} = (M, \tau)$.

(a) $\boldsymbol{M}$ is *torsion* if $\mathrm{Tor}(M) = M$.

(b) $\boldsymbol{M}$ is *torsion-free* if $\mathrm{Tor}(M) = 0$.

Proposition 5.5. *For every non-degenerate $\boldsymbol{R}$-module $\boldsymbol{M}$:*

(a) Tor *takes values in torsion* $\boldsymbol{R}$*-modules functorially.*

(b) $\mathrm{Tor}(\boldsymbol{M})$ *is the largest torsion* $\boldsymbol{R}$*-submodule of* $\boldsymbol{M}$*, and*

(c) $\boldsymbol{M}/\mathrm{Tor}(\boldsymbol{M})$ *is the largest torsion-free* $\boldsymbol{R}$*-module quotient of* $\boldsymbol{M}$*.*

Proof. (a): Clearly, we have $\mathrm{Tor}(\mathrm{Tor}(M)) = \mathrm{Tor}(M)$, so $\mathrm{Tor}(M)$ is a torsion R-module. We must check that $\mathrm{Tor}(M)$ is τ-stable: Consider $m_i \in \mathrm{Tor}(M_i)$, so there exists $0 \neq r \in R_i$ such that $r \cdot m_i = 0$. But then $0 = \tau(rm_i) = \sigma(r)\tau(m_i)$, and $0 \neq \sigma(r) \in R_{i-1}$ since σ is injective and fulfills Definition 5.1(b_2).

(b): By definition, $\mathrm{Tor}(\boldsymbol{M})$ is the largest torsion R-submodule of M, and item (a) shows that it is τ-stable.

(c): By the structure theory of modules over Dedekind domains, $\boldsymbol{M}/\mathrm{Tor}(\boldsymbol{M})$ is the largest torsion-free R-module quotient of M, and item (a) together with Proposition 5.3 implies that it is an $\boldsymbol{R}$-module quotient of $\boldsymbol{M}$. ∴

There are two particular (extremal) types of torsion modules:

Definition 5.6. Consider a finitely generated torsion $\boldsymbol{R}$-module $\boldsymbol{M} = (M, \tau)$.

(a) $\boldsymbol{M}$ is *bijective* if τ_{lin} is bijective. (Other authors call such a module *etale*).

(b) $\boldsymbol{M}$ is *nilpotent* if τ_{lin} is nilpotent (cf. Remark 4.3(b)).

To obtain a filtration of every (finitely generated) torsion module by bijectives and nilpotents, we need an additional assumption.

Definition 5.7. The global bold ring $\boldsymbol{R}$ *has a base field* if there exists a σ-stable subfield $K \subset R$ such that

(a) A finitely generated $\boldsymbol{R}$-module $\boldsymbol{M}$ is torsion if and only if $\dim_K(M)$ is finite.

(b) For every such torsion $\boldsymbol{R}$-module $\boldsymbol{M}$, the natural homomorphism $(\sigma_{|K})_* M \to \sigma_* M$ is an isomorphism.

Given a torsion $\boldsymbol{R}$-module $\boldsymbol{M}$, set $\boldsymbol{M}^{\mathrm{bij}} := \bigcap_{n\geq 0} \mathrm{im}(\tau_{\mathrm{lin}}^n)$ and $\boldsymbol{M}^{\mathrm{nil}} := \boldsymbol{M}/\boldsymbol{M}^{\mathrm{bij}}$.

Proposition 5.8. *Assume that* $\boldsymbol{R}$ *has a base field. For every finitely-generated torsion* $\boldsymbol{R}$*-module* $\boldsymbol{M}$*:*

(a) $(-)^{\mathrm{bij}}$ *takes values in bijective torsion* $\boldsymbol{R}$*-modules functorially.*

(b) $\boldsymbol{M}^{\mathrm{bij}}$ *is the largest bijective* $\boldsymbol{R}$*-submodule of* $\boldsymbol{M}$*, and*

(c) $\boldsymbol{M}^{\mathrm{nil}}$ *is the largest nilpotent* $\boldsymbol{R}$*-module quotient of* $\boldsymbol{M}$*.*

Proof. (a): We note that $\operatorname{im}(\tau_{\mathrm{lin}}^n) = R \cdot \tau^n(M)$. Since M has finite length, this chain of submodules is becomes stationary, and in particular $\boldsymbol{M}^{\mathrm{bij}} = R \cdot \tau^n(M)$ for some $n \gg 0$. Therefore,

$$\boldsymbol{M}^{\mathrm{bij}} = R \cdot \tau^n(M) = R \cdot \tau^{n+1}(M) = R \cdot \tau(R \cdot \tau^n(M)) = R \cdot \tau(\boldsymbol{M}^{\mathrm{bij}}),$$

that is, the restriction of τ_{lin} to $\sigma_* \boldsymbol{M}^{\mathrm{bij}}$ is surjective. If we show that $\dim_K(\sigma_* \boldsymbol{M}^{\mathrm{bij}}) = \dim_K(\boldsymbol{M}^{\mathrm{bij}})$, this implies that the restriction of τ_{lin} to $\sigma_* \boldsymbol{M}^{\mathrm{bij}}$ is bijective.

By Definition 5.7(a), we have an isomorphism $\boldsymbol{M}^{\mathrm{bij}} \cong K^{\oplus \dim_K \boldsymbol{M}^{\mathrm{bij}}}$, so by Definition 5.7(b) we have

$$\sigma_* M = (\sigma|K)_* M = (\sigma|K)_*(K^{\oplus \dim_K M}) = ((\sigma|K)_* K)^{\oplus \dim_K(M)} = K^{\oplus \dim_K(M))}.$$

(b): Clearly, $\boldsymbol{M}^{\mathrm{bij}}$ is the largest possible bijective $\boldsymbol{R}$-submodule of $\boldsymbol{M}$.

(c): $\boldsymbol{M}/\boldsymbol{M}^{\mathrm{bij}}$ is nilpotent if $\tau^n(\boldsymbol{M}/\boldsymbol{M}^{\mathrm{bij}}) = 0$ for $n \gg 0$, that is, if $\tau^n(\boldsymbol{M}) \subset \boldsymbol{M}^{\mathrm{bij}}$ for $n \gg 0$. In the proof of (a) we have seen that this is in fact the case. ∴

Remark 5.9. If $\boldsymbol{R}$ has no base field, then Proposition 5.8 need not be true. Here is an example: $R = \mathbb{F}_p[t]$, with $\sigma(r) := r^p$. Consider $M = \mathbb{F}_p = R/(t)$, equipped with $\tau([r]) := [\sigma(r)](= [r])$. Then $\tau_{\mathrm{lin}} : \sigma_* M \to M$ is surjective, but not injective, since $\dim_{\mathbb{F}_p}(\sigma_* M) = \dim_{\mathbb{F}_p}(\mathbb{F}_p[t]/\mathbb{F}_p[t^p]) = p > 1 = \dim_{\mathbb{F}_p}(M)$.

Remark 5.10. If $\boldsymbol{R}$ is a global bold ring with bijective σ, and $\boldsymbol{M}$ is a finite $\boldsymbol{R}$-module, then $\boldsymbol{M}/\boldsymbol{M}^{\mathrm{bij}} \cong \bigcup_{n \geq 1} \ker \tau^n$, so the filtration of Proposition 5.8 splits canonically. This is what is usually known as Fitting's Lemma.

If σ is not bijective, there is no splitting in general. This situation is formally dual to the connected-etale sequence for finite group schemes over a non-perfect field.

Example 5.11. Let $R = K$ be a non-perfect field of positive characteristic, equipped with $\sigma(r) = r^p$, and choose an element $u \in K$ which is not a p-th power. Consider $\boldsymbol{M} = (Km_1 \oplus Km_2, \tau)$, with $\tau = \begin{pmatrix} 1 & u \\ 0 & 0 \end{pmatrix} \sigma$. Then clearly $\boldsymbol{M}^{\mathrm{bij}} = (Km_1, \sigma)$ and $\boldsymbol{M}^{\mathrm{nil}} \cong (K, 0)$. However, there is no element $m = \lambda m_1 + \mu m_2 \in M$ with $\tau(m) = 0$, since else the calculation

$$0 = \tau(m) = \lambda^p \tau(m_1) + \mu^p \tau(m_2) = (\lambda^p + \mu^p u) m_1$$

would imply that u is a p-th power. Therefore, $\boldsymbol{M}$ contains no copy of $\boldsymbol{M}^{\mathrm{nil}}$, and $\boldsymbol{M}^{\mathrm{nil}} \ncong \cup_{n \geq 1} \ker \tau^n$.

One may contrast this with the example $\boldsymbol{N} = (Kn_1 \oplus Kn_2, \tau)$, where $\tau = \begin{pmatrix} 0 & u \\ 0 & 1 \end{pmatrix} \sigma$. In this case, we have $\boldsymbol{N}^{\mathrm{bij}} = (K \cdot (un_1 + n_2), \sigma)$ and $\boldsymbol{N}^{\mathrm{nil}} \cong (Kn_1, 0)$. It follows that $\boldsymbol{N} \cong \boldsymbol{N}^{\mathrm{bij}} \oplus \boldsymbol{N}^{\mathrm{nil}}$.

We leave the torsion modules and turn to the structure of the R-modules underlying torsion-free nondegenerate $\boldsymbol{R}$-modules.

Proposition 5.12. *Let $\boldsymbol{M} = (M, \tau)$ be a finitely generated torsion-free $\boldsymbol{R}$-module.*

(a) $\boldsymbol{M}$ is non-degenerate if and only if τ_{lin} is injective.

(b) In this case, M is a projective R-module of (finite) constant rank.

(c) If $\boldsymbol{R}$ is a local bold ring and $\boldsymbol{M}$ is non-degenerate, then M is even a free R-module.

Proof. We have $M = M_1 \times \cdots \times M_s$, with $M_i = R_i \otimes_R M$ a projective R_i-module of finite rank r_i.

(a,b): If $\boldsymbol{M}$ is nondegenerate, the kernel of τ_{lin} must be torsion. Since the M_i are torsion-free, this implies that this kernel vanishes, so τ_{lin} is injective.

Conversely, assume that τ_{lin} is injective. Now σ maps R_{i+1} to R_i, so τ_{lin} maps

$$(\sigma_* M)_i := R_i \otimes_R (\sigma_* M) = R_i \otimes_{\sigma, R_{i+1}} M_{i+1}$$

to M_i. Since τ_{lin} is injective, this shows that $r_{i+1} \leq r_i$ for all $i \in \mathbb{Z}/s$, which implies that all r_i are equal, and proves (b).

But an injective homomorphism between projective modules of equal constant rank must have torsion cokernel, which shows that $\boldsymbol{M}$ is nondegenerate.

(c): By [Eis95, Exercise 4.13], a finitely generated module over a Noetherian semilocal ring is free if and only if it is locally free of constant rank. By (b), we may apply this to each factor M_i. ∴

Definition 5.13. Let $\boldsymbol{M}$ be a torsion-free nondegenerate $\boldsymbol{R}$-module $\boldsymbol{M}$.

(a) The *rank* $\mathrm{rk}(\boldsymbol{M})$ of $\boldsymbol{M}$ is the rank of M as R-module. By Proposition 5.12(b), this is well-defined.

(b) The *determinant* $\det(\boldsymbol{M})$ of $\boldsymbol{M}$ is the (highest non-trival) exterior power $\Lambda^{\mathrm{rk}\,\boldsymbol{M}}\boldsymbol{M}$ of $\boldsymbol{M}$. By Proposition 5.3, this is a nondegenerate $\boldsymbol{R}$-module, and it is torsion-free of rank 1.

Remark 5.14. If $\boldsymbol{R}$ is a local bold ring, it follows that a torsion free nondegenerate $\boldsymbol{R}$-module is determined by a matrix $\Delta \in \mathrm{Mat}_{\mathrm{rk}(\boldsymbol{M})\times\mathrm{rk}(\boldsymbol{M})}(R)$ with $\det(\Delta)$ a non-zerodivisor of R. Conversely, such a matrix gives rise to a torsion-free nondegenerate $\boldsymbol{R}$-module. This will be rather useful in calculations!

Proposition 5.15. *The full subcategory of torsion-free nondegenerate $\boldsymbol{R}$-modules is a tensor category over R^σ. If $\boldsymbol{R}$ is a bold field, this category is abelian.*

Proof. Follows from Proposition 5.3 and the definitions. Note that if $\boldsymbol{R}$ is a bold field, then every nondegenerate $\boldsymbol{R}$-module is torsion-free. ∴

We wish to relate torsion $\boldsymbol{R}$-modules with certain homomorphisms of torsion-free nondegenerate $\boldsymbol{R}$-modules which are "close" to being isomorphisms.

Definition 5.16. A homomorphism $f : \boldsymbol{M} \longrightarrow \boldsymbol{N}$ of non-degenerate $\boldsymbol{R}$-modules is an *isogeny* if both $\ker(f)$ and $\operatorname{coker}(f)$ are torsion $\boldsymbol{R}$-modules.

Example 5.17. Given a non-degenerate $\boldsymbol{R}$-module $\boldsymbol{M}$, every non-zero element $r \in R^{\sigma}$ gives rise to an isogeny $[r]_{\boldsymbol{M}} : \boldsymbol{M} \to \boldsymbol{M}$ by left multiplication.

As in the proof of Proposition 5.12, we see that given two torsion-free nondegenerate $\boldsymbol{R}$-modules $\boldsymbol{M}, \boldsymbol{N}$, a homomorphism $f : \boldsymbol{M} \longrightarrow \boldsymbol{N}$ is an isogeny if and only $\boldsymbol{M}$ and $\boldsymbol{N}$ have equal rank and f is injective.

There are two particular (extremal) types of such isogenies:

Definition 5.18. Let $f : \boldsymbol{M} \longrightarrow \boldsymbol{N}$ be an isogeny of torsion-free non-degenerate $\boldsymbol{R}$-modules.

(a) f is *separable* if $\operatorname{coker}(f)$ is a bijective $\boldsymbol{R}$-module.

(b) f is *purely inseparable* if $\operatorname{coker} f$ is a nilpotent $\boldsymbol{R}$-module.

Remark 5.19. Let $f : \boldsymbol{M} \to \boldsymbol{N}$ be an isogeny of torsion-free non-degenerate $\boldsymbol{R}$-modules. If $\boldsymbol{R}$ has a base field, the filtration of Proposition 5.8 lets us split f as $f' \circ f^{\mathrm{sep}}$, where f' is purely inseparable, and f^{sep} is separable. Namely, let $\boldsymbol{N}'$ be the kernel of the composite homomorphism $\boldsymbol{M} \to \boldsymbol{N} \to \operatorname{coker}(f)^{\mathrm{nil}}$, and let $f' : \boldsymbol{N}' \to \boldsymbol{N}$ be the natural inclusion. By construction, it is purely inseparable. The universal property of kernels shows that f factors through $\boldsymbol{N}'$, this is our homomorphism f^{sep}. It follows that $\operatorname{coker}(f^{\mathrm{sep}}) = \operatorname{coker}(f)^{\mathrm{bij}}$, so f^{sep} is indeed separable.

We end this section with some definitions, starting with a subset of dualisable objects of the category of torsion-free non-degenerate $\boldsymbol{R}$-modules.

Definition 5.20. A $\boldsymbol{R}$-module $\boldsymbol{M}$ is *restricted* if it is finitely-generated, torsion-free and τ_{lin} is bijective.

In particular, a restricted $\boldsymbol{R}$-module is non-degenerate. For the next definition we keep in mind the following commutative diagram, associated to any restricted $\boldsymbol{R}$-module $\boldsymbol{M}$ and element $f \in \operatorname{Hom}_R(M, R)$:

$$\begin{array}{ccc} \sigma_* M & \xrightarrow[\sigma_* f]{} & \sigma_* R \\ \downarrow{\scriptstyle \tau_{\mathrm{lin}}} & & \downarrow{\scriptstyle \sigma_{\mathrm{lin}}} \\ M & \xrightarrow[f]{} & R \end{array}$$

Definition 5.21. Let $\boldsymbol{M}$ be a restricted $\boldsymbol{R}$-module. The *dual* $\boldsymbol{R}$-module $\boldsymbol{M}^\vee$ is the R-module $M^\vee := \mathrm{Hom}_R(M, R)$ equipped with the semilinear endomorphism

$$M^\vee \to M^\vee, \quad f \mapsto \sigma_{\mathrm{lin}} \circ \sigma_* f \circ \tau_{\mathrm{lin}}^{-1}.$$

It is again a restricted $\boldsymbol{R}$-module. If $\boldsymbol{N}$ is any other $\boldsymbol{R}$-module, set $\mathbf{Hom}(\boldsymbol{M}, \boldsymbol{N}) := \boldsymbol{M}^\vee \otimes_{\boldsymbol{R}} \boldsymbol{N}$, the *inner Hom* of $\boldsymbol{M}$ and $\boldsymbol{N}$.

Lemma 5.22. *Let $\boldsymbol{M}$ be a restricted $\boldsymbol{R}$-module. For every $\boldsymbol{R}$-module $\boldsymbol{N}$, we have the formula* $\mathrm{Hom}_{\boldsymbol{R}}(\boldsymbol{M}, \boldsymbol{N}) = \mathbf{Hom}(\boldsymbol{M}, \boldsymbol{N})^\tau$.

Proof. Follows directly from the definitions. ∴

Proposition 5.23. *The full subcategory of restricted $\boldsymbol{R}$-modules is a rigid tensor category over R^σ. If $\boldsymbol{R}$ is a bold field, it is also abelian.*

Proof. The category in question is a full subcategory of the category of torsion-free nondegenerate $\boldsymbol{R}$-modules, which is a tensor category over R^σ by Proposition 5.15. It contains $\mathbf{1} = (R, \sigma)$ and is closed under tensor products, so it is also a tensor category over R^σ.

For every given restricted $\boldsymbol{R}$-module, one checks that its dual $\boldsymbol{M}^\vee$ is a dual in the categorical sense of Definition 1.5, so our category is rigid.

If $\boldsymbol{R}$ is a bold field, then every subquotient of a restricted $\boldsymbol{R}$-module is a torsion-free non-degenerate $\boldsymbol{R}$-module by Proposition 5.15. An application of the Snake-Lemma to the sequence of respective τ_{lin}'s shows that such a subquotient is also restricted, so our category is abelian. ∴

Proposition 5.24. *A torsion-free nondegenerate $\boldsymbol{R}$-module is restricted if and only if its determinant is restricted.*

Proof. We prove this result only for local bold rings, as we will not use the general case. A modification of the following argument would prove the general case.

Let us use Remark 5.14, so we may assume that our torsion-free nondegenerate $\boldsymbol{R}$-module $\boldsymbol{M}$ is given by a matrix $\Delta \in \mathrm{Mat}_{\mathrm{rk}(\boldsymbol{M})\times\mathrm{rk}(\boldsymbol{M})}(R)$. Now $\det(\boldsymbol{M})$ is given by $\det\Delta \in R$, and both $\boldsymbol{M}$ and $\det(\boldsymbol{M})$ are restricted if and only if $\det\Delta$ is invertible in R. ∴

Proposition 5.25. *Let $\boldsymbol{M}$ be a restricted $\boldsymbol{R}$-module, and consider an $\boldsymbol{R}$-submodule $\boldsymbol{M}' \subset \boldsymbol{M}$. If $\boldsymbol{M}'$ is saturated in $\boldsymbol{M}$, that is, if $\boldsymbol{M}' = (\mathrm{Frac}(\boldsymbol{R}) \otimes_{\boldsymbol{R}} \boldsymbol{M}') \cap \boldsymbol{M}$ (intersection in $\mathrm{Frac}(\boldsymbol{R}) \otimes_{\boldsymbol{R}} \boldsymbol{M}$), then $\boldsymbol{M}'$ restricted.*

Proof. The saturation of $\boldsymbol{M}'$ in $\boldsymbol{M}$ implies that $\boldsymbol{M}'' := \boldsymbol{M}/\boldsymbol{M}'$ is torsion-free. We consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \sigma_* M' & \longrightarrow & \sigma_* M & \longrightarrow & \sigma_* M'' & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \tau'_{\mathrm{lin}}} & & \big\downarrow{\scriptstyle \tau_{\mathrm{lin}}} & & \big\downarrow{\scriptstyle \tau''_{\mathrm{lin}}} & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

We use the Snake Lemma: Since τ_{lin} is injective, so is τ'_{lin}. Therefore, by Proposition 5.12, M' has constant rank, whence also M''. Since τ_{lin} is surjective, τ''_{lin} is surjective, which implies that τ''_{lin} is injective since M'' is projective of constant rank (a surjective homomorphism of projective modules over the same Dedekind ring of equal rank must be injective). Therefore, τ'_{lin} is surjective! ∴

Definition 5.26. A *bold order* of a bold field $\boldsymbol{Q}$ is a bold subring $\boldsymbol{R} \subset \boldsymbol{Q}$ such that

(a) $\boldsymbol{R}$ is a global bold ring.

(b) The inclusion induces an isomorphism $\mathrm{Frac}(\boldsymbol{R}) \to \boldsymbol{Q}$ of bold rings.

A *local bold order* of a bold field $\boldsymbol{Q}$ is a bold order $\boldsymbol{R} \subset \boldsymbol{Q}$ for which $\boldsymbol{R}$ is a local bold ring.

Definition 5.27. A *bold place* of a global bold ring $\boldsymbol{R}$ is a local bold ring extension $\mathrm{Frac}(\boldsymbol{R}) \supset \boldsymbol{R}' \supset \boldsymbol{R}$.

Definition 5.28. Let $\boldsymbol{R}$ be a bold order of a bold field $\boldsymbol{Q}$. A finitely generated $\boldsymbol{Q}$-module $\boldsymbol{M}$ is *etale at* $\boldsymbol{R}$ (or $\boldsymbol{R}$-*etale*) if there exists a restricted $\boldsymbol{R}$-module $\boldsymbol{N}$ such that $\boldsymbol{M}$ is isomorphic to $\boldsymbol{Q} \otimes_{\boldsymbol{R}} \boldsymbol{N}$.

Definition 5.29. Let $\boldsymbol{R}'$ be a bold place of a global bold ring $\boldsymbol{R}$. A nondegenerate $\boldsymbol{R}$-module $\boldsymbol{M}$ is *etale at* $\boldsymbol{R}'$ (or $\boldsymbol{R}$-*etale*) if $\boldsymbol{R}' \otimes_{\boldsymbol{R}} \boldsymbol{M}$ is a restricted $\boldsymbol{R}'$-module.

Proposition 5.30. *Let $\boldsymbol{Q}$ be a bold field, and fix a local bold order $\boldsymbol{R}$ of $\boldsymbol{Q}$. The full subcategory of $\boldsymbol{R}$-etale $\boldsymbol{Q}$-modules is a rigid abelian tensor category over R^σ.*

Proof. By definition, the category in question contains the unit $\mathbf{1} = \boldsymbol{Q}$ of the enveloping rigid abelian tensor category over R^σ (Proposition 5.23) of restricted $\boldsymbol{Q}$-modules. It also contains the dual of every object.

It remains to show that it is closed under subquotients therein: Let $\boldsymbol{M}$ be an $\boldsymbol{R}$-etale $\boldsymbol{Q}$-module, so $\boldsymbol{M} = \boldsymbol{Q} \otimes_{\boldsymbol{R}} \boldsymbol{N}$ for some restricted $\boldsymbol{R}$-module $\boldsymbol{N}$. If $\boldsymbol{M}' \subset \boldsymbol{M}$ is a submodule, then $\boldsymbol{N}' := \boldsymbol{M}' \cap \boldsymbol{N}$ is a restricted $\boldsymbol{R}$-module, and $\boldsymbol{M}' = \boldsymbol{Q} \otimes_{\boldsymbol{R}} \boldsymbol{N}'$. If $\pi : \boldsymbol{M} \to \boldsymbol{M}''$ is a quotient $\boldsymbol{Q}$-module, then $\boldsymbol{N}'' := \pi(\boldsymbol{N})$ is a restricted $\boldsymbol{R}$-module, and $\boldsymbol{M}'' = \boldsymbol{Q} \otimes_{\boldsymbol{R}} \boldsymbol{N}''$. ∴

6 Examples of bold fields and their orders

Let k be a finite field of cardinality q.

Examples 6.1. Let F be a global field of positive characteristic, with constant field k. Let K be any other field containing k.

(a) Let $\boldsymbol{K}$ denote the bold ring consisting of K equipped with the k-linear Frobenius endomorphism σ of K, mapping $\lambda \in K$ to $\sigma(\lambda) := \lambda^q$. This is a bold field, and K itself is a base field of $\boldsymbol{K}$.

(b) Set $F_K := \mathrm{Frac}(F \otimes_k K)$. The endomorphism σ of $\boldsymbol{K}$ induces an injective endomorphism $\mathrm{id} \otimes \sigma$ of $F \otimes_k K$, which extends uniquely to an endomorphism, again denoted as σ, of F_K. Then $\boldsymbol{F_K} := (F_K, \sigma)$ is a bold field: By our assumptions, F_K is a field. Clearly, σ is injective and $(F_K)^\sigma = F$, so the remaining conditions are fulfilled.

(c) For any place $\mathfrak{P}$ of F_K, let $\mathscr{O}_{(\mathfrak{P}),K} \subset F_K$ be the valuation ring corresponding to $\mathfrak{P}$. In general, this is not a bold subring of $\boldsymbol{F_K}$. However, for any place $\mathfrak{p}$ of F, the intersection $\mathscr{O}_{(\mathfrak{p}),K}$ of the rings $\mathscr{O}_{(\mathfrak{P}),K}$ for all places $\mathfrak{P}$ of F_K lying over $\mathfrak{p}$ is σ-stable. Since these primes $\mathfrak{P}$ are finite in number, $\mathscr{O}_{(\mathfrak{p}),K}$ is a semilocal ring. Clearly, $\mathscr{O}^\sigma_{(\mathfrak{p}),K} = \mathscr{O}_{(\mathfrak{p})}$, the local ring of F at $\mathfrak{p}$, and $F \otimes_{\mathscr{O}_{(\mathfrak{p})}} \mathscr{O}_{(\mathfrak{p}),K} \to F_K$ is an isomorphism. All in all, the bold ring $\boldsymbol{\mathscr{O}_{(\mathfrak{p}),K}}$ is a local bold ring, and it is a bold order of the bold field $\boldsymbol{F_K}$. Furthermore, $K \subset \mathscr{O}_{(\mathfrak{p}),K}$ is a base field.

(d) For a finite non-empty set $\{\infty_1, \ldots, \infty_s\}$ of places of F, let A denote the ring of elements of F integral outside the ∞_i, and set $A_K := A \otimes_k K$. This is a Dedekind ring, and equipped with $\mathrm{id} \otimes \sigma$, we obtain the global bold ring $\boldsymbol{A_K}$. It is a bold order of the bold field $\boldsymbol{F_K}$. Furthermore, $K \subset A_K$ is a base field. For every maximal prime $\mathfrak{p}$ of A, the bold ring $\boldsymbol{\mathscr{O}_{(\mathfrak{p}),K}}$ is a bold place of $\boldsymbol{A_K}$.

For the remaining examples, we need a calculation. Fix a separable closure k^{sep} of k. For every $n \geq 1$ let k_n denote the subfield of k^{sep} of degree n over k. Sometimes, k_∞ will denote k^{sep}. For every field $K \supset k$ and element $x \in K$ we set $\sigma_k(x) := x^q$ and $\boldsymbol{K} := (K, \sigma_k)$.

Proposition 6.2. *For $m, n \geq 1$ let $\delta := \gcd(m, n)$ and $\mu := \mathrm{lcm}(m, n)$. The two following maps are ring isomorphisms:*

$$i_x : \; k_m \otimes_k k_n \longrightarrow k_\mu^{\times\delta} \quad , \qquad x \otimes y \mapsto \left(\sigma_k^i(x) \cdot y\right)_{i=0}^{\delta-1}, \quad \textit{and}$$
$$i_y : \; k_m \otimes_k k_n \longrightarrow k_\mu^{\times\delta} \quad , \qquad x \otimes y \mapsto \left(x \cdot \sigma_k^i(y)\right)_{i=0}^{\delta-1}.$$

We start by checking that the given homomorphism of bold rings is an isomorphism in two special cases.

Lemma 6.3. *If $\delta = 1$, the homomorphism*

$$i_y : \; k_m \otimes_k k_n \to k_{mn}, \quad x \otimes y \mapsto xy$$

is an isomorphism of rings.

Proof. Since both sides are finite, the given homomorphism is bijective if it is injective. For this, it suffices to show that any k-linearly independent set of elements $x_1, \ldots, x_r \in k_m$ remains k_n-linearly independent in k_{mn}. If not, choose a counterexample $\sum_{i=1}^r x_i y_i = 0$ in k_{mn} with $y_i \in k_n$ and $r \geq 1$ minimal. We may assume that $y_r = 1$. It follows that $\sum_i x_i \sigma^m(y_i) = 0$, so by subtraction $\sum_{i=1}^{r-1} x_i(\sigma^m(y_i) - y_i) = 0$. By minimality of r, we deduce $\sigma^m(y_i) = y_i$ for all i. So $y_i \in k_m \cap k_n = k$, a contradiction. ∴

Lemma 6.4. *If $\delta = n$, the homomorphism*

$$i_y : k_m \otimes_k k_n \to k_m^{\times n}, \quad x \otimes y \mapsto (x\sigma^i(y))_{i=0}^{n-1}$$

is an isomorphism of rings.

Proof. Again, the given map is an isomorphism if it is injective. We must show that if given $y_1, \ldots y_r \in k_n$ are k-linearly independent, then the set of vectors $\{(\sigma^i(y_j))_{i=0}^{\delta-1}\}_{j=1}^r$ is k_m-linearly independent. If not, there exist $x_1, \ldots, x_r \in k_m$ with

$$\sum_{j=1}^r x_j \sigma^i(y_j) = 0 \quad \text{for all } 0 \leq i < n.$$

We may assume that $r \geq 1$ is minimal, and that $x_r = 1$. Applying σ to these equations, and using that σ^n is the identity on k_n, we deduce that

$$\sum_{j=1}^r \sigma(x_j)\sigma^i(y_j) = 0 \quad \text{for all } 0 \leq i < n.$$

Hence we find that

$$\sum_{j=1}^{r-1} (\sigma(x_j) - x_j)\sigma^i(y_j) = 0 \quad \text{for all } 0 \leq i < n.$$

By minimality of r, we find that all x_j lie in k. So the $i = 0$ case of the original equation shows that the y_j are linearly dependent, a contradiction. ∴

Proof of Proposition 6.2. The given homomorphism $i_y : k_m \otimes_k k_n \to k_\mu^\delta$ coincides with the following composite isomorphism:

$$i_{m,n} : k_m \otimes_k k_n \cong k_m \otimes_{k_\delta} (k_\delta \otimes_k k_n) \xrightarrow{\text{Lemma 6.4}} k_m \otimes_{k_\delta} k_n^\delta \cong (k_m \otimes_{k_\delta} k_n)^\delta \xrightarrow{\text{Lemma 6.3}} k_\mu^\delta.$$

Therefore it is an isomorphism of rings. The proof that i_x is an isomorphism of rings is symmetrical. ∴

Proposition 6.5. *For $m, n \geq 1$ let $\delta := \gcd(m, n)$ and $\mu := \operatorname{lcm}(m, n)$. Choose integers a, b such that $am + bn = \delta$. We consider two ring endomomorphisms σ_x, σ_y of $k_\mu^{\times\delta}$ defined as follows.*

For $\boldsymbol{z} = (z_0, \ldots, z_d) \in k_\mu^{\times\delta}$, set

$$\sigma_x(\boldsymbol{z})_i := \begin{cases} z_{i+1}, & 0 \leq i < \delta - 1 \\ \sigma_k^{bn}(z_0), & i = \delta - 1 \end{cases}$$

and

$$\sigma_y(\boldsymbol{z})_i := \begin{cases} z_{i+1}, & 0 \leq i < \delta - 1 \\ \sigma_k^{am}(z_0), & i = \delta - 1 \end{cases}$$

Then i_x induces an isomorphism of bold rings $k_m \otimes_k \boldsymbol{k_n} \to (k_\mu^{\times\delta}, \sigma_x)$, and i_y induces an isomorphism of bold rings $k_m \otimes_k \boldsymbol{k_n} \to (k_\mu^{\times\delta}, \sigma_y)$.

In particular, $k_m \otimes_k \boldsymbol{k_n}$ is a bold field.

Proof. We start by remarking that $k_\mu^{\times\delta}$ equipped with either σ_x or σ_y is a bold field. By Proposition 6.2, i_y is an isomorphism of rings. It remains to check that i_y is σ-equivariant. It suffices to check that $i_y \circ (\mathrm{id} \otimes \sigma_k) = \sigma_y \circ i_y$ on elements of the form $x \otimes y \in k_m \otimes_k k_n$. We have

$$i_y(\mathrm{id} \otimes \sigma_k(x \otimes y)) = (x\sigma_k^{i+1}(y))_{i=0}^{\delta-1}$$

and

$$(\sigma_y(i_y(x \otimes y))_i = \begin{cases} x\sigma_k^{i+1}(y) & 0 \leq i \leq \delta - 2 \\ \sigma_k^{am}(xy) & i = \delta - 1 \end{cases}$$

We have equality for the first $\delta - 1$ components. The calculation $\sigma_k^{am}(xy) = \sigma_k^{am}(x)\sigma_k^{\delta-bn}(x) = x\sigma_k^{\delta}(y)$ shows that the last components also coincide. The proof that i_x is σ-equivariant is symmetrical. ∴

Corollary 6.6. *For $\boldsymbol{z} = (z_0, \ldots, z_{d-1}) \in k_\infty^{\times d}$ set $\sigma'(\boldsymbol{z})_i := z_{i+1}$. Then i_y induces an isomorphism $k_\infty \otimes_k \boldsymbol{k_d} \to (k_\infty^{\times d}, \sigma')$, whereas i_x induces an isomorphism $k_d \otimes_k \boldsymbol{k_\infty} \to (k_\infty^{\times d}, \sigma')$ of bold fields.*

Proof. We have $k_\infty \otimes \boldsymbol{k_d} = \bigcup_{d|m} k_m \otimes_k \boldsymbol{k_d}$. By Proposition 6.5, i_y is an isomorphism $k_m \otimes_k \boldsymbol{k_d} \cong (k_m^d, \sigma')$. It follows that i_y gives an isomorphism

$$k_\infty \otimes \boldsymbol{k_d} \cong \bigcup_{d|m} (k_m^d, \sigma') = (k_\infty^{\times d}, \sigma').$$

The case of $k_d \otimes \boldsymbol{k_\infty}$ is symmetrical. ∴

Remark 6.7. By Proposition 6.5 and Corollary 6.6 we now know that, for $1 \leq m, n \leq \infty$ with either $m < \infty$ or $n < \infty$, the bold ring $k_m \otimes_k \boldsymbol{k_n}$ is a bold field.

To leave the realm of finite fields more substantially, we quote the following results of [Jac90].

Proposition 6.8. *Consider two field extensions E_1, E_2 of k, and assume that k is algebraically closed in E_1 (i.e.: every element of $E_1 \smallsetminus k$ is transcendental over k).*

(a) The tensor product $E_1 \otimes_k E_2$ is a domain.

(b) If E_2 is a finite extension of k, then $E_1 \otimes_k E_2$ is a field.

Proof. [Jac90, Theorem 8.50] gives item (a), and [Jac90, Theorem 8.46(2)] gives item (b). ∴

Corollary 6.9. *For every field $K \supset k$ and every $d \geq 1$, the bold ring $(k_d \otimes_k K, \mathrm{id} \otimes \sigma_k)$ is a bold field. If K contains a copy of k_d, then this bold field is isomorphic to $(K^{\times d}, \sigma'')$, where $\sigma''(\boldsymbol{z})_i := \sigma_k(z_{i+1})$ for $z = (z_0, \ldots, z_{d-1}) \in K^{\times d}$.*

Proof. Let k_K denote the algebraic closure of k in K. By Proposition 6.5 (and its Corollary 6.6 in case k_K is infinite) the bold ring $(k_d \otimes_k k_K, \mathrm{id} \otimes \sigma_k)$ is isomorphic to (k_μ^δ, σ_x), for certain $1 \leq \mu \leq \infty$ and $\delta \mid d$. Set $r := [k_\mu : k_K] < \infty$. By Proposition 6.8(b), the ring $K_r := k_\mu \otimes_{k_K} K$ is a field. It follows K_r is a finite field extension of K of degree r. Therefore, we have

$$k_d \otimes_k \boldsymbol{K} \cong (k_\mu^{\times\delta}, \sigma_x) \otimes_{\boldsymbol{k_K}} \boldsymbol{K} \cong (K_r^\delta, \sigma_x \circ \sigma_k),$$

where $\sigma_x \circ \sigma_k$ is given by first applying σ_k componentwise, and then σ_x. This is indeed a bold field.

If K contains k_∞, then $\delta = d$ and $r = 1$, so $K_r^{\times\delta} = K^{\times d}$, as required. Moreover, one checks that in this case $\sigma_x \circ \sigma_k$ coincides with the endomorphism σ'' given in the statement of this corollary. ∴

Corollary 6.10. *Consider two field extension F, K of k. If either F or K contains only a finite number of roots of unity, then $\boldsymbol{F_K} := (\mathrm{Frac}(F \otimes_k K), \mathrm{Frac}(\mathrm{id} \otimes \sigma_k))$ is a bold field.*

Proof. Abusing notation a little, we set $k_F := k^{\mathrm{sep}} \cap F$ and $k_K := k^{\mathrm{sep}} \cap K$, the respective algebraic closures of k in F and K. Now $(k_F \otimes_k k_K, \mathrm{id} \otimes \sigma_k)$ is a bold field by Proposition 6.5 and Corollary 6.6, for certain $1 \leq \mu \leq \infty$ and $1 \leq \delta < \infty$. In particular,

$$F \otimes_k K = F \otimes_{k_F} (k_F \otimes_k k_K) \otimes_{k_K} K \cong (F \otimes_{k_F} k_\mu \otimes_{k_K} K)^{\times\delta}.$$

To show that $\boldsymbol{F_K}$ is a bold field, it is sufficient to show that $\mathrm{Frac}(F \otimes_{k_F} k_\mu \otimes_{k_K} K)$ is a field, which follows if we show that $F \otimes_{k_F} k_\mu \otimes_{k_K} K$ is a domain.

We do this in the case where F has a finite number of roots of unity, i.e. that k_F is finite; the other case is symmetrical. Applying Proposition 6.8(b) to $k_0 = k_F$, $E_1 = F$ and $E_2 = k_\mu$ shows that $F \otimes_{k_F} k_\mu$ is a field. So applying Proposition 6.8(a) to $k_0 := k_F$, $E_1 := K$ and $E_2 := F \otimes_{k_F} k_\mu$ shows that $F \otimes_{k_F} k_\mu \otimes_{k_K} K$ is a domain. ∴

We may now introduce further bold fields.

Examples 6.11. We continue to use the notation given in Examples 6.1.

(a) The bold ring $\boldsymbol{\mathscr{O}_{K,\mathfrak{p}}}$ is defined as $\varprojlim_n \boldsymbol{\mathscr{O}_{(\mathfrak{p}),K}}/\mathfrak{p}^n$, the "completion at $\mathfrak{p}$" of the bold ring $\boldsymbol{\mathscr{O}_{(\mathfrak{p}),K}}$. Let $k_{\mathfrak{p}} = \mathscr{O}_{(\mathfrak{p})}/\mathfrak{p}$ and choose a local parameter $t \in \mathscr{O}_{(\mathfrak{p})}$ at $\mathfrak{p}$. By the Chinese Remainder theorem we have an isomorphism

$$\mathscr{O}_{K,\mathfrak{p}} \xrightarrow{\sim} (k_{\mathfrak{p}} \otimes_k K)[\![t]\!].$$

The σ of $\boldsymbol{\mathscr{O}_{K,\mathfrak{p}}}$ induces a unique endomorphism of the right hand side: It acts as the identity on t, and as $\mathrm{id} \otimes \sigma_k$ on elements of $k_{\mathfrak{p}} \otimes_k K$. Now $k_{\mathfrak{p}} \otimes_k K$ decomposes as finite direct product of the pairwise-isomorphic fields $K_{\mathfrak{P}} = \mathscr{O}_{(\mathfrak{P}),K}/\mathfrak{P}$ for those places $\mathfrak{P}$ of F_K lying above $\mathfrak{p}$, and equipped with $\mathrm{id} \otimes \sigma_k$ it is a bold field (Corollary 6.9). We have $\mathscr{O}_{K,\mathfrak{p}}^{\sigma} = \mathscr{O}_{F_{\mathfrak{p}}}$, the valuation ring of $F_{\mathfrak{p}}$. All in all, $\boldsymbol{\mathscr{O}_{K,\mathfrak{p}}}$ is a local bold ring. The subfield $K \subset \mathscr{O}_{K,\mathfrak{p}}$ is a base field.

(b) Set $F_{K,\mathfrak{p}} := \mathrm{Frac}(\mathscr{O}_{K,\mathfrak{p}})$ and let $\boldsymbol{F_{K,\mathfrak{p}}}$ be this ring equipped with the unique extension of σ. By the preceding, we may identify $F_{K,\mathfrak{p}}$ with

$$(k_{\mathfrak{p}} \otimes_k K)(\!(t)\!) := (k_{\mathfrak{p}} \otimes_k K)[\![t]\!][t^{-1}].$$

We have $F_{K,\mathfrak{p}}^{\sigma} = F_{\mathfrak{p}}$. Again using (a), we see that $\boldsymbol{F_{K,\mathfrak{p}}}$ is a bold field, with bold order $\boldsymbol{\mathscr{O}_{K,\mathfrak{p}}}$.

(c) Let $\mathfrak{p}$ be a place of F, and denote by $F_{\mathfrak{p}}$ the completion of F at $\mathfrak{p}$. Then $\boldsymbol{F_{\mathfrak{p},K}} := (\mathrm{Frac}(F_{\mathfrak{p}} \otimes_k K), \mathrm{Frac}(\mathrm{id} \otimes \sigma_k))$ is a bold field by Corollary 6.10. Clearly, $F_{\mathfrak{p},K} \subset F_{K,\mathfrak{p}}$, but it is fundamental to note that this inclusion is *strict* except if K is finite. The main question in this context is how we can characterize this inclusion; we shall come back to this in Chapter V.

(d) Set $\boldsymbol{\mathscr{O}_{\mathfrak{p},K}} := \boldsymbol{F_{\mathfrak{p},K}} \cap \boldsymbol{\mathscr{O}_{K,\mathfrak{p}}}$. This a global bold ring. It is a bold order of $\boldsymbol{F_{\mathfrak{p},K}}$ and has $K \subset \mathscr{O}_{\mathfrak{p},K}$ as base field. Clearly, $\mathscr{O}_{\mathfrak{p},K}^{\sigma} = \mathscr{O}_{F_{\mathfrak{p}}}$. We note that we have inclusions $\mathscr{O}_{F_{\mathfrak{p}}} \otimes_k K \subset \mathscr{O}_{\mathfrak{p},K} \subset \mathscr{O}_{K,\mathfrak{p}}$, but these inclusions are *strict* in general.

Let us review the most important rings for the following chapters by means of a diagram, in the case where F is a global field with field of constants K, and K is a field extension of K. Let ∞, $\mathfrak{p}$ be two different places of F, and let A be the ring of elements of F integral outside ∞. Then we have inclusions

$$\begin{array}{ccccccc}
A_K & \longrightarrow & \mathscr{O}_{(\mathfrak{p}),K} & \longrightarrow & \mathscr{O}_{\mathfrak{p},K} & \longrightarrow & \mathscr{O}_{K,\mathfrak{p}} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & F_K & \longrightarrow & F_{\mathfrak{p},K} & \longrightarrow & F_{K,\mathfrak{p}}
\end{array}$$

where the upper row consists of “integral” rings, whereas the lower row consists of “rational” rings. The corresponding diagram of scalar rings is

$$\begin{array}{ccccccc} A & \longrightarrow & \mathscr{O}_{(\mathfrak{p})} & \longrightarrow & \mathscr{O}_{\mathfrak{p}} & \longrightarrow & \mathscr{O}_{\mathfrak{p}} \\ & & \downarrow & & \downarrow & & \downarrow \\ & & F & \longrightarrow & F_{\mathfrak{p}} & \longrightarrow & F_{\mathfrak{p}} \end{array}$$

7 Galois representations

Choose a field $K \supset k$ of positive characteristic, with fixed separable closure K^{sep} and absolute Galois group $\Gamma_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$.

Let F be a global field of positive characteristic with field of constants k. Fix a place $\mathfrak{p}$ of $F$and let $\mathscr{O}_{\mathfrak{p}}$ denote the valuation ring of $F_{\mathfrak{p}}$, the completion of F at $\mathfrak{p}$. Each of these rings has a natural topology – the discrete topology for k, and the metric topologies for $\mathscr{O}_{\mathfrak{p}}$ and $F_{\mathfrak{p}}$. In the following, let R denote any one of these three rings. Note the topology of R induces a unique natural topology on $\mathrm{Aut}_R(V)$ for every finitely generated R-module V.

Definition 7.1. A *Galois representation over* R is a pair $\boldsymbol{V} = (V, \rho)$ consisting of a finitely generated R-module V and a continuous group homomorphism

$$\rho : \Gamma_K \longrightarrow \mathrm{Aut}_R(V).$$

A *homomorphism* between two Galois representations over R is a homomorphism of the underlying R-modules commuting with the respective actions of Γ_K. We obtain $\mathrm{Rep}_R(\Gamma_K)$, an abelian tensor category over R.

We shall sometimes call such Galois representations: integral $\mathfrak{p}$-adic representations of Γ_K (if $R = \mathscr{O}_{\mathfrak{p}}$), or rational $\mathfrak{p}$-adic representations of Γ_K (if $R = F_{\mathfrak{p}}$).

We shall see that we can classify such representations in terms of semilinear algebra. For this we consider auxiliary bold rings $\boldsymbol{D}_R$ and $\boldsymbol{B}_R$ associated to R by means of the following table (cf. Examples 6.1 and 6.11):

R	$\boldsymbol{D}_R$	$\boldsymbol{B}_R$
k	$\boldsymbol{K} = (K, \sigma)$	$\boldsymbol{K}^{\mathbf{sep}} = (K^{\mathrm{sep}}, \sigma)$
$\mathscr{O}_{\mathfrak{p}}$	$\mathscr{O}_{\boldsymbol{K},\mathfrak{p}} = \varprojlim \mathscr{O}_{(\mathfrak{p}),\boldsymbol{K}}/\mathfrak{p}^n$	$\mathscr{O}_{\boldsymbol{K}^{\mathrm{sep}},\mathfrak{p}} = \varprojlim \mathscr{O}_{(\mathfrak{p}),\boldsymbol{K}^{\mathrm{sep}}}/\mathfrak{p}^n$
$F_{\mathfrak{p}}$	$\boldsymbol{F}_{\boldsymbol{K},\mathfrak{p}} = \mathrm{Frac}(\mathscr{O}_{\boldsymbol{K},\mathfrak{p}})$	$\boldsymbol{F}_{\boldsymbol{K}^{\mathrm{sep}},\mathfrak{p}} = \mathrm{Frac}(\mathscr{O}_{\boldsymbol{K}^{\mathrm{sep}},\mathfrak{p}})$

Note that in all three cases $\boldsymbol{B}_R$ is naturally equipped with an $\boldsymbol{D}_R$-module structure and an action of Γ_K which commute with each other. Furthermore, we have $\boldsymbol{B}_R^{\sigma} = R$ and $\boldsymbol{B}_R^{\Gamma_K} = \boldsymbol{D}_R$.

Definition 7.2. Let $\boldsymbol{M} = (M, \tau)$ be either a $\boldsymbol{D}_k$-module or a $\boldsymbol{D}_{\mathcal{O}_\mathfrak{p}}$-module. It is *representational* if is finitely generated and τ_{lin} is bijective. A $\boldsymbol{D}_{F_\mathfrak{p}}$-module is *representational* if it is $\mathcal{O}_{\boldsymbol{K},\mathfrak{p}}$-etale.

For every Galois representation $\boldsymbol{V}$ over R, set

$$D_R(\boldsymbol{V}) := (\boldsymbol{B}_R \otimes_R \boldsymbol{V})^{\Gamma_K},$$

taking invariants with respect to the diagonal action of Γ_K on $\boldsymbol{B}_R \otimes_R \boldsymbol{V}$.

Conversely, given a representational $\boldsymbol{R}$-module $\boldsymbol{M}$, set

$$V_R(\boldsymbol{M}) := (\boldsymbol{B}_R \otimes_{\boldsymbol{D}_R} \boldsymbol{M})^{\tau},$$

taking invariants with respect to the diagonal action of τ on $\boldsymbol{B}_R \otimes_{\boldsymbol{D}_R} \boldsymbol{M}$.

Proposition 7.3.

(a) D_R is an R-linear exact tensor functor with values in representational $\boldsymbol{D}_R$-modules.

(b) V_R is an R-linear exact tensor functor with values in Galois representations over R.

(c) For every Galois representation $\boldsymbol{V}$ over R, the following natural homomorphism is an isomorphism: $\boldsymbol{B}_R \otimes D_R(\boldsymbol{V}) \longrightarrow \boldsymbol{B}_R \otimes \boldsymbol{V}$. It commutes with the actions of both τ and Γ_K.

(d) For every representational $\boldsymbol{D}_R$-module $\boldsymbol{M}$, the following natural homomorphism is an isomorphism: $\boldsymbol{B}_R \otimes V_R(\boldsymbol{M}) \longrightarrow \boldsymbol{B}_R \otimes \boldsymbol{M}$. It commutes with the actions of both τ and Γ_K.

Proof. There are three cases:

$R = k$: This is [PiT04, Proposition 4.1].

$R = \mathcal{O}_\mathfrak{p}$: Follows from the case of $R = k$ by reduction modulo $\mathfrak{p}^n$ and naturality.

$R = F_\mathfrak{p}$: Follows from the case of $R = \mathcal{O}_\mathfrak{p}$ using that every Galois representation $\boldsymbol{V}$ over $F_\mathfrak{p}$ is isomorphic to $F_\mathfrak{p} \otimes_{\mathcal{O}_\mathfrak{p}} \boldsymbol{T}$ for some Galois representation over $\boldsymbol{T}$ (since Γ_K is compact), whereas every representational $\boldsymbol{F}_{\boldsymbol{K},\mathfrak{p}}$-module $\boldsymbol{M}$ is isomorphic to $\boldsymbol{F}_{\boldsymbol{K},\mathfrak{p}} \otimes_{\mathcal{O}_{\boldsymbol{K},\mathfrak{p}}} \boldsymbol{N} = F_\mathfrak{p} \otimes_{\mathcal{O}_\mathfrak{p}} \boldsymbol{N}$ for some representational $\mathcal{O}_{\boldsymbol{K},\mathfrak{p}}$-module $\boldsymbol{N}$ by definition. So given a local parameter $t \in F$ at $\mathfrak{p}$, we may write $F_\mathfrak{p} = \mathcal{O}_\mathfrak{p}[t^{-1}]$, $\boldsymbol{V} = \boldsymbol{T}[t^{-1}]$ and $\boldsymbol{M} = \boldsymbol{N}[t^{-1}]$. Furthermore, everything else "commutes with localisation at t", and then the statements for $\mathcal{O}_\mathfrak{p}$ imply those for $F_\mathfrak{p}$. $\therefore$

Theorem 7.4. *D_R and V_R are mutually quasi-inverse equivalences of abelian tensor categories over R. In the case $R = \mathcal{O}_\mathfrak{p}$, torsion-free Galois representations correspond to restricted $\boldsymbol{R}$-modules.*

Proof. (c) implies that $V_R(D_R\boldsymbol{V}) \cong \boldsymbol{V}$ for all representations by taking τ-invariants, so in particular V_R is essentially surjective, whereas (d) shows that $D_R(V_R\boldsymbol{M}) \cong \boldsymbol{M}$ for all representational modules by considering Γ_K-invariants, which shows that D_R is essentially surjective. Moreover, taking simultaneous τ- and Γ_K-invariants in (c) shows that $\mathrm{End}_{\boldsymbol{R}}(D_R\boldsymbol{V}) \cong \mathrm{End}_{\Gamma_K}(\boldsymbol{V})$, so D_R is fully faithful, whereas considering simultaneous τ- and Γ_K-invariants in (d) shows that V_R is fully faithful. $\therefore$

Chapter II

Abelian A-modules and A-motives

8 A-modules

Let K be a field containing a finite field k.

Definition 8.1. A *vector group over K* is an algebraic group over K whose base change to the algebraic closure of K is isomorphic to a finite product of copies of the additive group $\mathbb{G}_a$.

A *k-linear vector group over K* is a vector group G over K together with a homomorphism $k \longrightarrow \mathrm{End}_K(G)$ which induces on $\mathrm{Lie}(G)$ the same action as that via $k \hookrightarrow K$.

Let F be a global field with field of constants k. Fix a finite non-empty set $\{\infty_1, \ldots, \infty_s\}$ of places of F, and let A be the ring consisting of those elements of F that are integral outside the ∞_i. We wish to represent A as a ring of endomorphisms of a commutative group scheme G over K.

Fix a k-linear ring homomorphism $\iota : A \longrightarrow K$, which we will refer to as the *characteristic homomorphism* of K. To give ι is equivalent to giving a maximal ideal $\mathfrak{P}_0$ of degree one of $A_K := A \otimes_k K$, the *characteristic point* of K. We set $\mathfrak{p}_0 := \mathfrak{P}_0 \cap A = \ker \iota$, this is a prime ideal of A, the *"small" characteristic point* of K. One says that the *characteristic* of K is *generic* if $\mathfrak{p}_0 = 0$, and *special* otherwise.

Definition 8.2. An *A-module over K (of characteristic ι)* is a pair $\boldsymbol{G} = (G, \phi)$ consisting of a k-linear vector group G over K and a k-linear ring homomorphism $\phi : A \longrightarrow \mathrm{End}_K(G)$, such that

- for every $a \in A$, every eigenvalue of the induced action of $d\phi(a)$ on $\mathrm{Lie}\, G$ is equal to $\iota(a)$.

A *homomorphism* of two such A-modules (of equal characteristic) is a group homomorphism of the underlying vector groups which is compatible with the actions of A. We obtain the A-linear category of A-modules, denoted by $A\text{-Mod}_K$. The *dimension* $\dim \boldsymbol{G}$ of an A-module $\boldsymbol{G}$ is the dimension of the underlying vector group.

Remark 8.3. The final condition of Definition 8.2 can be interpreted as meaning that we are considering "deformations" of the infinitesimal scalar action of A along ι on $\operatorname{Lie} G$.

Example 8.4. Let $A = k[t]$, and consider any field extension $K \supset k$. A k-linear homomorphism $\iota : A \to K$ is specified by the value $\theta := \iota(t)$.

An example of a k-linear vector group over K is of course $G := \mathbb{G}_{a,K}^d$ for any $d \geq 0$. Then $\operatorname{End}_K(G) = \operatorname{Mat}_{r\times r}(K)\{\tau\}$, and a k-linear homomorphism $\phi : A \to \operatorname{End}_K(G)$ is specifed by the value

$$\phi(t) = T_0 + T_1\tau + \cdots T_s\tau^s,$$

with $T_i \in \operatorname{Mat}_{r\times r}(K)$. Then the condition on the eigenvalues of $\phi(a)$ for all a is equivalent to saying that

$$T_0 = \theta \cdot \mathbf{1}_{r\times r} + N$$

for some *nilpotent* matrix N. In this case, $\boldsymbol{G} := (G, \phi)$ is an A-module over K of characteristic ι.

9 Classification of generalised A-modules

As in the previous section, let k be a finite field, $K \supset k$ a field extension, F a global field with constant field k, and A the ring of elements of F integral outside a finite non-empty set $\{\infty_1, \ldots, \infty_s\}$ of places of F. The content of this section is the classification of group schemes "of Verschiebung zero" equipped with an action of A, in terms of semilinear algebra.

Remark 9.1. The interested reader may check that the results of this section hold more generally for all k-algebras A.

Fact 9.2. Let G be a group scheme over K. Let σ_p^*G denote the base change of G along the absolute Frobenius homomorphism of $\operatorname{Spec}(K)$; this is again a group scheme over K.

(a) The absolute Frobenius homomorphism of G induces a homomorphism $F_G : G \longrightarrow \sigma_p^*G$ of group schemes over K, the *relative* Frobenius homomorphism.

(b) If G is commutative, there is another canonical group homomorphism $V_G : \sigma_p^* G \longrightarrow G$, called the *Verschiebung* homomorphism (cf. [DeG70] or [SGA, III]), for which the following equations hold true:

$$V_G F_G = p \cdot \mathrm{id}_G, \qquad F_G V_G = p \cdot \mathrm{id}_{\sigma_p^* G} .$$

Remark 9.3. Let G be an affine commutative group scheme of finite type over K. Then G is unipotent if and only if V_G is nilpotent.

We start with the classification of affine commutative group schemes G over K (not necessarily of finite type!) with $V_G = 0$. For such a group, set

$$M_p(G) := \mathrm{Hom}_K(G, \mathbb{G}_{a,K}).$$

It is a K-vector space, and the absolute Frobenius homomorphism of $\mathbb{G}_{a,K}$ induces a σ_p-linear endomorphism τ_p of $M_p(G)$, making it a $\boldsymbol{K}_p := (K, \sigma_p)$-module.

Theorem 9.4. *M_p is a contravariant functor, and gives rise to an anti-equivalence of abelian categories*

$$M_p : \begin{pmatrix} \textit{affine commutative group schemes} \\ \textit{over } K \textit{ with } V_G = 0 \end{pmatrix} \longrightarrow \boldsymbol{K}_p\text{-Mod},$$

natural in K. Moreover, for any such group

(a) G is of finite type over $K \iff M_p(G)$ is finitely generated as $K\{\tau\}$-module.

(b) G is finite over $K \iff$ $\dim_K M_p(G) < \infty$.

Proof. [DeG70]. ∴

Remark 9.5. We can construct a quasi-inverse functor to M_p explicitly. A $\boldsymbol{K}_p$-module $\boldsymbol{M} = (M, \tau_p)$ may be considered as a commutative p-Lie algebra, with "p-power map" given by τ_p. Then we let $G_p(\boldsymbol{M})$ be the spectrum of the enveloping algebra of the dual p-Lie algebra of (M, τ_p), and obtain the functor G_p which is quasi-inverse to M_p.

Corollary 9.6. *Let G be an affine commutative group scheme over K. Then $V_G = 0$ if and only if there exists a closed embedding $G \hookrightarrow \prod_I \mathbb{G}_{a,K}$ for some index set I.*

Proof. Since $V_{\mathbb{G}_{a,K}} = 0$, a closed subgroup of a product of copies of $\mathbb{G}_{a,K}$ has Verschiebung zero.

Conversely, if G is an affine commutative group scheme over K of Verschiebung zero, then the $\boldsymbol{K}_p$-module $M_p(G)$ is a quotient of sum of, say, I copies of the free $\boldsymbol{K}_p$-module $K\{\tau_p\}$. By antiequivalence

$$G \cong G_p(M_p(G)) \longrightarrow G_p\Big(\bigoplus_I K\{\tau_p\}\Big) = \prod_I G_p(K\{\tau_p\}) = \prod_I \mathbb{G}_{a,K}$$

is a monomorphism. It is known that monomorphisms in the category of affine commutative group schemes are closed embeddings. ∴

Corollary 9.7. *Let G be an affine commutative group scheme of finite type over K with $V_G = 0$. The following are equivalent:*

(a) G is a vector group.

(b) $\overline{\boldsymbol{K}}_p \otimes_{\boldsymbol{K}_p} M_p(G)$ is free as $\overline{K}\{\tau_p\}$-module (in which case some authors say that $M_p(G)$ is potentially free*).*

(c) $M_p(G)$ is torsion-free as $K\{\tau_p\}$-module.

(d) G is smooth and connected.

Proof. (a) $\Longleftrightarrow$ (b): G is a vector group if and only if $G_{\overline{K}}$ is a product of copies of $\mathbb{G}_a$, which is true if and only if $M_p(G_{\overline{K}})$ is a free $\overline{K}\{\tau\}$-module. Since M_p is natural in K, we have $M_p(G_{\overline{K}}) \cong \overline{\boldsymbol{K}}_p \otimes_{\boldsymbol{K}_p} M_p(G)$.

(b) $\Longleftrightarrow$ (c): For any bold ring $\boldsymbol{R} = (R, \sigma)$ with underlying ring R a domain and any $\boldsymbol{R}$-module $\boldsymbol{M}$, set

$$\mathrm{Tor}_{R\{\tau\}}(\boldsymbol{M}) := \Big\{ m \in M : \ rm = 0 \text{ for some } 0 \neq r \in R\{\tau\} \Big\}.$$

Then since $\overline{K}\{\tau_p\}$ is a principal ideal domain and $M_p(G_{\overline{K}})$ is finitely generated over $\overline{K}\{\tau_p\}$, this latter module is $\overline{K}\{\tau_p\}$-free if and only if it is $\overline{K}\{\tau_p\}$-torsion-free. We have $\mathrm{Tor}_{\overline{K}\{\tau_p\}}(M_p(G_{\overline{K}})) \cong \overline{K} \otimes_k \mathrm{Tor}_{K\{\tau_p\}}(M_p(G))$, so $M_p(G_{\overline{K}})$ is torsion-free if and only if $M_p(G)$ is.

(c) $\Longleftrightarrow$ (d): G is smooth and connected if and only if it has no finite quotients. By Theorem 9.4(b), this is equivalent to $M_p(G)$ having no finite K-dimensional $\boldsymbol{K}_p$-submodules, which means that $M_p(G)$ is torsion-free as $K\{\tau_p\}$-module. ∴

Let us remark on the structure of vector groups.

Theorem 9.8 (Kambayashi, Miyanishi, Takeuchi)**.** *Let G be a d-dimensional vector group over K.*

(a) There exists an integer $r \geq 0$, a matrix $A_0 \in \mathrm{GL}_d(K)$ and matrices $A_1, \dots A_r \in \mathrm{Mat}_{d\times d}(K)$ such that G is isomorphic to the closed subgroup scheme of $\mathbb{G}_{a,K}^d \times \mathbb{G}_{a,K}^d$ given by the equations

$$(A_0 + A_1\tau + \cdots A_n\tau^n)x = \sigma^n(y), \quad (x, y) \in \mathbb{G}_{a,K}^d \times \mathbb{G}_{a,K}^d.$$

(b) If $d = 1$ then $\mathrm{End}_K(G)$ is a finite field if and only if G is not isomorphic to $\mathbb{G}_{a,K}$ over K.

Proof. This is [KMT74]. It is basically a calculation with $\boldsymbol{K}_p$-modules. Item (b) and the case $d = 1$ of item (a) were first shown in [Rus70]. ∴

We wish to enrich the equivalence M_p given by Theorem 9.4 with actions of A, taking into account the special role of k.

Definition 9.9. A *generalised A-module over K* is a pair $\boldsymbol{G} = (G, \phi)$ consisting of an affine commutative group scheme over K such that $V_G = 0$, and a ring homomorphism $\phi : A \longrightarrow \mathrm{End}_K(G)$.

Let $A \otimes_{\mathbb{F}_p} \boldsymbol{K}_p$ denote the bold ring $(A \otimes_{\mathbb{F}_p} K, \mathrm{id} \otimes \sigma_p)$. If $\boldsymbol{G}$ is a generalised A-module over K, then the action ϕ induces an A-module structure on $M_p(G)$, and makes the latter into a $A \otimes_{\mathbb{F}_p} \boldsymbol{K}_p$-module which we denote as $M_p(\boldsymbol{G})$.

Definition 9.10. We say that an $A \otimes_{\mathbb{F}_p} \boldsymbol{K}_p$-module $\boldsymbol{M}$ is *k-linear* if the two actions of k (considered first as a subring of A, secondly as a subring of K) on both $\ker \tau_{\mathrm{lin}}$ and $\operatorname{coker} \tau_{\mathrm{lin}}$ coincide.

For every $A \otimes_{\mathbb{F}_p} \boldsymbol{K}_p$-module $\boldsymbol{M}$, the largest k-linear $A \otimes_{\mathbb{F}_p} \boldsymbol{K}_p$-submodule is

$$\boldsymbol{M}^{k\text{-lin}} = \{m \in M : (1 \otimes \lambda)m = (\lambda \otimes 1)m \quad \forall\, \lambda \in k\}.$$

Let $\boldsymbol{K}$ be the bold ring consisting of K equipped with the $[k : \mathbb{F}_p]$-th power σ_k of σ_p. Let $\boldsymbol{A_K}$ denote the bold ring $(A \otimes_k K, \mathrm{id} \otimes \sigma_k)$.

Lemma 9.11 (Tamagawa)**.** *The natural functor*

$$\boldsymbol{A_K}\text{-Mod} \longrightarrow \left(\!\!\left(\; k\text{-linear } A \otimes_{\mathbb{F}_p} \boldsymbol{K}_p\text{-modules} \; \right)\!\!\right).$$

is an equivalence of abelian categories.

Proof. This is checked easily by a direct calculation. ∴

Definition 9.12. A *k-linear generalised A-module over K* is a generalised A-module $\boldsymbol{G}$ over K such that $M_p(\boldsymbol{G})$ is a k-linear $A \otimes_{\mathbb{F}_p} \boldsymbol{K}_p$-module.

A homomorphism of k-linear generalised A-modules over K is a k-linear A-equivariant homomorphism of the underlying group schemes over K. We denote the category of all k-linear generalised A-modules over K as $\mathscr{G}_{A,K}$.

We are now in the position to define the enriched version of the functor M_p. For every k-linear generalised A-module $\boldsymbol{G}$ over K, set

$$M(\boldsymbol{G}) := M_p(\boldsymbol{G})^{k\text{-lin}}.$$

By Lemma 9.11, $M(\boldsymbol{G})$ is an $\boldsymbol{A_K}$-module.

Remark 9.13. Equipping $\mathbb{G}_{a,K}$ with the natural scalar action of k, we have

$$M(\boldsymbol{G}) = \mathrm{Hom}_{K,k\text{-linear}}(G, \mathbb{G}_{a,K}),$$

where the latter K-vector space has the structure of an $\boldsymbol{A_K}$-module via the induced action of A and the σ_k-linear endomorphism given by the $[k : \mathbb{F}_p]$-th power of the absolute Frobenius homomorphism of $\mathbb{G}_{a,K}$.

Theorem 9.14. *The contravariant functor*

$$M : \mathscr{G}_{\mathrm{A},K} \longrightarrow \boldsymbol{A_K}\text{-Mod}$$

is an anti-equivalence of A-linear abelian categories, natural in both A and K. Moreover, the following dictionary between $\boldsymbol{G} = (G, \phi)$ *and* $\boldsymbol{M} = M(\boldsymbol{G})$ *holds:*

(a) G is of finite type over K if and only if $\boldsymbol{M}$ *is finitely generated as* $K\{\tau\}$*-module.*

(b) G is finite over K if and only if $\boldsymbol{M}$ *is finite-dimensional over K.*

(c) G is a (k-linear) vector group over K if and only if $\boldsymbol{M}$ *is torsion free as* $K\{\tau\}$*-module.*

(d) If G is a vector group over K, then M induces a natural isomorphism $(\mathrm{Lie}\, G)^\vee := \mathrm{Hom}_K(\mathrm{Lie}\, G, K) \longrightarrow \mathrm{Lie}^* M(G)$ *of* A_K*-modules.*

Proof. Items (a,b) follow from Theorem 9.4 using Lemma 9.11.

Item (c) follows from Corollary 9.7 using Lemma 9.11.

(d): If K is perfect, then $\boldsymbol{G} \cong \mathbb{G}_{a,K}^d$, and a direct calculation shows that the natural pairing

$$\mathrm{Lie}(G) \times \Big(M(\boldsymbol{G})/\tau_{\mathrm{lin}}(M(\boldsymbol{G}))\Big) \longrightarrow K, \quad (x, [f]) \mapsto \partial_x f$$

is non-degenerate, so it induces the desired homomorphism (cf. [And86]). In the general case, the pairing still exists, and since everything is natural under base change, we can check non-degeneracy after base change to a perfect field containing K, so we are done! ∴

10 *A*-motives

As in the previous two sections, let k be a finite field, $K \supset k$ a field extension, F a global field with constant field k, and A the ring of elements of F integral outside a finite non-empty set $\{\infty_1, \dots, \infty_s\}$ of places of F. We set $\boldsymbol{A_K} := (A \otimes_k K, \mathrm{id} \otimes \sigma_k)$, where σ_k is the k-linear Frobenius endomorphism of K.

Furthermore, a k-linear ring homomorphism $\iota : A \longrightarrow K$ is given, the *characteristic homomorphism* of K. Recall that to give ι is equivalent to giving a maximal ideal $\mathfrak{P}_0$ of degree one of $A_K := A \otimes_k K$, the *characteristic point* of K. We set $\mathfrak{p}_0 := \mathfrak{P}_0 \cap A = \ker \iota$, this is a prime ideal of A, the *"small" characteristic point* of K. One says that the *characteristic* of K is *generic* if $\mathfrak{p}_0 = 0$, and *special* otherwise.

Definition 10.1. An *A-motive over K (of characteristic ι)* is a non-degenerate torsion-free $\boldsymbol{A_K}$-module $\boldsymbol{M} = (M, \tau)$ such that

- $\operatorname{Supp}(\operatorname{Lie}^* \boldsymbol{M}) \subset \{\mathfrak{P}_0\}$.

A *homomorphism* of two such A-motives is homomorphism of $\boldsymbol{A_K}$-modules, that is, a homomorphism of the underlying A_K-modules compatible with the respective actions of τ. We shall only consider homomorphisms between A-motives with equal characteristic point $\mathfrak{P}_0$. The *rank* $\operatorname{rk} \boldsymbol{M}$ of an A-motive $\boldsymbol{M}$ is the rank of M over A_K (this is well-defined since A_K is a domain).

Remark 10.2. More generally, one could consider all nondegenerate torsion-free $\boldsymbol{A_K}$-modules $\boldsymbol{M}$ (these are called *τ-modules* in some parts of the literature). Then, $\operatorname{Lie}^* \boldsymbol{M}$ has finite support (as A_K-module). If the closed points of $\operatorname{Lie}^* \boldsymbol{M}$ each have degree 1, then one might call such a module $\boldsymbol{M}$ an A-motive with the "multiple characteristic points" corresponding to these points. If not, the base change of $\boldsymbol{M}$ to some finite extension $K' \supset K$ would be of this form.

On the other hand, there exist A-motives $\boldsymbol{M}$ over K with $\operatorname{Supp}(\operatorname{Lie}^* \boldsymbol{M}) = \varnothing$, for instance $\mathbf{1} := \boldsymbol{A_K}$. These A-motives are A-motives over K for every characteristic homomorphism, in [Tae07] they are called *interior motives* and studied in more detail.

Lemma 10.3. *Let* $\boldsymbol{M}$ *and* $\boldsymbol{N}$ *be two nondegenerate torsion-free* $\boldsymbol{A_K}$*-modules. The following natural homomorphism of* A_K*-modules is injective:*

$$K \otimes_k \operatorname{Hom}(\boldsymbol{M}, \boldsymbol{N}) \longrightarrow \operatorname{Hom}_{A_K}(M, N), \quad \lambda \otimes h \mapsto \lambda \cdot h.$$

Proof. [And86, Theorem 2]. ∴

Proposition 10.4. *Let* $\boldsymbol{M}$, $\boldsymbol{N}$ *be two A-motives over K. Then* $\operatorname{Hom}(\boldsymbol{M}, \boldsymbol{N})$ *is a projective A-module of rank bounded above by* $\operatorname{rk} \boldsymbol{M} \cdot \operatorname{rk} \boldsymbol{N}$.

Proof. By Lemma 10.3, $K \otimes_k \operatorname{Hom}(\boldsymbol{M}, \boldsymbol{N})$ is a submodule of a projective A_K-module of rank $\operatorname{rk} \boldsymbol{M} \cdot \operatorname{rk} \boldsymbol{N}$. This implies that $H := \operatorname{Hom}(\boldsymbol{M}, \boldsymbol{N})$ is a torsion-free A-module.

It also implies that H is a finitely generated A-module: Choose a finite set of generators $h'_1, \dots h'_r$ of $K \otimes_k H$. Each h'_i is a finite K-linear combination of elements of H, so we can find a finite set $h_1, \dots h_s$ of generators of $K \otimes_k H$ lying in H. Let H_0 denote the A-submodule of H generated by the h_i. The inclusion $H_0 \subset H$ induces an equality $K \otimes_k H_0 = K \otimes_k H$ by construction, so we deduce that $H = H_0$ is a finitely generated A-module.

Therefore, H is a projective A-module of finite rank, and using Lemma 10.3 we may bound this rank above by $\operatorname{rk} \boldsymbol{M} \cdot \operatorname{rk} \boldsymbol{N}$. ∴

Definition 10.5. An A-motive $\boldsymbol{M}$ over K is called *abelian* if M is finitely generated over $K\{\tau\}$. An A-module $\boldsymbol{G}$ over K is called *abelian* if $M(\boldsymbol{G})$ is finitely generated over A_K.

Remark 10.6. The A-motive $\mathbf{1} = \boldsymbol{A_K}$ is not abelian, and the A-module $(\mathbb{G}_{a,K}, \iota)$ is not abelian.

Proposition 10.7. *Let $\boldsymbol{M}$ be a finitely generated $\boldsymbol{A_K}$-module, finitely generated also as $K\{\tau\}$-module. Let* $\operatorname{Tor}(\boldsymbol{M})$ *denote the A_K-torsion submodule of M. Then:*

(a) $\operatorname{Tor}(\boldsymbol{M})$ *is an $\boldsymbol{A_K}$-submodule of $\boldsymbol{M}$.*

(b) $\operatorname{Tor}(\boldsymbol{M})$ *coincides with the set of $K\{\tau\}$-torsion elements of $\boldsymbol{M}$.*

Proof. [And86, Lemma 1.4.5]. ∴

Theorem 10.8. *The functor M of Theorem 9.14 restricts to an equivalence of A-linear categories*

$$M : \left(\!\!\left(\text{ abelian } A\text{-modules over } K \right)\!\!\right) \longrightarrow \left(\!\!\left(\text{ abelian } A\text{-motives over } K \right)\!\!\right).$$

Proof. Combine Theorem 9.14 with Proposition 10.7. ∴

Definition 10.9. (a) The *rank* of an abelian A-module $\boldsymbol{G}$ over K is the rank of its associated A-motive $M(\boldsymbol{G})$.

(b) The *dimension* of an abelian A-motive $\boldsymbol{M} \cong M(\boldsymbol{G})$ over K is the dimension of its associated A-module $\boldsymbol{G}$.

Remark 10.10. In [Tae03] it is shown how to reconstruct the rank of an abelian A-module (G, ϕ) over K directly in terms of the action ϕ. Since the rank of an abelian A-module is invariant under extension of K, one may assume that K is algebraically closed. Consider

$$\deg \det : \operatorname{End}(G) \subset \operatorname{End}_{\overline{K}}(G) = \operatorname{Mat}_{\dim G \times \dim G}(\overline{K}\{\tau\}) \longrightarrow \mathbb{N} \cup \{\infty\},$$

the composite of the Dieudonne determinant (cf. [Tae03]) with the degree function (degree in τ) on $\overline{K}\{\tau\}$. Then $\operatorname{rk}(\boldsymbol{G})$ is the unique integer such that $\deg \det \phi(a) = \operatorname{rk} \boldsymbol{G} \cdot \deg(a)$ for all $a \in A$.

Remark 10.11 (cf. Remark 8.3). The interpretation of the rank of an abelian A-module as in Remark 10.10 may be interpreted as meaning that if an A-module is abelian, then the action ϕ is a “nontrivial” deformation of the scalar action, in the sense that it involves “higher powers of τ”.

Lemma 10.12. *If $\boldsymbol{G} = (G, \phi)$ is an abelian A-module over K, then ϕ is injective.*

Proof. By definition, ϕ restricted to k is injective. If a is in $A \smallsetminus k$, then $\deg \det \phi(a) = \operatorname{rk} \boldsymbol{G} \deg(a)$ is non-zero, since $\operatorname{rk} \boldsymbol{G}$ is. In particular, $\phi(a)$ is non-zero. ∴

Definition 10.13. A *Drinfeld A-module* over K of is an abelian A-module over K with $G \cong \mathbb{G}_{a,K}$.

This is the classical definition. However:

Corollary 10.14. *All one-dimensional abelian A-modules over K are Drinfeld modules (and all one-dimensional abelian A-motives over K come from such).*

Proof. An action ϕ of an abelian A-module $\boldsymbol{G} = (G, \phi)$ is faithful by Lemma 10.12. By Theorem 9.8(b), this implies that G is a trivial form of $\mathbb{G}_{a,K}$, as required. ∴

Next, we will consider three closure properties of the category of A-motives over K in the category of all $\boldsymbol{A_K}$-modules.

Definition 10.15. Let R be a Dedekind ring, with quotient field L, and fix a projective R-module M. An R-submodule $M' \subset M$ is called *saturated in M* if $M' = M \cap (L \otimes_R M')$ as submodules of $L \otimes_R M$. Equivalently, if M'' is projective. We extend this definition to $\boldsymbol{A_K}$-modules by considering their underlying A_K-modules.

Proposition 10.16. *Consider a short exact sequence*

$$0 \longrightarrow \boldsymbol{M}' \longrightarrow \boldsymbol{M} \longrightarrow \boldsymbol{M}'' \longrightarrow 0$$

of $\boldsymbol{A_K}$-modules.

(a) *If $\boldsymbol{M}'$ and $\boldsymbol{M}''$ are A-motives over K of characteristic ι, then so is $\boldsymbol{M}$. In particular, A-Mot_K is closed under finite direct sums.*

(b) *Assume that M' is saturated in M. If $\boldsymbol{M}$ is an A-motive over K of characteristic ι then so are $\boldsymbol{M}'$ and $\boldsymbol{M}''$.*

Proof. Write $\boldsymbol{M}' = (M', \tau')$, $\boldsymbol{M} = (M, \tau)$ and $\boldsymbol{M}'' = (M'', \tau'')$, Note that both A_K and $K\{\tau\}$ are Noetherian. Hence, $\boldsymbol{M}$ is finitely generated over both A_K and $K\{\tau\}$ if and only if both $\boldsymbol{M}'$ and $\boldsymbol{M}''$ are. Furthermore, M' and M'' are A_K-projective of finite rank if and only if M is such and M' is saturated in M, since A_K is Dedekind. The interesting part of this proposition concerns nondegeneracy, and whether the characteristics turn out right. Let ι correspond to $\mathfrak{P}_0$.

Let us consider the Snake Lemma applied to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \sigma_* M' & \longrightarrow & \sigma_* M & \longrightarrow & \sigma_* M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tau'_{\mathrm{lin}}} & & \downarrow{\scriptstyle \tau_{\mathrm{lin}}} & & \downarrow{\scriptstyle \tau''_{\mathrm{lin}}} & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

(a): The assumptions imply first that $\boldsymbol{M}$ is nondegenerate, and then that $\operatorname{Lie}^* \boldsymbol{M}$ is composed of $\operatorname{Lie}^* \boldsymbol{M}'$ and $\operatorname{Lie}^* \boldsymbol{M}''$, in particular $\operatorname{Supp}\operatorname{Lie}^* \boldsymbol{M} \subset \{\mathfrak{P}_0\}$.

(b): The induced homomorphism $\operatorname{Lie}^* \boldsymbol{M} \longrightarrow \operatorname{Lie}^* \boldsymbol{M}''$ is surjective, hence $\operatorname{Supp}\operatorname{Lie}^* \boldsymbol{M}'' \subset \{\mathfrak{P}_0\}$ and $\operatorname{Lie}^* \boldsymbol{M}''$ is finite-dimensional over K. This last statement also implies that τ''_{lin} is injective, since $\sigma_* M''$ of equal rank as M''. So $\boldsymbol{M}''$ is an A-motive over K of required characteristic.

Clearly, τ'_{lin} is injective. By the injectivity of τ''_{lin} shown above, $\operatorname{Lie}^* \boldsymbol{M}' \longrightarrow \operatorname{Lie}^* \boldsymbol{M}$ is injective, so $\operatorname{Supp}\operatorname{Lie}^* \boldsymbol{M}' \subset \{\mathfrak{P}_0\}$. ∴

Remark 10.17. Let $\mathscr{A}$ denote the category of all torsion-free $\boldsymbol{A_K}$-modules. In $\mathscr{A}$, all kernels exist, and they agree with the kernel computed in the ambient abelian category of all $\boldsymbol{A_K}$-modules. Additionally, all cokernels exist in $\mathscr{A}$: The cokernel of a homomorphism $f : \boldsymbol{M} \to \boldsymbol{N}$ is the torsion-free $\boldsymbol{A_K}$-module $\boldsymbol{X}/\operatorname{Tor}(\boldsymbol{X})$, where $\boldsymbol{X} := \boldsymbol{N}/f(\boldsymbol{M})$; i.e., the quotient of $\boldsymbol{N}$ by the saturation of the set-theoretical image of f in $\boldsymbol{N}$. In general, this cokernel does not coincide with the cokernel $\boldsymbol{X}$ in the ambient category of all $\boldsymbol{A_K}$-modules, and $\mathscr{A}$ is *not* abelian: The categorical image and coimage differ in general by a torsion $\boldsymbol{A_K}$-module.

Corollary 10.18. *Let $\boldsymbol{M}$ be an A-motive over K, and consider two A-submotives $\boldsymbol{M}'$ and $\boldsymbol{M}''$ of $\boldsymbol{M}$. Then both $\boldsymbol{M}' + \boldsymbol{M}''$ and $\boldsymbol{M}' \cap \boldsymbol{M}''$ are A-motives over K of characteristic ι.*

Proof. Let us consider the following natural exact sequence of $\boldsymbol{A_K}$-modules.

$$0 \longrightarrow \boldsymbol{M}' \cap \boldsymbol{M}'' \longrightarrow \boldsymbol{M}' \oplus \boldsymbol{M}'' \longrightarrow \boldsymbol{M}' + \boldsymbol{M}'' \longrightarrow 0.$$

By Proposition 10.16(a), we have $\boldsymbol{M}' \oplus \boldsymbol{M}'' \in A\text{-}\mathrm{Mot}_K$. Now $M' + M''$ is A_K-projective, so $M' \cap M''$ is saturated in $M' \oplus M''$. Thus, by Proposition 10.16(b), both $\boldsymbol{M}' \cap \boldsymbol{M}''$ and $\boldsymbol{M}' + \boldsymbol{M}''$ lie in $A\text{-}\mathrm{Mot}_K$. ∴

As with all $\boldsymbol{A_K}$-modules, we may consider the tensor product of two A-motives.

Proposition 10.19. *Let $\boldsymbol{M}$, $\boldsymbol{N}$ be two A-motives over K of characteristic ι. Then so is their tensor product $\boldsymbol{M} \otimes_{\boldsymbol{A_K}} \boldsymbol{N}$.*

Proof. Clearly, $M \otimes_{A_K} N$ is a projective A_K-module, and $\tau_{\mathrm{lin}}^M \otimes \tau_{\mathrm{lin}}^N$ remains injective. One checks that the support of $\mathrm{Lie}^*(\boldsymbol{M} \otimes \boldsymbol{N})$ is contained in $\{\mathfrak{P}_0\}$. ∴

Proposition 10.20. *The category of A-motives over K is a tensor category over A.*

Proof. Proposition 10.19 shows that it is a tensor category. The unit is $\mathbf{1} = \boldsymbol{A_K}$, so since $A_K^\sigma = A$, our category is a tensor category over F. ∴

Chapter III

A-Isomotives

11 Isogeny

Definition 11.1.

(a) Let R be a Dedekind ring, and consider a homomorphism $f: M \longrightarrow N$ of R-modules. We say that f is an *isogeny* if both kernel and cokernel of f are torsion R-modules.

(b) A homomorphism of A-motives over K is called an *isogeny* if it is an isogeny of the underlying A_K-modules.

Note that if $M \xrightarrow{f} N$ is a homomorphism of finitely generated modules over a Dedekind ring and M is projective, then f is an isogeny if and only if M and N have equal rank and f is injective, which in turn is equivalent to $\mathrm{coker}(f)$ being a torsion module.

Example 11.2. Let $\boldsymbol{M}$ be an A-motive. Any $0 \neq a \in A$ gives rise to an isogeny $[a]_M: \boldsymbol{M} \longrightarrow \boldsymbol{M}$.

Definition 11.3. An isogeny $\boldsymbol{M} \to \boldsymbol{M}$ of the form $[a]_{\boldsymbol{M}}$ for $0 \neq a \in A$ is called a *standard isogeny*.

Proposition 11.4. *Let $\boldsymbol{M} \longrightarrow \boldsymbol{N}$ be an isogeny of A-motives. Then $\boldsymbol{M}$ and $\boldsymbol{N}$ have equal rank. Moreover, if one is abelian, then so is the other and both have equal dimension.*

Proof. A homomorphism of finitely generated projective modules over a Dedekind ring R is an isogeny if and only if the induced homomorphism over $\mathrm{Frac}(R)$ is an isomorphism – this follows from the structure theory of such modules. Moreover, the rank of such a module over R is equal to the dimension of the induced module over Frac(R). So we see that isogenous A-motives have equal rank.

Let $\boldsymbol{C} = (C, \tau^C)$ be the cokernel. It is clearly finitely generated over $K\{\tau\}$. Since $K\{\tau\}$ is Noetherian, this shows that M is finitely generated if and only if N is. The Snake Lemma shows that we have an exact sequence

$$0 \to \ker \tau^C_{\mathrm{lin}} \to \mathrm{Lie}^* \boldsymbol{M} \to \mathrm{Lie}^* \boldsymbol{N} \to \operatorname{coker} \tau^C_{\mathrm{lin}} \to 0,$$

where $\dim_K \ker \tau^C_{\mathrm{lin}} = \dim_K \operatorname{coker} \tau^C_{\mathrm{lin}}$ since $\dim_K \sigma_* C = \dim_K C$. Therefore, $\dim \boldsymbol{M} = \dim_K \mathrm{Lie}^* \boldsymbol{M} = \dim_K \mathrm{Lie}^* \boldsymbol{N} = \dim \boldsymbol{N}$. ∴

Definition 11.5. An isogeny $f : \boldsymbol{M} \longrightarrow \boldsymbol{N}$ of A-motives is called *separable* if it induces an isomorphism $\mathrm{Lie}^* \boldsymbol{M} \longrightarrow \mathrm{Lie}^* \boldsymbol{N}$. It is called *totally inseparable* if $\boldsymbol{N}/f(\boldsymbol{M})$ is a nilpotent $\boldsymbol{A_K}$-module.

Example 11.6. Let $\boldsymbol{M}$ be an A-motive. If an element $0 \neq a \in A$ is not divisible by $\mathrm{char}_A(K)$, then the standard isogeny $[a]_{\boldsymbol{M}}$ is separable.

Proof. The standard isogeny $[a]_{\boldsymbol{M}}$ induces on $\mathrm{Lie}^*(\boldsymbol{M})$ the homomorphism $[a]_{\mathrm{Lie}^* \boldsymbol{M}}$ given by left-multiplication by a. Since $\mathfrak{P}_0$ does not divide a and the support of $\mathrm{Lie}^* \boldsymbol{M}$ is contained in $\mathfrak{P}_0$, we see that $[a]_{\mathrm{Lie}^* \boldsymbol{M}}$ is invertible, and hence an isomorphism. ∴

Proposition 11.7. *Let $f : \boldsymbol{M} \longrightarrow \boldsymbol{N}$ be an isogeny of A-motives, and set $\boldsymbol{C} :=$* coker(f)*. Then f is separable if and only if $\boldsymbol{C}$ is a bijective $\boldsymbol{A_K}$-module.*

Proof. The definitions and the Snake Lemma applied to the following commutative diagram with exact rows show the stated equivalence.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \sigma_* M & \xrightarrow{\sigma_* f} & \sigma_* N & \longrightarrow & \sigma_* \operatorname{coker}(f) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tau^M_{\mathrm{lin}}} & & \downarrow{\scriptstyle \tau^M_{\mathrm{lin}}} & & \downarrow & & \\
0 & \longrightarrow & M & \xrightarrow{f} & N & \longrightarrow & \operatorname{coker}(f) & \longrightarrow & 0.
\end{array}$$

∴

Definition 11.8. Let R be a Dedekind ring, and let $M \cong R/\mathfrak{p}_1^{n_1} \oplus \cdots R/\mathfrak{p}_r^{n_r}$ be a torsion R-module (with not necessarily pairwise distinct $\mathfrak{p}_i$). The *characteristic ideal* of M is the ideal $\mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_r^{n_r}$ of R.

Definition 11.9. Let f be a separable isogeny of A-motives over K. Then $\boldsymbol{C} :=$ coker(f) is a representational $\boldsymbol{K}$-module in the sense of Definition 7.2. Therefore, $V_k(\boldsymbol{C})$ is a finite-dimensional k-vector space, and since V_k is natural (Proposition 7.3), we may consider $V_k(\boldsymbol{C})$ as a finitely-generated torsion A-module.

The *degree* of f is the characteristic ideal of A corresponding to $V_k(\boldsymbol{C})$.

Remark 11.10. In general, $\deg(f)$ is not principal, and even if it is, there is no canonical generator. However, for $A = k[t]$ and semisimple "pure" A-motives over finite fields, there is a canonical generator, cf. [Har06, Proposition 3.30].

We refer to [Har06, Definition 1.3.2] for a discussion of a suitable notion of degree for all isogenies, separable or not.

Proposition 11.11. *Let $\boldsymbol{M} \xrightarrow{f} \boldsymbol{N}$ be an isogeny of A-motives. Then there exists an A-motive $\boldsymbol{N}'$ and a factorisation $f = f' \circ f^{sep}$ such that $f^{sep} : \boldsymbol{M} \longrightarrow \boldsymbol{N}'$ is a separable isogeny and $f' : \boldsymbol{N}' \longrightarrow \boldsymbol{N}$ is a totally inseparable isogeny.*

Proof. Let $\boldsymbol{C} = (C, \tau)$ denote the cokernel of the given isogeny, By Proposition 5.8 we have an exact sequence

$$0 \longrightarrow \boldsymbol{C}^{\mathrm{bij}} \longrightarrow \boldsymbol{C} \longrightarrow \boldsymbol{C}^{\mathrm{nil}} \longrightarrow 0.$$

As in Remark 5.19, this gives the desired factorisation. ∴

Theorem 11.12. *In generic characteristic all isogenies of A-motives are separable.*

Proof. We show the following: If $\boldsymbol{M} = (M, \tau)$ is a finitely-generated torsion $\boldsymbol{A_K}$-module such that $\mathrm{Supp}(\ker \tau_{\mathrm{lin}}) \cup \mathrm{Supp}(\mathrm{coker}\, \tau_{\mathrm{lin}}) \subset \{\mathfrak{P}_0\}$, then $\boldsymbol{M}$ is bijective. By Proposition 11.7(a), this implies the statement of this theorem. I thank Gebhard Böckle for help in simplifying the following proof.

Since $\mathfrak{P}_0$ lies over the generic prime of A, we have:

$$\text{The prime ideals } \sigma_*^n(\mathfrak{P}_0) \text{ for } n \geq 0 \text{ are pairwise different.} \tag{11.13}$$

Set $X := \ker(\tau_{\mathrm{lin}})$ and $Y := \mathrm{coker}(\tau_{\mathrm{lin}})$. We consider the exact sequence

$$0 \longrightarrow X \longrightarrow \sigma_*(M) \xrightarrow{\tau_{\mathrm{lin}}} M \longrightarrow Y \longrightarrow 0.$$

To every torsion A_K-module $N \cong \bigoplus_{\mathfrak{a}} A_K/\mathfrak{a}$ we may associate its characteristic ideal $\chi(N) := \prod \mathfrak{a}$. We have $\dim_K X = \dim_K Y$, so $\chi(X) = \chi(Y) = \mathfrak{P}_0^n$ for some $n \geq 0$, and

$$\chi(\sigma_* M) = \chi(M). \tag{11.14}$$

Now (11.14) means that σ_* permutes the (finitely many) prime ideals lying in the support of M. Therefore, for every such prime ideal $\mathfrak{Q}$ in the support there exists an integer $m \geq 0$ such that $\sigma_*^m \mathfrak{Q} = \mathfrak{Q}$. Now (11.13) excludes the possibility that $\mathfrak{P}_0$ is contained in the support of M. It follow that both X and Y are zero, and so $\boldsymbol{M}$ is indeed bijective. ∴

Theorem 11.15. *Let $f : \boldsymbol{M} \longrightarrow \boldsymbol{N}$ be an isogeny of A-motives over K. Then* $\mathrm{Ann}_A(\mathrm{coker}\, f) \neq 0$.

Proof. By Proposition 11.11 and Theorem 11.12 we may assume that either f is separable, or that the characteristic is special and f is totally inseparable.

In the first case, $\deg(f) \subset \mathrm{Ann}_A(\mathrm{coker}(f))$, since $\deg(f)$ annihilates $V_k(\mathrm{coker}\, f)$ and V_k is natural (cf. Definition 11.9 and Proposition 7.3).

We turn to the second case. It follows if we show the following: Let $\boldsymbol{M} = (M, \tau)$ be a nilpotent finite $\boldsymbol{A_K}$-module such that $\mathrm{Supp}\ker\tau_{\mathrm{lin}} \cup \mathrm{Supp}\,\mathrm{coker}\,\tau_{\mathrm{lin}} \subset \{\mathfrak{P}_0\}$. Then a power of $\mathfrak{p}_0 = \mathfrak{P}_0 \cap A \neq 0$ annihilates M.

We proceed by induction in $n \geq 1$ over the following statement: A power of $\mathfrak{p}_0$ annihilates $\mathrm{coker}\,\tau_{\mathrm{lin}}^n$. Since $C = \mathrm{coker}\,\tau_{\mathrm{lin}}^n$ for $n \gg 0$, this proves what we have to prove.

For $n = 1$, it is true by assumption. For $n > 1$, we consider the short exact sequence

$$\mathrm{im}(\tau_{\mathrm{lin}}^n)/\,\mathrm{im}(\tau_{\mathrm{lin}}^{n+1}) \to M/\,\mathrm{im}(\tau_{\mathrm{lin}}^{n+1}) \to M/\,\mathrm{im}(\tau_{\mathrm{lin}}^n) \to 0 \tag{11.16}$$

Next, the kernel of the surjective composite homorphism

$$\sigma_*^n M \xrightarrow{\tau_{\mathrm{lin}}^n} \mathrm{im}(\tau_{\mathrm{lin}}^n) \to \mathrm{im}(\tau_{\mathrm{lin}}^n)/\,\mathrm{im}(\tau_{\mathrm{lin}}^{n+1})$$

contains $\sigma_*^n\,\mathrm{im}(\tau_{\mathrm{lin}})$, so the above composite homorphism factors through a homorphism

$$\sigma_*^n(\mathrm{coker}\,\tau_{\mathrm{lin}}) = \sigma_*^n M/\sigma_*^n(\mathrm{im}\,\tau_{\mathrm{lin}}) \to \mathrm{im}(\tau_{\mathrm{lin}}^n)/\,\mathrm{im}(\tau_{\mathrm{lin}}^{n+1}), \tag{11.17}$$

which is again surjective.

Splicing the homomorphisms (11.16) and (11.17) together, we obtain an exact sequence

$$\sigma_*^n(\mathrm{coker}\,\tau_{\mathrm{lin}}) \to \mathrm{coker}(\tau_{\mathrm{lin}}^{n+1}) \to \mathrm{coker}(\tau_{\mathrm{lin}}^n) \to 0.$$

By induction, a power of $\mathfrak{p}_0$ annihilates $\mathrm{coker}(\tau_{\mathrm{lin}}^n)$. Since $\mathrm{Supp}(\sigma_*^n(\mathrm{coker}\,\tau_{\mathrm{lin}}) \subset \{\sigma_*^n\mathfrak{P}_0\}$ and $\sigma_*^n\mathfrak{P}_0 \cap A = \mathfrak{p}_0$, a power of $\mathfrak{p}_0$ also annihilates $\sigma_*^n(\mathrm{coker}\,\tau_{\mathrm{lin}})$. Therefore we have proven our induction step! $\therefore$

Remark 11.18. Gebhard Böckle remarks that every totally inseparable isogeny may be filtered in such a way that τ_{lin} vanishes on the consecutive subquotients. This gives an alternative proof in the “second case” of the proof of Theorem 11.15.

Proposition 11.19. *Let $f : \boldsymbol{M} \longrightarrow \boldsymbol{N}$ be an isogeny of A-motives over K. There exists an element $0 \neq a \in A$, and an isogeny $f^\vee : \boldsymbol{N} \longrightarrow \boldsymbol{M}$ such that $f^\vee \circ f = [a]_{\boldsymbol{M}}$ and $f \circ f^\vee = [a]_{\boldsymbol{N}}$. In particular, every isogeny is a factor of a standard isogeny.*

Proof. Let $\boldsymbol{C} := \mathrm{coker}(f)$, a finite $\boldsymbol{A_K}$-module. By Theorem 11.15, there exists an element $0 \neq a \in A$ such that $a \cdot \boldsymbol{C} = 0$. Therefore, $a \cdot \boldsymbol{N}$ is contained in $f(\boldsymbol{M}) \cong \boldsymbol{M}$, so we obtain an isogeny $\boldsymbol{M} \xrightarrow{f^\vee} \boldsymbol{N}$ with $f \circ f^\vee = [a]_{\boldsymbol{N}}$. Since f is a homomorphism of $\boldsymbol{A_K}$-modules, we have $f \circ f^\vee \circ f = [a]_{\boldsymbol{N}} \circ f = f \circ [a]_{\boldsymbol{M}}$, so since f is injective we obtain $f^\vee \circ f = [a]_{\boldsymbol{M}}$. $\therefore$

Remark 11.20. In as much as for a given isogeny f the element $0 \neq a \in A$ annihilating $\mathrm{coker}(f)$ is not unique (cf. Remark 11.10 and the proof of Theorem 11.15), the same is true for "the" dual isogeny $f^\vee$ constructed in Proposition 11.19: It depends on the choice of such an $a \neq 0$.

Corollary 11.21. *The relation of isogeny is an equivalence relation on the category of A-motives.*

Proof. The relation is clearly reflexive and transitive. Proposition 11.19 shows that it is also symmetric. ∴

12 Inverting isogenies

In this section, we construct the category of A-motives "up to isogeny", by formally inverting all isogenies. We give it a simple, concrete interpretation and embed it into the category of restricted $\boldsymbol{F_K}$-modules by considering generic stalks. This implies that the category of A-motives "up to isogeny" is an F-finite abelian tensor category over F.

Definition 12.1. The category $A\text{-Isomot}_K^{\mathrm{eff}}$ of *A-motives over K up to isogeny* (or *effective A-isomotives over K*) is obtained by formally inverting all isogenies, i.e., by localizing $A\text{-Mot}_K$ with respect to the class of isogenies (cf. [Wei94]). It is a tensor category over F by Proposition 10.20

Let $A\text{-Mot}_K \odot_A F$ denote the additive scalar extension (Definition 1.3) of $A\text{-Mot}_K$ from A to F. It is also a tensor category over F.

Proposition 12.2. *The natural functor*

$$A\text{-Mot}_K \odot_A F \longrightarrow A\text{-Isomot}_K^{\mathrm{eff}}$$

is an equivalence of tensor categories over F.

Proof. The functor is well-defined since nonzero elements of A induce isogenies on $A\text{-Mot}_K$ (cf. Example 11.2). Conversely, we have seen in Theorem 11.15 that every isogeny of A-motives is a factor of a standard isogeny given by a non-zero element of A, hence our functor is full.

Both categories have the objects of $A\text{-Mot}_K$ as underying objects, so they are clearly tensor categories and the natural functor is a tensor functor. The unit object of the tensor category $A\text{-Mot}_K$ is $\mathbf{1} = (A_K, \sigma)$, so since $A_K^\sigma = A$ both $A\text{-Mot}_K \odot_A F$ and $A\text{-Isomot}_K^{\mathrm{eff}}$ are tensor categories over F. ∴

For the purposes of explicit calculations, we next wish to embed $A\text{-Isomot}_K^{\text{eff}}$ into another category of modules over a bold ring. To every A-motive $\boldsymbol{M}$ over K we associate its "generic stalk" $\widetilde{\boldsymbol{M}} := \boldsymbol{F_K} \otimes_{\boldsymbol{A_K}} \boldsymbol{M}$, a restricted $\boldsymbol{F_K}$-module.

Theorem 12.3. *The functor "generic stalk" gives rise to a fully faithful F-linear tensor functor*

$$A\text{-Isomot}_K^{\text{eff}} \longrightarrow \left(\!\!\left(\text{ restricted } \boldsymbol{F_K}\text{-modules } \right)\!\!\right),$$

and its essential image is closed under subquotients.

Proof. By construction and Proposition 12.2 the functor is clearly an F-linear faithful tensor functor.

To show that it is full, we apply Corollary 10.18 in the following way. Consider two A-motives $\boldsymbol{M}$ and $\boldsymbol{N}$ over K, and a homomorphism of $\boldsymbol{F_K}$-modules $\widetilde{f} : \widetilde{\boldsymbol{M}} \longrightarrow \widetilde{\boldsymbol{N}}$. Set $\boldsymbol{X} := \widetilde{f}(\boldsymbol{M}) \cap \boldsymbol{N}$. Now $\widetilde{f}(\boldsymbol{M})$ is an A-motive by Proposition 10.16, so $\boldsymbol{X}$ is also an A-motive by Corollary 10.18. The inclusion $\boldsymbol{X} \subset \widetilde{f}(\boldsymbol{M})$ is an isogeny of A-motives. Hence $\boldsymbol{M} \longrightarrow \widetilde{f}(\boldsymbol{M}) \supset \boldsymbol{X} \subset \boldsymbol{N}$ is a composite of homomorphisms and inverses of isogenies, as required.

The statement about the essential image follows from Proposition 10.16. ∴

Proposition 12.4. *The category of effective A-isomotives over K is an F-finite abelian tensor category over F.*

Proof. We already know that this category is a tensor category over F. Theorem 12.3 implies it is abelian, and together with Propositions 10.4 and 12.2 implies that it is F-finite. ∴

Proposition 12.5. *For every maximal ideal $\mathfrak{p} \neq \mathfrak{p}_0$ of A and every A-motive $\boldsymbol{M}$ over K, the $\mathscr{O}_{(\mathfrak{p}),K}$-module $\mathscr{O}_{(\mathfrak{p}),K} \otimes_{\boldsymbol{A_K}} \boldsymbol{M}$ is restricted. In particular, the functor "generic stalk" of Theorem 12.3 has values in $\mathscr{O}_{(\mathfrak{p}),K}$-etale $\boldsymbol{F_K}$-modules.*

Proof. Since $\text{Supp}(\text{Lie}^* \boldsymbol{M}) \subset \{\mathfrak{P}_0\}$ for every A-motive over K, the $\mathscr{O}_{(\mathfrak{p}),K}$-module $\mathscr{O}_{(\mathfrak{p}),K} \otimes_{\boldsymbol{A_K}} \boldsymbol{M}$ is restricted. So $\boldsymbol{F_K} \otimes_{\boldsymbol{A_K}} \boldsymbol{M} \cong \boldsymbol{F_K} \otimes_{\mathscr{O}_{(\mathfrak{p}),K}} (\mathscr{O}_{(\mathfrak{p}),K} \otimes_{\boldsymbol{A_K}} \boldsymbol{M})$, the generic stalk of $\boldsymbol{M}$, is $\mathscr{O}_{(\mathfrak{p}),K}$-etale. ∴

Definition 12.6. The category of *A-isomotives over K* (or *F-motives over K*) is the rigid abelian tensor subcategory generated by the image of the category of effective A-isomotives over K in the category of all restricted $\boldsymbol{F_K}$-modules. We denote it by $F\text{-Mot}_K$.

13 Semisimplicity

Definition 13.1. We say that an A-motive is *simple* (resp. *semisimple*) if it is simple (resp. semisimple) as an object of the category of A-motives up to isogeny.

Proposition 13.2. *Let $\boldsymbol{M}$ be an A-motive. Then $\boldsymbol{M}$ is simple if and only if every nonzero A-submotive of $\boldsymbol{M}$ is isogenous to $\boldsymbol{M}$.*

Proof. If $\boldsymbol{M} = (M, \tau)$ is simple, and $\boldsymbol{M}' = (M', \tau')$ is a nonzero submotive, then the inclusion $\boldsymbol{M}' \hookrightarrow \boldsymbol{M}$ induces an isomorphism in $A\text{-Isomot}_K^{\text{eff}}$ by assumption. In particular, M and M' have equal A_K-rank, so the inclusion $M' \subset M$ is an isogeny.

Conversely, assume that every nonzero A-submotive of $\boldsymbol{M} = (M, \tau)$ is isogenous to $\boldsymbol{M}$. Then, by the last sentence in Theorem 12.3, every subobject of $\boldsymbol{M}$ in $A\text{-Isomot}_K^{\text{eff}}$ is isomorphic to $\boldsymbol{M}$. ∴

Proposition 13.3. *Let $\boldsymbol{M}$ be an A-motive. Then $\boldsymbol{M}$ is semisimple if and only if it is isogenous to a direct sum of simple A-motives.*

Proof. The proof of this equivalence parallels the proof of Proposition 13.2. ∴

Corollary 13.4. *For every semisimple A-motive $\boldsymbol{M}$, the F-algebra $F \otimes_A \operatorname{End}(\boldsymbol{M})$ is finite-dimensional and semisimple*

Proof. By 12.4, the algebra in question is finite F-dimensional. Since $\boldsymbol{M}$ is semisimple of finite length, Schur's Lemma shows that this algebra is semisimple. ∴

Not every A-motive is semisimple, as the following example shows.

Example 13.5. We quote [Har06, Example 3.11]: Set $A = k[t]$, $K = k(\alpha)$ with α transcendental over k, and

$$\iota : A \to K, \quad t \mapsto \theta$$

k-linear. We consider the A-motive $\boldsymbol{M} = (M, \tau)$ of characteristic ι over K given by $M := A_K^{\oplus 2}$ and

$$\tau := (t-\theta)\begin{pmatrix} \alpha & 1 \\ 0 & 1 \end{pmatrix}\sigma$$

This is an A-motive of rank 2, and it may be checked that it is abelian of dimension 2. Obviously, it is not simple, since projection onto the second factor displays the quotient A-motive $\boldsymbol{M}'' := (A_K, (t-\theta)\sigma)$. Assume that $\boldsymbol{M}$ is semisimple. Then, by Theorem 12.3, the projection $\boldsymbol{M} \to \boldsymbol{M}''$ has a section after passage to the associated category of $\boldsymbol{F_K}$-modules. That is, there exists an element $f \in F_K = k(t)(\alpha)$ such that $(f, 1)^T \circ \tau_{\boldsymbol{M}''} = \tau_{\boldsymbol{M}} \circ (f, 1)^T$. This means that

$$\begin{pmatrix} f \\ 1 \end{pmatrix} \cdot (t-\theta) = (t-\theta)\begin{pmatrix} \alpha & 1 \\ 0 & 1 \end{pmatrix} \cdot \sigma\begin{pmatrix} f \\ 1 \end{pmatrix},$$

which is equivalent to the equation $f = \alpha\sigma(f) + 1$. Calculating in $k(t)(\!(\alpha)\!)$, we may write $f = \sum_{i=N}^{\infty} f_i\alpha^i$, with $f_i \in k(t)$, and we must solve

$$\sum_{i=N}^{\infty} f_i\alpha^i - \sum_{i=N}^{\infty} f_i\alpha^{qi+1} = 1.$$

It follows that $N = 0$, $f_0 = 1$, and then

$$f_i = \begin{cases} f_0 & \text{if } i = 1 + q + \cdots q^j \text{ for some } j \geq 1 \\ 0 & \text{else} \end{cases}$$

By a well-known characterisation, we have

$$k(t)(\alpha) = \left\{ \sum_{i=N}^{\infty} f_i\alpha^i \in k(t)(\!(\alpha)\!) \mid \exists\, m, n > 0 : \ f_{i+n} = f_i \ \forall\, i \geq n \right\}$$

Therefore the unique solution f lies in $k(t)(\!(\alpha)\!) \smallsetminus k(t)(\alpha)$, so there is no section of $\boldsymbol{M} \to \boldsymbol{M}''$ in the category of $\boldsymbol{F_K}$-modules, and $\boldsymbol{M}$ is not semisimple.

Remark 13.6. See also [PaR03], where extension groups of Drinfeld modules and certain more general A-motives are discussed.

On the other hand, one of the main results of [Har06] is: Every pure (cf. [And86] for the definition of purity) abelian A-motive over a *finite field* K becomes semisimple after a finite extension K'/K of the base field.

However, as do all objects of finite length of a given abelian category, every A-isomotive admits two canonical filtrations with semisimple subquotients.

Definition 13.7 (Socle and radical)**.** Let X be an object of finite length of an abelian category.

(a) The *socle* of X is the sum of all simple subobjects of X. This is the largest semisimple subobject of X, and we denote it by $\operatorname{soc}(X)$.

(b) Inductively, set $\operatorname{soc}^0(X) := 0$, $\operatorname{soc}^1(X) := \operatorname{soc}(X)$, and for $i \geq 2$ let $\operatorname{soc}^i(X) := \pi^{-1}(X/\operatorname{soc}^{i-1}(X))$, where π denotes the canonical projection $\pi : X \longrightarrow X/\operatorname{soc}^{i-1}(X)$. The collection of all $(\operatorname{soc}^i(X))_{i\geq 0})$ is called the *socle filtration* of X. The *socle length* of X is the smallest integer i such that $\operatorname{soc}^i(X) = X$.

(c) The *radical* of X is the intersection of all maximal subobjects of X. This is the kernel of the projection of X to its largest semisimple quotient, and we denote it by $\operatorname{rad}(X)$.

(d) Inductively, set $\mathrm{rad}_0(X) := X$, $\mathrm{rad}_1(X) := \mathrm{rad}(X)$, and for $i \geq 2$ let $\mathrm{rad}_i(X) := \mathrm{rad}(\mathrm{rad}_{i-1}(X))$. The collection of all $(\mathrm{rad}_i(X))_{i\geq 0})$ is called the *radical filtration* of X. The *radical length* of X is the smallest integer i such that $\mathrm{rad}_i(X) = 0$.

These socle and radical filtrations are functorial, and a given object X is semisimple if and only if $\mathrm{soc}(X) = X$, which in turn is equivalent to $\mathrm{rad}(X) = 0$. It can be shown that the socle and radical lengths coincide.

Given an A-motive $\boldsymbol{M}$ over K, the socle filtration $\{\mathrm{soc}^i(\widetilde{\boldsymbol{M}})\}_i$ of its generic fibre $\widetilde{\boldsymbol{M}}$ gives a canonical filtration $\mathrm{soc}^i(\boldsymbol{M}) := \boldsymbol{M} \cap \mathrm{soc}^i(\widetilde{\boldsymbol{M}})$ of $\boldsymbol{M}$ such that the successive subquotients are semisimple A-motives over K. The analogous statement is true for the radical filtration.

14 Tate modules

Let k, F, A, K, ι be as in Sections 6 and 8. We also fix a prime $\mathfrak{p} \neq \mathfrak{p}_0 = \ker\iota$ of A. Let $A_\mathfrak{p} = \varprojlim_n A/\mathfrak{p}^n$ denote the completion of the ring A at $\mathfrak{p}$, and let $F_\mathfrak{p}$ denote the completion of the field F at $\mathfrak{p}$.

We start within the setup of k-linear generalised A-modules over K. For any ideal $I \subset A$, and any $\boldsymbol{G} = (G, \phi) \in \mathscr{G}_{A,K}$, let

$$\boldsymbol{G}[I] := \bigcap_{a\in I} \ker\phi(a)$$

be the A-submodule scheme over K of I-torsion points of $\boldsymbol{G}$. We may then also consider $\boldsymbol{G}[I](K^{\mathrm{sep}})$, the points of $\boldsymbol{G}[I]$ in K^{sep}. This is an A-module on which Γ_K acts.

Definition 14.1. Let $\boldsymbol{G}$ be a k-linear generalised A-module over K.

(a) The *(integral) Tate module* at $\mathfrak{p}$ of $\boldsymbol{G}$ is the projective limit

$$\mathrm{T}_\mathfrak{p}(\boldsymbol{G}) := \varprojlim_n \boldsymbol{G}[\mathfrak{p}^n](K^{\mathrm{sep}}),$$

considered as an $A_\mathfrak{p}$-module on which Γ_K acts.

(b) The *(rational) Tate module* at $\mathfrak{p}$ of $\boldsymbol{G}$ is defined to be $\mathrm{V}_\mathfrak{p}(\boldsymbol{G}) := F_\mathfrak{p} \otimes_{A_\mathfrak{p}} \mathrm{T}_\mathfrak{p}(\boldsymbol{G})$.

Remark 14.2. In general, one cannot say whether $\mathrm{T}_\mathfrak{p}(\boldsymbol{G})$ is finitely-generated over $A_\mathfrak{p}$, nor even, if so, whether the ranks of these representations coincide for varying $\mathfrak{p}$ (cf. [Yu97]). The situation will improve for *abelian* A-modules over K.

Recall that, for R equal to $A_\mathfrak{p}$ or $F_\mathfrak{p}$, continuous representations of Γ_K over R are classified by representational $\boldsymbol{D}_R$-modules (Definition 7.2) via the functors V_R and D_R of Proposition 7.3.

Definition 14.3. (a) Let $\boldsymbol{M}$ be a restricted $\mathscr{O}_{(\mathfrak{p}),K}$-module. The *(integral) Tate module* at $\mathfrak{p}$ of $\boldsymbol{M}$ is the integral $\mathfrak{p}$-adic representation of Γ_K defined by

$$\mathrm{T}_\mathfrak{p}(\boldsymbol{M}) := V_{A_\mathfrak{p}}(\mathscr{O}_{\boldsymbol{K},\mathfrak{p}} \otimes_{\mathscr{O}_{(\mathfrak{p}),K}} \boldsymbol{M}).$$

(b) Let $\boldsymbol{M}$ be an $\mathscr{O}_{(\mathfrak{p}),K}$-etale $\boldsymbol{F_K}$-module. The *(rational) Tate module* at $\mathfrak{p}$ of $\boldsymbol{M}$ is the rational $\mathfrak{p}$-adic representation of Γ_K defined by

$$\mathrm{V}_\mathfrak{p}(\boldsymbol{M}) := V_{F_\mathfrak{p}}(\boldsymbol{F}_{\boldsymbol{K},\mathfrak{p}} \otimes_{\boldsymbol{F_K}} \boldsymbol{M}).$$

(c) If $\boldsymbol{M}$ is an $\mathscr{O}_{(\mathfrak{p}),K}$-etale $\boldsymbol{A_K}$-module, we set $\mathrm{T}_\mathfrak{p}(\boldsymbol{M}) := \mathrm{T}_\mathfrak{p}(\mathscr{O}_{(\mathfrak{p}),K} \otimes_{\boldsymbol{A_K}} \boldsymbol{M})$ and $\mathrm{V}_\mathfrak{p}(\boldsymbol{M}) := \mathrm{V}_\mathfrak{p}(\boldsymbol{F}_{\mathfrak{p},K} \otimes_{\boldsymbol{A_K}} \boldsymbol{M}) = F_\mathfrak{p} \otimes_{A_\mathfrak{p}} \mathrm{T}_\mathfrak{p}(\boldsymbol{M})$.

In particular, to every A-motive $\boldsymbol{M}$ over K we have an associated integral Tate module $\mathrm{T}_\mathfrak{p}(\boldsymbol{M})$ and an associated rational Tatel module $\mathrm{V}_\mathfrak{p}(\boldsymbol{M})$, since $\boldsymbol{M}$ is $\mathscr{O}_{(\mathfrak{p}),K}$-etale by Proposition 12.5.

By Proposition 7.3, we know that Tate module of a restricted $\mathscr{O}_{(\mathfrak{p}),K}$-module (resp. $\mathscr{O}_{(\mathfrak{p}),K}$-etale $\boldsymbol{F_K}$-module) $\boldsymbol{M}$ is a continuous Galois representation over $A_\mathfrak{p}$ (resp. $F_\mathfrak{p}$) of rank rk $\boldsymbol{M}$.

Let Ω_A denote the A-module of Kähler differentials of A over k. It is a locally free A-module of rank 1.

Proposition 14.4. *Let* $\boldsymbol{G}$ *be an abelian* A*-module over* K *with associated abelian* A*-motive* $\boldsymbol{M} := M(\boldsymbol{G})$ *over* K*. Then there exists a canonical* $A_\mathfrak{p}$*-bilinear* Γ_K*-equivariant pairing*

$$\mathrm{T}_\mathfrak{p}(\boldsymbol{G}) \times \mathrm{T}_\mathfrak{p}(\boldsymbol{M}) \longrightarrow \Omega_A \otimes_A A_\mathfrak{p}.$$

This pairing induces a canonical $F_\mathfrak{p}$*-bilinear* Γ_K*-equivariant pairing*

$$\mathrm{V}_\mathfrak{p}(\boldsymbol{G}) \times \mathrm{V}_\mathfrak{p}(\boldsymbol{F_K} \otimes_{\boldsymbol{A_K}} \boldsymbol{M}) \longrightarrow \Omega_A \otimes_A F_\mathfrak{p}.$$

Proof. We refer to [And86, Proposition 1.8.3] and [Gos96, Theorem 5.6.8]. $\therefore$

It follows that the diagram of categories and functors

$$\begin{array}{ccc}
\left(\!\!\left(\begin{matrix}\text{abelian}\\ A\text{-modules over } K\end{matrix}\right)\!\!\right)^{op} & \xrightarrow{\ \mathrm{T}_\mathfrak{p}\ } & \left(\!\!\left(\begin{matrix}\text{integral } \mathfrak{p}\text{-adic Galois}\\ \text{representations}\end{matrix}\right)\!\!\right)^{op} \\
\cong \big\downarrow M & & \cong \big\downarrow (-)^\vee \\
\left(\!\!\left(\begin{matrix}\text{abelian}\\ A\text{-motives over } K\end{matrix}\right)\!\!\right) & \xrightarrow{\ \mathrm{T}_\mathfrak{p}\ } & \left(\!\!\left(\begin{matrix}\text{integral } \mathfrak{p}\text{-adic Galois}\\ \text{representations}\end{matrix}\right)\!\!\right)
\end{array}$$

commutes up to a twist by $\Omega_A \otimes_A A_{\mathfrak{p}}$, and the study of the Tate modules of abelian A-modules over K is reduced to the study of the Tate modules of abelian A-motives over K.

An analogous diagram exists for effective A-isomotives arising from abelian A-motives, abelian A-modules “up to isogeny” and rational $\mathfrak{p}$-adic Galois representations, and the analogous remarks hold true.

Chapter IV

Scalar Extension of Restricted Modules

This chapter is inspired by [Bou81].

In this chapter, we assume that k is a finite field, $F \supset k$ is *any* field extension, and $K \supset k$ is a field extension such that K has a finite number of roots of unity.

Recall that by Definition 5.20 an $\boldsymbol{F_K}$-module $\boldsymbol{M} = (M, \tau)$ is *restricted* if M is finitely-generated torsion-free over F_K and τ_{lin} is bijective. We have two natural functors on (restricted) $\boldsymbol{F_K}$-modules, namely the functor of invariants mapping $\boldsymbol{M}$ to the F-module $\boldsymbol{M}^\tau = \{m \in M : \tau(m) = m\}$, and the socle functor mapping $\boldsymbol{M}$ to its largest semisimple submodule soc $\boldsymbol{M}$.

For a given field extension $F' \supset F$, we wish to understand the behaviour of the two functors “invariants” and “socle” with respect to scalar extension. We shall establish that, in a certain sense, they both commute with scalar extension.

15 Invariant computations

Let $F' \supset F$ be a *any* field extension. The main result of this section is:

Theorem 15.1. *Let $\boldsymbol{M}$ be a restricted $\boldsymbol{F_K}$-module. The following natural F'-module homomorphism is an isomorphism:*

$$F' \otimes_F \boldsymbol{M}^\tau \longrightarrow (\boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{M})^\tau.$$

Proposition 15.2. *Let $\boldsymbol{M}, \boldsymbol{N}$ be two restricted $\boldsymbol{F_K}$-modules. The following natural F'-module homomorphism is an isomorphism:*

$$F' \otimes_F \operatorname{Hom}_{\boldsymbol{F_K}}(\boldsymbol{M}, \boldsymbol{N}) \longrightarrow \operatorname{Hom}_{\boldsymbol{F'_K}}(\boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{M}, \boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{N}).$$

Proof. This follows by applying Theorem 15.1 to $\boldsymbol{X} := \boldsymbol{M}^\vee \otimes_{\boldsymbol{F_K}} \boldsymbol{N}$ and taking τ-invariants, since $\mathrm{Hom}_{\boldsymbol{F_K}}(\boldsymbol{M}, \boldsymbol{N}) = \boldsymbol{X}^\tau$ and $\mathrm{Hom}_{\boldsymbol{F'_K}}(\boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{M}, \boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{N}) = (\boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{X})^\tau$. ∴

Proof of Theorem 15.1. Let $\boldsymbol{M}' := \boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{M}$ denote the restricted $\boldsymbol{F'_K}$-module associated to $\boldsymbol{M}$. Since the homomorphism $F' \otimes_F F_K \hookrightarrow F'_K = \mathrm{Frac}(F' \otimes_F F_K)$ is injective and the functor $(-)^\tau$ is left-exact, it follows that the homomorphism

$$F' \otimes_F \boldsymbol{M}^\tau = (F' \otimes_F \boldsymbol{F_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{M})^\tau \hookrightarrow (\boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{M})^\tau$$

is injective. We must show that it is surjective!

On the other hand, the statement of the theorem is transitive in towers of field extensions $F'' \supset F' \supset F$, i.e., if the theorem is true for $F'' \supset F'$ and $F' \supset F$, then it is true for $F'' \supset F$.

Moreover, we may assume that $F' \supset F$ is finitely generated, since for every element $m' \in (\boldsymbol{M}')^\tau$ there exists a finitely generated field extension $F' \supset F^{fg} \supset F$ such that m' lies in $(\boldsymbol{M}^{\boldsymbol{fg}})^\tau$, where $\boldsymbol{M}^{\boldsymbol{fg}} := \boldsymbol{F}^{\boldsymbol{fg}}_{\boldsymbol{K}} \otimes_{\boldsymbol{F_K}} \boldsymbol{M}$ with $\boldsymbol{F}^{\boldsymbol{fg}}_{\boldsymbol{K}} := (\mathrm{Frac}(F^{fg} \otimes_F F_K, \mathrm{id} \otimes \sigma)$.

All in all, the theorem reduces to the two special cases of $F' \supset F$ finite, and $F' \supset F$ purely transcendental of degree 1. They are settled in the following lemma and proposition. ∴

Lemma 15.3. *Theorem 15.1 is true for $F' \supset F$ finite.*

Proof. If $F' \supset F$ is finite, we have $F' \otimes_F \boldsymbol{F_K} \cong \boldsymbol{F'_K}$, and hence

$$F' \otimes_F \boldsymbol{M}^\tau = (F' \otimes_F \boldsymbol{F_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{M})^\tau \cong (\boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{M})^\tau$$

as claimed. ∴

Proposition 15.4. *Theorem 15.1 is true for $F' = F(X)$ purely transcendental of degree 1 over F.*

For the proof of Proposition 15.4, we need to extend the notion of "denominator" of a rational function to slightly more general situation.

By Corollary 6.10, the ring F_K is a finite product $Q_1 \times \cdots \times Q_s$ of fields Q_i. We set $F_K(X) := \mathrm{Frac}(F(X) \otimes_F F_K) = Q_1(X) \times \cdots \times Q_s(X)$. For $f_i \in Q_i(X)$ the *denominator* $\mathrm{den}(f_i) \in Q_i[X]$ is defined, it is a monic polynomial. For $f = (f_i)_i \in F_K(X) = Q_1(X) \times \cdots \times Q_s(X)$, we set $\mathrm{den}(f) := (\mathrm{den}\, f_i)_i$.

Similarly, for $f_i, g_i \in Q_i[X]$ the *least common multiple* $\mathrm{lcm}(f_i, g_i) \in Q_i[X]$ is defined, it is a monic polynomial. For $f = (f_i), g = (g_i) \in F_K[X] := F[X] \otimes_F F_K = Q_1[X] \times \cdots \times Q_s[X]$, we set $\mathrm{lcm}(f, g) := (\mathrm{lcm}(f_i, g_i))_i$.

Note that for $f, g \in F_K(X)$, the following relation holds:

$$\mathrm{den}(f + g) \mid \mathrm{lcm}(\mathrm{den}\, f, \mathrm{den}\, g), \tag{15.5}$$

where $|$ denotes divisibility in $F_K[X]$.

We may now characterise the subring $F(X) \otimes_F F_K$ of $F_K(X)$. Note that an element $f = (f_i)_i$ of $F_K(X)$ is invertible if and only if all components f_i are non-zero.

Lemma 15.6. *We have*

$$F(X) \otimes_F F_K = \left\{ f \in F_K(X) : \begin{array}{c} \mathrm{den}(f) \mid g \\ \text{for some } g \in F[X] \setminus \{0\} \end{array} \right\},$$

Proof. "$\subset$": Assume that f' is an element of $F(X) \otimes_F F_K$. We may write $f' = \sum_{i=1}^m \frac{a_i}{b_i} \otimes \lambda_i$ for elements $\lambda_i \in F_K$ and $a_i, b_i \in F[X]$ with $b_i \neq 0$. Then $\mathrm{den}(f')$ divides $d := \prod_{i=1}^m b_i$, an element of $F[X] \setminus \{0\}$ as claimed.

"$\supset$": Assume that f' is an element of $F_K(X)$ which divides a non-zero element $g \in F[X]$. This means that there exists an element $h \in F_K[X]$ invertible in $F_K(X)$ such that $g = \mathrm{den}(f) \cdot h$. We have $f' = \frac{1}{\mathrm{den}(f')} f''$ with $f'' \in F_K[X] \subset F(X) \otimes_F F_K$. Therefore $f' = \frac{1}{\mathrm{den}(f')h} \cdot (f''h)$ with $\frac{1}{\mathrm{den}(f')h} = \frac{1}{g} \in F(X)$ and $f'' \cdot h \in F_K[X]$, which implies our claim that f' is an element of $F(X) \otimes_F F_K$. $\therefore$

Given a vector $\boldsymbol{x} = (x_j) \in F_K(X)^{\oplus r}$ for some $r \geq 1$, we set $\mathrm{den}(\boldsymbol{x}) = \mathrm{lcm}_j(\mathrm{den}\, x_j)$.

Lemma 15.7. *Fix two integers $m, n \geq 1$. For every matrix $A \in \mathrm{Mat}_{m\times n}(F_K)$ and every vector $\boldsymbol{x} \in F_K(X)^{\oplus n}$, we have*

$$\mathrm{den}\,(A\boldsymbol{x}) \mid \mathrm{den}(\boldsymbol{x}).$$

In particular, if $m = n$ and A is invertible, then $\mathrm{den}(A\boldsymbol{x}) = \mathrm{den}(\boldsymbol{x})$.

Proof. Case $m = n = 1$: For $x = (x_1, \dots, x_s) \in F_K(X)$ with $x_i \in Q_i(X)$ and $a = (a_1, \dots, a_n) \in F_K$ with $a_i \in Q_i$, one has

$$\mathrm{den}(a_i \cdot x_i) = \begin{cases} \mathrm{den}(x_i), & \text{if } a_i \neq 0 \\ 1, & \text{if } a_i = 0 \end{cases}$$

It follows that $\mathrm{den}(a_i \cdot x_i) \mid \mathrm{den}(x_i)$ in any case, so by definition $\mathrm{den}(a \cdot x) \mid \mathrm{den}(x)$.

Case $m = 1$, $n > 1$: We have $\boldsymbol{x} = (x_1, \dots, x_n)$ with $x_i \in F_K(X)$ and $A = (a_1, \dots, a_n)$ with $a_i \in F_K$. Therefore

$$\begin{array}{rcl} \mathrm{den}(A\boldsymbol{x}) & = & \mathrm{den}\left(\displaystyle\sum_{i=1}^n a_i \cdot x_i\right) \\ & \mid & \mathrm{lcm}(\mathrm{den}(a_i \cdot x_i)) \quad \text{by (15.5)} \\ & \mid & \mathrm{lcm}(x_i) \quad \text{by the case } m = n = 1 \\ & = & \mathrm{den}(\boldsymbol{x}) \quad \text{by definition} \end{array}$$

Case $m, n \geq 1$: Each entry of $\boldsymbol{y} := (y_1, \ldots, y_m) := A\boldsymbol{x}$ fulfills $\operatorname{den}(y_i) \mid \operatorname{den}(\boldsymbol{x})$ by the case $m = 1$, so by definition $\operatorname{den}(\boldsymbol{y}) \mid \operatorname{den}(\boldsymbol{x})$ as claimed.

"In particular": By what we have proven, $\operatorname{den}(A\boldsymbol{x}) \mid \operatorname{den}(\boldsymbol{x})$. Applied to $\boldsymbol{x}' := A\boldsymbol{x}$ and $A' := A^{-1}$, we obtain

$$\operatorname{den}(\boldsymbol{x}) = \operatorname{den}(A'\boldsymbol{x}') \mid \operatorname{den}(\boldsymbol{x}') = \operatorname{den}(Ax).$$

Since both $\operatorname{den}(\boldsymbol{x})$ and $\operatorname{den}(A\boldsymbol{x})$ have monic components in the decomposition $F_K(X) = Q_1(X) \times \cdots \times Q_s(X)$, this implies that $\operatorname{den}(A\boldsymbol{x}) = \operatorname{den}(\boldsymbol{x})$. $\therefore$

Proof of Proposition 15.4. Assume that $\boldsymbol{M}$ has rank r. By choosing a basis, $\boldsymbol{M}$ is isomorphic to the free F_K-module $\mathrm{Mat}_{r\times 1}(F_K)$ equipped with the mapping $\boldsymbol{m} = (m_i)_i \mapsto \Delta \cdot (\sigma(m_i)_i)$ for a certain matrix $\Delta = (\delta_{ij}) \in \mathrm{GL}_r(F_K)$. We assume that $\boldsymbol{M}$ is of this form. Note that $\sigma(\operatorname{den}(\boldsymbol{m})) = \operatorname{den}(\sigma(\boldsymbol{m}))$. We set $d := \operatorname{den}(\boldsymbol{m})$.

Assume that $m \in \boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{M}$ is τ-invariant, so $\boldsymbol{m} \in F_K(X)^{\oplus r}$ and $\boldsymbol{m} = \Delta \cdot \sigma(\boldsymbol{m})$. By Lemma 15.7 applied to $\boldsymbol{x} = \sigma(\boldsymbol{m})$ and the invertible matrix $A = \Delta$, we obtain that $d = \sigma(d)$ is an element of $F[X]$.

Now $\operatorname{den}(m_i) \mid d$ by definition, so Lemma 15.6 implies that $m_i \in F' \otimes_F F_K$ for all i. Hence $m \in F' \otimes_F \boldsymbol{M}^\tau$; we are done. $\therefore$

16 Radical computations

Definition 16.1. A field extension $F' \supset F$ is *separable* if for every field extension $F'' \supset F$ the ring $F' \otimes_F F''$ is reduced (contains no nilpotent elements).

Remark 16.2. This definition of separability for (possibly non-algebraic) field extensions is equivalent to various others, cf. [Bou81, VIII.§7.3, Théorème 1].

For instance, if $F' \supset F$ is an algebraic extension, then the above definition is equivalent to the usual definition of separability.

As in the algebraic case, in characteristic zero all field extensions are separable, and if F is a field of positive characteristic p, then all field extensions $F' \supset F$ are separable if and only if F is perfect (contains the p-th root of each of its elements).

The notion of nonalgebraic separable extensions is of interest to us because completions of global fields are separable extensions:

Proposition 16.3. *Every completion $F_\mathfrak{p}$ of any global field F is a separable field extension.*

Proof. We may assume that F is a global field of positive characteristic p. Let us start with the special case of $F = k(t)$ completed at $\mathfrak{p} = (t)$, so $F_\mathfrak{p} = k(\!(t)\!)$. By [Bou81, V.§15.4] it is sufficient to prove the following: If $f_1, \ldots, f_m \in k(\!(t)\!)$

are linearly independent over $k(t)$, then so are the f_i^p. Without loss of generality, assume that $f_i \in k[\![t]\!]$, and that for certain $g_i \in k[t]$ we have $\sum_i g_i f_i^p = 0$. We must show that all g_i are zero.

Since k is perfect, we may write $g_i =: \sum_{j=0}^{p-1} g_{ij}^p t^j$ for certain $g_{ij} \in k[t]$. These defining equations, together with $\sum_i g_i f_i^p = 0$, imply that for all j we have $\sum_i g_{ij}^p f_i^p = 0$. By extracting p-th roots of both sides we obtain $\sum_i g_{ij} f_i = 0$ for all j. By assumption the f_i are linearly independent, so we have $g_{ij} = 0$ for all i and j. Therefore all g_i are zero, as required.

Let us come back to the general setting. We choose a local parameter $t \in F$ at $\mathfrak{p}$. Denoting the residue field of F at $\mathfrak{p}$ by $k_\mathfrak{p}$, we have $F_\mathfrak{p} = k_\mathfrak{p}(\!(t)\!)$ and the following commutative diagram of inclusions:

$$\begin{array}{ccc} k(t) & \longrightarrow & F \\ \downarrow & & \downarrow \\ k(\!(t)\!) & \longrightarrow & k_\mathfrak{p}(\!(t)\!) \end{array}$$

We have just seen that $k(t) \subset k(\!(t)\!)$ is separable; clearly, so is $k(\!(t)\!) \subset k_\mathfrak{p}(\!(t)\!)$, hence $k(t) \subset F_\mathfrak{p}$ is separable. Moreover, $k(t) \subset F$ is separable algebraic since t is a local parameter. This implies that $F \subset F_\mathfrak{p}$ is separable by [Bou81, V.§15]. $\therefore$

The main result of this section is the following (cf. Definition 13.7 for the notion of socles of objects):

Theorem 16.4. *If $F' \supset F$ is a separable field extension, then for any restricted $\boldsymbol{F_K}$-module $\boldsymbol{M}$ we have*

$$\boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} \operatorname{soc}(\boldsymbol{M}) = \operatorname{soc}(\boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{M}).$$

In particular, $\boldsymbol{M}$ is semisimple if and only if $\boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{M}$ is semisimple.

This will be established after a sequence of lemmas.

We fix some notation. We assume that F, K are two field extensions of a given finite field k, of which K contains only a finite number of roots of unity. The letters $\boldsymbol{M}, \boldsymbol{N}, \boldsymbol{S}, \ldots$ denote $\boldsymbol{F_K}$-modules. Given a field extension $F' \supset F$, the letters $\boldsymbol{M}', \boldsymbol{N}', \ldots$ denote the $\boldsymbol{F'_K}$-modules induced by base extension $\boldsymbol{F'_K} \otimes_{\boldsymbol{F_K}} (-)$. We will sometimes deal with $\boldsymbol{F_K}$-modules using the language of noncommutative algebra, as explained in Remark 4.3(a).

Remark 16.5. A priori, for a given restricted $\boldsymbol{F_K}$-module $\boldsymbol{M}$, there are two possible socles we might consider: Considering $\boldsymbol{M}$ as an object of the finite abelian category of restricted $\boldsymbol{F_K}$-modules, we might let $\operatorname{soc}^{\mathrm{res}}(\boldsymbol{M})$ denote the sum of all simple *restricted* $\boldsymbol{F_K}$-submodules of $\boldsymbol{M}$. Considering $\boldsymbol{M}$ simply as a $\boldsymbol{F_K}$-module, we might let soc($\boldsymbol{M}$) denote the sum of *all* simple $\boldsymbol{F_K}$-submodules of $\boldsymbol{M}$ whatsoever. However, by Propositions 5.3 and 5.23 we know that the category of restricted $\boldsymbol{F_K}$-modules is closed under subquotients in the category of all $\boldsymbol{F_K}$-modules, so these two notions of socle coincide. Similarly, the two possible notions of radical coincide.

Proposition 16.6. *Let $\boldsymbol{R}$ be a bold ring, and let $\boldsymbol{M}$ be an $\boldsymbol{R}$-module.*

(a) If $\boldsymbol{M}$ is non-zero and finitely $\boldsymbol{R}$-generated (cf. Remark 4.3 for definitions), then $\operatorname{rad}\boldsymbol{M} \neq \boldsymbol{M}$.

(b) If $\boldsymbol{M} \xrightarrow{f} \boldsymbol{N}$ is a homomorphism of $\boldsymbol{R}$-modules, then $f(\operatorname{rad}\boldsymbol{M}) \subset \operatorname{rad}\boldsymbol{N}$.

(c) $\boldsymbol{M}$ embeds into a product of simple $\boldsymbol{R}$-modules if and only if $\operatorname{rad}\boldsymbol{M} = 0$.

(d) $\boldsymbol{M}$ is semisimple of finite length if and only if $\operatorname{rad}\boldsymbol{M} = 0$ *and $\boldsymbol{M}$ is artinian.*

(e) Assume that $\boldsymbol{M}$ admits a finite number of $\boldsymbol{R}$-generators $m_1, \dots, m_n$. Then an element $m \in M$ lies in $\operatorname{rad}\boldsymbol{M}$ *if and only if for all $r_1, \dots, r_n \in R\{\tau\}$ the elements $m_i + r_i m$ are $\boldsymbol{R}$-generators of $\boldsymbol{M}$.*

Proof. Viewing $\boldsymbol{R}$-modules as left $R\{\tau\}$-modules, these properties follow from general properties of the Jacobson radical of rings. Proofs may be found in [Bou81, VIII]. ∴

Lemma 16.7. *For any $\boldsymbol{F_K}$-module $\boldsymbol{M}$ we have*

$$\boldsymbol{M} \cap \operatorname{rad}(\boldsymbol{M}') \subset \operatorname{rad}(\boldsymbol{M}).$$

Proof. Assume first that $\boldsymbol{M}$ is simple, in particular $\operatorname{rad}\boldsymbol{M} = 0$. Any nonzero $m \in M$ generates $\boldsymbol{M}'$, hence $\boldsymbol{M}'$ is finitely generated and $\boldsymbol{M}' \neq \operatorname{rad}(\boldsymbol{M}')$ by Proposition 16.6(a). Combining these facts, we see that $\boldsymbol{M} \cap \operatorname{rad}(\boldsymbol{M}') = 0$, as required.

In the general case, for $m \in \boldsymbol{M} \cap \operatorname{rad}(\boldsymbol{M}')$ and $f : \boldsymbol{M} \longrightarrow \boldsymbol{S}$ a homomorphism with $\boldsymbol{S}$ a simple $\boldsymbol{F_K}$-module, we must show that $f(m) = 0$. The induced homomorphism $f' : \boldsymbol{M}' \longrightarrow \boldsymbol{S}'$ has the property that $f'(\operatorname{rad}(\boldsymbol{M}')) \subset \operatorname{rad}(\boldsymbol{S}')$ by Proposition 16.6(b). So $f(m) \in \boldsymbol{S} \cap \operatorname{rad}(\boldsymbol{S}')$, which is zero by the special case treated above. ∴

Note that Lemma 16.7 neither implies that $\boldsymbol{M}$ is semisimple if $\boldsymbol{M}'$ is, nor that $\boldsymbol{M}'$ is semisimple if $\boldsymbol{M}$ is. But we can do better!

Theorem 16.8. *For any field extension $F' \supset F$ and any restricted $\boldsymbol{F_K}$-module M we have*

$$\boldsymbol{M} \cap \operatorname{rad}(\boldsymbol{M}') = \operatorname{rad}(\boldsymbol{M}).$$

In particular, if $\boldsymbol{M}'$ is semisimple, then so is $\boldsymbol{M}$.

We start with two special kinds of field extensions, finite and transcendental of transcendence degree 1.

Lemma 16.9. *If $F' \supset F$ is a finite extension, then for any $\boldsymbol{F_K}$-module $\boldsymbol{M}$ we have*

$$\boldsymbol{M} \cap \operatorname{rad}(\boldsymbol{M}') = \operatorname{rad}(\boldsymbol{M}).$$

Proof. By Lemma 16.7 it is sufficient to show that $\operatorname{rad} \boldsymbol{M} \subset \operatorname{rad}(\boldsymbol{M}')$, i.e., that for any homomorphism $f' : \boldsymbol{M}' \longrightarrow \boldsymbol{S}'$ with $\boldsymbol{S}'$ a simple $\boldsymbol{F}'_{\boldsymbol{K}}$-module, we have $f'(\operatorname{rad} \boldsymbol{M}) = 0$.

The restriction of f' to M is an $\boldsymbol{F_K}$-module homomorphism to $\boldsymbol{S}'$, regarded as an $\boldsymbol{F_K}$-module. Setting $\boldsymbol{X} := \operatorname{rad}_{\boldsymbol{F_K}}(\boldsymbol{S}')$, we have $f'(\operatorname{rad} \boldsymbol{M}) \subset \boldsymbol{X}$ by Proposition 16.6(b), so it suffices to show that $\boldsymbol{X} = 0$.

We claim that $\boldsymbol{X}$ is an $\boldsymbol{F}'_{\boldsymbol{K}}$-submodule of $\boldsymbol{S}'$. Since $F' \supset F$ is finite, we have $F'_K = F' \otimes_F F_K$, so it suffices to show that $F'X \subset X$. Since elements $f' \in F'$ may be considered as $\boldsymbol{F_K}$-module endomorphisms of $\boldsymbol{S}'$, this follows from Proposition 16.6(b).

Since $\boldsymbol{S}'$ is a simple $\boldsymbol{F}'_{\boldsymbol{K}}$-module, $\boldsymbol{X}$ is either zero or $\boldsymbol{S}'$ itself. But since $F \subset F'$ is finite, $\boldsymbol{S}'$ is finitely generated over $\boldsymbol{F_K}$, so by Proposition 16.6(a) we may rule out the case $\boldsymbol{X} = \boldsymbol{S}'$. ∴

Lemma 16.10. *If $F' = F(X) \supset F$ is a purely transcendental field extension of transcendence degree 1, then for any restricted $\boldsymbol{F_K}$-module $\boldsymbol{M}$ we have an equivalence*

$$\boldsymbol{M} \textit{ is simple} \iff \boldsymbol{M}' \textit{ is simple}.$$

In particular, we have $\boldsymbol{M} \cap \operatorname{rad}(\boldsymbol{M}') = \operatorname{rad} \boldsymbol{M}$.

Proof. If $\boldsymbol{M}'$ is simple, then so is $\boldsymbol{M}$ – this is clear.

So let us assume that $\boldsymbol{M}$ is simple. Consider the decomposition

$$F_K = Q_1 \times \cdots \times Q_s.$$

Note that we have $F'_K = F_K(X) := Q_1(X) \times \cdots \times Q_s(X)$. We let $\boldsymbol{F_K}[X]$ be the bold ring consisting of the ring $F_K[X] = Q_1[X] \times \cdots \times Q_s[X]$ equipped with the restriction of the σ of $\boldsymbol{F}'_{\boldsymbol{K}}$, it acts as the identity on X. Note that the "model" $\mathscr{M} := \boldsymbol{F_K}[X] \otimes_{\boldsymbol{F_K}} \boldsymbol{M}$ of $\boldsymbol{M}' = \boldsymbol{F}'_{\boldsymbol{K}} \otimes_{\boldsymbol{F_K}} \boldsymbol{M}$ is a restricted $\boldsymbol{F_K}[X]$-module with $\mathscr{M}/(X) \cong \boldsymbol{M}$.

Assume that $\boldsymbol{M}'$ is not simple, so there exists a nontrivial $\boldsymbol{F}'_{\boldsymbol{K}}$-submodule $\boldsymbol{N}' \subsetneq \boldsymbol{M}'$. It follows that $\mathscr{N} := \mathscr{M} \cap \boldsymbol{N}'$ is a non-trivial $\boldsymbol{F_K}[X]$-submodule of $\mathscr{M}$ other than $\mathscr{M}$, and therefore that $\boldsymbol{N} := \mathscr{N}/(X)$ is a non-trivial $\boldsymbol{F_K}$-submodule of $\mathscr{M}/(X) \cong \boldsymbol{M}$ other than $\boldsymbol{M}$, in contradiction to the simplicity of $\boldsymbol{M}$.

Let us prove the statement of the last sentence of this lemma: By Lemma 16.7, it is sufficient to show that $\operatorname{rad}(\boldsymbol{M}) \subset \operatorname{rad}(\boldsymbol{M}')$. For this, we consider $\boldsymbol{M}/\operatorname{rad}(\boldsymbol{M})$.

It is a semisimple $\boldsymbol{F_K}$-module, so by what we have proven already, $(\boldsymbol{M}/\operatorname{rad}(\boldsymbol{M}))'$ is a semisimple $\boldsymbol{F'_K}$-module. However,

$$(\boldsymbol{M}/\operatorname{rad}(\boldsymbol{M}))' \cong \boldsymbol{M}'/\operatorname{rad}(\boldsymbol{M})'$$

must be a quotient of the largest semisimple quotient $\boldsymbol{F'_K}$-module $\boldsymbol{M}'/\operatorname{rad}(\boldsymbol{M}')$ of $\boldsymbol{M}'$, so we see that $(\operatorname{rad}\boldsymbol{M})' \subset \operatorname{rad}(\boldsymbol{M}')$. Since $\operatorname{rad}(\boldsymbol{M}) \subset (\operatorname{rad}\boldsymbol{M})'$ for trivial reasons, we are done. ∴

Proof of Theorem 16.8. By Lemma 16.7 it is sufficient to prove that $\operatorname{rad}\boldsymbol{M} \subset \operatorname{rad}(\boldsymbol{M}')$. Choose $\boldsymbol{F_K}$-generators $m_1, \dots, m_n$ of $\boldsymbol{M}$, and fix $m \in \operatorname{rad}\boldsymbol{M}$. The m_i are $\boldsymbol{F'_K}$-generators of $\boldsymbol{M}'$, so by Proposition 16.6(e) we have that $m \in \operatorname{rad}\boldsymbol{M}'$ if and only if for all $x_1, \dots, x_n \in \boldsymbol{F'_K}\{\tau\}$ the $m_i + x_i m$ are $\boldsymbol{F'_K}$-generators of $\boldsymbol{M}'$

Fix such x_i. There exists a finitely generated field extension $F' \supset F^{fg} \supset F$ such that $F^{fg}_K\{\tau\}$ contains all x_i. Set $\boldsymbol{F^{fg}_K} := \operatorname{Frac}(F^{fg} \otimes_F F_K, \operatorname{id}\otimes\sigma)$. Since F^{fg} is a finite algebraic extension of a purely transcendental extension of F of finite transcendence degree, Lemma 16.9 and and a repeated application of Lemma 16.10 show that $m \in \operatorname{rad}(\boldsymbol{M^{fg}})$, where $\boldsymbol{M^{fg}} := \boldsymbol{F^{fg}_K} \otimes_{\boldsymbol{F_K}} \boldsymbol{M}$. By Proposition 16.6(e), this shows that the $m_i + x_i m$ are $\boldsymbol{F^{fg}_K}$-generators of $\boldsymbol{M^{fg}}$. This then implies that they are $\boldsymbol{F'_K}$-generators of $\boldsymbol{M}'$, as required. All in all, $\operatorname{rad}\boldsymbol{M} \subset \operatorname{rad}(\boldsymbol{M}')$. ∴

Theorem 16.11. *For any separable field extension $F \subset F'$ and any restricted $\boldsymbol{F_K}$-module $\boldsymbol{M}$ we have*

$$\operatorname{rad}(\boldsymbol{M})' = \operatorname{rad}(\boldsymbol{M}').$$

Proof. "$\subset$": By Theorem 16.8 we have $\operatorname{rad}\boldsymbol{M} \subset \operatorname{rad}(\boldsymbol{M}')$, and hence $(\operatorname{rad}\boldsymbol{M})' = F'_K \cdot \operatorname{rad}\boldsymbol{M} \subset \operatorname{rad}(\boldsymbol{M}')$, since both sides are F'_K-modules.

"$\supset$": By Theorem 16.8 we have $\operatorname{rad}\boldsymbol{M} = \boldsymbol{M} \cap \operatorname{rad}(\boldsymbol{M}')$, so we may procede by showing that $\operatorname{rad}(\boldsymbol{M}') \subset (\boldsymbol{M} \cap \operatorname{rad}(\boldsymbol{M}'))'$. Fix $m' \in \operatorname{rad}(\boldsymbol{M}')$. Since we are proposing an inclusion of $\boldsymbol{F'_K}$-modules, we may multiply by the denominators of m' and assume that

$$m' = \sum_i f'_i m_i,$$

where the $f'_i \in F'$ are F-linearly independent, and the m_i are elements of M. We claim that $m_i \in \operatorname{rad}(\boldsymbol{M}')$ for all i, which implies that $m_i \in \operatorname{rad}\boldsymbol{M}$ for all i (by Theorem 16.8) and therefore that $m' \in \operatorname{rad}(\boldsymbol{M})'$, as required.

For this, let $\overline{F'}$ be an algebraic closure of F'. By Theorem 16.8 we have $m' \in \operatorname{rad}(\overline{\boldsymbol{M}'})$, where $\overline{\boldsymbol{M}'} = \overline{\boldsymbol{F'_K}} \otimes_{\boldsymbol{F_K}} \boldsymbol{M}$. Since $F \subset F'$ is separable and the f'_i are F-linearly independent, by [Bou83, V.§15.6, Théorème 4] there exist F-linear field automorphisms $g_1, \dots, g_n$ of $\overline{F'}$ such that the matrix $(g_i(f'_j))^n_{i,j=1}$ is invertible. Let $(h_{ij})_{i,j} \in \mathrm{GL}_n(\overline{F'})$ denote its inverse. Considering the g_i as automorphisms of $\overline{M'}$ (or, more precisely, as isomorphisms $g_i^*\overline{M'} \longrightarrow \overline{M'}$), Proposition 16.6(b) gives that $g_i(m') \in \operatorname{rad}(\overline{\boldsymbol{M'}})$. Hence $m_i = \sum_j h_{ij} g_j(m') \in \operatorname{rad}(\overline{\boldsymbol{M'}}) \supset \operatorname{rad}(\boldsymbol{M}')$, as claimed. ∴

Proof of Theorem 16.4. This follows directly from Theorem 16.11, since for every restricted $\boldsymbol{F_K}$-module $\boldsymbol{M}$ one has $\operatorname{soc}(\boldsymbol{M}) = (\boldsymbol{M}^\vee / \operatorname{rad}(\boldsymbol{M}^\vee))^\vee$, similarly $\operatorname{soc}(\boldsymbol{M}') = (\boldsymbol{M}'^{,\vee} / \operatorname{rad}(\boldsymbol{M}'^{,\vee}))^\vee$, and therefore

$$\operatorname{soc}(\boldsymbol{M})' = \Big(\big(\boldsymbol{M}^\vee / \operatorname{rad}(\boldsymbol{M}^\vee)\big)^\vee\Big)' = \big((\boldsymbol{M}')^\vee / \operatorname{rad}(\boldsymbol{M}')^\vee\big)^\vee = \operatorname{soc}(\boldsymbol{M}'),$$

where the middle identification uses Theorem 16.11 applied to $\boldsymbol{M}^\vee$. ∴

Chapter V

Tamagawa-Fontaine Theory

In this chapter, we assume that F is a global field with field of constants k, that $\mathfrak{p}$ is a place of F, and that the base field $K \supset k$ is *finitely generated* over its prime field. The idea of using the following theory and a sketch of its construction comes from [Tam95].

In this chapter, etale $\boldsymbol{F}_{\boldsymbol{K},\mathfrak{p}}$-module will mean $\mathscr{O}_{\boldsymbol{K},\mathfrak{p}}$-etale $\boldsymbol{F}_{\boldsymbol{K},\mathfrak{p}}$-module, and etale $\boldsymbol{F}_{\mathfrak{p},\boldsymbol{K}}$-module will mean $\mathscr{O}_{\mathfrak{p},\boldsymbol{K}}$-etale $\boldsymbol{F}_{\mathfrak{p},\boldsymbol{K}}$-module, as defined in Definition 5.29 and Examples 6.11.

In Section 7 we have classified rational $\mathfrak{p}$-adic Galois representations in terms of etale $\boldsymbol{F}_{\boldsymbol{K},\mathfrak{p}}$-modules. The content of what we term "Tamagawa-Fontaine theory" is to determine which of these representations arise from etale $\boldsymbol{F}_{\mathfrak{p},\boldsymbol{K}}$-modules by constructing a right-adjoint functor $\mathrm{Q}_{\mathfrak{p}}$ from $\mathrm{Rep}_{F_{\mathfrak{p}}}(\Gamma_K)$ to etale $\boldsymbol{F}_{\mathfrak{p},\boldsymbol{K}}$-modules. This functor $\mathrm{Q}_{\mathfrak{p}}$ determines, equivalently, which etale $\boldsymbol{F}_{\boldsymbol{K},\mathfrak{p}}$-modules arise from etale $\boldsymbol{F}_{\mathfrak{p},\boldsymbol{K}}$-modules, and shows that the base change from $\boldsymbol{F}_{\mathfrak{p},\boldsymbol{K}}$ to $\boldsymbol{F}_{\boldsymbol{K},\mathfrak{p}}$ is a fully faithful functor, the essential image of which is closed under subquotients.

17 The formal theory and its consequences

Recall that $\boldsymbol{F}_{\mathfrak{p},\boldsymbol{K}}$ is the bold ring consisting of $F_{\mathfrak{p},K} := \mathrm{Frac}(F_{\mathfrak{p}} \otimes_k K)$ equipped with the unique extension of σ of $\boldsymbol{K}$ acting as the identity on $F_{\mathfrak{p}}$. Set $\mathscr{O}_{\mathfrak{p},\boldsymbol{K}} := \boldsymbol{F}_{\mathfrak{p},\boldsymbol{K}} \cap \mathscr{O}_{\boldsymbol{K},\mathfrak{p}}$, in the representation of $F_{\mathfrak{p},K}$ as Laurent series with parameter a local uniformizer $t \in F$ at $\mathfrak{p}$, this is the subring of power series.

Recall that an $\boldsymbol{F}_{\mathfrak{p},\boldsymbol{K}}$-module $\boldsymbol{M}$ is called ($\mathscr{O}_{\mathfrak{p},\boldsymbol{K}}$-)*etale* if it is isomorphic to $\boldsymbol{F}_{\mathfrak{p},\boldsymbol{K}} \otimes_{\mathscr{O}_{\mathfrak{p},\boldsymbol{K}}} \mathscr{O}_{\boldsymbol{M}}$ for some restricted $\mathscr{O}_{\mathfrak{p},\boldsymbol{K}}$-module $\mathscr{O}_{\boldsymbol{M}}$. Then $\boldsymbol{F}_{\boldsymbol{K},\mathfrak{p}} \otimes_{\boldsymbol{F}_{\mathfrak{p},\boldsymbol{K}}} \boldsymbol{M}$ is an ($\mathscr{O}_{\boldsymbol{K},\mathfrak{p}}$-)etale $\boldsymbol{F}_{\boldsymbol{K},\mathfrak{p}}$-module, and

$$\mathrm{V}_{\mathfrak{p}}(\boldsymbol{M}) := V_{F_{\mathfrak{p}}}(\boldsymbol{F}_{\boldsymbol{K},\mathfrak{p}} \otimes_{\boldsymbol{F}_{\mathfrak{p},\boldsymbol{K}}} \boldsymbol{M})$$

is a rational $\mathfrak{p}$-adic Galois representation (cf. Definition 7.2 for the definition of $V_{F_\mathfrak{p}}$). Note that $\mathrm{V}_\mathfrak{p}$ is an $F_\mathfrak{p}$-linear tensor functor and preserves ranks (and is therefore exact), since it is composed of two functors with these properties.

Conversely, we say that a rational $\mathfrak{p}$-adic Galois representation is *quasigeometric* if it is isomorphic to a representation obtained in this fashion from an etale $\boldsymbol{F}_{\mathfrak{p},K}$-module.

Claim 17.1. *There exists a Γ_K-stable bold subring $\boldsymbol{B} \subset \boldsymbol{F}_{K^{\mathrm{sep}},\mathfrak{p}}$ (with scalar ring $\boldsymbol{B}^\sigma = F_\mathfrak{p}$) and the following properties:*

(a) $\boldsymbol{B}^{\Gamma_K} = \boldsymbol{F}_{\mathfrak{p},K}$.

(b) For every etale $\boldsymbol{F}_{\mathfrak{p},K}$-module $\boldsymbol{M}$ one has $\mathrm{V}_\mathfrak{p}(\boldsymbol{M}) \subset \boldsymbol{B} \otimes_{\boldsymbol{F}_{\mathfrak{p},K}} \boldsymbol{M}$.

Note that the existence of such a ring of periods is a matter of *construction*, since property (a) requires $\boldsymbol{B}$ to be "small enough" (as $(F_{K^{\mathrm{sep}},\mathfrak{p}})^{\Gamma_K} = F_{K,\mathfrak{p}}$, which contains $F_{\mathfrak{p},K}$ but is strictly larger than $F_{K,\mathfrak{p}}$ if K is not a finite field), whereas property (b) requires $\boldsymbol{B}$ to be "large enough" (as it must contain the Galois-invariant elements of $\boldsymbol{F}_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{\boldsymbol{F}_{\mathfrak{p},K}} \boldsymbol{M}$ for every etale $\boldsymbol{F}_{\mathfrak{p},K}$-module $\boldsymbol{M}$).

This claim will be justified in the next section (Definitions 18.6 and 18.10). For this section and the proof of its statements, *we assume that it holds true*. Clearly, the scalar ring of such a bold ring $\boldsymbol{B}$ must be $F_\mathfrak{p}$, since $\boldsymbol{B}$ contains $\boldsymbol{F}_{\mathfrak{p},K}$ and the scalar ring of $\boldsymbol{F}_{\mathfrak{p},K}$ is $F_\mathfrak{p}$. Let us look for the *smallest* possible ring meeting the requirements of Claim 17.1.

Definition 17.2. The *ring of periods* of an etale $\boldsymbol{F}_{\mathfrak{p},K}$-module $\boldsymbol{M}$ is the the $F_{\mathfrak{p},K}$-subalgebra of $F_{K^{\mathrm{sep}},\mathfrak{p}}$

$$P(M) := \bigcap \left\{ N \subset F_{K^{\mathrm{sep}},\mathfrak{p}} : \begin{array}{l} N \text{ an } F_{\mathfrak{p},K}\text{-vector subspace, and} \\ N \otimes_{F_{K,\mathfrak{p}}} M \supset (F_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{F_{K,\mathfrak{p}}} M)^\tau \end{array} \right\}.$$

We call it the *ring of periods* of $\boldsymbol{M}$. In terms of an $F_{\mathfrak{p},K}$-basis of M it is generated by the coefficients of all elements of $\mathrm{V}_\mathfrak{p}(\boldsymbol{M})$.

Now for a given etale $\boldsymbol{F}_{\mathfrak{p},K}$-module $\boldsymbol{M}$, item (b) of Claim 17.1 means precisely that that B contains the ring $P(\boldsymbol{M})$ of periods of $\boldsymbol{M}$.

Definition 17.3. The *ring $\boldsymbol{P}$ of quasigeometric periods* is the Γ_K-stable bold subring of $\boldsymbol{F}_{K^{\mathrm{sep}},\mathfrak{p}}$ generated by the elements of the rings of periods of all etale $\boldsymbol{F}_{\mathfrak{p},K}$-modules.

Lemma 17.4. *The bold ring $\boldsymbol{P}$ fulfills the requirements of Claim 17.1. It is the smallest such ring.*

Proof. By the preceding discussion, $\boldsymbol{P}$ is the smallest possible Γ_K-stable bold subring of $\boldsymbol{F}_{\boldsymbol{K}^{\mathbf{sep}},\mathfrak{p}}$ fulfilling the requirements of Claim 17.1(b).

It remains to show that $\boldsymbol{P}$ is "small enough" to fulfill Claim 17.1(a). By construction, $F_{\mathfrak{p},K} \subset P$, so it remains to show that $P^{\Gamma_K} \subset F_{\mathfrak{p},K}$. It is here that we use the assumption that a bold ring $\boldsymbol{B}$ as in Claim 17.1 exists: Since $\boldsymbol{B}$ fulfills Claim 17.1(b), we have $P \subset B$, so in particular

$$P^{\Gamma_K} \subset B^{\Gamma_K} = F_{\mathfrak{p},K}.$$

∴

What follows does not depend on our choice of $\boldsymbol{B}$. But we might as well choose $\boldsymbol{B} = \boldsymbol{P}$ in the following to make all definitions independent of this choice. So we do.

Lemma 17.5. *Let $\boldsymbol{M}$ be an etale $F_{\mathfrak{p},K}$-module. Then the natural comparison isomorphism $F_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{F_{\mathfrak{p}}} \mathrm{V}_{\mathfrak{p}}(\boldsymbol{M}) \to F_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{F_{\mathfrak{p},K}} \boldsymbol{M}$ of Theorem 7.4(c) descends to a Γ_K-equivariant isomorphism of $\boldsymbol{B}$-modules*

$$c_{\boldsymbol{M}} : \boldsymbol{B} \otimes_{F_{\mathfrak{p}}} \mathrm{V}_{\mathfrak{p}}(\boldsymbol{M}) \longrightarrow \boldsymbol{B} \otimes_{F_{\mathfrak{p},K}} \boldsymbol{M}$$

Proof. Claim 17.1(b) and the inclusion $P(\boldsymbol{M}) \subset \boldsymbol{B}$ imply that the given isomorphism descends to a Γ_K-equivariant *homomorphism* of $\boldsymbol{P}$-modules

$$c_{\boldsymbol{M}} : \boldsymbol{B} \otimes_{F_{\mathfrak{p}}} \mathrm{V}_{\mathfrak{p}}(\boldsymbol{M}) \longrightarrow \boldsymbol{B} \otimes_{F_{\mathfrak{p},K}} \boldsymbol{M}$$

by the definition of the objects involved. Since both sides are free B-modules of finite (constant) rank, it suffices to show that the determinant of $c_{\boldsymbol{M}}$ is an isomorphism. Since $\mathrm{V}_{\mathfrak{p}}$ is a tensor functor and the comparison isomorphism is compatible with tensor products, we have

$$\det(c_{\boldsymbol{M}}) = c_{\det(\boldsymbol{M})}.$$

Therefore we may reduce to the case where $\operatorname{rk} \boldsymbol{M} = 1$. In this case, choosing a basis for both $\mathrm{V}_{\mathfrak{p}}(\boldsymbol{M})$ and $\boldsymbol{M}$, we see that $c_{\boldsymbol{M}}$ is given by left multiplication by an element $c(\boldsymbol{M}) \in B$. Choosing the dual bases of $\mathrm{V}_{\mathfrak{p}}(\boldsymbol{M}^{\vee})$ and $\boldsymbol{M}^{\vee}$, analogously $c_{\boldsymbol{M}^{\vee}}$ is given by left multiplication by an element $c(\boldsymbol{M}^{\vee})$.

By Theorem 7.4(c), the element $c(\boldsymbol{M})$ is invertible in $F_{K^{\mathrm{sep}},\mathfrak{p}}$. By unraveling the definitions, one sees that its inverse $c(\boldsymbol{M})^{-1}$ coincides with $c(\boldsymbol{M}^{\vee})$. By Claim 17.1(b), both $c(\boldsymbol{M})$ and $c(\boldsymbol{M}^{\vee})$ lie in B, so $c_{\boldsymbol{M}}$ is indeed an isomorphism. ∴

Remark 17.6. Lemma 17.5 further substantiates our choice of calling P the ring of (quasigeometric) periods: It has become customary to call the entries of a matrix involved in a comparison isomorphism between two "(co)homology theories" periods, even if they are not given by integrals on complex varieties, as in the classical case of abelian varieties, Betti and singular homology.

We may continue to exploit the consequences of Claim 17.1, and obtain the "Tate Conjecture", proven independently in [Tag96] and [Tam94].

Theorem 17.7. *The functor* $\mathrm{V}_\mathfrak{p}$ *on etale* $F_{\mathfrak{p},K}$*-modules is fully faithful.*

Proof. Consider two etale $\boldsymbol{F}_{\mathfrak{p},K}$-modules $\boldsymbol{M}, \boldsymbol{N}$. By Lemma 17.5 we have a τ- and Γ_K-equivariant natural isomorphism

$$\boldsymbol{B} \otimes \boldsymbol{M}^\vee \otimes \boldsymbol{N} \longrightarrow \boldsymbol{B} \otimes \mathrm{V}_\mathfrak{p}(\boldsymbol{M}^\vee \otimes \boldsymbol{N}) = \boldsymbol{B} \otimes \mathrm{V}_\mathfrak{p}(\boldsymbol{M})^\vee \otimes \mathrm{V}_\mathfrak{p}(\boldsymbol{N}).$$

It follows that

$$(\boldsymbol{M}^\vee \otimes \boldsymbol{N})^\tau = (\boldsymbol{B} \otimes \boldsymbol{M}^\vee \otimes \boldsymbol{N})^{\Gamma,\tau} \cong (\boldsymbol{B} \otimes \mathrm{V}_\mathfrak{p}(\boldsymbol{M})^\vee \otimes \mathrm{V}_\mathfrak{p}(\boldsymbol{N}))^{\tau,\Gamma} = (\mathrm{V}_\mathfrak{p}(\boldsymbol{M})^\vee \otimes \mathrm{V}_\mathfrak{p}(\boldsymbol{N}))^\Gamma$$

Now $\mathrm{Hom}(\boldsymbol{M}, \boldsymbol{N}) = (\boldsymbol{M}^\vee \otimes \boldsymbol{N})^\tau$ and $\mathrm{Hom}(\mathrm{V}_\mathfrak{p}(\boldsymbol{M})^\vee, \mathrm{V}_\mathfrak{p}(\boldsymbol{N})) = (\mathrm{V}_\mathfrak{p}(\boldsymbol{M})^\vee \otimes \mathrm{V}_\mathfrak{p}(\boldsymbol{N}))^{\Gamma_K}$, so we see that $\mathrm{V}_\mathfrak{p}$ is indeed fully faithful. ∴

Definition 17.8. (a) For any rational $\mathfrak{p}$-adic Galois representation $\boldsymbol{V}$, we set

$$\mathrm{Q}_\mathfrak{p}(\boldsymbol{V}) := (\boldsymbol{B} \otimes_{F_\mathfrak{p}} \boldsymbol{V})^{\Gamma_K},$$

taking Galois-invariants along the diagonal action. Since $\boldsymbol{B} \supset \boldsymbol{F}_{\mathfrak{p},K}$ is an $\boldsymbol{F}_{\mathfrak{p},K}$-module and $\boldsymbol{B}^{\Gamma_K} = \boldsymbol{F}_{\mathfrak{p},K}$ by Claim 17.1(a), multiplication via the first factor gives $\mathrm{Q}_\mathfrak{p}(\boldsymbol{V})$ the structure of an $\boldsymbol{F}_{\mathfrak{p},K}$-module.

(b) Set $\mathcal{O}_{\boldsymbol{B}} := \boldsymbol{B} \cap \mathcal{O}_{\boldsymbol{K}^{\mathrm{sep}},\mathfrak{p}}$. For any integral $\mathfrak{p}$-adic Galois representation $\boldsymbol{T}$ which is torsion-free over $\mathcal{O}_\mathfrak{p}$, we set

$$\mathcal{O}_{\mathrm{Q}_\mathfrak{p}}(\boldsymbol{T}) := (\mathcal{O}_{\boldsymbol{B}} \otimes_{\mathcal{O}_\mathfrak{p}} \boldsymbol{T})^{\Gamma_K},$$

taking Galois-invariants along the diagonal action. This is an $\mathcal{O}_{\mathfrak{p},\boldsymbol{K}}$-module, since $\mathcal{O}_{\boldsymbol{B}}$ is an $\mathcal{O}_{\mathfrak{p},\boldsymbol{K}}$-module.

Lemma 17.9. *For every etale* $\boldsymbol{F}_{\mathfrak{p},K}$*-module* $\boldsymbol{M}$*, the comparison isomorphism* $c_{\boldsymbol{M}}$ *of Lemma 17.5 induces an isomorphism of* $\boldsymbol{F}_{\mathfrak{p},K}$*-modules*

$$\mathrm{Q}_\mathfrak{p}(\mathrm{V}_\mathfrak{p}\, \boldsymbol{M}) \xrightarrow{\cong} \boldsymbol{M}.$$

Proof. Take Γ_K-invariants! ∴

Proposition 17.10. *(a)* $\mathcal{O}_{\mathrm{Q}_\mathfrak{p}}$ *is an exact* $\mathcal{O}_\mathfrak{p}$*-linear tensor functor.*

(b) $\mathrm{Q}_\mathfrak{p}$ *is an exact* $F_\mathfrak{p}$*-linear tensor functor.*

Proof. By definition, $\mathscr{O}_{\mathrm{Q}_\mathfrak{p}}$ and $\mathrm{Q}_\mathfrak{p}$ are clearly left exact linear functors. Let us show that they are tensor functors, which we will deduce from the fact that the functors $D_{\mathscr{O}_\mathfrak{p}}$ and $D_{F_\mathfrak{p}}$ of Section 7 are such.

Let us do this for $\mathrm{Q}_\mathfrak{p}$, mutatis mutandis the proof is the same for $\mathscr{O}_{\mathrm{Q}_\mathfrak{p}}$. Consider a rational $\mathfrak{p}$-adic Galois representation $\boldsymbol{V}$. We have $D_{F_\mathfrak{p}}(\boldsymbol{V}) = (\boldsymbol{F}_{\boldsymbol{K}^{\mathrm{sep}},\mathfrak{p}} \otimes_{F_\mathfrak{p}} \boldsymbol{V})^{\Gamma_K}$ and $\mathrm{Q}_\mathfrak{p}(\boldsymbol{V}) = (\boldsymbol{B} \otimes_{F_\mathfrak{p}} \boldsymbol{V})^{\Gamma_K}$. Therefore, calculating inside $F_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{F_\mathfrak{p}} V$, we have $\mathrm{Q}_\mathfrak{p}(\boldsymbol{V}) = (\boldsymbol{B} \otimes_{F_\mathfrak{p}} \boldsymbol{V}) \cap D_{F_\mathfrak{p}}(\boldsymbol{V})$.

Given another rational $\mathfrak{p}$-adic Galois representation $\boldsymbol{W}$, we may apply these remarks to $\boldsymbol{V}$, $\boldsymbol{W}$ and $\boldsymbol{V} \otimes_{F_\mathfrak{p}} \boldsymbol{W}$, and calculate inside $F_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{F_\mathfrak{p}} V \otimes_{F_\mathfrak{p}} W$ to obtain:

$$\begin{aligned}
\mathrm{Q}_\mathfrak{p}(V \otimes_{F_\mathfrak{p}} W) &= (B \otimes_{F_\mathfrak{p}} V \otimes_{F_\mathfrak{p}} W) \cap D_{F_\mathfrak{p}}(V \otimes_{F_\mathfrak{p}} W) \\
&= \Big((B \otimes_{F_\mathfrak{p}} V) \otimes_B (B \otimes_{F_\mathfrak{p}} W)\Big) \cap \Big(D_{F_\mathfrak{p}}(V) \otimes_{F_{K,\mathfrak{p}}} D_{F_\mathfrak{p}}(W)\Big) \\
&= \Big((B \otimes_{F_\mathfrak{p}} V) \cap D_{F_\mathfrak{p}}(V)\Big) \otimes_{F_{\mathfrak{p},K}} \Big((B \otimes_{F_\mathfrak{p}} W) \cap D_{F_\mathfrak{p}}(W)\Big) \\
&= \mathrm{Q}_\mathfrak{p}(V) \otimes_{F_{\mathfrak{p},K}} \mathrm{Q}_\mathfrak{p}(W).
\end{aligned}$$

Finally, the (right) exactness of $\mathrm{Q}_\mathfrak{p}$ (and $\mathscr{O}_{\mathrm{Q}_\mathfrak{p}}$) follows formally from what we have proven. We do this again only for $\mathrm{Q}_\mathfrak{p}$, mutatis mutandis the proof is the same for $\mathscr{O}_{\mathrm{Q}_\mathfrak{p}}$. Since $\mathrm{Q}_\mathfrak{p}$ is a tensor functor and $\boldsymbol{V}$ admits a dual $\boldsymbol{V}^\vee$, the $\boldsymbol{F}_{\mathfrak{p},\boldsymbol{K}}$-module $\mathrm{Q}_\mathfrak{p}(\boldsymbol{V})$ admits a (functorial) dual, namely $\mathrm{Q}_\mathfrak{p}(\boldsymbol{V}^\vee)$. Therefore, if $\boldsymbol{V}' \to \boldsymbol{V} \to \boldsymbol{V}'' \to 0$ is a right exact sequence of rational $\mathfrak{p}$-adic Galois representations, then its image under $\mathrm{Q}_\mathfrak{p}$ coincides with the dual of the image of the left exact sequence $0 \to (\boldsymbol{V}'')^\vee \to \boldsymbol{V}^\vee \to (\boldsymbol{V}')^\vee$. Since $\mathrm{Q}_\mathfrak{p}$ is left exact, the image of this left exact sequence is left exact. So since dualisation is exact, the image of our original right exact sequence is right exact, and we are done. $\therefore$

Lemma 17.11. *(a) $\mathscr{O}_B$ is a projective $\mathscr{O}_{\mathfrak{p},K}$-module.*

(b) B is a projective $F_{\mathfrak{p},K}$-module.

Proof. By Corollary 6.10, $F_{\mathfrak{p},K} = Q_1 \times \cdots \times Q_s$ is a finite product of fields Q_i. Setting $B_i := Q_i \otimes_{F_{\mathfrak{p},K}} B$, wo obtain a decomposition $B = B_1 \oplus \cdots \oplus B_s$. Since the Q_i are fields, the B_i are free Q_i-modules, so B is a projective $F_{\mathfrak{p},K}$-module.

To show that this implies that $\mathscr{O}_B$ is a projective $\mathscr{O}_{\mathfrak{p},K}$-module, we need some more notation. Choose a local parameter $t \in F$ at $\mathfrak{p}$. We have $F_{\mathfrak{p},K} \subset F_{K,\mathfrak{p}}$, and the latter ring splits as $F_{K,\mathfrak{p}} = Q'_1 \times \cdots \times Q'_s$ where $Q'_i \cong K_r((t))$ for a finite field extension $K_r \supset K$ (use Corollary 6.9 and Example 6.11(b)). We may thus identify the fields Q_i with subfields of $K_r(\!(t)\!)$, for later use we note that Q_i *contains* t. Under this identification, setting $R_i := Q_i \cap K_r(\!(t)\!)$, we have $\mathscr{O}_{\mathfrak{p},K} = R_1 \times \cdots \times R_s$.

The ring B is a subring of

$$F_{K^{\mathrm{sep}},\mathfrak{p}} \cong (k_\mathfrak{p} \otimes_k K^{\mathrm{sep}})(\!(t)\!) = (k_\mathfrak{p} \otimes_k K \otimes_K K^{\mathrm{sep}})(\!(t)\!) = (K_r \otimes_K K^{\mathrm{sep}})^s(\!(t)\!),$$

with B_i contained in the i-th copy of $(K_r \otimes_K K^{\mathrm{sep}})(\!(t)\!)$. The ring $\mathcal{O}_B$ splits as $\mathcal{O}_{B,1} \times \cdots \times \mathcal{O}_{B,s}$, where $\mathcal{O}_{B,i} := \mathcal{O}_B \cap B_i$ is the ring consisting of those elements of B_i which, viewed as elements of the i-th copy of $(K_r \otimes_K K^{\mathrm{sep}})(\!(t)\!)$ in $F_{K^{\mathrm{sep}},\mathfrak{p}}$, are power series, that is, lie in $(K_r \otimes_K K^{\mathrm{sep}})[\![t]\!]$.

Let us show that $\mathcal{O}_{B,i}$ is a free R_i-module (this implies that $\mathcal{O}_B$ is a projective $\mathcal{O}_{\mathfrak{p},K}$-module). For this, we choose a Q_i-basis $\{b_{ij}\}_{j\in J_i}$ of B_i. Under the identifications given above, each b_{ij} corresponds to a Laurent series $\sum b_{ijn}t^n$ in $(K_r \otimes_K K^{\mathrm{sep}})(\!(t)\!)$. Now $K_r \otimes_K K^{\mathrm{sep}} \cong K^{\mathrm{sep},\rho}$ for some $\rho \geq 1$, whereby $1 \otimes 1$ corresponds to an element $(e_1, \ldots, e_\rho)$. By multiplying b_{ij} with a suitable element of the form $(e_1 t^{n(i,j,1)}, \ldots, e_\rho t^{n(i,j,\rho)})$, we may assume that $b_{ijn} = 0$ for $n < 0$ and that b_{ij0} is invertible in $K_r \otimes_K K^{\mathrm{sep}}$. And then under this assumption, one may check that $\{b_{ij}\}$ is indeed an R_i-basis of $\mathcal{O}_{B,i}$. ∴

Lemma 17.12.

(a) The natural homomorphism $\mathcal{O}_{K,\mathfrak{p}} \otimes_{\mathcal{O}_{\mathfrak{p},K}} \mathcal{O}_B \longrightarrow \mathcal{O}_{K^{\mathrm{sep}},\mathfrak{p}}$ *is injective.*

(b) The natural homomorphism $F_{K,\mathfrak{p}} \otimes_{F_{\mathfrak{p},K}} B \longrightarrow F_{K^{\mathrm{sep}},\mathfrak{p}}$ *is injective.*

Proof. Item (b) follows from item (a) by inverting any local parameter $t \in F$ at $\mathfrak{p}$.

(a): We will use the following facts from commutative algebra: Given an ideal $I \subset R$ of a commutative ring R such that $\bigcap I^n = 0$, the natural homomorphism $R \to \widehat{R}$ to the I-adic completion $\widehat{R} := \varprojlim R/I^n$ is injective. Furthermore, if M is a projective R-module, then the natural homomorphism

$$\widehat{R} \otimes_R M \longrightarrow \widehat{M} := \varprojlim M/I^n M \tag{17.13}$$

is also injective: It suffices to prove this for free R-modules M by the additivity of source and target of the homomorphism involved, but if $M \cong R^{\oplus J}$ for some set J, then the left hand side is isomorphic to $\left(\widehat{R}\right)^{\oplus J}$, whereas the right hand side is isomorphic to

$$\widehat{(R^{\oplus J})} = \varprojlim_n \left(\widehat{R}^{\oplus J}/I^n\right) = \varprojlim_n \left(R^{\oplus J}/I^n\right).$$

Hence the kernel is contained in $\bigcap_n I^n \left(\widehat{R}^{\oplus J}\right) = \left(\bigcap_n I^n \widehat{R}\right)^{\oplus J} = 0$.

We wish to apply this to $R = \mathcal{O}_{\mathfrak{p},K}$, $I = \mathfrak{p}$ (whence $\widehat{R} = \mathcal{O}_{K,\mathfrak{p}}$) and $M = \mathcal{O}_B$. By Lemma 17.11, $\mathcal{O}_B$ is a projective $\mathcal{O}_{\mathfrak{p},K}$-module. Next, since $\mathcal{O}_{K^{\mathrm{sep}},\mathfrak{p}}$ is $\mathfrak{p}$-adically complete, $\mathcal{O}_B \subset \mathcal{O}_{K^{\mathrm{sep}},\mathfrak{p}}$ and $\varprojlim$ is left-exact, we have $\widehat{\mathcal{O}_B} \subset \mathcal{O}_{K^{\mathrm{sep}},\mathfrak{p}}$. Together with (17.13) this completes our proof. ∴

Proposition 17.14. *(a) For every integral* $\mathfrak{p}$*-adic representation* $\boldsymbol{T}$*, the following natural map is injective:*

$$\mathcal{O}_{\boldsymbol{K},\boldsymbol{\mathfrak{p}}} \otimes_{\mathcal{O}_{\boldsymbol{\mathfrak{p}},\boldsymbol{K}}} \mathcal{O}_{\mathrm{Q}_\mathfrak{p}}(\boldsymbol{T}) \longrightarrow D_{F_\mathfrak{p}}(\boldsymbol{T})$$

(b) *For every rational* $\mathfrak{p}$*-adic representation* $\boldsymbol{V}$*, the following natural map is injective:*

$$\boldsymbol{F}_{\boldsymbol{K},\mathfrak{p}} \otimes_{\boldsymbol{F}_{\mathfrak{p},\boldsymbol{K}}} \mathrm{Q}_{\mathfrak{p}}(\boldsymbol{V}) \longrightarrow D_{F_{\mathfrak{p}}}(\boldsymbol{V})$$

Proof. (a): We calculate:

$$\begin{aligned}
\mathcal{O}_{K,\mathfrak{p}} \otimes_{\mathcal{O}_{\mathfrak{p},K}} \mathcal{O}_{\mathrm{Q}_{\mathfrak{p}}}(\boldsymbol{T}) &= \mathcal{O}_{K,\mathfrak{p}} \otimes_{\mathcal{O}_{\mathfrak{p},K}} \left(\mathcal{O}_B \otimes_{\mathcal{O}_{\mathfrak{p}}} T\right)^{\Gamma_K} \\
&= \left(\mathcal{O}_{K,\mathfrak{p}} \otimes_{\mathcal{O}_{\mathfrak{p},K}} \mathcal{O}_B \otimes_{\mathcal{O}_{\mathfrak{p}}} T\right)^{\Gamma_K} \\
&\subset \left(\mathcal{O}_{K^{\mathrm{sep}},\mathfrak{p}} \otimes_{\mathcal{O}_{\mathfrak{p}}} T\right)^{\Gamma_K} \qquad \text{by Lemma 17.12(a)} \\
&= D_{\mathcal{O}_{\mathfrak{p}}}(\boldsymbol{T}),
\end{aligned}$$

(b): We either repeat the calculation of (a), using Lemma 17.12(b), or write $\boldsymbol{V} = F_{\mathfrak{p}} \otimes_{\mathcal{O}_{\mathfrak{p}}} \boldsymbol{T}$ for some Γ_K-stable $\mathcal{O}_{\mathfrak{p}}$-lattice in $\boldsymbol{V}$ and note that the natural map under consideration is the localisation with respect to a local parameter $t \in F$ at $\mathfrak{p}$ of the respective natural map involving $\boldsymbol{T}$. ∴

Proposition 17.15. (a) *The functor* $\mathcal{O}_{\mathrm{Q}_{\mathfrak{p}}}$ *takes values in restricted* $\mathcal{O}_{\mathfrak{p},\boldsymbol{K}}$*-modules.*

(b) *The functor* $\mathrm{Q}_{\mathfrak{p}}$ *takes values in etale* $\boldsymbol{F}_{\mathfrak{p},\boldsymbol{K}}$*-modules.*

(c) *For every representation* $\boldsymbol{V}$*, one has* $\operatorname{rk} \mathrm{Q}_{\mathfrak{p}}(\boldsymbol{V}) \le \operatorname{rk} \boldsymbol{V}$.

Proof. For every rational representation $\boldsymbol{V}$ there exists an integral representation $\boldsymbol{T}$ such that $\boldsymbol{V} = F_{\mathfrak{p}} \otimes_{\mathcal{O}_{\mathfrak{p}}} \boldsymbol{T}$, and then $\mathrm{Q}_{\mathfrak{p}}(\boldsymbol{V}) = \boldsymbol{F}_{\mathfrak{p},\boldsymbol{K}} \otimes_{\mathcal{O}_{\mathfrak{p},\boldsymbol{K}}} \mathcal{O}_{\mathrm{Q}_{\mathfrak{p}}}(\boldsymbol{T})$. Therefore, it suffices to show that $\mathcal{O}_{\mathrm{Q}_{\mathfrak{p}}}(\boldsymbol{T})$ is a restricted $\mathcal{O}_{\mathfrak{p},\boldsymbol{K}}$-module of rank bounded above by $\operatorname{rk}(\boldsymbol{T})$.

By Proposition 17.14(a), $\mathcal{O}_{K,\mathfrak{p}} \otimes_{\mathcal{O}_{\mathfrak{p},K}} \mathcal{O}_{\mathrm{Q}_{\mathfrak{p}}}(\boldsymbol{T})$ is a submodule of $D_{\mathcal{O}_{\mathfrak{p}}}(\boldsymbol{T})$, which is a free $\mathcal{O}_{K,\mathfrak{p}}$-module of rank $\operatorname{rk}(\boldsymbol{T})$. Therefore, $\mathcal{O}_{\mathrm{Q}_{\mathfrak{p}}}(\boldsymbol{T})$ is a finitely generated torsion-free $\mathcal{O}_{\mathfrak{p},K}$-module. Since $D_{\mathcal{O}_{\mathfrak{p}}}(\boldsymbol{T})$ has bijective τ_{lin}, its submodule $\mathcal{O}_{K,\mathfrak{p}} \otimes_{\mathcal{O}_{\mathfrak{p},K}} \mathcal{O}_{\mathrm{Q}_{\mathfrak{p}}}(\boldsymbol{T})$ has injective τ_{lin}, and therefore $\mathcal{O}_{\mathrm{Q}_{\mathfrak{p}}}(\boldsymbol{T})$ has injective τ_{lin} as well. By Proposition 5.12, this implies that $\mathcal{O}_{\mathrm{Q}_{\mathfrak{p}}}(\boldsymbol{T})$ is free of *constant* rank, say $r := \operatorname{rk}_{\mathcal{O}_{\mathfrak{p},K}} \mathcal{O}_{\mathrm{Q}_{\mathfrak{p}}}(\boldsymbol{T}) \le \operatorname{rk} \boldsymbol{T}$. It remains to show that $\mathcal{O}_{\mathrm{Q}_{\mathfrak{p}}}(\boldsymbol{T})$ is restricted, i.e., that its τ_{lin} is bijective.

By Proposition 5.24, $\mathcal{O}_{\mathrm{Q}_{\mathfrak{p}}}(\boldsymbol{T})$ is restricted if and only if its determinant (i.e., its r-th exterior power) is restricted. By Proposition 17.10(a), $\mathcal{O}_{\mathrm{Q}_{\mathfrak{p}}}$ is a tensor functor, so we obtain an inclusion

$$\mathcal{O}_{K,\mathfrak{p}} \otimes_{\mathcal{O}_{\mathfrak{p},K}} \mathcal{O}_{\mathrm{Q}_{\mathfrak{p}}}(\Lambda^r \boldsymbol{T}) \subset D_{\mathcal{O}_{\mathfrak{p}}}(\Lambda^r \boldsymbol{T}),$$

where the right hand side is a restricted $\mathcal{O}_{\boldsymbol{K},\mathfrak{p}}$-module of rank ≥ 1. Tracing through the definitions, we see that the left hand side is saturated (cf. Proposition 5.25) in the right hand side. By Proposition 5.25, this implies that $\mathcal{O}_{K,\mathfrak{p}} \otimes_{\mathcal{O}_{\mathfrak{p},K}} \mathcal{O}_{\mathrm{Q}_{\mathfrak{p}}}(\Lambda^r \boldsymbol{T})$ is restricted as well. Now the equality $\mathcal{O}_{\mathfrak{p},K}^{\times} = \mathcal{O}_{K,\mathfrak{p}}^{\times} \cap \mathcal{O}_{\mathfrak{p},K}$ implies that $\mathcal{O}_{\mathrm{Q}_{\mathfrak{p}}}(\boldsymbol{T})$ is restricted. ∴

Proposition 17.16. *(a) $\boldsymbol{V}$ is quasigeometric if and only if* $\operatorname{rk}\mathrm{Q}_\mathfrak{p}(\boldsymbol{V}) = \operatorname{rk}(\boldsymbol{V})$.

(b) $\mathrm{V}_\mathfrak{p}(\mathrm{Q}_\mathfrak{p}(\boldsymbol{V}))$ *is the largest quasigeometric subrepresentation of* $\boldsymbol{V}$.

(c) Every subquotient of a quasigeometric representation is quasigeometric.

Proof. (a): Assume that $\boldsymbol{V} \cong \mathrm{V}_\mathfrak{p}(\boldsymbol{M})$ is quasigeometric. Consider the canonical isomorphism

$$c_{\boldsymbol{M}} : \boldsymbol{B} \otimes_{F_\mathfrak{p}} \mathrm{V}_\mathfrak{p}(\boldsymbol{M}) \longrightarrow \boldsymbol{B} \otimes_{\boldsymbol{F}_{\mathfrak{p},K}} \boldsymbol{M}$$

of Lemma 17.5. It implies that $\mathrm{Q}_\mathfrak{p}(\mathrm{V}_\mathfrak{p}(\boldsymbol{M})) \cong \boldsymbol{M}$ by restricting to Γ_K-invariants. Therefore, using the fact that $\mathrm{V}_\mathfrak{p}$ preserves ranks, we have

$$\operatorname{rk}\mathrm{Q}_\mathfrak{p}(\boldsymbol{V}) = \operatorname{rk}\mathrm{Q}_\mathfrak{p}\left(\mathrm{V}_\mathfrak{p}(\boldsymbol{M})\right) = \operatorname{rk}(\boldsymbol{M}) = \operatorname{rk}\mathrm{V}_\mathfrak{p}(\boldsymbol{M}) = \operatorname{rk}\boldsymbol{V},$$

as claimed.

Assume that we have an equality of ranks. By Proposition 17.14(b), the natural homomorphism $\boldsymbol{F}_{K,\mathfrak{p}} \otimes_{\boldsymbol{F}_{\mathfrak{p},K}} \mathrm{Q}_\mathfrak{p}(\boldsymbol{V}) \longrightarrow D_{F_\mathfrak{p}}(\boldsymbol{V})$ is injective. Since $D_{F_\mathfrak{p}}$ preserves ranks, both sides are free of equal finite rank over the semisimple commutative ring $F_{K,\mathfrak{p}}$. So the homomorphism is an isomorphism! We set $\boldsymbol{M} := \mathrm{Q}_\mathfrak{p}(\boldsymbol{V})$, an etale $\boldsymbol{F}_{\mathfrak{p},K}$-module by Proposition 17.15. Then the following isomorphisms shows that $\boldsymbol{V}$ is quasigeometric:

$$\boldsymbol{V} \cong V_{F_\mathfrak{p}}(D_{F_\mathfrak{p}}(\boldsymbol{V})) \cong V_{F_\mathfrak{p}}(\boldsymbol{F}_{K,\mathfrak{p}} \otimes_{\boldsymbol{F}_{\mathfrak{p},K}} \mathrm{Q}_\mathfrak{p}(\boldsymbol{V})) = \mathrm{V}_\mathfrak{p}(\mathrm{Q}_\mathfrak{p}(\boldsymbol{V})) = \mathrm{V}_\mathfrak{p}(\boldsymbol{M}).$$

(b): $\mathrm{V}_\mathfrak{p}(\mathrm{Q}_\mathfrak{p}\,\boldsymbol{V})$ is quasigeometric by Proposition 17.15(b). Proposition 17.14(b) and the exactness of $\mathrm{V}_\mathfrak{p}$ imply that $\mathrm{V}_\mathfrak{p}(\mathrm{Q}_\mathfrak{p}\,\boldsymbol{V})$ is a subrepresentation of $\boldsymbol{V}$. Let us show that it contains every other quasigeometric subrepresentation $\mathrm{V}_\mathfrak{p}(\boldsymbol{M}) \cong \boldsymbol{V}' \subset \boldsymbol{V}$. By restricting the isomorphism $c_{\boldsymbol{M}}$ of Lemma 17.5 to Γ_K-invariants, we have $\boldsymbol{M} = \mathrm{Q}_\mathfrak{p}(\mathrm{V}_\mathfrak{p}\,\boldsymbol{M})$. So using the left-exactness of $\mathrm{Q}_\mathfrak{p}$ given by Proposition 17.10(b), we see that

$$\boldsymbol{M} = \mathrm{Q}_\mathfrak{p}(\mathrm{V}_\mathfrak{p}\,\boldsymbol{M}) = \mathrm{Q}_\mathfrak{p}\,\boldsymbol{V}' \subset \mathrm{Q}_\mathfrak{p}\,\boldsymbol{V}.$$

In turn, since $\mathrm{V}_\mathfrak{p}$ is exact, this shows that $\boldsymbol{V}' = \mathrm{V}_\mathfrak{p}\,\boldsymbol{M} \subset \mathrm{V}_\mathfrak{p}(\mathrm{Q}_\mathfrak{p}\,\boldsymbol{V})$, as claimed.

(c): Let $0 \longrightarrow \boldsymbol{V}' \longrightarrow \boldsymbol{V} \longrightarrow \boldsymbol{V}'' \longrightarrow 0$ be an exact sequence of representations, of which $\boldsymbol{V}$ is quasigeometric. Consider the induced sequence

$$0 \longrightarrow \mathrm{Q}_\mathfrak{p}\,\boldsymbol{V}' \longrightarrow \mathrm{Q}_\mathfrak{p}\,\boldsymbol{V} \longrightarrow \mathrm{Q}_\mathfrak{p}\,\boldsymbol{V}' \longrightarrow 0 \tag{17.17}$$

It is exact by Proposition 17.10. Applying the exact functor $\mathrm{V}_\mathfrak{p}$, we obtain an exact sequence

$$0 \longrightarrow \mathrm{V}_\mathfrak{p}\,\mathrm{Q}_\mathfrak{p}\,\boldsymbol{V}' \longrightarrow \boldsymbol{V} \longrightarrow \mathrm{V}_\mathfrak{p}\,\mathrm{Q}_\mathfrak{p}\,\boldsymbol{V}'' \longrightarrow 0,$$

where $\boldsymbol{V} = \mathrm{V}_\mathfrak{p}\,\mathrm{Q}_\mathfrak{p}\,\boldsymbol{V}$ by item (b). Now

$$\operatorname{rk}\boldsymbol{V} = \operatorname{rk}\mathrm{V}_\mathfrak{p}\,\mathrm{Q}_\mathfrak{p}\,\boldsymbol{V}' + \operatorname{rk}\mathrm{V}_\mathfrak{p}\,\mathrm{Q}_\mathfrak{p}\,\boldsymbol{V}'' \leq \operatorname{rk}\boldsymbol{V}' + \operatorname{rk}\boldsymbol{V}'' = \operatorname{rk}\boldsymbol{V}$$

implies that $\operatorname{rk}\mathrm{V}_\mathfrak{p}\,\mathrm{Q}_\mathfrak{p}\,\boldsymbol{V}' = \operatorname{rk}\boldsymbol{V}'$ and $\operatorname{rk}\mathrm{V}_\mathfrak{p}\,\mathrm{Q}_\mathfrak{p}\,\boldsymbol{V}' = \operatorname{rk}\boldsymbol{V}'$, so $\boldsymbol{V}' = \mathrm{V}_\mathfrak{p}\,\mathrm{Q}_\mathfrak{p}\,\boldsymbol{V}'$ and $\boldsymbol{V}'' = \mathrm{V}_\mathfrak{p}\,\mathrm{Q}_\mathfrak{p}\,\boldsymbol{V}''$ are both quasigeometric. ∴

We collect our results in a categorical reformulation.

Theorem 17.18. *(a) The functor* $\mathrm{V}_\mathfrak{p}$ *is a semisimple fully faithful exact* $F_\mathfrak{p}$*-linear tensor functor.*

(b) The pair $(\mathrm{V}_\mathfrak{p}, \mathrm{Q}_\mathfrak{p})$ *is an adjoint pair of functors, that is, for every* $\boldsymbol{F}_{\mathfrak{p},K}$*-module* $\boldsymbol{M}$ *and rational* $\mathfrak{p}$*-adic Galois representation* $\boldsymbol{V}$ *there exists a natural isomorphism of* $F_\mathfrak{p}$*-vector spaces*

$$\operatorname{Hom}(\mathrm{V}_\mathfrak{p}(\boldsymbol{M}), \boldsymbol{V}) \longrightarrow \operatorname{Hom}(\boldsymbol{M}, \mathrm{Q}_\mathfrak{p}(\boldsymbol{V}))$$

(c) The unit $\mathrm{id} \Rightarrow \mathrm{Q}_\mathfrak{p} \circ \mathrm{V}_\mathfrak{p}$ *of this adjunction is an isomorphism (so* $\mathrm{Q}_\mathfrak{p}$ *is a "coreflection" of the "inclusion"* $\mathrm{V}_\mathfrak{p}$*).*

(d) The counit $\mathrm{V}_\mathfrak{p} \circ \mathrm{Q}_\mathfrak{p} \Rightarrow \mathrm{id}$ *of this adjunction is a monomorphism.*

Proof. (a): That $\mathrm{V}_\mathfrak{p}$ is an exact $F_\mathfrak{p}$-linear tensor functor has been proven elsewhere. It is fully faithful by Theorem 17.7. Proposition 17.16(c) implies that $\mathrm{V}_\mathfrak{p}$ it maps semisimple objects to semisimple objects, so it is a semisimple functor (cf. Definition 3.1).

(b): Let us construct the inverse of the adjunction isomorphism for given $\boldsymbol{M}$ and $\boldsymbol{V}$. Since $\mathrm{V}_\mathfrak{p}$ is fully faithful, we have a natural isomorphism

$$\mathrm{V}_\mathfrak{p} :\ \operatorname{Hom}(\boldsymbol{M}, \mathrm{Q}_\mathfrak{p}\,\boldsymbol{V}) \longrightarrow \operatorname{Hom}(\mathrm{V}_\mathfrak{p}\,\boldsymbol{M}, \mathrm{V}_\mathfrak{p}\,\mathrm{Q}_\mathfrak{p}\,\boldsymbol{V})$$

One the other hand, every homorphism $\mathrm{V}_\mathfrak{p}\,\boldsymbol{M} \to \boldsymbol{V}$ has a quasigeometric image by Proposition 17.16(c), which must lie in $\mathrm{V}_\mathfrak{p}\,\mathrm{Q}_\mathfrak{p}\,\boldsymbol{V}$ by Proposition 17.16(b). Therefore, $\operatorname{Hom}(\mathrm{V}_\mathfrak{p}\,\boldsymbol{M}, \mathrm{V}_\mathfrak{p}\,\mathrm{Q}_\mathfrak{p}\,\boldsymbol{V}) = \operatorname{Hom}(\mathrm{V}_\mathfrak{p}\,\boldsymbol{M}, \boldsymbol{V})$, and we are done.

(c,d): Both items follow from Proposition 17.16. ∴

18 Constructing a ring of periods

We assume that K is a *finitely generated* field extension of our finite field k with q elements, and identify K with the function field $k(X)$ of a proper normal variety X over k. For every finite Galois extension $K^{\mathrm{sep}} \supset L \supset K$, let X_L be the normalisation of X in L, this is a proper normal variety over L.

Let Σ_L be the set of prime (Weil) divisors of X_L. For every Galois tower

$$K^{\mathrm{sep}} \supset L' \supset L \supset K$$

we have a projection map $\mathrm{pr}_{L,L'} : \Sigma_{L'} \longrightarrow \Sigma_L$, so we may let

$$\Sigma^{\mathrm{sep}} := \varprojlim_{L \supset K} \Sigma_L$$

be the projective limit along the projections $\mathrm{pr}_{L',L}$. Given a Galois extension $L \supset K$, an element $x_L \in \Sigma_L$ and an element $x \in \Sigma^{\mathrm{sep}}$, we say that x *lies over* x_L if x_L is the L-th component of x.

For each $x = (x_L)_L \in \Sigma^{\mathrm{sep}}$, there is a unique associated valuation

$$v_x : K^{\mathrm{sep}} \longrightarrow \mathbb{Q} \cup \{\infty\}$$

extending the normalised valuation v_{x_K} of K associated to x_K. Explicitly, for $f \in K^{\mathrm{sep}}$ we may choose a finite Galois extension $K \subset L \subset K^{\mathrm{sep}}$ containing f, and set $v_x(f) := v_{x_L}(f)/e_{x_L}$, where v_{x_L} denotes the normalised valuation of L associated to x_L, and e_{x_L} is the index of $v_{x_L}(K^*)$ in $v_{x_L}(L^*) = \mathbb{Z}$.

Let F be a global field with field of constants k, and fix a place $\mathfrak{p}$ of degree $d := \deg \mathfrak{p}$ of F with residue field $k_\mathfrak{p}$. We wish to extend v_x to a function on $F_{K^{\mathrm{sep}},\mathfrak{p}}$. For calculational reasons, we choose a local parameter $t \in F$ at $\mathfrak{p}$ and obtain identifications $\mathscr{O}_{K^{\mathrm{sep}},\mathfrak{p}} = (k_\mathfrak{p} \otimes_k K^{\mathrm{sep}})[\![t]\!]$ and $F_{K^{\mathrm{sep}},\mathfrak{p}} = (k_\mathfrak{p} \otimes_k K^{\mathrm{sep}})(\!(t)\!) = \mathscr{O}_{K^{\mathrm{sep}},\mathfrak{p}}[t^{-1}]$. Recall that by Corollary 6.9 the homomorphism

$$(k_\mathfrak{p} \otimes_k K^{\mathrm{sep}}, \mathrm{id} \otimes \sigma) \longrightarrow \left((K^{\mathrm{sep}})^d, \sigma'\right) \tag{18.1}$$

mapping $x \otimes y$ to $(x \cdot \sigma^i(y))_{i=0}^{d-1}$ is an isomorphism of bold rings, where $\sigma(\lambda) = \lambda^q$ for $\lambda \in K^{\mathrm{sep}}$ and

$$\sigma'(z_0, \ldots, z_{d-1}) = (z_{d-1}^q, z_0^q, \ldots, z_{d-2}^q)$$

for $(z_0, \ldots, z_{d-1}) \in K^{\mathrm{sep}\,d}$. We will denote the action of σ' on $(K^{\mathrm{sep}})^d$ simply by σ. Writing an element $f \in F_{K^{\mathrm{sep}},\mathfrak{p}}$ as $f = \sum_{i \gg -\infty} f_i t^i$ with $f_i = (f_{ij})_j \in (K^{\mathrm{sep}})^d$, we set

$$v_x(f) := \inf_{i,j} v_x(f_{ij}) = \inf_i \min_j v_x(f_{ij}).$$

Moreover, for all $m, n \geq 1$ and $\Delta = (\delta_{ij}) \in \mathrm{Mat}_{m \times n}(F_{K^{\mathrm{sep}},\mathfrak{p}})$ we set

$$v_x(\Delta) := \min_{i,j} v_x(\delta_{ij}).$$

Proposition 18.2. *For each $x \in \Sigma^{\mathrm{sep}}$ and all $m, n \geq 1$, the function*

$$v_x : \mathrm{Mat}_{m \times n}(F_{K^{\mathrm{sep}},\mathfrak{p}}) \longrightarrow \mathbb{R} \cup \{\pm\infty\}$$

is well-defined and independent of the choices made. For $m = n = 1$ and all $f, g \in F_{K^{\mathrm{sep}},\mathfrak{p}}$ it has the following properties:

(a) $v_x(f+g) \geq \min\{v_x(f), v_x(g)\}$.

(b) $v_x(fg) \geq v_x(f) + v_x(g)$ *(using the convention* $-\infty + \infty = -\infty$*).*

(c) $v_x(\sigma(f)) = q \cdot v_x(f)$.

Proof. Since $v_x(k_\mathfrak{p}^*) = 0$, the choice of local parameter does not influence the definition of v_x. Now (a,b) follow from short calculations using the semicontinuity of infima, whereas (c) follows from (18.1). $\therefore$

Remark 18.3. Note that, in general, we do not have $v_x(fg) = v_x(f) + v_x(g)$.

Proposition 18.4. *For all integers* $m, n \geq 1$*, column vectors* $F \in \mathrm{Mat}_{n\times 1}(F_{K^{\mathrm{sep}},\mathfrak{p}})$ *and matrices* $\Delta \in \mathrm{Mat}_{n\times n}(F_{K^{\mathrm{sep}},\mathfrak{p}})$ *the equation* $\sigma^m(F) = \Delta \cdot F$ *implies the inequality*

$$v_x(F) \geq \frac{1}{q^m - 1} v_x(\Delta).$$

Proof. If $v_x(\Delta) = -\infty$, the inequality stated is tautological, so we assume that $C := v_x(\Delta) \neq -\infty$. By a matrix-version of Proposition 18.2, the equation $\sigma^m(F) = \Delta F$ would imply that $q^m \cdot v_x(F) \geq C + v_x(F)$. If also $v_x(F) \neq \pm\infty$, this would imply the claim of this Proposition. However, if $v_x(F) = -\infty$, there is a problem. The following proof deals with all cases at once!

Write $F = (f_i)$ and $\Delta = (\delta_{ij})$ with $f_i, \delta_{ij} \in F_{K^{\mathrm{sep}},\mathfrak{p}}$. Furthermore, write $f_i = \sum_r f_{ir} t^r$ and $\delta_{ij} = \sum_s h_{ijs} t^s$ for $f_{ir}, \delta_{ijs} \in k_\mathfrak{p} \otimes_k K^{\mathrm{sep}}$. By multiplying the entire equation by a suitable power of t, we may assume that these coefficients are zero for $r, s < 0$. By assumption we have $v_x(\delta_{ijs}) \geq C$, and by definition we have $v_x(f_{ir}) \neq -\infty$.

The equation $\sigma^m(F) = \Delta \cdot F$ means $\sigma^m(f_i) = \sum_{j=1}^n \delta_{ij} f_j$ for all i, and gives

$$\sum_{r\geq 0} \sigma^m(f_{ir}) t^r = \sum_{j=1}^{n} \sum_{a\geq 0} \sum_{b\geq 0} \delta_{ija} f_{jb} t^{a+b} = \sum_{r\geq 0} \left(\sum_{j=1}^{n} \sum_{l=0}^{r} \delta_{ijl} f_{j,r-l} \right) t^r.$$

From this we see that

$$\sigma^m(f_{ir}) = \sum_{j=1}^{n} \sum_{l=0}^{r} \delta_{ijl} f_{j,r-l} \tag{18.5}$$

and must prove that $v_x(f_{ir}) \geq C/(q^m - 1)$. Let us do this by induction on r.

If $r = 0$, then for all i we have $\sigma^m(f_{i0}) = \sum_{j=1}^n \delta_{ij0} f_{j0}$ which gives $q^m \cdot v_x(f_{i0}) \geq \min_{j=1}^n (C + v_x(f_{j0}))$. Choosing j such that the minimum is attained we get $q^m v_x(f_{j0}) \geq C + v_x(f_{j0})$ and hence $v_x(f_{j0}) \geq C/(q^m - 1)$. So by the choice of j, for all i we may deduce that $v_x(f_{i0}) \geq v_x(f_{j0}) \geq C/(q^m - 1)$.

For $r > 0$, Equation (18.5) gives $q^m v_x(f_{ir}) \geq \inf_{j\leq n, l\leq r}(C + v_x(f_{jl}))$, hence by the induction hypothesis for all $r' < r$

$$q^m v_x(f_{ir}) \geq \min\left(\frac{q^m}{q^m - 1}C, \min_{j=1}^{n}(C + v_x(f_{jr}))\right).$$

If $q^m C/(q^m - 1)$ is smaller, we obtain $v_x(f_{ir}) \geq C/(q^d - 1)$ for all i as in the case $r = 0$. Else, choosing j such that the inner minimum is attained, we get first $v_x(f_{jr}) \geq C/(q^m - 1)$ and then $v_x(f_{ir}) \geq C/(q^m - 1)$ for all i, as in the case $r = 0$. ∴

We now turn to the definition of our ring of periods.

Definition 18.6. Following [Tam95], we set

(a) $B^+ := \left\{ f \in F_{K^{\mathrm{sep}},\mathfrak{p}} : \begin{array}{ll} v_x(f) \neq -\infty & \text{for all } x \in \Sigma^{\mathrm{sep}} \\ v_x(f) \geq 0 & \text{for almost all } x \in \Sigma^{\mathrm{sep}} \end{array} \right\}$,
"almost all" meaning that the set of exceptions has finite image in Σ_K.

(b) $S := \{s \in \mathcal{O}^{\times}_{K^{\mathrm{sep}},\mathfrak{p}} : \frac{\sigma(s)}{s} \in F_{\mathfrak{p}} \otimes_k K\}$.

Lemma 18.7. *B^+ is a Γ_K-stable ring.*

Proof. The fact that B^+ is Γ_K-stable follows directly from its definition. That B^+ is a ring (closed under finite sums and products) follows from Proposition 18.2: Clearly, B^+ contains 1. For $f \in B^+$ let Σ_f denote the finite subset of those elements of Σ_K over which there lies an element $x \in \Sigma^{\mathrm{sep}}$ such that $v_x(f) < 0$.

Given two elements $f, g \in B^+$, for all $x \in \Sigma^{\mathrm{sep}}$ by Proposition 18.2(a) we have $v_x(f + g) \geq \min(v_x(f), v_x(g))$, which is not equal to $-\infty$, since this is such for both $v_x(f)$ and $v_x(g)$. For all x whose image in Σ_K does not lie in its the finite subset $\Sigma_f \cup \Sigma_g$ we even have $v_x(f + g) \geq 0$. Therefore, $f + g$ is an element of B^+.

A similar proof, using Proposition 18.2(b), shows that $f \cdot g$ is an element of B^+. All in all, B^+ is a ring. ∴

Lemma 18.8. $(B^+)^{\Gamma_K} = F_{\mathfrak{p}} \otimes_k K$.

Proof. We note that $(B^+)^{\Gamma_K} = B^+ \cap F_{K,\mathfrak{p}}$. So the desired equality $(B^+)^{\Gamma_K} = F_{\mathfrak{p}} \otimes_k K$ is an equality of subrings of $F_{K,\mathfrak{p}}$. By Corollary 6.9 and Example 6.11(b), we have $F_{K,\mathfrak{p}} = (K_r)^e(\!(t)\!)$ for a finite Galois extension $K_r \supset K$ (it is Galois since $k_{\mathfrak{p}} \supset k$ is Galois and $k_{\mathfrak{p}} \otimes_k K \cong (K_r)^e$). The inclusion $F_{K,\mathfrak{p}} \subset F_{K^{\mathrm{sep}},\mathfrak{p}}$ corresponds to a homomorphism $(K_r)^e(\!(t)\!) \hookrightarrow (K^{\mathrm{sep}})^d(\!(t)\!)$ mapping the i-the component of the source to d/e components of the target, according to the d/e different K-linear embeddings of K_r in K^{sep}. It follows that the image of this homomorphism lies in $(K_r)^d(\!(t)\!)$.

Given an element $f \in F_{K,\mathfrak{p}}$, we may write it as a Laurent series $\sum_i f_i t^i$, with coefficients $f_i = (f_{i1}, \ldots, f_{id}) \in K_r^d$. We let V_f denote the k-vector subspace of K_r

generated by the f_{ij}. Clearly, $F_\mathfrak{p} \otimes_k K$ consists of those elements of $F_{K,\mathfrak{p}}$ such that $\dim_k V_f$ is finite.

On the other hand, by definition $(B^+)^{\Gamma_K}$ consists of those elements of $F_{K,\mathfrak{p}}$ such that $v_{x_r}(f) \neq -\infty$ for all $x_r \in \Sigma_{K_r}$ and $v_{x_r}(f) \geq 0$ for all but a finite number of $x_r \in \Sigma_{K_r}$.

Now, if $f \in F_{K,\mathfrak{p}}$ is an element of $F_\mathfrak{p} \otimes_k K$, then $\dim_k V_f$ is finite, so the subset of Σ_{K_r} consisting of the poles of the (coefficients of the) elements of V_f is finite, so f is an element of B^+ by our above characterisation.

On the other hand, if $f \in F_{K,\mathfrak{p}}$ is an element of B^+, then we may choose a finite subset $\Sigma_0 \subset \Sigma_{K_r}$ such that $v_{x_r}(f) \geq 0$ for all $x_r \notin \Sigma_0$. For $x_r \in \Sigma_0$, we set $n(x_r) := -v_{x_r}(f)$, which is finite by assumption. Let X_r denote the proper normal variety over k corresponding to K_r. Since it is proper, the space of global sections of

$$\mathscr{O}_{X_r}\left(\sum_{x_r \in \Sigma_0} n(x_r)x_r\right)$$

is finite-dimensional. Since it contains V_f, this implies that $f \in F_\mathfrak{p} \otimes_k K$ by our above characterisation. ∴

Lemma 18.9. *S is a Γ_K-stable multiplicative subset of B^+.*

Proof. The fact that S is a Γ_K-stable multiplicative subset of $F_{K^{\mathrm{sep}},\mathfrak{p}}$ follows directly from its definition.

Let us show that S is contained in B^+. For $s \in S$ choose $f \in F_\mathfrak{p} \otimes_k K$ such that $\sigma(s) = f \cdot s$, such an f exists by definition of S. By Lemma 18.8 and Proposition 18.4, $v_x(s) \neq -\infty$ for all $x \in \Sigma^{\mathrm{sep}}$, and there exists a finite subset Σ_0 of Σ_K such that $v_x(f) \geq 0$ for all $x \in \Sigma^{\mathrm{sep}}$ not lying over Σ_0.

For all $x \in \Sigma^{\mathrm{sep}}$, Proposition 18.4 shows that $v_x(s) \geq v_x(f)/(q-1)$. So s has the required properties that $v_x(s) \neq -\infty$ for all $x \in \Sigma^{\mathrm{sep}}$ and $v_x(s) \geq 0$ for all $x \in \Sigma^{\mathrm{sep}}$ not lying over Σ_0, since this is the case for f. ∴

Definition 18.10. Following [Tam95], we let $B \subset F_{K^{\mathrm{sep}},\mathfrak{p}}$ be the ring obtained by inverting $S \subset B^+$, and set $\boldsymbol{B} = (B, \sigma)$, where σ is the given ring endomorphism of $\boldsymbol{F}_{\boldsymbol{K}^{\mathbf{sep}},\mathfrak{p}}$.

Lemma 18.11. *$\boldsymbol{B}$ is a bold ring with ring of scalars $F_\mathfrak{p}$.*

Proof. B is clearly σ-stable since B^+ and S are. Furthermore, since $F_\mathfrak{p} \subset B$ and $B^\sigma \subset F^\sigma_{K^{\mathrm{sep}},\mathfrak{p}} = F_\mathfrak{p}$, we have $B^\sigma = F_\mathfrak{p}$. ∴

We say that an element $f \in F_{K^{\mathrm{sep}},\mathfrak{p}}$ has *order* $n \in \mathbb{Z}$ if, writing f as $\sum f_i t^i \in (K^{\mathrm{sep}} \otimes_k k_\mathfrak{p})(\!(t)\!)$ we have $n = \inf\{i : f_i \neq 0\}$. We say that an element $f \in F_{K^{\mathrm{sep}},\mathfrak{p}}$ of order n has *invertible leading coefficient* if f_n is invertible in $k_\mathfrak{p} \otimes_k K^{\mathrm{sep}}$. If f

has order 0, then we will denote by $f(0)$ the leading coefficient of f. Note that the invertible elements of $\mathscr{O}_{K^{\mathrm{sep}},\mathfrak{p}}$ are precisely the elements of $F_{K^{\mathrm{sep}},\mathfrak{p}}$ of order 0 with invertible leading coefficient.

Remark 18.12. Let us set $t_i := e_i \cdot t \in F_{K^{\mathrm{sep}},\mathfrak{p}}$, where e_i is the standard basis vector of the i-th copy of K^{sep} in the product $(K^{\mathrm{sep}})^d$. Clearly, an element $f \in F_{K^{\mathrm{sep}},\mathfrak{p}}$ is invertible if and only if we can write

$$f = \left(\prod_{i=0}^{d-1} t_i^{n_i}\right) \cdot \widetilde{f},$$

for certain $n_i \in \mathbb{Z}$, where $\widetilde{f}$ is an element of $\mathscr{O}^{\times}_{K^{\mathrm{sep}},\mathfrak{p}}$.

Lemma 18.13. *Every element $f \in \mathscr{O}^{\times}_{K^{\mathrm{sep}},\mathfrak{p}}$ may be written as $f = \frac{\sigma(s)}{s}$ for some other element $s \in \mathscr{O}^{\times}_{K^{\mathrm{sep}},\mathfrak{p}}$.*

Proof. We write $f = \sum_{i\geq 0} f_i t^i$ and use the "ansatz" $s = \sum_{j\geq 0} s_j t^j$. This gives

$$\sum_r \sigma(s_r)t^r = \sigma(s) = sf = \sum_{i,j} f_i s_j t^{i+j} = \sum_r \left(\sum_{i=0}^{r} f_i s_{r-i}\right) t^r.$$

We proceed by induction. For $r = 0$, we must solve $\sigma(s_0) = f_0 s_0$. We write $f_0 = (f_{0,0}, \ldots, f_{0,d-1})$ and $s_0 = (s_{0,0}, \ldots, s_{0,d-1})$ for $f_{0,i}, s_{0,i} \in K^{\mathrm{sep}}$. Note that by assumption all $f_{0,i} \neq 0$. Since

$$\sigma(s_0) = (s_{0,d-1}^q, s_{0,0}^q, s_{0,1}^1, \ldots, s_{0,d-1}^q)$$

our equation $\sigma(s_0) = f_0 s_0$ is equivalent to the system of equations

$$s_{0,i}^q = f_{0,i+1} s_{0,i+1}, \qquad i \in \mathbb{Z}/d\mathbb{Z}.$$

This means, for instance, that $s_{0,0} = s_{0,d-1}^q / f_{0,0}$ and $s_{0,d-1} = s_{0,d-2}^q / f_{0,d-1}$, which gives

$$s_{0,0}^q = \frac{s_{0,d-1}^q}{f_{0,0}} = \frac{\left(s_{0,d-2}^q / f_{0,d-1}\right)^q}{f_{0,0}}.$$

Iterating this substitution, we obtain the equation

$$s_{0,0}^{q^d} - \left(f_{0,1}^{q^{d-1}} \cdot f_{0,2}^{q^{d-2}} \cdots f_{0,d-1}^q \cdot f_{0,0}\right) s_{0,0} = 0.$$

Since all the $f_{0,i} \neq 0$, the constant $\phi := f_{0,1}^{q^{d-1}} \cdot f_{0,2}^{q^{d-2}} \cdots f_{0,d-1}^q \cdot f_{0,0}$ is non-zero, so this is a separable equation for $s_{0,0}$ and hence has a non-trivial solution in K^{sep}.

The $s_{0,i}$ for $i \neq 0$ are then determined by the assignments $s_{0,i} := s_{0,i-1}^q / f_{0,i}$, they are non-trivial since $s_{0,0}$ and the $f_{0,i}$ are.

Let us consider the case $r > 0$, and write $s_r = (s_{r,0}, \ldots, s_{r,d-1})$ and $f_r = (f_{r,0}, \ldots, f_{r,d-1})$. In this case, the equation $\sigma(s_r) = \sum_{i=0}^r f_i s_{r-i}$ that we must solve is equivalent to the system of equations

$$s_{r,i+1}^q = \sum_{j=0}^{r} f_{i,0} s_{r-i,0} =: f_{0,i} s_{r,i} + C_{r,i},$$

where the $C_{r,i} \in K^{\text{sep}}$ are constants dependant only on f and the $s_{r'}$ for $r' < r$.

We may use the same type of replacement as before, and obtain an equation

$$s_{r,0}^{q^d} - \phi \cdot s_{r,0} = C_r$$

with $C_r \in K^{\text{sep}}$ a constant determined by the $C_{r,i}$. Again, this is a separable equation for $s_{r,0}$, so there exists a solution in K^{sep}. The $s_{r,i}$ for $i \neq 0$ are then determined by the equations $s_{r,i} = (s_{r,i+1}^q - C_{r,i})/f_{0,i}$.

Finally, since we may choose the $s_{0,i}$ to be non-zero, our solution s is in fact invertible in $\mathcal{O}_{K^{\text{sep}},\mathfrak{p}}$. ∴

Proposition 18.14. *B is a Γ_K-stable ring, and $B^{\Gamma_K} \supset F_{\mathfrak{p},K}$.*

Proof. B is clearly Γ_K-stable, since B^+ and S both are. We have $B^{\Gamma_K} = B \cap F_{K,\mathfrak{p}}$.

Let us show that $F_{\mathfrak{p},K} \subset B$. Consider $g/f \in F_{\mathfrak{p},K}$ with $f, g \in F_{\mathfrak{p}} \otimes_k K$. By Remark 18.12, we may assume that f is in $\mathcal{O}_{K^{\text{sep}},\mathfrak{p}}^\times$. By Lemma 18.13 there exists an element $s \in S$ with $f = \sigma(s)/s$. It follows that $g/f = gs/\sigma(s) \in B$, since $gs \in B^+$ by Lemma 18.7 and $\sigma(s) \in S$. ∴

We turn to the inclusion $B^{\Gamma_K} \subset F_{\mathfrak{p},K}$, which is more difficult. Consider $b = b^+/s \in B^{\Gamma_K}$, with $b^+ \in B^+$ and $s \in S \subset \mathcal{O}_{K^{\text{sep}},\mathfrak{p}}^\times$. We set $f := \sigma^d(s)/s$, which is an element of $F_{\mathfrak{p}} \otimes_k K$, and for $N \geq 0$ – following [Tam04] – we set

$$a_N := b \cdot f\left(t^{q^d}\right) \cdot f\left(t^{q^{2d}}\right) \cdots f\left(t^{q^{Nd}}\right) \in F_{K,\mathfrak{p}}.$$

Remark 18.15. Our goal is to show that for N large enough the element a_N lies in B^+. By Lemma 18.9 this will imply that $a_N \in F_{\mathfrak{p}} \otimes_k K$, and in particular that $b \in B$.

Lemma 18.16. *There exists a finite set $\Sigma_0 \subset \Sigma_K$ such that for all $N \geq 0$ and all $x \in \Sigma^{\text{sep}}$ not lying above Σ_N we have $v_x(a_N) \geq 0$.*

Proof. The idea is to use that b^+, s and f all lie in B^+, and then use Proposition 18.2(b). In order to handle $1/s$, which is not necessarily an element of B^+, we need some modifications. Let $s(0)$ denote the leading coefficient of s, and set

$\widetilde{s} := s/s(0)$. Clearly, $\widetilde{s}$ is an element of S with leading coefficient 1. Setting $\widetilde{f} := \sigma^d(\widetilde{s})/\widetilde{s}$, we have $\widetilde{f} \in F_{\mathfrak{p}} \otimes_k K$ and $f = \mu \cdot \widetilde{f}$ with $\mu := \sigma^d(s(0))/s(0)$ an invertible element of $k_{\mathfrak{p}} \otimes_k K$. Now by definition and Proposition 18.2(b), we have

$$\begin{aligned} v_x(a_N) &= v_x\left(\frac{b^+}{s} \cdot f\left(t^{q^d}\right) \cdots f\left(t^{q^{Nd}}\right)\right) \\ &= v_x\left(\frac{\mu^N}{s(0)} \cdot b^+ \cdot \frac{1}{\widetilde{s}} \cdot \widetilde{f}\left(t^{q^d}\right) \cdots \widetilde{f}\left(t^{q^{Nd}}\right)\right) \\ &\geq N \cdot v_x(\mu) + v_x\left(\frac{1}{s(0)}\right) + v_x(b^+) + v_x\left(\frac{1}{\widetilde{s}}\right) + N \cdot v_x(\widetilde{f}) \end{aligned}$$

Since $E := \{\mu, 1/s(0), b^+, \widetilde{f}\}$ is a finite subset of B^+, the set Σ_0' of those $x \in \Sigma^{\mathrm{sep}}$ for which there exists an $e \in E$ such that $v_x(e) < 0$ has finite image in Σ_K. Call this image Σ_0, and consider any $x \in \Sigma_0$. Proposition 18.4 implies that $v_x(\widetilde{s}) \geq v_x(\widetilde{f})/(q^d - 1) \geq 0$. Since $\widetilde{s}$ has leading coefficient 1, we may calculate $1/\widetilde{s}$ via the geometric series, and obtain $v_x(1/\widetilde{s}) \geq 0$, using Proposition 18.2. Therefore, $v_x(a_N)$ is bounded below by a finite sum of non-negative numbers, so $v_x(a_N) \geq 0$ for all x not lying above Σ_0. ∴

Lemma 18.17 (following [Tam94]). *Let $s \in \mathscr{O}^{\times}_{K^{\mathrm{sep}},\mathfrak{p}}$, $x \in \Sigma^{\mathrm{sep}}$ and $N \geq 0$ fulfill*

(a) $v_x(s) \geq 0$, and

(b) $v_x(s(0)) < q^N$.

Then, for every $a \in F_{K,\mathfrak{p}}$ we have an inequality

$$v_x\left(\sigma^N(a)\right) \geq \left\lfloor \frac{v_x\left(s \cdot \sigma^N(a)\right)}{q^N} \right\rfloor \cdot q^N,$$

where for $x \in \mathbb{R}$ the term $\lfloor x \rfloor$ denotes the largest integer smaller than x.

Proof. We write $s = \sum_{i \geq 0} s_i t^i$ and $b := \sigma^N(a) = \sum_i b_i t^i$ with coefficients $s_i \in k_{\mathfrak{p}} \otimes_k K^{\mathrm{sep}}$ and $b_i \in k_{\mathfrak{p}} \otimes_k K$. We may assume that $b_i = 0$ for $i < 0$. By assumption, $v_x(s_i) \geq 0$ for all i, and $v_x(s_0) < q^N$. Note that since s_0 is invertible, the inequality $v_x(s_0 \cdot b_i) \geq v_x(s_0) + v_x(b_i)$ is in fact an equality!

We set $C := \lfloor v_x(sb)/q^N \rfloor \cdot q^N$, must prove that $v_x(b_i) \geq C$ for all i, and do this by induction on i.

For $i = 0$, we consider the inequality $v_x(s_0) + v_x(b_0) = v_x(s_0 b_0) \geq C$. It implies that, $v_x(b_0) \geq C - v_x(s_0) > C - q^N$. However, by assumption the value of $v_x(b_0)$ lies in $q^N \cdot \mathbb{Z} \cup \{\infty\}$, and there exists no integral multiple of q^N strictly greater than $C - q^N$ and less than C. Therefore, we have $v_x(b_0) \geq C$.

For $i > 0$, we have $s_0 b_i = (sb)_i - \sum_{j=1}^{i} s_j b_{i-j}$. By induction, we deduce that

$$\begin{aligned} v_x(s_0 b_i) &= v_x\left((sb)_i - \sum_{j=1}^{i} s_j b_{i-j}\right) \\ &\geq \min\left(v_x\left((sb)_i\right), \min_{1\leq j\leq i}\left(v_x(s_j) + v_x(b_{i-j})\right)\right) \\ &\geq \min(C, \min(0 + C)) \geq C \end{aligned}$$

So $v_x(b_i) \geq C - v_x(s_0)$, which implies that $v_x(b_i) \geq C$ as in the case $i = 0$ since $v_x(b_i)$ is an integral multiple of q^N and $0 \leq v_x(s_0) < q^N$. ∴

Lemma 18.18. *There exists an $N_0 \geq 1$ such that for all $N \geq N_0$ and all $x \in \Sigma^{\mathrm{sep}}$ we have $v_x(a_N) \neq -\infty$.*

Proof. By Lemma 18.18, there exists a finite set $\Sigma_0 \subset \Sigma_K$ such that $v_x(a_N) \geq 0 > -\infty$ for all x not lying above Σ_0. Hence it suffices to prove that, for one given $x_K \in \Sigma_K$, there exists an integer $N_0 \geq 1$ such that for all $N \geq N_0$ and all x lying above x_K we have $v_x(a_N) \neq -\infty$. We fix such an $x_K \in \Sigma_0$.

Let π denote a local parameter of K at x_K. For all x over x_K, we have $v_x(s) \geq v_x(f)/(q^d - 1) > -\infty$ by Proposition 18.4, so that $s = \pi^{-n}\widetilde{s}$ for some $n \geq 0$ and $\widetilde{s} \in S$ satisfying $v_x(s) \geq 0$. As a first substep, we wish to show that it is sufficient to deal with the case $s = \widetilde{s}$. This will make our calculations easier!

If $n > 0$, then

$$\widetilde{f} := \frac{\sigma^d(\widetilde{s})}{\widetilde{s}} = \frac{\sigma^d(\pi^n)}{\pi^n} \cdot \frac{\sigma(s)}{s} = \pi^{n(q^d-1)} f \in F_{\mathfrak{p}} \otimes_k K,$$

and by setting $\widetilde{b^+} := \pi^n b^+ \in B^+$, we obtain $b = \widetilde{b^+}/\widetilde{s}$, so that

$$\begin{aligned} \widetilde{a_N} &:= b \cdot \widetilde{f}(t^{q^d}) \cdots \widetilde{f}(t^{q^{Nd}}) \\ &= b \cdot \pi^{n(q^d-1)} f(t^{q^d}) \cdots \pi^{n(q^d-1)} f(t^{q^{Nd}}) \\ &= \pi^{Nn(q^d-1)} a_N. \end{aligned}$$

In particular, $v_x(a_N) \neq -\infty$ if and only if $v_x(\widetilde{a_N}) \neq -\infty$, and we may assume in the following without loss of generality that the $s \in \mathscr{O}^{\times}_{K^{\mathrm{sep}},\mathfrak{p}}$ we are given fulfills $v_x(s) \geq 0$.

We remark that for all $g \in F_{K^{\mathrm{sep}},\mathfrak{p}}$ and $i \geq 0$ we have the formula

$$\sigma^{id}(g(t^{q^{id}})) = g^{q^{id}}, \tag{18.19}$$

in particular for our given $f \in F_{\mathfrak{p}} \otimes_k K$.

Secondly, note that from $b^+ = bs$ and $\sigma^d(s) = sf$ we obtain $\sigma^d(b^+) = \sigma^d(b)\sigma^d(s) = \sigma^d(b)sf$, and by induction for $N \geq 1$

$$\sigma^{Nd}(b^+) = \sigma^{Nd}(b)s \cdot \left(f \cdot \sigma^d(f) \cdots \sigma^{(N-1)d}(f)\right). \tag{18.20}$$

Hence,

$$\begin{aligned}
\sigma^{Nd}(a_N)s &= \sigma^{Nd}\left(b \cdot f(t^{q^d}) \cdots f(t^{q^{Nd}})\right) \cdot s \\
&= \sigma^N(b)s \cdot \sigma^{Nd}\left(f(t^{q^d}) \cdots f(t^{q^{Nd}})\right) \\
&= \sigma^N(b^+) \cdot \frac{\sigma^{Nd}\left(f(t^{q^d}) \cdots f(t^{q^{Nd}})\right)}{\sigma^{(N-1)d}(f) \cdots f} \quad \text{by Equation (18.20)} \\
&= \sigma^N(b^+) \cdot \prod_{i=1}^{N} \sigma^{(N-i)d}\left(\frac{\sigma^{id}(f(t^{q^{id}}))}{f}\right) \\
&= \sigma^N(b^+) \cdot \prod_{i=1}^{N} \sigma^{(N-i)d}\left(f^{q^{id}-1}\right) \quad \text{by Equation (18.19)} \\
&=: \sigma^N(b^+) \cdot \phi,
\end{aligned}$$

with $\phi \in F_{\mathfrak{p}} \otimes_k K$, so it follows that $v_x(\sigma^N(a_N)s) \geq q^N v_x(b^+) + v_x(\phi) \neq -\infty$.

Now if N is large enough, namely, $q^N > v_x(s(0))$, then Lemma 18.17 shows that $q^N v_x(a_N) = v_x(\sigma^N(a_N)) \neq -\infty$, so $v_x(a_N) \neq -\infty$ as required. $\therefore$

Proposition 18.21. *The ring B fulfills $B^{\Gamma_K} = F_{\mathfrak{p},K}$.*

Proof. By Proposition 18.14 it suffices to show that $B^{\Gamma_K} \subset F_{\mathfrak{p},K}$. For $b \in B^{\Gamma_K}$ and $N \geq 0$, define a_N as before Remark 18.15. Lemmas 18.16 and 18.18 show that for N large enough, a_N is an element of B^+. By construction, it is an Γ_K-invariant, so Lemma 18.7 shows that $a_N \in F_{\mathfrak{p}} \otimes_k K$. By definition, this shows that

$$b = \frac{a_N}{f\left(t^{q^d}\right) \cdot f\left(t^{q^{2d}}\right) \cdots f\left(t^{q^{Nd}}\right)}$$

is an element of $F_{\mathfrak{p},K}$, since both a_N and the denominator lie in $F_{\mathfrak{p}} \otimes_k K \subset F_{\mathfrak{p},K}$. $\therefore$

So far, we have shown that $\boldsymbol{B}$ is a well-defined Γ_K-stable bold ring with scalar ring $F_{\mathfrak{p}}$ and $B^{\Gamma_K} = F_{K,\mathfrak{p}}$. It remains to prove that B has property (b) of Claim 17.1.

Lemma 18.22. *Let $\boldsymbol{M}$ be a etale $\boldsymbol{F_{\mathfrak{p},K}}$-module. Then* $\mathrm{V}_{\mathfrak{p}}(\boldsymbol{M}) \subset \boldsymbol{B} \otimes_{\boldsymbol{F_{\mathfrak{p},K}}} \boldsymbol{M}$.

Proof. We may assume, by choosing a basis, that $\boldsymbol{M} = (F_{\mathfrak{p},K}^{\oplus n}, \tau)$ with $\tau(m) = \Delta\sigma(m)$ for some matrix $\Delta \in \mathrm{GL}_n(F_{\mathfrak{p},K})$ and all m.

Since $\mathrm{V}_{\mathfrak{p}}(\boldsymbol{M}) = (\boldsymbol{F}_{\boldsymbol{K}^{\mathrm{sep}},\mathfrak{p}} \otimes \boldsymbol{M})^{\tau}$, we have to prove that for all $m \in F_{K^{\mathrm{sep}},\mathfrak{p}}^{\oplus n}$ the equation $\Delta \cdot \sigma(m) = m$ implies that all entries of m lie in B.

Let us denote the inverse of Δ by $\Delta^{-1} = (g_{ij}/f_{ij})_{i,j}$, with $g_{ij} \in F_{\mathfrak{p}} \otimes_k K$ and $f_{ij} \in (F_{\mathfrak{p}} \otimes_k K) \cap \mathscr{O}_{K^{\mathrm{sep}},\mathfrak{p}}^{\times}$. Setting $f := \prod_{i,j} f_{ij}$, we see that $\Delta^{-1} = \frac{1}{f}\Delta'$ for some matrix Δ' with entries in $F_{\mathfrak{p}} \otimes_k K \subset B^{+}$.

By Lemma 18.13, we may write $f = \sigma(s)/s$ for some $s \in S$. For any element $m \in M$ write $m' := sm$. Now the equation $\tau(m) = m$ is equivalent to the equation $\sigma(m') = \Delta' \cdot m'$. By Proposition 18.4, this implies that m' has entries in B^{+}, so in particular $m = m'/s$ has entries in B, as claimed. $\therefore$

Chapter VI

Main Results – in down to earth terms

In this chapter, we assume that K is *finitely generated* over its prime field.

Let $\mathfrak{p}$ be a place of F. The letters $\boldsymbol{M}$, $\boldsymbol{N}$, ... denote restricted $\boldsymbol{F_K}$-modules which are etale at $\mathfrak{p}$, meaning that they are $\mathscr{O}_{(\mathfrak{p}),K}$-etale (cf. Definition 5.29 and Examples 6.1(b,c)).

The main examples are of course given by A-motives over K of characteristic unequal to $\mathfrak{p}$. Using the relation between the Tate modules of abelian A-modules over K and the Tate modules of the corresponding A-motives over K one obtains further versions of Theorems 19.1 and 20.1 for abelian A-modules and their Tate modules.

19 The Tate Conjecture

Theorem 19.1. *Let $\boldsymbol{M}$, $\boldsymbol{N}$ be two restricted $\boldsymbol{F_K}$-modules which are etale at $\mathfrak{p}$. Then the natural homomorphism*

$$F_{\mathfrak{p}} \otimes_F \mathrm{Hom}(\boldsymbol{M}, \boldsymbol{N}) \longrightarrow \mathrm{Hom}(\mathrm{V}_{\mathfrak{p}}(\boldsymbol{M}), \mathrm{V}_{\mathfrak{p}}(\boldsymbol{N}))$$

is an isomorphism.

Proof. Combine Proposition 15.2 and Theorem 17.7. ∴

Proposition 19.2. *Let $\boldsymbol{M}$, $\boldsymbol{N}$ be two A-motives over K of characteristic unequal to $\mathfrak{p}$. Then the following two natural homomorphisms are isomorphisms:*

(a) $A_{\mathfrak{p}} \otimes_A \mathrm{Hom}(\boldsymbol{M}, \boldsymbol{N}) \longrightarrow \mathrm{Hom}(\mathrm{T}_{\mathfrak{p}}\, \boldsymbol{M}, \mathrm{T}_{\mathfrak{p}}\, \boldsymbol{N})$

(b) $F_{\mathfrak{p}} \otimes_A \mathrm{Hom}(\boldsymbol{M}, \boldsymbol{N}) \longrightarrow \mathrm{Hom}(\mathrm{V}_{\mathfrak{p}}\, \boldsymbol{M}, \mathrm{V}_{\mathfrak{p}}\, \boldsymbol{N})$

Proof. (b): Combine Theorem 12.3 and Proposition 12.2 with Theorem 19.1.

(a): It is well-known and easy to see that the given homomorphism is injective and has saturated image. Therefore this item follows from item (b). ∴

20 The Semisimplicity Conjecture

Theorem 20.1. *Let $\boldsymbol{M}$ be a restricted $\boldsymbol{F_K}$-module which is etale at $\mathfrak{p}$. Then*

$$\mathrm{V}_{\mathfrak{p}}(\mathrm{soc}\,\boldsymbol{M}) = \mathrm{soc}(\mathrm{V}_{\mathfrak{p}}\,\boldsymbol{M}).$$

In particular, $\boldsymbol{M}$ is semisimple if and only if $\mathrm{V}_{\mathfrak{p}}(\boldsymbol{M})$ is semisimple.

Proof. By Theorem 16.4 and the separability of $F_{\mathfrak{p}}$ over F (Proposition 16.3) we see that $\boldsymbol{F_{\mathfrak{p},K}} \otimes_{\boldsymbol{F_K}} \mathrm{soc}(\boldsymbol{M}) = \mathrm{soc}(\boldsymbol{F_{\mathfrak{p},K}} \otimes_{\boldsymbol{F_K}} \boldsymbol{M})$, even with out the assumption that $\boldsymbol{M}$ is etale at $\mathfrak{p}$. However, this assumption shows that $\boldsymbol{F_{\mathfrak{p},K}} \otimes_{\boldsymbol{F_K}} \boldsymbol{M}$ is again etale at $\mathfrak{p}$. Hence we may apply the main Theorem 17.18(a) of Tamagawa-Fontaine theory. Together with Theorem 3.4, it shows that $\mathrm{soc}\,\mathrm{V}_{\mathfrak{p}}(\boldsymbol{F_{\mathfrak{p},K}} \otimes_{\boldsymbol{F_K}} \boldsymbol{M}) = \mathrm{V}_{\mathfrak{p}}\,\mathrm{soc}(\boldsymbol{F_{\mathfrak{p},K}} \otimes_{\boldsymbol{F_K}} \boldsymbol{M})$.

All in all, we see that $\mathrm{V}_{\mathfrak{p}}(\mathrm{soc}\,\boldsymbol{M}) = \mathrm{soc}(\mathrm{V}_{\mathfrak{p}}\,\boldsymbol{M})$, as required. ∴

21 A Tate conjecture for subobjects

Theorem 21.1 (Pink)**.** *Let $\boldsymbol{M}$ be a restricted $\boldsymbol{F_K}$-module which is etale at $\mathfrak{p}$. There exists a restricted $\boldsymbol{F_K}$-module $\boldsymbol{N}$ in $(\!(\boldsymbol{M})\!)_{\otimes}$ (necessarily etale at $\mathfrak{p}$) such that for all subrepresentations $\boldsymbol{V} \subset \mathrm{V}_{\mathfrak{p}}(\boldsymbol{M})$:*

(a) $\exists\, \phi \in \mathrm{Hom}(\boldsymbol{M}, \boldsymbol{N}) \otimes_F F_{\mathfrak{p}}$ *such that* $\boldsymbol{V} = \ker(\phi)$.

(b) $\exists\, \psi \in \mathrm{Hom}(\boldsymbol{N}, \boldsymbol{M}) \otimes_F F_{\mathfrak{p}}$ *such that* $\boldsymbol{V} = \mathrm{im}(\psi)$.

Proof. We shall repeatedly and without explicit mention use Theorems 19.1 and 20.1, the Tate and Semisimplicity conjectures. If $\boldsymbol{M}$ is semisimple, then so is $\mathrm{V}_{\mathfrak{p}}(\boldsymbol{M})$. Setting $\boldsymbol{N} := \boldsymbol{M}$ we see that there exist ϕ and ψ as required.

Let us prove that for every $\boldsymbol{M}$ and any *semisimple* subrepresentation $\boldsymbol{V}$ of $\mathrm{V}_{\mathfrak{p}}(\boldsymbol{M})$ the semisimplified module $\boldsymbol{N} := \boldsymbol{M}^{\mathrm{ss}}$ allows a homomorphism ψ as in item (b): There exists a projection $\mathrm{V}_{\mathfrak{p}}(\boldsymbol{N}) = \mathrm{V}_{\mathfrak{p}}(\boldsymbol{M})^{\mathrm{ss}} \longrightarrow \boldsymbol{V}$ whose composition with the inclusion $\boldsymbol{V} \subset \mathrm{V}_{\mathfrak{p}}(\boldsymbol{M})$ gives a homomorphism $\psi \in \mathrm{Hom}(\boldsymbol{N}, \boldsymbol{M}) \otimes_F F_{\mathfrak{p}}$ with $\boldsymbol{V} = \mathrm{im}(\psi)$.

Next we prove item (a). Since $\mathrm{rk}(\boldsymbol{M})$ is finite, we may make the additional assumption that $\mathrm{rk}(\boldsymbol{V}) = s$ for some fixed number $s \leq \mathrm{rk}\,\boldsymbol{M}$, because if $\boldsymbol{N}_s$ does what is required for all $\boldsymbol{V}$ of rank s, then $\boldsymbol{N} := \bigoplus_{s=1}^{\mathrm{rk}(\boldsymbol{M})} \boldsymbol{N}_s$ does what is required for all subrepresentations $\boldsymbol{V}$ of $\mathrm{V}_{\mathfrak{p}}(\boldsymbol{M})$.

Consider the homomorphism of rational $\mathfrak{p}$-adic Galois representations

$$\mathrm{V}_{\mathfrak{p}}(\boldsymbol{M}) \longrightarrow \mathbf{Hom}\left(\bigwedge^{s} \boldsymbol{V}, \bigwedge^{s+1} \mathrm{V}_{\mathfrak{p}}(\boldsymbol{M})\right)$$

mapping $v \in \mathrm{V}_{\mathfrak{p}} \boldsymbol{M}$ to the homomorphism mapping $x \in \bigwedge^s \boldsymbol{V} \subset \bigwedge^s \mathrm{V}_{\mathfrak{p}}(\boldsymbol{M})$ to $v \wedge x \in \bigwedge^{s+1} \mathrm{V}_{\mathfrak{p}}(\boldsymbol{M})$. The kernel of this homomorphism is $\boldsymbol{V}$.

Since $\bigwedge^s \boldsymbol{V}$ has rank 1 it is simple, so by the preparatory considerations, setting $\boldsymbol{N}_0 := (\bigwedge^s \boldsymbol{M})^{\mathrm{ss}}$, there exists a surjective homomorphism $\mathrm{V}_{\mathfrak{p}}(\boldsymbol{N}_0) \longrightarrow \bigwedge^s \boldsymbol{V}$. Therefore we still have $\boldsymbol{V} = \ker(\mathrm{V}_{\mathfrak{p}} \boldsymbol{M} \longrightarrow \mathbf{Hom}(\mathrm{V}_{\mathfrak{p}} \boldsymbol{N}_0, \bigwedge^{s+1} \mathrm{V}_{\mathfrak{p}} \boldsymbol{M}))$.

Set $\boldsymbol{N} := \boldsymbol{N}_0^{\vee} \otimes_{\boldsymbol{F_K}} \bigwedge^{s+1} \boldsymbol{M}$. Since $F_{\mathfrak{p}} \otimes_F \mathrm{Hom}(\boldsymbol{M}, \boldsymbol{N}) \cong \mathrm{Hom}(\mathrm{V}_{\mathfrak{p}} \boldsymbol{M}, \mathrm{V}_{\mathfrak{p}} \boldsymbol{N})$, we see that we have found a homomorphism $\phi \in \mathrm{Hom}(\boldsymbol{M}, \boldsymbol{N}) \otimes_F F_{\mathfrak{p}}$ such that $\boldsymbol{V} = \ker(\phi)$.

Finally we prove item (b). Instead of a direct construction, we reduce it to item (a). Set $\boldsymbol{V}' := (\mathrm{V}_{\mathfrak{p}}(\boldsymbol{M})/\boldsymbol{V})^{\vee}$. This is a subrepresentation of $\mathrm{V}_{\mathfrak{p}}(\boldsymbol{M}^{\vee})$. Thus by (a) applied to $\boldsymbol{M}^{\vee}$ we have a module $\boldsymbol{N}'$ and a homomorphism $\phi' : \mathrm{V}_{\mathfrak{p}}(\boldsymbol{M}^{\vee}) \longrightarrow \mathrm{V}_{\mathfrak{p}}(\boldsymbol{N})$ with kernel $\boldsymbol{V}'$, and therefore image $\boldsymbol{V}'^{\vee}$. Setting $\boldsymbol{N} := \boldsymbol{N}'^{,\vee}$ we see that $\psi := \phi'^{,\vee}$ is a homomorphism in $\mathrm{Hom}(\boldsymbol{N}, \boldsymbol{M}) \otimes_F F_{\mathfrak{p}}$ with image $\boldsymbol{V}$, as required. $\therefore$

Remark 21.2. One may view Theorem 21.1(b) as a "generalisation" of Theorem 19.1 (the Tate conjecture): Consider two restricted $\boldsymbol{F_{\mathfrak{p},K}}$-modules $\boldsymbol{M}_1, \boldsymbol{M}_2$ which are etale at $\mathfrak{p}$, and set $\boldsymbol{M} := \boldsymbol{M}_1^{\vee} \otimes \boldsymbol{M}_2$. Then $\boldsymbol{V} := \mathrm{Hom}_{\Gamma_K}(\mathrm{V}_{\mathfrak{p}} \boldsymbol{M}_1, \mathrm{V}_{\mathfrak{p}} \boldsymbol{M}_2) = (\mathrm{V}_{\mathfrak{p}} \boldsymbol{M})^{\Gamma_K}$ is the largest subrepresentation (i.e. Γ_K-stable sub-vector space) of $\mathrm{V}_{\mathfrak{p}} \boldsymbol{M}$ which is actually point-wise Γ_K-stable. Now the Tate conjecture for homomorphisms states that every element of $\boldsymbol{V}$ is the image of an $F_{\mathfrak{p}}$-linear combination of elements of $\mathrm{Hom}_{\boldsymbol{F_{\mathfrak{p},K}}}(\boldsymbol{M}_1, \boldsymbol{M}_2) = \boldsymbol{M}^{\tau}$.

On the other hand, The Tate conjecture for subobjects deals with all Γ_K-stable sub-vector spaces $\boldsymbol{W}$, even if they are not point-wise Γ_K-stable. Now we can no longer expect that such a subrepresentation is the image of an $F_{\mathfrak{p}}$-linear combination of elements of $\boldsymbol{M}^{\tau}$, but at least we find an $\mathscr{O}_{\boldsymbol{\mathfrak{p},K}}$-etale module $\boldsymbol{N}$ such that $\boldsymbol{W}$ is the image of an $F_{\mathfrak{p}}$-linear combination of elements of $(\boldsymbol{N}^{\vee} \otimes \boldsymbol{M})^{\tau}$.

Chapter VII

Scalar extension of F-finite abelian categories

22 Endomorphisms and semisimplification

Let $\mathscr{A}$ be a finite abelian category. In general, the functor $(-)^{\mathrm{ss}}$ mapping objects of $\mathscr{A}$ to their semisimplifications (cf. Definition 2.3) is *not* faithful, but we have the following:

Proposition 22.1. *For every object X, one has* $\dim_F \mathrm{End}(X) \leq \dim_F \mathrm{End}(X^{\mathrm{ss}})$.

For the proof, we use the following:

Lemma 22.2. *For every object X, and every semisimple object S, there exists an F-linear injection*

$$J_X : \mathrm{Hom}_{\mathscr{A}}(X, S) \hookrightarrow \mathrm{Hom}_{\mathscr{A}}(X^{\mathrm{ss}}, S).$$

Proof. Every homomorphism $X \to S$ factors through $X/\operatorname{rad}(X)$, since this is the largest semisimple quotient of object X. Therefore, $\mathrm{Hom}_{\mathscr{A}}(X/\operatorname{rad}(X), S) \cong \mathrm{Hom}_{\mathscr{A}}(X, S)$.

On the other hand, $\mathrm{Hom}_{\mathscr{A}}(X/\operatorname{rad}(X), S)$ embeds into $\mathrm{Hom}_{\mathscr{A}}(X^{\mathrm{ss}}, S)$, since there exists a projection $X^{\mathrm{ss}} \to X/\operatorname{rad}(X)$.

All in all, there exists an F-linear injection $\mathrm{Hom}_{\mathscr{A}}(X, S) \hookrightarrow \mathrm{Hom}_{\mathscr{A}}(X^{\mathrm{ss}}, S)$ as stated. ∴

Proof of Proposition 22.1. We will construct (non-functorial!) F-linear injective homomorphisms

$$I_X : \mathrm{End}_{\mathscr{A}}(X) \hookrightarrow \mathrm{End}_{\mathscr{A}}(X^{\mathrm{ss}})$$

by induction on $s := \operatorname{slg}(X)$. For $s = 0, 1$, we have $X = X^{\mathrm{ss}}$, so the proposition is trivial. For $s \geq 2$, we let K be the kernel of the homorphism $\mathrm{End}(X) \to$

$\operatorname{End}(\operatorname{soc} X) \oplus \operatorname{End}(X/\operatorname{soc} X)$ mapping $f \in \operatorname{End}(X)$ to the direct sum of its restriction to $\operatorname{soc}(X)$, and its equivalence class in $\operatorname{End}(X/\operatorname{soc} X)$. We choose a retraction ϕ_X of the inclusion $K \subset \operatorname{End}(X)$. For $f \in K$, we consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{soc}(X) & \longrightarrow & X & \longrightarrow & X/\operatorname{soc} X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0=f|_{\operatorname{soc} X}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle [f]_{\operatorname{soc} X}=0} & & \\ 0 & \longrightarrow & \operatorname{soc}(X) & \longrightarrow & X & \longrightarrow & X/\operatorname{soc}(X) & \longrightarrow & 0. \end{array}$$

The Snake Lemma provides us with homomorphism $\partial f : X/\operatorname{soc} X \to \operatorname{soc} X$ making the following sequence long exact:

$$0 \to \operatorname{soc} X \to \ker f \to X/\operatorname{soc} X \xrightarrow{\partial f} \operatorname{soc}(X) \to \operatorname{coker} f \to X/\operatorname{soc} X \to 0.$$

We see that $f = 0$ if and only if $\partial f = 0$. By the naturality of the connecting homorphism, we have obtained an F-linear injection

$$\partial : K \hookrightarrow \operatorname{Hom}_{\mathscr{A}}(X/\operatorname{soc} X, \operatorname{soc} X).$$

We now apply Lemma 22.2 to the pair $(X/\operatorname{soc} X, \operatorname{soc} X)$, and obtain an F-linear injection $J_{X/\operatorname{soc} X} : \operatorname{Hom}(X/\operatorname{soc} X, \operatorname{soc} X) \to \operatorname{Hom}((X/\operatorname{soc} X)^{\mathrm{ss}}, \operatorname{soc} X)$. Now the assignment

$$I_X(f) := \Big(f|_{\operatorname{soc} X}, J_{X/\operatorname{soc} X}(\partial(\phi_X(f))), I_{X/\operatorname{soc} X}([f]_{X/\operatorname{soc} X}) \Big)$$

gives our desired F-linear injection

$$\begin{aligned} \operatorname{End}(X) &\to \operatorname{End}(\operatorname{soc} X) \oplus \operatorname{Hom}((X/\operatorname{soc} X)^{\mathrm{ss}}, \operatorname{soc} X) \oplus \operatorname{End}((X/\operatorname{soc} X)^{\mathrm{ss}}) \\ &\subset \operatorname{End}(\operatorname{soc} X \oplus (X/\operatorname{soc} X)^{\mathrm{ss}}) = \operatorname{End}(X^{\mathrm{ss}}). \end{aligned}$$

∴

23 Scalar extension – definition and first properties

Let F be a field, and consider an F-linear abelian category $\mathscr{A}$.

Recall that we bypass set-theoretical difficulties by assuming the logical axiom of existence of universes, which is independent of (ZFC).

Definition 23.1. The category $\operatorname{ind} \mathscr{A}$ of *ind-objects* of $\mathscr{A}$ is the following. An object of $\operatorname{ind} \mathscr{A}$ is a filtered direct system $(X_i)_{i \in I}$ of objects of $\mathscr{A}$. Given two such objects $(X_i)_{i \in I}$ and $(Y_j)_{j \in J}$, we set

$$\operatorname{Hom}_{\operatorname{ind} \mathscr{A}}((X_i)_i, (Y_j)_j) := \varprojlim_i \varinjlim_j \operatorname{Hom}_{\mathscr{A}}(X_i, Y_j).$$

We have a natural functor $\mathscr{A} \to \operatorname{ind}\mathscr{A}$, mapping an object X of $\mathscr{A}$ to the object $(X_i)_{i\in I_\varnothing}$ given by $I_\varnothing := \{\varnothing\}$ and $X_\varnothing := X$.

Recall that $\mathscr{A}$ is called *cocomplete* if it contains all colimits, which is equivalent to requiring $\mathscr{A}$ to contain all direct sums.

Lemma 23.2. *(a)* $\operatorname{ind}\mathscr{A}$ *is a cocomplete F-linear abelian category.*

(b) The functor $\mathscr{A} \to \operatorname{ind}\mathscr{A}$ *is F-linear, exact, and fully faithful.*

(c) If $\mathscr{A}$ is Noetherian, then $\mathscr{A}$ is closed under subquotients in $\operatorname{ind}\mathscr{A}$*, and we may describe every object of* $\operatorname{ind}\mathscr{A}$ *as a union of objects in the essential image of* $\mathscr{A} \to \operatorname{ind}\mathscr{A}$.

Proof. [Del87, §4.1 and Lemme 4.2.1]. ∴

Let F'/F be a field extension.

Definition 23.3. An *F'-module in $\mathscr{A}$* is a pair $\boldsymbol{X} = (X, \phi)$ consisting of an object X of $\mathscr{A}$, and an F-linear ring homomorphism $\phi : F' \to \operatorname{End}_{\mathscr{A}}(X)$. Given two F'-modules $\boldsymbol{X}$ and $\boldsymbol{Y}$ in $\mathscr{A}$, we let $\operatorname{Hom}_{\mathscr{A}_{F'}}(\boldsymbol{X}, \boldsymbol{Y})$ be the subset of $\operatorname{Hom}_{\mathscr{A}}(X, Y)$ consisting of those homomorphisms that commute with the respective actions of F'. In this way, we obtain the F'-linear abelian category $\mathscr{A}_{F'}$ of F'-modules in $\mathscr{A}$.

Note that $\mathscr{A}_{F'}$ may consist only of trivial F'-modules, for instance if $\mathscr{A}$ is F-finite and $[F' : F]$ is infinite.

Definition 23.4. Consider an element $X \in \operatorname{ind}\mathscr{A}$, and let $E \subset \operatorname{End}_{\operatorname{ind}\mathscr{A}}(X)$ be a subring. For a free right E-module M, the *external tensor product* $M \otimes_E X$ is defined (abusing language slightly) to be "the" object representing the functor $Y \mapsto \operatorname{Hom}_E(M, \operatorname{Hom}_{\operatorname{ind}\mathscr{A}}(X, Y))$ on $\operatorname{ind}\mathscr{A}$, i.e., equipped with a natural isomorphism

$$\operatorname{Hom}_E(M, \operatorname{Hom}_{\operatorname{ind}\mathscr{A}}(X, Y)) \xrightarrow{\cong} \operatorname{Hom}_{\operatorname{ind}\mathscr{A}}(M \otimes_E X, Y).$$

It may be identified with a direct sum of $\operatorname{rk}_E(M)$ copies of X.

The external tensor product is an exact F-linear functor in its first variable if we fix X and E, and in its second variable if we let $E = F$ and fix M.

Remark 23.5. In the situation of Definition 23.4, if M is a free right E-module *of finite rank,* then $M \otimes_E X$ has a second universal property, namely it represents the functor $Z \mapsto V \otimes_E \operatorname{Hom}_{\operatorname{ind}\mathscr{A}}(X, Z)$ on $\operatorname{ind}\mathscr{A}$, so one has a natural isomorphism

$$M \otimes_E \operatorname{Hom}_{\operatorname{ind}\mathscr{A}}(Z, X) \xrightarrow{\cong} \operatorname{Hom}_{\operatorname{ind}\mathscr{A}}(Z, M \otimes_E X).$$

For every object $X \in \operatorname{ind}\mathscr{A}$, we may consider $F' \otimes_F X$ as an F'-module in $\operatorname{ind}\mathscr{A}$ by using the natural action of F' on itself by multiplication μ. In this way, we obtain an exact F-linear functor

$$t : \operatorname{ind}\mathscr{A} \longrightarrow (\operatorname{ind}\mathscr{A})_{F'}, \quad X \mapsto (F' \otimes_F X, \mu \otimes \mathrm{id}). \tag{23.6}$$

Lemma 23.7. *For every X in $\operatorname{ind}\mathscr{A}$ and $\boldsymbol{Y} = (Y, \psi)$ in $(\operatorname{ind}\mathscr{A})_{F'}$, the following natural homomorphism is an isomorphism:*

$$\operatorname{Hom}_{\operatorname{ind}\mathscr{A}}(X, Y) \longrightarrow \operatorname{Hom}_{(\operatorname{ind}\mathscr{A})_{F'}}(F' \otimes_F X, \boldsymbol{Y}).$$

In other words, the functor t of (23.6) is left adjoint to the forgetful functor from F'-modules in $\operatorname{ind}\mathscr{A}$ to $\operatorname{ind}\mathscr{A}$.

Proof. We start by making explicit the natural homomorphism in the statement of this lemma. An element $h \in \operatorname{Hom}(X, Y)$ is mapped to the unique homomorphism $e(h) \in \operatorname{Hom}_{(\operatorname{ind}\mathscr{A}_{F'}}(F' \otimes_F X, \boldsymbol{Y})$ which corresponds via the injection

$$\operatorname{Hom}_{(\operatorname{ind}\mathscr{A})_{F'}}(F' \otimes_F X, \boldsymbol{Y}) \subset \operatorname{Hom}_{\operatorname{ind}\mathscr{A}}(F' \otimes_F X, Y) \cong \operatorname{Hom}_F(F', \operatorname{Hom}_{\operatorname{ind}\mathscr{A}}(X, Y))$$

to the homomorphism mapping $f' \in F'$ to the homomorphism

$$X \xrightarrow{h} Y \xrightarrow{\psi(f')} Y.$$

By construction, $e(h)$ is a homomorphism of F'-modules.

The inverse to e is given by mapping an element of $\operatorname{Hom}_{(\operatorname{ind}\mathscr{A})_{F'}}(F' \otimes_F X, \boldsymbol{Y})$ to its restriction to X via the injection $X \cong F \otimes_F X \subset F' \otimes_F X$. ∴

Lemma 23.8. *If $\mathscr{A}$ is finite, then $t : \mathscr{A} \to (\operatorname{ind}\mathscr{A})_{F'}$ is F'/F-fully faithful.*

Proof. We must show that for $\mathscr{A}$ finite and $X, Y \in \mathscr{A}$ the natural homomorphism

$$F' \otimes_F \operatorname{Hom}_{\operatorname{ind}\mathscr{A}}(X, Y) \longrightarrow \operatorname{Hom}_{(\operatorname{ind}\mathscr{A})_{F'}}(F' \otimes_F X, F' \otimes_F Y)$$

is an isomorphism. By Lemma 23.7, the target of this isomorphism coincides with $\operatorname{Hom}_{\operatorname{ind}\mathscr{A}}(X, F' \otimes_F Y)$, so we must show that the natural homomorphism

$$F' \otimes_F \operatorname{Hom}_{\operatorname{ind}\mathscr{A}}(X, Y) \longrightarrow \operatorname{Hom}_{\operatorname{ind}\mathscr{A}}(X, F' \otimes_F Y)$$

is an isomorphism.

Injectivity: Given an element $\widetilde{h} \in F' \otimes_F \operatorname{Hom}(X, Y)$, there exists a finite F-dimensional subspace $V \subset F'$ such that $\widetilde{h}$ arises from an element of $V \otimes_F$

$\operatorname{Hom}(X, Y)$. By Remark 23.5, we have a natural isomorphism $V \otimes_F \operatorname{Hom}(X, Y) \cong \operatorname{Hom}(X, V \otimes_F Y)$. Now the commutative diagram

$$\begin{array}{ccc} V \otimes_F \operatorname{Hom}(X, Y) & \longrightarrow & \operatorname{Hom}(X, V \otimes_F Y) \\ \downarrow & & \downarrow \\ F' \otimes_F \operatorname{Hom}(X, Y) & \longrightarrow & \operatorname{Hom}(X, F' \otimes_F Y) \end{array} \tag{23.9}$$

shows that $\widetilde{h}$ is indeed mapped to a non-zero element of $\operatorname{Hom}(X, F' \otimes_F Y)$.

Surjectivity: Consider an element h of $\operatorname{Hom}(X, F' \otimes_F Y)$. Since X is finite, the image $\operatorname{im}(h)$ of h is finite. The object $F' \otimes_F Y$ is the union over all finite F-dimensional subspaces $V \subset F'$ of its subobjects $V \otimes_F Y$. It follows that $\operatorname{im}(h) \subset V \otimes_F Y$ for some finite F-dimensional $V \subset F'$.

Therefore, h lies in $\operatorname{Hom}(X, V \otimes_F Y)$. By Remark 23.5, we have a natural isomorphism $V \otimes_F \operatorname{Hom}(X, Y) \cong \operatorname{Hom}(X, V \otimes_F Y)$, so h arises from an element of $V \otimes_F \operatorname{Hom}(X, Y) \subset F' \otimes_F \operatorname{Hom}(X, Y)$ as desired, since again the diagram (23.9) commutes. $\therefore$

Remark 23.10. If $\mathscr{A}$ is not finite, then t need not be F'/F-fully faithful. Here is a counter-example: Let F be a field, and $\mathscr{A}$ the category of all F-vector spaces. Consider $X := \bigoplus_{j \in \mathbb{N}} F$ and $Y := F$. Choose a field extension $F' \supset F$ such that F' is isomorphic, as F-vector space, to $\bigoplus_{i \in \mathbb{N}} F$. We claim that the homomorphism

$$F' \otimes_F \operatorname{Hom}(X, Y) \to \operatorname{Hom}(X, F' \otimes_F Y)$$

is not surjective. Indeed, we have $F' \otimes_F \operatorname{Hom}(X, Y) \cong \bigoplus_{i \in \mathbb{N}} \prod_{j \in \mathbb{N}} F$, whereas $\operatorname{Hom}(X, F' \otimes_F Y) \cong \prod_{j \in \mathbb{N}} \bigoplus_{i \in \mathbb{N}} F$. The latter strictly contains the former.

Definition 23.11. (a) An object $X_0 \in \mathscr{A}$ *generates* an F'-module $\boldsymbol{X}$ in $\operatorname{ind} \mathscr{A}$ if there exists an epimorphism $F' \otimes_F X_0 \to \boldsymbol{X}$ of F'-modules.

(b) If $\mathscr{A}$ is F-finite, the *scalar extension* of $\mathscr{A}$ from F to F' is the full subcategory $\mathscr{A} \otimes_F F'$ of $(\operatorname{ind} \mathscr{A})_{F'}$ consisting of those F'-modules $\boldsymbol{X}$ in $\operatorname{ind} \mathscr{A}$ generated by objects of $\mathscr{A}$.

It is clear that $\mathscr{A} \otimes_F F'$ is an F'-linear additive category, and that the functor $\mathscr{A} \to (\operatorname{ind} \mathscr{A})_{F'}$ restricts to an exact F-linear functor

$$t : \mathscr{A} \to \mathscr{A} \otimes_F F', \quad X \mapsto F' \otimes_F X$$

which is F'/F-fully faithful by Lemma 23.8. But, whereas in $\mathscr{A} \otimes_F F'$ all cokernels exist by definition, the same is not true for kernels. Therefore, in general it is not clear whether $\mathscr{A} \otimes_F F'$ is abelian.

Lemma 23.12. *Let $\mathscr{A}$ be F-finite.*

(a) If $[F' : F]$ is finite, then $\mathscr{A} \otimes_F F' = \mathscr{A}_{F'}$ is an abelian category.

(b) Every object of $(\operatorname{ind}\mathscr{A})_{F'}$ is the union of subobjects lying in $\mathscr{A} \otimes_F F'$.

Proof. (a): This is clear from the definitions. We state it for clarification.

(b): We consider an object $\boldsymbol{X} = (X, \phi)$ of $(\operatorname{ind}\mathscr{A})_{F'}$. By Lemma 23.2(c), we may write $X = \bigcup_{i\in I} X_i$ for objects $X_i \in \mathscr{A}$. To prove our claim, it suffices to find objects $\boldsymbol{Y}_i$ of $\mathscr{A} \otimes_F F'$ such that $\boldsymbol{X} = \bigcup \boldsymbol{Y}_i$. We can achieve this as follows: We put

$$Y_i := \sum_{f'\in F'} \phi(f')(X_i),$$

this is an object of $\operatorname{ind}\mathscr{A}$. By definition of Y_i, the action ϕ of $\boldsymbol{X}$ maps Y_i into itself, so we have found objects $\boldsymbol{Y}_i := (Y_i, \phi\,|_{Y_i})$ of $(\operatorname{ind}\mathscr{A})_{F'}$ such that $\boldsymbol{X} = \bigcup \boldsymbol{Y}_i$.

It remains to show that each $\boldsymbol{Y}_i$ is an object of $\mathscr{A} \otimes_F F'$. However, the inclusion $X_i \subset Y_i$ induces an epimorphism $F' \otimes_F X_i \to Y_i$ by the very definition of Y_i, which shows that X_i generates $\boldsymbol{Y}_i$. We are done. ∴

Before we can study the question of whether or not $\mathscr{A} \otimes_F F'$ is abelian, we intersperse a discussion of the semisimplicity of $\mathscr{A} \to (\operatorname{ind}\mathscr{A})_{F'}$.

Definition 23.13. Given $X \in \operatorname{ind}\mathscr{A}$, an object $Y \in \operatorname{ind}\mathscr{A}$ is called *X-isotypic* if Y is isomorphic to a direct sum of copies of X.

Lemma 23.14. *For $X \in \operatorname{ind}\mathscr{A}$ and $E := \operatorname{End}_{\operatorname{ind}\mathscr{A}}(X)$, the functor $- \otimes_E X$ gives rise to an equivalence of categories between the category of free right E-modules and category of X-isotypic objects of $\operatorname{ind}\mathscr{A}$.*

Proof. For any index set I let $(-)^{(I)}$ denote the direct sum of I copies of $-$. We first show that $- \otimes_E X$ is well-defined. Since M is a free right E-module, $M \cong E^{(I)}$ for some index set I. Then

$$M \otimes_E X \cong E^{(I)} \otimes_E X \cong (E \otimes_E X)^{(I)} \cong X^{(I)},$$

so $M \otimes_E X$ is X-isotypic. We claim that $\operatorname{Hom}_{\operatorname{ind}\mathscr{A}}(X, -)$ is a quasi-inverse functor, and start by showing that this functor is well-defined: If $Y \cong X^{(I)}$ is X-isotypic, then

$$\operatorname{Hom}_{\operatorname{ind}\mathscr{A}}(X, Y) \cong \operatorname{Hom}_{\operatorname{ind}\mathscr{A}}(X, X^{(I)}) \cong \operatorname{Hom}_{\operatorname{ind}\mathscr{A}}(X, X)^{(I)} \cong E^{(I)}$$

is a free right E-module.

Similar calculations show that $\operatorname{Hom}_{\operatorname{ind}\mathscr{A}}(X, M \otimes_E X) \cong M$ if M is a free right E-module, and if Y is X-isotypic then $\operatorname{Hom}_{\operatorname{ind}\mathscr{A}}(X, Y) \otimes_E X \cong Y$.

Clearly both $- \otimes_E X$ and $\mathrm{Hom}_{\mathrm{ind}\,\mathscr{A}}(X, -)$ are additive functors. It remains to show that they are fully faithful. However, if $M \cong E^{(I)}$ and $N \cong E^{(J)}$ are two free right E-modules, then commutativity of the following natural diagram (which is easily checked) shows that $- \otimes_E X$ is fully faithful, and a similar argument shows that $\mathrm{Hom}_{\mathrm{ind}\,\mathscr{A}}(X, -)$ is fully faithful:

$$\begin{array}{ccc} \mathrm{Hom}_E(M,N) & \longrightarrow & \mathrm{Hom}_{\mathrm{ind}\,\mathscr{A}}(M \otimes_E V, N \otimes_E V) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{Mat}_{J\times I}(E) & \xrightarrow{\mathrm{id}} & \mathrm{Mat}_{J\times I}(E). \end{array}$$

∴

Proposition 23.15. *Let X be a simple object of $\mathscr{A}$, and set $E := \mathrm{End}_{\mathscr{A}}(X)$. Then the functor $- \otimes_E X$ gives rise to an inclusion preserving bijection between the set of right ideals of $F' \otimes_F E$ and the set of subobjects of $F' \otimes_F X$ in $(\mathrm{ind}\,\mathscr{A})_{F'}$.*

Proof. We set $E' := F' \otimes_F E$ and $X' := F' \otimes_F X$. Since X is simple, E is a skew field over F. Note that we may regard X' as an X-isotypic element of $\mathrm{ind}\,\mathscr{A}$, and that E' is a free right E'-module.

Consider the following diagram of lattices:

$$\begin{array}{ccc} \{\text{right } E\text{-submodules of } E'\} & \longleftrightarrow & \left\{\begin{array}{c} X\text{-isotypic} \\ \text{subobjects of } X' \end{array}\right\} \\ & & \\ \left\{\begin{array}{c} F'\text{-stable right} \\ E\text{-submodules of } E' \end{array}\right\} & \longleftrightarrow & \left\{\begin{array}{c} F'\text{-stable } X\text{-isotypic} \\ \text{subobjects of } X' \end{array}\right\} \\ \| & & \| \\ \{\text{right ideals of } E'\} & \longleftrightarrow & \{\text{subobjects of } X'\} \end{array}$$

The upper row is a bijection by Lemma 23.14 and it preserves inclusions by construction. The second row corresponds to the F'-stable objects in the upper row, using the operations of F' on E' and V', respectively. Since the bijection in the first row is functorial, it induces a bijection of the second row. Finally, we may clearly identify the objects of the second row with the objects of the third row. ∴

Definition 23.16. A semisimple F-algebra E is *separable* if for every simple F-algebra direct summand $E' \subset E$ the center of E' is a separable field extension of F (cf. Definition 16.1 for the general notion of separable field extensions).

Remark 23.17. This definition of separability for algebras is equivalent to various others, cf. [Bou81, VIII.§7.5, Définition 1 and Proposition 6, Corollaire].

Proposition 23.18. *Let $X \in \mathscr{A}$ be a semisimple object of finite length such that* $\dim_F \operatorname{End}_{\mathscr{A}}(X) < \infty$.

(a) $F' \otimes_F X$ *has finite length in* $(\operatorname{ind}\mathscr{A})_{F'}$.

(b) If F'/F *is a separable field extension, or* $\operatorname{End}_{\mathscr{A}}(X)$ *is a separable* F*-algebra, then* $F' \otimes_F X$ *is semisimple.*

Proof. We may assume that X is simple by applying the following proofs to each direct summand of X separately.

(a): Set $E := \operatorname{End}_{\mathscr{A}}(X)$. Since E is finite F-dimensional, $F' \otimes_F E$ is finite F'-dimensional, and the lattice of right ideals of $F' \otimes_F E$ has finite length. By Proposition 23.15, this implies that the lattice of subobjects of $F' \otimes_F X$ has finite length, so $F' \otimes_F X$ has finite length.

(b): Since X is semisimple of finite length, E is a finite-dimensional semisimple F-algebra. Now [Bou81, §7, no. 6, Corollaire 3] proves that this, together with either the separability of F'/F or E/F, implies that $F' \otimes_F E$ is a semisimple algebra. This implies that the radical of the lattice of right ideals of E', i.e., the intersection of its maximal subobjects, is zero. Therefore, again by Proposition 23.15, the radical of $F' \otimes_F X$ is zero. Since $F' \otimes_F X$ has finite length by (a), this shows that $F' \otimes_F X$ is semisimple. ∴

Theorem 23.19. *Assume that $\mathscr{A}$ is F-finite.*

(a) The objects of $\mathscr{A} \otimes_F F'$ *are precisely the* F'*-modules in* $\operatorname{ind}\mathscr{A}$ *of finite length.*

(b) $\mathscr{A} \otimes_F F'$ *is a finite abelian category.*

(c) If F'/F *is separable, then* $\mathscr{A}$ *is* F'*-finite and* $\mathscr{A} \to \mathscr{A} \otimes_F F'$ *is semisimple.*

Remark 23.20. If $\mathscr{A}$ is a finite F-linear abelian category, but not F-finite, then $\mathscr{A} \otimes_F F'$ may contain objects of infinite length. For instance, if F'/F is an infinite field extension, consider the category $\mathrm{Vec}_{F'}$ of finite-dimensional F'-vector spaces, with F'-linear homomorphisms. It is obviously F'-finite abelian, so it is a finite F-linear abelian category. The object $F' \otimes_F F'$ is an object of $(\mathrm{Vec}_{F'}) \otimes_F F'$ of infinite length, as may be verified using Proposition 23.15.

Remark 23.21. Following discussions with Richard Pink, I am convinced that with a little more effort, dealing with inseparability, one should be able to show that $\mathscr{A} \otimes_F F'$ is F'-finite for every F-finite abelian category $\mathscr{A}$.

Proof. (a,b): We first show that all objects of $\mathscr{A} \otimes_F F'$ have finite length. It is sufficient to show this for objects of the form $F' \otimes_F X$ with $X \in \mathscr{A}$, since every object of $\mathscr{A} \otimes_F F'$ is a quotient of such an object. We may also assume that X is simple, since X has finite length. Then Proposition 23.18(a) shows that $X \otimes_F F'$ has finite length.

Conversely (and here we paraphrase parts of [Del87, Lemme 4.5]), let $\boldsymbol{X}$ be a finite-length object of $(\operatorname{ind}\mathscr{A})_{F'}$. By Lemma 23.12(b), $\boldsymbol{X}$ is a union of subobjects $\boldsymbol{Y}$ lying in $\mathscr{A} \otimes_F F'$. Since $\boldsymbol{X}$ has finite length, it equals one of these subobjects, so we have $\boldsymbol{X} \in \mathscr{A} \otimes_F F'$.

Clearly, the full subcategory of (the abelian category) $(\operatorname{ind}\mathscr{A})_{F'}$ consisting of those objects having finite length is abelian, so $\mathscr{A} \otimes_F F'$ is a finite abelian category.

(c): The idea of the proof of F'-finiteness is the following: Given $\boldsymbol{X}, \boldsymbol{Y} \in \mathscr{A} \otimes_F F'$, choose $X_0 \in \mathscr{A}$ and an epimorpism $\pi : F' \otimes_F X_0 \twoheadrightarrow \boldsymbol{X}$. If we can find an object $Y_0 \in \mathscr{A}$ and a monomorphism $\iota : \boldsymbol{Y} \hookrightarrow F' \otimes_F Y_0'$, then the assignment $f \mapsto \iota \circ f \circ \pi$ gives rise to an F'-linear monomorphism

$$\operatorname{Hom}_{\mathscr{A}\otimes_F F'}(\boldsymbol{X}, \boldsymbol{Y}) \hookrightarrow \operatorname{Hom}_{\mathscr{A}\otimes_F F'}(F' \otimes_F X_0, F' \otimes_F Y_0) = F' \otimes_F \operatorname{Hom}_{\mathscr{A}}(X_0, Y_0),$$

which is a finite-dimensional F'-vectorspace since $\mathscr{A}$ is F-finite.

Assume that F'/F is separable. It is sufficient to show that $\operatorname{End}_{\mathscr{A}\otimes_F F'}(\boldsymbol{X})$ is finite F'-dimensional for every $\boldsymbol{X} \in \mathscr{A} \otimes_F F'$. By Proposition 22.1, we may assume that $\boldsymbol{X}$ is semisimple. Let an object $X_0 \in \mathscr{A}$ and an epimorphism $\pi : F' \otimes_F X_0 \twoheadrightarrow \boldsymbol{X}$ be chosen. Since $\boldsymbol{X}$ is semisimple, $\operatorname{rad}(F' \otimes_F X_0) \subset \ker(\pi)$, so π induces an epimorphism

$$\varpi : F' \otimes_F X_0 / \operatorname{rad}(F' \otimes_F X_0') \twoheadrightarrow \boldsymbol{X}.$$

Since F'/F is separable, by Proposition 23.18(b) the functor $\mathscr{A} \to \mathscr{A} \otimes_F F'$ is semisimple, so by Theorem 3.4(c) we have $\operatorname{rad}(F' \otimes_F X_0) = F' \otimes_F \operatorname{rad}(X_0)$. So $\boldsymbol{X}$ is a quotient of $F' \otimes_F (X_0 / \operatorname{rad} X_0)$, a semisimple object of $\mathscr{A} \otimes_F F'$ since $X_0/\operatorname{rad}(X_0)$ is semisimple and $\mathscr{A} \to \mathscr{A} \otimes_F F$ is semisimple. Hence ϖ splits, we can choose an embedding $\iota : \boldsymbol{X} \hookrightarrow (X_0/\operatorname{rad} X_0)$, and may follow the method of proof given above. ∴

Example 23.22. (a) If F'/F is any field extension, and Vec_F is the category of finite-dimensional F-vector spaces, then $\mathrm{Vec}_F \otimes_F F'$ is the category $\mathrm{Vec}_{F'}$ of finite-dimensional F'-vector spaces.

(b) If G is an affine group scheme over F, and $\mathrm{Rep}_F\, G$ is the category of finite-dimensional representations of G over F, then $(\mathrm{Rep}_F\, G) \otimes_F F'$ is the category $\mathrm{Rep}_{F'}(G_{F'})$ of finite-dimensional representations of $G_{F'}$ over F'. This follows, for example, from [Wat79, Theorem 3.5].

24 Universal property of scalar extension

Let F'/F be a field extension, and let $\mathscr{A}$ be an F-linear abelian category.

If $\mathscr{A}$ is F-finite, Theorem 23.19 gives us a finite F'-linear abelian category $\mathscr{A} \otimes_F F'$ and an F-linear exact functor $\mathscr{A} \to \mathscr{A} \otimes_F F'$. The goal of this section is to show that this functor is "universal" among right-exact F'-linear functors to F'-linear abelian categories. By this we mean that every such functor $V : \mathscr{A} \to \mathscr{B}$ "factors" through $\mathscr{A} \otimes_F F'$ via a right-exact F'-linear functor $V : \mathscr{A} \otimes_F F' \to \mathscr{B}$, and does so "uniquely". Since we are working with functors, we have to be more precise in stating this universal property.

Theorem 24.1. *Assume that $\mathscr{A}$ is F-finite. Let $\mathscr{B}$ be an F'-linear abelian category, and let $V : \mathscr{A} \to \mathscr{B}$ be a right-exact F-linear functor. Then:*

(a) There exists a right-exact F'-linear functor $V' : \mathscr{A} \otimes_F F \to \mathscr{B}$ and an isomorphism of functors $\alpha : V \Rightarrow V' \circ (F' \otimes_F -)$.

(b) If (V_1', α_1) and (V_2', α_2) both have the properties stated in (a), then there exists a unique isomorphism of functors $\beta' : V_1' \Rightarrow V_2'$ such that $\alpha_{2,X} = \beta'_{F' \otimes_F X} \circ \alpha_{1,X}$ for every $X \in \mathscr{A}$.

Remark 24.2. For F-finite abelian $\mathscr{A}$ let us set $\mathscr{A} \otimes_F^l F' := (\mathscr{A}^{\mathrm{op}} \otimes_F F')^{\mathrm{op}}$. Then, by categorical nonsense, $\mathscr{A} \otimes_F^l F'$ has a universal property as well, namely the one obtained by replacing right-exactness by left-exactness in the statement of Theorem 24.1. If $\mathscr{A}$ is a rigid tensor category over F, we will see in the next section that $\mathscr{A} \otimes_F^l F'$ and $\mathscr{A} \otimes_F F'$ coincide.

Following discussions with Richard Pink, I am convinced that, if for every object $\boldsymbol{X}$ of $\mathscr{A} \otimes_F F'$ there exists an object $X^0 \in \mathscr{A}$ and an *inclusion* $\boldsymbol{X} \hookrightarrow F' \otimes_F X^0$, then the category $\mathscr{A} \otimes_F F'$ should have two universal properties, namely the one stated in Theorem 24.1 and the universal property of $\mathscr{A} \otimes_F^l F'$ as stated above.

However, if a functor V as in Theorem 24.1 happens to be exact, in general the induced right-exact functor V' need not be left-exact, as examples show. So if the conviction stated in the previous paragraph turns out to be justified, then an exact functor V would have two extension to $\mathscr{A} \otimes_F F'$, a right-exact functor V_r' and a left-exact functor V_l', but these two functors would differ in general.

The idea of the proof of Theorem 24.1 is to use the purported right-exactness of V' for the proof of its existence. After all, for every $\boldsymbol{X} \in \mathscr{A} \otimes_F F'$ by Definition 23.11, Proposition 23.19 and Lemma 23.8 we have a presentation

$$F' \otimes_F X_1 \xrightarrow{\sum_i' \lambda_i \otimes f_i} F' \otimes_F X_0 \to \boldsymbol{X} \to 0,$$

with $X_0, X_1 \in \mathscr{A}$, and finitely many $\lambda_i \in F'$ and $f_i \in \mathrm{Hom}_{\mathscr{A}}(X_1, X_0)$. Therefore, by right-exactness and F'-linearity of V', we should have

$$V'(\boldsymbol{X}) \cong \mathrm{coker}\Big(V(X_1) \xrightarrow{\sum_i' \lambda_i V(f_i)} V(X_0) \Big).$$

However, since there is no canonical such presentation, it seems difficult to verify that this idea gives us a well-defined functor V' directly. Hence, we take a detour through the respective ind-categories.

We begin by supplementing Lemma 23.2. Remember that we have made a choice of an exact fully faithful F-linear functor $\mathscr{A} \to \operatorname{ind}\mathscr{A}$ after Definition 23.1, which simplifies the statement of the following lemma.

Definition 24.3. Let $\mathscr{B}$ be an F-linear abelian category, and let $V : \mathscr{A} \to \mathscr{B}$ be an F-linear functor. The *ind-extension* of V is the F-linear functor $\operatorname{ind}V : \operatorname{ind}\mathscr{A} \to \operatorname{ind}\mathscr{B}$ mapping an object $(X_i)_{i\in I}$ of $\operatorname{ind}\mathscr{A}$ to $\operatorname{ind}V((X_i)_{i\in I}) := (VX_i)_{i\in I}$ in $\operatorname{ind}\mathscr{B}$, and a homomorphism $f = \varprojlim_i \varinjlim_j f_{ij}$ in $\operatorname{Hom}_{\operatorname{ind}\mathscr{A}}((X_i)_{i\in I},(Y_j)_{j\in J}) = \varprojlim_i \varinjlim_j \operatorname{Hom}_{\mathscr{A}}(X_i, Y_j)$ to $\operatorname{ind}V(f) := \varprojlim_i \varinjlim_j V(f_{ij})$.

Lemma 24.4. *(a)* $\operatorname{ind}(V)$ *is a functor extending* V *and functorial in* V.

(b) If V *is right-exact, then* $\operatorname{ind}(V)$ *is right exact.*

Proof. [SGA, 4.8] ∴

Lemma 24.5. *Every* F'*-module* $\boldsymbol{X}$ *in* $\operatorname{ind}\mathscr{A}$ *has a functorial presentation*

$$\Pi(\boldsymbol{X}) : \quad F' \otimes_F X_1 \xrightarrow{d_1} F' \otimes_F X_0 \xrightarrow{d_0} \boldsymbol{X} \to 0$$

using objects in the image of $\mathscr{A}$ *under* $F' \otimes_F -$.

Proof. First, for every object $\boldsymbol{X} = (X, \phi)$ of $(\operatorname{ind}\mathscr{A})_{F'}$, let $\widetilde{\phi}$ denote the homomorphism $F' \otimes_F X \to X$ corresponding to ϕ via the correspondence

$$\operatorname{Hom}_{\mathscr{A}}(F' \otimes_F X, X) \cong \operatorname{Hom}_F(F', \operatorname{End}_{\mathscr{A}}(X))$$

given by Definition 23.4. We remark that $\widetilde{\phi}$ is actually a homomorphism of F'-modules, if we equip $F' \otimes_F X$ with the action given by $\ell \otimes \operatorname{id}_X$, where ℓ is the natural action of F' on itself by left multiplication. Moreover, $\widetilde{\phi}$ is an epimorphism.

We may now define our presentation: Given $\boldsymbol{X}$ as above, we set $X_0 := X$, and $d_0 := \widetilde{\phi}$. Then $\ker(d_0)$ is an F'-module (X_1, ϕ_1), and we set $d_1 := \widetilde{\phi_1}$. We obtain an exact sequence

$$F' \otimes_F X_1 \xrightarrow{d_1} F' \otimes_F X_0 \xrightarrow{d_0} \boldsymbol{X} \to 0$$

in $\mathscr{A}_{F'}$, which we denote as $\Pi(\boldsymbol{X})$.

Let us show that $\Pi(\boldsymbol{X})$ is functorial in $\boldsymbol{X}$: Given another F'-module $\boldsymbol{Y} = (Y, \psi)$ and a homomorphism $f : \boldsymbol{X} \to \boldsymbol{Y}$, we set $f_0 := \operatorname{id} \otimes f$ and $f_1 := \operatorname{id} \otimes (f_0|_{(X_1,\phi_1)})$. We obtain a diagram

$$\begin{array}{llllllll}
\Pi(\boldsymbol{X}) : & F' \otimes_F X_1 & \longrightarrow & F' \otimes_F X_0 & \longrightarrow & \boldsymbol{X} & \longrightarrow & 0 \\
 & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \\
\Pi(\boldsymbol{Y}) : & F' \otimes_F Y_1 & \longrightarrow & F' \otimes_F Y_0 & \longrightarrow & \boldsymbol{Y} & \longrightarrow & 0
\end{array}$$

This diagram commutes by definition, so we have constructed a canonical homomorphism $\Pi(f) : \Pi(\boldsymbol{X}) \to \Pi(\boldsymbol{Y})$. ∴

Lemma 24.6. *Let* $\operatorname{ind} V : \operatorname{ind} \mathscr{A} \to \operatorname{ind} \mathscr{B}$ *be a right-exact* F*-linear functor. There exists a right-exact* F'*-linear functor*

$$\operatorname{ind} V' : (\operatorname{ind} \mathscr{A})_{F'} \to \operatorname{ind} \mathscr{B}$$

and an isomorphism of functors $\operatorname{ind} \alpha : \operatorname{ind} V \Rightarrow (\operatorname{ind} V') \circ (F' \otimes_F -)$.

Proof. We note first that there exists a unique F'-linear functor

$$\operatorname{ind} \widetilde{V} : (F' \otimes_F -)(\operatorname{ind} \mathscr{A}) \to \operatorname{ind} \mathscr{B}$$

such that $\operatorname{ind} \widetilde{V} \circ (F' \otimes_F -) = \operatorname{ind} V$; it fulfills $\operatorname{ind} \widetilde{V}(F' \otimes_F X) = \operatorname{ind} V(X)$ on objects $X \in \operatorname{ind} \mathscr{A}$, and is the F'-linear extension of $\operatorname{ind} V$ on homomorphisms.

In particular, given an F'-module $\boldsymbol{X} = (X, \phi)$ in $\operatorname{ind} \mathscr{A}$, we may apply $\operatorname{ind} \widetilde{V}$ to the portion $F' \otimes_F X_1 \xrightarrow{d_1} F' \otimes_F X_0$ of the presentation $\Pi(\boldsymbol{X})$ given by Lemma 24.5, and set

$$\operatorname{ind} V'(\boldsymbol{X}) := \operatorname{coker}\left(\operatorname{ind} V(X_1) \xrightarrow{\operatorname{ind} \widetilde{V}(d_1)} \operatorname{ind} V(X_0)\right).$$

Given a second object $\boldsymbol{Y}$ and a homomorphism $f : \boldsymbol{X} \to \boldsymbol{Y}$ in of F'-modules, we may apply $\operatorname{ind} \widetilde{V}$ to the portion

$$\begin{array}{ccc} F' \otimes_F X_1 & \longrightarrow & F' \otimes_F X_0 \\ \downarrow f_1 & & \downarrow f_0 \\ F' \otimes_F Y_1 & \longrightarrow & F' \otimes_F Y_0 \end{array}$$

of the homomorphism $\Pi(f)$ of presentations given by Lemma 24.5. Now the universal property of cokernels implies that there is exactly one homomorphism $\operatorname{ind} V'(f) : \operatorname{ind} V'(\boldsymbol{X}) \to \operatorname{ind} V'(\boldsymbol{Y})$ completing the image of the above commutative square under $\operatorname{ind} \widetilde{V}$ to a commutative diagram

$$\begin{array}{ccccccc} \operatorname{ind} \widetilde{V}(F' \otimes_F X_1) & \longrightarrow & \operatorname{ind} \widetilde{V}(F' \otimes_F X_0) & \longrightarrow & \operatorname{ind} V'(\boldsymbol{X}) & \longrightarrow & 0 \\ \downarrow \operatorname{ind} \widetilde{V}(f_1) & & \downarrow \operatorname{ind} \widetilde{V}(f_0) & & \downarrow \operatorname{ind} V'(f) & & \\ \operatorname{ind} \widetilde{V}(F' \otimes_F Y_1) & \longrightarrow & \operatorname{ind} \widetilde{V}(F' \otimes_F Y_0) & \longrightarrow & \operatorname{ind} V'(\boldsymbol{Y}) & \longrightarrow & 0 \end{array}$$

The universal property of cokernels also shows that $\operatorname{ind} V'(\mathrm{id}_{\boldsymbol{X}}) = \mathrm{id}_{\operatorname{ind} V'(\boldsymbol{X})}$ for all $\boldsymbol{X}$, and that $\operatorname{ind} V'(gf) = \operatorname{ind} V'(g) \operatorname{ind} V'(f)$ for all pairs of composable arrows $\boldsymbol{X} \xrightarrow{f} \boldsymbol{Y} \xrightarrow{g} \boldsymbol{Z}$, so $\operatorname{ind} V'$ is indeed a functor $(\operatorname{ind} \mathscr{A})_{F'} \to \operatorname{ind} \mathscr{B}$.

Let us prove that $\operatorname{ind} V'$ is right-exact, so let $\boldsymbol{X} \xrightarrow{f} \boldsymbol{Y} \xrightarrow{g} \boldsymbol{Y} \to 0$ be a right-exact sequence in $(\operatorname{ind} \mathscr{A})_{F'}$. We obtain the following commutative diagram:

$$
\begin{array}{ccccccc}
\operatorname{ind} V(X_1) & \longrightarrow & \operatorname{ind} V(X_0) & \longrightarrow & \operatorname{ind} V'(\boldsymbol{X}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \operatorname{ind} V'(f)} & & \\
\operatorname{ind} V(Y_1) & \longrightarrow & \operatorname{ind} V(Y_0) & \longrightarrow & \operatorname{ind} V'(\boldsymbol{Y}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \operatorname{ind} V'(g)} & & \\
\operatorname{ind} V(Z_1) & \longrightarrow & \operatorname{ind} V(Z_0) & \longrightarrow & \operatorname{ind} V'(\boldsymbol{Z}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}
$$

The rows are the sequences defining $\operatorname{ind} V'$ on objects, so they are exact by definition. Since V is right-exact, the first two columns are exact. Hence, by the 3×3-Lemma, the remaining column is exact, which is what we had to prove.

Finally, let us construct an isomorphism $\operatorname{ind} \alpha : \operatorname{ind} V \Rightarrow (\operatorname{ind} V') \circ (F' \otimes_F -)$ of functors. We let K be the kernel of the multiplication μ of F', so we have an exact sequence of F'-vector spaces

$$0 \to K \to F' \otimes_F F' \xrightarrow{\mu} F' \to 0.$$

For every object X of $\operatorname{ind} \mathscr{A}$, this induces an exact sequence

$$0 \to K \otimes_{F'} \operatorname{ind} V(X) \to (F' \otimes_F F') \otimes_{F'} \operatorname{ind} V(X) \to F' \otimes_{F'} \operatorname{ind} V(X) \to 0 \quad (24.7)$$

in $\operatorname{ind} \mathscr{B}$. We use this observation to construct the following diagram:

$$
\begin{array}{ccccccc}
\operatorname{ind} \widetilde{V}(F' \otimes_F X_1) & \xrightarrow{\operatorname{ind} \widetilde{V}(d_1)} & \operatorname{ind} \widetilde{V}(F' \otimes_F X_0) & \longrightarrow & \operatorname{ind} V'(F' \otimes_F X) & \longrightarrow & 0 \\
\| & & \| & & & & \\
\operatorname{ind} V(K \otimes_F X) & \longrightarrow & \operatorname{ind} V(F' \otimes_F X) & & & & \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & & & \\
K \otimes_F \operatorname{ind} V(X) & \longrightarrow & F' \otimes_F \operatorname{ind} V(X) & & & & \\
\downarrow & & \downarrow{\scriptstyle \cong} & & & & \\
K \otimes_{F'} \operatorname{ind} V(X) & \longrightarrow & (F' \otimes_F F') \otimes_{F'} \operatorname{ind} V(X) & \longrightarrow & F' \otimes_{F'} \operatorname{ind} V(X) & \longrightarrow & 0
\end{array}
$$

The first row is the definition of $\operatorname{ind} V'(F' \otimes_F X)$, which we unravel in the second row. The isomorphisms connecting the second and third row are canonical, as are the epimorphism and the isomorphism connecting the third row with the fourth, which is the exact sequence (24.7). One can check that this diagram commutes, so by the Five Lemma we obtain a canonical isomorphism $\operatorname{ind} V'(F' \otimes_F X) \to F' \otimes_{F'} \operatorname{ind} V(X)$. Precomposing the inverse of this isomorphism with the canonical isomorphism $F' \otimes_{F'} \operatorname{ind} V(X) \to \operatorname{ind} V(X)$, we obtain an isomorphism

$$\operatorname{ind} \alpha_X : \quad V(X) \xrightarrow{\cong} V'(F' \otimes_F X),$$

as desired. By construction, $\operatorname{ind} \alpha_X$ is natural in X, so $\operatorname{ind} \alpha$ is a homomorphism of functors. Therefore $\operatorname{ind} \alpha$ is an isomorphism of functors, since we have already seen that $\operatorname{ind} \alpha_X$ is an isomorphism for each $X \in \mathscr{A}$. ∴

Lemma 24.8. *Let $\mathscr{A}$ be an F-finite abelian category, let $V : \mathscr{A} \to \mathscr{B}$ be a right-exact F-linear functor. Let* $\operatorname{ind} V'$ *be the right-exact F'-linear functor associated to* $\operatorname{ind} V$ *via Lemma 24.6. There exists a functor*

$$V' : \mathscr{A} \otimes_F F' \longrightarrow \mathscr{B}$$

such that V' fulfills the requirements of Theorem 24.1(a) and the following diagram commutes:

$$\begin{array}{ccc} (\operatorname{ind} \mathscr{A})_{F'} & \xrightarrow{\operatorname{ind} V'} & \operatorname{ind} \mathscr{B} \\ \uparrow & & \uparrow \\ \mathscr{A} \otimes_F F' & \xrightarrow{V'} & \mathscr{B} \end{array}$$

Proof. By Lemma 24.4, V induces a right-exact F-linear functor $\operatorname{ind} V : \operatorname{ind} \mathscr{A} \to \operatorname{ind} \mathscr{B}$. By Lemma 24.6, $\operatorname{ind} V$ induces a right-exact F'-linear functor $\operatorname{ind} V' : (\operatorname{ind} \mathscr{A})_{F'} \to \operatorname{ind} \mathscr{B}$. We obtain the following diagram, which commutes up to isomorphism of functors:

$$\begin{array}{ccccc} \mathscr{A} & \longrightarrow & \operatorname{ind} \mathscr{A} & \longrightarrow & (\operatorname{ind} \mathscr{A})_{F'} \\ {\scriptstyle V}\downarrow & & \downarrow{\scriptstyle \operatorname{ind} V} & \swarrow{\scriptstyle \operatorname{ind} V'} & \\ \mathscr{B} & \longrightarrow & \operatorname{ind} \mathscr{B} & & \end{array}$$

We let V' be the restriction of $\operatorname{ind} V'$ to $\mathscr{A} \otimes_F F' \subset (\operatorname{ind} \mathscr{A})_{F'}$. If we prove that the image of V' lies in the essential image of $\mathscr{B}$ in $\operatorname{ind} \mathscr{B}$, then we will have shown that the following diagram commutes up to isomorphism of functors:

$$\begin{array}{ccccc} \mathscr{A} & \longrightarrow & \mathscr{A} \otimes_F F' & \longrightarrow & (\operatorname{ind} \mathscr{A})_{F'} \\ & {\scriptstyle V}\searrow & \vdots{\scriptstyle V'} & & \downarrow{\scriptstyle \operatorname{ind} V'} \\ & & \mathscr{B} & \longrightarrow & \operatorname{ind} \mathscr{B} \end{array}$$

So let us do this: Given $\boldsymbol{X}$ in $\mathscr{A} \otimes_F F'$, by Definition 23.11, Proposition 23.19 and Lemma 23.8 there exists a right exact sequence

$$F' \otimes_F X_1 \xrightarrow{\varpi} F' \otimes_F X_0 \to \boldsymbol{X} \to 0$$

in $(\operatorname{ind}\mathscr{A})_{F'}$, with $X_0, X_1 \in \mathscr{A}$ and $\varpi \in F' \otimes \operatorname{Hom}_{\mathscr{A}}(X_0, Y_0)$. Since $\operatorname{ind} V'$ is right-exact, and its restriction to $\mathscr{A}$ is isomorphic to V, the induced sequence

$$V(X_1) \xrightarrow{(\operatorname{ind} V')(\varpi)} V(X_0) \to V'(\boldsymbol{X}) \to 0$$

is exact in $\operatorname{ind}\mathscr{B}$. Since $\mathscr{B} \to \operatorname{ind}\mathscr{B}$ is exact, it follows that $V'(\boldsymbol{X})$ is isomorphic to the cokernel of the homomorphism $\operatorname{ind} V'(\varpi)$ calculated in $\mathscr{B}$. ∴

We turn to the unicity of our extensions $\operatorname{ind} V'$ and V'.

Lemma 24.9. *Let* $\operatorname{ind} V_1$, $\operatorname{ind} V_2 : \operatorname{ind}\mathscr{A} \to \operatorname{ind}\mathscr{B}$ *be two right-exact F-linear functors. Let* $(\operatorname{ind} V_1', \operatorname{ind}\alpha_1)$ *be an extension of* $\operatorname{ind} V_1$ *and* $(\operatorname{ind} V_2', \operatorname{ind}\alpha_2)$ *an extension of* $\operatorname{ind} V_2$ *(each as in as in Lemma 24.6).*

For every homomorphism of functors $\operatorname{ind}\beta : \operatorname{ind} V_1 \Rightarrow \operatorname{ind} V_2$ *there exists a unique homomorphism of functors* $\operatorname{ind}\beta' : \operatorname{ind} V_1' \Rightarrow \operatorname{ind} V_2'$ *such that* $\operatorname{ind}\alpha_{2,X} \circ \operatorname{ind}\beta_X = \operatorname{ind}\beta'_{F'\otimes_F X} \circ \operatorname{ind}\alpha_{1,X}$ *for all* $X \in \operatorname{ind}\mathscr{A}$.

Moreover, $\operatorname{ind}\beta$ *is a monomorphism (resp. epimorphism, resp. isomorphism) if and only if* $\operatorname{ind}\beta'$ *is.*

Proof. For $\boldsymbol{X} \in (\operatorname{ind}\mathscr{A})_{F'}$, the sequences $\operatorname{ind} V_i'(\Pi(\boldsymbol{X}))$ are both exact, since both $\operatorname{ind} V_i'$ are right-exact by assumption. They are connected by means of the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\operatorname{ind} V_1'(F'\otimes_F X_1) & \longrightarrow & \operatorname{ind} V_1'(F'\otimes_F X_0) & \longrightarrow & \operatorname{ind} V_1'(\boldsymbol{X}) & \longrightarrow & 0 \\
\downarrow{\scriptstyle (\operatorname{ind}\alpha_{1,X_1})^{-1}} & & \downarrow{\scriptstyle (\operatorname{ind}\alpha_{1,X_0})^{-1}} & & & & \\
\operatorname{ind} V(X_1) & \longrightarrow & \operatorname{ind} V(X_0) & & & & \\
\downarrow{\scriptstyle \operatorname{ind}\beta_{X_1}} & & \downarrow{\scriptstyle \operatorname{ind}\beta_{X_0}} & & & & \\
\operatorname{ind} V(X_1) & \longrightarrow & \operatorname{ind} V(X_0) & & & & \\
\downarrow{\scriptstyle \operatorname{ind}\alpha_{2,X_1}} & & \downarrow{\scriptstyle \operatorname{ind}\alpha_{2,X_0}} & & & & \\
\operatorname{ind} V_2'(F'\otimes_F X_1) & \longrightarrow & \operatorname{ind} V_2'(F'\otimes_F X_0) & \longrightarrow & \operatorname{ind} V_2'(\boldsymbol{X}) & \longrightarrow & 0
\end{array}$$

By the universal property of cokernels, we obtain a unique homomorphism $\operatorname{ind}\beta'_{\boldsymbol{X}} : V_1'(\boldsymbol{X}) \to V_2'(\boldsymbol{X})$ completing the diagram to a homomorphism of right-exact sequences. By the Five Lemma, $\operatorname{ind}\beta'_{\boldsymbol{X}}$ is a monomorphism (resp. epimorphism, resp. isomorphism) if and only if $\operatorname{ind}\beta$ is. Now by construction $\operatorname{ind}\beta'_{\boldsymbol{X}}$ is natural in $\boldsymbol{X}$, so $\operatorname{ind}\beta' : \operatorname{ind} V_1' \Rightarrow \operatorname{ind} V_2'$ is a homomorphism of functors, which is a monomorphism (resp. epimorphism, resp. isomorphism) if and only if $\operatorname{ind}\beta$ is.

The same diagram shows that any homomorphism $\operatorname{ind} V_1' \Rightarrow \operatorname{ind} V_2'$ which restricts to $(\operatorname{ind}\alpha_2)\circ(\operatorname{ind}\beta)\circ(\operatorname{ind}\alpha_1)^{-1}$ on the image of $\operatorname{ind}\mathscr{A}$ under $F'\otimes_F -$ must coincide with $\operatorname{ind}\beta'$.

It remains to show that $\operatorname{ind}\beta'$ restricts in such a way. But this again follows from the same diagram, since if $\boldsymbol{X} = F' \otimes_F \widetilde{X}$ for $\widetilde{X} \in \operatorname{ind}\mathscr{A}$, then $\operatorname{ind}\alpha_{2,\widetilde{X}} \circ \operatorname{ind}\beta_X \circ (\operatorname{ind}\alpha_{1,\widetilde{X}})^{-1}$ fits in the same place as $\operatorname{ind}\beta'_{F'\otimes_F\widetilde{X}}$, so the two homomorphisms must coincide by the universal property of cokernels. ∴

Lemma 24.10. *Given two pairs (V_i', α_i), (V_2', α_2) extending V as in Theorem 24.1(a), there exists a unique isomorphism of functors $\beta' : V_1' \Rightarrow V_2'$ such that $\beta'_{F'\otimes_F X} \circ \alpha_{1,X} = \alpha_{2,X}$ for all $X \in \mathscr{A}$.*

Proof. Given two such pairs of data (V_i', α_i), by Lemma 24.4 we obtain two pairs of data $(\operatorname{ind} V_i', \operatorname{ind}\alpha_i)$ extending $\operatorname{ind} V$ as in Lemma 24.6, so Lemma 24.9 (applied to $\operatorname{ind}\beta = \mathrm{id}$) shows that there exists an isomorphism of functors $\operatorname{ind}\beta' : \operatorname{ind} V_1' \Rightarrow \operatorname{ind} V_2'$ such that $\operatorname{ind}\beta'_{F'\otimes_F X} \circ \operatorname{ind}\alpha_{1,X} = \operatorname{ind}\alpha_{2,X}$ for all $X \in \operatorname{ind}\mathscr{A}$. The restriction β' of $\operatorname{ind}\beta'$ to $\mathscr{A} \otimes_F F' \subset (\operatorname{ind}\mathscr{A})_{F'}$ is then an isomorphism of functors $V_1' \Rightarrow V_2'$ with the required properties.

Let us show that this β' is unique. Given two isomorphisms of functors $\beta_1', \beta_2' : V_1' \Rightarrow V_2'$ with an identification of isomorphisms

$$\beta_1' \mid_{(F'\otimes_F -)(\mathscr{A})} = \alpha_2 \circ \alpha_1^{-1} = \beta_2' \mid_{(F'\otimes_F -)(\mathscr{A})} : \ V_1' \Rightarrow V_2',$$

applying $\operatorname{ind}(-)$ gives us an identification of isomorphisms

$$\operatorname{ind}\beta_1' \mid_{(F'\otimes_F -)(\operatorname{ind}\mathscr{A})} = \operatorname{ind}(\alpha_2 \circ \alpha_1^{-1}) = \operatorname{ind}\beta_2' \mid_{(F'\otimes_F -)(\operatorname{ind}\mathscr{A})} : \ \operatorname{ind} V_1' \Rightarrow \operatorname{ind} V_2'$$

by Lemma 24.4, in which $\operatorname{ind}(\alpha_2 \circ \alpha_1^{-1}) = \operatorname{ind}\alpha_2 \circ \operatorname{ind}\alpha_1^{-1}$. Lemma 24.9 shows that $\operatorname{ind}\beta_1' = \operatorname{ind}\beta_2'$, so restricting to $F' \otimes_F \mathscr{A}$ we obtain

$$\beta_1' = \operatorname{ind}\beta_1' \mid_{\mathscr{A}\otimes_F F'} = \operatorname{ind}\beta_2' \mid_F = \beta_2',$$

as desired. ∴

Proof of Theorem 24.1. Lemma 24.8 proves item (a), whereas Lemma 24.10 proves item (b). ∴

Remark 24.11. In Theorem 24.1(a), V' may be chosen such that $\alpha = \mathrm{id}_V$. This is not a terribly 2-categorical way of viewing things, but we state it all the same: Choose $(\widetilde{V'}, \alpha)$ as in Theorem 24.1(a). Let us define V' on objects first: If $\boldsymbol{X} \in \mathscr{A} \otimes_F F'$ is of the form $F' \otimes_F X_0$ for some $X_0 \in \mathscr{A}$, then we set $V'(\boldsymbol{X}) := V(X_0)$ and $\beta_{\boldsymbol{X}} := \alpha_{X_0}$. Otherwise, we set $V'(\boldsymbol{X}) := \widetilde{V'}(\boldsymbol{X})$ and $\beta_{\boldsymbol{X}} := \mathrm{id}_{V'(\boldsymbol{X})}$.

Given two objects $\boldsymbol{X}, \boldsymbol{Y}$ of $\mathscr{A} \otimes_F F'$, we define V' on homomorphisms by letting V' map $f \in \operatorname{Hom}_{\mathscr{A}\otimes_F F'}(\boldsymbol{X}, \boldsymbol{Y})$ to $\beta_{\boldsymbol{Y}}^{-1} \circ \left(\widetilde{V'}(f)\right) \circ \beta_{\boldsymbol{X}}$, which by construction is an element of $\operatorname{Hom}_{\mathscr{B}}(V'(\boldsymbol{X}), V'(\boldsymbol{Y}))$.

Proposition 24.12. *Assume that $\mathscr{A}$ is F-finite. Let $\mathscr{B}$ be an F'-linear abelian category, and let $V'_1, V'_2 : \mathscr{A} \otimes_F F' \to \mathscr{B}$ be two right-exact F'-linear functors. Then for every homomorphism of functors $\beta : V'_1|_{\mathscr{A}} \Rightarrow V'_2|_{\mathscr{A}}$ there exists a unique homomorphism of functors $\beta' : V'_1 \Rightarrow V'_2$ extending β.*

Moreover, β is a monomorphism (resp. epimorphism, resp. isomorphism) if and only if β' is such.

Proof. This may be deduced from Lemma 24.9 as in the proof of Lemma 24.10. $\therefore$

25 Tensor products

Let F'/F be a field extension, and let $\mathscr{A}$ be an F-finite abelian category. We denote the canonical functor $\mathscr{A} \to \mathscr{A} \otimes_F F'$, $X \mapsto F' \otimes_F X$ by t, and for any category $\mathscr{C}$ we let s denote the functor $\mathscr{C} \times \mathscr{C} \to \mathscr{C} \times \mathscr{C}$, $(X, Y) \mapsto (Y, X)$.

Recall that an abelian tensor category over F is a datum consisting of an F-linear abelian category $\mathscr{A}$, an F-bilinear functor $\otimes : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ which is right-exact in both variables, further data (an associativity constraint ϕ and a commutativity constraint ψ), and these data together must satisfy certain axioms.

Given an F-finite such abelian tensor category $(\mathscr{A}, \otimes)$ over F, we wish to equip $\mathscr{A} \otimes_F F'$ with a “unique” structure of abelian tensor category over F', “extending” the tensor product of $\mathscr{A}$. For this we must first state a multilinear version of Theorem 24.1.

Theorem 25.1. *Let $\mathscr{A}$ be an F-finite abelian category, and $n \geq 1$ an integer.*

(a) Let $M : \mathscr{A}^{\times n} \to \mathscr{A}$ be an F-multilinear functor right-exact in each variable. Then there exists an F'-multilinear functor $M' : (\mathscr{A} \otimes_F F')^{\times n} \to \mathscr{A} \otimes_F F'$ right-exact in each variable, and an isomorphism $\alpha : t \circ M \Rightarrow M' \circ (t^{\times n})$ of functors.

(b) Let $M_1, M_2 : \mathscr{A}^{\times n} \to \mathscr{A}$ be two F-multilinear functors exact in each variable, and let (M'_1, α_1), (M'_2, α_2) be extensions as in (a) of M_1, M_2 respectively. Then, for every homomorphism of functors $\beta : M_1 \Rightarrow M_2$ there exists a unique homomorphism of functors $\beta' : M'_1 \Rightarrow M'_2$ such that $\beta' \circ \alpha_1 = \alpha_2 \circ t\beta$ in the sense that for every n-tuple of objects $(X_1, \dots X_n) \in \mathscr{A}^{\times n}$ the following diagram commutes:

$$\begin{array}{ccc} M'_1(F' \otimes_F X_1, \dots, F' \otimes_F X_n) & \xrightarrow{\alpha_{1,(X_1,\dots,X_n)}} & F' \otimes_F M_1(X_1, \dots, X_n) \\ \downarrow{\scriptstyle \beta'_{(F' \otimes_F X_1, \dots, F' \otimes_F X_n)}} & & \downarrow{\scriptstyle \mathrm{id} \otimes \beta_{(X_1,\dots,X_n)}} \\ M'_2(F' \otimes_F X_1, \dots, F' \otimes_F X_n) & \xrightarrow{\alpha_{2,(X_1,\dots,X_n)}} & F' \otimes_F M_1(X_1, \dots, X_n) \end{array}$$

Proof. This is one of the proofs in mathematics which does not become much clearer by writing it down in detail. The case $n = 1$ follows from Theorem 24.1(a) and Proposition 24.12 applied to $t \circ V$. We settle for a sketch of the construction of M' in the case $n = 2$. We set $\otimes := M$ and will denote the desired extension M' by $\otimes'$. Let us abbreviate notation by setting $\mathscr{A}' := \mathscr{A} \otimes_F F'$.

For every $Y \in \mathscr{A}$, let

$$- \otimes' tY := (t \circ (- \otimes Y))' : \mathscr{A}' \to \mathscr{A}'$$

denote the scalar extension of $t \circ (- \otimes Y)$ as in Theorem 24.1(a). It is an F'-linear right-exact functor. It is also functorial in Y, since a homomorphism $f : Y_1 \to Y_2$ induces a homomorphism of functors $t \circ (- \otimes Y_1) \Rightarrow t \circ (- \otimes Y_2)$ given for $X \in \mathscr{A}$ by $\mathrm{id} \otimes f : X \otimes Y_1 \to X \otimes Y_2$, which by Proposition 24.12 induces a unique homomorphism of functors $- \otimes' tY_1 \Rightarrow - \otimes' tY_2$. Therefore, we obtain a functor

$$- \otimes' t- : \mathscr{A}' \times \mathscr{A} \to \mathscr{A}', \quad (\boldsymbol{X}, Y) \mapsto \boldsymbol{X} \otimes' tY$$

which is F'-linear in the first variable, F-linear in the second, and right-exact in both variables.

For every $\boldsymbol{X} \in \mathscr{A}'$, let

$$\boldsymbol{X} \otimes' - := ((\boldsymbol{X} \otimes' -) \circ t)' : \mathscr{A}' \to \mathscr{A}'$$

denote the scalar extension of $(\boldsymbol{X} \otimes' -) \circ t$ as in Theorem 24.1(a). It is an F'-linear right-exact functor. By similar reasoning as before, it is functorial in $\boldsymbol{X}$. Therefore, we obtain a functor

$$- \otimes' - : \mathscr{A}' \times \mathscr{A}' \to \mathscr{A}', \quad (\boldsymbol{X}, \boldsymbol{Y}) \mapsto \boldsymbol{X} \otimes' \boldsymbol{Y},$$

which is F'-bilinear and right-exact in both variables. It fulfills what is required in item (a). ∴

Now if $(\mathscr{A}, \otimes, \phi, \psi)$ is an F-finite abelian tensor category over F, then $\mathscr{A}' := \mathscr{A} \otimes_F F'$ is a finite F'-linear abelian category by Theorem 23.19, and we may choose an extension $\otimes'$ of $\otimes$ to $\mathscr{A}'$ by Theorem 25.1(a) for $n = 2$. Then the associativity constraint $\phi : \otimes \circ (\mathrm{id} \times \otimes) \Rightarrow \otimes \circ (\otimes \times \mathrm{id})$ has a unique extension to an isomorphism of functors $\phi' : \otimes' \circ (\mathrm{id} \times \otimes') \Rightarrow \otimes \circ (\otimes' \times \mathrm{id})$ by Theorem 25.1(a) for $n = 3$, and the commutativity constraint $\psi : \otimes \Rightarrow \otimes \circ s$ has a unique extension to an isomorphism of functors $\psi' : \otimes' \Rightarrow \otimes' \circ s$ by Theorem 25.1(a) for $n = 2$.

Theorem 25.2. *Let $(\mathscr{A}, \otimes, \phi, \psi)$ be an F-finite abelian tensor category over F.*

(a) $(\mathscr{A} \otimes_F F', \otimes', \phi', \psi')$ *is a finite abelian tensor category over* F'.

(b) *t induces a tensor functor* $(\mathscr{A}, \otimes) \longrightarrow (\mathscr{A}', \otimes')$.

Proof. (a): $\mathscr{A}' := \mathscr{A} \otimes_F F'$ is a finite F'-linear abelian category by Theorem 23.19, and $\otimes' : \mathscr{A}' \times \mathscr{A}' \to \mathscr{A}'$ is an F-bilinear functor right-exact in both variables by Theorem 25.1(a). It remains to check that three relations hold among ϕ' and ψ' ($\psi' \circ \psi' = \mathrm{id}$, the Pentagon Axiom and the Hexagon Axiom), and that there exists a unit object $\mathbf{1}' \in \mathscr{A}'$ for which $F' \to \mathrm{End}_{\mathscr{A}'}(\mathbf{1}')$ is an isomorphism.

Each of these three relations states that certain homomorphisms (constructed using ϕ' and ψ') of certain functors $\mathscr{A}'^{\times n} \to \mathscr{A}'$ (constructed using $\otimes'$) are equal. The first states that $\psi'_{\boldsymbol{Y},\boldsymbol{X}} \circ \psi'_{\boldsymbol{X},\boldsymbol{Y}} = \mathrm{id}_{\boldsymbol{X} \otimes' \boldsymbol{Y}}$. The Pentagon Axiom states that $\phi \circ \phi = (\phi \otimes \mathrm{id}) \circ \phi \circ (\mathrm{id} \otimes \phi)$ in the sense that for every quadruple $(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{T})$ of objects of $\mathscr{A}'$ the following diagram commutes:

$$\begin{array}{ccccc}
\boldsymbol{X} \otimes' (\boldsymbol{Y} \otimes' (\boldsymbol{Z} \otimes' \boldsymbol{T})) & \longrightarrow & (\boldsymbol{X} \otimes' \boldsymbol{Y}) \otimes' (\boldsymbol{Z} \otimes' \boldsymbol{T}) & \longrightarrow & ((\boldsymbol{X} \otimes' \boldsymbol{Y}) \otimes' \boldsymbol{Z}) \otimes' \boldsymbol{T} \\
\downarrow & & & & \uparrow \\
\boldsymbol{X} \otimes' ((\boldsymbol{Y} \otimes' \boldsymbol{Z}) \otimes' \boldsymbol{T}) & & \longrightarrow & & (\boldsymbol{X} \otimes' (\boldsymbol{Y} \otimes' \boldsymbol{Z})) \otimes' \boldsymbol{T}
\end{array}$$

The Hexagon Axiom states that $\phi \circ \psi \circ \phi = (\psi \otimes \mathrm{id}) \circ \phi \circ (\mathrm{id} \otimes \psi)$ in the sense that for every triple $(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z})$ of objects of $\mathscr{A}'$ the following diagram commutes:

$$\begin{array}{ccccc}
\boldsymbol{X} \otimes' (\boldsymbol{Y} \otimes' \boldsymbol{Z}) & \longrightarrow & (\boldsymbol{X} \otimes' \boldsymbol{Y}) \otimes' \boldsymbol{Z} & \longrightarrow & \boldsymbol{Z} \otimes' (\boldsymbol{X} \otimes' \boldsymbol{Y}) \\
\downarrow & & & & \downarrow \\
\boldsymbol{X} \otimes' (\boldsymbol{Z} \otimes' \boldsymbol{Y}) & \longrightarrow & (\boldsymbol{X} \otimes' \boldsymbol{Z}) \otimes' \boldsymbol{Y} & \longrightarrow & (\boldsymbol{Z} \otimes' \boldsymbol{X}) \otimes' \boldsymbol{Y}
\end{array}$$

In all cases, Theorem 25.1(b) and the assumption that $\mathscr{A}$ is a tensor category shows that the stated relations hold. Let us prove the first relation $\psi' \circ \psi' = \mathrm{id}$. The left hand side is a homomorphism of functors $\otimes' \to \otimes'$. Its restriction to $\otimes$ is equal to $\psi \circ \psi$ by definition, and is equal to the identity endomorphism of $\otimes$, since ψ' extends ψ and $\mathscr{A}$ is a tensor category. So $\psi' \circ \psi'$ is an extension of the identity endomorphism of $\otimes$. Since the identity endomorphism of $\otimes'$ is another extension of the identity endomorphism of $\otimes$, Theorem 25.1(b) shows that $\psi' \circ \psi'$ and the identity endomorphism of $\otimes'$ coincide!

The proof of the Pentagon and Hexagon axioms is similar, if somewhat more involved notationally.

It remains to show that there exists a unit object of $(\mathscr{A}', \otimes')$ with endomorphism ring F', and we claim that $t(\mathbf{1})$ is one for every unit object $\mathbf{1}$ of $(\mathscr{A}, \otimes)$. To say that $\mathbf{1}$ is a unit object means that there exists an isomorphism $u : \mathbf{1} \to \mathbf{1} \otimes \mathbf{1}$ and that $\mathbf{1} \otimes -$ is an equivalence of categories $\mathscr{A} \to \mathscr{A}$.

Now $t(u) : t(\mathbf{1}) \to t(\mathbf{1} \otimes \mathbf{1}) = t(\mathbf{1}) \otimes' t(\mathbf{1})$ is an isomorphism since t is a functor. Let V be a quasi-inverse of the restriction $\mathbf{1} \otimes -$ of the functor $t(\mathbf{1}) \otimes' -$. Then $(t \circ V)'$, the scalar extension of $t \circ V$, is a quasi-inverse of the functor $t(\mathbf{1}) \otimes' -$, this may again be proved using Theorem 25.1(b). Finally, $F' \to \mathrm{End}_{\mathscr{A}'}(t(\mathbf{1}))$ is an isomorphism since $\mathbf{1}$ has endomorphism ring F and t is F'/F-fully faithful.

(b): This statement is true by construction, since we have given $\mathscr{A}'$ a structure of tensor category extending that of $\mathscr{A}$. ∴

Lemma 25.3. *Every finite rigid abelian tensor category over a field F is F-finite.*

Proof. [Del02, Proposition 1.1], one uses the finitude of the length of $\mathbf{Hom}(X, Y) = X^\vee \otimes Y$ to bound the dimension of $\mathrm{Hom}(X, Y) = \mathrm{Hom}(\mathbf{1}, \mathbf{Hom}(X, Y))$ above. ∴

Proposition 25.4. *Let $\mathscr{A}$ be a finite rigid abelian tensor category over F. Then $\mathscr{A} \otimes_F F'$ is a finite rigid abelian tensor category over F'.*

Proof. By Lemma 25.3, $\mathscr{A}$ is F-finite, so Theorem 25.2 applies and shows that $\mathscr{A} \otimes_F F'$ is a finite abelian tensor category over F'.

For every object $\boldsymbol{X} \in \mathscr{A} \otimes_F F'$, we may choose a presentation

$$F' \otimes_F X_1 \xrightarrow{\varpi} F' \otimes_F X_0 \to \boldsymbol{X} \to 0.$$

Since $\mathscr{A}$ is rigid, the X_i are both dualisable. Since t is a tensor functor, the $F' \otimes_F X_i$ are both dualisable, namely $F' \otimes_F (X_i^\vee)$ is a dual of $F' \otimes_F X_i$. But every object of a tensor category which is presented by dualisable objects is dualisable, namely $\boldsymbol{X}^\vee := \ker(\varpi^\vee)$ is a dual of $\mathrm{coker}(\varpi)$. ∴

Theorem 25.5. *Let $\mathscr{B}$ be an abelian F'-linear tensor category, and consider a right-exact F'-linear functor $V' : \mathscr{A}' \to \mathscr{B}$ such that $V := V' \circ t$ is a tensor functor.*

(a) V' is a tensor functor.

(b) If $\mathscr{A}$ is rigid, then V' is exact if and only if V is exact.

Proof. (a): The proof is similar to the proof of Theorem 25.2(a), using Theorem 25.1(b) and the precise definition of tensor functors. We suppress it.

(b): Both V and V' are right-exact. If V' is exact, then so is V as a composition of exact functors.

Every tensor functor commutes with duals. If $0 \to \boldsymbol{X}' \to \boldsymbol{X} \to \boldsymbol{X}''$ is a left exact sequence, then

$$0 \to V'(\boldsymbol{X}') \to V'(\boldsymbol{X}) \to V'(\boldsymbol{X}'')$$

is exact. Why? It suffices to show that its dual is exact. But this dual is the image of the exact sequence $\boldsymbol{X}''^\vee \to \boldsymbol{X}^\vee \to \boldsymbol{X}'^\vee \to 0$ under the right-exact functor V', so it is exact. ∴

Theorem 25.6. *Let $\mathscr{B}$ be a rigid abelian F'-linear tensor category, and consider an exact F'-linear tensor functor $V' : \mathscr{A}' \to \mathscr{B}$. Set $V := V' \circ t$.*

(a) If $\mathscr{B} \neq 0$, then both V and V' are faithful.

(b) V' is fully faithful if and only if V is F'/F-fully faithful.

(c) If F'/F is separable and V if F'/F-fully faithful, then V' is semisimple if and only if V is semisimple.

Proof. (a): [Del82, Proposition 1.19] An exact functor is faithful if and only if it maps non-zero objects to non-zero objects. A dualisable object $X \in \mathscr{A}$ is non-zero if and only if $X \otimes X^\vee \to \mathbf{1}$ is surjective, and this criterion is respected by right-exact tensor functors, so if $\mathscr{B} \neq 0$, that is, if $\mathbf{1}_{\mathscr{B}} \not\cong 0$, then both V and V' are automatically faithful.

(b): If V' is fully faithful, then its restriction $V = V' \circ t$ is F'/F-fully faithful since t is F'/F-fully faithful by Lemma 23.8.

Conversely, let us assume that V is F'/F-fully faithful. We first prove that for every $\boldsymbol{X} \in \mathscr{A} \otimes_F F'$ and every $Y \in \mathscr{A}$, the homomorphism

$$V' : \mathrm{Hom}_{\mathscr{A}'}(\boldsymbol{X}, F' \otimes_F Y) \longrightarrow \mathrm{Hom}_{\mathscr{B}}(V'(\boldsymbol{X}), V(Y))$$

is an isomorphism. We choose a presentation

$$F' \otimes_F X_1 \to F' \otimes_F X_0 \to \boldsymbol{X} \to 0 \tag{25.7}$$

of $\boldsymbol{X}$. Applying $\mathrm{Hom}(-, F' \otimes_F Y)$ to this sequence, and applying $\mathrm{Hom}(-, VY)$ to the right exact sequence which is the image of (25.7) under V', we obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Hom}(\boldsymbol{X}, F' \otimes_F Y) & \longrightarrow & \mathrm{Hom}(F' \otimes_F X_0, F' \otimes_F Y) & \longrightarrow & \mathrm{Hom}(F' \otimes_F X_1, F' \otimes_F Y) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & \mathrm{Hom}(V'\boldsymbol{X}, VY) & \longrightarrow & \mathrm{Hom}(VX_0, VY) & \longrightarrow & \mathrm{Hom}(VX_1, VY) \end{array}$$

The two last vertical arrows are isomorphisms since both $F' \otimes_F -$ and V are F'/F-fully faithful functors. By the Five Lemma, the first vertical arrow is an isomorphism, as claimed.

In general, consider $\boldsymbol{X}$ and $\boldsymbol{Y}$ in $\mathscr{A} \otimes_F F'$. The dual of a presentation of $\boldsymbol{Y}^\vee$ gives us a copresentation

$$0 \to \boldsymbol{Y} \to F' \otimes_F Y^0 \to F' \otimes_F Y^1 \tag{25.8}$$

of $\boldsymbol{Y}$. Applying $\mathrm{Hom}(\boldsymbol{X}, -)$ to this sequence, and applying $\mathrm{Hom}(-, V'\boldsymbol{Y})$ to the left exact sequence which is the image of (25.8) under V', we obtain a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Hom}(\boldsymbol{X}, \boldsymbol{Y}) & \longrightarrow & \mathrm{Hom}(\boldsymbol{X}, F' \otimes Y^0) & \longrightarrow & \mathrm{Hom}(\boldsymbol{X}, F' \otimes_F Y^1) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Hom}(V'\boldsymbol{X}, V'\boldsymbol{Y}) & \longrightarrow & \mathrm{Hom}(V'\boldsymbol{X}, VY^0) & \longrightarrow & \mathrm{Hom}(V'\boldsymbol{X}, VY^1) \end{array}$$

By what we have already proven, the last two vertical arrows are isomorphisms, so by the Five Lemma so is the first, and we have shown that V' is fully faithful.

(c): If V' is semisimple, then V is semisimple as a composition of the semisimple functors V' and $F' \otimes_F -$, the latter being semisimple by Theorem 23.19(b). Conversely, assume that V' restricted to $\mathscr{A}$ is semisimple. Let $\boldsymbol{X}$ be a semisimple object of $\mathscr{A} \otimes_F F'$, we must show that $V'(\boldsymbol{X})$ is semisimple. There exists an object X_0 of $\mathscr{A}$, and an epimorphism $F' \otimes_F X_0 \to \boldsymbol{X}$. As in the proof of Theorem 23.19(b), we may assume that X_0 itself is semisimple, since our given epimorphism factors through $F' \otimes_F (X_0/\operatorname{rad} X_0)$. Hence $V'(\boldsymbol{X})$, being a quotient of the semisimple object $V(X_0)$, is semisimple. ∴

Proposition 25.9. *Let $\mathscr{B}$ be a rigid abelian F'-linear tensor category, and consider an exact F'-linear tensor functor $V' : \mathscr{A}' \to \mathscr{B}$. Let $\eta : V' \Rightarrow V'$ be an automorphism of functors. Then η is a tensor automorphism of V if and only if its restriction to V is a tensor automorphism.*

Proof. Again, as in Theorems 25.2(a) and 25.5(a), this is a matter of checking that certain natural transformations are equal, and we suppress it. ∴

26 Tannakian categories

In this section, we use the results of the previous sections in order to discuss non-neutral Tannakian categories using only the neutral flavour of Tannakian categories – groups, not groupoids.

Let F be a field extension.

Definition 26.1. (a) A *pre-Tannakian category over F* is a finite rigid abelian tensor category over F.

(b) A subcategory $\mathscr{S}$ of a pre-Tannakian category $\mathscr{T}$ over F is a *strictly full pre-Tannakian subcategory* if it is a full subcategory closed under tensor products, duals, and all subquotients in $\mathscr{T}$.

(c) Given a set S of objects of a pre-Tannakian category $\mathscr{T}$ over F, we let $\langle\!\langle S \rangle\!\rangle_\otimes$ denote the smallest strictly full pre-Tannakian subcategory of $\mathscr{T}$ containing all objects of S.

(d) A *fibre functor* over some field extension $F' \supset F$ of a pre-Tannakian category $\mathscr{T}$ is an F-linear exact faithful tensor functor on $\mathscr{T}$ with values in the category of finite-dimensional F'-vector spaces.

(e) A *Tannakian category over F* is a pre-Tannakian category over F for which there exists a fibre functor over some field extension F' of F.

(f) A Tannakian category over F is *neutral* if there exists a fibre functor over F itself.

For the rest of this section, we fix a Tannakian category $\mathscr{T}$ over F.

Definition 26.2. The *monodromy group* of $\mathscr{T}$ with respect to a given fibre functor ω of $\mathscr{T}$ over a field extension F' is the functor

$$G_\omega(\mathscr{T}) : F'\text{-Algebras} \longrightarrow \text{Groups}$$

mapping an F'-algebra R' to the group of tensor automorphisms of the tensor functor $R' \otimes_{F'} \omega(-)$ which maps $X \in \mathscr{T}$ to the R'-module $R' \otimes_{F'} \omega(X)$.

The *monodromy group* $G_\omega(X)$ of an object X of $\mathscr{T}$ is the monodromy group of the strictly full Tannakian subcategory $\langle\!\langle X \rangle\!\rangle_\otimes$ of $\mathscr{T}$ with respect to ω (cf. Definitions 1.6 and 26.1).

From the literature on Tannakian categories, we use (only) the following two theorems:

Theorem 26.3. *Let G be an algebraic group over F. The monodromy group of* $\mathrm{Rep}_F(G)$ *with respect to the forgetful functor* $\mathrm{Rep}_F(G) \to \mathrm{Vec}_F$ *is G.*

Proof. [Del82, Theorem 2.8]. ∴

Theorem 26.4. *Assume that $\mathscr{T}$ is neutral, and fix a fibre functor ω over F.*

(a) $G_\omega(\mathscr{T})$ is an affine group scheme over F. It is of finite type if and only if $\mathscr{T}$ is finitely generated.

(b) ω induces an equivalence of categories $\mathscr{T} \longrightarrow \mathrm{Rep}_F(G_\omega(\mathscr{T}))$.

Proof. [Saa72] or [Del82, Theorem 2.11]. ∴

Remark 26.5. In the situation of Theorem 26.4, if the Tannakian category $\mathscr{T}$ is finitely generated then for every $M \in \mathscr{T}$ with $\langle\!\langle M \rangle\!\rangle_\otimes = \mathscr{T}$ the vector space $\omega(M)$ gives rise to a *faithful* representation of $G_\omega(\mathscr{T})$.

We complement it in the non-neutral case by the following:

Theorem 26.6. *Fix a fibre functor ω over some field extension F' of F.*

(a) *$G_\omega(\mathscr{T})$ is an affine group scheme over F'. It is of finite type if and only if $\mathscr{T}$ is finitely generated.*

(b) *ω induces an equivalence of categories $\mathscr{T} \otimes_F F' \longrightarrow \mathrm{Rep}_{F'}(G_\omega(\mathscr{T}))$.*

Remark 26.7. The general theory of Tannaka categories associates to a pair $(\mathscr{T}, \omega)$ – consisting of a Tannakian category $\mathscr{T}$ over F and a fibre functor ω over F' – an affine *groupoid* scheme $\mathscr{G}_\omega(\mathscr{T})$, the definition of which we suppress, and shows that ω induces an equivalence of categories from $\mathscr{T}$ to the category of finite-dimensional representations of the groupoid scheme $\mathscr{G}_\omega(\mathscr{T})$.

Note that the original reference [Saa72] is faulty in the non-neutral case. For this, [Del90] is the correct place to look. For even further generality, see [Del02].

Proof of Theorem 26.6. The category $\mathscr{T} \otimes_F F'$ is pre-Tannakian over F' by the results of Sections 23 and 25. Using Corollary 24.11, we may choose an extension $\omega' : \mathscr{T} \otimes_F F' \to \mathrm{Vec}_{F'}$ of ω, which is a fibre functor of $\mathscr{T} \otimes_F F'$ over F' by the results of Sections 24 and 25. So $\mathscr{T} \otimes_F F'$ is a neutral Tannakian category, and Theorem 26.6 applies to it.

It remains to show that $G_\omega(\mathscr{T})$ and $G_{\omega'}(\mathscr{T} \otimes_F F')$ coincide. But given an F'-algebra R, Theorem 24.1(b) shows that the restriction map

$$\mathrm{Aut}\,((R \otimes_{F'} -) \circ \omega') \longrightarrow \mathrm{Aut}\,((R \otimes_{F'} -) \circ \omega)$$

is a bijection, which implies by Proposition 25.9 that its restriction

$$G_{\omega'}(\mathscr{T} \otimes_F F')(R) = \mathrm{Aut}^{\otimes}\,((R \otimes_{F'} -) \circ \omega') \to \mathrm{Aut}^{\otimes}\,((R \otimes_{F'} -) \circ \omega) = G_\omega(\mathscr{T})(R)$$

to tensor automorphisms is a bijection, so we are done. ∴

Proposition 26.8. *Let $\mathscr{S}$ be a Tannakian category over F, let $\mathscr{T}$ be a neutral Tannakian category over F, and let $V : \mathscr{S} \to \mathscr{T}$ be an exact fully faithful F-linear tensor functor. Then V is semisimple if and only if the essential image of V is closed under subquotients in $\mathscr{T}$.*

Proof. If the essential image of V is subquotient-closed in $\mathscr{T}$, then V is semisimple: This has been proven more generally in Proposition 3.6.

Conversely, let us assume that V is semisimple. We show first that the essential image of V is closed under subobjects in $\mathscr{T}$, that is, for every object X of $\mathscr{S}$ and every subobject $Y' \subset V(X)$ there exists an object X' of $\mathscr{S}$ with $Y' \cong V(X')$. Since $\mathscr{T}$ is a neutral Tannakian category over F, it is equivalent to the category of finite-dimensional representations of a group scheme over F by Theorem 26.4, and the usual rules of the machinery of exterior algebra apply.

In particular, in the situation $Y' \subset V(X)$ there is a well-defined rank $r := \mathrm{rk}(Y') \geq 1$ of Y', and Y' coincides with the kernel of the homomorphism

$$V(X) \longrightarrow \mathbf{Hom}\,(\Lambda^r Y', \Lambda^{r+1}V(X)), \quad v \mapsto (x \mapsto v \wedge x),$$

as in the proof of Theorem 21.1(a).

Now $\Lambda^r Y'$ has rank 1, so it is simple, and so there exists a projection of the semisimplification $(\Lambda^r V(X))^{\mathrm{ss}}$ of $\Lambda^r V(X)$ onto $\Lambda^r Y'$. Since V is semisimple, we may identify $(\Lambda^r V(X))^{\mathrm{ss}}$ with $\Lambda^r V(X^{\mathrm{ss}})$. Therefore, the displayed homomorphism of the previous paragraph induces a homomorphism

$$g:\ V(X) \longrightarrow \mathbf{Hom}\left(\Lambda^r V(X^{\mathrm{ss}}), \Lambda^{r+1}V(X)\right) =: V(X'')$$

with kernel Y', where $X'' := (\Lambda^r X^{\mathrm{ss}})^\vee \otimes \Lambda^{r+1}X$.

Since V is full, the above homomorphism $g : V(X) \to V(X'')$ is induced by a homomorphism $f : X \to X''$. Setting $X' := \ker f$, since V is exact we have $V(X') = V(\ker f) = \ker(Vf) = \ker(g) = Y'$, which is what we had to prove.

That the essential image of V is closed under quotients in $\mathscr{T}$ follows formally from the above: If $V(X) \to Y'' \to 0$ is an exact sequence in $\mathscr{T}$ with $X \in \mathscr{S}$, then the above applied to the dual exact sequence $0 \to (Y'')^\vee \to V(X^\vee)$ gives an object $X' \in \mathscr{S}$ with $(Y'')^\vee \cong V(X')$, and so $Y'' \cong (Y'')^{\vee,\vee} \cong V(X'^{,\vee})$. ∴

Theorem 26.9. *Let F'/F be a separable field extension, let $\mathscr{T}$ be a Tannakian category over F, and let $\mathscr{T}'$ be a neutral Tannakian category over F'.*

Assume that $V :\ \mathscr{T} \to \mathscr{T}'$ is an exact F-linear tensor functor which is both F'/F-fully faithful and semisimple. Then V induces an equivalence of Tannakian categories

$$V' :\ \mathscr{T} \otimes_F F' \longrightarrow \langle\!\langle V\mathscr{T} \rangle\!\rangle_\otimes,$$

where $\langle\!\langle V\mathscr{T} \rangle\!\rangle_\otimes$ denotes the strictly full pre-Tannakian subcategory of $\mathscr{T}'$ generated by the image of $\mathscr{T}$ under V.

Proof. By Theorem 25.6, the exact functor $V' :\ \mathscr{T} \otimes_F F' \to \mathscr{T}'$ induced by V is fully faithful and semisimple. We must show that its essential image coincides with the strictly full Tannakian subcategory of $\mathscr{T}'$ generated by $V\mathscr{T}$.

On the one hand, we have $V'(\mathscr{T} \otimes_F F') \subset \langle\!\langle V\mathscr{T} \rangle\!\rangle_\otimes$, since every object of $\mathscr{T} \otimes_F F'$ has a presentation by objects arising from $\mathscr{T}$, V' extends V and is exact.

On the other hand, we must show that $\langle\!\langle V\mathscr{T} \rangle\!\rangle_\otimes \subset V'(\mathscr{T} \otimes_F F')$. Clearly, the essential image of V' is closed under direct sums, and also under tensor products and duals since V' is a tensor functor by 25.5. We need to show that this essential image is closed under subquotients. And this follows from Proposition 26.8. ∴

Proposition 26.10. *In the situation of Theorem 26.9, let ω' be a fibre functor of $\mathscr{T}'$ over F'. For every object X of $\mathscr{T}$, the monodromy groups $G_{\omega' \circ V}(X)$ and $G_{\omega'}(VX)$ coincide.*

Proof. By Theorems 26.3 and 26.6, the monodromy group $G_{\omega' \circ V}(X)$ coincides with the monodromy group of $F' \otimes_F X$ as calculated in $\mathscr{T} \otimes_F F'$.

Applying Theorem 26.9 to the Tannakian categories $\widetilde{\mathscr{T}} := \langle\!\langle X \rangle\!\rangle_\otimes$ and $\widetilde{\mathscr{T}'} := \langle\!\langle VX \rangle\!\rangle_\otimes$, we obtain an equivalence of categories $\widetilde{\mathscr{T}} \otimes_F F' \cong \widetilde{\mathscr{T}'}$, which clearly implies that the monodromy group of $F' \otimes_F X$ as calculated in $\mathscr{T} \otimes_F F'$ coincides with the monodromy group of $V(X)$.

Taken together, the two previous paragraphs prove the statement of this Proposition. $\therefore$

Chapter VIII

Main Results – in Tannakian terms

27 Representation valued fibre functors

In this section, let F be a global field, $F' \supset F$ a local field arising by completing F at some place (recall that by Proposition 16.3 this extension is separable), and let Γ be a profinite group. Let $\mathscr{T}$ be a Tannakian category over F, and let $\mathrm{Rep}_{F'}\,\Gamma$ denote the category of continuous finite-dimensional representations of Γ over F'.

We assume that we are given a faithful exact F-linear tensor functor

$$V:\ \mathscr{T} \longrightarrow \mathrm{Rep}_{F'}\,\Gamma,$$

a "representation valued fibre functor", which is *additionally* both F'/F-fully faithful and semisimple.

For every object X of $\mathscr{T}$, let $\Gamma(X)$ denote the image of Γ in $\mathrm{Aut}_{F'}(VX)$, and let $G(X)$ denote the algebraic monodromy group (cf. Definition 26.2) of X with respect to the fibre functor on $\mathscr{T}$ arising by postcomposing V with the forgetful functor.

There exists a unique reduced algebraic subgroup of $\mathrm{GL}(VX)$ which has as set of F'-rational points the Zariski closure of Γ in $\mathrm{GL}(VX)(F')$, and it is natural to hope that this group coincides with $G(X)$:

Theorem 27.1. *(a)* $\Gamma(X)$ *is canonically a Zariski-dense subgroup of* $G(X)(F')$.

(b) If X is semisimple and $\mathrm{End}_{\mathscr{T}}(X)$ *is a separable F-algebra, then* $G(X)^\circ$*, the identity component of* $G(X)$*, is a reductive group.*

We will prove this theorem using our results on scalar extension of abelian categories.

Proposition 27.2. *Let V be a finite-dimensional F'-vector space, and consider an algebraic subgroup $G \subset \mathrm{GL}(V)$ together with a (Zariski) dense subgroup $\Gamma \subset G(F')$ of its F'-rational points. Then:*

(a) A linear subspace $V' \subset V$ is G-stable if and only if it is Γ-stable.

(b) We have $\mathrm{End}_G(V) = \mathrm{End}_\Gamma(V)$.

Proof. (a): Given a linear subspace $V' \subset V$ the stabiliser $G' := \mathrm{Stab}_G(V')$ is an algebraic subgroup of G. If V' is G-stable, then the F'-valued points of $G' = G$ contain Γ, so V' is Γ-stable.

Conversely, if V' is Γ-stable, then $G'(F')$ contains Γ. Since Γ is dense in $G(F')$, this implies that $G' = G$, and so V' is G-stable.

(b): We note that $\mathrm{End}_G(V)$ is the maximal G-stable subspace of $V^\vee \otimes V$ on which G acts trivially, and similarly $\mathrm{End}_\Gamma(V)$ is the maximal Γ-stable subspace on which Γ acts trivially. By a similar argument as in (a), these two spaces must coincide. ∴

Theorem 27.3. *Let V be a finite-dimensional F'-vector space, consider a subgroup $\Gamma \subset \mathrm{GL}(V)(F')$ with associated algebraic group $G := \overline{\Gamma}^{Zar} \subset \mathrm{GL}(V)$. Let V^{cont} represent V considered as a continuous representation of Γ over F', and let V^{alg} represent V considered as a representation of G over F'.*

(a) The natural functor

$$\langle\!\langle V^{alg} \rangle\!\rangle_\otimes \longrightarrow \langle\!\langle V^{cont} \rangle\!\rangle_\otimes$$

between the strictly full Tannakian subcategories of $\mathrm{Rep}_{F'} G$ resp. of $\mathrm{Rep}_{F'} \Gamma$ generated by V^{alg} and V^{cont} is an equivalence of Tannakian categories.

(b) In particular, the algebraic monodromy group of V^{cont} coincides with G.

Proof. (a): Any object of $\langle\!\langle V^{alg} \rangle\!\rangle_\otimes$ yields a continuous representation of Γ, and this gives rise to the desired exact F'-linear tensor functor; let us denote it by C. We wish to employ Theorem 26.9 to conclude that C is an equivalence of Tannakian categories, so we must show that C is fully faithful and semisimple, let us do this.

Consider $W \in \langle\!\langle V^{alg} \rangle\!\rangle_\otimes$, let G_W denote the image of G in $\mathrm{GL}(W)$ and let Γ_W denote the image of Γ in $G_W(F')$. By continuity, Γ_W is dense in $G_W(F')$, so Proposition 27.2(b) shows that $\mathrm{End}_G(W) = \mathrm{End}_\Gamma(CW)$. Since this is true for all W, we conclude that C is fully faithful. If W is simple, Proposition 27.2(a) shows that CW is simple. By Theorem 3.4(d), we conclude that C is semisimple.

(b): It is well-known (cf. [Wat79, Theorem 3.5]) that $\langle\!\langle V^{alg} \rangle\!\rangle_\otimes$ is equivalent to $\mathrm{Rep}_{F'}(G)$. Thus, by Theorem 26.3, G is the algebraic monodromy group of V^{alg}, and so by (a) G is also the algebraic monodromy group of V^{cont}. ∴

Theorem 27.4. *Let V be a finite-dimensional F'-vector space, and consider a closed algebraic subgroup $G \subset \mathrm{GL}(V)$. If V is semisimple as a representation of G, and $\mathrm{End}_G(V)$ is a separable F-algebra, then the identity component G° is a reductive group.*

Proof. Let $\overline{F}$ be an algebraic closure of F. By [Bou81, no. 7, §5, Proposition 6, Corollaire] the $\overline{F}$-algebra $E \otimes_F \overline{F}$ is semisimple, since E is both semisimple and separable over F. By Corollary 23.18(b) applied to Example 23.22(b), $V \otimes_F \overline{F}$ is a semisimple representation of $G_{\overline{F}}$, the base change of G to $\overline{F}$. Therefore we may assume that F is algebraically closed.

Let U be the unipotent radical of G, and let $V^U \subset V$ denote the sub-vector space consisting of those elements fixed (pointwise) by U. Since U is normal in G, V^U is a G-stable subspace of V. We claim that $V^U = V$. If not, since V is semisimple, we may write $V = V^U \oplus V'$ for some G-stable complement V' of V^U. Since U operates unipotently on V', it follows that $(V')^U \neq 0$, which is a contradiction to the definition of V' as a complement of V^U. Therefore $V^U = V$. Since G operates faithfully on V, it follows that $U = 1$, which means that G° is reductive. ∴

Proof of Theorem 27.1. (a): By Theorem 24.1 we may choose a factorisation of V as

$$\mathscr{T} \xrightarrow{F' \otimes_F -} \mathscr{T} \otimes_F F' \xrightarrow{V'} \mathrm{Rep}_{F'} \Gamma,$$

where V' is an F'-linear functor which is an exact tensor functor by Theorem 25.5, and both fully faithful and semisimple by Theorem 25.6 and the fact that F'/F is a separable field extension. Given $X \in \mathscr{T}$, we consider the following diagram:

$$(\!(X)\!)_\otimes \xrightarrow{F' \otimes_F -} (\!(X)\!)_\otimes \otimes_F F' \xrightarrow{V'} (\!(VX)\!)_\otimes \xrightarrow{U} \mathrm{Vec}_{F'}.$$

By Theorem 26.6, the monodromy group $G(X) = G_{U \circ V' \circ (F' \otimes_F -)}(X)$ of X is isomorphic to the monodromy group $G_{U \circ V'}(F' \otimes_F X)$ of $F' \otimes_F X$ with respect to $U \circ V'$.

By Theorem 26.9, the functor V' is an equivalence of categories

$$(\!(X)\!)_\otimes \otimes_F F' \xrightarrow{V'} (\!(VX)\!)_\otimes,$$

so in particular the group monodromy group $G_{U \circ V'}(F' \otimes_F X)$ is isomorphic to the monodromy group $G_U(VX)$ of $V(X)$ with respect to the forgetful functor U.

Finally, by Theorem 27.3, the monodromy group $G_U(VX)$ may be identified with the Zariski closure of $\Gamma(X)$ inside $\mathrm{GL}(VX)$, as claimed.

(b): By our assumptions, $G(X)$ is a closed algebraic subgroup of $\mathrm{GL}(VX)$, and $V(X)$ is semisimple as a representation of $G(X)$, since both X and V are semisimple. Since V is F'/F-fully faithful, $\mathrm{End}(VX) = F' \otimes_F \mathrm{End}(X)$, which is a separable F'-algebra since $\mathrm{End}(X)$ is a separable F-algebra. Therefore, the assumptions of Theorem 27.4 hold true, and $G(X)^\circ$ is a reductive group. ∴

28 Monodromy groups of A-motives

Theorem 28.1. *Let $\boldsymbol{M}$ be an A-isomotive over K, and choose a maximal prime $\mathfrak{p}$ of A. If K is finitely generated over its prime field and $\mathfrak{p}$ is not equal to the characteristic of K, then:*

(a) The image $\Gamma_{\boldsymbol{M}}$ of Γ_K in $\mathrm{Aut}_{F_\mathfrak{p}}(\mathrm{V}_\mathfrak{p}\,\boldsymbol{M})$ is Zariski-dense in the algebraic monodromy group $G_\mathfrak{p}(\boldsymbol{M})$ of $\boldsymbol{M}$.

(b) If $\boldsymbol{M}$ semisimple and $\mathrm{End}(\boldsymbol{M})$ is separable, then $G_\mathfrak{p}(\boldsymbol{M})^0$ is a reductive group over $F_\mathfrak{p}$.

Proof. This follows from Theorem 27.1, using Theorems 19.1 and 20.1 to show that $\mathrm{V}_\mathfrak{p}$ is $F_\mathfrak{p}/F$-fully faithful and semisimple. ∴

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