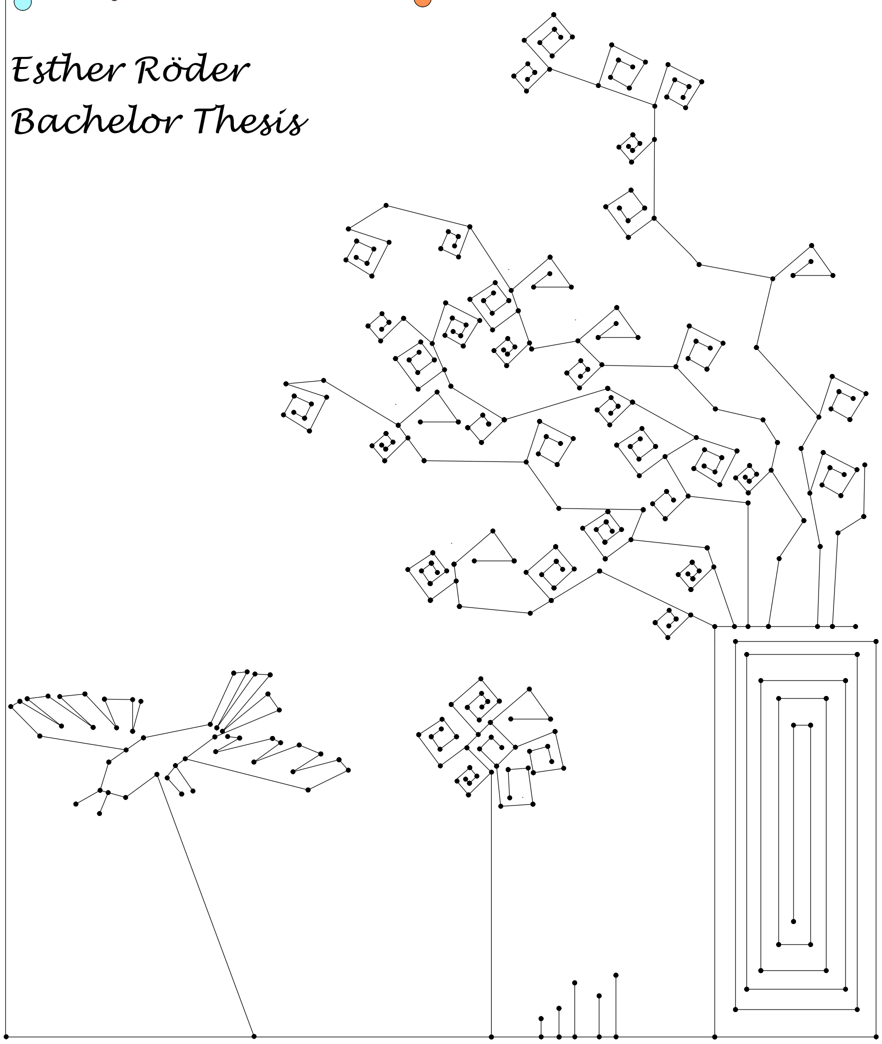
Dessins d´enfants - Trees

Esther Röder

Bachelor Thesis



Dessins d'enfants - trees

Bachelor Thesis of Esther Röder

Supervised by Prof. Richard Pink and Patrik Hubschmid

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Contents

0 Introduction

The absolut Galois group of $\mathbb{Q}$ is of big interest in mathematics. In his Esquisse d'un Programme, Alexander Grothendieck developed a correspondence between $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and special types of graphs drawn on surfaces, which he called dessin d'enfants. One of the most important result is that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the set of dessins in genus 1, on the set of dessins in genus 0 and even on the set of trees. This thesis eplains the proof of the faithfulness of the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the set of trees.
In section 1 we assemble the basics to show that there is a bijection between a class of holomorphic coverings of $\mathbb{P}^1\mathbb{C}$ and dessin d' enfants, namely the Grothendieck correspondence. Section 2 deals with Belyi's theorem and Belyi morphisms. Here we show that an algebraic curve A over $\mathbb{C}$ is defined over $\overline{\mathbb{Q}}$ if and only if there exists a holomorphic covering $p : A \to \mathbb{P}^1\mathbb{C}$ which is unramified outside $\{0, 1, \infty\}$. Such a covering is called a Belyi morphism. In section 3 we define dessin d'enfants and the cartographical group of a dessin. Here we will prove the Grothendieck correspondence. Section 4 deals with trees, which are embedded, connected graphs without cycles, on $\mathbb{P}^1\mathbb{C}$. We give an algorithm which computes a Belyi polynomial in $\overline{\mathbb{Q}}[z]$, whose ramification orders over 1 are all exactly 2, for every abstract tree. We will show that for every abstract tree $\mathscr{I}(T)$ there exists at least one Belyi polynomial β such that $(\beta^{-1}(0), \beta^{-1}[0,1], \mathbb{P}^1\mathbb{C}) \in \mathscr{I}(T)$. In general, there is no reason why this algorithm should give a Belyi polynomial defined over the smallest possible field. But in some cases it does. We found some of them, called them special trees, and prove the statement for them in section 5. Here we also define the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on abstract trees. In addition we give some invariants of this action, one of them is the conjugacy class of the cartographical group of a dessin. Yet, there are no invariants known which solve the problem of determining the Galois orbits of dessins d'enfants completely. For example the cartographical groups do not separate the trees from the orbit of *Leila's flower*[5]. In section 6 we will give two examples of the Galois orbits of abstract trees. In the first example we will obtain a set of abstract trees with two distinct Galois orbits, the second example gives a set of abstract dessins which are all contained in one Galois orbit.

Due to the proofs in this thesis, we are able to compute the degree of a Belyi polynomial which corresponds to the tree! on the cover sheet. We only have to sum up the number of incident edges of the black points . If we do so, we obtain that a Belyi polynomial corresponding to the dessin on the cover sheet has degree 772.

I would like to thank Prof. Richard Pink and Patrik Hubschmid for their supervision, which couldn't have been better.

1 About coverings

Theorem 1.1. [2, Theorem 2.1, page 10] *(Local behaviour of holomorphic maps) Suppose X and Y are Riemann surfaces and $f : X \to Y$ is a non-constant holomorphic map. Let $a \in X$ and $b := f(a)$. There exist an integer $k \geq 1$ and charts $\varphi : U \to V$ on X and $\psi : U' \to V'$ on Y with the following properties:*

(i) $a \in U, \varphi(a) = 0$; $b \in U', \psi(b) = 0$.

(ii) $f(U) \subset U'$.

(iii) The map $F := \psi \circ f \circ \varphi^{-1} : V \to V'$ is given by $F(z) = z^k$ for all $z \in V$.

We say f has **multiplicity** k at the point $a = \varphi^{-1}(0)$ and k is the multiplicity of a. We call a a **critical point** of f if its multiplicity is greater than one.

Definition 1.2. *Let x be a critical point of $f : X \to Y$, we call $f(x)$ a* ***critical value*** *of f.*

Definition 1.3. [2, Definition 4.11, page 24] *Suppose X and Y are topological spaces. A continuous map $p : X \to Y$ is called a* ***covering map*** *if the following holds. Every point $y \in Y$ has an open neighborhood U_y such that its preimage $p^{-1}(U_y)$ can be represented as*

$$p^{-1}(U_y) = \bigcup_{j \in J} V_j$$

where the $V_j, j \in J$, are disjoint subsets of X, and all the maps $p_{|V_j} : V_j \to U_y$ are homeomorphisms. In particular, p is a local homeomorphism.

Theorem 1.4. [2, Theorem 4.16, page 26] *Suppose X and Y are Hausdorff spaces with Y path-connected and $p : X \to Y$ is a covering map. Then for any two points $y_0, y_1 \in Y$ the sets $p^{-1}(y_0)$ and $p^{-1}(y_1)$ have the same cardinality.*

The common cardinality of $p^{-1}(y)$ for $y \in Y$ is called the **number of sheets** of the covering.

1.1 Ramified coverings of Riemann surfaces

Definition 1.5. [2, Definition 4.3, page 21] *Suppose X and Y are Riemann surfaces and $p : X \to Y$ is a non-constant holomorphic map. A point $x \in X$ is called a* ***ramification point of p****, if there is no neighborhood U of x such that $p|_U$ is injective. We say p is ramified over $p(x)$. The map p is called unramified if it has no ramification points.*

Theorem 1.1 implies

Corollary 1.6. *A point x is a ramification point of p if and only if the multiplicity k of x is greater than 1.*

This is equivalent to:

Corollary 1.7. *$\{x \in X|$ x is a ramification point of $p\}$=$\{$critical points of $p\}$*

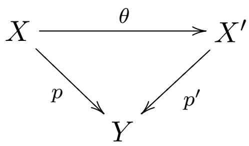
Definition 1.8. [2, Definition 4.20, page 28] *A continuous map $f : X \to Y$ between two locally compact spaces is called **proper** if the preimage of every compact set is compact. This is always the case if X is compact.*

Theorem and Definition 1.9. [2, Theorem and Definition 4.23, page 29] *Suppose X and Y are Riemann surfaces and $f : X \to Y$ is a proper, non-constant, holomorphic map. It follows from Theorem 1.1 that the set A of ramification points of f is closed and discrete. Since f is proper, the set $B := f(A)$ of critical values is also closed and discrete. Let $Y' := Y \backslash B$ and $X' := X \backslash f^{-1}(B) \subset X \backslash A$. Then $f_{|X'} : X' \to Y'$ is a proper unramified holomorphic covering. If A is non empty then we call $f : X \to Y$ a **ramified holomorphic covering**.*

1.2 Classification of coverings of a connected Riemann surface

Let X, X' and Y be compact connected Riemann surfaces.

Definition 1.10. *Two proper holomorphic coverings $p : X \to Y$ and $p' : X' \to Y$ are called **isomorphic coverings**, if there exists a biholomorphic map $\theta : X \to X'$ which is fiber-preserving, so the following diagram has to be commutative:*



Let B be a closed discrete subset of Y. Let $p : X \to Y$ and $p' : X' \to Y$ be proper holomorphic coverings ramified over a subset of B. Choose a basepoint y_0 in $Y \backslash B$.

Theorem and Definition 1.11. [6, Theorem and Definition 6.3.1, page 155, adapted to ramified coverings] *Let x_0 be in the fiber of y_0. The unramified covering*

$$p : (X \backslash p^{-1}(B), x_0) \to (Y \backslash B, y_0)$$

induces an injective group homomorphism

$$p_* : \pi_1(X \backslash p^{-1}(B), x_0) \to \pi_1(Y \backslash B, y_0).$$

*The subgroup $p_* \pi_1(X \backslash p^{-1}(B), x_0)$ is called the **characteristic subgroup in $\pi_1(Y \backslash B, y_0)$ of the covering** $p : X \to Y$, corresponding to the point x_0.The index of $p_* \pi_1(X \backslash p^{-1}(B), x_0)$ in $\pi_1(Y \backslash B, y_0)$ is equal to the number of sheets of the covering p.*

Theorem and Definition 1.12. [6, Theorem and Definition 6.3.3, page 156, adapted to ramified coverings] *Let $p : X \to Y$ be a holomorphic covering ramified over a subset of B. Then $\mathcal{C}(X,p)_B := \{p_* \pi_1(X \backslash p^{-1}(B), x_0) \mid x_0 \in p^{-1}(y_0)\}$ is a class of conjugated subgroups of $\pi_1(Y \backslash B, y_0)$; it is called the **characteristic conjugation class in $\pi_1(Y \backslash B, y_0)$ of the covering** $p : X \to Y$.*

Theorem 1.13. [6, Theorem 6.3.4, page 155, adapted to ramified coverings using analytic continuation] *(criteria of isomorphism) Two proper holomorphic coverings* $p : X \to Y$ *and* $p' : X' \to Y$*, both ramified over a subset of* B*, are isomorphic if and only if* $\mathcal{C}(X,p)_B$ *and* $\mathcal{C}(X',p')_B$ *coincide in* $\pi_1(Y \backslash B, y_0)$*.*

Theorem 1.14. [6, Theorem 6.6.2, page 165, adapted to ramified coverings] *If we assign to any holomorphic covering* $p : X \to Y$*, ramified over a subset of* B *its characteristic conjugation class* $\mathcal{C}(X,p)_B$ *in* $\pi_1(Y \backslash B, y_0)$*, we obtain a bijection between the set of isomorphism classes of holomorphic coverings of* Y *which are ramified over a subset of* B *and the set of conjugation classes in* $\pi_1(Y \backslash B, y_0)$ *of subgroups of* $\pi_1(Y \backslash B, y_0)$*.*

2 Belyi's theorem and Belyi morphism

2.1 Belyi's theorem

Choose a basepoint y_0 in $\mathbb{P}^1\mathbb{C}\backslash\{0,1,\infty\}$.

Proposition 2.1. $\pi_1 := \pi_1(\mathbb{P}^1\mathbb{C}\backslash\{0,1,\infty\}, y_0) = \langle l_0, l_1, l_\infty \mid l_0 l_1 l_\infty = 1\rangle \cong F_2$*, where* l_i *is the loop around* i *and* F_2 *is the free group with two generators.*

Theorem 2.2. [5, Theorem 1.2, page 49] *Let* X *be a smooth projective algebraic curve defined over* $\mathbb{C}$*. Then* X *is defined over* $\bar{\mathbb{Q}}$ *if and only if there exists a non-constant holomorphic function* $f : X \to \mathbb{P}^1\mathbb{C}$ *such that all critical values of* f *lie in the set* $\{0,1,\infty\}$*.*

2.2 Unramified coverings of $\mathbb{P}^1\mathbb{C}\backslash\{0,1,\infty\}$ and their extensions to $\mathbb{P}^1\mathbb{C}$

Theorem 2.3. [2, Theorem 8.4, page 51] *Suppose* Y *is a Riemann surface,* $A \subset Y$ *is a closed discrete subset and let* $Y' = Y \backslash A$*. Suppose* X' *is another Riemann suface and* $p' : X' \to Y'$ *is a proper unramified holomorphic covering. Then* p' *extends to a possibly ramified covering of* Y *i.e., there exists a Riemann surface* X *and a proper holomorphic map* $p : X \to Y$ *and a fiber-preserving biholomorphic map* $\theta : X \backslash p^{-1}(A) \to X'$*.*

Proposition 2.4. [1] *Let* X *be a smooth connected projective algebraic curve defined over the field* $\mathbb{C}$*. Then* X *is a compact connected Riemann surface. The converse is also true: Any compact connected Riemann surface is obtained from some smooth connected projective algebraic curve.*

Proposition 2.5. [5, Proof of Lemma 1.1, page 48] *Let* $f : X \to \mathbb{P}^1\mathbb{C}\backslash\{0,1,\infty\}$ *be a proper unramified holomorphic cover and let* G *be an element of the corresponding conjugacy class of subgroup of* π_1*. Let* $f' : X' \to \mathbb{P}^1\mathbb{C}$ *be the extended covering of* $\mathbb{P}^1\mathbb{C}$*, possibly ramified only over 0, 1 and* ∞*. Then the ramification indices over 0, 1 and* ∞ *are the lengths of the orbits in* π_1/G *under the action of* l_0, l_1 *and* l_∞ *respectively by left multiplication.*

Let L_1 be the minimal normal subgroup of π_1 which contains $\langle {l_1}^2 \rangle$. Let $\pi_1' := \pi_1/L_1$. As a consequence of Theorem 1.14 and Prop. 2.5 we have

Corollary 2.6. [5, Corollary (to Lemma 1.1), page 48] *There is a bijection between the set of conjugacy classes of subgroups of* π_1' *of finite index and the set of isomorphism classes of holomorphic coverings of* $\mathbb{P}^1\mathbb{C}$*, ramified only over* 0*,* 1 *and* ∞*, such that the ramification over* 1 *is of degree at most* 2*.*

2.3 Belyi morphism

Let X be a compact connected Riemann surface or equivalently a smooth connected projective algebraic curve defined over $\mathbb{C}$.

Definition 2.7. [5, Definition 1, page 50] *A non-constant holomorphic map* $\beta : X \to \mathbb{P}^1\mathbb{C}$ *all of whose critical values lie in* $\{0, 1, \infty\}$ *is called a* ***Belyi morphism****. We call* β *a* ***pre-clean*** *Belyi morphism if all the ramification degrees over* 1 *are less than or equal to* 2 *and* ***clean*** *if they are all exactly* 2*.*

Corollary 2.8. [5, Corollary (to Lemma 1.2), page 50]*A smooth algebraic curve* X *defined over* $\mathbb{C}$ *is defined over* $\overline{\mathbb{Q}}$ *if and only if there exists a clean Belyi morphism* $\beta : X \to \mathbb{P}^1\mathbb{C}$*.*

Proof. From Theorem 2.2 we know that X is defined over $\overline{\mathbb{Q}}$ if and only if there exists a Belyi morphism $\alpha : X \to \mathbb{P}^1\mathbb{C}$. Let $\alpha : X \to \mathbb{P}^1\mathbb{C}$ be a Belyi morphism, we will show that $\beta = 4\alpha(1-\alpha)$ is a clean one.
Let α' and β' be the derivation of α respective β. Let x be a point in X, then the following equation holds:

$$\beta'(x) = 4\left(\alpha'(x)\left(1-\alpha(x)\right) - \alpha(x)\alpha'(x)\right) = 4\alpha'(x)(1-2\alpha(x)).$$

We conclude that x is a critical point of β if and only if x is a critical point of α or $\alpha(x) = 1/2$. We see that x is a zero of β if and only if x is a zero of α or $(1-\alpha)$. If $\alpha(x) = \infty$ then $\beta(x) = \infty$. By construction of β we know that $\beta(x) = 1$ if and only if $\alpha(x) = 1/2$. Hence all critical values of β are contained in the set $\{0, 1, \infty\}$ and the ramification order of β over 1 is 2. Therefore β is a clean Belyi morphism. □

If X is a smooth algebraic curve defined over $\bar{\mathbb{Q}}$ and β is a Belyi morphism on it, we call the couple (X, β) a **Belyi pair**. Two Belyi pairs (X, β) and (Y, γ) are said to be isomorphic if there is a biholomorphic map $\theta : X \to Y$ such that $\beta = \gamma \circ \theta$. We call (X, β) a **pre-clean** Belyi pair respective **clean** Belyi pair if β is a pre-clean resp. clean Belyi morphism.We denote the isomorphism class of a Belyi pair (X, β) by $\mathscr{I}(X, \beta)$.

3 Dessin d'enfants

Definition 3.1. *Let* X_1 *be a union of one-cells,* $X_0 \subset X_1$ *a subset of* X_1 *consisting only zero-cells. A* ***line segment*** *in* $X_1 \backslash X_0$ *is a connected component of* $X_1 \backslash X_0$ *which is either the homeomorphic image of the open unit interval* $(0, 1) \subset \mathbb{R}$ *or the half open unit interval* $(0, 1] \subset \mathbb{R}$*.*

Definition 3.2. [5, Definition 2, page 50] *A* ***pre-clean Grothendieck dessin*** *is a triple $X_0 \subset X_1 \subset X_2$ where X_2 is a compact connected Riemann surface, X_0 is a non-empty finite set of points, $X_1 \backslash X_0$ is a finite disjoint union of line segments and $X_2 \backslash X_1$ is a finite disjoint union of open cells.*

The last condition and the requirement on X_2 to be connected implies that X_1 is connected. In the following we will always mean pre-clean Grothendieck dessins when we talk about dessins. We denote a dessin by $D = (X_0, X_1, X_2)$.

Definition 3.3. [5, Definition 3, page 50] *Two dessins $D = (X_0, X_1, X_2)$ and $D' = (X'_0, X'_1, X'_2)$ are* ***isomorphic*** *if there exists a homeomorphism γ from X_2 into X'_2 inducing a homeomorphism from X_1 to X'_1 and from X_0 to X'_0. We say γ is an* ***isomorphism between D and D'****. An isomorphism class of dessins is called an* ***abstract dessin****. We denote the isomorphism class of D by $\mathscr{I}(D)$.*

Definition 3.4. *The* ***valency*** *of a vertex $x_0 \in X_0$ is the number of incident edges.*

Definition 3.5. *A dessin $D = (X_0, X_1, X_2)$ is called* ***clean****, if every line segment in $X_1 \backslash X_0$ is open, i.e., they are all homeomorphic to $(0, 1) \subset \mathbb{R}$.*

3.1 The cartographical group of a dessin

Let $D = (X_0, X_1, X_2)$ be a dessin, we mark D according to the following description. Considering a line segment x_1 in $X_1 \backslash X_0$ which is homeomorphic to $(0, 1) \subset \mathbb{R}$ via some homeomorphism γ_{x_1}, then we mark $\gamma_{x_1}^{-1}(1/2)$ with $\circ$ and consider the new edges $\gamma_{x_1}^{-1}(0, 1/2) \subset X_1 \backslash X_0$ and $\gamma_{x_1}^{-1}(1/2, 1) \subset X_1 \backslash X_0$. Since there are several homeomorphism from x_1 to $(0, 1)$ the marking is not unique, but we always choose a point in between the endpoints of the closure of x_1.
If x_1 is a line segment homeomorphic to $(0, 1]$ via a homeomorphism γ_{x_1}, then we mark $\gamma_{x_1}^{-1}(1)$ with $\circ$ and consider the new edge $\gamma_{x_1}^{-1}(0, 1) \subset X_1 \backslash X_0$. So we mark the endpoint of the closure of x_1 which is not contained in X_0.

We use the notation $\bullet$ for points in X_0 and $\circ$ for marked points in $X_1 \backslash X_0$. Enumerate the new edges with integers $1, \ldots, n$. Therefore every edge has a $\bullet$-point and a $\circ$-point as endpoints. We define $\deg(D) := n$, this is well defined.

Let x_{0_i} be a vertex in X_0. Due to our numbering above all edges which incident with x_{0_i} are labeled with a unique number of the set $\{1, \ldots, n\}$. We define s_i to be the cycle $(k_{i1} \ldots\ k_{il_i})$, which we obtain by rotating all incident edges of x_{0_i} counterclockwise. Because of the marking above we can see that an edge k_i, which incidents with a vertex x_{0_i}, does not incident with any other $x_{0_l} \in X_0$. Therefore, for $i \neq l$, the cycles s_i and s_l are disjoint. We define δ to be the composition of the disjoint cycles $s_1 \ldots s_j$ in S_n, where j is the number of $\bullet$ points. We define σ similarly for the $\circ$ vertices.

Definition 3.6. *The* ***cartographical group*** *G_D is the subgroup of S_n generated by δ and σ.*

By construction, the cartographical group of a dessin is not unique, as it depends on the numeration of the new line segments of the dessin. Therefore it is defined up to simultaneous conjugation of the generators σ and δ by an element $g \in S_n$.

Here are some examples:

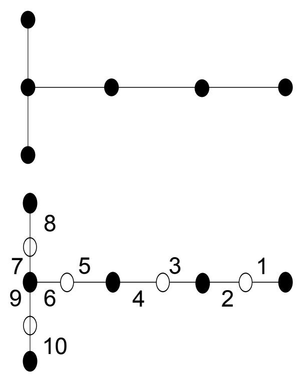


Figure 1: a clean dessin before and after marking

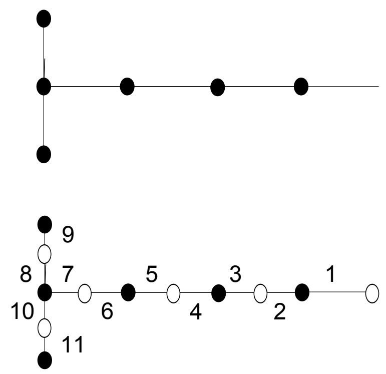


Figure 2: Another clean dessin before and after marking. Its cartographical group is $\langle(12)(34)(56)(7\ 8\ 10), (23)(45)(67)(89)(10\ 11)\rangle$.

Observe that the cycle decomposition of σ consists only of transpositions. The number of fixed points of σ is equal to the number of semi closed line segments of D. So, if D is a clean dessin then there is no fix point in σ. The cartographical group G_D of a dessin D of degree n is a transitive subgroup of S_n, as X_1 is connected.

Lemma 3.7. *Let D and D' be two isomorphic dessins of degree n with cartographical groups $G_D = \langle \delta, \sigma \rangle$ respectively $G_{D'} = \langle \delta', \sigma' \rangle$, then $\delta' = g \circ \delta \circ g^{-1}$ and $\sigma' = g \circ \sigma \circ\ g^{-1}$, for some $g \in S_n$.*

Definition 3.8. *Let D be a dessin and $\mathscr{I}(D)$ its isomorphism class. We define the **class of cartographical groups** of $\mathscr{I}(D)$ by $\mathscr{C}(\mathscr{I}(D)) = \{G_{D'} \mid D' \in \mathscr{I}(D)\}$.*

Observe that the class of cartographical groups of an abstract dessin of degree n is a conjugacy class of subgroups in S_n.

3.2 Scheduling the Grothendieck correspondence

With Theorem1.14 and Theorem 2.3 follow:

Theorem 3.9. [4, 1.8.14, page 74] *Let $B \subset \mathbb{P}^1\mathbb{C}$ be a finite subset. Choose a basepoint y_0 in $\mathbb{P}^1\mathbb{C}\backslash B$. There is a natural bijection between the following sets:*

(1) *Isomorphism classes of possibly ramified holomorphic n-sheeted coverings $f : C \to \mathbb{P}^1\mathbb{C}$, where C is a compact connected Riemann surface, such that the set of*

critical values of f is contained in B.

(2) *Conjugacy classes of homomorphisms $\theta : \pi_1(\mathbb{P}^1\mathbb{C}\backslash B, y_0) \to S_n$ such that the image of θ is transitive.*

Here we say that
$\theta : \pi_1(\mathbb{P}^1\mathbb{C}\backslash B, y_0) \to S_n$ and $\theta' : \pi_1(\mathbb{P}^1\mathbb{C}\backslash B, y_0) \to S_n$ are conjugated, if there is some $\tau \in S_n$ such that $\theta(\gamma) = \tau\theta'(\gamma)\tau^{-1}$ for all $\gamma \in \pi_1(\mathbb{P}^1\mathbb{C}\backslash B, y_0)$.

We can describe this assignement more precisely. Let $\mathscr{I}(C, f)$ be an isomorphism class of n-sheeted holomorphic coverings, such that C is a compact connected Riemann surface and all critical values of f are contained in B. We consider $\mathcal{C}(C, f)_B$ and for every element $G \in \mathcal{C}(C, f)_B$ there is a conjugacy class of homomorphisms $\theta_G : \pi_1(\mathbb{P}^1\mathbb{C}\backslash B, y_0) \to S_n$ obtained by left multiplication of $\pi_1(\mathbb{P}^1\mathbb{C}\backslash B, y_0)$ on the left cosets of G. The bijection is now given by assigning $\mathscr{I}(C, f)$ to the conjugacy class of θ_G.
Let B be the set $\{x_1, \ldots, x_k\}$ where $x_i \in \mathbb{P}^1\mathbb{C}$ and $l_i \in \pi_1(\mathbb{P}^1\mathbb{C}\backslash B, y_0)$ is the loop around x_i. For any two conjugate homomorphisms θ and θ' we conclude that $\mathrm{im}(\theta) = \langle\theta(l_1), \ldots, \theta(l_k)\rangle$ and $\mathrm{im}(\theta') = \langle\theta'(l_1), \ldots, \theta'(l_k)\rangle$ are generated by simultaneously conjugate elements.

Definition 3.10. *The* ***class of monodromy groups*** *of $\mathscr{I}(C, f)$ is the set of the images of the conjugacy class of homeomorphisms which is assigned to $\mathscr{I}(C, f)$ by the bijection above.*

Recall that $\pi_1' = \pi_1/L_1$. Let $f : C \to \mathbb{P}^1\mathbb{C}$ be a holomorphic covering of degree n of $\mathbb{P}^1\mathbb{C}$, where C is a compact connected Riemann surface. Suppose that f is unramified outside $\{0, 1, \infty\}$ and the ramification orders over 1 are at most 2. For $G \in \mathcal{C}(C, f)_{\{0,1,\infty\}}$ let $\theta_G : \pi_1 \to S_n$ be the homomorphism, given by the left multiplication of π_1 on the left cosets of G. As the ramification order of f over 1 is less than or equa tol 2, we know that $\langle l_1^2\rangle \subset \ker(\theta_G)$ and θ_G factors through π_1'. On the other hand, if θ_G factors through π_1', we know that ${l_1}^2 \in \ker(\theta_G)$. Therefore, by Proposition 2.5 we know that the ramification orders of f over 1 are at most 2.

Corollary 3.11. *There is a bijection between*

(1) *Isomorphism classes of n-sheeted holomorphic maps $f : C \to \mathbb{P}^1\mathbb{C}$, where C is a compact connected Riemann surface, such that the set of critical values of f is contained in $\{0, 1, \infty\}$ and the ramification order over 1 is at most two.*

(2) *Conjugacy classes of homomorphisms $\theta : \pi_1' \to S_n$ whose image is transitive.*

3.3 The Grothendieck correspondence

Theorem 3.12. *There is a bijection between abstract pre-clean dessins and isomorphism classes of pre-clean Belyi pairs.*

Proof. We define

$$\Lambda : \{\mathscr{I}(X,\beta) \mid (X,\beta) \text{ pre-clean Belyi pair}\} \to \{\mathscr{I}(D) \mid D \text{ pre-clean dessin}\}$$

by

$$\Lambda(\mathscr{I}(X,\beta)) := \{(\beta'^{-1}(0), \beta'^{-1}[0,1], X') \mid (X',\beta') \in \mathscr{I}(X,\beta)\}$$

which is well defined. Let D' and D'' be two dessins which are contained in $\Lambda(\mathscr{I}(X,\beta))$, hence they correspond to isomorphic Belyi pairs (X',β') resp. (X'',β''). Therefore there exists an isomorphism $\theta : X \to X'$ such that $\beta' = \beta'' \circ \theta$. Hence D' and D'' are isomorphic via θ. Observe: By construction the class of cartographical groups corresponding to $\Lambda(\mathscr{I}(X,\beta))$ is the class of monodromy groups of $\mathscr{I}(X,\beta)$. Suppose there exist two distinct isomorphism classes of Belyi pairs $\mathscr{I}(X,\beta)$, $\mathscr{I}(X',\beta')$ such that $\Lambda(\mathscr{I}(X,\beta)) = \Lambda(\mathscr{I}(X',\beta'))$. Corollary 3.11 and the observation above tell us that $\mathscr{I}(X,\beta)$ has to be $\mathscr{I}(X',\beta')$. Hence Λ is injective.
Surjectivity follows by considering a dessin $D \in \mathscr{I}(D)$ and the class of groups, generated by the simultaneuous conjugation of its cartographical group. Corollary 3.11 gives us an isomorphism class of pre-clean Belyi pairs $\mathscr{I}(X_D,\beta_D)$ having class of monodromy groups the latter ones. $\mathscr{I}(D)$ will be the image of $\mathscr{I}(X_D,\beta_D)$ under Λ. □

Corollary 3.13. *There is a bijection between abstract clean dessins and isomorphism class of clean Belyi pairs.*

Proof. We show that the bijection restricted on isomorphism class of clean Belyi pairs gives a bijection between abstract clean dessins and isomorphism class of clean Belyi pairs. Let $\mathscr{I}(X,\beta)$ be an isomorphism class of clean Belyi pairs of degree n. Therefore the ramification orders over 1 are all exactly 2. Let M be in the class of monodromy groups of $\mathscr{I}(X,\beta)$ such that $M = im(\theta)$ for some homomorphism $\theta : \pi_1' \to S_n$ in the corresponding class of homomorphisms. Hence $M = \langle \theta(l_0 L_1), \theta(l_1 L_1) \rangle$. Since $\mathscr{I}(X,\beta)$ is an isomorphism class of clean Belyi pairs, the cycle decomposition of $\theta(l_1/L_1)$ is the composition of $n/2$ transpositions. Hence $\Lambda(\mathscr{I}(X,\beta))$ is an isomorphism class of clean dessins. On the other hand, if $\mathscr{I}(D)$ is an abstract clean dessin we can see that every element in the class of monodromy groups of the corresponding ismorphism class of Belyi pairs is generated by some elements $g_0 := \theta(l_0 L_1)$ and $g_1 := \theta(l_1 L_1)$ such that θ is in the corresponding class of homomorphism between π_1' and S_n. The assumption of $\mathscr{I}(D)$ to be a clean abstract dessin gives us that the cycle decomposition of g_1 is the composition of $n/2$ transpositions. But as the lengths of cycles in g_1 are exactly the ramification orders over 1 of elements in $\Lambda^{-1}(\mathscr{I}(D))$, we see that $\Lambda^{-1}(\mathscr{I}(D))$ is an isomorphism class of clean Belyi pairs. □

Note: Let (X,β) be a Belyi pair and $D = (\beta^{-1}(0), \beta^{-1}[0,1], X)$ be the corresponding dessin. Then $\beta^{-1}(\infty)$ gives a point in every open cell of $X \backslash \beta^{-1}[0,1]$.

A similar approach to the Grothendieck correspondence can be found in [4]

4 Trees

Definition 4.1. *A **tree** is a clean dessin $T = (X_0, X_1, X_2)$ such that $X_2 \backslash X_1$ consists of exactly one open cell.*

Therefore the eqivalence class of a tree is an abstract tree and not a tree in terms of graph theory.
Denote by v the number of elements in X_0 and define e to be the number of the old line segments in $X_1 \backslash X_0$. We denote the number of open cells in $X_2 \backslash X_1$ by o. As T is a tree, the relation $e = v - 1$ holds. If we denote the genus of X_2 by g and consider the Euler charcteristic, we get the equation

$$2g - 2 = e - v - o = (v - 1) - v - 1 = -2$$

which tells us that the genus g of X_2 is zero.

Remark: Let $(\mathbb{P}^1\mathbb{C}, \beta)$ be a Belyi pair, then $(\mathbb{P}^1\mathbb{C}, \beta')$ is in the same isomorphism class if and only if there exists a $\sigma \in PSL_2(\mathbb{C})$ so that $\beta = \beta' \circ \sigma$. Remark that for genus $g = 0$, up to biholomorphic mappings, there exists only one compact connected Riemann surface, namely $\mathbb{P}^1\mathbb{C}$. Let $\mathscr{I}(T)$ be an abstract tree and $\mathscr{I}(\mathbb{P}^1\mathbb{C}, \beta)$ the corresponding Belyi pair, such that $T = (\beta^{-1}(0), \beta^{-1}[0, 1], \mathbb{P}^1\mathbb{C})$. From the remark above that $\beta^{-1}(\infty)$ gives a point in each open cell of the tree T, we see that $\beta^{-1}(\infty)$ consists of only one point. Precombining β with a suitable transformation in PSL_2, sending $\beta^{-1}(\infty)$ to ∞ and sending the set $\{0, 1\}$ onto itself, we see that $\mathscr{I}(T) = \Lambda(\mathbb{P}^1\mathbb{C}, \beta')$, where β' is a polynomial which has two finite critical values, namely 0 and 1.
The aim of the following section is to show that for every abstract tree $\mathscr{I}(T)$ there exists a clean Belyi polynomial $\beta : \mathbb{P}^1\mathbb{C} \to \mathbb{P}^1\mathbb{C} \in \overline{\mathbb{Q}}[z]$ such that $\Lambda(\mathscr{I}(\mathbb{P}^1\mathbb{C}, \beta)) = \mathscr{I}(T)$.

4.1 Chebyshev polynomials and clean Belyi polynomials

Definition 4.2. [5, Definition 10, page 67] *A polynomial $P \in \mathbb{C}[z]$ is called a **generalized Chebyshev polynomial** if there exist c_1 and $c_2 \in \mathbb{C}$ such that for all z_0 such that $P'(z_0) = 0$ we have either $P(z_0) = c_1$ or $P(z_0) = c_2$, i.e. P has at most 2 critical values. If the critical values of P are exactly ± 1 we say that P is **normalized**.*

Lemma 4.3. [5, Lemma 3.4, page 67]
(i) Let $P(z)$ be a normalized generalized Chebyshew polynomial, and let $\beta(z) = 1 - P(z)^2$. Then $\beta(z)$ is a clean Belyi polynomial and the dessin given by $D = \left(\beta^{-1}(0), \beta^{-1}[0, 1], \mathbb{P}^1\mathbb{C}\right)$ is a tree with ∞ in its open cell.

(ii) Let $\mathscr{I}(T)$ be an abstract tree. Then there is a normalized generalized Chebyshev polynomial $P(z)$ such that $\mathscr{I}(T) = \Lambda(\mathscr{I}(\mathbb{P}^1\mathbb{C}, \beta))$ for $\beta(z) = 1 - P(z)^2$.

Proof. (*i*) If $\beta(z) = 1 - P(z)^2$ then $\beta'(z) = -2P(z)P'(z)$. Hence z_0 is a critical point of β if and only if z_0 is a root or a critical point of P. We see that $\beta(z) = 1$ if and only if $P(z) = 0$. Since P is normalized, we conclude that 0 is no critical value of P. Therefore the ramification order of β over 1 is 2. From assuming P to be a normalized generalized

Chebyhev polynomial, we see that all critical values of β are contained in the set $\{0,1\}$. Thus β is a clean Belyi function and since it has only one pole $\beta^{-1}([0,1])$ must be a tree.

(ii) By Theorem 3.12 and Corollary 3.13 we know that for a given abstract tree $\mathscr{I}(T)$ there exists a Belyi pair $(\mathbb{P}^1\mathbb{C},\beta)$ such that $\Lambda(\mathbb{P}^1\mathbb{C},\beta) = \mathscr{I}(T)$. Since $\mathscr{I}(T)$ is an abstract tree, β has only one pole. We may suppose that this pole is at ∞, so β is a Belyi polynomial whose only critical values are at 0 and 1. Moreover because we assume that β is clean, we must have $\beta(z)-1 = cQ(z)^2$ for some constant c and some polynomial Q having distinct roots. The critical points of β are the roots and critical points of Q. Moreover β can only have 0 and 1 as critical values and 1 can only occur at the roots of Q, so at a critical point z_0 of Q which is not a root we must have $1 + cQ(z_0)^2 = 0$ so $Q(z_0) = \pm\sqrt{-1/c}$. Set $P(z) = \sqrt{-c}Q(z)$. Then $\beta(z) = 1 - P(z)^2$ and the critical values of P are ± 1.

□

4.2 Construction of clean Belyi polynomials corresponding to valency lists

For further information on this subsection, see [5, page 68-69]

Let $\mathscr{I}(T)$ be the isomorphism class of a tree $T = (X_0, X_1, X_2)$, such that X_0 contains at least three elements. A bipartite structure on a tree is the assignment of $*$ or $**$ to each vertex in such a way that every edge connects a $*$ vertex with a $**$ vertex. The bipartite structure is unique up to global change of $*$ with $**$. Let n be the highest valency of any $*$ vertex and m the highest valency of any $**$ vertex. Let $V^* = (u_1, \dots, u_n)$ be the $*$ valency list of T, where u_i is the number of $*$ vertices having valency i. Let $V^{**} = (v_1, \dots, v_m)$ be the $**$ valency list of T, so v_j is the number of $**$ vertices having valency j. Every tree $T' \in \mathscr{I}(T)$ has valency lists V^* and V^{**}. Hence we say V^*, V^{**} are the valency lists of $\mathscr{I}(T)$.

Definition 4.4. *Let $\mathscr{I}(T)$ be an isomorphism class of trees, $V^* = (u_1, \dots, u_n)$ and $V^{**} = (v_1, \dots, v_m)$ its valency lists. If $u_i \leq 1$ for all $i = 1, \dots, n$ and in addition there exist distinct integers i_0 and i_1 such that $u_{i_0} = u_{i_1} = 1$ and $v_{i_0} = v_{i_1} = 0$, then we say $\mathscr{I}(T)$ is a* ***special abstract tree****. We call V^* and V^{**}* ***special valency lists****.*

Let $V^* = (u_1, \dots, u_n)$ and $V^{**} = (v_1, \dots, v_m)$ be the valency lists of an abstract tree $\mathscr{I}(T)$.
For $1 \leq i \leq n$ set

$$\tilde{P}_i(z) := z^{u_i} + C_{i,u_i-1}z^{u_i-1} + \dots + C_{i,1}z + C_{i,0}$$

and for $1 \leq j \leq m$ set

$$\tilde{Q}_j(z) := z^{v_j} + D_{j,v_j-1}z^{v_j-1} + \dots + D_{j,1}z + D_{j,0}$$

where the $C_{i,k}$ and the $D_{j,l}$ are indeterminates. The set $\{\tilde{P}_i, \tilde{Q}_j\}$ only depends on the valency lists V^* and V^{**} and therefore corresponds to a finite number of abstract trees as there are only finitely many disjoint abstract trees having valency lists V^* and V^{**}. We assumed that the isomorphism classes of trees which correspond to V^*, V^{**} have at least three vertices. Therefore either $\sum_{j=1}^{m} v_j \geq 2$ or $\sum_{i=1}^{n} u_i \geq 2$. Without loss of generality we can assume $\sum_{i=1}^{n} u_i \geq 2$ otherwise we change $*$ and $**$.
Case (1): If there exists some $i_0 \in \{1, \ldots, n\}$ such that $u_{i_0} \geq 2$ choose one of them and set

$$C_{i_0,0} := 0, \qquad C_{i_0,1} := 1.$$

For all $1 \leq i \leq n$ set $P_i(z) := \tilde{P}_i(z)$ and for $1 \leq j \leq m$ set $Q_j(z) := \tilde{Q}_j(z)$.
Case(2): If all $u_i \leq 1$, choose two distinct elements i_0 and i_1 of the set $\{1, \ldots, n\}$ such that $u_{i_0} = u_{i_1} = 1$ and set

$$P_{i_0}(z) := \tilde{P_{i_0}}(z) - C_{i_0,0} = z, \qquad P_{i_1}(z) := \tilde{P_{i_1}}(z) - C_{i_1,0} + 1 = z + 1.$$

For all $i \neq i_0, i_1$ set $P_i(z) := \tilde{P}_i(z)$ and for $1 \leq j \leq m$ set $Q_j(z) := \tilde{Q}_j(z)$. If $\mathscr{I}(T)$ is a special abstract tree, choose i_0, i_1 such that in addition $v_{i_0} = v_{i_1} = 0$.

Theorem 4.5. *Let $\mathscr{I}(T)$ be an abstract tree and $V^* = (u_1, \ldots, u_n)$ and $V^{**} = (v_1, \ldots, v_m)$ the corresponding valency lists. Set*

$$P(z) := \prod_{j=1}^{m} Q_j(z)^j \tag{1}$$

and let $S_{V^*,V^{**}}$ be the set of polynomial equations obtained by comparing coefficients on both sides of the following equation:

$$P(z) - P(0) = \prod_{i=1}^{n} P_i(z)^i \tag{2}$$

under the side condition

$$P(0) \neq 0. \tag{3}$$

We have:
(i) For each solution s of the system $S_{V^*,V^{**}}$, let $R_s(z)$ be the normalized generalized Chebyshev polynomial given by replacing the indeterminates in the polynomial $\frac{2}{P(0)}P(z) - 1$ by the values of s. Set $\beta_s(z) = 1 - R_s(z)^2$. Then $\beta_s(z)$ is a clean Belyi polynomial.

(ii)Let $\mathscr{I}(T) = \Lambda(\mathscr{I}(\mathbb{P}^1\mathbb{C}, \beta_s))$ for some solution s of $S_{V^*,V^{**}}$, then $\mathscr{I}(T)$ has valency lists V^* and V^{**}.

(iii) For every abstract tree $\mathscr{I}(T)$ with valency lists V^* and V^{**} there is at least one solution s in $S_{V^*,V^{**}}$ such that $\mathscr{I}(T) = \Lambda(\mathscr{I}(\mathbb{P}^1\mathbb{C}, \beta_s))$.

(iv) The system $S_{V^*,V^{**}}$ admits only a finite number of solutions, all defined over $\overline{\mathbb{Q}}$. In particular, all the $\beta_s(z)$ are defined over $\overline{\mathbb{Q}}$.

In the proof below P_s, $P_{i,s}$ and $Q_{j,s}$ are the polynomials obtained by replacing the indeterminates in P, P_i and Q_j by a solution s of $S_{V^*,V^{**}}$.

Proof. (i) First of all we have to show that for each solution s of $S_{V^*,V^{**}}$, the polynomial $R_s(z) := \frac{2}{P_s(0)}P_s(z) - 1$ is a normalized, generalized Chebyshev polynomial. Hence we have to show that if z_0 is a critical point of R_s, then $R_s(z_0) = \pm 1$. As

$$\frac{d}{dz}R(z) = \frac{2}{P_s(0)}\left(\frac{d}{dz}P(z)\right)$$

the relations

$$\frac{d}{dz}R_s(z) = 0 \Leftrightarrow \frac{d}{dz}P_s(z) = 0 \Leftrightarrow \frac{d}{dz}\left(P_s(z) - P(0)\right) = 0$$

hold. By the construction of the valency lists, we know that $\sum_{i=1}^{n} u_i + \sum_{j=1}^{m} v_j =: v$ is the number of vertices and $\sum_{i=1}^{n} iu_i = \sum_{j=1}^{m} jv_j =: e$ the number of edges of any tree T having valency lists V^* and V^{**}. Using equation (1) and equation (2) we get

$$\frac{d}{dz}P_s(z) = \left(\prod_{j=1}^{m} Q_{j,s}(z)^{j-1}\right)\cdot\left(\sum_{j=1}^{m} j\frac{d}{dz}Q_{j,s}(z)\prod_{i\neq j} Q_{i,s}(z)\right) \tag{4}$$

and

$$\frac{d}{dz}P_s(z) = \frac{d}{dz}(P_s(z) - P(0)) = \left(\prod_{i=1}^{n} P_{i,s}(z)^{i-1}\right)\cdot\left(\sum_{i=1}^{n} i\frac{d}{dz}P_{i,s}(z)\prod_{j\neq i} P_{j,s}(z)\right). \tag{5}$$

Lemma 4.6. *Let z^* be a root of $\frac{d}{dz}R_s(z)$ such that in addition z^* is a root of $Q_{j,s}$, for some $j \in \{1, \ldots, m\}$. Then $P_{i,s}(z^*) \neq 0$ for all $i \in \{1, \ldots, n\}$.*

Proof. Let $Q_{j,s}(z^*) = 0$ for some $j \in \{1, \ldots, m\}$. So z^* is a root of P_s and we conclude

$$\prod_{i=1}^{n} P_{i,s}(z^*)^i = -P_s(0) \neq 0.$$

But this means that $P_{i,s}(z^*) \neq 0$ $\forall i \in \{1, \ldots, n\}$ □

Therefore Lemma 4.6 tells us that the first factors of equation (4) and (5) are coprime. Thus the product of those is a divisor of $\frac{d}{dz}P_s(z)$ and therefore of $\frac{d}{dz}R_s(z)$. Since

$$\deg(\frac{d}{dz}R_s(z)) = \deg(\prod_{j=1}^{m} Q_{j,s}(z)^{j-1}\cdot\prod_{i=1}^{n} P_{i,s}(z)^{i-1})$$

we conclude that there exist an element $c \in \mathbb{C}$ such that

$$\begin{aligned}\frac{d}{dz}R_s(z) &= \frac{2}{P_s(0)}\left(\prod_{j=1}^{m} Q_{j,s}(z)^{j-1}\right)\cdot\left(\sum_{j=1}^{m} j\frac{d}{dz}Q_{j,s}(z)\prod_{i\neq j} Q_{i,s}(z)\right)\\ &= \frac{2}{P_s(0)}\left(\prod_{i=1}^{n} P_{i,s}(z)^{i-1}\right)\cdot\left(\sum_{i=1}^{n} i\frac{d}{dz}P_{i,s}(z)\prod_{j\neq i} P_{j,s}(z)\right)\\ &= \frac{2c}{P_s(0)}\prod_{j=1}^{m} Q_{j,s}(z)^{j-1}\prod_{i=1}^{n} P_{i,s}(z)^{i-1}.\end{aligned}$$

Note: By Lemma 4.6 and the equations above we can now conclude that for $j \in \{1,\dots,m\}$ and $i \in \{1,\dots,n\}$ the polynomials $Q_{j,s}$ and $P_{i,s}$ have distinct roots of multiplicity one, otherwise there would be a root of

$$\sum_{j=1}^{m} j\frac{d}{dz}Q_{j,s}(z)\prod_{i\neq j} Q_{i,s}(z) = c\prod_{i=1}^{n} P_{i,s}(z)^{i-1}$$

respective of

$$\sum_{i=1}^{n} i\frac{d}{dz}P_{i,s}(z)\prod_{j\neq i} P_{j,s}(z) = c\prod_{j=1}^{m} Q_{j,s}(z)^{j-1}$$

which is a root of $\prod_{j=1}^{m} Q_{j,s}(z)^{j-1}$ respective $\prod_{i=1}^{n} P_{i,s}(z)^{i-1}$, in contradiction to Lemma 4.6 . So if $\frac{d}{dz}R_s(z) = 0$ either $P_s(z) = 0$ and hence $R_s(z) = -1$ or $P_s(z) = P_s(0)$ and therefore $R_s(z) = 1$. But this means that $R_s(z)$ is a normalized, generalized Chebyshev polynomial and by Lemma 4.3 $\beta_s(z)$ is a clean Belyi polynomial.
By construction

$$\begin{aligned}\beta_s(z) &= 1 - R_s(z)^2\\ &= 1 - \left(\frac{2}{P_s(0)}P_s(z) - 1\right)^2\\ &= 1 - \left(\left(\frac{2}{P_s(0)}P_s(z)\right)^2 - \frac{4}{P_s(0)}P_s(z) + 1\right)\\ &= -\frac{4}{P_s(0)}P_s(z)\left(\frac{P_s(z)}{P_s(0)} - 1\right)\\ &= -\frac{4}{P_s(0)^2}\prod_{j=1}^{m} Q_{j,s}(z)^{j}\prod_{i=1}^{n} P_{i,s}(z)^{i}\end{aligned}$$

(ii) So, from above, we can see that the roots of β_s are exactly the roots of the polynomials $Q_{j,s}$ for $j \in \{1,\dots,m\}$ and those of the polynomials $P_{i,s}$ for $i \in \{1,\dots,n\}$. Thus

for all $j \in \{1, \ldots, m\}$ and $i \in \{1, \ldots, n\}$ the polynomial $\beta_s(z)$ has exactly v_j respectively u_i roots of multiplicity j, resp. i, which are all distinct. Now

$$\Lambda(\mathscr{I}(\mathbb{P}^1\mathbb{C}, \beta_s)) = \{(\beta'^{-1}(0), \beta'^{-1}[0,1], \mathbb{P}^1\mathbb{C}) \mid (\mathbb{P}^1\mathbb{C}, \beta') \in \mathscr{I}(\mathbb{P}^1\mathbb{C})\}$$

is by construction an abstract tree with valency list V* and V**.

(iii) and *(iv)* see [5, page 67/69] □

Let $\mathscr{I}(T)$ be a special abstract tree.

Theorem 4.7. *Let V^* and V^{**} be the special valency list of $\mathscr{I}(T)$. Let β_s and $\beta_{s'}$ be the Belyi polynomials corresponding to two different solutions $s, s' \in S_{V^*,V^{**}}$. Then $\mathscr{I}(\mathbb{P}^1\mathbb{C}, \beta_s) \neq \mathscr{I}(\mathbb{P}^1\mathbb{C}, \beta_{s'})$.*

Proof. Let $V^* = (u_1, \ldots, u_n)$ and $V^{**} = (v_1 \ldots, v_m)$ be the special valency lists. Hence there exist distinct elements i_0 and $i_1 \in \{1, \ldots, n\}$, such that $u_{i_0} = u_{i_1} = 1$ and $v_{i_0} = v_{i_1} = 0$, for which we set $P_{i_0}(z) = z$ resp. $P_{i_1}(z) = z + 1$ in the construction above. For any solutions s and s' of $S_{V^*,V^{**}}$, we obtain the equations

$$P_{i_0,s}(z) = z = P_{i_0,s'}(z) \tag{6}$$

and

$$P_{i_1,s}(z) = z + 1 = P_{i_1,s'}(z) \tag{7}$$

If $\mathscr{I}(\mathbb{P}^1\mathbb{C}, \beta_s) = \mathscr{I}(\mathbb{P}^1\mathbb{C}, \beta_{s'})$ we know that $\beta_s(z) = \beta_{s'}(\frac{az+b}{cz+d})$ for some elements $a, b, c, d \in \mathbb{C}$. Since β_s and $\beta_{s'}$ are both non-constant polynomials we must have $c = 0$ and w.l.o.g. we can assume $d = 1$, otherwise we replace a and b by a/d and b/d. Now the equation

$$\beta_s(z) = \beta_{s'}(az + b)$$

leads to

$$\frac{4}{P_s(0)^2} \left(\prod_{j=1}^{m} Q_{j,s}(z)^j\right) \left(\prod_{i=1}^{n} P_{i,s}(z)^i\right) = \frac{4}{P_{s'}(0)^2} \left(\prod_{j=1}^{m} Q_{j,s'}(az+b)^j\right) \left(\prod_{i=1}^{n} P_{i,s'}(az+b)^i\right).$$

The multiplicity of roots of a polynomial are invariant under linear transformations. Since $v_{i_0} = v_{i_1} = 0$ we conclude that the roots of the left and right side which have multiplicity i_0 are exactly the roots of $P_{i_0,s}$ and $P_{i_0,s'}$. This leads to

$$P_{i_0,s}(z) = c_{i_0} P_{i_0,s'}(az + b) \tag{8}$$

for some constant $c_{i_0} \in \mathbb{C}\backslash\{0\}$. As the roots of the left and right side with multiplicity i_1 are exactly the roots of $P_{i_1,s}$ and $P_{i_1,s'}$ we know that

$$P_{i_1,s}(z) = c_{i_1} P_{i_1,s'}(az + b) \tag{9}$$

for some constant $c_{i_1} \in \mathbb{C}\backslash\{0\}$. Equation (8) is equivalent to

$$z = c_{i_0}az + c_{i_0}b$$

hence we know $b = 0$. Therefore Equation (9) is equivalent to

$$z + 1 = c_{i_1}(az + 1) = c_{i_1}az + c_{i_1}$$

so we conclude that $c_{i_1} = 1$ and therefore $a = 1$. Therefore $\beta_s(z) = \beta_{s'}(z)$, in contradiction to the assumption $s \neq s'$. $\square$

5 The action of Gal($\overline{\mathbb{Q}}/\mathbb{Q}$) on abstract clean trees

Let $\beta(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial in $\overline{\mathbb{Q}}[z]$. Let σ be an element of Gal $(\overline{\mathbb{Q}}/\mathbb{Q})$. Define

$$\beta^\sigma(z) := (a_n)^\sigma z^n + (a_{n-1})^\sigma z^{n-1} + \cdots + (a_1)^\sigma z + (a_0)^\sigma$$

Let $\mathscr{I}(T)$ be an abstract tree. According to the last section there exists at least one clean Belyi polynomial β such that $\beta \in \overline{\mathbb{Q}}[z]$ and $\Lambda(\mathscr{I}(\mathbb{P}^1\mathbb{C}, \beta)) = \mathscr{I}(T)$. Therefore the set

$$\mathscr{B}(\mathscr{I}(T)) := \{(\mathbb{P}^1\mathbb{C}, \beta) \mid \Lambda(\mathscr{I}(\mathbb{P}^1\mathbb{C}, \beta)) = \mathscr{I}(T) \text{ and } \beta(z) \in \overline{\mathbb{Q}}[z]\} \subset \Lambda^{-1}(\mathscr{I}(T))$$

is not empty. Let β_1 and β_2 be two polynomials, such that $(\mathbb{P}^1\mathbb{C}, \beta_1)$ and $(\mathbb{P}^1\mathbb{C}, \beta_2)$ are both contained in $\mathscr{B}(\mathscr{I}(T))$. Hence there exist some elements a and b in $\overline{\mathbb{Q}}$ such that

$$\beta_1(z) = \beta_2(az + b) \tag{10}$$

therefore

$$\beta_1^\sigma(z) = \beta_2^\sigma(a^\sigma z + b^\sigma). \tag{11}$$

This leads to the following definition:

Definition 5.1. *Let $\mathscr{I}(T)$ be an abstract tree and let $(\mathbb{P}^1\mathbb{C}, \beta)$ be an element of $\mathscr{B}(\mathscr{I}(T))$. Let σ be an element of* Gal$(\overline{\mathbb{Q}}/\mathbb{Q})$. *Define $(\mathscr{I}(T))^\sigma := \Lambda(\mathscr{I}(\mathbb{P}^1\mathbb{C}, \beta^\sigma))$. We say $\mathscr{I}(T)$ and $(\mathscr{I}(T))^\sigma$ are **Galois conjugated abstract trees**.*

5.1 Properties of the action of Gal($\overline{\mathbb{Q}}/\mathbb{Q}$) on abstract trees

[4, 2.4.1.2, page 117] Set $\Gamma := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. The most important observation is the fact that all orbits of the action of Γ on abstract trees are finite, we will prove this below. Let $\mathscr{I}(T)$ be an abstract tree. Consider its stabilizer $\Gamma_{\mathscr{I}(T)} \leq \Gamma$. Due to the fact that the orbit of $\mathscr{I}(T)$ is finite, the subgroup $\Gamma_{\mathscr{I}(T)}$ is of finite index in Γ. Let $H \leq \Gamma_{\mathscr{I}(T)}$ be the maximal normal subgroup of Γ contained in $\Gamma_{\mathscr{I}(T)}$ and let $\mathscr{M}(T)$ be the fixed field of the normal subgroup $H \trianglelefteq \Gamma$. Note that the group H is the pointwise stabilizer of all elements of the orbit.

Definition 5.2. [4, Definition 2.4.3, page 117] *The field $\mathscr{K}(T)$ which is the fixed field of $\Gamma_{\mathscr{I}(T)}$ is the* ***field of moduli of the abstract tree*** *$\mathscr{I}(T)$. The field $\mathscr{M}(T)$ is called the* ***field of moduli of the orbit*** *of $\mathscr{I}(T)$.*

The field of moduli of the orbit of $\mathscr{I}(T)$ is the minimal field extension K over $\mathscr{K}(T)$, such that $K/\mathbb{Q}$ is galois.

Theorem 5.3. *The action of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *on the set of abstract trees is faithful.*

Proof. [5, Proof of Theorem 2.4, page 58] Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. We will exhibit an abstract tree such that the action of σ on it is non-trivial. Let K be a number field and let α be a primitive element for the extension $K/\mathbb{Q}$ such that the action of σ on α is non-trivial. In order to show that there is an abstract tree on which σ acts non-trivially it suffices to show that there is an abstract tree whose field of moduli is equal to K. So we show that there exists an isomorphism class of clean Belyi functions $\mathscr{I}(\mathbb{P}^1\mathbb{C}, \beta)$ corresponding to an abstract tree such that $\beta(z)$ is defined over K and $\beta^\sigma(z)$ is not equal to $\beta(\frac{az+b}{cz+d})$ for all $a, b, c, d \in \mathbb{P}^1\mathbb{C}$ i.e. $\mathscr{I}(\mathbb{P}^1\mathbb{C}, \beta) \neq \mathscr{I}(\mathbb{P}^1\mathbb{C}, \beta^\sigma)$.
A rational Belyi function β corresponds to a tree if ∞ has exactly one pre-image under β, as a tree is a dessin on $\mathbb{P}^1\mathbb{C}$ with a unique open cell. In particular this will be the case whenever $\beta(z)$ is a clean Belyi polynomial, in which case the unique point over ∞ will be ∞. A clean Belyi polynomial β corresponds to a tree whose unique open cell contains ∞. If a polynomial satisfies $\beta^\sigma(z) = \beta(\frac{az+b}{cz+d})$ then we must have $c = 0$ and w.l.o.g. we can assume $d = 1$. So we will exhibit a clean Belyi polynomial $\beta(z)$ defined over K such that $\beta^\sigma(z) \neq \beta(az + b)$ for all $a, b \in \mathbb{P}^1\mathbb{C}$.
We construct such a $\beta(z)$ explicitly as follows. Let $f_\alpha(z) \in K[z]$ be a polynomial whose derivative is

$$f_\alpha{}'(z) = z^3(z-1)^2(z-\alpha).$$

The proof of [5, Theorem 1.2, page 49] and the proof of Corollary 2.8 state that there exists a polynomial $f(z) \in \mathbb{Q}[z]$ such that $g_\alpha := f \circ f_\alpha$ is a clean Belyi polynom. Let $\gamma = \alpha^\sigma$ (by assumption, $\gamma \neq \alpha$). Since f is defined over $\mathbb{Q}$ we obtain another clean Belyi polynomial $g_\gamma = f \circ f_\gamma$ where $f_\gamma = f_\alpha{}^\sigma$. Let $\mathscr{I}(T_\alpha)$ be the abstract clean tree corresponding to $\mathscr{I}(\mathbb{P}^1\mathbb{C}, g_\alpha)$ and let $\mathscr{I}(T_\gamma)$ be the abstract clean tree corresponding to $\mathscr{I}(\mathbb{P}^1\mathbb{C}, g_\gamma)$, so that $\mathscr{I}(T_\gamma) = \mathscr{I}(T_\alpha{}^\sigma)$. In order to prove that σ acts non-trivially on $\mathscr{I}(T_\alpha)$ we must show that $\mathscr{I}(T_\alpha)$ and $\mathscr{I}(T_\gamma)$ are distinct. As mentioned above, this is equivalent to showing that we cannot have $g_\gamma(z) = g_\alpha(az + b)$ for any constants a, b. Suppose we do have such a and b. Then $g_\gamma(z) = g_\alpha(az+b)$, i.e. $f(f_\gamma(z)) = f(f_\alpha(az+b))$.

Lemma 5.4. [5, Lemma 2.3, page 57] *Let G, H, $\tilde{G}$ and $\tilde{H}$ be non-constant polynomials such that $G \circ H = \tilde{G} \circ \tilde{H}$ and $deg(H) = deg(\tilde{H})$. Then there exist constants c and d such that $\tilde{H} = cH + d$.*

Applying this lemma with $G = \tilde{G} = f(z)$ and $H = f_\alpha(az + b)$ and in addition $\tilde{H} = f_\gamma(z)$, we see that there exist constants c and d such that $f_\alpha(az + b) = cf_\gamma(z) + d$. Consider the critical points of both these functions. The right-hand function has the same critical points as f_γ, namely the point 0 (of order 3), the point 1 (of order 2) and the point γ (of order 1). The left-hand function has three critical points $x_i, i = 1, 2, 3$, where each x_i is of order i and $ax_1 + b = \alpha, ax_2 + b = 1$ and $ax_3 + b = 0$. Since $az + b$ must take these three critical points to the critical points of f_α, respecting their orders. By equality of the two sides we must have $x_1 = \gamma$, $x_2 = 1$ and $x_3 = 0$. But the two equations $ax_2 + b = 1$ and $ax_3 + b = 0$ then give $a = 1$ and $b = 0$, so the equation $ax_1 + b = \alpha$ shows $\gamma = \alpha$, contrary to the assumption $\gamma \neq \alpha$. Therefore we cannot have $g_\gamma(z) = g_\alpha(az + b)$ for any constants $a, b \in \mathbb{P}^1\mathbb{C}$. Hence $\mathscr{I}(\mathbb{P}^1\mathbb{C}, g_\alpha) \neq \mathscr{I}(\mathbb{P}^1\mathbb{C}, g_\gamma)$ and therefore $\mathscr{I}(T_\alpha)$ and $\mathscr{I}(T_\gamma)$ are distinct. □

If $V*$ and $V**$ are special valency lists Theorem 4.7 tells us that the algorithm above gives exactly one solution for each abstract tree which has valency lists V^* and V^{**}. For a given abstract special tree $\mathscr{I}(T)$ let $\beta_{s,\mathscr{I}(T)}$ be the solution which corresponds to $\mathscr{I}(T)$.

Theorem 5.5. *The field of moduli of a special abstract tree $\mathscr{I}(T)$ is the field of definition of $\beta_{s,\mathscr{I}(T)}$.*

Proof. We know that $\beta_{s,\mathscr{I}(T)} \in \overline{\mathbb{Q}}[z]$. Let F be the field of definition of $\beta_{s,\mathscr{I}(T)}$ and Γ_F be the subgroup of Γ with fixed field equal to F. Let σ be an element of $\Gamma_{\mathscr{I}(T)}$ acting non-trivially on

$$\beta_{s,\mathscr{I}(T)} = -\frac{4}{P_{s,\mathscr{I}(T)}(0)^2} \prod_{j=1}^{m} (Q_{j,s,\mathscr{I}(T)})^j \prod_{i=1}^{n} (P_{i,s,\mathscr{I}(T)})^i. \tag{12}$$

Hence

$$\beta^\sigma{}_{s,\mathscr{I}(T)} = -\frac{4}{P_{s,\mathscr{I}(T)}(0)^2} \prod_{j=1}^{m} (Q^\sigma_{j,s,\mathscr{I}(T)})^j \prod_{i=1}^{n} (P^\sigma_{i,s,\mathscr{I}(T)})^i \tag{13}$$

is the clean Belyi polynomial corresponding to a different solution of (2). Since $\mathscr{I}(T)$ is special we can conclude that $\mathscr{I}(\mathbb{P}^1\mathbb{C}, \beta_{s,\mathscr{I}(T)}) \neq \mathscr{I}(\mathbb{P}^1\mathbb{C}, \beta^\sigma{}_{s,\mathscr{I}(T)})$ which is in contradiction to the assumption that σ is an element of the stabilizer of $\mathscr{I}(T)$. Therefore $\Gamma_{\mathscr{I}(T)}$ is a subgroup Γ_F. Let γ be an element of Γ_F. Let $(\mathbb{P}^1\mathbb{C}, \beta(z))$ be a clean Belyi pair contained in $\mathscr{I}(\mathbb{P}^1\mathbb{C}, \beta_{\mathscr{I}(T)})$. Hence there exist elements a and b in $\mathbb{C}$ such that $\beta(az + b) = \beta_{s,\mathscr{I}(T)}(z)$. Therefore $\beta^\gamma(a^\gamma z + b^\gamma) = \beta_{s,\mathscr{I}(T)}(z)$ and we conclude that $\gamma \in \Gamma_{\mathscr{I}(T)}$. Since γ was chosen arbitrary Γ_F is a subgroup of $\Gamma_{\mathscr{I}(T)}$. So the two groups are equal and the field of moduli of $\mathscr{I}(T)$ is the field of definition of $\beta_{s,\mathscr{I}(T)}$. □

Let $\mathscr{I}(T)$ be a special abstract tree. Let $(\mathbb{P}^1\mathbb{C}, \beta)$ be a clean Belyi pair in $\Lambda^{-1}(\mathscr{I}(T))$.

Corollary 5.6. *The field of definition of β contains the field of definition of $\beta_{s,\mathscr{I}(T)}$.*

Proposition 5.7. [4, Remark 2.4.5, page 117] *For a Galois orbit having N elements the subgroup $\Gamma_{\mathscr{I}(T)}$ is of index N and therefore the field of moduli of the orbit is generated by the roots of an irreducible polynomial $p \in \mathbb{Q}[z]$ of degree N. In particular, if the orbit contains a single element, its field of moduli is $\mathbb{Q}$ itself.*

5.2 Combinatorial Invariants of the Galois action on abstract clean dessins

Corollary 5.8. *The valency lists of Galois conjugated abstract trees are equal.*

Proof. Let σ be an element of Γ. Let $\mathscr{I}(T) = \Lambda(\mathscr{I}(\mathbb{P}^1\mathbb{C}, \beta))$, let $(\mathbb{P}^1\mathbb{C}, \beta) \in \mathscr{B}(\mathscr{I}(T))$. Hence $(\mathscr{I}(T))^\sigma = \Lambda(\mathscr{I}(\mathbb{P}^1\mathbb{C}, \beta^\sigma))$. Since the multiplicity of roots are invariant under the action of σ on a polynomial and as σ is an automotphism, we assume that the valency lists of $\mathscr{I}(T)$ and $(\mathscr{I}(T))^\sigma$ have to be equal. □

Corollary 5.9. *The Galois orbit of an abstract tree is finite*

Proof. From Corollary 5.8 we know that the valency list of an abstract tree is preserved under Galois conjugation. But there are only finitely many different abstract trees having a given valency list. □

Theorem 5.10. [3, Theorem, page 27] *Let $\mathscr{I}(T)$ be an abstract tree of degree n and G an element of $\mathscr{C}(\mathscr{I}(T))$. Consider an element $\sigma \in \Gamma$ and $(\mathscr{I}(T))^\sigma$. Then, for any $G^\sigma \in \mathscr{C}((\mathscr{I}(T))^\sigma)$ the groups G and G^σ are conjugated in S_n.*

6 Examples of the method

Example 1: We consider the valency lists $V^* = (3, 2)$ and $V^{**} = (1, 1, 0, 1)$. There are exactly four distinct abstract trees with valency lists $(3, 2)$ and $(1, 1, 0, 1)$:

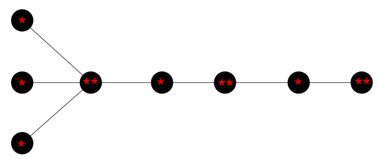


Figure 3: $\mathscr{I}(T_1)$

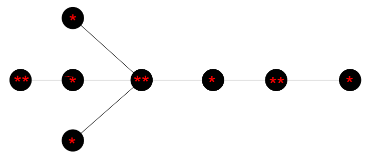


Figure 4: $\mathscr{I}(T_2)$

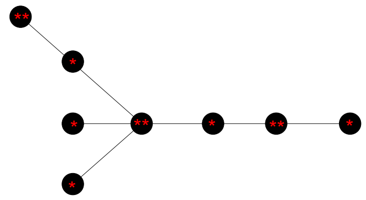


Figure 5: $\mathscr{I}(T_3)$

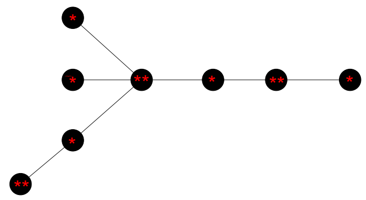


Figure 6: $\mathscr{I}(T_4)$

Even without computing the corresponding Belyi polynomials of the abstract trees above, we can say something about their Galois orbits. We can compute the order of their cartographical groups with maple. If we do so, we see that $\mathscr{C}(\mathscr{I}(T_1))$ and

$\mathscr{C}(\mathscr{I}(T_2))$ consist of groups of order 1270080. But the order of the groups in $\mathscr{C}(\mathscr{I}(T_3))$ and $\mathscr{C}(\mathscr{I}(T_4))$ is 56448. Hence Theorem 5.10 leads to the conclusion that the Galois orbits of $\mathscr{I}(T_1)$ and $\mathscr{I}(T_2)$ are subsets of $\{\mathscr{I}(T_1), \mathscr{I}(T_2)\} := \mathscr{O}_1$ resp. the Galois orbits of $\mathscr{I}(T_3)$ and $\mathscr{I}(T_4)$ are subsets of $\{\mathscr{I}(T_3), \mathscr{I}(T_4)\} := \mathscr{O}_2$.

The algorithm gives us four distinct solutions s_1, s_2, s_3, s_4 with corresponding Belyi polynomials:

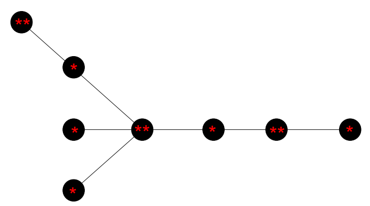
$$\beta_{s_1}(x) = 1 - \frac{56296884765625(-1+x)^2x^8(27(31-4\sqrt{21})+150(3-2\sqrt{21})x+625x^2)^2}{47775744(3-2\sqrt{21})^4(987-208\sqrt{21})^2}$$

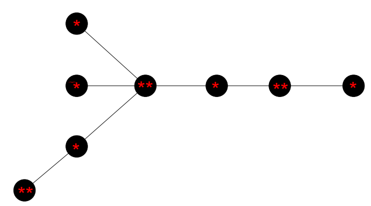
$$\beta_{s_2}(x) = 1 - \frac{56296884765625(-1+x)^2x^8(27(31+4\sqrt{21})+150(3+2\sqrt{21})x+625x^2)^2}{47775744(3+2\sqrt{21})^4(987+208\sqrt{21})^2}$$

$$\beta_{s_3}(x) = 1 + \frac{823543(-1+x)^2x^8(-3-i\sqrt{7}+(2-2i\sqrt{7})x+2x^2)^2}{262144}$$

$$\beta_{s_4}(x) = 1 + \frac{823543(-1+x)^2x^8(-3+i\sqrt{7}+(2+2i\sqrt{7})x+2x^2)^2}{262144}$$

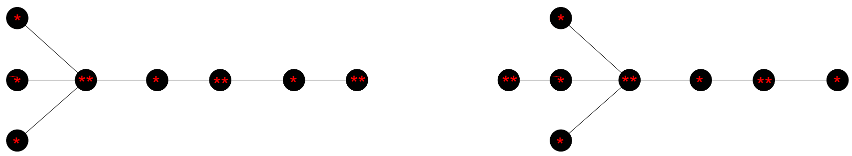
Consider $\sigma_1 \in \Gamma$ which sends $i\sqrt{7}$ onto $-i\sqrt{7}$. We have $\mathrm{Gal}(\mathbb{Q}(i\sqrt{7})/\mathbb{Q}) = \langle {\sigma_1}_{|\mathbb{Q}(i\sqrt{7})} \rangle$, where $({\sigma_1}_{|\mathbb{Q}(i\sqrt{7})})^2 = 1$. We see that β_{s_3} and β_{s_4} are both contained in $\mathbb{Q}(i\sqrt{7})[x]$ and in addition $\beta_{s_3}^{\sigma_1}(x) = \beta_{s_4}(x)$ and $\beta_{s_4}^{\sigma_1}(x) = \beta_{s_3}(x)$. Hence we conclude that $\mathscr{O}_2$ is a Galois orbit.
Consider $\sigma_2 \in \Gamma$ which sends $\sqrt{21}$ onto $-\sqrt{21}$. We have $\mathrm{Gal}(\mathbb{Q}(\sqrt{21})/\mathbb{Q}) = \langle {\sigma_2}_{|\mathbb{Q}(\sqrt{21})} \rangle$, where $({\sigma_2}_{|\mathbb{Q}(\sqrt{21})})^2 = 1$. We see that β_{s_1} and β_{s_2} are both contained in $\mathbb{Q}(\sqrt{21})[x]$ and in addition $\beta_{s_1}^{\sigma_2}(x) = \beta_{s_2}(x)$ and $\beta_{s_2}^{\sigma_2}(x) = \beta_{s_1}(x)$. Hence we conclude that the $\mathscr{O}_1$ is a Galois orbit.
If we note that complex conjugation corresponds to reflecting a plane tree over the real line, we are able to conclude that the abstract trees which correspond to $\mathscr{O}_2$ are:





Since there are two elements in $\mathscr{O}_2$, we see that the field of moduli of the orbit $\mathscr{O}_2$ is a field extension of $\mathbb{Q}$ of degree 2. The field of moduli of any abstract tree $\mathscr{I}(T)$ which is contained in $\mathscr{O}_2$ is a field extension of the field of moduli of $\mathscr{O}_2$. The field of definition of any Belyi polynomial β such that $\Lambda(\mathscr{I}(\mathbb{P}^1\mathbb{C}, \beta)) = \mathscr{I}(T)$ is an extension field of the

field of moduli of $\mathscr{I}(T)$. Hence the field of moduli of $\mathscr{I}(T_3)$ and of $\mathscr{I}(T_4)$ is the field of moduli of $\mathscr{O}_2$ and all of them are equal to $\mathbb{Q}(i\sqrt{7})$.
The abstract trees which correspond to $\mathscr{O}_1$ are:



Again we are able to conclude that the field of moduli of the two abstract trees above is $\mathbb{Q}(\sqrt{21})$, which is also the field of moduli of $\mathscr{O}_1$.

Example 2: Let us now consider the special valency lists $V^* = (0, 1, 1, 1)$ and $V^{**} = (5, 2)$. The algorithm produces six distict solutions $s_1, s_2, s_3, s_4, s_5, s_6$, such that the field of definition of β_{s_i} is contained in $\mathbb{Q}(r_i)$ where $r_1, \ldots, r_6$ are the distinct roots of the irreducible polynomial

$$8192x^6 - 9216x^5 - 76032x^4 - 162432x^3 + 62208x^2 + 1571724x - 1750329$$

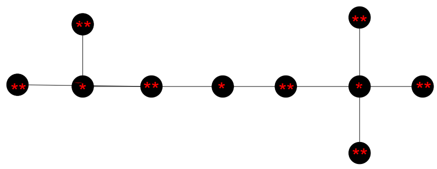


Figure 7: a tree with valency lists $(0, 1, 1, 1)$ and $(5, 2)$.

Since $(0, 1, 1, 1)$ and $(5, 2)$ are special valency lists and as all solutions β_{s_i} are Galois conjugated, because the polynomial above is irreducible, we conclude that there are exactly six distinct abstract trees with valency lists $(0, 1, 1, 1)$ and $(5, 2)$ which are all in one Galois orbit.

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