Lecture 9

December 16, 2004
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(§16 was also presented on that day, but moved to its proper place in the text.)

§22 Finite Witt group schemes

From now on we abbreviate $W := W_k$, restoring the index $_k$ only when the dependence on the field k is discussed. Also, we will no longer underline points in W or in quotients thereof.

For any integer $n \geq 1$ we let $W_n \cong W/V^nW$ denote the additive group scheme of Witt vectors of length n over k. Truncation induces natural epimorphisms $r\colon W_{n+1} \twoheadrightarrow W_n$, and Verschiebung induces natural monomorphisms $v\colon W_n \hookrightarrow W_{n+1}$, such that $rv = vr = V$. For any n, $n' \geq 1$ they induce a short exact sequence

$$0 \longrightarrow W_{n'} \xrightarrow{v^n} W_{n+n'} \xrightarrow{r^{n'}} W_n \longrightarrow 0.$$

(The exactness can be deduced from the fact that $r^{n'}$ possesses the scheme theoretic splitting $x \mapsto (x, 0, \ldots, 0)$, although we have not proved in this course that the category of all affine commutative group schemes is abelian.) Together with the natural isomorphism $W_1 \cong \mathbb{G}_a$, these exact sequences describe W_n as a successive extension of n copies of $\mathbb{G}_a$.

For any integers n, $m \geq 1$ we let W_n^m denote the kernel of F^m on W_n. As above, truncation induces natural epimorphisms $r : W_{n+1}^m \twoheadrightarrow W_n^m$, and Verschiebung induces natural monomorphisms $v\colon W_n^m \hookrightarrow W_{n+1}^m$, such that $rv = vr = V$. Similarly, the inclusion induces natural monomorphisms $i\colon W_n^m \hookrightarrow W_n^{m+1}$, and Frobenius induces natural epimorphisms $f\colon W_n^{m+1} \twoheadrightarrow W_n^m$, such that $if = fi = F$. For any n, n', m, $m' \geq 1$ they induce short exact sequences

$$0 \longrightarrow W_{n'}^m \xrightarrow{v^n} W_{n+n'}^m \xrightarrow{r^{n'}} W_n^m \longrightarrow 0,$$
$$0 \longrightarrow W_n^m \xrightarrow{i^{m'}} W_n^{m+m'} \xrightarrow{f^m} W_n^{m'} \longrightarrow 0.$$

Together with the natural isomorphism $W_1^1 \cong \boldsymbol{\alpha}_p$, these exact sequences describe W_n^m as a successive extension of nm copies of $\boldsymbol{\alpha}_p$. For later use note the following fact:

Lemma 22.1. Let G be a finite commutative group scheme with $F_G^m = 0$ and $V_G^n = 0$. Then any homomorphism $\varphi : G \to W_{n'}^{m'}$ with $m' \geq m$ and $n' \geq n$ factors uniquely through the embedding $i^{m'-m}v^{n'-n}\colon W_n^m \hookrightarrow W_{n'}^{m'}$.

Proof. By the functoriality of Frobenius from Proposition 14.1, the assumption implies that $F^m_{W^{m'}_{n'}} \circ \varphi = \varphi^{(p^m)} \circ F^m_G = 0$. Thus φ factors through the kernel of F^m on $W^{m'}_{n'}$, which is the image of $i^{m'-m} : W^m_{n'} \hookrightarrow W^{m'}_{n'}$. The analogous argument with V^n_G in place of F^m_G shows the rest. $\square$

We will show that all commutative finite group schemes of local-local type can be constructed from the *Witt group schemes* W^m_n. The main step towards this is the following result on extensions:

Proposition 22.2. For any short exact sequence $0 \to W^m_n \to G \to \boldsymbol{\alpha}_p \to 0$ there exists a homomorphism φ making the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W^m_n & \longrightarrow & G & \longrightarrow & \boldsymbol{\alpha}_p & \longrightarrow & 0 \\ & & {\scriptstyle iv}\big\downarrow & \swarrow{\scriptstyle \varphi} & & & & & \\ & & W^{m+1}_{n+1} & & & & & & \end{array}$$

Note. In more highbrow language this means that the homomorphism induced by iv on the Yoneda Ext groups $\mathrm{Ext}^1(\boldsymbol{\alpha}_p, W^m_n) \to \mathrm{Ext}^1(\boldsymbol{\alpha}_p, W^{m+1}_{n+1})$ is zero. I prefer to stay as down to earth as possible in this course.

Lemma 22.3. Proposition 22.2 holds in the case $n = m = 1$.

Proof. As a preparation let U denote the kernel of the epimorphism $rf : W^2_2 \twoheadrightarrow W^1_1 = \boldsymbol{\alpha}_p$. Then r and f induce epimorphisms

$$\begin{array}{rcl} r' : U & \twoheadrightarrow & \ker(f\colon W^2_1 \twoheadrightarrow W^1_1) \cong W^1_1 = \boldsymbol{\alpha}_p, \\ f' : U & \twoheadrightarrow & \ker(r\colon W^1_2 \twoheadrightarrow W^1_1) \cong W^1_1 = \boldsymbol{\alpha}_p, \end{array}$$

which together yield a short exact sequence

$$0 \longrightarrow \boldsymbol{\alpha}_p = W^1_1 \xrightarrow{iv} U \xrightarrow{(r',f')} \boldsymbol{\alpha}_p^{\oplus 2} \longrightarrow 0.$$

Since $F = V = 0$ on $\boldsymbol{\alpha}_p$, one easily shows that F_U and V_U are induced from

$$k^{\oplus 2} \;\cong\; \mathrm{Hom}(\boldsymbol{\alpha}_p^{\oplus 2}, \boldsymbol{\alpha}_p) \;\hookrightarrow\; \mathrm{Hom}(U, U).$$

In fact, going through the construction one finds that F_U and V_U correspond to the elements $(0,1)$ and $(1,0)$ of $k^{\oplus 2}$, respectively. Essentially the proof will show that U represents the universal extension of $\boldsymbol{\alpha}_p$ with $\boldsymbol{\alpha}_p$.

For any short exact sequence $0 \to \boldsymbol{\alpha}_p \to G \xrightarrow{\pi} \boldsymbol{\alpha}_p \to 0$ we define a group scheme G' such that the upper left square in the following commutative diagram with exact rows and columns is a pushout:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \boldsymbol{\alpha}_p & \longrightarrow & G & \longrightarrow & \boldsymbol{\alpha}_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & U & \longrightarrow & G' & \xrightarrow{\pi'} & \boldsymbol{\alpha}_p & \longrightarrow & 0 \\
 & & {\scriptstyle (r',f')}\downarrow & & {\scriptstyle \rho'}\downarrow & & & & \\
 & & \boldsymbol{\alpha}_p^{\oplus 2} & = & \boldsymbol{\alpha}_p^{\oplus 2} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

By looking at the induced short exact sequence

$$0 \longrightarrow \boldsymbol{\alpha}_p \longrightarrow G' \xrightarrow{(\pi',\rho')} \boldsymbol{\alpha}_p^{\oplus 3} \longrightarrow 0$$

one shows as above that $F_{G'}$ and $V_{G'}$ are induced from

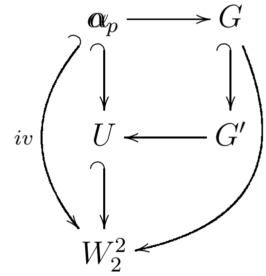
$$k^{\oplus 3} \;\cong\; \mathrm{Hom}(\boldsymbol{\alpha}_p^{\oplus 3}, \boldsymbol{\alpha}_p) \;\hookrightarrow\; \mathrm{Hom}(G', G').$$

In fact, comparison with the result for U shows that $F_{G'}$ and $V_{G'}$ correspond to triples $(x, 0, 1)$ and $(y, 1, 0)$, respectively, for certain elements x, $y \in k$. Define a subgroup scheme $G'' \subset G'$ as the pullback in the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \boldsymbol{\alpha}_p & \longrightarrow & G' & \longrightarrow & \boldsymbol{\alpha}_p^{\oplus 3} & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow {\scriptstyle (1,-y,-x)} & & \\
0 & \longrightarrow & \boldsymbol{\alpha}_p & \longrightarrow & G'' & \longrightarrow & \boldsymbol{\alpha}_p & \longrightarrow & 0
\end{array}$$

Then by construction one finds that $F_{G''} = 0$ and $V_{G''} = 0$. (In fact, G'' is just the right Baer linear combination of the extension G with the two basic extensions W_2^1 and W_1^2 which enjoys this property.) Thus Proposition 16.2 implies that $G'' \cong \boldsymbol{\alpha}_p^{\oplus 2}$ is split. This splitting yields an embedding $\iota\colon \boldsymbol{\alpha}_p \hookrightarrow G'$ satisfying $\pi'\iota = \mathrm{id}$, which in turn splits the extension $0 \to U \to G' \to \boldsymbol{\alpha}_p \to 0$.

Finally, the resulting homomorphism $G' \to U$ yields a composite arrow making the following diagram commute:

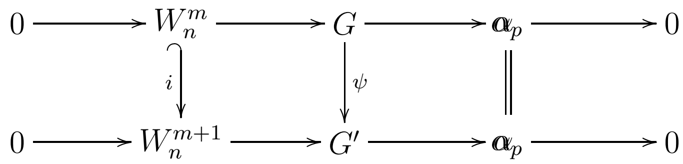


as asserted by Proposition 22.2. $\square$

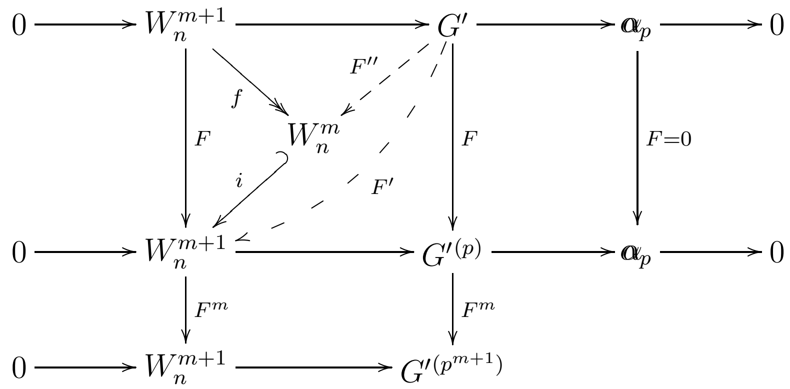
Lemma 22.4. (a) Fix $n \geq 1$. If Proposition 22.2 holds for this n and $m = 1$, then it holds for this n and all $m \geq 1$.

(b) Fix $m \geq 1$. If Proposition 22.2 holds for this m and $n = 1$, then it holds for this m and all $n \geq 1$.

Proof. For any short exact sequence $0 \to W_n^m \to G \to \boldsymbol{\alpha}_p \to 0$, define G' such that the left square in the following commutative diagram with exact rows is a pushout:



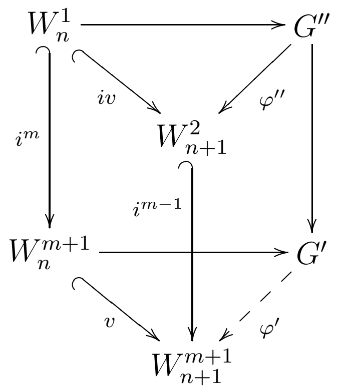
As $F = 0$ on $\boldsymbol{\alpha}_p$, and $F^m = 0$ on W_n^m, one easily shows that $F^{m+1} = 0$ on G. Thus F^{m+1} vanishes on $W_n^{m+1} \oplus G$, and since G' can be constructed as a quotient thereof, also on G'. Consider the following commutative diagram with exact rows, where the dashed arrows are not yet defined:



The dashed arrow F' is obtained from the fact that the upper right square commutes and that $F = 0$ on $\boldsymbol{\alpha}_p$. Looking at the lower left part of the diagram, the fact that $F^m \circ F = F^{m+1} = 0$ on G' implies that F' factors through the kernel of F^m on W_n^{m+1}. But this kernel is just the image of W_n^m under i, which yields the dashed arrow F'' making everything commute. Since the oblique arrow f is an epimorphism, the same holds a fortiori for F''. Setting $G'' := \ker F''$ we obtain a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & W_n^1 & \longrightarrow & G'' & \longrightarrow & \boldsymbol{\alpha}_p & \longrightarrow & 0\\
 & & {\scriptstyle i^m}\downarrow & (*) & \downarrow & & \| & & \\
0 & \longrightarrow & W_n^{m+1} & \longrightarrow & G' & \longrightarrow & \boldsymbol{\alpha}_p & \longrightarrow & 0\\
 & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F''} & & & & \\
 & & W_n^m & = & W_n^m & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Here by diagram chasing we find that the square marked $(*)$ is a pushout. By assumption we may apply Proposition 22.2 to G'', obtaining a homomorphism φ'' making the upper triangle of the following Toblerone diagram commute:

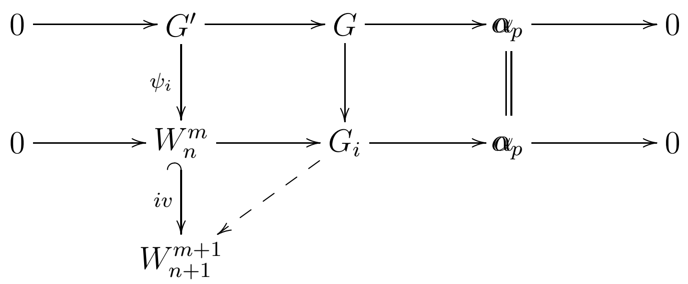


Since $(*)$ is a pushout, this commutative diagram can be completed by the dashed homomorphism φ' at the lower right. Altogether, the composite homomorphism $\varphi := \varphi'\psi : G \to G' \to W_{n+1}^{m+1}$ has the desired properties, proving (a). The proof of (b) is entirely analogous, with V in place of F. $\square$

Proof of Proposition 22.2. By Lemma 22.3 the proposition holds in the case $n = m = 1$. By Lemma 22.4 (a) the proposition follows whenever $n = 1$, and from this it follows in general by Lemma 22.4 (b). $\square$

Proposition 22.5. Every commutative finite group scheme of local-local type can be embedded into $(W_n^m)^{\oplus r}$ for some n, m, and r.

Proof. To prove this by induction on $|G|$, we may consider a short exact sequence $0 \to G' \to G \to \boldsymbol{\alpha}_p \to 0$ and assume that there exists an embedding $\psi = (\psi_1, \ldots, \psi_r) : G' \hookrightarrow (W_n^m)^{\oplus r}$. For $1 \leq i \leq r$ define G_i such that the upper left square in the following commutative diagram with exact rows is a pushout:



The dashed arrows, which exist by Proposition 22.2, determine an extension of the composite embedding $iv\psi : G' \hookrightarrow (W_{n+1}^{m+1})^{\oplus r}$ to a homomorphism $G \to (W_{n+1}^{m+1})^{\oplus r}$. The direct sum of this with the composite homomorphism $G \twoheadrightarrow \boldsymbol{\alpha}_p = W_1^1 \hookrightarrow W_{n+1}^{m+1}$ is an embedding $G \hookrightarrow (W_{n+1}^{m+1})^{\oplus r+1}$. $\square$

Proposition 22.6. Every commutative finite group scheme G with $F_G^m = 0$ and $V_G^n = 0$ possesses a copresentation (i.e., an exact sequence) for some r, s

$$0 \longrightarrow G \longrightarrow (W_n^m)^{\oplus r} \longrightarrow (W_n^m)^{\oplus s}.$$

Proof. By Proposition 22.5 there exists an embedding $G \hookrightarrow (W_{n'}^{m'})^{\oplus r}$ for some n', m', and r. After composing it in each factor with the embedding $iv\colon W_{n'}^{m'} \hookrightarrow W_{n'+1}^{m'+1}$, if necessary, we may assume that $n' \geq n$ and $m' \geq m$. Then Lemma 22.1 implies that the embedding factors through a homomorphism $G \to (W_n^m)^{\oplus r}$, which is again an embedding. Let H denote its cokernel. Since $F^m = 0$ and $V^n = 0$ on W_n^m, the same is true on $(W_n^m)^{\oplus r}$ and hence on H. Repeating the first part of the proof with H in place of G, we therefore find an embedding $H \hookrightarrow (W_n^m)^{\oplus s}$ for some s. The proposition follows. $\square$