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Fundamental Groups of Schemes

Bachelorarbeit

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Contents

Introduction

The text at hand is a first look at the theory of fundamental groups of schemes. As the name suggests, this theory has many similarities with the theory of fundamental groups in topology. On the other hand, it also encompasses classical Galois theory, thereby generalizing it to arbitrary arithmetic schemes.

We briefly recall the topological theory. Let X be a connected topological space, and let $x \in X$ be a point. Let F_x be the functor from the category of covers of X to the category of sets which sends a cover $Y \to X$ to its fiber over x. Each fiber is a $\pi_1(X, x)$-set via the monodromy action. If X has a universal cover $\widetilde{X} \to X$, then F_x is represented by $\widetilde{X}$ and factors through an equivalence of categories between the category of covers of X and the category of $\pi_1(X, x)$-sets. In this way, $\pi_1(X, x)$ completely classifies the covers of X. By the Yoneda Lemma, the automorphism group of F_x is isomorphic to $\mathrm{Aut}_X(\widetilde{X})^{\mathrm{op}}$, which in turn is isomorphic to $\pi_1(X, x)$. Hence we recover the fundamental group of X as the automorphism group of F_x.

The fundamental group of a connected scheme S is defined using an analogous framework. The first step is to identify the class of morphisms replacing topological covers; these are the finite étale covers of S. The base scheme S is equipped with a geometric base point $\overline{s}$, i.e. a morphism from the spectrum of an algebraically closed field. Working with geometric fibers, we then construct a functor $F_{\overline{s}}$ from the category of finite étale covers of S to the category of sets. Reversing the topological situation, the fundamental group $\pi_1(S, \overline{s})$ of S with base point $\overline{s}$ is *defined* to be the automorphism group of $F_{\overline{s}}$.

After setting up the general theory, we discuss the classification theorem: $F_{\overline{s}}$ induces an equivalence of categories with the category of finite continuous $\pi_1(S, \overline{s})$-sets. There are two aspects of the theory which facilitate this discussion. Firstly, a special role is played by connected finite étale Galois covers, which are those connected finite étale covers whose automorphism group acts transitively on geometric fibers. Every finite étale cover is an intermediate cover of a Galois cover, and we can describe automorphisms of $F_{\overline{s}}$ as compatible families of automorphisms of the Galois covers. Secondly, because of the finiteness condition placed on finite étale covers, $F_{\overline{s}}$ takes values in the category of finite sets; hence $\pi_1(S, \overline{s})$ has a natural profinite structure. Chapter 3 develops the necessary notions for this point of view.

The analogy drawn above can be made precise using the axiomatic framework of Galois categories, as developed by Grothendieck [1]. Although we do not introduce this notion, it will be clear that the proofs only use formal properties of finite étale covers and the functor $F_{\bar{s}}$.

The second part of the text is devoted to the study of finite étale covers of schemes which are locally of finite type over the complex numbers $\mathbb{C}$. We associate with such a scheme S a topological space S^{an}, called the analytification of S, whose topology is obtained by gluing the topologies locally inherited from the analytic topology on $\mathbb{C}^m$. This construction is functorial, and transforms finite étale covers of S into topological covers of S^{an}. The natural question is now whether any topological cover of S^{an} arises from a finite étale cover of S.

Perhaps surprisingly, this is the case. For smooth projective curves and their associated compact Riemann surfaces, this question was already studied by Riemann. Grothendieck [1] gave a proof in the general setting introduced above. More precisely, the functor which maps a finite étale cover of S to the associated topological cover of S^{an} with finite fibers is an equivalence of categories. It follows formally, using the material developed in Chapter 3 and the classification theorem in the topological setting, that there is an isomorphism of topological groups between the fundamental group of S and the profinite completion of the fundamental group of S^{an}.

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Chapter 1

Finite Étale Morphisms

We define finite étale morphisms and explore some of their properties, starting from the case where the base scheme is the spectrum of a field. The development of the theory in this and the next chapter follows Lenstra [4, Chapters 4 and 5] and Szamuely [7, Chapters 5.2 and 5.3].

1.1 Finite étale schemes over a field

Definition 1.1. *Let k be a field. A k-algebra A is called* étale over k *if it is isomorphic to a finite product of finite separable field extensions of k.*

Proposition 1.2. *Let k be a field, let Ω be an algebraically closed field containing k, and let A be a k-algebra. The following conditions are equivalent:*

(a) *A is étale over k,*

(b) *$A \otimes_k \Omega$ is isomorphic to a finite product of copies of Ω,*

(c) *$A \otimes_k \Omega$ is reduced and finite-dimensional over Ω.*

Proof. Assume first that (a) holds. Since the functor $- \otimes_k \Omega$ preserves finite products, we may assume that A is a finite separable field extension of k. By the primitive element theorem, A is isomorphic over k to $k[T]/(f)$ for a monic irreducible separable polynomial $f \in k[T]$. Let $f = \prod_{i=1}^m (T - a_i)$ be its factorization in $\Omega[T]$, where the factors $T - a_i$ are distinct because f is separable. Then

$$k[T]/(f) \otimes_k \Omega \cong \prod_{i=1}^{m} \Omega[T]/(T - a_i) \cong \Omega^m$$

by the Chinese Remainder Theorem.

It is clear that (b) implies (c). Assume now that (c) holds. Let m be the dimension of $A \otimes_k \Omega$ over Ω. It coincides with the dimension of A over k, so A is finite-dimensional. Hence it is Artinian, which in turn implies that it is isomorphic to a finite product of Artinian local k-algebras. Because $A \otimes_k \Omega$ is reduced, so is A. Thus A is in fact isomorphic to a finite product $k_1 \times \cdots \times k_r$ of finite field extensions of k. Each k_i is separable, because $k_i \otimes_k \Omega$ is reduced. Hence A is étale over k. □

Proposition 1.3. *If k is a field and A is an étale k-algebra, then the module of relative differentials $\Omega_{A/k}$ of A over k is zero.*

Proof. By Proposition 1.2 and the compatibility of relative differentials with base change, it suffices to consider the case where $A = k^m$ for a nonnegative integer m. Let M be an A-module, and let $d\colon A \to M$ be a k-derivation. Denote by $e_1, \ldots, e_m$ the canonical basis of A over k. Note that each e_i is idempotent; we claim that this implies $d(e_i) = 0$. Indeed, applying d to both sides of the equation $e_i^2 = e_i$ yields $2e_i d(e_i) = d(e_i)$. Multiplying by e_i on both sides turns this into $2e_i d(e_i) = e_i d(e_i)$, so $e_i d(e_i) = 0$. Thus $d(e_i) = 2e_i d(e_i) = 0$. Because $\{e_1, \ldots, e_m\}$ spans A as a vector space over k, it follows that $d = 0$. □

1.2 Finite locally free morphisms

Definition 1.4. *A morphism of schemes $\varphi\colon X \to S$ is called* finite locally free *if it is affine and $\varphi_* \mathcal{O}_X$ is a finite locally free $\mathcal{O}_S$-module.*

Proposition 1.5. *The image of a finite locally free morphism of schemes $\varphi\colon X \to S$ is open and closed.*

Proof. Since φ is finite, $\varphi(X)$ is closed. Hence $\varphi(X) = \operatorname{supp}(\varphi_* \mathcal{O}_X)$, which is open because $\varphi_* \mathcal{O}_X$ is finite locally free. □

Corollary 1.6. *If $\varphi\colon X \to S$ is a finite locally free morphism of schemes to a connected scheme S, then φ is surjective if and only if X is nonempty.*

Proof. The image of φ is open and closed in S by Proposition 1.5. Since S is connected, this means that φ is surjective if and only if its image is nonempty. □

Definition 1.7. *Let $\varphi\colon X \to S$ be a finite locally free morphism of schemes, and let $s \in S$ be a point. Since φ is finite locally free, the stalk $(\varphi_* \mathcal{O}_X)_s$ is free of finite rank over $\mathcal{O}_{S,s}$. Its rank is called the* degree of φ at s, *and is denoted by* $\deg_s(\varphi)$.

Proposition 1.8. *The degree of a finite locally free morphism of schemes $\varphi\colon X \to S$ is a locally constant function of $s \in S$. If S is connected, then the degree of φ is constant.*

Proof. Every point $s \in S$ has an open neighborhood U such that $(\varphi_* \mathcal{O}_X)|_U$ is free of rank $\deg_s(\varphi)$ over $\mathcal{O}_S|_U$. Then the stalk of $\varphi_* \mathcal{O}_X$ at every point of U is free of rank $\deg_s(\varphi)$ over the stalk of $\mathcal{O}_S$ at that point, so the degree of φ is constant on U. The second assertion is a direct consequence of the first assertion and the definition of connectedness. □

Lemma 1.9. *A finite locally free morphism of schemes $\varphi\colon X \to S$ is an isomorphism if and only if its degree at every point of S is* 1*.*

Proof. Being an isomorphism is local on the target, and a ring homomorphism $A \to B$ is an isomorphism if and only if it makes B a free A-module of rank 1. □

1.3 Finite étale morphisms

Definition 1.10. *A morphism of schemes $\varphi\colon X \to S$ is called* finite étale *if it is finite locally free and for every point $s \in S$ the fiber X_s of φ over s is the spectrum of an étale $\kappa(s)$-algebra, where $\kappa(s)$ denotes the residue field of s. A surjective finite étale morphism $X \to S$ is also called a* finite étale cover *of S.*

Let S be a scheme. We denote by $\mathbf{Sch}/S$ the category of S-schemes, and by $\mathbf{Fin\acute{E}t}/S$ its full subcategory whose objects are the finite étale morphisms $X \to S$. A *geometric point of S* is a morphism of schemes $\bar{s}\colon \operatorname{Spec}(\Omega) \to S$, where Ω is an algebraically closed field. The image of $\bar{s}$ consists of a single point s; we say that $\bar{s}$ *lies over* s. The *geometric fiber over* $\bar{s}$ of a morphism $X \to S$ is $X_{\bar{s}} := X \times_S \operatorname{Spec}(\Omega)$.

Definition 1.11. *Let S be a scheme, and let $\bar{s}\colon \operatorname{Spec}(\Omega) \to S$ be a geometric point. We view $\operatorname{Spec}(\Omega)$ as an S-scheme via $\bar{s}$. The* fiber functor associated with $\bar{s}$ *is the functor*

$$\begin{aligned} F_{\bar{s}}\colon \mathbf{Fin\acute{E}t}/S &\longrightarrow \mathbf{Set}, \\ (X \to S) &\longmapsto \operatorname{Mor}_S(\operatorname{Spec}(\Omega), X), \\ \psi &\longmapsto (\bar{x} \mapsto \psi \circ \bar{x}). \end{aligned}$$

Proposition 1.12. *Let $\varphi\colon X \to S$ be a finite étale morphism of schemes, and let $\bar{s}\colon \operatorname{Spec}(\Omega) \to S$ be a geometric point of S. Then $F_{\bar{s}}(X)$ is in natural bijection with the underlying set of $X_{\bar{s}}$.*

Proof. By Proposition 1.2, all points of $X_{\bar{s}}$ are Ω-rational; hence the underlying set of $X_{\bar{s}}$ is in natural bijection with $\operatorname{Mor}_\Omega(\operatorname{Spec}(\Omega), X_{\bar{s}})$. The claim follows from the natural bijection $\operatorname{Mor}_\Omega(\operatorname{Spec}(\Omega), X_{\bar{s}}) \cong \operatorname{Mor}_S(\operatorname{Spec}(\Omega), X)$. □

Definition 1.13. *Let $\varphi\colon X \to S$ be a finite étale morphism of schemes, and let $\bar{s}$ be a geometric point of S. The* degree of φ at $\bar{s}$ *is the degree of φ at the point s over which $\bar{s}$ lies, and is denoted by $\deg_{\bar{s}}(\varphi)$.*

Remark 1.14. Let $\varphi\colon X \to S$ be a finite étale morphism of schemes, and let $\bar{s}$ be a geometric point of S. It follows from Proposition 1.2 that the degree of φ at $\bar{s}$ is equal to the number of points of $X_{\bar{s}}$. By Proposition 1.12, it is therefore also equal to the cardinality of $F_{\bar{s}}(X)$.

Example 1.15. A locally closed embedding is finite étale if and only if it is an open and closed embedding.

Example 1.16. Let $K \subset L$ be a finite field extension. The corresponding morphism $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$ is finite locally free of degree $\dim_K(L)$; it is finite étale if and only if L is separable over K.

Example 1.17. Let A be a ring, and let $f \in A[T]$ be a monic polynomial of degree m such that $(f, \partial f/\partial T) = (1)$ in $A[T]$, where $\partial f/\partial T$ is the formal derivative of f with respect to T. Because f is monic, $A[T]/(f)$ is free of rank m over A. If $\mathfrak{p}$ is a prime ideal of A and Ω is an algebraic closure of its residue field $\kappa(\mathfrak{p})$, then $A[T]/(f) \otimes_A \Omega \cong \Omega^m$ as Ω-algebras by the Chinese Remainder Theorem and the fact that f splits into distinct linear factors over Ω. Hence the canonical morphism $A \to A[T]/(f)$ induces a finite étale morphism of degree m on spectra.

Example 1.18. Let k be a field, let $A = k[T, T^{-1}]$, and let $\mathbb{G}_{m,k} = \mathrm{Spec}(A)$ be the multiplicative group over k. For every nonzero integer n, the morphism of k-algebras $\psi_n\colon A \to A$ with $\psi_n(T) = T^n$ corresponds to a surjective finite locally free morphism of schemes $\varphi_n\colon \mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$. If $n > 0$, then ψ_n is isomorphic to the canonical morphism $A \to A[U]/(U^n - T)$. The formal derivative of $U^n - T$ with respect to U is nU^{n-1}. If the characteristic of k does not divide n, then $(U^n - T, nU^{n-1}) = (T) = (1)$ in $A[U]$; hence φ_n is finite étale of degree n by Example 1.17. If $n < 0$, then φ_n is the composite of φ_{-n} and the automorphism φ_{-1}. Provided that n is not divisible by the characteristic of k, the morphism φ_n is also finite étale of degree $-n$ in that case.

Example 1.19. Let k be a field, and let $n > 1$ be an integer. Consider the morphism of k-algebras $\vartheta_n\colon k[T] \to k[T]$ with $\vartheta_n(T) = T^n$, and the corresponding morphism $\mathbb{A}^1_k \to \mathbb{A}^1_k$. The latter is finite locally free of degree n, but not finite étale. Indeed, its fiber over the origin consist of a single nonreduced point.

Example 1.20. Let p be a prime number, and let A be an $\mathbb{F}_p$-algebra. Given an element $a \in A$, consider the polynomial $f = T^p - T - a \in A[T]$ and the scheme $X = \mathrm{Spec}(A[T]/(f))$. The formal derivative of f with respect to T is -1, so the canonical morphism $A \to A[T]/(f)$ corresponds to a finite étale morphism $X \to \mathrm{Spec}(A)$ of degree p by Example 1.17.

1.4 Permanence properties

We now discuss permanence properties of finite locally free and finite étale morphisms, such as being stable under composition and base change. As a technical tool, we need the following algebraic result.

Proposition 1.21. *Let A be a ring. For every A-module M the following conditions are equivalent:*

(a) *M is finitely generated and projective,*

(b) *M is finitely presented and $M_{\mathfrak{p}}$ is free for every $\mathfrak{p} \in \operatorname{Spec}(A)$,*

(c) *M is finite locally free.*

Proof. We indicate the main steps in the proof, and refer to Lenstra [4, Section 4.6] for a more complete explanation. That (a) implies (b) follows from the fact that a finitely generated projective module over a local ring is free. Suppose that M satisfies (b), and let $\mathfrak{p} \in \operatorname{Spec}(A)$. A straightforward calculation shows that any basis of $M_{\mathfrak{p}}$ over $A_{\mathfrak{p}}$ lifts to a basis of M_f over A_f for some $f \in A \smallsetminus \mathfrak{p}$. Hence M satisfies (c). Finally, in order to show that (c) implies (a), first prove that M is finitely presented. Consequently, the functors $\operatorname{Hom}_A(M, -)_f$ and $\operatorname{Hom}_{A_f}(M_f, (-)_f)$ are isomorphic for every $f \in A$. Since M is locally projective, it follows that M is projective. □

Proposition 1.22. (a) *The composite of two finite locally free morphisms of schemes $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ is finite locally free.*

(b) *Let $\varphi\colon X \to S$ and $\psi\colon Y \to S$ be morphisms of schemes. If φ is finite locally free, the so is the base change $\operatorname{pr}_2\colon X \times_S Y \to Y$ of φ by ψ.*

Proof. Because affine morphisms are stable under composition and base change, we may reduce to the affine case. Both assertions then follow from the equivalence of conditions (a) and (c) in Proposition 1.21 and the characterization of projective modules as direct summands of free modules. □

Proposition 1.23. (a) *The composite of two finite étale morphisms of schemes $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ is finite étale.*

(b) *Let $\varphi\colon X \to S$ and $\psi\colon Y \to S$ be morphisms of schemes. If φ is finite étale, the so is the base change $\operatorname{pr}_2\colon X \times_S Y \to Y$ of φ by ψ.*

Proof. (a) The morphism $\psi \circ \varphi$ is finite locally free by Proposition 1.22. In order to show that it is finite étale, we use Proposition 1.2. Let $z \in Z$ be a point, let Ω be an algebraic closure of $\kappa(z)$, and let $\overline{z}\colon \operatorname{Spec}(\Omega) \to Z$ be the resulting geometric point lying over z. We need to show that the geometric fiber $X_{\overline{z}}$ is isomorphic to the spectrum of a finite product of copies of Ω.

Note that $X_{\overline{z}}$ is naturally isomorphic to $X \times_Y Y_{\overline{z}}$ over $Y_{\overline{z}}$, so in particular over Ω. Since ψ is finite étale, $Y_{\overline{z}}$ is isomorphic over Ω to the spectrum of a finite product of copies of Ω; since φ is finite étale and fiber products of schemes commute with coproducts, so is $X_{\overline{z}}$.

(b) The morphism pr_2 is finite locally free by Proposition 1.22. As in the proof of part (a), choose a point $y \in Y$ and a geometric point $\overline{x}\colon \mathrm{Spec}(\Omega) \to Y$ lying over y. We need to show, by Proposition 1.2, that the geometric fiber $(X \times_S Y)_{\overline{x}}$ is isomorphic to the spectrum of a finite product of copies of Ω. Note that $(X \times_S Y)_{\overline{x}}$ is naturally isomorphic to $X_{\psi\circ\overline{x}} = X \times_S \mathrm{Spec}(\Omega)$ over Ω. Because φ is finite étale, $X_{\psi\circ\overline{x}}$ is isomorphic to the spectrum of a finite product of copies of Ω by Proposition 1.2; the claim follows. □

Corollary 1.24. *Let $\varphi\colon X \to Y$ and $\psi\colon Y \to Z$ be finite locally étale morphisms of schemes. If Y and Z are connected, then the degree of $\psi \circ \varphi$ is equal to the product of the degrees of φ and ψ.*

Proof. See the proof of part (a) of the preceding proposition, which gives a formula for the geometric fibers of $\psi \circ \varphi$. □

Remark 1.25. Let S be a scheme. For every finite family of schemes $X_1, \ldots, X_r$ which are finite étale over S, their coproduct $\coprod_{i=1}^{r} X_i$ in the category $\mathbf{Sch}/S$ is finite étale over S. Hence it is also their coproduct in the full subcategory $\mathbf{Fin\acute{E}t}/S$. The same thing is true for their product $X_1 \times_S \cdots \times_S X_r$ by Proposition 1.23. Note that the fiber functor preserves both finite coproducts and finite products.

Proposition 1.26. *Let $\varphi\colon X \to S$ be a finite étale morphism to a connected scheme S.*

(a) *The number of connected components of X is less than or equal to the degree of φ.*

(b) *Every connected component of X is open.*

Proof. Assertion (b) is a purely topological consequence of (a), since the degree of φ is finite. We now prove (a). Let $\overline{s}$ be a geometric point of S. We induct on $\deg_{\overline{s}}(\varphi)$, which is independent of the choice of $\overline{s}$ because S is connected. If $\deg_{\overline{s}}(\varphi) = 0$, then X is empty; if $\deg_{\overline{s}}(\varphi) = 1$, then φ is an isomorphism. Assume now that $\deg_{\overline{s}}(\varphi) > 1$. If X is connected, then the claim holds. Otherwise X is the union of two disjoint nonempty open and closed subsets U_1 and U_2. Being composites of finite étale morphisms, the restrictions $\varphi|_{U_1}$ and $\varphi|_{U_2}$ are finite étale; hence they are surjective by Corollary 1.6. Since $F_{\overline{s}}$ preserves coproducts, $F_{\overline{s}}(X)$ is the disjoint union of the nonempty sets $F_{\overline{s}}(U_1)$ and $F_{\overline{s}}(U_1)$. In particular the degrees of $\varphi|_{U_1}$ and $\varphi|_{U_2}$ are strictly smaller than that of φ. By induction the claim holds for U_1 and U_2; but then it holds for X. □

Proposition 1.27. *If $\varphi\colon X \to S$ is a finite étale morphism, then the sheaf of relative differentials $\Omega_{X/S}$ of X over S is zero.*

Proof. We may assume that $X = \operatorname{Spec}(B)$ and $S = \operatorname{Spec}(A)$ are affine, in which case $\Omega_{X/S}$ is the quasi-coherent $\mathcal{O}_X$-module associated with $\Omega_{B/A}$. Since φ is of finite type, $\Omega_{B/A}$ is finitely generated. Let $\mathfrak{q}$ be a prime ideal of B, and let $\mathfrak{p} = \varphi(\mathfrak{q})$. By Nakayama's Lemma, $(\Omega_{B/A})_{\mathfrak{q}} = 0$ if and only if $\Omega_{B/A} \otimes_B \kappa(\mathfrak{q}) = 0$. The latter is isomorphic to $\Omega_{B\otimes_A\kappa(\mathfrak{p})/\kappa(\mathfrak{p})} \otimes_{B\otimes_A\kappa(\mathfrak{p})} \kappa(\mathfrak{q})$, which is zero by Proposition 1.3. Thus $\Omega_{B/A} = 0$. □

Proposition 1.28. *If $\varphi\colon X \to S$ is a finite étale morphism, then the diagonal morphism $\Delta_\varphi\colon X \to X \times_S X$ is an open and closed embedding.*

Proof. Since φ is affine, it is separated, so Δ_φ induces an isomorphism of X with a closed subscheme Y of $X \times_S X$. We now wish to show that Y is an open subscheme of $X \times_S X$. We may assume that $X = \operatorname{Spec}(B)$ and $S = \operatorname{Spec}(A)$ are affine. Let I be the kernel of the codiagonal $B \otimes_A B \to B$. It is finitely generated because B is finitely generated over A, and the associated quasi-coherent ideal $\mathcal{I}$ of $\mathcal{O}_{X\times_S X}$ is the ideal of definition of Y. The quotient I/I^2 is isomorphic to $\Omega_{B/A}$, which is zero by Proposition 1.27. Let $\mathfrak{p}$ be a prime ideal of $B \otimes_A B$ containing I. Then $I_{\mathfrak{p}}$ is contained in the unique maximal ideal of $(B \otimes_A B)_{\mathfrak{p}}$. Since $I_{\mathfrak{p}}^2 = I_{\mathfrak{p}}$, we must have $I_{\mathfrak{p}} = 0$ by Nakayama's Lemma. In other words, the stalk of $\mathcal{I}$ at every point $y \in Y$ is trivial; because it is finitely generated, $\mathcal{I}$ vanishes on an open neighborhood of y. Thus Y is an open subscheme of $X \times_S X$. □

Proposition 1.29. *Let $\varphi\colon X \to S$ and $\psi\colon Y \to X$ be morphisms of schemes. If $\varphi \circ \psi$ and φ are finite étale, then so is ψ.*

Proof. The graph morphism $\Gamma_\psi\colon Y \to Y \times_S X$ is the base change of the diagonal morphism Δ_φ by $\psi \times_S \mathrm{id}_X$, and $\psi = \mathrm{pr}_2 \circ \Gamma_\psi$. As the diagonal morphism is an open and closed embedding by Proposition 1.28, it is finite étale. But then so is Γ_ψ, since finite étale morphisms are stable under base change by Proposition 1.23. Similarly, pr_2 is finite étale as the base change of $\varphi\circ\psi$ by φ. Thus ψ is finite étale as a composite of finite étale morphisms. □

Proposition 1.30. *Let Y be a connected S-scheme, and let $\varphi_1, \varphi_2\colon Y \rightrightarrows X$ be S-morphisms to a finite étale S-scheme X. If there exists a nonempty S-scheme T and an S-morphism $\psi\colon T \to Y$ such that $\varphi_1 \circ \psi = \varphi_2 \circ \psi$, then $\varphi_1 = \varphi_2$.*

Proof. Denote by $\operatorname{eq}(\varphi_1, \varphi_2)$ the equalizer of φ_1 and φ_2 in the category of S-schemes. The following diagram is easily checked to be cartesian:

$$\begin{array}{ccc} \operatorname{eq}(\varphi_1,\varphi_2) & \xrightarrow{\;j\;} & Y \\ \downarrow & & \downarrow{\scriptstyle(\varphi_1,\varphi_2)} \\ X & \xrightarrow{\;\Delta\;} & X \times_S X, \end{array}$$

where Δ is the diagonal morphism of X over S, j is the canonical embedding, and $\mathrm{eq}(\varphi_1, \varphi_2) \to X$ is the composite $\varphi_1 \circ j = \varphi_2 \circ j$. Since Δ is an open and closed embedding by Proposition 1.28, so is j. But Y is connected and $\mathrm{eq}(\varphi_1, \varphi_2)$ is nonempty by assumption, which means that j must be an isomorphism. Thus $\varphi_1 = \varphi_2$. □

Chapter 2

Galois Covers

2.1 Galois covers

Having introduced finite étale morphisms, we now study their automorphism groups.

Construction 2.1. Given a finite étale morphism $X \to S$ and a geometric point $\overline{s}$ of S, there is a canonical left action of $\mathrm{Aut}_S(X)$ on $F_{\overline{s}}(X)$. Namely, $f \in \mathrm{Aut}_S(X)$ acts on $\overline{x} \in F_{\overline{s}}(X)$ by $f \cdot \overline{x} := F_{\overline{s}}(f)(\overline{x})$.

Proposition 2.2. *Let $X \to S$ be a connected finite étale cover, and let $\overline{s}$ be a geometric point of S. Then the left action of $\mathrm{Aut}_S(X)$ on $F_{\overline{s}}(X)$ as defined in Construction 2.1 is free, and the cardinality of $\mathrm{Aut}_S(X)$ is less than or equal to the degree of φ.*

Proof. Suppose that $f \in \mathrm{Aut}_S(X)$ and $\overline{x} \in F_{\overline{s}}(X)$ are such that $F_{\overline{s}}(f)(\overline{x}) = \overline{x}$. Then $f \circ \overline{x} = \mathrm{id}_X \circ \overline{x}$, so $f = \mathrm{id}_X$ by Proposition 1.30. Hence the action is free. Since $X \to S$ is surjective, there is a point $\overline{x} \in F_{\overline{s}}(X)$. We have the injective map

$$\mathrm{Aut}_S(X) \hookrightarrow F_{\overline{s}}(X), \quad g \mapsto F_{\overline{s}}(g)(\overline{x});$$

hence the cardinality of $\mathrm{Aut}_S(X)$ is at most that of $F_{\overline{s}}(X)$. The second assertion follows from this and the natural bijection between $F_{\overline{s}}(X)$ and $X_{\overline{s}}$, see Proposition 1.12. □

Proposition 2.3. *Let $\varphi\colon X \to S$ be a connected finite étale cover. Then the following conditions are equivalent:*

(a) *The order of $\mathrm{Aut}_S(X)$ is equal to the degree of φ,*

(b) *$\mathrm{Aut}_S(X)$ acts transitively on $F_{\overline{s}}(X)$ for every geometric point $\overline{s}$ of S,*

(c) *$\mathrm{Aut}_S(X)$ acts transitively on $F_{\overline{s}}(X)$ for one geometric point $\overline{s}$ of S.*

Proof. Assume first that (a) holds. Let $\overline{s}$ be a geometric point of S, and let $\overline{x} \in F_{\overline{s}}(X)$ be a lift of $\overline{s}$. By Proposition 2.2, the action of $\mathrm{Aut}_S(X)$ on $F_{\overline{s}}(X)$ is free, so the map

$$u\colon \mathrm{Aut}_S(X) \to F_{\overline{s}}(X), \quad g \mapsto F_{\overline{s}}(g)(\overline{x})$$

is injective. Because the degree of φ is equal to the cardinality of $F_{\overline{s}}(X)$, the map u is a bijection. Therefore (a) implies (b). Since S is nonempty, (b) implies (c). Suppose that (c) holds, so $\mathrm{Aut}_S(X)$ acts transitively on $F_{\overline{s}}(X)$ for a geometric point $\overline{s}$ of S. Choose a lift $\overline{x} \in F_{\overline{s}}(X)$ and define u as above. Because the action of $\mathrm{Aut}_S(X)$ on $F_{\overline{s}}(X)$ is free and transitive, u is a bijection. Hence (a) follows. □

Definition 2.4. *A morphism of schemes $X \to S$ is called a* connected finite étale Galois cover *if it is a connected finite étale cover and satisfies the equivalent conditions of Proposition 2.3.*

Remark 2.5. Given a finite étale morphism of schemes $X \to S$ and a geometric point $\overline{s}$ of S, one can also consider the left action of $\mathrm{Aut}_S(X)$ on $X_{\overline{s}}$ arising by base change of its left action on X. Under the bijection between $X_{\overline{s}}$ with $F_{\overline{s}}(X)$ from Proposition 1.12, it corresponds to left action of $\mathrm{Aut}_S(X)$ on $F_{\overline{s}}$.

Example 2.6. If K is a field, then a connected finite étale Galois cover of $\mathrm{Spec}(K)$ is a morphism $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$ corresponding to a finite Galois extension $K \subset L$.

Example 2.7. Let us reexamine Example 1.18. Let n be a positive integer not divisible by the characteristic of k, let $A = B = k[T, T^{-1}]$, and view B as an A-algebra via ψ_n. There is an isomorphism of groups

$$u\colon \mathrm{Aut}_A(B) \xrightarrow{\sim} \mu_n(k), \quad f \mapsto f(T)/T,$$

where $\mu_n(k)$ denotes the group of nth roots of unity in k. Hence the corresponding connected finite étale cover $\varphi_n\colon \mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$ is Galois if and only if k contains a primitive nth root of unity.

Proof. Any $f \in \mathrm{Aut}_A(B)$ satisfies $\psi_n \circ f = \psi_n$, so $f(T)^n = T^n$. Hence $f(T)/T$ is an nth root of unity in k. We construct an inverse v of u. Given $\zeta \in \mu_n(k)$, define $v(\zeta)$ to be the morphism of k-algebras $k[T, T^{-1}] \to k[T, T^{-1}]$ sending T to ζT. Then $\psi_n \circ v(\zeta) = \psi_n$, since

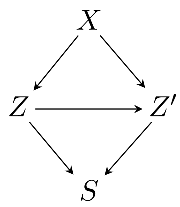
$$\psi_n(v(\zeta)(T)) = (\zeta T)^n = T^n.$$

The map v is clearly a group homomorphism and an inverse of u. The last assertion follows from the above isomorphism and Proposition 2.3. □

Proposition 2.8. *If $\varphi: X \to S$ is a connected finite étale Galois cover, then every S-endomorphism of X is an automorphism.*

Proof. Let $\overline{s}$ be a geometric point of S, let $\overline{x} \in F_{\overline{s}}(X)$ be a lift of $\overline{s}$, and let f be an S-endomorphism of X. Because φ is Galois, there exists an S-automorphism g of X such that $F_{\overline{s}}(g)(\overline{x}) = F_{\overline{s}}(f)(\overline{x})$. But then $g = f$ by Proposition 1.30. □

Definition 2.9. *Let $\varphi: X \to S$ be a finite étale cover. An* intermediate cover of φ *is a factorization $X \to Z \to S$ of φ. A* morphism of intermediate covers $(X \to Z \to S) \to (X \to Z' \to S)$ *is a morphism $Z \to Z'$ such that the diagram*



commutes.

As in Galois theory, there is a correspondence between intermediate covers of a connected finite étale Galois cover and subgroups of its automorphism group. In order to state the correspondence, we need to introduce quotients of schemes by groups of automorphisms.

2.2 Quotients of schemes

Definition 2.10. *Let $\mathcal{C}$ be a category, let X be an object of $\mathcal{C}$, and let G be a subgroup of* $\mathrm{Aut}(X)$. *A* quotient of X by G *is an object $G\backslash X$ of $\mathcal{C}$ together with a universal G-invariant morphism $\pi: X \to G\backslash X$, i.e. for every G-invariant morphism $\psi: X \to Y$ there is a unique morphism $\psi': G\backslash X \to Y$ satisfying $\psi' \circ \pi = \psi$.*

Remark 2.11. If $X \to S$ is a morphism in $\mathcal{C}$ and G is a subgroup of $\mathrm{Aut}_S(X)$ such that the quotient $G\backslash X$ exists, then $X \to S$ factors through the canonical morphism $X \to G\backslash X$.

Remark 2.12. Let $X = \mathrm{Spec}(A)$ be an affine scheme, and let G be a subgroup of $\mathrm{Aut}(A)$. The functor Spec induces a bijection of G with a subgroup G' of $\mathrm{Aut}(X)$. Denote by A^G the ring of invariants of the canonical left action of G on A. The affine scheme $\mathrm{Spec}(A^G)$ together with the morphism $X \to \mathrm{Spec}(A^G)$ corresponding to the inclusion $A^G \subset A$ is a quotient of X by G' in the category of affine schemes.

Construction 2.13. Let X be a scheme, and let G be a subgroup of $\mathrm{Aut}(X)$. Consider the quotient space $G\backslash X$, whose points are the orbits of the points of X under the canonical left action of G, and let $\pi\colon X \to G\backslash X$ be the canonical projection. Since $\pi_*\mathcal{O}_X = \pi_* g_* \mathcal{O}_X$ for every $g \in G$, there is a canonical right action of G on $\pi_*\mathcal{O}_X$. Defining $\mathcal{O}_{G\backslash X}$ to be the sheaf of G-invariant sections of $\pi_*\mathcal{O}_X$, we obtain the ringed space $(G\backslash X, \mathcal{O}_{G\backslash X})$.

Proposition 2.14. *Let $\varphi\colon X \to S$ be an affine morphism of schemes, and let G be a finite subgroup of $\mathrm{Aut}_S(X)$. Then the ringed space $G\backslash X$ as defined in Construction 2.13 is an S-scheme, and is a quotient of X by G in the category* **Aff**$/S$ *of schemes which are affine over S.*

Proof. See Szamuely [7, Proposition 5.3.6] for the first assertion, and Lenstra [4, Paragraph 5.18] for the second. □

Proposition 2.15. *Let $\varphi\colon X \to S$ be a finite étale morphism of schemes, and let G be a subgroup of $\mathrm{Aut}_S(X)$. Let $X \to G\backslash X$ be a quotient of X by G in* **Aff**$/S$. *Then both morphisms in the factorization $X \to G\backslash X \to S$ of φ are finite étale.*

Proof. See Lenstra [4, Proposition 5.20]. □

Corollary 2.16. *Let $\varphi\colon X \to S$ be a finite étale morphism of schemes, and let G be a subgroup of $\mathrm{Aut}_S(X)$. Then a quotient of X by G in the category* **FinÉt**$/S$ *exists.*

Proof. This is immediate from Proposition 2.15, because **FinÉt**$/S$ is a full subcategory of **Aff**$/S$. □

Proposition 2.17. *Let $\varphi\colon X \to S$ be a finite étale morphism of schemes, let G be a subgroup of $\mathrm{Aut}_S(X)$, and let $Z \to S$ be a morphism of schemes. Then the canonical morphism $G\backslash(X \times_S Z) \to (G\backslash X) \times_S Z$ is an isomorphism.*

Proof. See Lenstra [4, Proposition 5.21]. □

Proposition 2.18. *Let $\varphi\colon X \to S$ be a connected finite étale cover, and let G be a subgroup of $\mathrm{Aut}_S(X)$. Then $X \xrightarrow{\pi} G\backslash X \to S$ is a connected intermediate cover of φ, and $\mathrm{Aut}_{G\backslash X}(X) = G$.*

Proof. The first part follows from Proposition 2.15. Since π is G-invariant, we have $G \subset \mathrm{Aut}_{G\backslash X}(X)$. The degree of finite étale morphisms is multiplicative by Corollary 1.24, so π has degree equal to the order G. Hence we must have $G = \mathrm{Aut}_{G\backslash X}(X)$ by Proposition 2.2. □

Proposition 2.19. *A connected finite étale cover $\varphi\colon X \to S$ is Galois if and only if the canonical morphism $\mathrm{Aut}_S(X)\backslash X \to S$ is an isomorphism.*

Proof. Let $\overline{s}$ be a geometric point of S. The morphism $\mathrm{Aut}_S(X)\backslash X \to S$ is an isomorphism if and only if it is of degree 1. By Proposition 2.17, we have $\mathrm{Aut}_S(X)\backslash X_{\overline{s}} \cong (\mathrm{Aut}_S(X)\backslash X)_{\overline{s}}$. Hence the degree of $\mathrm{Aut}_S(X)\backslash X \to S$ is 1 if and only if $\mathrm{Aut}_S(X)$ acts transitively on $X_{\overline{s}}$. By Remark 2.5, this is the case if and only if $\mathrm{Aut}_S(X)$ acts transitively on $F_{\overline{s}}(X)$. □

2.3 Galois correspondence

Theorem 2.20 (Galois correspondence)**.** *Let $\varphi\colon X \to S$ be a connected finite étale Galois cover. The assignments*

$$\begin{aligned} (X \to Z \to S) &\longmapsto \mathrm{Aut}_Z(X) \\ (X \to H\backslash X \to S) &\longleftarrow\!\shortmid H. \end{aligned}$$

extend to an equivalence of categories between the category of intermediate covers of φ and the category of subgroups of $\mathrm{Aut}_S(X)$*.*

Proof. Given a morphism of intermediate covers

$$(X \to Z \to S) \to (X \to Z' \to S),$$

the inclusion $\mathrm{Aut}_Z(X) \subset \mathrm{Aut}_{Z'}(X)$ holds. Conversely, an inclusion $H \subset H'$ among subgroups of $\mathrm{Aut}_S(X)$ yields a factorization $X \to H\backslash X \to H'\backslash X$ of the canonical morphism $X \to H'\backslash X$, which is readily seen to be a morphism of intermediate covers

$$(X \to H\backslash X \to S) \to (X \to H'\backslash X \to S).$$

That these constructions are inverse to each other is immediate from Propositions 2.18 and 2.19. □

Proposition 2.21. *Let $\varphi\colon X \to S$ be a connected finite étale Galois cover, and let*

$$X \xrightarrow{\pi} Z \xrightarrow{\psi} S$$

be an intermediate cover of φ such that ψ is a connected finite étale Galois cover. Then there is surjective group homomorphism

$$u\colon\ \mathrm{Aut}_S(X) \to \mathrm{Aut}_S(Z)$$

with kernel $\mathrm{Aut}_Z(X)$*.*

Proof. Let $\overline{s}$ be a geometric point of S, and let $\overline{x} \in F_{\overline{s}}(X)$ be a lift of $\overline{s}$. Given $f \in \mathrm{Aut}_S(X)$, we want to find $u(f) \in \mathrm{Aut}_S(Z)$ making the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{\ u(f)\ } & Z \end{array}$$

commute. By Proposition 1.30, this is equivalent to

$$F_{\overline{s}}(u(f))(F_{\overline{s}}(\pi)(\overline{x})) = F_{\overline{s}}(\pi \circ f)(\overline{x}).$$

Since ψ is Galois, there exists a unique such $u(f)$. By uniqueness of $u(f)$, the resulting map $u\colon \operatorname{Aut}_S(X) \to \operatorname{Aut}_S(Z)$ is a group homomorphism. Now let us show that u is surjective. Note that $F_{\overline{s}}(\pi)$ is surjective. Because φ is Galois, this implies that for every $g \in \operatorname{Aut}_S(Z)$ there exists $f \in \operatorname{Aut}_S(X)$ such that $F_{\overline{s}}(\pi \circ f)(\overline{x}) = F_{\overline{s}}(g \circ \pi)(\overline{x})$. Then $u(f) = g$ by construction, so u is surjective. The kernel of u consists of those $f \in \operatorname{Aut}_S(X)$ satisfying $\pi = \pi \circ f$, i.e. $f \in \operatorname{Aut}_Z(X)$. □

Example 2.22. By Examples 1.16 and 2.6, the Galois correspondence for a Galois extension $K \subset L$ is a special case of Theorem 2.20.

Example 2.23. We return to Example 2.7. Assume that there exists a primitive nth root of unity ζ in k. Then the automorphism group of ψ_n is cyclic with generator $v(\zeta)$, and its subgroups are generated by powers of $v(\zeta)$. Let H be such a subgroup, say generated by $v(\zeta)^d$ for a divisor d of n. Denote by H' the corresponding group of automorphisms of $\mathbb{G}_{m,k}$. By Remark 2.12, $H' \backslash \mathbb{G}_{m,k}$ is given by the spectrum of A^H. The latter consist of all Laurent polynomials $f = \sum_{i=-r}^{r} a_i T^i$ such that $f(\zeta^d T) = f$, i.e. a_i is nonzero only if di is divisible by n. Hence $A^H = k[T^{n/d}, T^{-n/d}]$. The intermediate cover $\mathbb{G}_{m,k} \to H' \backslash \mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$ of φ_n corresponds to the canonical factorization $A \to A^H \subset A$ of ψ_n. Its isomorphism class is the same as that of the intermediate cover

$$\mathbb{G}_{m,k} \xrightarrow{\varphi_{n/d}} \mathbb{G}_{m,k} \xrightarrow{\varphi_d} \mathbb{G}_{m,k}.$$

By Theorem 2.20, every connected intermediate cover of φ_n lies in such an isomorphism class.

We now show that every connected finite étale cover of S is an intermediate cover of a connected finite étale Galois cover of S. The proof is taken from Szamuely [7, Proposition 5.3.9].

Proposition 2.24 (Galois closure)**.** *Let $\varphi\colon X \to S$ be a connected finite étale cover. Then there exists a connected finite étale Galois cover $P \to S$ which factors through φ.*

Proof. Let $\overline{s}$ be a geometric point of S, and let m be the degree of φ. Choose an enumeration $F_{\overline{s}}(X) = \{\overline{x}_1, \dots, \overline{x}_m\}$. We denote by X^m the m-fold product $X \times_S \cdots \times_S X$. By the universal property of the fiber product, there is a natural bijection

$$u\colon F_{\overline{s}}(X^m) \xrightarrow{\sim} F_{\overline{s}}(X)^m, \quad \overline{x} \mapsto (\mathrm{pr}_1 \circ \overline{x}, \dots, \mathrm{pr}_m \circ \overline{x}).$$

Let $\overline{x}$ be the element of $F_{\overline{s}}(X^m)$ corresponding to $(\overline{x}_1, \ldots, \overline{x}_m)$ under u, and let P be the connected component of X^m over which $\overline{x}$ lies. Let π be the composite of the embedding $P \hookrightarrow X^m$ with the projection $\mathrm{pr}_1 \colon X^m \to X$. By Proposition 1.23, both π and $\varphi \circ \pi$ are finite étale.

We now show that the image of $F_{\overline{s}}(P)$ in $F_{\overline{s}}(X)^m$ consists of tuples with pairwise distinct entries. Suppose that there exists $\overline{x}' \in F_{\overline{s}}(P)$ such that $u(\overline{x}') = (\overline{x}'_1, \ldots, \overline{x}'_m)$ has entries $\overline{x}'_i = \overline{x}'_j$ for distinct indices i and j. By Proposition 1.30, this implies that the projections $\mathrm{pr}_i, \mathrm{pr}_j \colon P \rightrightarrows X$ are equal. Since the entries of $u(\overline{x})$ are pairwise distinct, this is impossible.

We prove that every $\overline{x}' \in F_{\overline{s}}(P)$ lies in the $\mathrm{Aut}_S(P)$-orbit of $\overline{x}$. By the above, $u(\overline{x}') = (\overline{x}_{\sigma(1)}, \ldots, \overline{x}_{\sigma(m)})$ for a permutation $\sigma \in S(\{1, \ldots, m\})$. This permutation induces an automorphism f of X^m by permuting the factors. Then $f(P)$ is a connected component of X; the point $p \in P$ over which $\overline{x}'$ lies is contained in both P and $f(P)$, so we must have $f(P) = P$. Hence f restricts to an automorphism f' of P such that $F_{\overline{s}}(f')(\overline{x}) = \overline{x}'$. □

Corollary 2.25. *Let S be a connected scheme, let $\overline{s}$ be a geometric point of S, let $\varphi \colon X \to S$ be a finite étale cover, and let $\overline{x} \in F_{\overline{s}}(X)$. There exists a connected finite étale Galois cover $P \to S$, an S-morphism $\pi \colon P \to X$, and $\overline{p} \in F_{\overline{s}}(P)$ such that $F_{\overline{s}}(\pi)(\overline{p}) = \overline{x}$.*

Proof. Let Z be the connected component of X over which $\overline{x}$ lies. It is open and closed by Proposition 1.26, so the canonical embedding $j \colon Z \hookrightarrow X$ is finite étale. Applying Proposition 2.24 to $\varphi \circ j$ yields a connected finite étale Galois cover $P \to S$ which factors through a finite étale cover $\pi' \colon P \to Z$. Define $\pi = j \circ \pi'$. Since $F_{\overline{s}}(\pi')$ is surjective, $\overline{x}$ lies in the image of $F_{\overline{s}}(\pi)$. □

Chapter 3

Profinite groups

3.1 Continuous group actions

A left action of a topological group G on a set E without a topology is called *continuous* if the corresponding map $G \times E \to E$ is continuous, where E is equipped with the discrete topology. This is the case if and only if the stabilizer of every $x \in E$ is open in G.

Remark 3.1. If E is finite and the symmetric group $S(E)$ is equipped with the discrete topology, then an action $G \times E \to E$ is continuous if and only if the corresponding group homomorphism $G \to S(E)$ is continuous.

3.2 Profinite groups

By a *cofiltered diagram* in a category $\mathcal{C}$ we mean a functor $P\colon \mathcal{I} \to \mathcal{C}$, where $\mathcal{I}$ is a small cofiltered category. In order to simplify our notation, we will write $i \in \mathcal{I}$ for $i \in \mathrm{Ob}(\mathcal{I})$. A topological group G is called *profinite* if it is a limit of a cofiltered diagram of finite discrete topological groups. In particular G is quasi-compact, Hausdorff, and totally disconnected. Profinite groups form a full subcategory of **TopGrp**, the category of topological groups. The inclusion functor from profinite to topological groups has a left adjoint, which we construct in several steps.

Construction 3.2. Let G be a topological group. Define $\mathcal{I}$ to be the category whose objects are the open normal subgroups of G of finite index, with a unique morphism $M \to N$ if $M \subset N$, and no morphism from M to N otherwise. With the obvious composition of morphisms, $\mathcal{I}$ is a small cofiltered category. There is a functor $P\colon \mathcal{I} \to$ **TopGrp** that maps an object M of $\mathcal{I}$ to the finite discrete quotient group G/M. Given a morphism $M \to N$ in $\mathcal{I}$, define $P(M \to N)$ to be the unique morphism $G/M \to G/N$ that is compatible with the projections from G. The *profinite completion* $\widehat{G}$ of G is the limit of P; it is a profinite group, and comes with a natural morphism $\eta_G \colon G \to \widehat{G}$.

Lemma 3.3. *The image of* η_G *is dense in* $\widehat{G}$.

Proof. Let $h = (h_M M)_{M \in \mathcal{I}}$ be an element of $\widehat{G}$, and let $U = h \ker(\mathrm{pr}_N)$ be a fundamental open neighborhood of h; see Lemma 3.4. Then $\eta_G(h_N) \in U$, so $\eta_G(G)$ is dense in $\widehat{G}$. □

Lemma 3.4. *Let* G *be the limit of a cofiltered diagram* $P \colon \mathcal{I} \to \mathbf{TopGrp}$ *of finite discrete groups. For every* $i \in \mathcal{I}$*, denote by* $\mathrm{pr}_i \colon G \to P(i)$ *the canonical projection. The open normal subgroups* $\ker(\mathrm{pr}_i)$ *of* G *form a fundamental system of open neighborhoods of the identity element* $e \in G$.

Proof. Let U be an open neighborhood of e in G. By definition of the topology of the limit, there is a nonnegative integer m and objects $i(1), \ldots, i(r) \in \mathcal{I}$ such that $\bigcap_{k=1}^{r} \ker(\mathrm{pr}_{i(k)})$ is an open neighborhood of e in U. Because $\mathcal{I}$ is cofiltered, there exists $j \in \mathcal{I}$ such that there is a morphism $j \to i(k)$ in $\mathcal{I}$ for every $k \in \{1, \ldots, r\}$. Then $\ker(\mathrm{pr}_j)$ is an open neighborhood of e in U. □

Proposition 3.5. *The assignment* $G \mapsto \widehat{G}$ *extends to a functor from the category of topological groups to the category of profinite groups.*

Proof. Let $\varphi \colon G \to H$ be a morphism of topological groups. For every open normal subgroup N of H of finite index, $\varphi^{-1}(N)$ is an open normal subgroup of G of finite index. The family of morphisms

$$\widehat{G} \to G/\varphi^{-1}(N) \hookrightarrow H/N,$$

where the first morphism is the canonical projection, induces a morphism $\widehat{\varphi} \colon \widehat{G} \to \widehat{H}$ by the universal property of the limit. The diagram

$$\begin{array}{ccc} G & \xrightarrow{\eta_G} & \widehat{G} \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \widehat{\varphi}} \\ H & \xrightarrow{\eta_H} & \widehat{H} \end{array}$$

commutes; since the image of $G \to \widehat{G}$ is dense by Lemma 3.3 and $\widehat{G}$ is Hausdorff, this uniquely characterizes $\widehat{\varphi}$. It follows that

$$\begin{aligned} G &\longmapsto \widehat{G}, \\ \varphi &\longmapsto \widehat{\varphi} \end{aligned}$$

is a functor. □

Proposition 3.6. *A topological group* G *is profinite if and only if* η_G *is an isomorphism.*

Proof. The condition is clearly sufficient, so we only need to prove necessity. Suppose that G is a profinite group. The kernel of η_G is trivial by Lemma 3.4 and the fact that G is Hausdorff. Because G is quasi-compact and $\widehat{G}$ is Hausdorff, η_G is a closed map. Its image is also dense by Lemma 3.3, so it is a homeomorphism. □

Proposition 3.7. *The profinite completion of a topological group G has the following universal property: given a profinite group H and a morphism $\varphi\colon G \to H$, there exists a unique morphism $\varphi'\colon \widehat{G} \to H$ such that $\varphi = \varphi' \circ \eta_G$.*

Proof. By Proposition 3.5 there exists a unique morphism $\widehat{\varphi}\colon \widehat{G} \to \widehat{H}$ such that $\widehat{\varphi} \circ \eta_G = \eta_H \circ \varphi$. Since η_H is an isomorphism, the claim follows. □

It follows formally that the functor $G \mapsto \widehat{G}$ is left adjoint to the inclusion functor from profinite groups to topological groups.

Corollary 3.8. *Let G be a profinite group, and let H be a closed subgroup of G, equipped with the induced topology. Then H is a profinite group.*

Proof. We apply Proposition 3.6. Since H is quasi-compact and $\widehat{H}$ is Hausdorff, the morphism η_H is closed and surjective. Consider the commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{\eta_H} & \widehat{H} \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \widehat{j}} \\ G & \xrightarrow{\eta_G} & \widehat{G}, \end{array}$$

where j is the inclusion. Since η_G is an isomorphism and j is injective, η_H is injective. Thus it is an isomorphism. □

Proposition 3.9. *Let G be a topological group. There is a natural isomorphism of categories between the category of finite continuous G-sets and the category of finite continuous $\widehat{G}$-sets.*

Proof. Let E be a finite set. By Remark 3.1, continuous left actions of G on E are in natural bijection with morphisms $G \to S(E)$. By Proposition 3.7, these are in natural bijection with morphisms $\widehat{G} \to S(E)$. Applying Remark 3.1 again yields the desired isomorphism on objects. The correspondence on morphisms is immediate. □

3.3 Automorphism groups of functors

Construction 3.10. Let $\mathcal{C}$ be a small category, and let $F\colon \mathcal{C} \to \mathbf{FinSet}$ be a functor to the category of finite sets. For every $E \in \mathcal{C}$, equip $\mathrm{Aut}(F(E))$ with the discrete topology. Let $\mathcal{I}$ be the category whose objects are the finite subsets of $\mathrm{Ob}(\mathcal{C})$, with a unique morphism $A \to A'$ if $A' \subset A$, and no morphism from A to A' otherwise. Consider the cofiltered diagram

$$\begin{aligned} P\colon \mathcal{I} &\longrightarrow \mathbf{TopGrp}, \\ A &\longmapsto \prod_{E \in A} \mathrm{Aut}(F(E)), \end{aligned}$$

which sends a morphism $A \to A'$ to the canonical projection

$$\prod_{E \in A} \mathrm{Aut}(F(E)) \to \prod_{E' \in A'} \mathrm{Aut}(F(E')).$$

Then $\prod_{E \in \mathcal{C}} \mathrm{Aut}(F(E))$ is a limit of P, so in particular a profinite group. Note that $\mathrm{Aut}(F)$ is a closed subgroup of this product, because the groups $\mathrm{Aut}(F(E))$ are Hausdorff. We equip $\mathrm{Aut}(F)$ with the induced topology, which makes it a profinite group by Corollary 3.8.

Proposition 3.11. *In the situation of Construction 3.10, the canonical left action of* $\mathrm{Aut}(F)$ *on* E *is continuous for every* $E \in \mathcal{C}$.

Proof. The left action $\mathrm{Aut}(F) \times E \to E$ factors as

$$\mathrm{Aut}(F) \times E \xrightarrow{\mathrm{pr}_E \times \mathrm{id}_E} \mathrm{Aut}(F(E)) \times E \longrightarrow E,$$

where the second map is the canonical left action of $\mathrm{Aut}(F(E))$ on E. This action is continuous because the topology of $\mathrm{Aut}(F(E))$ is discrete; the map $\mathrm{pr}_E \times \mathrm{id}_E$ is continuous by definition of the topology of $\mathrm{Aut}(F)$. □

Proposition 3.12. *For every topological group* G*, the category* **FinCont**-G-**Set** *of finite continuous* G*-sets is essentially small, i.e. equivalent to a small category.*

Proof. Let E be a finite continuous G-set, and let $E_1, \ldots, E_r$ be its G-orbits. For every $i \in \{1, \ldots, r\}$, choose $x_i \in E_i$ and denote by U_i its stabilizer. Since U_i is open, the canonical left action of G on G/U_i is continuous. Since E_i is a transitive G-set, the map $g \mapsto g \cdot x_i$ induces an isomorphism between G/U_i and E_i. Hence E is isomorphic to $\coprod_{i=1}^{r} G/U_i$. Such G-sets form a set, so **FinCont**-G-**Set** has a full subcategory whose inclusion functor is an equivalence of categories. □

Whenever necessary, e.g. when applying Construction 3.10, we replace **FinCont**-G-**Set** by the full subcategory described in the preceding proof.

Proposition 3.13. *Let* G *be a profinite group, and let*

$$U\colon \mathbf{FinCont}\text{-}G\text{-}\mathbf{Set} \to \mathbf{FinSet}$$

be the forgetful functor. Endow $\mathrm{Aut}(U)$ *with the topology from Construction 3.10. There is a natural isomorphism of topological groups*

$$u\colon G \xrightarrow{\sim} \mathrm{Aut}(U)$$

mapping $g \in G$ *to the automorphism* $u(g)$ *of* U *whose component at* $E \in \mathcal{C}$ *is*

$$u(g)_E\colon U(E) \to U(E), \quad x \mapsto g \cdot x.$$

Proof. Because the morphisms in **FinCont-G-Set** are G-equivariant maps, u is well-defined. It follows from the definition of a left action that u is a group homomorphism.

We now construct an inverse of u. For every open normal subgroup M of G we have the continuous map

$$\mathrm{Aut}(U) \to G/M, \quad f \mapsto f_{G/M}(M).$$

The family of these maps is compatible with the projection $G/N' \to G/N$ for every inclusion $N' \subset N$ among open normal subgroups of G, so it induces a continuous map

$$v\colon\ \mathrm{Aut}(U) \to \widehat{G}.$$

Since $\mathrm{Aut}(U)$ is quasi-compact and $\widehat{G}$ is Hausdorff, v is closed. It is immediate from the respective constructions that $v \circ u = \eta_G$, which is an isomorphism by Proposition 3.6. Hence $(\eta_G^{-1} \circ v) \circ u = \mathrm{id}_G$.

It remains to show that $u \circ (\eta_G^{-1} \circ v) = \mathrm{id}_{\mathrm{Aut}(U)}$. Let f be an automorphism of U, and let M be an open normal subgroup of G. For every $g \in G$ there is a G-equivariant map

$$\omega_g\colon G/M \to G/M, \quad g'M \mapsto g'gM.$$

This is well-defined because M is normal. Since f is a morphism of functors,

$$f_{G/M}(gM) = f_{G/M}(\omega_g(M)) = \omega_g(f_{G/M}(M)) = f_{G/M}(M)gM.$$

Hence $f_{G/M}$ is just left-multiplication by $f_{G/M}(M)$. By construction of u and v, this implies

$$u(\eta_G^{-1}(v(f)))_{G/M} = f_{G/M}.$$

In order to finish the proof, it suffices to show that f is already determined by the components $f_{G/N}$, where N ranges over all open normal subgroups of G. Let E be a finite continuous G-set. By Proposition 3.12, we may suppose that E is of the form $\coprod_{i=1}^{r} G/H_i$ for open subgroups H_i of G. Denote by j the inclusion of G/H_i into E. Since f is a morphism of functors,

$$f_E|_{G/H_i} = f_E \circ U(j) = U(j) \circ f_{G/H_i}.$$

Hence we further reduce to the case that $E = G/H$ for an open subgroup H of E. By Lemma 3.4, H contains a normal open subgroup N of G. Then the diagram

$$\begin{array}{ccc} U(G/N) & \twoheadrightarrow & U(G/H) \\ \downarrow{\scriptstyle f_{G/N}} & & \downarrow{\scriptstyle f_{G/H}} \\ U(G/N) & \twoheadrightarrow & U(G/H) \end{array}$$

commutes, so $f_{G/H}$ can be recovered from $f_{G/N}$. Thus $u \circ (\eta_G^{-1} \circ v) = \mathrm{id}_{\mathrm{Aut}(U)}$, as desired. $\square$

Chapter 4

Fundamental Group

4.1 Definition

Fix a scheme S and a geometric point $\overline{s}$ of S. In order to avoid set-theoretic difficulties, we prove that $\mathbf{Fin\acute{E}t}/S$ is essentially small. Whenever necessary, we replace $\mathbf{Fin\acute{E}t}/S$ by the full subcategory described in the proof.

Proposition 4.1. *The category* $\mathbf{Fin\acute{E}t}/S$ *is essentially small.*

Proof. Up to isomorphism, a finite locally free morphism $X \to S$ is determined by

(a) an affine open covering $(V_i)_{i\in I}$ of S consisting of pairwise distinct sets,

(b) a nonnegative integer m_i for every $i \in I$,

(c) the structure of an $\mathcal{O}_S(V_i)$-algebra on every module $\mathcal{O}_S(V_i)^{m_i}$,

(d) and gluing data for the schemes $U_i := \operatorname{Spec}(\mathcal{O}_S(V_i)^{m_i})$, i.e. open subschemes $U_{ij} \subset U_i$ for all $i, j \in I$ and S-isomorphisms $\varphi_{ij} : U_{ij} \xrightarrow{\sim} U_{ji}$ satisfying the cocycle conditions.

The schemes obtained by this process form a set, which implies that the category of finite locally free morphisms $X \to S$ is essentially small. Hence so is the full subcategory of finite étale morphisms $X \to S$. □

Definition 4.2. *The* fundamental group of S with base point $\overline{s}$ *is defined to be the automorphism group of $F_{\overline{s}}$, equipped with the topology from Construction 3.10, and is denoted by $\pi_1(S, \overline{s})$.*

4.2 Profinite structure of the fundamental group

From now on, we assume that the base scheme S is connected.

Definition 4.3. *Let $\mathcal{C}$ be a category. A functor $F\colon \mathcal{C} \to \mathbf{Set}$ is called* pro-representable *if there exists a cofiltered diagram $P\colon \mathcal{I} \to \mathcal{C}$ together with an isomorphism of functors*

$$\operatorname*{colim}_{i\in\mathcal{I}} \mathrm{Mor}_{\mathcal{C}}(P(i), -) \xrightarrow{\sim} F.$$

If this is the case, then we say that P pro-represents *F.*

Construction 4.4. We construct a cofiltered diagram pro-representing $F_{\overline{s}}$. The index category $\mathcal{I}$ is a subcategory of $\mathbf{Fin\acute{E}t}/S$ whose objects are the connected finite étale Galois covers of S. For every $i \in \mathcal{I}$, choose $\overline{p}_i \in F_{\overline{s}}(i)$. Given $i, j \in \mathcal{I}$, there is at most one S-morphism $\varphi_{ij}\colon i \to j$ satisfying

$$F_{\overline{s}}(\varphi_{ij})(\overline{p}_i) = \overline{p}_j.$$

We define $\mathrm{Mor}_{\mathcal{I}}(i, j) = \{\varphi_{ij}\}$ if φ_{ij} exists, and $\mathrm{Mor}_{\mathcal{I}}(i, j) = \varnothing$ otherwise. Denote the inclusion functor $\mathcal{I} \to \mathbf{Fin\acute{E}t}/S$ by P.

Given $i, j \in \mathcal{I}$, the canonical morphism $P(i) \times_S P(j) \to S$ is finite étale. By Corollary 2.25, there exists $k \in \mathcal{I}$ and an S-morphism $\pi\colon P(k) \to P(i) \times_S P(j)$ such that $F_{\overline{s}}(\pi)(\overline{p}_k) = (\overline{p}_i, \overline{p}_j)$. Composing π with the respective projections, we obtain morphisms $k \to i$ and $k \to j$ in $\mathcal{I}$. Hence $\mathcal{I}$ is cofiltered.

We retain the diagram P and the points $\overline{p}_i$ for the rest of this chapter. The following proof is taken from Szamuely [7, Proposition 5.4.6].

Proposition 4.5. *The cofiltered diagram P defined in Construction 4.4 pro-represents $F_{\overline{s}}$.*

Proof. For every $i \in \mathcal{I}$ there is a morphism of functors

$$\eta_i\colon \mathrm{Mor}_S(P(i), -) \to F_{\overline{s}}$$

whose component at an object X of $\mathbf{Fin\acute{E}t}/S$ is

$$(\eta_i)_X\colon \mathrm{Mor}_S(P(i), X) \to F_{\overline{s}}(X), \quad g \mapsto F_{\overline{s}}(g)(\overline{p}_i).$$

For every morphism $\varphi_{ij}\colon i \to j$ in $\mathcal{I}$ we have $\eta_i \circ \mathrm{Mor}_S(\varphi_{ij}, -) = \eta_j$, since

$$F_{\overline{s}}(g \circ \varphi_{ij})(\overline{p}_i) = F_{\overline{s}}(g)(\overline{p}_j)$$

for every $g \in \mathrm{Mor}_S(P(i), X)$. The universal property of

$$G := \operatorname*{colim}_{i\in\mathcal{I}} \mathrm{Mor}_S(P(i), -)$$

yields a unique morphism $\eta\colon G \to F_{\overline{s}}$ induced by the morphisms η_i.

We construct an inverse of η. Given a finite étale cover $X \to S$ and $\overline{x} \in F_{\overline{s}}(X)$, there exists $i \in \mathcal{I}$ and an S-morphism $\pi\colon P(i) \to X$ such that

$$F_{\overline{s}}(\pi)(\overline{p}_i) = \overline{x}$$

by Corollary 2.25. Let $\vartheta_X\colon F_{\overline{s}}(X) \to G(X)$ be the map that sends $\overline{x}$ to the equivalence class of π; it is well-defined because $\mathcal{I}$ is cofiltered. We claim that ϑ_X is an inverse of η_X. On the one hand, we have

$$(\eta_X \circ \vartheta_X)(\overline{x}) = F_{\overline{s}}(\pi)(\overline{p}_i) = \overline{x};$$

hence $\eta_X \circ \vartheta_X$ is the identity map. On the other hand, if $\pi'\colon P(i') \to X$ represents an element of $G(X)$, then

$$(\vartheta_X \circ \eta_X)([\pi']) = \vartheta_X(F_{\overline{s}}(\pi')(\overline{p}_{i'})) = [\pi'].$$

Note that ϑ_X is natural in X, so we have a morphism of functors $\vartheta\colon F_{\overline{s}} \to G$ with component ϑ_X at X. It follows from the above calculations that ϑ is the desired inverse of η. □

Given a group G, we write G^{op} for the group with the same underlying set and the opposite group law; it is naturally isomorphic to G via $x \mapsto x^{-1}$.

Construction 4.6. Let $\varphi_{ij}\colon i \to j$ be a morphism in the category $\mathcal{I}$, i.e. $P(j) \to S$ is an intermediate connected finite étale Galois cover of $P(i) \to S$. By Proposition 2.21, there is a surjective group homomorphism

$$u_{ij}\colon \operatorname{Aut}_S(P(i))^{\mathrm{op}} \to \operatorname{Aut}_S(P(j))^{\mathrm{op}};$$

it maps $g_i \in \operatorname{Aut}_S(P(i))^{\mathrm{op}}$ to the unique $g_j \in \operatorname{Aut}_S(P(j))^{\mathrm{op}}$ satisfying

$$g_j \circ \varphi_{ij} = \varphi_{ij} \circ g_i.$$

Note that u_{ii} is the identity and $u_{jk} \circ u_{ij} = u_{ik}$ for all $i, j, k \in \mathcal{I}$, so we have a functor

$$\begin{aligned} \mathcal{I} &\longrightarrow \mathbf{TopGrp}, \\ i &\longmapsto \operatorname{Aut}_S(P(i))^{\mathrm{op}}, \\ \varphi_{ij} &\longmapsto u_{ij}, \end{aligned}$$

where the finite groups $\operatorname{Aut}_S(P(i))^{\mathrm{op}}$ are equipped with the discrete topology.

Proposition 4.7. *There is an isomorphism of topological groups*

$$u\colon \pi_1(S, \overline{s}) \xrightarrow{\sim} \lim_{i \in \mathcal{I}} \operatorname{Aut}_S(P(i))^{\mathrm{op}}$$

induced by the morphisms

$$u_i\colon \pi_1(S, \overline{s}) \to \operatorname{Aut}_S(P(i))^{\mathrm{op}}$$

such that

$$F_{\overline{s}}(u_i(f))(\overline{p}_i) = f_{P(i)}(\overline{p}_i)$$

for every $f \in \pi_1(S, \overline{s})$.

Proof. We construct an inverse w of u. As in the proof of Proposition 4.5, denote by G the functor $\operatorname{colim}_{i\in\mathcal{I}} \operatorname{Mor}_S(P(i), -)$ and by η the isomorphism $G \xrightarrow{\sim} F_{\overline{s}}$. The diagram

$$i \longmapsto \operatorname{Mor}_S(P(i), -)$$

is the composite of P with the contravariant Yoneda embedding

$$P(i) \longmapsto \operatorname{Mor}_S(P(i), -)$$

and $\operatorname{Aut}(P)^{\mathrm{op}} = \lim_{i\in\mathcal{I}} \operatorname{Aut}_S(P(i))^{\mathrm{op}}$, so there is a canonical group homomorphism

$$v\colon \lim_{i\in\mathcal{I}} \operatorname{Aut}_S(P(i))^{\mathrm{op}} \to \operatorname{Aut}(G).$$

Let w be the composite of v with

$$\operatorname{Aut}(G) \xrightarrow{\sim} \pi_1(S, \overline{s}), \quad h \mapsto \eta \circ h \circ \eta^{-1}.$$

Let us check that u and w are inverse to each other. Given $f \in \pi_1(S, \overline{s})$, let $X \to S$ be a finite étale morphism of schemes, and $\overline{x} \in F_{\overline{s}}(X)$. By Corollary 2.25, there exists $i \in \mathcal{I}$ and an S-morphism $\pi\colon P(i) \to X$ such that $F_{\overline{s}}(\pi)(\overline{p}_i) = \overline{x}$. Then

$$\begin{aligned}
w(u(f))_X(\overline{x}) &= w(u(f))_X(F_{\overline{s}}(\pi)(\overline{p}_i)) \\
&= F_{\overline{s}}(\pi)(w(u(f))_{P(i)}(\overline{p}_i)) \\
&= F_{\overline{s}}(\pi)(f_{P(i)}(\overline{p}_i)) \\
&= f_X(F_{\overline{s}}(\pi)(\overline{p}_i)) \\
&= f_X(\overline{x}).
\end{aligned}$$

Hence $(w \circ u)(f) = f$. On the other hand, starting out with an element $g = (g_i)_{i\in\mathcal{I}}$ of $\lim_{i\in\mathcal{I}} \operatorname{Aut}_S(P(i))^{\mathrm{op}}$, we have

$$\begin{aligned}
F_{\overline{s}}(u_i(w(g)))(\overline{p}_i) &= w(g)_{P(i)}(\overline{p}_i) \\
&= (\eta \circ v(g) \circ \eta^{-1})_{P(i)}(\overline{p}_i) \\
&= \eta_{P(i)}(v(g)_{P(i)}([\mathrm{id}_{P(i)}])) \\
&= \eta_{P(i)}(g_i) \\
&= F_{\overline{s}}(g_i)(\overline{p}_i);
\end{aligned}$$

by Proposition 1.30, this implies $u_i(w(g)) = g_i$.

Thus u is a bijective group homomorphism. It is also continuous, because each u_i is continuous. Since $\pi_1(S, \overline{s})$ is quasi-compact and $\lim_{i\in\mathcal{I}} \operatorname{Aut}_S(P(i))^{\mathrm{op}}$ is Hausdorff, u is also a closed map. Hence it is a homeomorphism. □

The following technical result shows that the morphisms u_i are surjective.

Lemma 4.8. *Let G be the limit of a cofiltered diagram $P\colon \mathcal{I} \to$ **TopGrp** of quasi-compact Hausdorff topological groups with surjective transition morphism. Suppose moreover that between any two objects of $\mathcal{I}$ there are only finitely many parallel morphisms. Then the projection $\mathrm{pr}_j\colon G \to P(j)$ is surjective for every $j \in \mathcal{I}$.*

Proof. Let $h_j \in P(j)$. For every $i \in \mathcal{I}$, define E_i to be the subset of $\prod_{i' \in \mathcal{I}} P(i')$ consisting of all elements $(g_{i'})_{i' \in \mathcal{I}}$ such that $g_j = h_j$ and $P(\varphi)(g_i) = g_{i'}$ for every $i' \in \mathcal{I}$ and every morphism $\varphi\colon i \to i'$. Then $\mathrm{pr}_j^{-1}(h_j) = \bigcap_{i \in \mathcal{I}} E_i$, and each E_i is closed because the groups $P(i')$ are Hausdorff. Since $\prod_{i' \in \mathcal{I}} P(i')$ is quasi-compact, it suffices to show that the family $(E_i)_{i \in \mathcal{I}}$ has the finite intersection property. To prove that each E_i is nonempty, use that $\mathcal{I}$ is cofiltered, that there are only finitely many parallel morphisms between any two objects of $\mathcal{I}$, and that the transition morphisms are surjective. Given finitely many objects $i(1), \ldots, i(r)$ of $\mathcal{I}$, there exists $l \in \mathcal{I}$ such that there is a morphism $l \to i(k)$ in $\mathcal{I}$ for every $k \in \{1, \ldots, r\}$. Then E_l is contained in $\bigcap_{k=1}^{r} E_{i(k)}$, so the latter is nonempty. □

4.3 Classification theorem

Let f be an automorphism of $F_{\overline{s}}$, and let $\psi\colon X \to Y$ be a morphism in **FinÉt**$/S$. Then $f_Y \circ F_{\overline{s}}(\psi) = F_{\overline{s}}(\psi) \circ f_X$, i.e. $F_{\overline{s}}(\psi)$ is $\pi_1(S, \overline{s})$-equivariant. Thus $F_{\overline{s}}$ factors through the functor

$$\begin{aligned} \mathrm{Fib}_{\overline{s}}\colon \mathbf{Fin\acute{E}t}/S &\longrightarrow \mathbf{FinCont}\text{-}\pi_1(S, \overline{s})\text{-}\mathbf{Set}, \\ (X \to S) &\longmapsto F_{\overline{s}}(X), \\ \psi &\longmapsto F_{\overline{s}}(\psi). \end{aligned}$$

Proposition 4.9. *The group $\pi_1(S, \overline{s})$ acts transitively on $F_{\overline{s}}(X)$ for every connected finite étale cover $X \to S$.*

Proof. Let $\pi\colon Q \to X$ be a Galois closure of $X \to S$, and $\overline{q} \in F_{\overline{s}}(Q)$. We show that every point of $F_{\overline{s}}(X)$ lies in the orbit of $\overline{x} := F_{\overline{s}}(\pi)(\overline{q})$. Given $\overline{x}' \in F_{\overline{s}}(X)$, let $\overline{q}' \in F_{\overline{s}}(Q)$ be such that $F_{\overline{s}}(\pi)(\overline{q}') = \overline{x}'$. Since π is Galois, it has an automorphism h taking $\overline{q}$ to $\overline{q}'$. Combining Proposition 4.7 with Lemma 4.8, we see that h can be lifted to $\pi_1(S, \overline{s})$, i.e. there exists $f \in \pi_1(S, \overline{s})$ such that $f_Q(\overline{q}) = \overline{q}'$. Since f is a morphism of functors, we have

$$\begin{aligned} f_X(\overline{x}) &= f_X(F_{\overline{s}}(\pi)(\overline{q})) \\ &= F_{\overline{s}}(\pi)(f_Q(\overline{q})) \\ &= \overline{x}'. \end{aligned}$$

Hence $\pi_1(S, \overline{s})$ acts transitively on $F_{\overline{s}}(X)$. □

We are now in a position to prove that the fundamental group of S classifies its finite étale covers. More precisely, we have the following statement:

Theorem 4.10. *The functor* $\mathrm{Fib}_{\overline{s}}\colon \mathbf{Fin\acute{E}t}/S \to \mathbf{FinCont}\text{-}\pi_1(S,\overline{s})\text{-}\mathbf{Set}$ *is an equivalence of categories.*

Proof. We begin by checking that $\mathrm{Fib}_{\overline{s}}$ is faithful. Let $\psi_1, \psi_2\colon X \rightrightarrows Y$ be morphisms in $\mathbf{Fin\acute{E}t}/S$ such that $F_{\overline{s}}(\psi_1) = F_{\overline{s}}(\psi_2)$. Let $j\colon Z \hookrightarrow X$ be the embedding of a connected component of X. Then $F_{\overline{s}}(\psi_1 \circ j) = F_{\overline{s}}(\psi_2 \circ j)$, so $\psi_1 \circ j = \psi_2 \circ j$ by Proposition 1.30. Since this is the case for every connected component of X, we deduce $\psi_1 = \psi_2$.

Now let us show that $\mathrm{Fib}_{\overline{s}}$ is full. Let $X \to S$ and $Y \to S$ be finite étale morphisms, and let $\omega\colon F_{\overline{s}}(X) \to F_{\overline{s}}(Y)$ be a $\pi_1(S,\overline{s})$-equivariant map. Since $F_{\overline{s}}$ preserves finite coproducts, it suffices to consider the case where X is connected. Let $Y_1, \ldots, Y_r$ be the connected components of Y. Then $F_{\overline{s}}(Y)$ is canonically isomorphic to $\coprod_{i=1}^r F_{\overline{s}}(Y_i)$, and each $F_{\overline{s}}(Y_i)$ is a transitive $\pi_1(S,\overline{s})$-set by Proposition 4.9. As $F_{\overline{s}}(X)$ is transitive for the same reason, ω factors through $F_{\overline{s}}(Y_i)$ for some $i \in \{1,\ldots,r\}$. Hence we may assume that X and Y are connected. Choose a point $\overline{x} \in F_{\overline{s}}(X)$. Since $\pi_1(S,\overline{s})$ acts transitively on $F_{\overline{s}}(X)$ and $F_{\overline{s}}(Y)$, the map ω is already determined by $\overline{y} := \omega(\overline{x})$. Arguing as in Construction 4.4, we find a connected finite étale Galois cover $Q \to S$, a point $\overline{q} \in F_{\overline{s}}(Q)$, and S-morphisms $\pi_X\colon Q \to X$ and $\pi_Y\colon Q \to Y$ satisfying

$$F_{\overline{s}}(\pi_X)(\overline{q}) = \overline{x} \quad \text{and} \quad F_{\overline{s}}(\pi_Y)(\overline{q}) = \overline{y}.$$

We claim that π_Y factors through π_X. For every $h \in \mathrm{Aut}_X(Q)$, there exists $f \in \pi_1(S,\overline{s})$ such that $f_Q(\overline{q}) = F_{\overline{s}}(h)(\overline{q})$ by an argument as in the proof of Proposition 4.9. Straightforward calculations using naturality of f and the definition of $\overline{x}$ show that

$$F_{\overline{s}}(\pi_Y \circ h)(\overline{q}) = F_{\overline{s}}(\pi_Y)(\overline{q}).$$

By Proposition 1.30, this implies $\pi_Y \circ h = \pi_Y$. Since X together with π_X is a quotient of Q by $\mathrm{Aut}_X(Q)$, there exists a unique S-morphism $\psi\colon X \to Y$ such that $\pi_Y = \psi \circ \pi_X$. By construction, $F_{\overline{s}}(\psi)(\overline{x}) = \overline{y}$, so $F_{\overline{s}}(\psi) = \omega$.

Finally, let us show that $\mathrm{Fib}_{\overline{s}}$ is essentially surjective. Let E be a finite continuous $\pi_1(S,\overline{s})$-set. Decomposing E into its $\pi_1(S,\overline{s})$-orbits and using the fact that $\mathrm{Fib}_{\overline{s}}$ preserves finite coproducts, we may assume that E is a transitive $\pi_1(S,\overline{s})$-set. Then E is isomorphic to $\pi_1(S,\overline{s})/H$ for an open subgroup H of $\pi_1(S,\overline{s})$ of finite index. Let $i \in \mathcal{I}$ be such that $\ker(u_i) \subset H$, see Lemma 3.4, and let $U = u_i(H)$. Consider the $\pi_1(S,\overline{s})$-equivariant map

$$\omega\colon \pi_1(S,\overline{s}) \to F_{\overline{s}}(P(i)), \quad f \mapsto f_{P(i)}(\overline{p}_i),$$

which is surjective by Proposition 4.9. If $f \in \pi_1(S,\overline{s})$, then $\omega^{-1}(U\omega(f))$ consist of all $g \in \pi_1(S,\overline{s})$ for which there exists $h \in H$ such that

$$g_{P(i)}(\overline{p}_i) = F_{\overline{s}}(u_i(h))(f_{P(i)}(\overline{p}_i)),$$

since $\mathrm{Aut}_S(P(i))$ acts on $F_{\overline{s}}(P(i))$ via $F_{\overline{s}}$. But

$$F_{\overline{s}}(u_i(h))(f_{P(i)}(\overline{p}_i)) = (f \circ h)_{P(i)}(\overline{p}_i),$$

so $(f \circ h)^{-1} \circ g \in \ker(u_i)$ and therefore $g \in fH$. This shows $\omega^{-1}(U\omega(f)) \subset fH$; the reverse inclusion is immediate. Hence ω induces an isomorphism of $\pi_1(S, \overline{s})$-sets

$$\pi_1(S, \overline{s})/H \xrightarrow{\sim} U \backslash F_{\overline{s}}(P(i)).$$

Since

$$U \backslash F_{\overline{s}}(P(i)) \cong F_{\overline{s}}(U \backslash P(i))$$

by Proposition 2.17, this implies $E \cong F_{\overline{s}}(U \backslash P(i))$. □

Chapter 5

Analytic Topology

The material in this chapter is adapted from Neeman [6, Chapter 4]. From now on we work in the category **Sch**/$\mathbb{C}$ of schemes over the field of complex numbers $\mathbb{C}$. Recall that for every scheme X locally of finite type over $\mathbb{C}$ there is a natural bijection between the set $X(\mathbb{C})$ of $\mathbb{C}$-rational points of X and the set of closed points of X; the bijection maps a morphism $\mathrm{Spec}(\mathbb{C}) \to X$ to the unique point in its image.

Construction 5.1. Let X be an affine scheme of finite type over $\mathbb{C}$. Choose a closed embedding $\varphi\colon X \hookrightarrow \mathbb{A}^m_{\mathbb{C}}$; it induces an injective map

$$\varphi(\mathbb{C})\colon X(\mathbb{C}) \hookrightarrow \mathbb{A}^m_{\mathbb{C}}(\mathbb{C}), \quad t \mapsto \varphi \circ t.$$

Denote by u_m the natural bijection $\mathbb{A}^m_{\mathbb{C}}(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}^m$. We equip $\mathbb{C}^m$ with the analytic topology, and $X(\mathbb{C})$ with the initial topology induced by $u_m \circ \varphi(\mathbb{C})$; the resulting topological space is denoted by X^{an}. As a formal consequence of the fact that the analytic topology on $\mathbb{C}^m$ is finer than the Zariski topology, it follows that the topology of X^{an} is finer than the induced topology. Hence the canonical map $X^{\mathrm{an}} \to X$ is continuous.

Proposition 5.2. *The topology of X^{an} is independent of φ.*

Proof. Let $\varphi\colon X \hookrightarrow \mathbb{A}^m_{\mathbb{C}}$ and $\psi\colon X \hookrightarrow \mathbb{A}^n_{\mathbb{C}}$ be closed embeddings, with corresponding surjective morphisms of $\mathbb{C}$-algebras

$$\alpha\colon \mathbb{C}[X_1, \dots, X_m] \twoheadrightarrow \mathcal{O}_X(X) \quad \text{and} \quad \beta\colon \mathbb{C}[Y_1, \dots, Y_n] \twoheadrightarrow \mathcal{O}_X(X).$$

For every $i \in \{1, \dots, m\}$, choose a polynomial $p_i \in \mathbb{C}[Y_1, \dots, Y_n]$ such that $\alpha(X_i) = \beta(p_i)$. Let

$$\omega\colon \mathbb{C}[X_1, \dots, X_m] \longrightarrow \mathbb{C}[Y_1, \dots, Y_n]$$

be the morphism of $\mathbb{C}$-algebras with $\omega(X_i) = p_i$. Then $\beta \circ \omega = \alpha$, so passing to spectra we have the commutative diagram

$$\begin{array}{ccc} X & \overset{\varphi}{\hookrightarrow} & \mathbb{A}^m_{\mathbb{C}} \\ \Big\| \, \mathrm{id}_X & & \Big\uparrow \, \mathrm{Spec}(\omega) \\ X & \underset{}{\overset{\psi}{\hookrightarrow}} & \mathbb{A}^n_{\mathbb{C}}. \end{array}$$

Now pass to $\mathbb{C}$-rational points; $\mathrm{Spec}(\omega)(\mathbb{C})\colon \mathbb{A}^n_{\mathbb{C}}(\mathbb{C}) \to \mathbb{A}^m_{\mathbb{C}}(\mathbb{C})$ corresponds to the map

$$\mathbb{C}^n \to \mathbb{C}^m, \quad y \mapsto (p_1(y), \ldots, p_m(y)),$$

which is continuous with respect to the analytic topologies. By commutativity of the above diagram, the topology on $X(\mathbb{C})$ induced by φ is coarser than the topology induced by ψ. Since the argument is symmetric in φ and ψ, the claim follows. □

Definition 5.3. *Let X be a scheme locally of finite type over $\mathbb{C}$. We define the* analytic topology *on $X(\mathbb{C})$ to be the final topology with respect to the canonical injective maps $U^{\mathrm{an}} \hookrightarrow X(\mathbb{C})$, where U ranges over all affine open subschemes of X. The resulting topological space is denoted by X^{an} and is called the* analytification *of X.*

Remark 5.4. Since the analytic topology on the affine open subschemes is finer than the induced topology, the canonical map $X^{\mathrm{an}} \to X$ is continuous.

Construction 5.5. The assignment $X \longmapsto X^{\mathrm{an}}$ extends to a functor from the category of schemes locally of finite type over $\mathbb{C}$ to the category of topological spaces, which we call the *analytification functor*. In fact, equipping $\mathbb{C}^m$ with the sheaf of holomorphic functions, X^{an} locally inherits the structure of a ringed space. Thus X^{an} becomes an *analytic space*; see Grothendieck [2, Définition 2.1]. Given a morphism $f\colon X \to Y$, we have

$$f^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}, \quad t \mapsto f \circ t.$$

Continuity of f^{an} in the affine case follows by drawing a diagram as in the proof of Proposition 5.2, replacing the upper X by Y and id_X by f. The general case can then be reduced to this, because continuity is local.

Instead of defining the analytic topology as the final topology with respect to all embeddings of affine open subschemes, we could choose an affine open covering and try to glue the analytic topologies of the members of the covering. The next two results show that this procedure works and yields the same topology.

Lemma 5.6. *Let X be an affine scheme of finite type over $\mathbb{C}$. For every $f \in \mathcal{O}_X(X)$, the canonical open embedding $i\colon D(f) \hookrightarrow X$ of the distinguished open subscheme $D(f)$ induces an open embedding $i^{\mathrm{an}}\colon D(f)^{\mathrm{an}} \hookrightarrow X^{\mathrm{an}}$.*

Proof. Suppose first that $X = \mathbb{A}^m_{\mathbb{C}}$ with coordinates $X_1, \ldots, X_m$. Given $f \in \mathbb{C}[X_1, \ldots, X_m]$, we have the commutative diagram

$$\begin{array}{ccc} & \mathbb{A}^m_{\mathbb{C}} & \\ {\scriptstyle i}\nearrow & & \nwarrow{\scriptstyle \pi} \\ D(f) & \xhookrightarrow{\;j\;} & \mathbb{A}^{m+1}_{\mathbb{C}}, \end{array}$$

where j embeds $D(f)$ as the closed subscheme $V(fX_{m+1} - 1)$ of $\mathbb{A}^{m+1}_{\mathbb{C}}$, and π is the projection onto the first m coordinates. Passing to $\mathbb{C}$-rational points and using the bijections $u_k\colon \mathbb{A}^k_{\mathbb{C}}(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}^k$, we need to show that the projection

$$V(f(\underline{x})x_{m+1} - 1) \to \mathbb{C}^m, \quad (\underline{x}, x_{m+1}) \mapsto \underline{x}$$

from the vanishing locus $V(f(\underline{x})x_{m+1} - 1) \subset \mathbb{C}^{m+1}$ is an open embedding for the complex topologies; but this is clear, as it induces a homeomorphism with the open subspace

$$(\mathbb{C}^m)_f := \{\underline{x} \in \mathbb{C}^m \mid f(\underline{x}) \neq 0\} \subset \mathbb{C}^m.$$

In the general case, choose a closed embedding $X \hookrightarrow \mathbb{A}^m_{\mathbb{C}}$ corresponding to a surjective morphism $\alpha\colon \mathbb{C}[X_1, \ldots, X_m] \twoheadrightarrow \mathcal{O}_X(X)$. Let $g \in \alpha^{-1}(f)$. By the universal property of localization, there is a unique morphism $\beta\colon \mathbb{C}[X_1, \ldots, X_m]_g \to \mathcal{O}_X(X)_f$ such that the diagram

$$\begin{array}{ccc} \mathbb{C}[X_1, \ldots, X_m] & \xrightarrow{\;\alpha\;} & \mathcal{O}_X(X) \\ \downarrow & & \downarrow \\ \mathbb{C}[X_1, \ldots, X_m]_g & \xrightarrow{\;\beta\;} & \mathcal{O}_X(X)_f \end{array}$$

commutes, where the unlabeled morphisms are the canonical ones. Since α is surjective, so is β. Passing to spectra, then to $\mathbb{C}$-rational points and using the first step, we obtain the commutative diagram

$$\begin{array}{ccc} \mathbb{C}^m & \hookleftarrow & X^{\mathrm{an}} \\ \uparrow & & \uparrow {\scriptstyle i^{\mathrm{an}}} \\ (\mathbb{C}^m)_g & \hookleftarrow & D(f)^{\mathrm{an}}, \end{array}$$

where all morphisms except for i^{an} are known to be embeddings; hence i^{an} is also an embedding. Its image corresponds to the intersection of $D(f)$ with the set of closed points of X, so we conclude that i^{an} is an open embedding. □

Proposition 5.7. *Let X be a scheme locally of finite type over $\mathbb{C}$, and let $j\colon U \hookrightarrow X$ be the embedding of an affine open subscheme. The associated continuous map $j^{\mathrm{an}}\colon U^{\mathrm{an}} \to X^{\mathrm{an}}$ is an open embedding.*

Proof. The map j^{an} is injective, and its image is the inverse image of the open set U under the canonical continuous map $c\colon X^{\mathrm{an}} \to X$. It remains to show that j^{an} is an open map. Let A be an open subset U^{an}; we need to show that for every affine open subscheme V of X, the inverse image B of $c(j^{\mathrm{an}}(A))$ under the map $V^{\mathrm{an}} \to X$ is an open subset of V^{an}. Take a point $x \in B$, and let W be an affine open neighborhood of $c(x)$ in $U \cap V$ which is simultaneously distinguished in U and V. By Lemma 5.6, we have canonical open embeddings $W^{\mathrm{an}} \hookrightarrow U^{\mathrm{an}}$ and $W^{\mathrm{an}} \hookrightarrow V^{\mathrm{an}}$. Hence B is open at x. □

Lemma 5.8. *Let $X \to S$ and $Y \to S$ be morphisms of schemes locally of finite type over $\mathbb{C}$. The natural bijection*

$$u\colon (X \times_S Y)(\mathbb{C}) \xrightarrow{\sim} X(\mathbb{C}) \times_{S(\mathbb{C})} Y(\mathbb{C}), \quad t \mapsto (\mathrm{pr}_1 \circ t, \mathrm{pr}_2 \circ t).$$

induces a homeomorphism $(X \times_S Y)^{\mathrm{an}} \xrightarrow{\sim} X^{\mathrm{an}} \times_{S^{\mathrm{an}}} Y^{\mathrm{an}}$.

Proof. Because of the gluing construction of fiber products and Proposition 5.7, we may assume that X, Y, and S are affine. Choosing appropriate generators of the corresponding rings, we find a closed embedding $\chi\colon S \hookrightarrow \mathbb{A}^l_{\mathbb{C}}$ as well as closed embeddings $\varphi\colon X \hookrightarrow \mathbb{A}^m_{\mathbb{C}}$ and $\psi\colon Y \hookrightarrow \mathbb{A}^n_{\mathbb{C}}$ with $m \geqslant l$ and $n \geqslant l$ fitting into commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & \mathbb{A}^m_{\mathbb{C}} \\ \downarrow & & \downarrow \\ S & \xrightarrow{\chi} & \mathbb{A}^l_{\mathbb{C}} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} Y & \xrightarrow{\psi} & \mathbb{A}^n_{\mathbb{C}} \\ \downarrow & & \downarrow \\ S & \xrightarrow{\chi} & \mathbb{A}^l_{\mathbb{C}}, \end{array}$$

where the morphisms $\mathbb{A}^m_{\mathbb{C}} \to \mathbb{A}^l_{\mathbb{C}}$ and $\mathbb{A}^n_{\mathbb{C}} \to \mathbb{A}^l_{\mathbb{C}}$ are the projections. Then $\varphi \times_\chi \psi$ can be used to define the analytic topology on $(X \times_S Y)(\mathbb{C})$ by Proposition 5.2. The claim follows from naturality of u. □

A continuous map $\varphi\colon X \to Y$ of topological spaces is *separated* if the diagonal of $X \times_Y X$ is closed. Equivalently, any two distinct points of X lying over the same point of Y have disjoint open neighborhoods in X.

Proposition 5.9. *Let $\varphi\colon X \to S$ be a morphism of schemes locally of finite type over $\mathbb{C}$. If φ is a separated morphism of schemes, then φ^{an} is a separated continuous map.*

Proof. Denote by Δ the diagonal morphism $X \to X \times_S X$, which is a closed embedding since φ is separated. The diagonal map

$$\delta\colon X^{\mathrm{an}} \to X^{\mathrm{an}} \times_{S^{\mathrm{an}}} X^{\mathrm{an}}, \quad t \mapsto (t, t)$$

is the composite of Δ^{an} with the homeomorphism u from Lemma 5.8. By Remark 5.4, the image of Δ^{an} is closed in $(X \times_S X)^{\mathrm{an}}$. Hence the image of δ, which is precisely the diagonal of $X^{\mathrm{an}} \times_{S^{\mathrm{an}}} X^{\mathrm{an}}$, is closed. □

Proposition 5.10. *A scheme X locally of finite type over $\mathbb{C}$ is connected if and only if its analytification X^{an} is connected.*

Proof. See Grothendieck [1, Exposé XII, Proposition 2.4]. □

Chapter 6

Comparison Theorem

6.1 Topological theory

Let X be a connected topological space, and let $x \in X$ be a point. A *cover of* X is a continuous map $\varphi\colon Y \to X$ such that every point $x \in X$ has an *evenly covered open neighborhood*, i.e. an open neighborhood U in X such that $\varphi^{-1}(U)$ is a disjoint union of open subsets of Y each of which is mapped homeomorphically onto U by φ. Note that we do not assume that φ is surjective. As for schemes, we define a fiber functor F_x from the category **Cov**$/X$ of covers of X to the category of sets. It maps a cover $\varphi\colon Y \to X$ to the fiber $\varphi^{-1}(x)$, and a morphism of covers, i.e. a continuous map over X, to the induced map on the fibers. A *universal cover of* X is a simply connected cover of X; it exists for example if X is locally contractible.

Theorem 6.1. *If X has a universal cover, then the fiber functor F_x induces an equivalence of categories*

$$\mathbf{Cov}/X \simeq \pi_1(X,x)\text{-}\mathbf{Set},$$

which sends a cover to its fiber over x equipped with the monodromy action.

Proof. See Szamuely [7, Theorem 2.3.4]. □

We view $\pi_1(X,x)$ as a discrete topological group, and denote its profinite completion by $\widehat{\pi}_1(X,x)$.

Corollary 6.2. *If X has a universal cover, then F_x induces an equivalence of categories*

$$\mathbf{FinCov}/X \simeq \mathbf{FinCont}\text{-}\widehat{\pi}_1(X,x)\text{-}\mathbf{Set}.$$

Proof. The equivalence of categories from Theorem 6.1 restricts to an equivalence of categories

$$\mathbf{FinCov}/X \simeq \mathbf{Fin}\text{-}\pi_1(X,x)\text{-}\mathbf{Set}.$$

Since $\pi_1(X,x)$ is discrete, every finite $\pi_1(X,x)$-set is continuous. Now apply Proposition 3.9. □

6.2 Analytification of finite étale covers

We again work in the category **Sch**/$\mathbb{C}$. Fix a connected scheme S locally of finite type over $\mathbb{C}$ and a geometric point $\bar{s} \in S^{\mathrm{an}}$. As a first step toward the comparison theorem, we prove that the analytification of a finite étale cover of S is a cover of S^{an} with finite fibers.

Definition 6.3. *A morphism of affine schemes $X \to Y$ is called* standard étale *if it is isomorphic to a morphism of the form* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *with*

$$B = (A[Y]/(g))_h$$

for $g \in A[Y]$ monic and $h \in A[Y]/(g)$ such that $\partial g/\partial Y$ becomes invertible in $(A[Y]/(g))_h$.

Theorem 6.4. *A morphism of schemes $\varphi\colon X \to Y$ is finite étale if and only if it is finite and for every point $x \in X$ there exist affine open neighborhoods U of x and V of $\varphi(x)$ such that $\varphi(U) \subset V$ and $\varphi|_U\colon U \to V$ is standard étale.*

Proof. See Milne [5, Chapter I, Theorem 3.14]. □

Lemma 6.5. *Let $f_1, \dots, f_r \in \mathbb{C}[X_1, \dots, X_m]$ and $g \in \mathbb{C}[X_1, \dots, X_m, Y]$ be such that*

$$\frac{\partial g}{\partial Y}(\underline{x}, y) \in \mathbb{C}$$

is invertible at a point $(\underline{x}, y)$ of the vanishing locus $V(f_1, \dots, f_r, g) \subset \mathbb{C}^m \times \mathbb{C}$. Then the projection

$$\pi\colon V(f_1, \dots, f_r, g) \to V(f_1, \dots, f_r), \quad (\underline{x}', y') \mapsto \underline{x}'$$

is a homeomorphism at $(\underline{x}, y)$.

Proof. Consider the holomorphic map

$$F\colon \mathbb{C}^m \times \mathbb{C} \to \mathbb{C}, \quad (\underline{x}', y') \mapsto g(\underline{x}', y'),$$

which satisfies $V(f_1 \dots, f_r) \cap F^{-1}(0) = V(f_1, \dots, f_r, g)$. Since $(\partial g/\partial Y)(\underline{x}, y)$ is invertible, the Implicit Function Theorem applies. Thus there are open sets $U \subset \mathbb{C}^m$ and $V \subset \mathbb{C}$ satisfying $(\underline{x}, y) \in U \times V$ together with a holomorphic map $G\colon U \to V$ such that $(U \times V) \cap F^{-1}(0)$ is the graph of G. In particular, the restriction of π to

$$(U \times V) \cap V(f_1, \dots, f_r, g) \to U \cap V(f_1, \dots, f_r)$$

has the inverse $\underline{x}' \mapsto (\underline{x}', G(\underline{x}'))$. Hence π is a homeomorphism at $(\underline{x}, y)$. □

Proposition 6.6. *Let $\varphi\colon X \to Y$ be a morphism of affine schemes of finite type over $\mathbb{C}$. If φ is standard étale, then its analytification $\varphi^{\mathrm{an}}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ is a local homeomorphism.*

Proof. Assume that $X = \operatorname{Spec}(B)$ and $Y = \operatorname{Spec}(A)$ are as in Definition 6.3. As Y is of finite type over $\mathbb{C}$, we can replace A by $\mathbb{C}[X_1, \ldots, X_m]/(f_1, \ldots, f_r)$ for certain $f_1, \ldots, f_r \in \mathbb{C}[X_1, \ldots, X_m]$. Lifting g to $\mathbb{C}[X_1, \ldots, X_m, Y]$, we may assume

$$B = (\mathbb{C}[X_1, \ldots, X_m, Y]/(f_1, \ldots, f_r, g))_h$$

for $h \in \mathbb{C}[X_1, \ldots, X_m, Y]/(f_1, \ldots, f_r, g)$ such that $\partial g/\partial Y$ becomes invertible in B. Then φ^{an} corresponds to the composite

$$V(f_1, \ldots, f_r, g)_h \hookrightarrow V(f_1, \ldots, f_r, g) \xrightarrow{\pi} V(f_1, \ldots, f_r),$$

where π is as in Lemma 6.5. Because of our assumption on $\partial g/\partial Y$ and h, $(\partial g/\partial Y)(x)$ is invertible at every point $x \in V(f_1, \ldots, f_r, g)_h$. Combining Lemma 6.5 with Lemma 5.6 shows that φ^{an} is a local homeomorphism. □

The *degree* of a continuous map $\varphi\colon X \to Y$ of topological spaces at a point $y \in Y$ is the cardinality of the fiber $\varphi^{-1}(y)$.

Proposition 6.7. *Let $\varphi\colon X \to Y$ be a separated continuous map of topological spaces. If φ is a local homeomorphism with finite fibers whose degree is a locally constant function of $y \in Y$, then φ is a cover.*

Proof. Let y be a point of Y, and let $\varphi^{-1}(y) = \{x_1, \ldots, x_m\}$ be an enumeration of the fiber of φ over y. If it is empty, then y has an open neighborhood whose inverse image under φ is empty, because the degree of φ is locally constant. Hence we may assume that $\varphi^{-1}(y)$ is nonempty. Since φ is separated, there are pairwise disjoint open neighborhoods $U_1, \ldots, U_m$ of $x_1, \ldots, x_m$ in X. After shrinking them, we may assume that there exists an open neighborhood V of y on which the degree of φ is constant and such that each U_i maps homeomorphically onto V. Then V is an evenly covered open neighborhood of y, because $\varphi^{-1}(V) = U_1 \cup \cdots \cup U_m$. □

Proposition 6.8. *If $\varphi\colon X \to S$ is a finite étale morphism, then its analytification $\varphi^{\mathrm{an}}\colon X^{\mathrm{an}} \to S^{\mathrm{an}}$ is a cover with finite fibers.*

Proof. The map φ^{an} is a local homeomorphism by Propositions 6.3 and 6.6, and separated by Proposition 5.9. Its fibers are in bijection with geometric fibers of φ, so they are finite and their cardinality is locally constant. The claim follows from Proposition 6.7. □

6.3 Comparison theorem

Conversely, every cover of S^{an} with finite fibers arises as the analytification of a finite étale cover. This is a much deeper result, which was already studied by Riemann in the case of smooth projective curves and their associated compact Riemann surfaces.

Theorem 6.9. *The functor*

$$\begin{aligned} \mathbf{Fin\acute{E}t}/S &\longrightarrow \mathbf{FinCov}/S^{\mathrm{an}}, \\ (X \to S) &\longmapsto (X^{\mathrm{an}} \to S^{\mathrm{an}}), \\ \psi &\longmapsto \psi^{\mathrm{an}} \end{aligned}$$

is an equivalence of categories.

Proof. See Grothendieck [1, Exposé XII, Théorème 5.1]. □

Using this theorem, we are now going to compare the fundamental group of S with the topological fundamental group of S^{an}. In order to use the topological theory, we need the following result.

Theorem 6.10. *The topological space S^{an} is locally contractible.*

Proof. As a special case of Hironaka [3, Theorem 1], S^{an} locally admits the structure of a simplicial complex, so it is locally contractible. □

Theorem 6.11. *There is an isomorphism of topological groups*

$$\widehat{\pi}_1(S^{\mathrm{an}}, \overline{s}) \xrightarrow{\sim} \pi_1(S, \overline{s}).$$

Proof. We have the following diagram:

$$\begin{array}{ccc} \mathbf{Fin\acute{E}t}/S & \xrightarrow[\simeq]{A} & \mathbf{FinCov}/S^{\mathrm{an}} \\ \downarrow{\scriptstyle F_{\overline{s}}} & & {\scriptstyle \simeq}\downarrow{\scriptstyle T} \\ \mathbf{Set} & \xleftarrow{U} & \mathbf{FinCont}\text{-}\widehat{\pi}_1(S^{\mathrm{an}}, \overline{s})\text{-}\mathbf{Set}, \end{array}$$

where $F_{\overline{s}}$ is the fiber functor, A is the analytification functor, T is the equivalence of categories from Corollary 6.2, and U is the forgetful functor. If $\varphi\colon X \to S$ is a finite étale cover, then $(\varphi^{\mathrm{an}})^{-1}(\overline{s})$ coincides with $F_{\overline{s}}(X)$. Hence $U \circ T \circ A = F_{\overline{s}}$, i.e. the above diagram commutes. Since $T \circ A$ is an equivalence of categories, it induces an equivalence $- \circ (T \circ A)$ between the category of set-valued functors on $\mathbf{FinCont}\text{-}\widehat{\pi}_1(S^{\mathrm{an}}, \overline{s})\text{-}\mathbf{Set}$ and the category of set-valued functors on $\mathbf{Fin\acute{E}t}/S$. Explicitly, $- \circ (T \circ A)$ is given by

$$\begin{aligned} F &\longmapsto F \circ (T \circ A), \\ (f\colon F \to G) &\longmapsto f \circ \mathrm{id}_{T \circ A}, \end{aligned}$$

where $f \circ \mathrm{id}_{T \circ A}$ is the horizontal composite of $\mathrm{id}_{T \circ A}$ and f. Thus we have an isomorphism of topological groups

$$\mathrm{Aut}(U) \xrightarrow{\sim} \mathrm{Aut}(F_{\overline{s}}), \quad f \mapsto f \circ \mathrm{id}_{T \circ A}.$$

Composing this with the isomorphism of topological groups

$$\widehat{\pi}_1(S^{\mathrm{an}}, \overline{s}) \xrightarrow{\sim} \mathrm{Aut}(U)$$

from Proposition 3.13 yields the claim. □

Before departing from the reader, we give some applications of the results in this chapter.

Example 6.12. Let S be the affine line over $\mathbb{C}$; then S^{an} is homeomorphic to $\mathbb{C}$. Because every cover of a simply connected topological space is trivial, Theorem 6.9 implies that every finite étale morphism $X \to S$ is isomorphic to one of the form $\coprod_{i=1}^{n} S \to S$ for some nonnegative integer n. In particular, $\pi_1(S, \overline{s})$ is trivial.

Example 6.13. Continuing the story of Example 1.18, let S be the multiplicative group over $\mathbb{C}$. Its analytification S^{an} is homeomorphic to the subspace $\mathbb{C} \setminus \{0\}$ of $\mathbb{C}$, so $\pi_1(S^{\mathrm{an}}, \overline{s}) \cong \mathbb{Z}$. By Theorems 6.9 and 6.1, we have $\mathbf{Fin\acute{E}t}/S \simeq \mathbf{Fin}\text{-}\pi_1(S^{\mathrm{an}}, \overline{s})\text{-}\mathbf{Set}$. Up to isomorphism, every finite transitive $\mathbb{Z}$-set is of the form $\mathbb{Z}/(n)$ for some nonzero integer n. It follows that every connected finite étale cover of S is isomorphic to φ_n for some nonzero integer n. An arbitrary finite étale morphism $X \to S$ is a finite coproduct of such covers.

Example 6.14. Let $S = \mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$ be the projective line minus three points. Its analytification S^{an} is homeomorphic to $\mathbb{C} \setminus \{0, 1\}$, so $\pi_1(S^{\mathrm{an}}, \overline{s})$ is a free group on two generators. Equip $\pi_1(S^{\mathrm{an}}, \overline{s})$ with the discrete topology. By Proposition 3.7, morphisms from $\widehat{\pi}_1(S^{\mathrm{an}}, \overline{s})$ to a profinite group H are in natural bijection with morphisms $\pi_1(S^{\mathrm{an}}, \overline{s}) \to H$. Such morphisms in turn correspond to elements of $H \times H$. Hence $\widehat{\pi}_1(S^{\mathrm{an}}, \overline{s})$ is a free profinite group on two generators.

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