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Geometrically Reductive Groups and Finitely Generated Rings of Invariants

Bachelor Thesis

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Abstract

We discuss when rings of invariants of algebras over an algebraically closed field are finitely generated. We prove Nagata's Theorem, which states that geometrically reductive groups have finitely generated rings of invariants. Along the way, we introduce linear algebraic groups, their representations, and different reductivity conditions.

Contents

Introduction

Consider the polynomial ring $R := k[X_1, \ldots, X_n]$ over an algebraically closed field k and a group $G \subset \mathrm{GL}_n(k)$ acting on R by linear substitutions of the variables X_i. We want to study the polynomials which are invariant under G. Observe that the set of invariant polynomials is a subring of R, aptly named the *ring of invariants* R^G. So we can describe the invariants by finding generators of R^G as a k-algebra. But first, it would be nice to know if and when R^G is finitely generated over k.

More generally, we want to answer the following question.

Question 0.1. *What conditions on the group $G \subset \mathrm{GL}_n(k)$ and the action of G on a finitely generated k-algebra R are sufficient to guarantee that the ring of invariants R^G is finitely generated over k?*

A *linear algebraic group* is a subgroup of GL_n admitting the structure of an algebraic variety compatible with its group law. Examples include GL_n, SL_n and finite groups. By taking the Zariski-closure of $G \subset \mathrm{GL}_n$, we can assume our group G in Question 0.1 is a linear algebraic group (see Section 2.3).

For the set of invariants to even be a sub-k-algebra, the group G should definitely act via k-algebra automorphisms. Actually, the group G needs to learn some manners and act nicer, namely *rationally.* Rational actions are actions which respect the additional structure linear algebraic groups carry. An example of a rational action is the action by linear substitutions of the opening paragraph. We assume for the rest of the introduction that G is a linear algebraic group acting rationally on R.

A first answer to Question 0.1 was given by P. Gordan [4], who constructed a finite set of generators for certain rings of invariants of $\mathrm{SL}_2(\mathbb{C})$. Hilbert was able to improve Gordan's result to $\mathrm{SL}_n(\mathbb{C})$ in his famous paper *Über die Theorie der algebraische Formen* [6]. Hilbert's idea was to construct a certain $\mathrm{SL}_n(\mathbb{C})$-equivariant projection $R \twoheadrightarrow R^{\mathrm{SL}_n(\mathbb{C})}$, called a *Reynolds Operator*, and then deduce finite generatedness with the help of his Basissatz. Using this method we generalize Hilbert's result to the following theorem (see the beginning of Chapter 6).

Theorem 0.2 (Hilbert). *If all nonzero finite dimensional rational representations of G are semisimple, i.e., every subrepresentation has a G-stable linear complement, then the ring of invariants R^G is finitely generated over k.*

However, this is a weak statement in positive characteristic, as not many matrix groups (not even SL_n, GL_n, or all finite groups!) over a field of positive characteristic can guarantee the semisimplicity of their representations. So we define a weaker condition on G called *geometric reductivity*. Geometric reductivity is sufficient for finite generation of finitely generated rings of invariants, a fact first proved by Nagata in [11].

Theorem 0.3 (Nagata). *If G is a geometrically reductive group, then the ring of invariants R^G is finitely generated over k.*

In order to justify that Theorem 0.3 is indeed satisfactory, we should show that SL_n and GL_n are geometrically reductive. Although this is beyond the scope of this thesis, in Section 5.3 we outline a method of proving geometric reductivity for certain groups including SL_n and GL_n.

For the most part we are guided by Chapters 1 and 2 of the book *Invariant Theory* by T.Springer [15]. Inspiration from Chapter 4 in S. Mukai's book *An Introduction to Invariants and Moduli* [9] is scattered throughout (especially in the section on linear reductivity). I was introduced to Nagata's Theorem and linear algebraic groups in Chapter 3 of the book *Lectures on Invariant Theory* by I. Dolgachev [2]. Other sources of ideas are mentioned in passing.

Prerequistes for this thesis are a good knowledge of commutative algebra and basic algebraic geometry over an algebraically closed field. Previous exposure to representation theory is helpful.

I would like to thank Prof. Richard Pink for supervising this thesis and for many helpful discussions.

Table 1: An Incomplete History of Finitely Generated Rings of Invariants

1868	Gordan proves finiteness for certain rings of invariants of $\mathrm{SL}_2(\mathbb{C})$ [4]
1890	Hilbert proves finiteness for certain rings of invariants of $\mathrm{SL}_n(\mathbb{C})$ [6]
1900	Hilbert poses his 14th problem, conjecturing finiteness for all subgroups $G \subset \mathrm{GL}_n(\mathbb{C})$ acting by linear substitutions on a polynomial [15, page 37]
1916	Noether proves finiteness for the case of finite groups, with a bound on the number of generators [8, page 14]
1950s	The theory of linear algebraic groups is developed leading to the notion of linearly reductive groups and Theorem 0.2 [15, page 37]
1959	Nagata gives a counterexample to Hilbert's 14th problem [15, page 37]
1964	Nagata proves finiteness for geometrically reductive groups [11]
1964	Oda proves SL_2 is geometrically reductive in characteristic 2, a consequence being that SL_2 has finitely generated rings of invariants in characterstic 2 [5, page 67]
1969	Seshadri proves that GL_2 and SL_2 are geometrically reductive in all characteristics, a consequence being that SL_2 has finitely generated rings of invariants in all characterstics [5, page 67]
1975	Haboush proves that reductive groups are geometrically reductive, incidentally proving finiteness for many groups including $\mathrm{GL}_n(k), \mathrm{SL}_n(k)$ in any characteristic. [5]
1979	Popov proves equivalence of having finitely generated ring of invariants with geometric reductivity [14]

Conventions

Throughout k is an algebraically closed field. Everything is over k, in particular, all vector spaces are over k.

The dual of a vector space V is denoted by $V^\vee$ and its elements are called *linear forms*. For a basis $\mathcal{B} = \{b_i\}_i$ of a vector space we denote the basis dual to $\mathcal{B}$ by $\mathcal{B}^\vee = \{b_i^\vee\}_i$.

A *finite R-module* is an R-module M admitting a surjective R-linear map $R^n \twoheadrightarrow M$. We never call such modules "finitely generated".

All topological notions refer to the Zariski-topology.

All representations are finite dimensional.

All actions are left actions.

Facts are true statements we do not prove. This does not mean there is nothing to prove. The reader might wish to verify facts for themselves.

1 Some Commutative Algebra

The following facts, concerning finiteness properties of rings, are essential tools for proving that rings of invariants are finitely generated.

Theorem 1.1 (Hilbert's Basissatz). *A ring R is noetherian if and only if the polynomial ring $R[X]$ is noetherian.*

Corollary 1.2. *Every finitely generated k-algebra is noetherian.*

Proposition 1.3. *For any $\mathbb{Z}^{\geqslant 0}$-graded ring $R = \bigoplus_{d \geqslant 0} R_d$ the augmentation ideal $R_+ = \bigoplus_{d>0} R_d$ is finitely generated if and only if R is a finitely generated R_0-algebra.*

Proof. First we assume R is finitely generated over R_0. Let $a_1, \ldots, a_n \in R_+$ be generators of positive degree. Let $x \in R_+$. Write $x = \sum_{\underline{i} \in \mathbb{Z}^{\geqslant 0}} \lambda_{\underline{i}} a_1^{i_1} \cdots a_n^{i_n}$ for certain $\lambda_{\underline{i}} \in R_0$. Since x lies in the augmentation ideal R_+ and all $\lambda_{\underline{i}}$ have degree 0, we deduce that every nonzero summand must contain a nonzero power of at least one of the a_i. Hence, the element x lies in $(a_1, \ldots, a_n)$. Varying x, we get the equality $R_+ = (a_1, \ldots, a_n)$, proving the "if" part of the proposition.

Conversely, assume $R_+ = (a_1, \ldots, a_n)$ for some $a_1, \ldots, a_n \in R_+$. Let $x \in R$. Write $x = x_+ + x_0$ for certain elements $x_+ \in R_+$ and $x_0 \in R_0$. The summand x_+ is an R-linear combination of the a_i, say $x_+ = \sum_{i=1}^n x_i a_i$ for certain $x_i \in R$. Since the a_i have positive degree, we see that the x_i have degree strictly smaller than x_+. By induction on the degree, we can assume, without loss of generality, that x_+ is an R_0-linear combination of the a_i, whence follows that x is a linear polynomial in the a_i with coefficients in R_0. Varying x proves the "only if" part of the claim. $\square$

Proposition 1.4. *Let $R \subset S$ be an integral extension of k-algebras. If S is finitely generated over k, then R is finitely generated over k and S is a finite R-module.*

Proof. Write $S = k[b_1, \ldots, b_n]$. Let $f_i \in R[X]$ be monic polynomials such that $f_i(b_i) = 0$ for $i = 1, \ldots, n$. Let $A \subset R$ be the k-algebra finitely generated by the coefficients of all the f_i. The b_i generate S as an A-algebra and S is integral over A. So S is a finite and therefore noetherian A-module. In particular, S is finite over R. As a submodule of S, we find R to be a finite A-module. It follows that R is finitely generated over k. $\square$

Theorem 1.5 (Emmy Noether). *For any finitely generated k-algebra R which is an integral domain, the integral closure $\tilde{R}$ of R in a finite field extenstion L of* $\mathrm{Quot}(R)$ *is a finite R-module. In particular, $\tilde{R}$ is a finitely generated k-algebra.*

Proof. See Corollary 13.13 in [3, page 297] □

2 Linear Algebraic Groups

The goal of this chapter is to introduce linear algebraic groups, which are the groups we will be dealing with.

2.1 Varieties without choosing coordinates

Not all vector spaces appearing in this thesis come with a canonical basis. So we want to define varieties without choosing coordinates.

Let V be a finite dimensional vector space.

Definition 2.1. A *polynomial function* f on V is a map $f\colon V \to k$ such that for some basis $b_1, \ldots, b_n$ and every vector $v = \sum_{i=1}^n \beta_i b_i \in V$ the value $f(v)$ is a polynomial in the coefficients β_i, i.e., there exists a polynomial $\tilde{f}(X_1, \ldots, X_n) \in k[X_1, \ldots, X_n]$ such that $f(\sum_{i=1}^n \beta_i b_i) = \tilde{f}(\beta_1, \ldots, \beta_n)$.

Polynomial functions do not care about our choice of basis.

Lemma 2.2. *Let $b_1, \ldots, b_n$ and $c_1, \ldots, c_n$ be bases of V and $f\colon V \to k$ a map. Then for all vectors $v = \sum_{i=1}^n \beta_i b_i = \sum_{j=1}^n \gamma_j c_j \in V$ the value $f(v)$ is a polynomial in the β_i if and only if it is a polynomial in the γ_j.*

Proof. First we notice that, by symmetry, both directions are identical up to change of notation. Suppose $f(v)$ is a polynomial in the β_i. Let $M = (\mu_{ij})_{1 \leqslant i,j \leqslant n} \in \mathrm{GL}_n(k)$ be the base change matrix changing (c_j) to (b_i). We have $\beta_i = \sum_{j=1}^n \mu_{ij}\gamma_j$. The value $f(v)$ is a polynomial in the β_i and each β_i is a polynomial in the γ_j, hence $f(v)$ is also a polynomial in the γ_j. □

Example 2.3. The determinant $\det\colon \mathrm{End}(V) \to k$ is a polynomial function on the vector space $\mathrm{End}(V)$.

Definition 2.4. The *coordinate ring* $k[V]$ of the vector space V is the ring of polynomial functions on V (with pointwise addition and multiplication).

Remark 2.5. A polynomial function $f\colon V \to k$ is *homogenous of degree d* if $f(\lambda v) = \lambda^d f(v)$ for all $\lambda \in k$ and $v \in V$. Setting $k[V]_d$ as the set of all homogenous polynomial functions of degree d definies the $\mathbb{Z}^{\geqslant 0}$-grading $k[V] = \bigoplus_{d \geqslant 0} k[V]_d$ with $k[V]_0 = k$. For any basis $b_1, \dots, b_n$ of V its dual basis $b_1^\vee, \dots, b_n^\vee \in V^\vee \cong k[V]_1$ is a set of degree 1 homogeneous generators of the coordinate ring $k[V]$. In summary, $k[V]$ is a finitely generated graded k-algebra.

Fact 2.6. *For every basis $b_1, \dots b_n$ of V the ring homomorphism $k[V] \to \mathrm{Sym}_k(V^\vee)$ given by the assignments $b_i^\vee \mapsto b_i^\vee$ is an isomorphism of graded rings.*

Remark 2.7. A group $G \subset \mathrm{GL}(V)$ acts on $k[V]$ by ${}^g f(v) := f(g^{-1}v)$. Since g^{-1} corresponds to a base change, we say that *G acts on $k[V]$ by linear substituions*. The *ring of invariants under linear substitutions* is $k[V]^G := \{f \in k[V] \mid \forall g \in G : {}^g f = f\}$; its elements are *$G$-invariants*.

Proposition-Definition 2.8. (i) For every ideal $\mathfrak{a} \subset k[V]$ the set $\mathcal{V}(\mathfrak{a}) = \{v \in V \mid \forall f \in \mathfrak{a} : f(v) = 0\}$ is the *vanishing set of* $\mathfrak{a}$. The vanishing sets of ideals of $k[V]$ are the closed sets of the *Zariski-topology* on V.

(ii) An *affine algebraic variety* is a closed subset of V. Each affine algebraic variety $X := \mathcal{V}(\mathfrak{a}) \subset V$ has a *coordinate ring* $k[X] := k[V]/\mathfrak{a}$.

(iii) A *standard open subset* is an open subset of the form $\mathcal{D}_f := \{v \in V \mid f(v) \neq 0\}$ for some $f \in k[V]$. Every standard open subset $\mathcal{D}_f$ is an affine algebraic variety with coordinate ring $k[V][f^{-1}]$.

Proof. Omitted. □

Remark 2.9. We endow all finite dimensional vector spaces with the Zariski-topology. In particular, the vector space $\mathrm{End}(V)$ carries the Zariski-topology.

Fact 2.10. *Closed subsets of affine algebraic varieties are affine algebraic varieties.*

Fact 2.11. *Every finite subset of a finite dimensional vector space is closed.*

Definition 2.12. Let W be a finite dimensional vector space. Let $X \subset V$ and $Y \subset W$ be affine algebraic varieties. A map $\varphi \colon X \to Y$ is a *morphism of algebraic varieties* if there exists a ring homomorphism $\varphi^\flat \colon k[Y] \to k[X]$ such that for every linear form $l \in W^\vee$ and every $x \in X$ we have $l(\varphi(x)) = \varphi^\flat(l)(x)$. An *isomorphism of algebraic varieties* is a morphism of algebraic varieties possessing a two sided inverse that is also a morphism of algebraic varieties.

Fact 2.13. *Let W be a finite dimensional vector space. Let $X \subset V$ and $Y \subset W$ be affine algebraic varieties and let $\varphi \colon X \to Y$ be a morphism of algebraic varieties.*

(i) For all $x \in X$ the image $\varphi(x)$ is equal to $\sum_{i=1}^n \varphi^\flat(b_i^\vee)(x)$ for any basis $b_1, \dots, b_n$ of W.

(ii) For every $g \in k[Y]$ we have $g(\varphi(x)) = \varphi^\flat(g)(x)$.

Proposition 2.14. *Every morphism of algebraic varieties $\varphi \colon X \to Y$ is continuous.*

Proof. We show that for every ideal $\mathfrak{b} \subset k[Y]$ the preimage of the closed set $\mathcal{V}(\mathfrak{b}) \subset Y$ is equal to the closed set $\mathcal{V}(\varphi^\flat(\mathfrak{b})k[X])$. Let $x \in \varphi^{-1}(\mathcal{V}(\mathfrak{b}))$ and $f \in \varphi^\flat(\mathfrak{b})$. Let $g \in \mathfrak{b}$ such that $\varphi^\flat(g) = f$. We have $f(x) = \varphi^\flat(g)(x) = g(\varphi(x)) = 0$. So x lies in $\mathcal{V}(\varphi^\flat(\mathfrak{b})k[X])$. Varying x shows the inclusion $\varphi^{-1}(\mathcal{V}(\mathfrak{b})) \subset \mathcal{V}(\varphi^\flat(\mathfrak{b})k[X])$. Furthermore, for all $x \in \mathcal{V}(\varphi^\flat(\mathfrak{b})k[X])$ and for all $g \in \mathfrak{b}$ we have $g(\varphi(x)) = \varphi^\flat(g)(x) = 0$, yielding the other inclusion. □

Fact 2.15. *The general linear group* $\mathrm{GL}(V)$ *is the standard open subset* $\mathcal{D}_{\det} \subset \mathrm{End}(V)$. *In particular,* $\mathrm{GL}(V)$ *is an affine algebraic variety with coordinate ring*

$$k[\mathrm{GL}(V)] \cong k[\{X_{i,j}\}_{i,j=1}^n, \det((X_{ij})_{i,j})^{-1}].$$

2.2 Definition

Definition 2.16. A *linear algebraic group (over k)* is a Zariski-closed subgroup of $\mathrm{GL}(V)$ for some finite dimensional vector space V.

Special Case 2.17. If V is taken to be k^n then we know $\mathrm{End}(V)$ is canonically isomorphic to $\mathrm{Mat}_n(k) = k^{n^2}$. Similarly, we view a linear algebraic group $G \subset \mathrm{GL}(k^n)$ as a matrix group which is closed in $\mathrm{GL}_n(k) \subset \mathrm{Mat}_n(k)$.

Important Fact 2.18. Every linear algebraic group is an affine algebraic variety

Lemma 2.19. *For any subgroup* $G \subset \mathrm{GL}(V)$ *inversion* $G \to G$, $g \mapsto g^{-1}$ *and multiplication by a fixed element* $G \to G$, $g \mapsto hg$ *(or* $g \mapsto gh$*) are homeomorphisms. Moreover, if* G *is a linear algebraic group, then* $g \mapsto g^{-1}$ *and* $g \mapsto hg$ *are automorphisms of algebraic varieties.*

Proof. Since multiplication, resp. inversion, in G is the restriction of multiplication, resp. inversion, in $\mathrm{GL}(V)$ it suffices to consider $G = \mathrm{GL}(V)$. By choosing a basis we further reduce to $G = \mathrm{GL}_n(k)$. Note that, by Proposition 2.14, all isomorphisms of algebraic varieties are homeomorphisms.

The map $g \mapsto h^{-1}g$ is the inverse of $g \mapsto hg$. So to prove that the maps $g \mapsto hg$, $h \in G$ are isomorphisms of algebraic varieties, it suffices to prove that the maps $g \mapsto hg$ are morphisms.

For every $g \in G$ the matrix coordinates of hg are linear combinations of the matrix coordinates of g. Hence, the k-algebra homomorphism $\varphi^\flat \colon k[\mathrm{GL}_n] \to k[\mathrm{GL}_n]$ given by $X_{ij} \mapsto X_{ij} \circ (h\cdot)$ is well-defined. Since $X_{ij}(hg) = \varphi^\flat(X_{ij})(g)$, it follows from Fact 2.13 that $g \mapsto hg$ is a morphism of algebraic varieties.

Since the map $g \mapsto g^{-1}$ is its own inverse, it suffices to show that it is a morphism. The inverse of a matrix $M \in \mathrm{GL}_n$ is given by $\det(M)^{-1}M^{\mathrm{adj}}$, where M^{adj} is the adjunct of M. Therefore, the matrix coordinates of g^{-1} are polynomials in $\det(g)^{-1}$ and the matrix coordinates of g. So we obtain a k-algebra homomorphism $k[\mathrm{GL}_n] \to k[\mathrm{GL}_n]$ by $X_{ij} \mapsto X_{ij} \circ (\cdot)^{-1}$. We conclude, as before, that $g \mapsto g^{-1}$ is a morphism. □

In fact, a group is linear algebraic if and only if it is an affine algebraic variety such that the inversion map and multiplication are morphisms of algebraic varieties (see [7, page 63]).

Definition 2.20. Let G and H be linear algebraic groups. An *algebraic group homomorphism* $G \to H$ is a group homomorphism which is a morphism of algebraic varieties.

Special Case 2.21. If G and H are closed matrix groups, then a group homomorphism $\varphi\colon G \to H$ is an *algebraic group homomorphism* if and only if there exist polynomial functions $f_{ij} \in k[G]$ such that the matrix coordinate $\varphi(g)_{ij}$ is equal to $f_{ij}(g)$ for all i, j.

2.3 Invariants cannot tell the difference between a group and its closure

Let V be a finite dimensional vector space.

Proposition 2.22. *The Zariski-closure $\overline{G}$ of a subgroup $G \subset \mathrm{GL}(V)$ is a linear algebraic group.*

Proof. We have to show that $\overline{G}$ is closed under inversion and multiplication.

For all elements $h \in G$ the set $h\overline{G}$ contains G and is Zariski-closed as the preimage of a Zariski-closed set under the continuous map $\mathrm{GL}(V) \to \mathrm{GL}(V)$, $g \to h^{-1}g$ (Lemma 2.19). Since the Zariski-closure of G is minimal among all Zariski-closed sets containing G, we have $\overline{G} \subset h\overline{G}$ for all $h \in G$. Since we also have $\overline{G} \subset h^{-1}\overline{G}$, we deduce the equality $\overline{G} = h\overline{G}$ for every $h \in G$. Altogether we conclude $\overline{G} = G\overline{G}$. Furthermore,

$$\overline{G} = G\overline{G} = \bigcup_{g \in \overline{G}} Gg = \overline{\bigcup_{g \in G} Gg} \supset \bigcup_{g \in \overline{G}} \overline{Gg} \overset{\text{continuity}}{=} \bigcup_{g \in \overline{G}} \overline{G}g = \overline{G}\,\overline{G}.$$

Hence, the set $\overline{G}$ is closed under multiplication.

By continuity of inversion, for all Zariski-closed sets $C \subset \mathrm{GL}(V)$ containing G the set C^{-1} is also a Zariski-closed set containg G. Therefore, $\overline{G} \subset C^{-1}$ or, equivalently, $\overline{G}^{-1} \subset C$ for all Zariski-closed sets $C \subset \mathrm{GL}(V)$. The equality $\overline{G}^{-1} = \overline{G}$ follows. $\square$

Proposition 2.23. *The ring of invariants under linear substitutions $k[V]^G$ of a subgroup G of $\mathrm{GL}(V)$ (see Remark 2.7) is equal to the ring of invariants $k[V]^{\overline{G}}$ of its closure $\overline{G}$.*

Proof. We have the inlcusion $k[V]^{\overline{G}} \subset k[V]^G$ because all elements invariant under $\overline{G}$ are invariant under the smaller group G.

Note that the closure of a subset $S \subset \mathrm{End}(V)$ is given by $\mathcal{V}(\mathcal{I}(S))$ for the ideal $\mathcal{I}(S) := \{f \in k[\mathrm{End}(V)] \,|\, \forall g \in S : f(g) = 0\}$. Let $f \in k[V]^G$ be a G-invariant and $v \in V$. Consider the polynomial function $\hat{f}_v\colon \mathrm{End}[V] \to k$, $g \mapsto gf(v) - f(v)$. Observe that $\hat{f}_v$ vanishes on G and is in $\mathcal{I}(G)$. Since $\overline{G} = \mathcal{V}(\mathcal{I}(G))$, the function $\hat{f}_v$ must also vanish on $\overline{G}$ which is the same as saying $gf(v) = f(v)$ for every $g \in G$. Varying v shows that f is a $\overline{G}$-invariant. Varying f shows the equality $k[V]^G = k[V]^{\overline{G}}$. $\square$

Remark 2.24. By adapting the proof, we can show that a function $k[V]$ is $\overline{G}$-invariant if and only if it is invariant under a Zariski-dense subset of $\overline{G}$.

2.4 Examples

Example 2.25. The special linear group $\mathrm{SL}(V)$ is equal to $\mathcal{V}(\det -1) \subset \mathrm{GL}(V)$ and is therefore a linear algebraic group.

Example 2.26. The multiplicative group of k is the linear algebraic group $\mathrm{GL}_1(k)$. We often denote the multiplicative group of k by $k^\times$ or, if we want to emphasize its algebraic structure, by $\mathbb{G}_m$.

Example 2.27. The additive group of k is viewed as a linear algebraic group via the obvious isomorphism to the matrix group

$$\mathbb{G}_a := \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \,\middle|\, \lambda \in k \right\} = \mathcal{V}(X_{11} - 1, X_{21}, X_{22} - 1) \subset \mathrm{SL}_2(k).$$

Example 2.28. A *torus* is a group isomorphic to a finite power of the multiplicative group $\mathbb{G}_m$. Tori are linear algebraic groups. Indeed,

$$\mathbb{G}_m^n \cong \{\mathrm{diag}(\lambda_1, \ldots, \lambda_n) \mid \lambda_i \in k^\times\} = \mathcal{V}(\{X_{ij} \mid i \neq j\}) \cap \mathrm{GL}_n(k).$$

Example 2.29 (Finite groups are linear algebraic). The symmetric group S_n is isomorphic to the group P_n of $n \times n$ permutation matrices. Cayley's Theorem states that every finite group is a subgroup of some S_n or, equivalently, of some P_n. So finite groups are closed subgroups of GL_n.

3 Rational Representations

In this chapter we look at *rational representations*, which are representations of linear algebraic groups that keep track of the additional algebraic data. Those familiar with representation theory will recognize that many of the notions discussed here are the same as those of abstract representations up to the adjective "rational".

Fix a linear algebraic group G.

Definition 3.1. A *rational representation of* G is an algebraic group homomorphism $\rho : G \to \mathrm{GL}(V)$ for some finite dimensional vector space V.

Remark 3.2. A rational representation $\rho\colon G \to \mathrm{GL}(V)$ induces a linear action on V. The *G-translates* of a vector $v \in V$ are elements of the *G-orbit* $Gv := \{\rho(g)(v) \mid g \in G\}$ of v. When introducing a rational representation $\rho : G \to \mathrm{GL}(V)$ we often just specify the vector space V instead of the homomorphism ρ. In this case the G-translates of a vector v are written either as gv or ${}^{g}v$.

Example 3.3. If $G \subset \mathrm{GL}(V)$, then the inclusion $G \hookrightarrow \mathrm{GL}(V)$ is a rational representation.

Fact 3.4. *Every representation of a finite group is rational.*

Definition 3.5. Let V be a rational representation of G.

(i) A subset $S \subset V$ is *G-stable* if the set $GS := \bigcup_{s\in S} Gs$ lies in S.

(ii) A *subrepresentation* of V is a G-stable subspace.

Fact 3.6. *Every subrepresentation of a rational representation is also a rational representation.*

Example 3.7. For any rational representation V of G we *space of invariants* as $V^G := \{v \in V \mid \forall g \in G : gv = v\} = \{v \in V \mid Gv = \{v\}\}$. Its elements are called *(G-)invariants.* Every subset of V^G is G-stable.

Definition 3.8. (i) A rational representation of G is *simple* if it has exactly two subrepresentations.

(ii) A rational representation of G is *semisimple* if every subrepresentation possesses a G-stable linear complement.

Fact 3.9. *A rational representation is semisimple if and only if it is completely reducible, i.e., if it can be written as a finite direct sum of simple subrepresentations.*

Definition 3.10. A *G-linear map* is a linear map $\varphi : V \to W$ of rational representations of G which is G-equivariant, i.e., $\varphi(gv) = g\varphi(v)$ for all $g \in G$ and $v \in V$. A *G-isomorphism* is a G-linear map possessing a G-linear two-sided inverse.

Fact 3.11. *A G-linear map is a G-isomorphism if and only if it is an isomorphism of vector spaces.*

We create new rational representations from old ones by applying constructions from linear algebra.. Let $\rho\colon G \to \mathrm{GL}(V)$ be a rational representation.

Example 3.12. Define the *contragredient representation* $\rho^\vee\colon G \to \mathrm{GL}(V^\vee)$ by

$$\rho^\vee(g)(l) := {}^g l := l \circ \rho(g^{-1})$$

for all $g \in G$ and $l \in V^\vee$. The representation $\rho^\vee$ is rational. If we let G act trivially on k, then the G-invariants of $V^\vee$ are precisely the G-linear forms.

Proof. By choosing a basis, we can assume $\mathrm{GL}(V) = \mathrm{GL}_n(k)$. For every $g \in G$ the matrix $\rho^\vee(g)$ is equal to $(\rho(g)^{-1})^t = \rho(g^{-1})^t$. Since ρ is a rational representation and inverting is an automorphism of the algebraic variety G (Lemma 2.19), we conclude that the matrix coordinates of $\rho^\vee(g)$ are given by functions in $k[G]$. So $\rho^\vee$ is a rational representation.

If $l \in (V^\vee)^G$ is a G-invariant linear form, then we have $l(gv) = {}^{g^{-1}}l(v) = l(v) = gl(v)$ for all $g \in G$ and $v \in V$, that is, l is G-linear. On the other hand, if $l \in V^\vee$ is G-linear, then ${}^g l(v) = l(g^{-1}v) = g^{-1}l(v) = l(v)$, that is, l is G-invariant. $\square$

Remark 3.13. We endow all dual spaces of rational representations with the contragredient representation.

Example 3.14. Let $\sigma\colon G \to \mathrm{GL}(W)$ be an additional rational representation. Then the vector space $\mathrm{Hom}_k(V, W)$ is a rational representation, called the Hom-*representation*, via the linear action of G given by ${}^g\varphi := \sigma(g) \circ \varphi \circ \rho(g^{-1})$. We might remember this definition by writing $\rho(g)(v) = gv$ and $\sigma(g)(w) = gw$ and then noticing ${}^g\varphi(v) = g\varphi(g^{-1}v)$ looks like conjugation. The G-invariants of $\mathrm{Hom}_k(V, W)$ are precisely the G-linear maps $V \to W$. Example 3.12 is really the special case $\mathrm{Hom}_k(V, k)$.

Example 3.15. Let $U \subset V$ be a subrepresentation of V. There is a unique rational representation $\rho_{V/U}\colon G \to \mathrm{GL}(V/U)$, the *quotient representation*, such that the projection $\pi\colon V \twoheadrightarrow V/U$ is G-linear. The quotient representation $\rho_{V/U}\colon G \to \mathrm{GL}(V/U)$ is defined by $\rho_{V/U}(g)(v + U) := \rho(g)(v) + U$ for $g \in G$ and $v \in V$.

Proof. By cleverly choosing bases of V and V/U we can assume that the matrix representation of $\rho_{V/U}(g)$ is a block on the diagonal of the block-triangular matrix representation of $\rho(g)$. So the fact that ρ is an algebraic group homomorphism immediately implies that $\rho_{V/U}$ is also an algebraic group homomorphism. The proof of the G-linearity of $V \twoheadrightarrow V/U$ and uniqueness are immediate verifications. $\square$

4 Rational Actions on Algebras

Fix a linear algebraic group G and a finitely generated k-algebra R.

Definition 4.1. The group G acts *rationally* on R if it acts as a group of k-linear k-algebra automorphisms and if R is a sum of finite dimensional rational representations V_i of G such that the action of G on the V_i coincides with the action on R.

Fact 4.2. *If G acts rationally on R, then the vector space spanned by the G-orbit of any element $a \in R$ is finite dimensional and every G-stable finite dimensional subspace $V \subset R$ is a rational representation.*

Example 4.3 (Regular action). Let V be a rational representation of G. Define an action $\alpha\colon G \times k[V] \to k[V]$ by ${}^g f(v) = f(g^{-1}v)$ for $v \in V$ and $f \in k[V]$. The action α preserves the grading on $k[V]$: for all $f \in k[V]_d$, $g \in G$, $v \in V$, and $\lambda \in k$ we have ${}^g f(\lambda v) = f(g^{-1}\lambda v) = f(\lambda g^{-1} v) = \lambda^d({}^g f(v))$. Hence, the k-algebra $k[V]$ is the union of the rational representations $\bigoplus_{d=0}^n k[V]_d$, $n > 0$. So α is a rational action, called the *regular action* of G on $k[V]$.

Definition 4.4. Let G act rationally on R.

(i) An element $x \in R$ is *G-invariant* if $gx = x$ for all $g \in G$.

(ii) A subset $B \subset R$ is *G-stable* if $gb \in B$ for all $b \in B$.

(iii) The *ring of invariants* of R is the subring, denoted R^G, of G-invariant elements of R.

Example 4.5. Let V be a rational representation of G. Since the regular action on $k[V]$ respects the grading, we have $k[V]_d^G = k[V]_d \cap k[V]^G$. Therefore, the ring of invariants $k[V]^G = \bigoplus_{d\geqslant 0} k[V]_d^G$ is $\mathbb{Z}^{\geqslant 0}$-graded.

Definition 4.6. Let G act rationally on R and on a finitely generated k-algebra S. A k-algebra homomorphism $\varphi\colon R \to S$ is a *G-homomorphism* if it is G-equivariant. A *G-isomorphism* is a G-homomorphism with a G-equivariant two-sided inverse.

Fact 4.7. *(i) The kernel of a G-homomorphism is G-stable.*

(ii) If G acts rationally on R and $\mathfrak{a} \subset R$ is a G-stable ideal, then there is a unique rational action of G on $R/\mathfrak{a}$, given by $g(a + \mathfrak{a}) := ga + \mathfrak{a}$, such that the projection $R \twoheadrightarrow R/\mathfrak{a}$ is G-equivariant.

(iii) A G-homomorphism is a G-isomorphism if and only if it is an isomorphism of k-algebras.

Proposition 4.8. *If G acts rationally on R, then there exists a vector space V and a surjective G-homomorphism $k[V^\vee] \twoheadrightarrow R$. In particular, the k-algebra R is G-isomorphic to $k[V^\vee]/\ker(\varphi)$.*

Proof. Let $V \subset R$ be a finite dimensional G-stable subspace that generates R as a k-algebra. (For example the space spanned by generators $a_1, \dots, a_n$ of R.) Note that $V \hookrightarrow k[V^\vee]$ and that G acts rationally on $k[V^\vee]$ via the regular action. The k-algebra $k[V^\vee]$ is naturally isomorphic to the symmetric algebra $\operatorname{Sym}_k V = \bigoplus_{d \geqslant 0} \operatorname{Sym}_k^d V$ and satisfies a universal property. By this universal property, there exists a unique k-algebra homomorphism $\varphi\colon k[V^\vee] \to R$ such that $\varphi|_V$ is the inclusion $V \hookrightarrow R$. Additionally, the map $\varphi|_V$ is G-equivariant and V generates $k[V^\vee]$ as a k-algebra. Hence, the homomorphism φ is a G-homomorphism and surjects onto R. $\square$

5 Reductivity Conditions

In this chapter we define what a geometrically reductive group is. We motivate the definition by first studying groups whose nonzero rational representations are semisimple, the so-called linearly reductive groups. Throughout the discussion we mention some groups which do or do not satisfy the reductivity conditions.

5.1 Linearly reductive groups

Definition 5.1. A linear algebraic group G is *linearly reductive* if for every rational representation V and every nonzero G-invariant vector $v \in V^G \smallsetminus \{0\}$ there exists a G-invariant linear form $l \in (V^\vee)^G$ that does not vanish at v, i.e., $l(v) \neq 0$.

Proposition 5.2. *For any linear algebraic group G the following three statements are equivalent.*

(i) The group G is linearly reductive.

(ii) Every rational representation of G is semisimple.

(iii) For every surjective G-linear map $\varphi\colon V \twoheadrightarrow W$ of rational representations the induced map of subspaces of invariants $\varphi\colon V^G \to W^G$ is surjective.

Proof. We show the sufficiently many implications *"(i) $\Longrightarrow$ (ii)", "(ii) $\Longrightarrow$ (iii)"* and *"(iii) $\Longrightarrow$ (i)"*.

We repeatedly use Examples 3.12, 3.14, and 3.15. Throughout the proof we identify finite dimensional vectorspaces V with their double dual $V^{\vee\vee} := (V^\vee)^\vee$ via the canonical isomorphism

$$\mathrm{eval}\colon V \xrightarrow{\sim} V^{\vee\vee},\ v \mapsto \mathrm{eval}_v\colon l \mapsto l(v).$$

Moreover, if V is a rational representation of G and $V^{\vee\vee}$ is endowed with the contragredient representation of $V^\vee$, then this isomorphism is a G-isomorphism.

Let V be an arbitrary nonzero rational representation of G.

"(iii) $\Longrightarrow$ (i)": Let G act trivially on k. Let $v \in V^G \smallsetminus \{0\}$ be a nonzero G-invariant. Then $\mathrm{eval}_v\colon V^\vee \twoheadrightarrow k$ is a surjective G-equivariant linear form. Applying *(iii)* to eval_v yields the surjective map $\mathrm{eval}_v\colon (V^\vee)^G \twoheadrightarrow k^G = k$. A preimage $l \in (V^\vee)^G$ of $1 \in k$ is a G-invariant linear form which does not vanish at v.

"(ii) $\Longrightarrow$ (iii)": Let $\varphi\colon V \twoheadrightarrow W$ be a surjective G-linear map. Applying V's assumed semisimplicity, we find a G-stable linear complement V' of $\ker(\varphi) \subset V$. The representation V' is G-isomorphic to W. Hence, the space of invariants $(V')^G$ is G-isomorphic to W^G. Since V^G contains $(V')^G$, we conclude that V^G surjects onto W^G.

"(i) $\Longrightarrow$ (ii)": The idea for this implication is the following. We want a G-stable linear complement of a proper subrepresentation $U \subset V$. Each linear complement is the image of some linear section of the projection $V \twoheadrightarrow V/U$, and if a linear section is G-equivariant, then the image is G-stable. So all we need to do is find a G-linear section.

First we fix some notation. Let $U \subset V$ be a proper subrepresentation. We combine Examples 3.14 and 3.15 to obtain a rational representation $\operatorname{Hom}(V/U, V)$. Let $\pi\colon V \twoheadrightarrow V/U$ be the canonical projection and $\sigma\colon V/U \to V$ a linear section of π, i.e., $\sigma \in \operatorname{Hom}(V/U, V)$ such that $\pi \circ \sigma = \mathrm{id}_{V/U}$. Let $T := \operatorname{span}_k(G\sigma)$ be the subrepresentation of $\operatorname{Hom}(V/U, V)$ spanned by the G-orbit of σ. Finally, let $T' \subset T$ to be the subspace spanned by the set $\{{}^g\sigma - \sigma \mid g \in G\}$.

For every $g \in G$ and $v \in V$ we have

$$\begin{aligned}\pi \circ ({}^g\sigma - \sigma)(v + U) &= \pi(g\sigma(g^{-1}v + U)) - \pi(\sigma(v + U)) = \\ &= g\pi(\sigma(g^{-1}v + U)) - \pi(\sigma(v + U)) = gg^{-1}v - v + U = 0.\end{aligned}$$

In other words, for every $g \in G$ the image of ${}^g\sigma - \sigma$ is contained in the kernel of π. If σ were in T' then $\pi \circ \sigma = 0$, but this is impossible, because U is a proper subrepresentation and thus $\mathrm{id}_{V/U} \neq 0$. So σ is in $T \smallsetminus T'$ and $T = T' + \operatorname{span}_k(\sigma)$. We legally choose a nonzero linear form $l\colon T \to k$ such that $l|_{T'} = 0$. The differences ${}^gl - l$ vanish on all of T, meaning that l is G-invariant. By invoking G's linear reductivity, we find a G-invariant vector $\tau \in T^G = (T^{\vee\vee})^G$ such that $l(\tau) \neq 0$. We know τ cannot be in T', thus when we write $\tau = a\sigma + \sum_{g\in G} a_g({}^g\sigma - \sigma)$ for some $a, a_g \in k$, that is, as a k-linear combination of σ and ${}^g\sigma - \sigma$, we have $a \neq 0$. Replace τ with $a^{-1}\tau$. We check that the G-linear map $\tau\colon V/U \to V$ is a section of π. Indeed,

$$\pi \circ \tau = \pi \circ \tau = \pi \circ (\sigma + \sum_{g\in G} a_g({}^g\sigma - \sigma)) = \pi \circ \sigma = \mathrm{id}_{V/U}\,.$$

□

Example 5.3. Let G be a finite group. It is known from the representation theory of finite groups that if $\operatorname{char}(k)$ does not divide the order of G, then every nonzero finite dimensional representation of G is semisimple or, equivalently, the group G is linearly reductive. If $\operatorname{char}(k)$ divides G, then G need not be linearly reductive.

Theorem 5.4. *Tori are linearly reductive.*

Proof. First we prove that $\mathbb{G}_m$ is linearly reductive by showing that every rational representation of $\mathbb{G}_m$ is semisimple.

By choosing a basis, it suffices to consider a rational representation $\rho\colon \mathbb{G}_m \to \mathrm{GL}_n$. Since $k[\mathbb{G}_m] = k[X^{\pm 1}]$, for every $t \in \mathbb{G}_m$ the matrix coordinates of $\rho(t)$ are equal to $f_{ij}(t)$

for certain $f_{ij}(X) \in k[X^{\pm 1}]$. In particular, the matrix coordinates of $\rho(t)$ are k-linear combinations of integer powers of t. (Here we crucially use the rationality of ρ.) Choose matrices $M_i \in \mathrm{Mat}_n(k)$, almost all equal to zero, such that $\rho(t) = \sum_{i\in\mathbb{Z}} t^i M_i$ for all $t \in \mathbb{G}_m$. For all $t, s \in \mathbb{G}_m$ we have

$$\sum_i t^i s^i M_i = \rho(ts) = \rho(t)\rho(s) = \sum_{i,j} t^i s^j M_i M_j. \tag{5.1}$$

Exploiting the linear independence of the set of maps $\{t \mapsto t^i \,|\, i \in \mathbb{Z}\}$, we deduce from (5.1) the equality

$$s^i = \sum_j s^j M_i M_j \tag{5.2}$$

for every i. Applying linear independence to (5.2) yields the matrix equalities

$$M_i M_j = 0 \tag{5.3}$$
$$M_i^2 = M_i \tag{5.4}$$

for every i and j. Set $V_i := \mathrm{im}(M_i)$ for every i. If a vector v is in the intersection $V_i \cap V_j$, then $v = M_i v_i = M_j v_j$ for some $v_i, v_j \in V$. Multiplying v with M_i yields

$$v = M_i v_i \overset{(5.4)}{=} M_i^2 v_i = M_i v \overset{(5.3)}{=} M_i M_j v_i = 0.$$

Consequently we have $V_i \cap V_j = 0$. Furthermore, every $v \in V$ is equal to $\rho(1)v = \sum_i M_i v$, which lies in $\bigoplus_i V_i$. Since almost all M_i are zero, altogether we have shown that the representation ρ is completely reducible or, equivilantely (Fact 3.9), that ρ is semisimple.

Now we show that if $\mathbb{G}_m^{n-1}$ is linearly reductive, then so is $\mathbb{G}_m^n$. Whence the theorem follows inductively. We view $\mathbb{G}_m^n$ as the group G of diagonal $n \times n$-matrices, the group $\mathbb{G}_m^{n-1}$ as the group H of diagonal matrices with first diagonal entry equal to 1, and $\mathbb{G}_m$ as the group N of diagonal matrices with last $(n-1)$-diagonal entries equal to 1. The groups H and N are (normal) subgroups of G.

Let $\varphi\colon V \twoheadrightarrow W$ be a surjective G-linear map. Since H is assumed to be linearly reductive, Proposition 5.2 *(iii)* tells us that the induced H-linear map $\varphi'\colon V^H \twoheadrightarrow W^H$ is surjective. Since G is commutative, the group N acts on V^H and W^H. Also, the map $\varphi'\colon V^H \twoheadrightarrow W^H$ is N-linear. We have already shown that N is linearly reductive, so the induced map $\overline{\varphi}\colon (V^H)^N \twoheadrightarrow (W^H)^N$ is surjective.

We claim $(V^H)^N = V^G$. The G-invariants are also invariants of any subgroup of G, so we have the inclusion $V^G \subset (V^H)^N$. Let $v \in (V^H)^N$ and $g \in G$. By definition of the groups G, H and N, we find certain $g_N \in N$ and $g_H \in H$ such that $g = g_N g_H$. Hence $gv = g_N g_H v = g_N v = v$. Varying g and v the inclusion $(V^H)^N \subset V^G$ follows. Similarly, we have $(W^H)^N = W^G$.

Remark that, in addition to being surjective, the map $\overline{\varphi}\colon V^G \twoheadrightarrow W^G$ agrees with φ. So the statement *(iii)* in Proposition 5.2 is true for the group $G = \mathbb{G}_m^n$. □

Example 5.5. Assume $\mathrm{char}(k) = 2$. Consider the rational representation $\rho\colon \mathrm{SL}_2(k) \to \mathrm{GL}_3(k)$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 1 & ac & bd \\ 0 & a^2 & b^2 \\ 0 & c^2 & d^2 \end{pmatrix}.$$

From this block triangular form we deduce that ρ is reducible and that the subspace of k^3 spanned by $(1, 0, 0)^t$ is G-stable. If ρ is semsimple, then there exists a basis of k^3 such that with regard to this basis $\rho(g)$, $g \in \mathrm{SL}_2(k)$, has block diagonal form with first block equal to $1 \in k$. Since in characteristic 2 the expressions ac and bd are not linear polynomials in a^2, b^2, c^2 and d^2, such a basis does not exist. Consequently, the group $\mathrm{SL}_2(k)$ is not linearly reductive.

Remark 5.6. Example 5.5 was taken from [10], where Nagata proves that in positive characteristic the only connected linearly reductive groups are subgroups of tori.

5.2 Geometrically reductive groups

Definition 5.7. A linear algebraic group G is *geometrically reductive* if for every rational representation V of G and every nonzero G-invariant vector $v \in V^G \setminus \{0\}$ there exists a nonconstant G-invariant homogeneous polynomial function $f \in k[V]_d^G \setminus k$ that does not vanish at v, i.e., $f(v) \neq 0$.

Fact 5.8. *Every linearly reductive group is geometrically reductive.*

Proposition 5.9. *All finite groups are geometrically reductive.*

Proof. Let G be a finite group and V a representation of G. Let $v \in V^G \setminus \{0\}$ be a nonzero G-invariant vector. Choose a linear form $l \in V^\vee$ such that $l(v) \neq 0$. We check that the G-invariant degree $|G|$ homogeneous polynomial function $f := \prod_{g \in G} {}^g l$ is the one we want. Indeed,

$$f(v) = \prod_{g \in G} {}^g l(v) = \prod_{g \in G} l(g^{-1}v) \overset{v \text{ invariant}}{=} l(v)^{|G|} \neq 0. \qquad \square$$

Proposition 5.10. *If* $\mathrm{char}(k) = 0$*, then a linear algebraic group over k is geometrically reductive if and only if it is linearly reductive.*

Proof. Let G be a geometrically reductive group and V a rational representation. Let $w \in V^G \setminus \{0\}$ be a nonzero invariant and $f \in k[V]_d^G$ be a nonconstant homogeneous G-invariant polynomial function which does not vanish at w. We can write for any $v_1, \dots, v_n \in V$ and $\lambda_1, \dots, \lambda_d \in k$

$$f(\lambda_1 v_1 + \ldots + \lambda_d v_d) = \sum_{i_1 + \ldots + i_d = d} \lambda_1^{i_1} \cdots \lambda_d^{i_d} f_{i_1, \ldots, i_d}(v_1, \ldots, v_d) \tag{5.5}$$

such that each function $f_{i_1,\ldots,i_d}\colon V^d \to k$, $(v_1,\ldots,v_d) \mapsto f_{i_1,\ldots,i_d}(v_1,\ldots,v_d)$ is *multihomogeneous* of *multidegree* $(i_1,\ldots,i_d)$, i.e., $f(\lambda_1 v_1,\ldots,\lambda_d v_d) = \lambda_1^{i_1}\cdots\lambda_d^{i_d} f(v_1,\ldots,v_d)$ for all $\lambda_1,\ldots,\lambda_d \in k$, and the function $f_{1,\ldots,1}$ is multilinear.

Plugging $v_1 := \ldots := v_d := v$ into (5.5) yields

$$\begin{aligned}\sum_{i_1+\ldots+i_d=d} \lambda_1^{i_1}\cdots\lambda_d^{i_d} f_{i_1,\ldots,i_d}(v,\ldots,v) &= \sum_{i_1+\ldots+i_d=d} f_{i_1,\ldots,i_d}(\lambda_1 v,\ldots,\lambda_d v)\\ &= f(\lambda_1 v+\ldots+\lambda_d v)\\ &= (\lambda_1+\ldots+\lambda_d)^d f(v)\\ &= (\lambda_1^d+\ldots+d!\lambda_1\cdots\lambda_d)f(v).\end{aligned}$$

By comparing coefficients of the monomials $\lambda_1^{i_1}\cdots\lambda_n^{i_n}$, we deduce the equality

$$f_{1,\ldots,1}(v,\ldots,v) = d!f(v) \tag{5.6}$$

for all $v \in V$. Define $l\colon V \to k$ by $l(v) := f_{1,\ldots,1}(v,w,\ldots,w)$. The multilinearity of $f_{1,\ldots,1}$ implies that l is a linear form. Since f is G-invariant and the action of G respects the multidegree, the function $f_{1,\ldots,1}$ is G-invariant, i.e., we have ${}^g f_{1,\ldots,1}(v_1,\ldots,v_n) := f_{1,\ldots,1}(g^{-1}v_1,\ldots,g^{-1}v_n) = f_{1,\ldots,1}(v_1,\ldots,v_n)$. The G-invariance of w and $f_{1,\ldots,1}$ imply that l is G-invariant. Note that $d! \neq 0$ because $\mathrm{char}(k) = 0$. So we have

$$l(w) \overset{(5.6)}{=} d!f(w) \neq 0.$$

Varying w and V shows that G is linearly reductive. □

This proof is inspired by the basic properties of *polarisation*, a concept from Classical Invariant Theory (see [8, page 34]).

5.3 Outlook: A criterion for geometric reductivity

To show that groups such as $\mathrm{SL}(V)$ and $\mathrm{GL}(V)$ are geometrically reductive, we use facts about *reductive groups*. In particular, we apply Haboush's Theorem (Theorem 5.14), which asserts that every reductive group is geometrically reductive. We do not give proofs of these facts. See [7] for a thorough introduction to reductive groups. Proposition 5.18 and its proof are taken from [13].

Let G be a linear algebraic group.

Definition 5.11. A *unipotent element* of a linear algebraic group $G \subset \mathrm{GL}(V)$ is an element which is the sum of the identity and a nilpotent endomorphism of V.

Proposition-Definition 5.12. (i) A largest connected normal solvable subgroup of G exists and is called the *radical* of G and is denoted by $\mathrm{rad}(G)$.

(ii) The subgroup of unipotent elements in $\mathrm{rad}(G)$ is normal in G and is called the *unipotent radical* of G and is denoted by $\mathrm{rad}_u(G)$.

Definition 5.13. (i) A *semisimple group* is a nontrivial linear algebraic group with trivial radical.

(ii) A *reductive group* is a nontrivial linear algebraic group with trivial unipotent radical.

Theorem 5.14 (Haboush [5]). *Every reductive group is geometrically reductive.*

Theorem 5.15 (Miyata, Nagata [12]). *Every geometrically reductive group is reductive.*

Lemma 5.16. *Let G be a linear algebraic group and $N \subset G$ a normal closed subgroup. For every rational representation V of G the subspace of N-invariants V^N is a subrepresentation of V*

Proof. Let $v \in V^N$ and $g \in G$ and $n \in N$. Then for all $n \in N$ we have

$$ngv = g\underbrace{(g^{-1}ng)}_{\in N}v = gv.$$

Hence, gv is N-invariant and lies in V^N. So V^N is G-stable. □

Lemma 5.17. *The subspace of invariants $V^{\mathrm{rad}_u(G)}$ of any nonzero finite dimensional representation V of the unipotent radical $\mathrm{rad}_u(G)$ is nonzero.*

Proof. Consider a unipotent element $u = 1 + \varphi_u \in \mathrm{rad}_u(G)$. Then for all $v \in V$ we have $uv = v + \varphi_u(v)$. Since the kernel of φ_u is nonempty there must be a vector v fixed by u. So we would need to find a vector which is simultaneaously killed by all φ_u, for $u \in \mathrm{rad}_u(G)$. Doing this requires hard facts from representation theory. See [7, page 112] for a theorem which implies this lemma. □

Proposition 5.18. *If a linear algebraic group G possesses a simple faithful representation $\rho\colon G \hookrightarrow \mathrm{GL}(V)$ then G is reductive.*

Proof. The hard work is done by Lemma 5.17. The unipotent radical $\mathrm{rad}_u(G)$ is normal. Therefore, the space $V^{\mathrm{rad}_u(G)}$ is a nonzero subrepresentation of V. Since V is simple we must have $V = V^{\mathrm{rad}_u(G)}$. So $\mathrm{rad}_u(G)$ acts trivially on V, that is, $\rho(\mathrm{rad}_u(G)) = \{\mathrm{id}_V\}$. Since ρ is faithful, it follows that $\mathrm{rad}_u(G)$ is trivial. □

Together, Proposition 5.18 and Theorem 5.14 give us a method of proving geometric reductivity.

Example 5.19 ($\mathrm{SL}(V)$ and $\mathrm{GL}(V)$ are geometrically reductive). The group $\mathrm{SL}(V)$ acts faithfully on V. For every pair of vectors $v, v' \in V$ there exists an element $g \in \mathrm{SL}(V)$ such that $gv \in \mathrm{span}_k(v')$. In particular, no proper nonzero subspace can be $\mathrm{SL}(V)$-stable. Therefore V is a simple representation of $\mathrm{SL}(V)$. Now, Proposition 5.18 tells us that $\mathrm{SL}(V)$ is reductive and, by Haboush's Theorem, geometrically reductive. The same argument shows that $\mathrm{GL}(V)$ is geometrically reductive.

6 Nagata's Theorem and its Proof

First, we prove two special cases of finiteness: finite groups and linearly reductive groups. Then, using a lot of commutative algebra, we stitch together the ideas of the simpler cases to create the proof of Nagata's Theorem, Theorem 6.5.

The proof of Theorem 6.3 is taken from [8]. The proof of Theorem 6.5 follows the argument in the original paper [11] with the help of recreations in [2, page 43] and [15, page 24].

Proposition 6.1. *If a finite group G acts rationally on a finitely generated k-algebra R, then the ring of invariants R^G is a finitely generated k-algebra.*

Proof. For every $x \in R$ the monic G-invariant polynomial $f_x(X) := \prod_{g \in G}(X - gx)$ vanishes at x. Hence the ring extenstion $R^G \subset R$ is integral. We conclude, by Proposition 1.4, that R^G is finitely generated over k. □

Lemma 6.2. *For every rational representation V of a linearly reductive group G there exists a unique G-linear projection $\pi : k[V] \twoheadrightarrow k[V]^G$ such that $\pi(hf) = h\pi(f)$ for every invariant $h \in k[V]^G$ and polynomial $f \in k[V]$. This projection is called a* Reynolds-Operator*.*

Proof. Consider the rational representations $V_n := \bigoplus_{d=0}^n k[V]_d$. Applying G's linear reductivity, we choose G-stable linear complements V_n' of the subspaces of G-invariants $V_n^G = \bigoplus_{d=0}^n k[V]_d^G \subset V_n$. Let $\pi_n \colon V_n \twoheadrightarrow V_n/V_n' \cong V_n^G$ be the canonical G-linear projections. Observe that for $n < m$ the projections π_n and π_m agree on V_n. We get a linear projection $\pi \colon k[V] \twoheadrightarrow k[V]^G$ which agrees with π_n on V_n by defining $\pi(f) := \pi_{\deg(f)}(f)$ for every $f \in k[V]$.

Choose N large enough such that $hf \in V_N$. Write $f = \pi_N(f) + f'$ for some $f' \in V_N'$. We have $\pi(hf) = \pi_N(h\pi_N(f) + hf') = h\pi_N^2(f) + 0 = h\pi(f)$. So π is the desired Reynolds-Operator. Uniqueness follows from the fact that any such projection must agree with π_n on V_n. □

Theorem 6.3. *If a linearly reductive group G acts rationally on a finitely generated k-algebra, then the ring of invariants R^G is a finitely generated k-algebra.*

Proof. We first prove the theorem for $R = k[V]$ for some rational representation V of G. In this case, we have a Reynolds-Operator, call it $\pi \colon k[V] \twoheadrightarrow k[V]^G$. Consider the augmentation ideal of the ring of invariants $R^G_+ = \bigoplus_{d>0} k[V]^G_d$. By Hilbert's Basissatz, the ideal $\mathfrak{a} \subset R$ generated by $R^G_+ \subset R$ is finitely generated. Choose homogeneous generators $f_1, \dots, f_n \in R^G_+$ of $\mathfrak{a}$. For all $h \in R^G_+$ we have $h = \sum_i a_i f_i$ for some $a_i \in R$ and

$$h = \pi(h) = \sum_i \underbrace{\pi(a_i)}_{\in R^G} f_i,$$

thus $h \in \sum_i f_i R^G$. Hence the f_i generate R^G_+ as an ideal of R^G. Using Proposition 1.3 we deduce that R^G is finitely generated over $R^G_0 = k$.

Now let R be arbitrary. By Proposition 4.8, there exists a surjective G-homomorphism $k[V] \twoheadrightarrow R$. Using linear reductivity in the form of Proposition 5.2 *(iii)*, we conclude that the induced map $k[V]^G \twoheadrightarrow R^G$ is surjective. Since we already proved that $k[V]^G$ is finitely generated, it follows that R^G is finitely generated. □

Lemma 6.4 (Main Step). *Let G be a geometrically reductive group acting rationally on the finitely generated k-algebras R and S. If $\varphi\colon R \twoheadrightarrow S$ is a surjective G-homomorphism, then the induced map of invariants $\varphi\colon R^G \to S^G$ is integral. Additionally, if G is linearly reductive, then $\varphi\colon R^G \to S^G$ is surjective.*

Proof. It suffices to show that for all invariants $b \in S^G$ there exists a positive integer $d > 0$ such that $b^d \in \varphi(R^G)$. This clearly holds for $b = 0$, so we assume for the rest of the proof that $b \neq 0$.

Let $a \in R$ be a preimage of b. Consider the vector space V spanned by the G-orbit of a. The vector space V is finite dimensional because G acts rationally on R. Since b is an invariant, we have $\varphi(ga - a) = gb - b = 0$ for all $g \in G$ or, in other words, $ga - a \in \ker(\varphi)$ for all $g \in G$. Set $U := V \cap \ker(\varphi)$ and observe that U is G-stable as the intersection of two G-stable sets. Choose a basis $a_1, \dots, a_n$ of U. Since $b \neq 0$, its preimage a is not in U. Furthermore, the expression $ga = ga - a + a$ shows that every G-translate lies in $U + \operatorname{span}(a)$. In summary, the set $\{a, a_1, \dots, a_n\}$ is a basis of V.

We claim that $a^\vee\colon V \to k$ is a G-invariant linear form. Note that $a^\vee$ vanishes on U. Let $v \in V$. We can write $v = \lambda a + u$ for certain $\lambda \in k$ and $u \in U$. For all $g \in G$ we have

$${}^g a^\vee(v) = a^\vee(g^{-1}v) = a^\vee(\lambda g^{-1}a + g^{-1}v').$$

Using the G-invariance of U we have

$${}^g a^\vee(v) = a^\vee(\lambda g^{-1}a) = a^\vee(\lambda a + \underbrace{\lambda(g^{-1}a - a)}_{\in U}) = a^\vee(\lambda a) = a^\vee(v),$$

which shows our claim.

We use G's geometric reductivity to get a nonconstant homogeneous G-invariant polynomial $f \in k[X, X_1, \dots, X_n]^G_d = k[V^\vee]^G_d$ such that $f(a^\vee) = f(1, 0, \dots, 0) \neq 0$. By scaling f, we can assume that f is monic in the variable X. Therefore, the difference $f(a, a_1, \dots, a_n) - a^d$ lies in the ideal $(a_1, \dots, a_n)$, which itself is contained in $\ker(\varphi)$. We have

$$b^d = \varphi(a^d) = \varphi(a^d + f(a, a_1, \dots, a_n) - a^d) = \varphi(f(a_1, \dots, a_n)).$$

Because f is G-invariant, the integer d does the job.

If G is linearly reductive, then we can take $d = 1$ and surjectivity follows. $\square$

It is worth comparing the above proof to the proof of Proposition 5.2.

Theorem 6.5. *If a geometrically reductive group G acts rationally on the finitely generated k-algebra S, then the ring of invariants S^G is a finitely generated k-algebra.*

Proof. Applying Proposition 4.8, we reduce to the case $S = k[V]/\mathfrak{b}$ for some G-stable ideal $\mathfrak{b} \subset S$.

Step 1. Assume $\mathfrak{b}$ is homogeneous.

Consider the set of ideals

$$\mathcal{S}_1 := \left\{ \mathfrak{a}' \subset k[V] \;\middle|\; \begin{matrix} \mathfrak{a}' \, G\text{-stable homogenous ideal with } \mathfrak{b} \subset \mathfrak{a}', \\ (k[V]/\mathfrak{a}')^G \text{ not finitely generated over } k \end{matrix} \right\}.$$

Showing that S^G is finitely generated over k is the same as showing that $\mathcal{S}_1$ is empty. Assume, by contradiction, that $\mathcal{S}_1$ is not empty. As a nonempty set of ideals of a noetherian ring, the set $\mathcal{S}_1$ has a maximal element $\mathfrak{a}$. Let R be the graded k-algebra $k[V]/\mathfrak{a}$. We obtain a contradiction by showing that the ring of invariants R^G is finitely generated.

Claim 1. For every nonzero G-stable homogenous ideal $\mathfrak{a}' \subset R$ the ring $R^G/(\mathfrak{a}' \cap R^G)$ is finitely generated over k and $(R/\mathfrak{a}')^G$ is a finite $R^G/(\mathfrak{a}' \cap R^G)$-module.

The maximality of $\mathfrak{a}$ implies the finite generatedness of $(R/\mathfrak{a}')^G$ over k. Lemma 6.4 implies that the extension $R^G/(\mathfrak{a}' \cap R^G) \subset (R/\mathfrak{a}')^G$ is integral. Applying Propositon 1.4 we deduce that $R^G/(\mathfrak{a}' \cap R^G)$ is finitely generated over k and that $(R/\mathfrak{a}')^G$ is a finite $R^G/(\mathfrak{a}' \cap R^G)$-module, which proves Claim 1.

Assume that R is an integral domain. Let $f \in R^G$ be a homogeneous invariant of positive degree d. Since R is an integral domain and $f \neq 0$, all $x \in R$ satisfying $g(fx) - fx = 0$ for all $g \in G$ are G-invariant. Equivalently, we have $fR \cap R^G = fR^G$. By Claim 1, the ring $R^G/(fR \cap R^G) = R^G/fR^G$ is finitely generated over k. From Proposition 1.3 we know that R^G_+/fR^G is finitely generated over k, say by the elements $f_1 + fR^G, \ldots, f_n + fR^G$. Hence R^G_+ is generated by the finitely many elements $f, f_1, \ldots, f_n$. Using Proposition 1.3, we deduce the finite generatedness of R^G over $R^G_0 = k$. This is a contradiction to $\mathfrak{a} \in \mathcal{S}_1$.

Now assume that R has zero divisors. If there is no invariant zero divisor, then we may procced as in the previous paragraph. So we choose an invariant zero divisor $f \in R^G$. We can choose f to be homogeneous. Indeed, if there is an invariant zero divisor, then it can be written as the sum of homogeneous invariants, which must be zerodiviors. Furthermore, since $R_0 = k$ and f is a zero divisor, the degree of f is positive.

Consider the annihilator $\operatorname{Ann}(f) := \{x \in R | fx = 0\}$, which is a homogeneous ideal. As f is G-invariant, for all $x \in \operatorname{Ann}(f)$ we have $fgx = gfgx = g(fx) = 0$. So $\operatorname{Ann}(f)$ is G-stable. By Claim 1, we know that $(R/\operatorname{Ann}(f))^G$ is a finite $R_1 := R^G/\operatorname{Ann}(f) \cap R^G$-module and that both R_1 and $R_2 := R^G/fR \cap R^G$ are finitely generated k-algebras.

Let $A \subset R^G$ be the k-algebra generated by representatives of generators of both R_1 and R_2. Since A surjects onto R_1, the ring $(R/\operatorname{Ann}(f))^G$ is integral over $A/\operatorname{Ann}(f) \cap A$. So $(R/\operatorname{Ann}(f))^G$ is a finite $A/\operatorname{Ann}(f) \cap A$-module. Choose representatives $b_1, \ldots, b_n \in R$ of $A/\operatorname{Ann}(f) \cap A$-module generators of $(R/\operatorname{Ann}(f))$. Observe that $gb_i - b_i + \operatorname{Ann}(f) = 0$ for all i and $g \in G$, which implies that fb_i is G-invariant for all i.

We claim that R^G is finitely generated over A. Let $x \in R^G$. The ring A surjects onto R_2. Choose an $a \in A$ such that $x = a \mod fR$. Let $r \in R$ such that $x - a = fr$. The G-invariance of $x - a$ implies that the difference $r - gr$ annihilates f or, equivalently, $r + \operatorname{Ann}(f) \in (R/\operatorname{Ann}(f))^G$. Since $(R/\operatorname{Ann}(f))^G$ is a finite $A/\operatorname{Ann}(f) \cap A$-module, we write $r + \operatorname{Ann}(f) = \sum_i a_i b_i + \operatorname{Ann}(f)$ for some $a_i \in A$. Hence $x = a + \sum_i a_i f b_i$ and, therefore, the element x lies in $A[fb_1, \ldots, fb_n]$. Varying x proves our claim. As A is a finitely generated k-algebra, it follows that R^G is a finitely generated k-algebra, which is a contradiction to $\mathfrak{a} \in \mathcal{S}_1$. We have done Step 1.

Step 2. The general case.

We want to show that the set

$$\mathcal{S}_2 := \left\{ \mathfrak{a}' \subset k[V] \;\middle|\; \begin{array}{c} \mathfrak{a}' \; G\text{-stable ideal with } \mathfrak{b} \subset \mathfrak{a}', \\ (k[V]/\mathfrak{a}')^G \text{ not finitely generated over k} \end{array} \right\}$$

is empty. By contradiction assume $\mathcal{S}_2$ is not empty and pick a maximal element $\mathfrak{a}$. Set $R := k[V]/\mathfrak{a}$. As before, we find a contradiction by showing that R^G is finitely generated over k.

Claim 2. For all nonzero G-stable ideals $\mathfrak{a}' \subset R$ the ring $R^G/(\mathfrak{a}' \cap R^G)$ is finitely generated over k and $(R/\mathfrak{a}')^G$ is a finite $R^G/(\mathfrak{a} \cap R^G)$-module.

The proof is analagous to that of Claim 1.

If R^G contains zero diviors, then we argue as in Step 1 by dropping the word "homogeneous" and replacing references to Claim 1 with references to Claim 2.

Assume that R^G is an integral domain. By Lemma 6.4, the surjective G-homomorphism $k[V] \twoheadrightarrow R$ induces the integral extension

$$R' := k[V]^G/(\mathfrak{a}_0 \cap k[V]^G) \hookrightarrow R^G.$$

Applying Step 1 to the graded ring $k[V]$ proves that R' is finitely generated over k. Let K be the quotient field of R^G and let K' be the quotient field of R'.

We claim that K is a finitely generated field extension of k. Consider the multiplicative subset $U \subset R$ of all nonzero zero divisors in R. (In the case that R is an integral domain, the ring $U^{-1}R$ is the quotient field of R.) Choose a maximal ideal $\mathfrak{m} \subset U^{-1}R$. All nonzero invariants of R are not zero divisors. Hence, the nonzero invariants are units in $U^{-1}R$. In particular, we have $R^G \cap \mathfrak{m} = (0)$. Consequently, K is a subfield of the field $L := U^{-1}R/\mathfrak{m}$. As the quotient field of the finitely generated k-algebra $R/\mathfrak{m}$, the field L is finitely generated as a field over k. It follows that K is finitely generated as a field over K'.

Since $R' \subset R^G$ is integral, the field extension K/K' is algebraic and, therefore, finite. Applying Noether's Theorem, Theorem 1.5, we find that the integral closure $\tilde{R}'$ of R' in K is finitely generated over k. Use Proposition 1.4 together with the fact that $R^G \subset \tilde{R}'$ is an integral extension to conclude that R^G is finitely generated over k. This is our desired contradiction. □

Remark 6.6. The assumption that G is geometrically reductive is only used when applying Lemma 6.4.

Remark 6.7. Nagata's Theorem guarantees existence of a finite generating set but does not tell us how to find such a set. In the book *Computational Invariant Theory*, Harm Derksen and Gregor Kemper [1] discuss algorithms and their implementation for finding generating sets in different situations such as when G is finite, linearly reductive, or geometrically reductive.

Remark 6.8. By a theorem of Popov [14], a linear algebraic group is geometrically reductive if and only if for every finitely generated k-algebra R on which G acts rationally the rings of invariants R^G is finitely generated over k.

We conclude with some fun examples of finite generating sets of certain rings of invariants. The examples are taken from [8, pages 4-8].

Example 6.9. A standard example of a finitely generated ring of invariants is the ring of symmetric functions $k[X_1, \ldots, X_n]^{S_n} = k[s_1, \ldots, s_n]$, where the s_r are the elementary symmetric functions, that is, $s_r = \sum_{1 \leqslant i_1 < \ldots < i_r \leqslant n} X_{i_1} \cdots X_{i_r}$.

Example 6.10. Let $\mathrm{GL}_n(k)$ act by conjugation on $\mathrm{Mat}_n(k)$. Then $k[\mathrm{Mat}_n(k)]^{\mathrm{GL}(V)} = k[s_1, \ldots, s_n]$, where $s_i \in k[\mathrm{Mat}_n(k)]$ such that for all $A \in \mathrm{Mat}_n(k)$ we can write the characteristic polynomial of A as $\det(X \cdot \mathrm{Id}_n - A) = X^N + \sum_{i=1}^n (-1)^n s_i(A) X^{n-i}$. Furthermore, the $s_i(A)$ are the elementary symmetric functions in the eigenvalues of A.

Proof. We claim that the invariant polynomial functions $f \in k[\mathrm{Mat}_n(k)]^{\mathrm{GL}_n(k)}$ are determined by their values on diagonal matrices. Since f is invariant under conjugation, it suffices to show that the set of diagonalizable matrices is Zariski-dense in $\mathrm{Mat}_n(k)$ (Remark 2.24). Note that the open set $\bigcap_{i,j=1}^n \mathcal{D}_{X_{ii} - X_{jj}}$ of matrices with pairwise distinct entries in their diagonals is dense in $\mathrm{Mat}_n(k)$. In particular, every upper triangular matrix has an open neighborhood containing a diagonalizable matrix. Every matrix is conjugate to an upper triangular matrix because we are working over an algebraically closed field. Since conjugation is Zariski-continuous, we have proved our claim.

Let $D \subset \mathrm{Mat}_n(k)$ be the subset of diagonal matrices. An invariant function $f \in k[\mathrm{Mat}_n(k)]^{\mathrm{GL}_n}$ is symmetric in the entries of the diagonal. Hence $f|_D \colon D \to k$ is a symmetric function in the eigenvalues of D and, by Example 6.9, of the form $g(s_1, \ldots, s_n)$ for some polynomial $g \in k[Y_1, \ldots, Y_n]$. By the previous paragraph, we conclude $f = g(s_1, \ldots, s_n)$. □

For every rational representation V of a linear algebraic group G we define a rational representation $G \to \mathrm{GL}(V \oplus V^\vee)$ by $g(v, l) := (gv, {}^g l)$.

Example 6.11. Let the torus $T := \mathbb{G}_m^n$ act on $V := k^n$ as the group of invertible diagonal $n \times n$-matrices. The ring of invariants $k[V \oplus V^\vee]^T$ is equal to $k[e_1^\vee e_1^{\vee\vee}, \ldots, e_n^\vee e_n^{\vee\vee}]$, where $e_1, \ldots, e_n$ is the standard basis of V.

Proof. We identify $k[V \oplus V^\vee] = k[x_1, \ldots, x_n, y_1, \ldots y_n] =: R$ via the T-isomorpihsm given by $e_i^\vee \mapsto x_i$ and $e_j^{\vee\vee} \mapsto y_j$, where the action of T on R is given by $\operatorname{diag}(a_1, \ldots, a_n)x_i = a_i^{-1}x_i$ and $\operatorname{diag}(a_1, \ldots, a_n)y_j = a_j y_j$. We want to use the fact that the action of T preserves the multidegree. Let $f \in R$ be multihomogenous of degree $(d_1, \ldots, d_n, c_1, \ldots, c_n)$. Then $\operatorname{diag}(a_1, \ldots, a_n)f = (a_1^{-d_1} \cdots a_n^{-d_n})(a_1^{c_1} \cdots a_n^{c_n})f$ for all $a_1, \ldots, a_n \in k$. Hence, the element f is T-invariant if and only if $d_1 = c_1, \ldots, d_n = c_n$. The equality $k[x_i, y_j]^T = k[x_1y_1, \ldots, x_ny_n]$ follows. □

Example 6.12. Let V be a finite dimensional vector space. Consider the linear form $\langle \cdot, \cdot \rangle \colon V \oplus V^\vee \to k$, $(v, l) \mapsto \langle v, l \rangle := l(v)$. The ring of invariants $k[V \oplus V^\vee]^{\mathrm{GL}(V)}$ is equal to $k[\langle \cdot, \cdot \rangle]$. This is a special case of the First Fundamental Theorem for $\mathrm{GL}(V)$ (see [8, §2]) which describes the $\mathrm{GL}(V)$-invariants of $k[V^n \oplus (V^\vee)^m]$.

Proof. First we check that $\langle \cdot, \cdot \rangle$ is G-invariant. Indeed, for all $g \in \mathrm{GL}(V)$ we have $g\langle v, l \rangle = {}^g l(gv) = l(g^{-1}gv) = \langle v, l \rangle$. We get the inclusion $k[\langle \cdot, \cdot \rangle] \subset k[V \oplus V^\vee]^{\mathrm{GL}(V)}$

Consider the standard open set $U := \mathcal{D}_{\langle \cdot, \cdot \rangle} \subset V \oplus V^\vee$. Let $w \in V \setminus \{0\}$ be a nonzero vector. For every $(v, l) \in U$ choose $g_{v,l} \in \mathrm{GL}(V)$ such that $g_{v,l}(v, l) = (w, \langle v, l \rangle w^\vee)$. Let $f \in k[V \oplus V^\vee]_d^{\mathrm{GL}(V)}$ be a homogenous invariant of degree d. For all $(v, l) \in U$ we have

$$
\begin{aligned}
f(v, l) = g_{v,l}^{-1} f(v, l) &= f(g_{v,l}(v, l)) = f(w, \langle v, l \rangle w^\vee) = \\
&= f(\frac{\langle v, l \rangle}{\langle v, l \rangle} w, \langle v, l \rangle w^\vee) = \langle v, l \rangle^d f(\frac{1}{\langle v, l \rangle} w, w^\vee).
\end{aligned}
$$

We have written $f|_U$ as the product of a power of $\langle \cdot, \cdot \rangle$ and a rational function $f' := f(\frac{1}{\langle \cdot, \cdot \rangle} w, w^\vee) \in k[\langle \cdot, \cdot \rangle^{-1}]$, where $\deg(f') \geqslant -d$. This implies that on the open subset U the function $\langle \cdot, \cdot \rangle$ divides f or f is constant. Since U is Zariski-dense in $V \oplus V^\vee$, we deduce $f \in k[\langle \cdot, \cdot \rangle]$. We have shown the inclusion $k[V \oplus V^\vee]^{\mathrm{GL}(V)} \subset k[\langle \cdot, \cdot \rangle]$. □

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