Hodge Structures over Function Fields

by

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Abstract

We develop a general theory of mixed Hodge structures over local or global function fields which in many ways resembles the formalism of classical Hodge structures. Our objects consist of a finite dimensional vector space together with a weight filtration, but instead of a Hodge filtration we require finer information. In order to obtain a reasonable category we impose a semistability condition in the spirit of invariant theory and prove that the resulting category is tannakian. This allows us to define and analyze Hodge groups and determine them in some cases.

The analogies with classical mixed Hodge structures range from the role of semistability to the fact that both objects arise from the analytic behavior of motives. The precise relation of our objects with the analytic uniformization of Anderson's t-motives will be the subject of a separate paper. For Hodge structures arising from Drinfeld modules we can combine the present results with a previous one on Galois representations, obtaining a precise analogue of the Mumford-Tate conjecture.

Contents

0 Introduction

Among the many interesting invariants of an algebraic variety in characteristic zero is the rational mixed Hodge structure on its singular cohomology. More precisely, one must embed the ground field into the field of complex numbers; the Hodge structure then depends only on the resulting complex analytic space. In other words, the Hodge structure reflects certain aspects of the analytic behavior of the algebraic variety over the complex numbers. There are several interesting relations, proved or unproved, between the Hodge structure and other invariants and properties of the algebraic variety, such as comparison isomorphisms, the Hodge conjecture, the Mumford-Tate conjecture, etc. This interplay between analytic, algebraic, and arithmetic phenomena was one of the origins of Grothendieck's concept of motives.

Since Drinfeld [6] and Anderson [1] one disposes of a theory of motives over function fields in positive characteristic, which in many ways resembles the formalism in characteristic zero. It encompasses among others a theory of analytic uniformization, but so far it had not been clarified which analogues of Hodge structures might arise in this way. The aim of this article is to develop a comprehensive theory of what I call *Hodge structures over function fields* which is intended to fill this gap. We shall concentrate on the purely abstract properties of these objects; their relation with analytic uniformization will be the subject of a separate paper [26]. An overview of the main results was given in [25].

Origins: In order to motivate the basic definitions we shall nevertheless examine roughly how these objects arise. Let K be a global function field in one variable over a finite field $\mathbb{F}_q$ with q elements, and let $\hat{K}$ denote the completion of K at a fixed place ∞. Choosing a local parameter, we may identify $\hat{K}$ with the field of Laurent series in one variable $k(\!(z)\!)$, where k is a finite extension of $\mathbb{F}_q$. Let $A \subset K$ be the ring of functions which are regular outside ∞, and note that this is a discrete subring of $\hat{K}$. The standard example is $A = \mathbb{F}_q[t]$ and $\hat{K} = \mathbb{F}_q(\!(t^{-1})\!)$. In the following any finitely generated discrete A-submodule of a topological $\hat{K}$-vector space will be called an *A-lattice.*

Now let $\mathbb{C}_q$ denote the completion of the algebraic closure of $\hat{K}$. This field is the basis for non-archimedean analysis in equal characteristic. It has infinite degree over $\hat{K}$, so it contains A-lattices Λ of arbitrarily large rank r. Following Drinfeld [6] §3 one may formally algebraize the quotient $\mathbb{C}_q/\Lambda$, obtaining what is today called a *Drinfeld module* of rank r over $\mathbb{C}_q$. To understand the functorial properties of this description it is best to look at the more general case of Anderson's t-motives.

Anderson made the fundamental and rather subtle observation that the two distinct roles of the elements of A, on one side as elements of the coefficient ring A, on the other as elements of the base field $\mathbb{C}_q$, ought to be separated. So let us change notation and suppose that $\mathbb{C}_q$ is any fixed algebraically closed complete normed field given together with a continuous field homomorphism $\iota : \hat{K} \hookrightarrow \mathbb{C}_q$. Abbreviating $\zeta := \iota(z)$, it is easy to see that ι lifts to a unique homomorphism $\hat{K} \hookrightarrow \mathbb{C}_q[\![z-\zeta]\!]$ which maps z to itself (cf. Proposition 3.1). Choose a quadratic matrix Z of arbitrary size $d \times d$ with coefficients in $\mathbb{C}_q$, whose only eigenvalue is ζ, i.e. with

$$(0.1) \qquad (Z-\zeta)^n = 0 \qquad \text{for all } n \gg 0.$$

Then we can define an action of $\hat{K}$ on the vector space $\mathbb{C}_q^d$ by letting the residue field k act by scalars and z act through Z. Consider an A-lattice $\Lambda \subset \mathbb{C}_q^d$ of arbitrary rank with respect to this twisted action. In the case $d > 1$ there is no direct way to algebraize the quotient $\mathbb{C}_q^d/\Lambda$; instead we simply assume that Λ comes from a *uniformizable t-motive* over $\mathbb{C}_q$ in the sense of Anderson.

One vital feature of t-motives is that one can form *tensor products* and describe the effect on the associated lattices. To explain this effect note that by Condition 0.1 there is a natural map on the right hand side of the sequence

$$0 \longrightarrow \mathfrak{q} \longrightarrow \Lambda \otimes_A \mathbb{C}_q[\![z-\zeta]\!] \longrightarrow \mathbb{C}_q^d \longrightarrow 0\,.$$

This map is surjective (see [1]), so the main information lies in the kernel $\mathfrak{q}$. By construction this is a $\mathbb{C}_q[\![z-\zeta]\!]$*-lattice* in $\Lambda \otimes_A \mathbb{C}_q(\!(z-\zeta)\!)$, that is, a finitely generated $\mathbb{C}_q[\![z-\zeta]\!]$-submodule containing a basis. Anderson proved that the tensor product of any two uniformizable t-motives is again uniformizable and that the associated lattices Λ and $\mathfrak{q}$ are obtained by tensoring the original ones over A, respectively over $\mathbb{C}_q[\![z-\zeta]\!]$.

Another aspect of t-motives is that they may be *pure* of some weight $\mu \in \mathbb{Q}$. The tensor product of two pure motives of weights μ_1, μ_2 is pure of weight $\mu_1 + \mu_2$. By analogy with the characteristic zero case we generalize this a little and consider *mixed* motives, i.e. t-motives with an increasing weight filtration, indexed by rational numbers, such that each graded piece of weight μ is a pure t-motive of weight μ. If the t-motive is mixed and uniformizable, the lattice Λ inherits a weight filtration, denoted W.

To summarize, any uniformizable mixed t-motive determines a triple $(\Lambda, W, \mathfrak{q})$, whose formation is functorial and compatible with tensor product. It turns out that this data contains enough information to serve as an analogue of classical Hodge structures. In the following we prefer a K-linear theory, so we replace Λ by the vector space $H := \Lambda \otimes_A K$, which depends only on the t-motive up to isogeny.

Main concepts: The fundamental definition reads:

Definition 0.2 *(compare Definition 9.1) A* mixed K-∞-pre-Hodge structure *is a triple* $\underline{H} = (H, W, \mathfrak{q}_H)$ *where*

(a) H *is a finite dimensional* K*-vector space,*

(b) $W = (W_\mu H)_{\mu \in \mathbb{Q}}$ *is an increasing filtration by* K*-subspaces of* H*, called the* weight filtration*, and*

(c) $\mathfrak{q}_H$ *is a* $\mathbb{C}_q[\![z-\zeta]\!]$*-lattice in* $H \otimes_K \mathbb{C}_q(\!(z-\zeta)\!)$.

Homomorphisms of such objects are homomorphisms of the underlying K-vector spaces that are compatible with the rest of the data. The resulting category is K-linear but not abelian; this forces us to restrict attention to a suitable subcategory. Recall that, when $\underline{H}$ comes from a uniformizable mixed t-motive, we have not yet used the fact that $\Lambda \subset \mathbb{C}_q^d$ is discrete. It is easy to see that this discreteness is equivalent to the condition $(H \otimes_K \hat{K}) \cap \mathfrak{q}_H = \{0\}$. Speaking heuristically we can say that the lattice $\mathfrak{q}_H$ should be sufficiently far away from any non-zero $\hat{K}$-subspace of $\hat{H} := H \otimes_K \hat{K}$. We shall strengthen this condition as follows.

For every $\hat{K}$-subspace $\hat{H}' \subset \hat{H}$ consider the $\mathbb{C}_q[\![z-\zeta]\!]$-lattices

$$\begin{aligned} \mathfrak{p}_{\hat{H}'} &:= \hat{H}' \otimes_{\hat{K}} \mathbb{C}_q[\![z-\zeta]\!] , \\ \mathfrak{q}_{\hat{H}'} &:= \mathfrak{q}_H \cap \big(\hat{H}' \otimes_{\hat{K}} \mathbb{C}_q(\!(z-\zeta)\!)\big) , \end{aligned}$$

and define (compare 4.1):

$$\deg_{\mathfrak{q}}(\underline{\hat{H}}') := \dim_{\mathbb{C}_q}\Big(\frac{\mathfrak{q}_{\hat{H}'}}{\mathfrak{p}_{\hat{H}'} \cap \mathfrak{q}_{\hat{H}'}}\Big) - \dim_{\mathbb{C}_q}\Big(\frac{\mathfrak{p}_{\hat{H}'}}{\mathfrak{p}_{\hat{H}'} \cap \mathfrak{q}_{\hat{H}'}}\Big) .$$

This number measures the size of $\mathfrak{q}_{\hat{H}'}$. On the other hand set

$$\deg^W(\underline{\hat{H}}') := \sum_{\mu \in \mathbb{Q}} \mu \cdot \dim_{\hat{K}} \mathrm{Gr}^W_\mu(\hat{H}') .$$

Definition 0.3 *(compare Definitions 4.5 and 9.2) A mixed* K*-*∞*-pre-Hodge structure* $\underline{H}$ *is called* locally semistable *if and only if for every* $\hat{K}$*-subspace* $\hat{H}' \subset \hat{H}$ *we have*

$$\deg_{\mathfrak{q}}(\underline{\hat{H}}') \le \deg^W(\underline{\hat{H}}') ,$$

with equality whenever $\hat{H}' = W_\mu \hat{H}$ *for some* $\mu \in \mathbb{Q}$. *A locally semistable mixed* K-∞-*pre-Hodge structure is called a* mixed K-∞-Hodge structure.

Note that in the pure case this condition is rather similar to the usual semistability condition for vector bundles. The category of all mixed K-∞-Hodge structures is denoted $\mathscr{H}\!odge_{K,\infty}$. The first main result of this paper is:

Theorem 0.4 *(compare 9.3) The category* $\mathscr{H}\!odge_{K,\infty}$ *together with the forgetful functor* $\underline{H} \mapsto H$ *is a neutral tannakian category over* K.

In [26] we shall prove that the above construction yields a fully faithful tensor functor from the category of uniformizable mixed t-motives up to isogeny over $\mathbb{C}_q$ to the category $\mathscr{H}\!odge_{K,\infty}$. This will amount to an analogue of the Hodge conjecture.

Hodge groups: The assertion of Theorem 0.4 matches a well-known result for classical mixed Hodge structures and allows us to define Hodge groups. For any object $\underline{H}$ of $\mathscr{H}\!odge_{K,\infty}$ let $\langle\!\langle \underline{H} \rangle\!\rangle$ denote the strictly full tannakian subcategory of $\mathscr{H}\!odge_{K,\infty}$ which is generated by $\underline{\hat{H}}$. Let $\omega_{\underline{H}}$ denote the forgetful functor $\langle\!\langle \underline{H} \rangle\!\rangle \to \mathscr{V}\!ec_K$ which to any object associates its underlying K-vector space. The *Hodge group* of $\underline{H}$ is then defined by general tannakian theory as $G_{\underline{H}} := \underline{\mathrm{Aut}}^{\otimes}(\omega_{\underline{H}})$. It can be interpreted as an algebraic subgroup of the general linear group $\mathrm{Aut}_K(H)$.

Various properties of $\underline{H}$ correspond to properties of $G_{\underline{H}}$, and a part of this paper is devoted to studying this interrelation from different angles. This material is developed with an eye towards the problem of determining Hodge groups explicitly. For instance, for every object $\underline{H}$ in a certain subcategory $\mathscr{H}\!odge^{\mathrm{sha}}_{K,\infty} \subset \mathscr{H}\!odge_{K,\infty}$ we prove that $G_{\underline{H},K^{\mathrm{sep}}}$ is generated by certain one-parameter subgroups characterized by Hodge numbers, called *Hodge cocharacters* (see Definition 9.7 and Theorem 9.11). Among other things we show that $\mathscr{H}\!odge^{\mathrm{sha}}_{K,\infty}$ is a strictly full tannakian subcategory and contains all those objects whose Hodge group is reductive (Proposition 9.8 and Theorem 9.10).

Hodge structures of Drinfeld module type: As an application we determine the Hodge group for the following kind of objects:

Definition 0.5 *(compare Definition 10.1) A mixed* K-∞-*Hodge structure* $\underline{H}$ *is called* of Drinfeld module type *if and only if it is pure and has* $\mathfrak{q}_H \subset \mathfrak{p}_H$ *of* $\mathbb{C}_q$-*codimension* 1.

In fact, all objects that arise from the analytic uniformization of Drinfeld modules are of this type. Suppose now that $\underline{H}$ has rank r, and choose a basis of H. The following result solves a problem raised in [23] Guess 0.5.

Theorem 0.6 *(compare Theorem 10.3) The Hodge group of an object* $\underline{H}$ *of Drinfeld module type of rank* r *is*

$$G_{\underline{H}} = \mathrm{Cent}_{\mathrm{GL}_{r,K}}(\mathrm{End}(\underline{H})) \ .$$

The full thrust of this statement is produced only in combination with a previous result on Galois representations. Suppose that $\underline{H}$ comes from a Drinfeld module φ over a subfield $F \subset \mathbb{C}_q$ which is finitely generated over $\mathbb{F}_q$, and let F^{sep} denote the separable closure of F in $\mathbb{C}_q$. Let K_λ denote the completion of K at an arbitrary place $\lambda \neq \infty$. Then the Galois action on the λ-adic Tate module of φ corresponds to a continuous representation

$$\rho_\lambda : \text{Gal}(F^{\text{sep}}/F) \longrightarrow \text{GL}_r(K_\lambda) .$$

Let us suppose for simplicity that all endomorphisms of φ are defined over F; we then have a canonical isomorphism $\text{End}_F(\varphi) \otimes_A K \cong \text{End}(\underline{H})$.

Theorem 0.7 *(Pink [23] Thm. 0.2) Under the above assumptions the image of ρ_λ is an open subgroup of* $\text{Cent}_{\text{GL}_r(K_\lambda)}(\text{End}_F(\varphi))$.

Combining Theorems 0.6 and 0.7 we obtain a precise analogue of the Mumford-Tate conjecture for abelian varieties:

Corollary 0.8 *Under the above assumptions the image of ρ_λ is an open compact subgroup of $G_{\underline{H}}(K_\lambda)$.*

The beauty of such a statement lies in the fact that it relates two groups which are constructed in completely different ways and thus reflect very different properties of the underlying motive, in this case arithmetic and analytic ones. The author expects that a generalization to arbitrary uniformizable mixed t-motives will be possible.

Outline: Now we briefly discuss the contents of the individual sections. The material is developed independently of any t-motives, which will not even be mentioned again outside this introduction.

In Section 1 we review a number of general constructions with filtered vector spaces which are rather elementary, except the concept of Kempf filtration which we quote without proof.

Section 2 discusses Hodge structures in characteristic zero. It is not needed in the rest of the article and was included mainly for motivation. We show that the usual characterization of rational mixed Hodge structures can be rephrased elegantly in terms of a semistability condition. Thus the concepts of rational mixed Hodge structures, of filtered modules according to Fontaine, and of our Hodge structures over function fields share the same basic formalism. Given the fact that each of these concepts arises from some variant of Hodge or de Rham theory, this is perhaps not so surprising, but it would be nice to have a deeper explanation for this phenomenon.

The remainder of the article deals with (pre-)Hodge structures over local and global function fields. We essentially keep the notations of the introduction, except that we abbreviate $\mathbb{C}_q$ by $\mathbb{C}$. The concept of mixed $\hat{K}$-pre-Hodge structure is obtained from Definition 0.2 by simply starting with a $\hat{K}$-vector space instead of one over K. The reason for studying these objects first is that most notions, especially that of semistability, occur already at the local level.

Section 3 gives the basic definition and elementary properties of mixed $\hat{K}$-pre-Hodge structures. Section 4 discusses semistability and mixed $\hat{K}$-Hodge structures, which are defined as in 0.3. Using standard arguments we show that the category $\mathscr{Hodge}_{\hat{K}}$ of mixed $\hat{K}$-Hodge structures is abelian. The main point in proving the local analogue of Theorem 0.4 (see Corollary 5.7) is to show that semistability is invariant under tensor product. This is done in Section 5. The proof is modeled on similar statements for vector bundles or filtered modules. An indispensable preliminary step for this as well as for later arguments is the *Frobenius functoriality*, i.e. the statement that semistability is invariant under Frobenius pullback (Proposition 5.5).

The next two sections are devoted to the Hodge group. Section 6 contains an assortment of general properties, including a discussion of *polygons*. The numerical invariants of the weight filtration are encoded in the weight polygon, which gives rise to certain one-parameter subgroups of the Hodge group, called *weight (quasi-)cocharacters* (see Proposition 6.5). We also have a Hodge polygon, which is determined by the elementary divisors relating the lattices $\mathfrak{p}_{\hat{H}}$ and $\mathfrak{q}_{\hat{H}}$ (cf. Definition 0.3 above). But in general this polygon is not additive in short exact sequences, so the information carried by it cannot be interpreted directly in terms of a cocharacter.

Section 7 deals with this problem by describing the largest strictly full tannakian subcategory $\mathscr{Hodge}_{\hat{K}}^{\text{sha}} \subset \mathscr{Hodge}_{\hat{K}}$ on which the Hodge polygon is additive. We characterize its objects precisely (Theorem 7.8) and show among others that every object with reductive Hodge group is in it (Proposition 7.3). On this subcategory we can then define Hodge cocharacters whose numerics are based on Hodge polygons, just as for the weight cocharacter, and prove that the Hodge group is generated by all conjugates of Hodge cocharacters (Theorem 7.11).

Let us mention on the side that the problem is related to the following phenomenon. The lattice $\mathfrak{q}_{\hat{H}}$ determines a Hodge filtration of $\hat{H}_{\mathbb{C}}$ (see Definition 3.5), in happy analogy with classical Hodge structures. But $\mathfrak{q}_{\hat{H}}$ contains strictly more information. In fact, in positive characteristic it is not even possible to build a reasonable theory based on weight and Hodge filtrations alone (see Example 5.16).

Section 8 treats questions of describing the totality of mixed $\hat{K}$-Hodge structures of a given type. It presents a few qualitative results on period spaces, extensions, obstructions, and infinitesimal deformations.

With Section 9 we return to the global case, reviewing the main concepts and results of the earlier sections in that setting. Everything follows either by direct application of the respective local result or by the analogous arguments.

Finally, in Section 10 we work out the Hodge group for all Hodge structures of Drinfeld module type, proving Theorem 0.6.

Possible generalization: A natural topic which is not covered in this article is that of functoriality with respect to arbitrary finite extensions of local or global function fields. The reason is that the behavior of the place ∞ under finite extensions would require a generalization of the concept of Hodge structures right from the start. Instead of one completion $\hat{K}$ of K, and one embedding $\hat{K} \hookrightarrow \mathbb{C}$, one would have to consider an arbitrary finite number of them and accordingly finitely many lattices $\mathfrak{q}_H$, et cetera. Although this would lead to little more than notational complications and might be useful for some purpose, we have avoided it in the interest of clarity.

1 Filtered vector spaces

We recall a number of general concepts related to filtrations and fix some notations. We also quote some results from Totaro [28] on filtrations and subspaces of tensor products, which will be used in Section 5.

Filtrations: Consider a finite dimensional vector space V over a field K. A *decreasing ($\mathbb{Q}$-)filtration of* V is a collection of subspaces $F = (F^iV)_{i\in\mathbb{Q}}$ such that $F^iV \subset F^jV$ whenever $i \geq j$. Any such filtration determines a $\mathbb{Q}$-graded vector space

$$\mathrm{Gr}_F V \;:=\; \bigoplus_{i\in\mathbb{Q}} \mathrm{Gr}_F^i V \;:=\; \bigoplus_{i\in\mathbb{Q}} \big(F^iV / \textstyle\bigcup_{j>i} F^jV\big)$$

whose total dimension satisfies

$$\dim_K(\mathrm{Gr}_F V) \leq \dim_K V.$$

We shall always require that this inequality is an equality. This means several things: First, the filtration is *separated*, that is $F_iV = 0$ for all $i \gg 0$. Second, the filtration is *exhaustive*, that is $F_iV = V$ for all $i \ll 0$. There is also a semi-continuity condition at each $i \in \mathbb{Q}$ and a continuity condition on $\mathbb{R} \smallsetminus \mathbb{Q}$, which the reader is invited to write out for himself. If $\mathrm{Gr}_F^i V = 0$ for all $i \notin \mathbb{Z}$, the whole information is contained in $(F^iV)_{i\in\mathbb{Z}}$. Thus our discussion includes $\mathbb{Z}$-filtrations as a special case. The filtration is called *trivial* if and only if $\mathrm{Gr}_F^0 V \cong V$.

Similarly, an *increasing filtration of* V is a collection of subspaces $F = (F_iV)_{i\in\mathbb{Q}}$ such that $F_iV \subset F_jV$ whenever $i \leq j$. As before, the dimension of its associated graded vector space

$$\mathrm{Gr}^F V \;:=\; \bigoplus_{i\in\mathbb{Q}} \mathrm{Gr}_i^F V \;:=\; \bigoplus_{i\in\mathbb{Q}} \big(F_iV / \textstyle\bigcup_{j<i} F_jV\big)$$

is assumed to be equal to that of V. Throughout we shall use upper indices for decreasing filtrations and lower indices for increasing ones. Every assertion for one type of filtrations has an analogue for the other; mostly we state just one of them.

Induced filtrations: Consider a decreasing filtration F of V and a subspace $V' \subset V$. The *induced filtration* of V' is defined by $F^iV' := V' \cap F^iV$. Dually, the induced filtration of the factor space $V'' := V/V'$ is defined by $F^iV'' := (V' + F^iV)/V'$. Combining the two constructions yields a natural filtration of any subquotient of V. Actually, there are two ways to achieve this: Given subspaces $V_2 \subset V_1 \subset V$ one may first restrict the filtration to V_1 and then project it to V_1/V_2, or one may first project F to V/V_2 and then restrict it to the subspace V_1/V_2. Happily, the result is independent of the order of operations:

$$\frac{V_2 + (V_1 \cap F^i V)}{V_2} = \frac{V_1}{V_2} \cap \frac{V_2 + F^i V}{V_2} .$$

The induced filtration will be denoted $F|(V_1/V_2)$, or just F if confusion is unlikely. It is important to keep in mind that it may depend on the precise subspaces $V_2 \subset V_1$. Namely, if $V_2' \subset V_1' \subset V$ are other subspaces with $V_1' \subset V_1$ and $V_2' \subset V_2$, such that the map $V_1'/V_2' \longrightarrow V_1/V_2$ is an isomorphism, the induced filtrations of these subquotients do not necessarily correspond.

Let W be another finite dimensional K-vector space with a decreasing filtration F. The induced filtration on the tensor product is defined by

$$F^i(V \otimes_K W) := \sum_{j+k=i} F^j V \otimes_K F^k W.$$

This definition ensures

$$\mathrm{Gr}_F^i(V \otimes_K W) \cong \bigoplus_{j+k=i} \mathrm{Gr}_F^j V \otimes_K \mathrm{Gr}_F^k W.$$

For higher tensor products one uses iteration. For symmetric (resp. alternating) powers the induced filtration is obtained via the canonical surjection $V^{\otimes r} \twoheadrightarrow \mathrm{Sym}^r V$ (resp. $V^{\otimes r} \twoheadrightarrow \Lambda^r V$). Furthermore:

$$F^i \operatorname{Hom}_K(V, W) := \left\{\varphi \in \operatorname{Hom}_K(V, W) \;\middle|\; \forall j:\ \varphi(F^j V) \subset F^{i+j} W\right\} .$$

In particular, for the dual space we have

$$F^i V^\vee := \left\{\, w \in V^\vee \;\middle|\; \forall j < i:\ w|F^{-j} V = 0 \,\right\} .$$

Homomorphisms: A K-linear homomorphism $\varphi : V \to W$ is *compatible with* two given filtrations F if and only if $\varphi(F^i V) \subset F^i W$ for every i. The homomorphism is called *strictly compatible with* the filtrations, or *strict* for short, if and only if $\varphi(F^i V) = \varphi(V) \cap F^i W$ for every i. The filtered finite dimensional K-vector spaces together with all compatible morphisms form a K-linear category. It is important to note that this category is not abelian: Although any morphism possesses a kernel and a cokernel, the morphism $\mathrm{coim}(\varphi) \longrightarrow \mathrm{im}(\varphi)$ is an isomorphism if and only if φ is strict. This causes a number of difficulties. Even with strict morphisms one must be careful: a composite of strict morphisms need not be strict. When a more complicated category involving filtered objects turns out to be abelian, the crucial point is often to prove that all morphisms are strict.

Numerical invariants: The indexing of a filtration can be interpreted as endowing each non-trivial filtration step with a certain weight. By summing up with the appropriate multiplicities we obtain the *(total) degree of V with respect to F*:

$$\deg_F V := \sum_{i \in \mathbb{Q}} i \cdot \dim_K \mathrm{Gr}_F^i V . \tag{1.1}$$

When $V \neq 0$, we can also take its average:

$$\mu_F V := \frac{\deg_F V}{\dim_K V} . \tag{1.2}$$

Interchanging upper and lower indices gives the corresponding definitions for increasing filtrations.

Proposition 1.3 *Consider an isomorphism $\varphi : V \xrightarrow{\sim} W$ that is compatible with given filtrations on V and W, both denoted F. Then we have $\deg_F(V) \leq \deg_F(W)$ if the filtrations are decreasing, and $\deg^F(V) \geq \deg^F(W)$ if they are increasing. In either case we have equality if and only if φ is strict.*

This as well as the following properties vis-à-vis induced filtrations are easy to verify and left to the reader.

Proposition 1.4 *(a) $\deg_F V = \deg_F(\Lambda^r V)$ if $r = \dim_K V$.*

(b) $\deg_F V = \deg_F V' + \deg_F V''$ if $V' \subset V$ and $V'' := V/V'$.

(c) $\mu_F(V \otimes_K W) = \mu_F V + \mu_F W$ provided V, $W \neq 0$.

(d) $\mu_F \operatorname{Hom}_K(V, W) = \mu_F W - \mu_F V$ provided V, $W \neq 0$.

(e) $\deg_F V^\vee = -\deg_F V$.

Comparison of two filtrations: Suppose now that V is endowed with two arbitrary decreasing filtrations F and Φ. The following elementary and well-known fact is very useful:

Lemma 1.5 *(Bruhat) There exists a basis of V such that each filtration step of F or Φ is generated by a subset of this basis.*

Definition 1.6 *We say that F and Φ are* comparable *if and only if for every i and j we have $F^iV \subset \Phi^jV$ or $\Phi^jV \subset F^iV$.*

Linear combinations: Scaling with a rational factor $m > 0$ yields a new filtration $m\Phi$ by the formula $(m\Phi)^iV := \Phi^{i/m}V$. Clearly it satisfies $\deg_{m\Phi}(V) = m \cdot \deg_\Phi(V)$. There is also a sum:

Proposition 1.7 *There exists a unique decreasing filtration $F+\Phi$ of V such that for all $i \in \mathbb{Q}$*

$$(F + \Phi)^iV = \sum_{j+k=i} F^jV \cap \Phi^kV = \bigcap_{j+k=i} F^jV + \Phi^kV .$$

Furthermore we have $\deg_{F+\Phi}(V) = \deg_F(V) + \deg_\Phi(V)$.

Proof. Clearly the assertion is invariant under direct sums of bi-filtered vector spaces. Thus by Lemma 1.5 we are reduced to the one-dimensional case, where the assertion is seen by inspection. **q.e.d.**

The point of these constructions is the following fact.

Proposition 1.8 *The filtrations Φ and $F + m\Phi$ are comparable if $m \gg 0$.*

Proof. Choose $N \geq 0$ such that $F^N V = 0$ and $F^{-N}V = V$, and take $\epsilon > 0$ such that for every i at least two of $\Phi^{i-\epsilon}V \subset \Phi^i V \subset \Phi^{i+\epsilon}V$ are equal. Assume that $m \geq \frac{N}{\epsilon}$. If $\Phi^i V = \Phi^{i+\epsilon}V$, one easily shows that $\Phi^i V = (F + m\Phi)^{mi}V$. Otherwise one deduces $\Phi^i V = \Phi^{i-\epsilon}V = (F + m\Phi)^{m(i-\epsilon)}V$. Since $F + m\Phi$ is a filtration, this implies the desired assertion. **q.e.d.**

Two filtrations and a subspace: Suppose now that in addition to F and Φ we are given a subspace $V' \subset V$. Then Φ induces a filtration $\Phi|V'$ of V', whose associated graded pieces $\mathrm{Gr}^j_\Phi V' := \mathrm{Gr}^j_{\Phi|V'} V'$ are canonically isomorphic to subspaces of $\mathrm{Gr}^j_\Phi V$. Thus F induces a filtration of $\mathrm{Gr}^j_\Phi V'$ in two different ways, depending on whether one induces first to V' or to $\mathrm{Gr}^j_\Phi V$. In general the resulting filtrations are different. However:

Proposition 1.9 *If F and Φ are comparable, the filtrations $(F|V')|(\mathrm{Gr}^j_\Phi V')$ and $(F|\mathrm{Gr}^j_\Phi V)|(\mathrm{Gr}^j_\Phi V')$ coincide, and we have*

$$\deg_F(V') = \sum_{j \in \mathbb{Q}} \deg_{(F|\mathrm{Gr}^j_\Phi V)}(\mathrm{Gr}^j_\Phi V') \ .$$

Proof. By definition the i^{th} step of the filtration $(F|V')|(\mathrm{Gr}^j_\Phi V')$ is

$$\frac{(V' \cap F^i V \cap \Phi^j V) + (V' \cap \Phi^{j'} V)}{V' \cap \Phi^{j'} V} \ , \tag{1.10}$$

where $j' > j$ is sufficiently close. The corresponding step of $(F|\mathrm{Gr}^j_\Phi V)|(\mathrm{Gr}^j_\Phi V')$ is

$$\frac{(V' \cap \Phi^j V) + \Phi^{j'} V}{\Phi^{j'} V} \cap \frac{(F^i V \cap \Phi^j V) + \Phi^{j'} V}{\Phi^{j'} V} \tag{1.11}$$

Now by assumption one of the subspaces $F^i V$ and $\Phi^{j'} V$ is contained in the other. If $F^i V \subset \Phi^{j'} V$, both filtration steps are zero. If $\Phi^{j'} V \subset F^i V$, then 1.11 is equal to

$$\frac{(V' \cap F^i V \cap \Phi^j V) + \Phi^{j'} V}{\Phi^{j'} V} \cong \frac{V' \cap F^i V \cap \Phi^j V}{V' \cap \Phi^{j'} V}$$

which is equal to 1.10, as desired. This proves the first assertion. The second assertion follows from the first and the general formula

$$\deg_{F|V'}(V') = \sum_{j \in \mathbb{Q}} \deg_{(F|V')|(\mathrm{Gr}^j_\Phi V')}(\mathrm{Gr}^j_\Phi V')$$

which is a consequence of Proposition 1.4 (b). **q.e.d.**

Combining Propositions 1.8 and 1.9 we obtain:

Proposition 1.12 *For every $m \gg 0$ we have*

$$\deg_{F+m\Phi}(V') = \sum_{j \in \mathbb{Q}} \deg_{(F+m\Phi)|\mathrm{Gr}^j_\Phi V}(\mathrm{Gr}^j_\Phi V') \ .$$

Kempf filtration: Now consider two finite dimensional K-vector spaces V_1, V_2. By a *filtration of* (V_1, V_2) we shall mean a pair of decreasing filtrations of V_1 and V_2, both denoted by the same symbol, say F. The induced filtration of $V_1 \otimes_K V_2$ and of any subspace is then again denoted by F.

Definition 1.13 *A non-zero subspace $V' \subset V_1 \otimes_K V_2$ is called* $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$*-semistable if and only if for every filtration F of (V_1, V_2) we have $\mu_F(V') \leq \mu_F(V_1 \otimes_K V_2)$.*

To study a non-semistable subspace one looks for a filtration which maximizes the defect in this inequality. First one defines a kind of scalar product on the set of all filtrations.

Definition 1.14 *For any two filtrations F, F' of (V_1, V_2) we let*

$$\langle F, F' \rangle := \sum_{i=1}^{2} \sum_{j \in \mathbb{Q}} \sum_{j' \in \mathbb{Q}} j \cdot j' \cdot \dim_K(\mathrm{Gr}_F^j(\mathrm{Gr}_{F'}^{j'}(V_i)))$$

and

$$|F| := \sqrt{\langle F, F \rangle} = \Big(\sum_{i=1}^{2} \sum_{j \in \mathbb{Q}} j^2 \cdot \dim_K(\mathrm{Gr}_F^j(V_i)) \Big)^{1/2} .$$

Clearly the "norm" $|F|$ is positive unless F is trivial on both V_1 and V_2. Lemma 1.5 implies that the pairing $\langle F, F' \rangle$ is symmetric.

Proposition 1.15 *(Totaro [28] Prop. 1) Consider a non-zero subspace $V' \subset V_1 \otimes_K V_2$ which is not* $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$*-semistable. Then the function*

$$F \mapsto \frac{\mu_F(V') - \mu_F(V_1 \otimes_K V_2)}{|F|}$$

on the set of all non-trivial filtrations of (V_1, V_2) attains its maximum. The maximizing filtration Φ is called the Kempf filtration associated to V'. *It is unique up to scaling and satisfies $\mu_\Phi(V_i) = 0$.*

Proposition 1.16 *(Totaro [28] Prop. 2) Let V' and Φ be as in 1.15. Then for every filtration F of (V_1, V_2) we have*

(a) $\mu_F(V') - \mu_F(V_1 \otimes_K V_2) \leq \dfrac{\mu_\Phi(V') - \mu_\Phi(V_1 \otimes_K V_2)}{|\Phi|^2} \cdot \langle \Phi, F \rangle$, *and*

(b) $\langle \Phi, F \rangle = \displaystyle\sum_{i=1}^{2} \int_{\mathbb{R}} \big(\mu_F(\Phi^\ell V_i) - \mu_F(V_i) \big) \dim(\Phi^\ell V_i) d\ell$.

Universal Kempf filtration: For any field extension L/K and any K-vector space V we abbreviate $V_L := V \otimes_K L$. A filtration of V evidently induces one of V_L, which will be denoted by the same symbol. All the above constructions are compatible with base extension; in particular the numerical invariants are preserved. But neither the property of $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$-semistability nor the Kempf filtration are invariant under base extension, because there exist more filtrations over L than over K. We stabilize these concepts as follows.

Definition 1.17 *A non-zero subspace $V' \subset V_1 \otimes_K V_2$ is called* universally $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$-semistable *if and only if $V'_L \subset V_{1,L} \otimes_L V_{2,L}$ is $\mathrm{GL}(V_{1,L}) \times \mathrm{GL}(V_{2,L})$-semistable for every extension L/K.*

Consider a non-zero subspace $V' \subset V_1 \otimes_K V_2$ which is not universally $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$-semistable.

Definition 1.18 *A filtration of (V_1, V_2) is called* universal Kempf filtration associated to V' *if and only if for every extension L/K the induced filtration of $(V_{1,L}, V_{2,L})$ is a Kempf filtration associated to V'_L.*

Proposition 1.19 *When K is perfect, there exists a universal Kempf filtration associated to V'.*

Proof. First note that the set of all filtrations of (V_1, V_2) is a countable union of $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$-orbits and thus has a natural algebraic structure over K, which is locally of finite type. The invariants $|F|$ and $\mu_F(V_1 \otimes_K V_2)$ are locally constant, and $\mu_F(V')$ is upper semicontinuous in F for the Zariski topology. Thus the expression which is maximized in Proposition 1.15 is also upper semicontinuous in F.

Now let $\bar{K}$ denote an algebraic closure of K and Φ a Kempf filtration associated to $V'_{\bar{K}}$. By the preceding remarks Φ achieves the maximum among all non-trivial filtrations defined over all extension fields of K. By assumption $\bar{K}/K$ is Galois, and the unicity of the Kempf filtration implies that Φ is invariant under $\mathrm{Gal}(\bar{K}/K)$. Thus Φ descends to K, where it is the desired universal Kempf filtration. **q.e.d.**

For general K Proposition 1.19 still implies that a universal Kempf filtration exists over a finite purely inseparable extension of K. For a case where it is not defined over K see Example 5.16 below.

2 Hodge structures in characteristic zero

In this section we briefly discuss two kinds of mathematical objects which involve finite dimensional vector spaces with filtration data. They both arise from some variant of Hodge or de Rham theory, but we concentrate purely on their formal similarities. Our aim is to elucidate how the notion of semistability, which originates in geometric invariant theory, gives rise to an abelian and tannakian category. A similar formalism will occur in the rest of this article, but the material of the present section will not be needed itself and was included mainly for motivation. First we shall look at the concept of rational mixed Hodge structures from a slightly different angle. The second topic concerns the filtered modules according to Fontaine.

Rational mixed Hodge structures: We begin by recalling the standard definition (cf. Deligne [3] Déf. 2.3.1).

Definition 2.1 *A* rational mixed Hodge structure *is a triple (H, W, F) where* *(a) H is a finite dimensional $\mathbb{Q}$-vector space,*

(b) $W = (W_n H)_{n\in\mathbb{Z}}$ is an increasing filtration by $\mathbb{Q}$-subspaces of H, called the weight filtration, *and*

(c) $F = (F^p H_{\mathbb{C}})_{p\in\mathbb{Z}}$ is a decreasing filtration by $\mathbb{C}$-subspaces of $H_{\mathbb{C}}$, called the Hodge filtration,

such that the following condition is satisfied:

Condition 2.2 *For each $n \in \mathbb{Z}$ there is a decomposition of $\mathbb{C}$-vector spaces*

$$\mathrm{Gr}_n^W H_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q}$$

such that

(a) the filtration induced by F is given for all $p \in \mathbb{Z}$ by

$$F^p(\mathrm{Gr}_n^W H_{\mathbb{C}}) = \bigoplus_{\substack{p'+q'=n \\ p'\geq p}} H^{p',q'} ,$$

(b) $\overline{H^{p,q}} = H^{q,p}$ for all $p, q \in \mathbb{Z}$.

Local semistability: Let (H, W, F) be as in Definition 2.1 but not necessarily satisfying Condition 2.2. We shall show that this condition is equivalent to the following following *local semistability* condition over $\mathbb{R}$. It is related to the concept of semistability of a pair of filtrations (cf. Faltings [7].)

Condition 2.3 *For every $\mathbb{R}$-subspace $H'_{\mathbb{R}} \subset H_{\mathbb{R}}$ we have*

$$\deg_F(H'_{\mathbb{C}}) \leq \frac{1}{2} \cdot \deg^W(H'_{\mathbb{R}}),$$

with equality whenever $H'_{\mathbb{R}} = W_n H_{\mathbb{R}}$ for some $n \in \mathbb{Z}$.

Here the factor $\frac{1}{2}$ is related to the fact that the algebraic closure of $\mathbb{R}$ has degree 2 over $\mathbb{R}$. By renormalizing the weight filtration using half-integers this factor could be removed.

Proposition 2.4 *Conditions 2.2 and 2.3 are equivalent.*

Proof. We first prove the implication 2.2⇒2.3. Assume the former and consider a subspace $H'_{\mathbb{R}} \subset H_{\mathbb{R}}$, say of dimension r. It is well-known and easy to check that $\Lambda^r H$ with its induced filtrations again satisfies Condition 2.2. On the other hand by Proposition 1.4 (a) the inequality in Condition 2.3 is equivalent to the corresponding inequality for the subspace $\Lambda^r H'_{\mathbb{R}} \subset \Lambda^r H_{\mathbb{R}}$. Moreover, for any $n \in \mathbb{Z}$ we clearly have $\Lambda^r(W_n H_{\mathbb{R}}) = W_m \Lambda^r H_{\mathbb{R}}$ for some $m \in \mathbb{Z}$. Thus after replacing everything by its r^{th} alternating power we are reduced to the case $r = 1$.

In this case put $n := \deg^W(H'_{\mathbb{R}})$ and $p := \deg_F(H'_{\mathbb{C}})$. Then $\mathrm{Gr}_n^W H'_{\mathbb{C}}$ is a one-dimensional subspace of $\mathrm{Gr}_n^W H_{\mathbb{C}}$ which satisfies

$$\mathrm{Gr}_n^W H'_{\mathbb{C}} \subset F^p(\mathrm{Gr}_n^W H_{\mathbb{C}}) = \bigoplus_{\substack{p'+q'=n \\ p'\geq p}} H^{p',q'} .$$

As the subspace on the left hand side is defined over $\mathbb{R}$, we deduce

$$0 \neq \mathrm{Gr}_n^W H'_{\mathbb{C}} \subset F^p(\mathrm{Gr}_n^W H_{\mathbb{C}}) \cap \overline{F^p(\mathrm{Gr}_n^W H_{\mathbb{C}})} = \bigoplus_{\substack{p'+q'=n \\ p',q'\geq p}} H^{p',q'} .$$

Thus we must have $H^{p',q'} \neq 0$ for some pair of integers p', $q' \geq p$ with $p'+q' = n$. This implies $n \geq 2p$ and hence the desired inequality in Condition 2.3. Now suppose $H'_{\mathbb{R}} = W_n H_{\mathbb{R}}$. Since this has dimension 1, Condition 2.2 leaves only the possibility $\mathrm{Gr}_n^W H_{\mathbb{C}} = H^{p,p}$ with $n = 2p$. This proves the equality in Condition 2.3, as desired.

To show the converse we now assume Condition 2.3. Applying it to all subspaces $W_{n-1}H_{\mathbb{R}} \subset H'_{\mathbb{R}} \subset W_n H_{\mathbb{R}}$ and using the additivity of Proposition 1.4 (b) we easily deduce that Condition 2.3 also holds for $\mathrm{Gr}_n^W H$ with its induced filtrations. As Condition 2.2 depends only on these graded pieces, we may restrict ourselves to the case $H = \mathrm{Gr}_n^W H$. Using the Bruhat lemma 1.5 we choose a basis $\{h_1, \ldots, h_d\}$ of $H_{\mathbb{C}}$ such that each $F^p H_{\mathbb{C}}$ and each $\overline{F^q H_{\mathbb{C}}}$ is generated by a subset. Put $p_i := \max\{p \in \mathbb{Z} \mid h_i \in F^p H_{\mathbb{C}}\}$ and $q_i := \max\{q \in \mathbb{Z} \mid h_i \in \overline{F^q H_{\mathbb{C}}}\}$. Then Condition 2.2 is a direct consequence of the following lemma.

Lemma 2.5 *For every i we have $p_i + q_i = n$.*

Proof. First we prove $p_i + q_i \leq n$. For this observe that the subspace of $H_{\mathbb{C}}$ generated by h_i and its complex conjugate $\bar{h}_i$ descends to a subspace $H'_{\mathbb{R}} \subset H_{\mathbb{R}}$. If $\bar{h}_i$ and h_i are linearly dependent, we have $\dim_{\mathbb{R}}(H'_{\mathbb{R}}) = 1$ and therefore

$$p_i = q_i = \deg_F(H'_{\mathbb{C}}) \overset{2.3}{\leq} \frac{1}{2} \cdot \deg^W(H'_{\mathbb{R}}) = \frac{n}{2} ,$$

whence $p_i + q_i \leq n$, as desired. If $\bar{h}_i$ and h_i are linearly independent, we similarly deduce

$$p_i + q_i = \deg_F(H'_{\mathbb{C}}) \overset{2.3}{\leq} \frac{1}{2} \cdot \deg^W(H'_{\mathbb{R}}) = n ,$$

as desired. For the reverse inequality note that

$$\sum_{i=1}^{d} p_i = \deg_F(H_{\mathbb{C}}) = \deg_{\bar{F}}(H_{\mathbb{C}}) = \sum_{i=1}^{d} q_i ,$$

where $\bar{F}$ denotes the complex conjugate filtration. Together with Condition 2.3 this implies

$$\begin{aligned} \sum_{i=1}^{d} (p_i + q_i - n) &= \deg_F(H_{\mathbb{C}}) + \deg_{\bar{F}}(H_{\mathbb{C}}) - d \cdot n \\ &= 2 \cdot \deg_F(H_{\mathbb{C}}) - \deg^W H \\ &\overset{2.3}{=} 0 . \end{aligned}$$

Since all summands on the left hand side are ≤ 0, they must be zero, as desired. This proves Lemma 2.5 and thus finishes the proof of Proposition 2.4. **q.e.d.**

The category of rational mixed Hodge structures: A *morphism* $H_1 \to H_2$ of rational mixed Hodge structures is a $\mathbb{Q}$-linear map which is compatible with both filtrations. The fact that the Hodge and weight filtrations are sufficiently opposite implies the following crucial result (see Deligne [3] Th. 2.3.5).

Proposition 2.6 *Every morphism of rational mixed Hodge structures is strictly compatible with the Hodge and weight filtrations. The category of rational mixed Hodge structures is abelian.*

The Hodge and weight filtrations of a tensor product or Hom of two rational mixed Hodge structures are defined as in Section 1. It is easy to see that Condition 2.2 is invariant under these operations. It is much harder to prove this invariance directly in terms of Condition 2.3! In any case, we deduce:

Proposition 2.7 *The category of rational mixed Hodge structures together with the forgetful functor into the category of $\mathbb{Q}$-vector spaces is a neutral tannakian category.*

This result enables one to express information on a Hodge structure in terms of the algebraic group associated to it by tannakian theory, and vice versa.

Filtered modules and semistability: Consider a rational prime p and a finite extension K of $\mathbb{Q}_p$, which for simplicity we assume unramified. The Frobenius substitution of K over $\mathbb{Q}_p$ is denoted σ. We are interested in the following objects (cf. Fontaine [8] 1.2, [9] 1.1).

Definition 2.8 *A* filtered module over K *is a triple* $(M, \varphi, \mathrm{Fil})$ *where*

(a) M *is a finite dimensional* K*-vector space,*

(b) $\varphi : M \xrightarrow{\sim} M$ *is a* σ*-linear automorphism, i.e. an automorphism of additive groups satisfying* $\varphi(xm) = \sigma(x)\varphi(m)$ *for all* $x \in K$ *and* $m \in M$*, called* Frobenius*, and*

(c) $\mathrm{Fil} = (\mathrm{Fil}^i M)_{i\in\mathbb{Z}}$ *is a decreasing filtration by* K*-subspaces of* M*, called the* Hodge filtration.

A *morphism* of filtered modules is a K-linear map which is compatible with the Hodge filtration and commutes with Frobenius. The category of filtered modules is $\mathbb{Q}_p$-linear, has an obvious tensor product and inner Hom, but it is not abelian unless the objects are suitably rigidified. The naturally occurring condition has two quite different formulations. Historically that of semistability type came first.

Let $m := [K/\mathbb{Q}_p]$, then φ^m is a K-linear automorphism of M. Let ord_p be the normalized valuation on $\bar{\mathbb{Q}}_p$ with $\mathrm{ord}_p(p) = 1$. There is a unique φ^m-invariant $\mathbb{Q}$-grading $M = \bigoplus_{i\in\mathbb{Q}} M_i$ of K-vector spaces such that all eigenvalues of φ^m on M_i have normalized valuation mi. We set

$$\deg_\varphi(M) := \sum_{i\in\mathbb{Q}} i \cdot \dim_K M_i \,.$$

Definition 2.9 *(Fontaine [8] 4.1) A filtered module M over K is called* weakly admissible *if and only if for every filtered submodule $M' \subset M$ we have*

$$\deg_{\mathrm{Fil}}(M') \leq \deg_{\varphi}(M') \ ,$$

with equality whenever $M' = M$.

For the second formulation let $\mathcal{O}_K$ denote the ring of integers in K. By a *lattice in M* we mean a finitely generated $\mathcal{O}_K$-submodule that generates M over K.

Definition 2.10 *([9] 1.1, [20] 3.1) A lattice $\Lambda \subset M$ is called* strongly divisible *if and only if*

$$\varphi\Big(\sum_{i\in\mathbb{Z}} p^{-i} \cdot (\Lambda \cap \mathrm{Fil}^i M)\Big) = \Lambda \ .$$

The following equivalence is highly non-trivial:

Proposition 2.11 *(Laffaille [20] Th. 3.2) The following two assertions are equivalent:*

(a) M possesses a strongly divisible lattice.

(b) M is weakly admissible.

Both formulations are equally suited to show:

Proposition 2.12 *(Fontaine [8] Prop. 4.2.1) Every morphism of weakly admissible filtered modules over K is strictly compatible with the Hodge filtration. The category of weakly admissible filtered modules over K is abelian.*

In terms of strongly divisible lattices it is now easy to prove that weak admissibility is invariant under tensor product, and hence:

Proposition 2.13 *(Laffaille [20] §4) The category of weakly admissible filtered modules over K is tannakian.*

One can also prove this directly in terms of Definition 2.9, as done by Faltings [7] resp. Totaro [28]. These proofs are based on invariant theoretic methods.

3 Pre-Hodge structures over local function fields

Now we introduce an analogue of Hodge structures over a non-archimedean local field of equal characteristic. Such an object consists of a finite dimensional vector space together with a weight filtration, but instead of merely a Hodge filtration we require finer information.

Notations: We fix a perfect field k of arbitrary characteristic including, possibly, characteristic zero and consider the field of Laurent series in one variable $\hat{K} := k(\!(z)\!)$. We also fix a non-discrete algebraically closed complete normed field over k such that the elements of $k^\times$ have norm 1. For lack of a better notation, and because it plays a role analogous to that of the field of complex numbers, we shall denote this field by $\mathbb{C}$. (The symbol $\mathbb{R}$ will continue to denote the field of real numbers.) Fix an element $\zeta \in \mathbb{C}$ satisfying $0 < \|\zeta\| < 1$. The evaluation at ζ defines a continuous homomorphism of k-fields $\iota : \hat{K} \hookrightarrow \mathbb{C}$. The image of ι is, of course, just the field of Laurent series $k(\!(\zeta)\!)$, and the completion of its algebraic closure is the main example for $\mathbb{C}$ that we have in mind. But in order to better separate the roles played by z and ζ we have preferred to characterize $\mathbb{C}$ abstractly. Note the following elementary fact.

Proposition 3.1 *There is a natural injective algebra homomorphism* $\hat{K} \hookrightarrow \mathbb{C}[\![z-\zeta]\!]$ *which maps* z *to itself and on* k *coincides with* ι.

Proof. For every Laurent series $\sum_k a_k z^k \in \hat{K}$ we calculate formally

$$\begin{aligned}\sum_k i(a_k)\cdot z^k &= \sum_k i(a_k)\cdot(z-\zeta+\zeta)^k \\ &= \sum_k i(a_k)\cdot \sum_{\ell\geq 0}\tbinom{k}{\ell}\cdot(z-\zeta)^\ell\cdot\zeta^{k-\ell} \\ &= \sum_{\ell\geq 0}(z-\zeta)^\ell\cdot\sum_k i(a_k)\cdot\tbinom{k}{\ell}\cdot\zeta^{k-\ell}\,.\end{aligned}$$

The term $i(a_k)\cdot\binom{k}{\ell}$ has norm ≤ 1, so the inner sum in the last line converges in $\mathbb{C}$ for every $\ell \in \mathbb{Z}$. This defines the desired map, which one easily shows to be an algebra homomorphism. **q.e.d.**

We identify $\hat{K}$ with its image in $\mathbb{C}[\![z-\zeta]\!]$. Clearly the composite homomorphism $\hat{K} \hookrightarrow \mathbb{C}[\![z-\zeta]\!] \twoheadrightarrow \mathbb{C}$ coincides with ι.

The basic definition: By a *lattice in* a finite dimensional $\mathbb{C}(\!(z-\zeta)\!)$-vector space we mean a finitely generated $\mathbb{C}[\![z-\zeta]\!]$-submodule containing a $\mathbb{C}(\!(z-\zeta)\!)$-basis. For any finite dimensional $\hat{K}$-vector space $\hat{H}$ the embedding 3.1 provides us with a canonical lattice $\mathfrak{p}_{\hat{H}} := \hat{H} \otimes_{\hat{K}} \mathbb{C}[\![z-\zeta]\!]$.

Definition 3.2 *A* mixed $\hat{K}$-pre-Hodge structure *is a triple* $\underline{\hat{H}} = (\hat{H}, W, \mathfrak{q}_{\hat{H}})$ *where*

(a) $\hat{H}$ *is a finite dimensional* $\hat{K}$*-vector space,*

(b) $W = (W_\mu \hat{H})_{\mu\in\mathbb{Q}}$ *is an increasing filtration by* $\hat{K}$*-subspaces of* $\hat{H}$*, called the* weight filtration, *and*

(c) $\mathfrak{q}_{\hat{H}}$ *is a lattice in* $\hat{H} \otimes_{\hat{K}} \mathbb{C}(\!(z-\zeta)\!)$.

Definition 3.3 *A mixed* $\hat{K}$*-pre-Hodge structure* $\underline{\hat{H}}$ *is called* pure of weight $\mu \in \mathbb{Q}$ *if and only if* $\mathrm{Gr}^W_\mu \hat{H} \cong \hat{H}$.

These objects come with a Hodge filtration. Abbreviate

$$\hat{H}_{\mathbb{C}} := \mathfrak{p}_{\hat{H}}/(z-\zeta)\mathfrak{p}_{\hat{H}} \cong \hat{H} \otimes_{\hat{K},\iota} \mathbb{C}\,. \tag{3.4}$$

Definition 3.5 *For any mixed $\hat{K}$-pre-Hodge structure $\underline{\hat{H}}$ and any $i \in \mathbb{Z}$ let $F^i \hat{H}_{\mathbb{C}}$ denote the image of $\mathfrak{p}_{\hat{H}} \cap (z-\zeta)^i \mathfrak{q}_{\hat{H}}$ in $\hat{H}_{\mathbb{C}}$. This defines a decreasing filtration F of $\hat{H}_{\mathbb{C}}$, called the* Hodge filtration.

As in the case of classical mixed Hodge structures the weight and Hodge filtrations go in opposite directions. It is also instructive to view $\underline{\hat{H}}$ as a filtered vector bundle on the diagram of schemes

$$\begin{array}{ccc} \mathbf{Spec}\,\mathbb{C}(\!(z-\zeta)\!) & \hookrightarrow & \mathbf{Spec}\,\mathbb{C}[\![z-\zeta]\!] \\ \downarrow & \swarrow & \\ \mathbf{Spec}\,\hat{K} & & \end{array} \tag{3.6}$$

without the dotted arrow. Since by Proposition 3.1 the dotted arrow exists and makes the diagram commutative, the diagram without it behaves like a non-separated scheme. Although one could develop the theory from this point of view without any significant problems, we have here preferred the more down-to-earth approach.

The category of mixed $\hat{K}$-pre-Hodge structures:

Definition 3.7 *(a) A* morphism $\varphi : \underline{\hat{H}}_1 \to \underline{\hat{H}}_2$ *of mixed $\hat{K}$-pre-Hodge structures is a homomorphism of the underlying $\hat{K}$-vector spaces $\hat{H}_1 \to \hat{H}_2$ which is compatible with the weight filtrations and maps $\mathfrak{q}_{\hat{H}_1}$ into $\mathfrak{q}_{\hat{H}_2}$.*

(b) A morphism $\varphi : \underline{\hat{H}}_1 \to \underline{\hat{H}}_2$ of mixed $\hat{K}$-pre-Hodge structures is called strict *if and only if it is strictly compatible with the weight filtrations and satisfies*

$$\varphi(\mathfrak{q}_{\hat{H}_1}) = \mathfrak{q}_{\hat{H}_2} \cap \big(\varphi(\hat{H}_1) \otimes_{\hat{K}} \mathbb{C}(\!(z-\zeta)\!)\big) \, .$$

Clearly any morphism is also compatible with the Hodge filtrations (cf. Definition 3.5). One should be aware, however, that a strict morphism in the above sense is not necessarily strictly compatible with the Hodge filtrations (compare Example 6.14 below). The category of mixed $\hat{K}$-pre-Hodge structures with all morphisms (not necessarily strict) is $\hat{K}$-linear, but for the same reasons as in Section 1 it is not abelian. Also, a composite of strict morphisms need not be strict.

By a *subobject* we mean a morphism $\underline{\hat{H}}' \to \underline{\hat{H}}$ whose underlying homomorphism of $\hat{K}$-vector spaces is the inclusion of a subspace $\hat{H}' \hookrightarrow \hat{H}$. The subobject is called *strict* if and only if the morphism $\underline{\hat{H}}' \to \underline{\hat{H}}$ is strict. For any $\underline{\hat{H}}$ we can endow any subspace $\hat{H}' \subset \hat{H}$ with a unique structure of strict subobject $\underline{\hat{H}}'$. Dually, the factor space $\hat{H}'' := \hat{H}/\hat{H}'$ carries a unique mixed $\hat{K}$-pre-Hodge structure such that the map $\hat{H} \to \hat{H}''$ extends to a strict morphism $\underline{\hat{H}} \to \underline{\hat{H}}''$. This $\underline{\hat{H}}''$ is called a *strict factor object of* $\underline{\hat{H}}$, also denoted $\underline{\hat{H}}/\underline{\hat{H}}'$. The sequence $0 \to \underline{\hat{H}}' \to \underline{\hat{H}} \to \underline{\hat{H}}/\underline{\hat{H}}' \to 0$ and any sequence isomorphic to it is called *strict exact*. Combining these constructions we obtain a natural mixed $\hat{K}$-pre-Hodge structure on any subquotient $\hat{H}_1/\hat{H}_2$ of $\hat{H}$. As in the case of filtrations this induced structure does not depend on the order of operations; however, it does depend on the subspaces $\hat{H}_2 \subset \hat{H}_1 \subset \hat{H}$.

Applying these construction to any (say decreasing) filtration Φ of $\hat{H}$ we obtain canonical strict subobjects $\Phi^i\underline{\hat{H}}$ and subquotients $\mathrm{Gr}^i_\Phi\underline{\hat{H}}$. The analogous notations are used for increasing filtrations. Of particular interest are $W_\mu\underline{\hat{H}}$ and $\mathrm{Gr}^W_\mu\underline{\hat{H}}$.

The *tensor product* $\underline{\hat{H}}_1 \otimes \underline{\hat{H}}_2$ of two mixed $\hat{K}$-pre-Hodge structures is the tensor product of the underlying $\hat{K}$-vector spaces together with the induced weight filtration and the lattice $\mathfrak{q}_{\hat{H}_1} \otimes_{\mathbb{C}[[z-\zeta]]} \mathfrak{q}_{\hat{H}_2}$. All other standard tensor constructions extend to mixed $\hat{K}$-pre-Hodge structures in the obvious way. Thus the lattice for the symmetric power $\mathrm{Sym}^r\,\underline{\hat{H}}$ (resp. alternating power $\Lambda^r\underline{\hat{H}}$) is just the symmetric (resp. alternating) power of $\mathfrak{q}_{\hat{H}}$ taken over $\mathbb{C}[[z-\zeta]]$. Similarly the object $\underline{\mathrm{Hom}}(\underline{\hat{H}}_1, \underline{\hat{H}}_2)$ has underlying vector space $\mathrm{Hom}_{\hat{K}}(\hat{H}_1, \hat{H}_2)$ and lattice $\mathrm{Hom}_{\mathbb{C}[[z-\zeta]]}(\mathfrak{q}_{\hat{H}_1}, \mathfrak{q}_{\hat{H}_2})$. In particular, the dual object $\underline{\hat{H}}^\vee$ has lattice $\mathfrak{q}_{\hat{H}^\vee} := \mathrm{Hom}_{\mathbb{C}[[z-\zeta]]}(\mathfrak{q}_{\hat{H}}, \mathbb{C}[[z-\zeta]])$. One easily checks that the Hodge filtration of a tensor product $\underline{\hat{H}}_1 \otimes \underline{\hat{H}}_2$ coincides with the tensor product of filtrations as defined in Section 1. Similarly, the Hodge filtration commutes with $\underline{\mathrm{Hom}}$, Sym, and Λ, but not with subquotients (cf. Section 7). Finally, let $\mathbb{1}_{\hat{K}}$ denote the object with underlying vector space $\hat{K}$, lattice $\mathfrak{q}_{\hat{K}} = \mathfrak{p}_{\hat{K}}$, and which is pure of weight 0. There is an obvious functorial isomorphism $\underline{\hat{H}} \otimes \mathbb{1}_{\hat{K}} \cong \underline{\hat{H}}$, therefore $\mathbb{1}_{\hat{K}}$ is called the *unit object.*

4 Semistability and Hodge structures

To rigidify our objects we shall now impose a semistability condition in the spirit of invariant theory. We first discuss the necessary numerical invariants.

Numerical invariants: The *rank* of a mixed $\hat{K}$-pre-Hodge structure $\underline{\hat{H}}$ is simply $\mathrm{rank}(\underline{\hat{H}}) := \dim_{\hat{K}}(\hat{H})$. Recall that the weight filtration determines a total degree $\deg^W(\underline{\hat{H}})$, as well as an average weight $\mu^W(\underline{\hat{H}})$ provided that $\hat{H} \neq 0$ (cf. 1.1–1.2). Note that $\mu^W(\underline{\hat{H}}) = \mu$ if $\underline{\hat{H}}$ is pure of weight μ. Similarly the Hodge filtration determines invariants which can be expressed directly in terms of the lattice:

$$(4.1)\qquad \deg_{\mathfrak{q}}(\underline{\hat{H}}) := \dim_{\mathbb{C}}\Big(\frac{\mathfrak{q}_{\hat{H}}}{\mathfrak{p}_{\hat{H}} \cap \mathfrak{q}_{\hat{H}}}\Big) - \dim_{\mathbb{C}}\Big(\frac{\mathfrak{p}_{\hat{H}}}{\mathfrak{p}_{\hat{H}} \cap \mathfrak{q}_{\hat{H}}}\Big) = \deg_F(\hat{H}_{\mathbb{C}}) \,,$$

and, if $\underline{\hat{H}} \neq 0$:

$$(4.2)\qquad \mu_{\mathfrak{q}}(\underline{\hat{H}}) := \frac{\deg_{\mathfrak{q}}(\underline{\hat{H}})}{\mathrm{rank}(\underline{\hat{H}})} = \mu_F(\hat{H}_{\mathbb{C}}) \,.$$

The functorial behavior of these invariants is as in Proposition 1.4.

Semistability: We begin with the following observation.

Proposition 4.3 *Consider a morphism of mixed $\hat{K}$-pre-Hodge structures $\varphi : \underline{\hat{H}}_1 \to \underline{\hat{H}}_2$ whose underlying homomorphism of vector spaces is an isomorphism. Then we have $\deg^W(\underline{\hat{H}}_1) \geq \deg^W(\underline{\hat{H}}_2)$ and $\deg_{\mathfrak{q}}(\underline{\hat{H}}_1) \leq \deg_{\mathfrak{q}}(\underline{\hat{H}}_2)$. Moreover φ is an isomorphism if and only if both these inequalities are equalities.*

Proof. One easily shows that the isomorphy of lattices $\mathfrak{q}_{\hat{H}_1} \xrightarrow{\sim} \mathfrak{q}_{\hat{H}_2}$ is equivalent to the isomorphy between Hodge filtrations. Thus all the assertions depend only on the filtrations and thus follow directly from Proposition 1.3.

q.e.d.

Proposition 4.4 *The following conditions on a mixed $\hat{K}$-pre-Hodge structure $\underline{\hat{H}}$ are equivalent:*

(a) *For every subobject $\underline{\hat{H}}'$ of $\underline{\hat{H}}$ we have $\deg_{\mathfrak{q}}(\underline{\hat{H}}') \leq \deg^W(\underline{\hat{H}}')$, with equality whenever $\underline{\hat{H}}' = W_\mu\underline{\hat{H}}$ for some $\mu \in \mathbb{Q}$.*

(a′) *For every strict subobject $\underline{\hat{H}}'$ of $\underline{\hat{H}}$ we have $\deg_{\mathfrak{q}}(\underline{\hat{H}}') \leq \deg^W(\underline{\hat{H}}')$, with equality whenever $\underline{\hat{H}}' = W_\mu\underline{\hat{H}}$ for some $\mu \in \mathbb{Q}$.*

(b) *For every factor object $\underline{\hat{H}}''$ of $\underline{\hat{H}}$ we have $\deg_{\mathfrak{q}}(\underline{\hat{H}}'') \geq \deg^W(\underline{\hat{H}}'')$, with equality whenever $\underline{\hat{H}}'' = \underline{\hat{H}}/W_\mu\underline{\hat{H}}$ for some $\mu \in \mathbb{Q}$.*

(b′) *For every strict factor object $\underline{\hat{H}}''$ of $\underline{\hat{H}}$ we have $\deg_{\mathfrak{q}}(\underline{\hat{H}}'') \geq \deg^W(\underline{\hat{H}}'')$, with equality whenever $\underline{\hat{H}}'' = \underline{\hat{H}}/W_\mu\underline{\hat{H}}$ for some $\mu \in \mathbb{Q}$.*

Proof. The direction (a)⇒(a′) is obvious. For its converse consider an arbitrary subobject $\underline{\hat{H}}'$ of $\underline{\hat{H}}$ and let $\underline{\hat{H}}^+$ be the strict subobject of $\underline{\hat{H}}$ with the same underlying vector space as $\underline{\hat{H}}'$. Then the inclusion morphism factors as $\underline{\hat{H}}' \to \underline{\hat{H}}^+ \hookrightarrow \underline{\hat{H}}$. Using Proposition 4.3 and condition (a′) we deduce $\deg_{\mathfrak{q}}(\underline{\hat{H}}') \leq \deg_{\mathfrak{q}}(\underline{\hat{H}}^+) \leq \deg^W(\underline{\hat{H}}^+) \leq \deg^W(\underline{\hat{H}}')$, as desired. The equivalence (b)⇔(b′) is proved in the same way. The equivalence (a′)⇔(b′) follows from the additivity of both $\deg_{\mathfrak{q}}$ and $\deg^W$ in strict exact sequences. **q.e.d.**

Definition 4.5 *A mixed $\hat{K}$-pre-Hodge structure is called* semistable *if and only if it satisfies the equivalent conditions in Proposition 4.4. A semistable mixed $\hat{K}$-pre-Hodge structure is called a* mixed $\hat{K}$-Hodge structure. *Likewise a semistable pure $\hat{K}$-pre-Hodge structure of weight μ is called a* pure $\hat{K}$-Hodge structure of weight μ.

First properties: The dual formulations in Proposition 4.4 are very useful. For instance, using the fact that all degrees change their sign under dualization we obtain at once:

Proposition 4.6 *A mixed $\hat{K}$-pre-Hodge structure $\underline{\hat{H}}$ is semistable if and only if its dual $\underline{\hat{H}}^\vee$ is semistable.*

Next consider any morphism of mixed $\hat{K}$-pre-Hodge structures $\varphi : \underline{\hat{H}}_1 \to \underline{\hat{H}}_2$. Let $\underline{\hat{H}}_1' \subset \underline{\hat{H}}_1$ and $\underline{\hat{H}}_2' \subset \underline{\hat{H}}_2$ denote the strict subobjects whose underlying vector spaces are the kernel, respectively the image, of the underlying homomorphism of vector spaces. Put $\underline{\hat{H}}_1'' := \underline{\hat{H}}_1/\underline{\hat{H}}_1'$ and $\underline{\hat{H}}_2'' := \underline{\hat{H}}_2/\underline{\hat{H}}_2'$. Clearly $\underline{\hat{H}}_1'$ and $\underline{\hat{H}}_2''$ are a kernel, respectively a cokernel, of φ in the category of mixed $\hat{K}$-pre-Hodge structures. The following fact is crucial:

Proposition 4.7 *Assume that both $\underline{\hat{H}}_1$ and $\underline{\hat{H}}_2$ are semistable. Then*

(a) *$\underline{\hat{H}}_1'$, $\underline{\hat{H}}_1''$, $\underline{\hat{H}}_2'$, and $\underline{\hat{H}}_2''$ are semistable.*

(b) The natural morphism $\underline{\hat{H}}_1'' \xrightarrow{\sim} \underline{\hat{H}}_2'$ *is an isomorphism.*

(c) φ *is strict in the sense of Definition 3.7 (b).*

Proof. By construction φ factors through a morphism $\psi : \underline{\hat{H}}_1'' \to \underline{\hat{H}}_2'$ which is an isomorphism on the underlying vector spaces. Thus we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{\hat{H}}_1' & \longrightarrow & \underline{\hat{H}}_1 & \longrightarrow & \underline{\hat{H}}_1'' & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & \\ 0 & \longleftarrow & \underline{\hat{H}}_2'' & \longleftarrow & \underline{\hat{H}}_2 & \longleftarrow & \underline{\hat{H}}_2' & \longleftarrow & 0 \end{array} \tag{4.8}$$

whose rows are strict exact. We deduce

$$\deg_{\mathfrak{q}}(\underline{\hat{H}}_1'') \overset{4.3}{\leq} \deg_{\mathfrak{q}}(\underline{\hat{H}}_2') \overset{4.4\ (a)}{\leq} \deg^W(\underline{\hat{H}}_2') \overset{4.3}{\leq} \deg^W(\underline{\hat{H}}_1'') \overset{4.4\ (b)}{\leq} \deg_{\mathfrak{q}}(\underline{\hat{H}}_1'') \,.$$

Thus these inequalities are equalities, hence Proposition 4.3 shows that ψ is an isomorphism. This means that φ is strict and hence proves (b) and (c). To show (a) we first look at $\underline{\hat{H}}_1'$ and $\underline{\hat{H}}_2'$ and use the formulation 4.4 (a). The necessary inequality for any subobject follows directly from the corresponding characterization of the semistability of $\underline{\hat{H}}_1$ and $\underline{\hat{H}}_2$. It remains to prove the equality for a filtration step W_μ. By (c) we know already that φ is strictly compatible with the weight filtrations. In particular the natural homomorphism $W_\mu \hat{H}_1'' \to W_\mu \hat{H}_2'$ is an isomorphism, so we have a diagram just like 4.8:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W_\mu\underline{\hat{H}}_1' & \longrightarrow & W_\mu\underline{\hat{H}}_1 & \longrightarrow & W_\mu\underline{\hat{H}}_1'' & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longleftarrow & W_\mu\underline{\hat{H}}_2'' & \longleftarrow & W_\mu\underline{\hat{H}}_2 & \longleftarrow & W_\mu\underline{\hat{H}}_2' & \longleftarrow & 0 \end{array} \tag{4.9}$$

From Condition 4.4 (a) it is clear that $W_\mu\underline{\hat{H}}_1$ and $W_\mu\underline{\hat{H}}_2$ are semistable, so we may apply the above arguments to Diagram 4.9. It follows that the vertical arrow on the right hand side is an isomorphism and that $\deg_{\mathfrak{q}}(W_\mu\underline{\hat{H}}_2') = \deg^W(W_\mu\underline{\hat{H}}_2')$. This already proves the semistability of $\underline{\hat{H}}_2'$. Next we deduce the equality $\deg_{\mathfrak{q}}(W_\mu\underline{\hat{H}}_1') = \deg^W(W_\mu\underline{\hat{H}}_1')$ from the corresponding equality for $W_\mu\underline{\hat{H}}_1$ and the fact that both degrees are additive in the strict exact sequence $0 \to W_\mu\underline{\hat{H}}_1' \to W_\mu\underline{\hat{H}}_1 \to W_\mu\underline{\hat{H}}_2' \to 0$. This proves the semistability of $\underline{\hat{H}}_1'$. Finally, for $\underline{\hat{H}}_2''$ one may repeat the same arguments using factor objects and the formulation 4.4 (b). **q.e.d.**

Semistability and extensions: Let us fix a strict exact sequence of mixed $\hat{K}$-pre-Hodge structures

$$0 \to \underline{\hat{H}}' \to \underline{\hat{H}} \to \underline{\hat{H}}'' \to 0 \,. \tag{4.10}$$

If $\underline{\hat{H}}$ and one of $\underline{\hat{H}}'$, $\underline{\hat{H}}''$ are semistable, Proposition 4.7 (a) implies that all three are semistable. What can we deduce when $\underline{\hat{H}}'$ and $\underline{\hat{H}}''$ are semistable?

Proposition 4.11 *Assume*

(a) $\underline{\hat{H}}'$ *and* $\underline{\hat{H}}''$ *are semistable, and*

(b) for all $\mu \in \mathbb{Q}$ we have $W_\mu \underline{\hat{H}} \subset \underline{\hat{H}}'$ or $\underline{\hat{H}}' \subset W_\mu \underline{\hat{H}}$.

Then $\underline{\hat{H}}$ is semistable.

Proof. Consider any subobject $\underline{\hat{H}}_1$ of $\underline{\hat{H}}$. Let $\hat{H}' \subset \hat{H} \supset \hat{H}_1$ be the vector spaces underlying the objects $\underline{\hat{H}}' \subset \underline{\hat{H}} \supset \underline{\hat{H}}_1$, and let $\underline{\hat{H}}_1'$ be the strict subobject of $\underline{\hat{H}}_1$ with underlying vector space $\hat{H}_1 \cap \hat{H}'$. Then we have a strict exact sequence of subobjects of those in 4.10:

$$0 \to \underline{\hat{H}}_1' \to \underline{\hat{H}}_1 \to \underline{\hat{H}}_1'' \to 0 \,.$$

Using the additivity of degrees in strict exact sequences and the semistability assumption for $\underline{\hat{H}}'$ and $\underline{\hat{H}}''$ in the form of 4.4 (a) we calculate

$$\begin{aligned} \deg_{\mathfrak{q}}(\underline{\hat{H}}_1) &= \deg_{\mathfrak{q}}(\underline{\hat{H}}_1') + \deg_{\mathfrak{q}}(\underline{\hat{H}}_1'') \\ &\leq \deg^W(\underline{\hat{H}}_1') + \deg^W(\underline{\hat{H}}_1'') \\ &= \deg^W(\underline{\hat{H}}_1) \,. \end{aligned}$$

This is the inequality needed in 4.4 (a) for $\underline{\hat{H}}$. It remains to show that this is an equality whenever $\underline{\hat{H}}_1 = W_\mu \underline{\hat{H}}$. If $W_\mu \underline{\hat{H}} \subset \underline{\hat{H}}'$ this equality follows from the semistability of $\underline{\hat{H}}'$. If $\underline{\hat{H}}' \subset W_\mu \underline{\hat{H}}$ the construction implies $(W_\mu \underline{\hat{H}})/\underline{\hat{H}}' \cong W_\mu \underline{\hat{H}}''$, hence

$$\begin{aligned} \deg_{\mathfrak{q}}(W_\mu \underline{\hat{H}}) &= \deg_{\mathfrak{q}}(\underline{\hat{H}}') + \deg_{\mathfrak{q}}(W_\mu \underline{\hat{H}}'') \\ &= \deg^W(\underline{\hat{H}}') + \deg^W(W_\mu \underline{\hat{H}}'') \\ &= \deg^W(W_\mu \underline{\hat{H}}) \end{aligned}$$

using the semistability of $\underline{\hat{H}}''$, as desired. **q.e.d.**

Corollary 4.12 *In the strict exact sequence 4.10 assume that $\underline{\hat{H}}$ is pure of some weight. If two of $\underline{\hat{H}}'$, $\underline{\hat{H}}$, $\underline{\hat{H}}''$ are semistable, then so is the third.*

Next we can show that semistability depends only on the pure constituents, in the following sense:

Proposition 4.13 *The following assertions for a mixed $\hat{K}$-pre-Hodge structure $\underline{\hat{H}}$ are equivalent.*

(a) $\underline{\hat{H}}$ is semistable.

(b) $W_\mu \underline{\hat{H}}$ is semistable for every $\mu \in \mathbb{Q}$.

(c) $\underline{\hat{H}}/W_\mu \underline{\hat{H}}$ is semistable for every $\mu \in \mathbb{Q}$.

(d) $\mathrm{Gr}_\mu^W \underline{\hat{H}}$ is semistable for every $\mu \in \mathbb{Q}$.

Proof. The equivalence (a)⇔(b), respectively (a)⇔(c), follows at once from the formulation in Proposition 4.4 (a), resp. (b). The equivalence (b)⇔(d) follows by induction from Propositions 4.7 (a) and 4.11, applied to the exact sequence

$$0 \to W_{\mu'} \underline{\hat{H}} \to W_\mu \underline{\hat{H}} \to \mathrm{Gr}_\mu^W \underline{\hat{H}} \to 0$$

for all sufficiently near $\mu' < \mu$. **q.e.d.**

For split extensions we have:

Proposition 4.14 *Two given mixed $\hat{K}$-pre-Hodge structures $\underline{\hat{H}}_1$, $\underline{\hat{H}}_2$ are semistable if and only if their direct sum $\underline{\hat{H}}_1 \oplus \underline{\hat{H}}_2$ is semistable.*

Proof. Since Gr_μ^W commutes with $\oplus$, Proposition 4.13 reduces the problem to the case that $\underline{\hat{H}}_1$ and $\underline{\hat{H}}_2$ are pure of the same weight. When the $\underline{\hat{H}}_i$ are semistable, the semistability of $\underline{\hat{H}}_1 \oplus \underline{\hat{H}}_2$ then follows by applying Corollary 4.12 to the obvious strict exact sequence

$$0 \to \underline{\hat{H}}_1 \to \underline{\hat{H}}_1 \oplus \underline{\hat{H}}_2 \to \underline{\hat{H}}_2 \to 0 .$$

Conversely, if $\underline{\hat{H}}_1 \oplus \underline{\hat{H}}_2$ is pure and semistable, the inequality in 4.4 (a) follows at once for any subobject of either $\underline{\hat{H}}_i$, since that is also a subobject of $\underline{\hat{H}}_1 \oplus \underline{\hat{H}}_2$. The remaining equalities follow from the additivity of degrees. **q.e.d.**

The category of mixed $\hat{K}$-Hodge structures: Let $\mathscr{H}\!odge_{\hat{K}}$ denote the category of mixed $\hat{K}$-Hodge structures with all morphisms of Definition 3.7 (a). Propositions 4.7 and 4.14 imply:

Theorem 4.15 *The category $\mathscr{H}\!odge_{\hat{K}}$ is abelian.*

A non-zero object of $\mathscr{H}\!odge_{\hat{K}}$ has many subobjects in the category of mixed $\hat{K}$-pre-Hodge structures, but only few of them are again in $\mathscr{H}\!odge_{\hat{K}}$. In fact, any subobject in $\mathscr{H}\!odge_{\hat{K}}$ must be strict, hence it is determined uniquely by the underlying $\hat{K}$-subspace, but not every $\hat{K}$-subspace is possible. As the underlying vector spaces have finite dimension, this shows in particular that every object of $\mathscr{H}\!odge_{\hat{K}}$ has finite length. Proposition 4.13 has the following consequences.

Corollary 4.16 *Any simple object of $\mathscr{H}\!odge_{\hat{K}}$ is pure of some weight.*

Corollary 4.17 *For every $\mu \in \mathbb{Q}$ we have an exact functor*

$$\mathrm{Gr}_\mu^W : \mathscr{H}\!odge_{\hat{K}} \to \mathscr{H}\!odge_{\hat{K}}, \quad \underline{\hat{H}} \mapsto \mathrm{Gr}_\mu^W \underline{\hat{H}}.$$

5 Semistability and tensor product

For the category of mixed $\hat{K}$-Hodge structures to be tannakian we must prove that semistability is invariant under tensor product. This is done in the present section. The problem can be approached in several different ways. Our argument is a modification of Totaro's proof for filtered modules [28], which in turn is based on ideas of Kempf and Ramanan-Ramanathan [27]. This proof relies on explicit calculations of degrees of various filtrations and is therefore rather elementary. Our only additional ingredient is the passage from lattices to Hodge filtrations and back, inserted at strategically chosen places. For an explanation of Totaro's method and further comments see [28]. We have quoted the necessary results in Section 1. We also need the following preparations.

Frobenius functoriality: Consider a positive integer q which is a power of $\mathrm{char}(\hat{K})$. There is no need to exclude the case $q = 1$, which works in any characteristic. On any commutative $\hat{K}$-algebra we have a Frobenius endomorphism $\mathrm{Frob}_q : x \mapsto x^q$, and we are interested in its effect on mixed pre-Hodge structures.

Definition 5.1 *(Frobenius pullback) For any mixed $\hat{K}$-pre-Hodge structure $\underline{\hat{H}} = (\hat{H}, W, \mathfrak{q}_{\hat{H}})$ we define $\mathrm{Frob}_q^*\underline{\hat{H}} := (\hat{H}', W', \mathfrak{q}_{\hat{H}'})$, where*

(a) $\hat{H}' := \hat{H} \otimes_{\hat{K},\mathrm{Frob}_q} \hat{K}$,

(b) $W'_\mu \hat{H}' := W_{\mu/q}\hat{H} \otimes_{\hat{K},\mathrm{Frob}_q} \hat{K}$ *for every* $\mu \in \mathbb{Q}$, *and*

(c) $\mathfrak{q}_{\hat{H}'} := \mathfrak{q}_{\hat{H}} \otimes_{\mathbb{C}[\![z-\zeta]\!],\mathrm{Frob}_q} \mathbb{C}[\![z-\zeta]\!]$.

Definition 5.2 *(Frobenius pushforward) For any mixed $\hat{K}$-pre-Hodge structure $\underline{\hat{H}} = (\hat{H}, W, \mathfrak{q}_{\hat{H}})$ we define $\mathrm{Frob}_{q,*}\underline{\hat{H}} := (\hat{H}', W', \mathfrak{q}_{\hat{H}'})$, where*

(a) $\hat{H}' := \hat{H}$ *on which* $\hat{K}$ *acts through* Frob_q,

(b) $W'_\mu \hat{H}' := W_{\mu q}\hat{H}$ *for every* $\mu \in \mathbb{Q}$, *and*

(c) $\mathfrak{q}_{\hat{H}'} := \mathfrak{q}_{\hat{H}}$.

Using the canonical isomorphism

$$\mathrm{Frob}_q \otimes \mathrm{id} : \ \mathbb{C}[\![z-\zeta]\!] \otimes_{\hat{K},\mathrm{Frob}_q} \hat{K} \xrightarrow{\sim} \mathbb{C}[\![z-\zeta]\!]$$

one easily finds that the triples $\mathrm{Frob}_q^*\underline{\hat{H}}$ and $\mathrm{Frob}_{q,*}\underline{\hat{H}}$ are again mixed $\hat{K}$-pre-Hodge structures. Moreover, both prescriptions extend to functors of mixed $\hat{K}$-pre-Hodge structures in the obvious way, and Frob_q^* is left adjoint to $\mathrm{Frob}_{q,*}$. Furthermore, we have:

Proposition 5.3 *(a)* $\mathrm{Frob}_{q,*}\mathrm{Frob}_q^*\underline{\hat{H}} \cong \underline{\hat{H}}^{\oplus q}$.

(b) $\mathrm{Frob}_q^*\mathrm{Frob}_{q,*}\underline{\hat{H}}$ *is a successive extension of q copies of* $\underline{\hat{H}}$.

Proof. The isomorphism in (a) is induced by any basis of $\hat{K}$ over $\hat{K}^q := \mathrm{Frob}_q(\hat{K})$. In (b) the vector space underlying $\mathrm{Frob}_q^*\mathrm{Frob}_{q,*}\underline{\hat{H}}$ is

$$\hat{H} \otimes_{\hat{K}^q} \hat{K} \cong \hat{H} \otimes_{\hat{K}} (\hat{K} \otimes_{\hat{K}^q} \hat{K}) \cong \hat{H} \otimes_{\hat{K}} \hat{K}[t]/(t^q)\ .$$

The adjunction morphism $\mathrm{Frob}_q^*\mathrm{Frob}_{q,*}\underline{\hat{H}} \twoheadrightarrow \underline{\hat{H}}$ and multiplication by powers of t yield the desired description. **q.e.d.**

The behavior of the numerical invariants is read off easily from the definitions:

Proposition 5.4 *For any mixed $\hat{K}$-pre-Hodge structure $\underline{\hat{H}}$ we have*

(a) *(i)* $\mathrm{rank}(\mathrm{Frob}_q^*\underline{\hat{H}}) = \mathrm{rank}(\underline{\hat{H}})$,

(ii) $\deg^W(\mathrm{Frob}_q^*\underline{\hat{H}}) = q \cdot \deg^W(\underline{\hat{H}})$,

(iii) $\deg_{\mathfrak{q}}(\mathrm{Frob}_q^*\underline{\hat{H}}) = q \cdot \deg_{\mathfrak{q}}(\underline{\hat{H}})$,

(b) (i) $\mathrm{rank}(\mathrm{Frob}_{q,*}\underline{\hat{H}}) = q \cdot \mathrm{rank}(\underline{\hat{H}})$,

(ii) $\deg^W(\mathrm{Frob}_{q,*}\underline{\hat{H}}) = \deg^W(\underline{\hat{H}})$,

(iii) $\deg_\mathfrak{q}(\mathrm{Frob}_{q,*}\underline{\hat{H}}) = \deg_\mathfrak{q}(\underline{\hat{H}})$.

Now we can prove:

Proposition 5.5 *For any mixed $\hat{K}$-pre-Hodge structure $\underline{\hat{H}}$ the following assertions are equivalent:*

(a) $\underline{\hat{H}}$ is semistable.

(b) $\mathrm{Frob}_q^*\underline{\hat{H}}$ is semistable.

(c) $\mathrm{Frob}_{q,*}\underline{\hat{H}}$ is semistable.

Proof. Suppose first that $\mathrm{Frob}_q^*\underline{\hat{H}}$ is semistable. Any subobject $\underline{\hat{H}}_1 \subset \underline{\hat{H}}$ determines a subobject $\mathrm{Frob}_q^*\underline{\hat{H}}_1 \subset \mathrm{Frob}_q^*\underline{\hat{H}}$, and the (in-)equalities characterizing the semistability of $\underline{\hat{H}}$ follow at once from Proposition 5.4 and the corresponding (in-)equalities for $\mathrm{Frob}_q^*\underline{\hat{H}}$. Thus $\underline{\hat{H}}$ is semistable, proving the implication (b)⇒(a). In the same way one proves the direction (c)⇒(a).

For the reverse directions note first that with Proposition 4.13 the problem reduces to the pure case. So assume that $\underline{\hat{H}}$ is semistable and pure. Then by Corollary 4.12 and Proposition 5.3 both $\mathrm{Frob}_{q,*}\mathrm{Frob}_q^*\underline{\hat{H}}$ and $\mathrm{Frob}_q^*\mathrm{Frob}_{q,*}\underline{\hat{H}}$ are semistable. Applying the former directions to $\mathrm{Frob}_q^*\underline{\hat{H}}$ resp. $\mathrm{Frob}_{q,*}\underline{\hat{H}}$ in place of $\underline{\hat{H}}$, we deduce that these objects are semistable, as desired. **q.e.d.**

The main result:

Theorem 5.6 *The tensor product of any two semistable mixed $\hat{K}$-pre-Hodge structures is semistable.*

This is the remaining ingredient for the following result. Let $\mathscr{V}ec_{\hat{K}}$ denote the tensor category of finite dimensional $\hat{K}$-vector spaces, and consider the forgetful functor

$$\omega : \mathscr{H}odge_{\hat{K}} \to \mathscr{V}ec_{\hat{K}}, \ \underline{\hat{H}} \mapsto \hat{H} .$$

Corollary 5.7 *The category of mixed $\hat{K}$-Hodge structures $\mathscr{H}odge_{\hat{K}}$ together with the fiber functor ω is a neutral tannakian category.*

Proof. By Theorem 4.15 $\mathscr{H}odge_{\hat{K}}$ is an $\hat{K}$-linear abelian category, and by Theorem 5.6 it is invariant under tensor product. By Proposition 4.6 it is also invariant under dualization, and hence under inner $\underline{\mathrm{Hom}}$. Furthermore it possesses a unit object $\mathbb{1}_{\hat{K}}$. Therefore $\mathscr{H}odge_{\hat{K}}$ is a rigid tensor category. Clearly ω is a fiber functor. **q.e.d.**

Proof of Theorem 5.6: This will take the rest of the section. Consider arbitrary mixed $\hat{K}$-pre-Hodge structures $\underline{\hat{H}}_1$ and $\underline{\hat{H}}_2$. Recall that any decreasing filtrations Φ of the underlying vector spaces $\hat{H}_i$ determine filtrations by strict subobjects $\Phi^\ell\underline{\hat{H}}_i$, as well as strict subquotients $\mathrm{Gr}^\ell_\Phi\underline{\hat{H}}_i$. The same holds for the associated total filtration of $\hat{H}_1 \otimes_{\hat{K}} \hat{H}_2$, again denoted Φ.

Proposition 5.8 *There is a natural isomorphism*

$$\mathrm{Gr}^{\ell}_{\Phi}(\underline{\hat{H}}_1 \otimes \underline{\hat{H}}_2) \cong \bigoplus_{\ell_1+\ell_2=\ell} (\mathrm{Gr}^{\ell_1}_{\Phi}\underline{\hat{H}}_1) \otimes (\mathrm{Gr}^{\ell_2}_{\Phi}\underline{\hat{H}}_2) .$$

Proof. On the underlying $\hat{K}$-vector spaces the isomorphism is standard. More generally, note that a canonical isomorphism of this form exists for all objects in an additive tensor category which are filtered by direct summands. Applied to the underlying vector spaces together with their weight filtrations, the Bruhat lemma 1.5 shows that Φ induces filtrations by direct summands. Thus the above isomorphism is compatible with the weight filtrations. For the lattices $\mathfrak{q}$ we have a similar decomposition, which by canonicity coincides with that coming from the underlying $\hat{K}$-vector spaces after tensoring with $\mathbb{C}(\!(z-\zeta)\!)$. This means that we have an isomorphism of mixed $\hat{K}$-pre-Hodge structures, as desired. **q.e.d.**

The analogue of Proposition 5.8 holds for increasing filtrations. Applying it to the weight filtrations we can reduce ourselves to the pure case:

Lemma 5.9 *If Theorem 5.6 holds whenever $\underline{\hat{H}}_1$ and $\underline{\hat{H}}_2$ are pure, then it holds in general.*

Proof. Combine Propositions 4.13, 4.14, and 5.8. **q.e.d.**

For the rest of the proof we may assume that $\underline{\hat{H}}_1$ and $\underline{\hat{H}}_2$ are semistable and pure, say of weights μ_1 resp. μ_2. Then $\underline{\hat{H}} := \underline{\hat{H}}_1 \otimes \underline{\hat{H}}_2$ is pure of weight $\mu := \mu_1 + \mu_2$. To prove its semistability we shall use the formulation in 4.4 (a). Let us fix any subobject $\underline{\hat{H}}'$ of $\underline{\hat{H}}$. We assume $\underline{\hat{H}}' \neq 0$ since otherwise the desired assertion is obvious. Note that the semistability of the $\underline{\hat{H}}_i$ implies

$$\mu_{\mathfrak{q}}(\underline{\hat{H}}) = \mu_{\mathfrak{q}}(\underline{\hat{H}}_1) + \mu_{\mathfrak{q}}(\underline{\hat{H}}_2) = \mu^W(\underline{\hat{H}}_1) + \mu^W(\underline{\hat{H}}_2) = \mu_1 + \mu_2 = \mu ,$$

so that $\mu^W(\underline{\hat{H}}') = \mu = \mu_{\mathfrak{q}}(\underline{\hat{H}})$. Thus we must prove

$$\mu_{\mathfrak{q}}(\underline{\hat{H}}') \leq \mu_{\mathfrak{q}}(\underline{\hat{H}}) . \tag{5.10}$$

Lemma 5.11 *The inequality 5.10 holds when $\hat{H}'$ is universally* $\mathrm{GL}(\hat{H}_1) \times \mathrm{GL}(\hat{H}_2)$*-semistable.*

Proof. For clarity let F denote the Hodge filtration of $\hat{H}_{\mathbb{C}}$ determined by $\underline{\hat{H}}$ and F' the Hodge filtration of $\hat{H}'_{\mathbb{C}}$ determined by $\underline{\hat{H}}'$, according to Definition 3.5. The functoriality of the Hodge filtration means that

$$F'^i\hat{H}'_{\mathbb{C}} \subset \hat{H}'_{\mathbb{C}} \cap F^i\hat{H}_{\mathbb{C}}$$

for all i. Since $\hat{H}'_{\mathbb{C}}$ is $\mathrm{GL}(\hat{H}_{1,\mathbb{C}}) \times \mathrm{GL}(\hat{H}_{2,\mathbb{C}})$-semistable by assumption, we deduce

$$\mu_{\mathfrak{q}}(\underline{\hat{H}}') = \mu_{F'}(\hat{H}'_{\mathbb{C}}) \leq \mu_F(\hat{H}'_{\mathbb{C}}) \leq \mu_F(\hat{H}_{\mathbb{C}}) = \mu_{\mathfrak{q}}(\underline{\hat{H}}) .$$

This is the desired inequality 5.10. **q.e.d.**

For the rest of the proof we assume that $\hat{H}'$ is not universally $\mathrm{GL}(\hat{H}_1) \times \mathrm{GL}(\hat{H}_2)$-semistable.

Lemma 5.12 *It suffices to prove the inequality 5.10 whenever $\hat{H}'$ possesses a universal Kempf filtration over $\hat{K}$.*

Proof. By Proposition 1.19 there exists a universal Kempf filtration over a finite purely inseparable extension of $\hat{K}$, say of degree q. Since $\hat{K}$ is a complete local field with perfect residue field, this extension must be generated by a q^{th} root of z. It follows that the universal Kempf filtration exists for $\text{Frob}_q^* \underline{\hat{H}}' \subset (\text{Frob}_q^* \underline{\hat{H}}_1) \otimes (\text{Frob}_q^* \underline{\hat{H}}_2)$. By Proposition 5.5 we may replace all our objects by their pullbacks under Frob_q. Afterwards $\hat{H}'$ possesses a universal Kempf filtration over $\hat{K}$. **q.e.d.**

By Lemma 5.12 we may now assume that Φ is a universal Kempf filtration associated to $\hat{H}'$. For any $j \in \mathbb{Q}$ let $F_{(j)}$ resp. $F'_{(j)}$ denote the Hodge filtrations of $\text{Gr}^j_\Phi \hat{H}_\mathbb{C}$ resp. $\text{Gr}^j_\Phi \hat{H}'_\mathbb{C}$ determined by the $\hat{K}$-pre-Hodge structures $\text{Gr}^j_\Phi \underline{\hat{H}}$ and $\text{Gr}^j_\Phi \underline{\hat{H}}'$. Next we fix a filtration $\check{F}$ of $\hat{H}_\mathbb{C}$ whose induced filtration of $\text{Gr}^j_\Phi \hat{H}_\mathbb{C}$ is equal to $F_{(j)}$. (Caution: In general the restriction of $F_{(j)}$ to $\text{Gr}^j_\Phi \hat{H}'_\mathbb{C}$ does not coincide with $F'_{(j)}$. Similarly, for $\check{F}$ we cannot necessarily take the Hodge filtration associated to the $\hat{K}$-pre-Hodge structure $\underline{\hat{H}}$.) We calculate

$$\begin{aligned}
\deg_\mathsf{q}(\underline{\hat{H}}') &= \sum_{j\in\mathbb{Q}} \deg_\mathsf{q}(\text{Gr}^j_\Phi \underline{\hat{H}}') \\
&= \sum_{j\in\mathbb{Q}} \deg_{F'_{(j)}}(\text{Gr}^j_\Phi \hat{H}'_\mathbb{C}) \\
&\leq \sum_{j\in\mathbb{Q}} \deg_{F_{(j)}}(\text{Gr}^j_\Phi \hat{H}'_\mathbb{C}) \\
&= \sum_{j\in\mathbb{Q}} \deg_{\check{F}|\text{Gr}^j_\Phi \hat{H}_\mathbb{C}}(\text{Gr}^j_\Phi \hat{H}'_\mathbb{C}) \\
&= \sum_{j\in\mathbb{Q}} \Big(\deg_{(\check{F}+m\Phi)|\text{Gr}^j_\Phi \hat{H}_\mathbb{C}}(\text{Gr}^j_\Phi \hat{H}'_\mathbb{C}) - \deg_{(m\Phi)|\text{Gr}^j_\Phi \hat{H}_\mathbb{C}}(\text{Gr}^j_\Phi \hat{H}'_\mathbb{C})\Big) \\
&= \deg_{\check{F}+m\Phi}(\hat{H}'_\mathbb{C}) - \deg_{m\Phi}(\hat{H}'_\mathbb{C})\ .
\end{aligned}$$

Here the first equality follows from the additivity of $\deg_\mathsf{q}$ in strict exact sequences. The second line is equation 4.1, and the third follows from the functoriality of the Hodge filtration. The fourth and fifth hold by construction, and the last one follows from Proposition 1.12, where m must be sufficiently large. Dividing everything by $\text{rank}(\underline{\hat{H}}')$ we deduce

$$\mu_\mathsf{q}(\underline{\hat{H}}') \leq \mu_{\check{F}+m\Phi}(\hat{H}'_\mathbb{C}) - \mu_{m\Phi}(\hat{H}')\ . \tag{5.13}$$

Next let us abbreviate

$$c := \frac{\mu_\Phi(\hat{H}') - \mu_\Phi(\hat{H})}{|\Phi|^2}\ .$$

By construction this number is positive. Moreover, by Proposition 1.15 we have $\mu_\Phi(\hat{H}) = \mu_\Phi(\hat{H}_1) + \mu_\Phi(\hat{H}_2) = 0$, and therefore

$$\mu_{m\Phi}(\hat{H}') = m \cdot \mu_\Phi(\hat{H}') = m \cdot c \cdot |\Phi|^2\ . \tag{5.14}$$

On the other hand we have

$$\begin{aligned} \mu_{\mathfrak{q}}(\underline{\hat{H}}) &= \mu_{\check{F}}(\hat{H}_{\mathbb{C}}) \\ &= \mu_{\check{F}+m\Phi}(\hat{H}_{\mathbb{C}}) - \mu_{m\Phi}(\hat{H}_{\mathbb{C}}) \\ &= \mu_{\check{F}+m\Phi}(\hat{H}_{\mathbb{C}}) \; . \end{aligned} \tag{5.15}$$

Combining 5.13, 5.14, and 5.15 we deduce

$$\begin{aligned} \mu_{\mathfrak{q}}(\underline{\hat{H}}') - \mu_{\mathfrak{q}}(\underline{\hat{H}}) &\leq \Big(\mu_{\check{F}+m\Phi}(\hat{H}'_{\mathbb{C}}) - \mu_{\check{F}+m\Phi}(\hat{H}_{\mathbb{C}})\Big) - m \cdot c \cdot |\Phi|^2 \\ &\overset{1.16\ (a)}{\leq} c \cdot \langle \Phi, \check{F} + m\Phi \rangle - m \cdot c \cdot |\Phi|^2 \\ &= c \cdot \langle \Phi, \check{F} \rangle \\ &\overset{1.16\ (b)}{=} c \cdot \sum_{i=1}^{2} \int_{\mathbb{R}} \big(\mu_{\check{F}}(\Phi^j \hat{H}_{i,\mathbb{C}}) - \mu_{\check{F}}(\hat{H}_{i,\mathbb{C}})\big) \dim(\Phi^j \hat{H}_i) dj \\ &= c \cdot \sum_{i=1}^{2} \int_{\mathbb{R}} \big(\mu_{\mathfrak{q}}(\Phi^j \underline{\hat{H}}_i) - \mu_{\mathfrak{q}}(\underline{\hat{H}}_i)\big) \dim(\Phi^j \hat{H}_i) dj \; . \end{aligned}$$

By the semistability of $\underline{\hat{H}}_i$ the integrand is everywhere ≤ 0. The desired inequality 5.10 follows. This finishes the proof of Theorem 5.6. **q.e.d.**

Aliter: Using the interpretation of mixed $\hat{K}$-pre-Hodge structures as filtered vector bundles on the diagram 3.6 one can prove Theorem 5.6 also along the more algebro-geometric lines of Ramanan-Ramanathan [27]. In the pure case our concept of semistability coincides with the usual semistability of vector bundles. Thus the proof of [27] Thms. 3.18 and 3.23 translates into our setting with no difficulty. The main difference to the case of vector bundles on a projective curve is the fact that in our case the notion of semistability is always invariant under Frobenius pullback.

Example 5.16 We close this section with an example where the universal Kempf filtration exists only over an inseparable extension. For simplicity assume $\mathrm{char}(\hat{K}) = 2$. Put $\hat{H} := \hat{K}^2$, make it pure of weight $\frac{1}{2}$, and let

$$\mathfrak{q}_{\hat{H}} := \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ \sqrt{\zeta} \end{pmatrix} \cdot (z-\zeta)^{-1} \right\rangle_{\mathbb{C}[[z-\zeta]]} .$$

The Hodge filtration comes out to be

$$F^i \hat{H}_{\mathbb{C}} = \begin{cases} 0 & \text{if } i > 1, \\ \langle \binom{1}{\sqrt{\zeta}} \rangle_{\mathbb{C}} & \text{if } i = 1, \\ \hat{H}_{\mathbb{C}} & \text{if } i \leq 0. \end{cases}$$

The semistability condition follows easily from the fact that the middle step is not defined over $\iota(\hat{K})$; so this defines a simple pure $\hat{K}$-Hodge structure $\underline{\hat{H}}$ of weight $\frac{1}{2}$. Its endomorphism ring

$$\mathrm{End}(\underline{\hat{H}}) = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} \right\rangle_{\hat{K}}$$

is an inseparable quadratic extension of $\hat{K}$, which is the cause of problems. Let us identify the tensor square $\hat{H} \otimes_{\hat{K}} \hat{H}$ with the space of 2×2-matrices over $\hat{K}$, so that

$$\begin{aligned}\mathfrak{q}_{\hat H\otimes\hat H} &= \Big\langle \begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}1&\sqrt{\zeta}\\0&0\end{pmatrix}\cdot(z-\zeta)^{-1},\\ &\qquad \begin{pmatrix}1&0\\\sqrt{\zeta}&0\end{pmatrix}\cdot(z-\zeta)^{-1}, \begin{pmatrix}1&\sqrt{\zeta}\\\sqrt{\zeta}&\zeta\end{pmatrix}\cdot(z-\zeta)^{-2}\Big\rangle_{\mathbb{C}[\![z-\zeta]\!]}\,.\end{aligned}$$

Its Hodge filtration can be written as

$$F^i(\hat H\otimes\hat H)_{\mathbb{C}} = \begin{cases} 0 & \text{if } i>2,\\ \Big\langle\begin{pmatrix}1&\sqrt{\zeta}\\\sqrt{\zeta}&\zeta\end{pmatrix}\Big\rangle_{\hat K} & \text{if } i=2,\\ \Big\langle\begin{pmatrix}1&0\\\sqrt{\zeta}&0\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}1&0\\0&\zeta\end{pmatrix}\Big\rangle_{\hat K} & \text{if } i=1,\\ (\hat H\otimes\hat H)_{\mathbb{C}} & \text{if } i\le 0.\end{cases}$$

Now consider the subspace

$$\hat H' := \langle\begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}1&0\\0&z\end{pmatrix}\rangle_{\hat K} \subset \hat H\otimes_{\hat K}\hat H\,.$$

As the reader may verify for herself, this subspace is $\mathrm{GL}(\hat H)\times\mathrm{GL}(\hat H)$-semistable in the sense of Definition 1.13. On the other hand, the Hodge filtration makes this subspace non-semistable over $\mathbb{C}$, since $\mu_{F|\hat H'_{\mathbb{C}}}(\hat H'_{\mathbb{C}}) = \frac{3}{2}$ is greater than $\mu_F((\hat H\otimes_{\hat K}\hat H)_{\mathbb{C}}) = 1$. The universal Kempf filtration exists over $\hat K(\sqrt{z})$, where it is given by

$$\Phi^i\hat H\otimes_{\hat K}\hat K(\sqrt{z}) = \begin{cases} 0 & \text{if } i>1,\\ \langle\binom{1}{\sqrt{z}}\rangle_{\hat K(\sqrt{z})} & \text{if } 1\ge i>-1,\\ \hat H\otimes_{\hat K}\hat K(\sqrt{z}) & \text{if } i\le -1.\end{cases}$$

Note that this example also shows that in positive characteristic it is not possible to build a reasonable theory on the basis of Hodge filtrations instead of lattices. Namely, while $\hat H$ together with its Hodge filtration would constitute a semistable pure object in a suitable abelian category, its tensor square would not, because the subspace $\hat H'$ would violate the semistability condition.

6 Hodge groups: general properties

For any mixed $\hat K$-Hodge structure $\underline{\hat H}$ let $\langle\!\langle\underline{\hat H}\rangle\!\rangle$ denote the strictly full tannakian subcategory of $\mathscr{H}\!odge_{\hat K}$ which is generated by $\underline{\hat H}$, i.e. the smallest abelian full subcategory containing $\underline{\hat H}$ which is invariant by taking subquotients, tensor product, and duals. Let $\omega_{\underline{\hat H}} : \langle\!\langle\underline{\hat H}\rangle\!\rangle \to \mathscr{V}\!ec_{\hat K}$ denote the restriction to $\langle\!\langle\underline{\hat H}\rangle\!\rangle$ of the forgetful functor ω.

Definition 6.1 *The group* $G_{\underline{\hat H}} := \underline{\mathrm{Aut}}^{\otimes}(\omega_{\underline{\hat H}})$ *is called the* Hodge group of $\underline{\hat H}$.

The Hodge group may be viewed as an algebraic subgroup scheme of the general linear group $\mathrm{Aut}_{\hat K}(\hat H)$. By the fundamental theorem on neutral tannakian categories (Deligne-Milne [4] Th. 2.11, or Deligne [5] Th.1.12) the category $\langle\!\langle\underline{\hat H}\rangle\!\rangle$ is tensor equivalent to the category $\mathscr{R}\!ep_{G_{\underline{\hat H}}}$ of finite dimensional representations of $G_{\underline{\hat{H}}}$ over $\hat{K}$. Thus certain properties of mixed $\hat{K}$-Hodge structures correspond to properties of their associated Hodge groups. Some of these relations are discussed in the present section and the next. The theory is developed with an eye towards the problem of determining Hodge groups explicitly.

Basic properties: The first result concerns the abstract form of $G_{\underline{\hat{H}}}$:

Proposition 6.2 *The Hodge group $G_{\underline{\hat{H}}}$ is connected and reduced.*

Proof. It suffices to prove that any finite factor group scheme G_1 of $G_{\underline{\hat{H}}}$ is trivial. Consider a faithful representation of G_1 and let $\underline{\hat{H}}_1$ be its corresponding object of $\langle\!\langle \underline{\hat{H}} \rangle\!\rangle$. Then we have $G_{\underline{\hat{H}}_1} \cong G_1$ and $\langle\!\langle \underline{\hat{H}}_1 \rangle\!\rangle \cong \mathscr{R}ep_{G_1}$. As G_1 is a finite group scheme, there exists an object $\underline{\hat{H}}_2 \in \langle\!\langle \underline{\hat{H}}_1 \rangle\!\rangle$ such that any object of $\langle\!\langle \underline{\hat{H}}_1 \rangle\!\rangle$ is a quotient of $\underline{\hat{H}}_2^{\oplus n}$ for some n (see [4] Prop. 2.20). Choose integers $e_+ \geq e_-$ such that

$$(z-\zeta)^{e_+}\mathfrak{p}_{\hat{H}_2} \subset \mathfrak{q}_{\hat{H}_2} \subset (z-\zeta)^{e_-}\mathfrak{p}_{\hat{H}_2} .$$

Then for every $r \geq 0$ we have

$$(z-\zeta)^{e_+}\mathfrak{p}_{\hat{H}_1}^{\otimes r} \subset \mathfrak{q}_{\hat{H}_1}^{\otimes r} \subset (z-\zeta)^{e_-}\mathfrak{p}_{\hat{H}_1}^{\otimes r} ,$$

which is possible only if $\mathfrak{q}_{\hat{H}_1} = \mathfrak{p}_{\hat{H}_1}$. By a similar argument, or using the semistability condition, we deduce that $\underline{\hat{H}}_1$ is pure of weight 0. It follows that $\underline{\hat{H}}_1$ is a direct sum of copies of the unit object $\mathbb{1}_{\hat{K}}$, and hence $G_1 = G_{\underline{\hat{H}}_1} = 1$, as desired. **q.e.d.**

Functorial description of the lattice: The main data determining an object $\underline{\hat{H}}$ is its lattice $\mathfrak{q}_{\hat{H}}$. It can be described in terms of the Hodge group, as follows:

Proposition 6.3 *Consider any object $\underline{\hat{H}}$ of $\mathscr{H}odge_{\hat{K}}$ and let ρ denote the representation of $G_{\underline{\hat{H}}}$ on the underlying vector space $\hat{H}$.*

(a) There exists an element $\gamma \in G_{\underline{\hat{H}}}(\mathbb{C}(\!(z-\zeta)\!))$ such that $\mathfrak{q}_{\hat{H}} = \rho(\gamma)\mathfrak{p}_{\hat{H}}$.

(b) Consider any γ as in (a) and any object $\underline{\hat{H}}_1$ of $\langle\!\langle \underline{\hat{H}} \rangle\!\rangle$. Let ρ_1 denote the associated representation of $G_{\underline{\hat{H}}}$ on the underlying vector space $\hat{H}_1$. Then we have $\mathfrak{q}_{\hat{H}_1} = \rho_1(\gamma)\mathfrak{p}_{\hat{H}_1}$.

Proof. First we use general tannakian theory. Consider the fiber functor from $\langle\!\langle \underline{\hat{H}} \rangle\!\rangle$ to the category of finitely generated free $\mathbb{C}[\![z-\zeta]\!]$-modules which is given by $\underline{\hat{H}}_1 \mapsto \mathfrak{q}_{\hat{H}_1}$. By the fundamental theorem on tannakian categories (Deligne [5] Th.1.12) it becomes isomorphic to $\omega_{\underline{\hat{H}}}$ over some faithfully flat extension of $\mathbb{C}[\![z-\zeta]\!]$. Clearly such an isomorphism exists already over $\mathbb{C}[\![z-\zeta]\!]$, i.e., we have a functorial isomorphism

$$\gamma:\ \mathfrak{p}_{\hat{H}_1} = \hat{H}_1 \otimes_{\hat{K}} \mathbb{C}[\![z-\zeta]\!] \xrightarrow{\sim} \mathfrak{q}_{\hat{H}_1}$$

for all $\underline{\hat{H}}_1$ in $\langle\!\langle \underline{\hat{H}} \rangle\!\rangle$, which commutes with tensor product. Recall that we already have a tautological isomorphism of this form over $\mathbb{C}(\!(z-\zeta)\!)$. Thus we may interpret γ as an element of $\underline{\mathrm{Aut}}^{\otimes}(\omega_{\underline{\hat{H}}} \otimes_{\hat{K}} \mathbb{C}(\!(z-\zeta)\!)) = G_{\underline{\hat{H}}}(\mathbb{C}(\!(z-\zeta)\!))$. By construction this element enjoys property (b). This proves (a), but we still have to show that (b) holds for every other element γ as in (a).

So fix γ as in (a) and consider the collection of all $\underline{\hat{H}}_1$ which have the desired property. One easily finds that it is invariant under taking subquotients, direct sums, tensor product, and dualization, since all these constructions are equivariant under $G_{\underline{\hat{H}}}$. The collection also contains $\underline{\hat{H}}$ by assumption, and hence all objects of $\langle\!\langle \underline{\hat{H}} \rangle\!\rangle$, as desired. **q.e.d.**

Frobenius pullback: As earlier let $q \geq 1$ be a power of $\mathrm{char}(\hat{K})$. From Definition 5.1 and Proposition 5.5 it is clear that Frob_q^* defines a tensor functor from $\mathscr{H}\!\mathit{odge}_{\hat{K}}$ to itself.

Proposition 6.4 *For any object $\underline{\hat{H}}$ of $\mathscr{H}\!\mathit{odge}_{\hat{K}}$ the Hodge group $G_{\mathrm{Frob}_q^*\underline{\hat{H}}}$ is canonically isomorphic to* $\mathrm{Frob}_q^* G_{\underline{\hat{H}}} := G \times_{\hat{K},\mathrm{Frob}_q} \hat{K}$.

Proof. For any tannakian category $\mathscr{C}$ over a field F and any finite extension E/E there is a tannakian category $\mathscr{C} \otimes_F E$, as follows (compare Deligne [5] §5). An object of $\mathscr{C} \otimes_F E$ consists of an object X of $\mathscr{C}$ and a homomorphism of F-algebras $E \to \mathrm{End}_{\mathscr{C}}(X)$. Morphisms in this category are morphisms in $\mathscr{C}$ which commute with the additional E-action. There is an obvious E-linear tensor product in $\mathscr{C} \otimes_F E$ which makes this a tannakian category over E. Furthermore, any fiber functor $\omega : \mathscr{C} \to \mathscr{V}\!\mathit{ec}_F$ induces a fiber functor $\omega_E : \mathscr{C} \otimes_F E \longrightarrow \mathscr{V}\!\mathit{ec}_E$. Putting $G := \underline{\mathrm{Aut}}^{\otimes}(\omega)$ this construction yields an isomorphism

$$\mathscr{C} \otimes_F E \cong \mathscr{R}\!\mathit{ep}_G \otimes_F E \cong \mathscr{R}\!\mathit{ep}_{G \times_F E}\,.$$

Thus we obtain a canonical isomorphism $\underline{\mathrm{Aut}}^{\otimes}(\omega_E) \cong G \times_F E$.

Let us apply these general remarks to the category $\mathscr{C} := \langle\!\langle \underline{\hat{H}} \rangle\!\rangle$ and the embedding of fields $\mathrm{Frob}_q : \hat{K} \hookrightarrow \hat{K}$. The functor $\mathrm{Frob}_q^*|\langle\!\langle \underline{\hat{H}} \rangle\!\rangle$ factors through obvious tensor functors

$$\langle\!\langle \underline{\hat{H}} \rangle\!\rangle \longrightarrow \langle\!\langle \underline{\hat{H}} \rangle\!\rangle \otimes_{\hat{K},\mathrm{Frob}_q} \hat{K} \longrightarrow \langle\!\langle \mathrm{Frob}_q^* \underline{\hat{H}} \rangle\!\rangle\,.$$

It suffices to prove that the second functor is an equivalence of categories. From the definition of Frob_q^* it is clear that the functor is fully faithful and that its image is invariant under taking subquotients. Since the image contains the tensor generator $\mathrm{Frob}_q^* \underline{\hat{H}}$, the functor is essentially surjective, hence an equivalence of categories, as desired. **q.e.d.**

Cocharacters and $\mathbb{Z}$-gradings: Consider an algebraic group G over a field F. A *cocharacter of* G is a homomorphism of algebraic groups $\lambda : \mathbb{G}_{m,F} \to G$, where $\mathbb{G}_{m,F}$ denotes the multiplicative group over F. For any cocharacter λ and any algebraic representation V of G we have a natural $\mathbb{Z}$-grading $V = \bigoplus_{i \in \mathbb{Z}} V_i$, where V_i denotes the weight space of weight i under λ, that is, the subspace on which $\lambda(x)$ acts by multiplication with x^i for every $x \in F^{\times}$.

If λ is fixed, this grading is functorial in V and compatible with tensor products and duals. Conversely, suppose that for each V we are given a $\mathbb{Z}$-grading of V which is functorial in V and compatible with tensor products and duals. Then this data can be interpreted as a F-linear tensor functor $\mathscr{R}ep_G \to \mathscr{R}ep_{\mathbb{G}_{m,F}}$, so it comes from a unique cocharacter of G (compare [4] Example 2.30). In other words, the cocharacter and the associated grading determine each other.

Quasi-cocharacters and $\mathbb{Q}$-gradings: The following terminology extends this to arbitrary rational weights. For every integer $n > 0$ consider the group $G_n := \mathbb{G}_{m,F}$, and for any $n|n'$ consider the homomorphism $G_{n'} \to G_n$, $x \mapsto x^{n'/n}$. This defines an inverse system of linear algebraic groups, whose limit $\hat{\mathbb{G}}_{m,F} := \varprojlim G_n$ is the affine group scheme $\mathbf{Spec}\, F[X^r|_{r\in\mathbb{Q}}]$. A homomorphism of algebraic groups $\lambda : \hat{\mathbb{G}}_{m,F} \to G$ is called a *quasi-cocharacter* of G. Pulling back by the natural map $\hat{\mathbb{G}}_{m,F} \to \mathbb{G}_{m,F}$, any cocharacter can be viewed as a quasi-cocharacter. Conversely, any quasi-cocharacter factors through some G_n, so it can be viewed as an n^{th} root of a usual cocharacter.

Most properties of cocharacters extend naturally to quasi-cocharacters. For instance, the above correspondence between cocharacters and compatible systems of $\mathbb{Z}$-gradings extends in a natural way to a correspondence between quasi-cocharacters and compatible systems of $\mathbb{Q}$-gradings. Namely, if some positive power λ^n of a quasi-cocharacter λ is a usual cocharacter, the weight space of weight $i \in \mathbb{Q}$ for λ is just the weight space of weight ni for λ^n.

Weight (quasi-)cocharacter: Consider a mixed $\hat{K}$-Hodge structure $\underline{\hat{H}}$ which is isomorphic to the direct sum of its pure constituents $\mathrm{Gr}^W \underline{\hat{H}}$. Then this isomorphism is canonical, that is, the weight filtration of the underlying vector space $\hat{H}$ is split by a canonical $\mathbb{Q}$-grading. The same properties hold for any object of $\langle\!\langle \underline{\hat{H}} \rangle\!\rangle$, and the resulting gradings are functorial and compatible with tensor products and duals. Therefore the grading of $\hat{H}$ corresponds to a unique quasi-cocharacter of the Hodge group $w_{\underline{\hat{H}}} : \hat{\mathbb{G}}_{m,\hat{K}} \to G_{\underline{\hat{H}}}$. We shall call it the *weight quasi-cocharacter of* $G_{\underline{\hat{H}}}$, or *weight cocharacter* for short. Clearly its image is contained in the center of $G_{\underline{\hat{H}}}$.

Now let $\underline{\hat{H}}$ be an arbitrary object of $\mathscr{H}odge_{\hat{K}}$. Then the image of $G_{\underline{\hat{H}}}$ in its representation on $\mathrm{Gr}^W \hat{H}$ is just the Hodge group $G_{\mathrm{Gr}^W \underline{\hat{H}}}$. Let $U^-_{\underline{\hat{H}}}$ denote the kernel of the epimorphism $G_{\underline{\hat{H}}} \twoheadrightarrow G_{\mathrm{Gr}^W \underline{\hat{H}}}$. Clearly this is a unipotent group.

Proposition 6.5 *(a) The weight quasi-cocharacter of $G_{\mathrm{Gr}^W \underline{\hat{H}}}$ lifts to a quasi-cocharacter $w_{\underline{\hat{H}}} : \hat{\mathbb{G}}_{m,\hat{K}} \to G_{\underline{\hat{H}}}$, called* weight (quasi-)cocharacter of $G_{\underline{\hat{H}}}$.

(b) This lift is unique up to conjugation by $U^-_{\underline{\hat{H}}}(\hat{K})$.

(c) The weight cocharacter acts by non-positive weights on the Lie algebra $\mathrm{Lie}\, G_{\underline{\hat{H}}}$. *Its weights on* $\mathrm{Lie}\, U^-_{\underline{\hat{H}}}$ *and* $\mathrm{Lie}\, G_{\mathrm{Gr}^W \underline{\hat{H}}}$ *are negative, resp. zero.*

Proof. Assertion (c) is clear by construction, once the other statements have been proved. For these choose $n \geq 1$ such that $w^n_{\mathrm{Gr}^W \underline{\hat{H}}}$ is an honest cocharacter $\mathbb{G}_{m,\hat{K}} \longrightarrow G_{\mathrm{Gr}^W \underline{\hat{H}}}$. Define an algebraic group P by pullback so that we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
1 & \longrightarrow & U^-_{\underline{\hat{H}}} & \longrightarrow & G_{\underline{\hat{H}}} & \longrightarrow & G_{\mathrm{Gr}^W \underline{\hat{H}}} & \longrightarrow & 1 \\
 & & \Big\| & & \Big\uparrow & \square & \Big\uparrow {\scriptstyle w^n_{\mathrm{Gr}^W \underline{\hat{H}}}} & & \\
1 & \longrightarrow & U^-_{\underline{\hat{H}}} & \longrightarrow & P & \longrightarrow & \mathbb{G}_{m,\hat{K}} & \longrightarrow & 1
\end{array}$$

The problem then amounts to the existence and uniqueness up to conjugation of a Levi decomposition of P over $\hat{K}$. Since P is solvable, these properties are guaranteed by Borel [2] Thm. 18.2 (i) and Thm. 19.2, respectively. **q.e.d.**

Polygons: In the following by a polygon we shall mean the graph in $\mathbb{R}^2$ of a piecewise linear convex function $[0, n] \to \mathbb{R}$ for some integer n which starts at $(0, 0)$. All slopes of this function are assumed to be rational numbers, and the length of the subinterval on which the function has a given slope $i \in \mathbb{Q}$ is assumed to be an integer, called the *multiplicity of* i. The convexity means that the slopes are arranged in ascending order. The starting point $(0, 0)$, the endpoint, and any point where the slope changes is called a *break point* of the polygon.

To any finite dimensional $\mathbb{Q}$-graded vector space V over a field F is associated the unique polygon for which the multiplicity of each slope $i \in \mathbb{Q}$ is equal to $\dim_F(V_i)$. Given a linear algebraic group G and a quasi-cocharacter $\hat{\mathbb{G}}_{m,F} \to G$, we have a whole collection of polygons associated to the induced gradings on every finite dimensional representation of G. Note that these polygons do not change under semisimplification; one could say: they are additive in short exact sequences. This fact is a fundamental restriction on the possible collections of polygons which come from a cocharacter of G.

Given two polygons P and Q of the same length n, we say that P *lies above* Q if and only if P is on or above Q at every point of the interval of definition $[0, n]$. We say that P *lies strictly above* Q if and only if in addition they meet at most at the starting point and the endpoint.

Weight polygon: For any mixed $\hat{K}$-pre-Hodge structure $\underline{\hat{H}}$, not necessarily semistable, the polygon determined by the weight filtration is called the *weight polygon of* $\underline{\hat{H}}$. It has the following basic property, which is a direct consequence of the definition of strictness:

Proposition 6.6 *For any strict exact sequence of mixed* $\hat{K}$*-pre-Hodge structures* $0 \to \underline{\hat{H}}' \to \underline{\hat{H}} \to \underline{\hat{H}}'' \to 0$ *the weight polygons of* $\underline{\hat{H}}$ *and* $\underline{\hat{H}}' \oplus \underline{\hat{H}}''$ *coincide.*

When $\underline{\hat{H}}$ is semistable, any choice of weight cocharacter $w_{\underline{\hat{H}}}$ determines a splitting of the weight filtration, and its associated polygon is equal to the weight polygon of $\underline{\hat{H}}$. Thus for objects of $\mathscr{H}\!odge_{\hat{K}}$ the additivity assertion 6.6 also reflects the existence of the weight cocharacter. We shall next look at the state of affairs for the Hodge filtration.

Hodge polygon: The polygon determined by the Hodge filtration of $\hat{H}_{\mathbb{C}}$ is called the *Hodge polygon of* $\underline{\hat{H}}$. Its slopes can be viewed equivalently as the elementary divisors relating the lattices $\mathfrak{p}_{\hat{H}}$ and $\mathfrak{q}_{\hat{H}}$. The equivalent conditions for semistability from Proposition 4.4 can be rephrased once more in terms of polygons (compare Fontaine [8] Prop. 4.3.3):

Proposition 6.7 *A mixed $\hat{K}$-pre-Hodge structure $\underline{\hat{H}}$ is semistable if and only if*

(c) For every subobject $\underline{\hat{H}}' \subset \underline{\hat{H}}$ the weight polygon is above the Hodge polygon, and the endpoints coincide whenever $\underline{\hat{H}}' = W_\mu \underline{\hat{H}}$ for some $\mu \in \mathbb{Q}$.

Proof. The part of (c) relating to the endpoints is just the condition (a) of Proposition 4.4. Thus it remains to prove the first part of (c) under the assumption that the assertion for the endpoints is true for all subobjects $\underline{\hat{H}}_1 \subset \underline{\hat{H}}$. Assume that (c) fails, and let $s > 0$ be the smallest integer where the Hodge polygon of $\underline{\hat{H}}'$ is strictly above its weight polygon. If μ is the last slope of its weight polygon to the left of the point s, the last slope of its Hodge polygon must be greater than μ. Suppose that the slope μ of the weight polygon extends until the point $r \geq s$ but not further. Then the Hodge polygon is strictly above the weight polygon at the point r. Now r is just the rank of $\underline{\hat{H}}_1' := W_\mu \underline{\hat{H}}'$, whose weight polygon is an initial segment of the weight polygon of $\underline{\hat{H}}'$. The smallest Hodge slopes of $\underline{\hat{H}}_1'$ are $\geq$ the smallest Hodge slopes of $\underline{\hat{H}}'$, so the Hodge polygon of $\underline{\hat{H}}_1'$ is above the corresponding initial segment of the Hodge polygon of $\underline{\hat{H}}'$. It follows that the Hodge polygon of $\underline{\hat{H}}_1'$ is strictly above its weight polygon at their endpoint r. This contradicts assertion 4.4 (a), as desired. **q.e.d.**

By construction the slopes of the Hodge polygon are integers. This implies:

Proposition 6.8 *If $\underline{\hat{H}}$ is semistable, all break points of its weight polygon have integral coordinates.*

Proof. Any initial segment of the weight polygon of $\underline{\hat{H}}$ which ends in a break point is the weight polygon of some $W_\mu \underline{\hat{H}}$. By Proposition 6.7 this break point is the endpoint of some Hodge polygon and therefore has integral coordinates. **q.e.d.**

Next consider again the strict exact sequence 4.10.

Proposition 6.9 *The Hodge polygon of $\underline{\hat{H}}' \oplus \underline{\hat{H}}''$ lies above that of $\underline{\hat{H}}$ and has the same end-point.*

Proof. This assertion is similar to that of Katz [19] Lemma 1.2.3. To prove it consider a point (r, e) on the Hodge polygon of $\underline{\hat{H}}$ with $r \in \mathbb{Z}$. Then e is the smallest slope of the Hodge polygon of $\Lambda^r \underline{\hat{H}}$. By the definition of the Hodge filtration this implies

$$(z - \zeta)^{-e} \mathfrak{p}_{\Lambda^r \hat{H}} \subset \mathfrak{q}_{\Lambda^r \hat{H}} \ . \tag{6.10}$$

Observe that this property is inherited by any strict subquotient of $\Lambda^r \underline{\hat{H}}$. Now the given exact sequence induces a canonical filtration of $\Lambda^r \underline{\hat{H}}$, whose associated graded object is isomorphic to

$$\bigoplus_{r'+r''=r} \Lambda^{r'} \underline{\hat{H}}' \otimes \Lambda^{r''} \underline{\hat{H}}'' \cong \Lambda^r (\underline{\hat{H}}' \otimes \underline{\hat{H}}'') \ .$$

(Compare Proposition 5.8 and its proof.) It follows that the analogue of Formula 6.10 holds for $\Lambda^r(\underline{\hat{H}}' \otimes \underline{\hat{H}}'')$ in place of $\Lambda^r \underline{\hat{H}}$. In other words the smallest slope of the Hodge polygon of the former is $\geq e$, hence the Hodge polygon of $\underline{\hat{H}}' \otimes \underline{\hat{H}}''$ lies on or above the point (r, e), as desired. **q.e.d.**

Proposition 6.11 *The following statements are equivalent:*

(a) The Hodge polygons of $\underline{\hat{H}}$ *and* $\underline{\hat{H}}' \oplus \underline{\hat{H}}''$ *coincide.*

(b) The injection $\hat{H}'_{\mathbb{C}} \hookrightarrow \hat{H}_{\mathbb{C}}$ *is strictly compatible with the Hodge filtrations.*

(c) The surjection $\hat{H}_{\mathbb{C}} \twoheadrightarrow \hat{H}''_{\mathbb{C}}$ *is strictly compatible with the Hodge filtrations.*

(d) For every $i \in \mathbb{Z}$ *the following sequence is exact:*

$$0 \to F^i \hat{H}'_{\mathbb{C}} \to F^i \hat{H}_{\mathbb{C}} \to F^i \hat{H}''_{\mathbb{C}} \to 0 \ .$$

(e) For every $i \in \mathbb{Z}$ *the following sequence is exact:*

$$0 \to \mathrm{Gr}^i_F \hat{H}'_{\mathbb{C}} \to \mathrm{Gr}^i_F \hat{H}_{\mathbb{C}} \to \mathrm{Gr}^i_F \hat{H}''_{\mathbb{C}} \to 0 \ .$$

Proof. First we show the equivalence (b)⇔(c). To clarify notations let us distinguish the Hodge filtrations of $\hat{H}_{\mathbb{C}}$, $\hat{H}'_{\mathbb{C}}$, $\hat{H}''_{\mathbb{C}}$ as F, F', F'', respectively. The functoriality of the Hodge filtration implies

$$\deg_{F'}(\hat{H}'_{\mathbb{C}}) \leq \deg_{F|\hat{H}'_{\mathbb{C}}}(\hat{H}'_{\mathbb{C}}) \ , \tag{6.12}$$

with equality if and only if $F' = F|\hat{H}'_{\mathbb{C}}$, i.e. if the embedding $\hat{H}'_{\mathbb{C}} \hookrightarrow \hat{H}_{\mathbb{C}}$ is strictly compatible with the Hodge filtrations. The analogous remark applies to the inequality

$$\deg_{F''}(\hat{H}''_{\mathbb{C}}) \geq \deg_{F|\hat{H}''_{\mathbb{C}}}(\hat{H}''_{\mathbb{C}}) \ . \tag{6.13}$$

Now the calculation

$$\begin{aligned} \deg_{F'}(\hat{H}'_{\mathbb{C}}) + \deg_{F''}(\hat{H}''_{\mathbb{C}}) &= \deg_{\mathfrak{q}}(\underline{\hat{H}}') + \deg_{\mathfrak{q}}(\underline{\hat{H}}'') \\ &= \deg_{\mathfrak{q}}(\underline{\hat{H}}) \\ &= \deg_F(\hat{H}_{\mathbb{C}}) \\ &= \deg_{F|\hat{H}'_{\mathbb{C}}}(\hat{H}'_{\mathbb{C}}) + \deg_{F|\hat{H}''_{\mathbb{C}}}(\hat{H}''_{\mathbb{C}}) \end{aligned}$$

shows that 6.12 is an equality if and only if 6.13 is one. This implies the desired equivalence (b)⇔(c).

Next, (d) is equivalent to the conjunction of (b) and (c). The equivalence (d)⇔(e) follows by induction on i using the 3×3-lemma. The direction (e)⇒(a) is obvious, so to finish the proof it suffices to show the implication (a)⇒(d). For this we consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{p}_{\hat{H}'} \cap (z-\zeta)^{i-1}\mathfrak{q}_{\hat{H}'} & \longrightarrow & \mathfrak{p}_{\hat{H}} \cap (z-\zeta)^{i-1}\mathfrak{q}_{\hat{H}} & \longrightarrow & \mathfrak{p}_{\hat{H}''} \cap (z-\zeta)^{i-1}\mathfrak{q}_{\hat{H}''} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (z-\zeta)} & & \downarrow{\scriptstyle (z-\zeta)} & & \downarrow{\scriptstyle (z-\zeta)} & & \\
0 & \longrightarrow & \mathfrak{p}_{\hat{H}'} \cap (z-\zeta)^{i}\mathfrak{q}_{\hat{H}'} & \longrightarrow & \mathfrak{p}_{\hat{H}} \cap (z-\zeta)^{i}\mathfrak{q}_{\hat{H}} & \longrightarrow & \mathfrak{p}_{\hat{H}''} \cap (z-\zeta)^{i}\mathfrak{q}_{\hat{H}''} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F^i \hat{H}'_{\mathbb{C}} & \longrightarrow & F^i \hat{H}_{\mathbb{C}} & \longrightarrow & F^i \hat{H}''_{\mathbb{C}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Its columns are exact by construction, and we shall use downward induction on i to show that its rows are exact. For $i \gg 0$ the middle row is equal to

$$0 \to (z-\zeta)^i \mathfrak{q}_{\hat{H}'} \to (z-\zeta)^i \mathfrak{q}_{\hat{H}} \to (z-\zeta)^i \mathfrak{q}_{\hat{H}''} \to 0\ ,$$

which is exact by strictness. So for the induction step we may assume that the middle row is exact. In the bottom row we always have $F^i \hat{H}'_{\mathbb{C}} \subset F^i \hat{H}_{\mathbb{C}}$. On the other hand, the exactness of the middle row implies the surjectivity of the map $F^i \hat{H}_{\mathbb{C}} \to F^i \hat{H}''_{\mathbb{C}}$. Furthermore, the assumption (a) implies that $\dim(F^i \hat{H}_{\mathbb{C}}) = \dim(F^i \hat{H}'_{\mathbb{C}}) + \dim(F^i \hat{H}''_{\mathbb{C}})$. Thus the bottom row is exact. Now the 3×3-lemma implies the exactness of the top row, thus finishing the proof.

q.e.d.

Non-exactness of the Hodge filtration: Both F^i and Gr^i_F define additive functors $\mathscr{H}\!odge_{\hat{K}} \to \mathscr{V}\!ec_{\mathbb{C}}$. But they are not exact. As the following example shows, such non-exactness occurs when the extension is in some sense too non-trivial on the lattices $\mathfrak{q}$. By restricting the permitted extensions one can expect to obtain an exact functor on a suitable subcategory. In the next section we show that there is a unique largest strictly full tannakian subcategory $\mathscr{H}\!odge^{\mathrm{sha}}_{\hat{K}}$ of $\mathscr{H}\!odge_{\hat{K}}$ on which all functors F^i and Gr^i_F are exact. On this subcategory we then have a theory of cocharacters whose numerics are based on Hodge polygons, just as for the weight cocharacter.

Example 6.14 Put $\hat{H} := \hat{K}^2$, make it pure of weight 0, and let

$$\mathfrak{q}_{\hat{H}} := \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} (z-\zeta)^{-e} \\ 1 \end{pmatrix} \right\rangle_{\mathbb{C}[\![z-\zeta]\!]} .$$

This defines a pure $\hat{K}$-pre-Hodge structure $\underline{\hat{H}}$ of weight 0. Clearly the strict subobject $\underline{\hat{H}}'$ with underlying vector space $\langle \binom{1}{0} \rangle_{\hat{K}}$ is isomorphic to the unit object $\mathbb{1}_{\hat{K}}$, and so is the factor object $\underline{\hat{H}}/\underline{\hat{H}}'$. Thus using Corollary 4.12 we deduce that $\underline{\hat{H}}$ is semistable. Assume $e > 0$, then the Hodge filtration comes out to be

$$F^i\hat{H}_{\mathbb{C}} = \begin{cases} 0 & \text{if } i > e, \\ \hat{H}'_{\mathbb{C}} & \text{if } e \geq i > -e, \\ \hat{H}_{\mathbb{C}} & \text{if } -e \geq i. \end{cases}$$

Since the Hodge filtration for $\underline{\hat{H}}'$ is trivial, it follows that the embedding $\hat{H}'_{\mathbb{C}} \hookrightarrow \hat{H}_{\mathbb{C}}$ is not strictly compatible with the Hodge filtrations. The equivalent condition of Proposition 6.11 fail here, hence the functors F^i and Gr^i_F are not exact. Note that the Hodge group of this $\underline{\hat{H}}$ is non-trivial unipotent.

7 Hodge groups and Hodge cocharacters

Throughout this section we work inside the tannakian category $\mathscr{H}\!odge_{\hat{K}}$, i.e. any object $\underline{\hat{H}}$ is assumed semistable. Its semisimplification will be denoted $\underline{\hat{H}}^{\mathrm{ss}}$.

Definition 7.1 *(a) $\underline{\hat{H}}$ is called* Hodge additive *if and only if its Hodge polygon coincides with that of $\underline{\hat{H}}^{\mathrm{ss}}$.*

(b) $\underline{\hat{H}}$ is called strongly Hodge additive *if and only if every object $\underline{\hat{H}}_1$ of $\langle\!\langle \underline{\hat{H}} \rangle\!\rangle$ is Hodge additive.*

Clearly $\underline{\hat{H}}$ is strongly Hodge additive if and only if the equivalent conditions in Proposition 6.11 are satisfied for any short exact sequence in $\langle\!\langle \underline{\hat{H}} \rangle\!\rangle$. Our first aim is to characterize this property in a simpler way.

Proposition 7.2 *$\underline{\hat{H}}$ is strongly Hodge additive if and only if $\underline{\hat{H}}$ is Hodge additive and $\underline{\hat{H}}^{\mathrm{ss}}$ strongly Hodge additive.*

Proof. The "only if" part follows at once from the inclusion $\langle\!\langle \underline{\hat{H}}^{\mathrm{ss}} \rangle\!\rangle \subset \langle\!\langle \underline{\hat{H}} \rangle\!\rangle$. For the "if" part note that any object of $\langle\!\langle \underline{\hat{H}} \rangle\!\rangle$ is a subquotient of an object of the form

$$\underline{\hat{H}}^{(\underline{m},\underline{n})} := \bigoplus_{i=1}^{r} \underline{\hat{H}}^{\otimes m_i} \otimes (\underline{\hat{H}}^{\vee})^{\otimes n_i} .$$

It suffices to show that $\underline{\hat{H}}^{(\underline{m},\underline{n})}$ is Hodge additive, since this property is clearly inherited by subquotients. Now the Hodge polygon of any tensor product $\underline{\hat{H}}_1 \otimes \underline{\hat{H}}_2$ depends only on the Hodge polygons of the factors; more precisely: its slopes are all sums of a slope of one polygon with a slope of the other, taken with the appropriate multiplicities. Thus if $\underline{\hat{H}}$ is Hodge additive, the Hodge polygon of $\underline{\hat{H}}^{(\underline{m},\underline{n})}$ does not change when $\underline{\hat{H}}$ is replaced by $\underline{\hat{H}}^{\mathrm{ss}}$. If, moreover, $\underline{\hat{H}}^{\mathrm{ss}}$ is strongly Hodge additive, it follows that $\underline{\hat{H}}^{(\underline{m},\underline{n})}$ is Hodge additive, as desired. **q.e.d.**

Next recall that a linear algebraic group over an algebraically closed field is called *reductive* if and only if its unipotent radical is trivial. For example, any algebraic group possessing a faithful semisimple representation is reductive. A linear algebraic group G over an arbitrary field F is reductive if and only if $G_{\bar{F}}$ is reductive.

Proposition 7.3 *If $G_{\underline{\hat{H}}}$ is reductive, then $\underline{\hat{H}}$ is strongly Hodge additive.*

Proof. Using the formulation 6.11 (b) we must prove that for any pair of objects $\underline{\hat{H}}_2 \subset \underline{\hat{H}}_1$ in $\langle\!\langle \underline{\hat{H}} \rangle\!\rangle$ the injection $\hat{H}_{2,\mathbb{C}} \hookrightarrow \hat{H}_{1,\mathbb{C}}$ is strictly compatible with the Hodge filtrations. One easily checks that this is equivalent to the strictness of $\Lambda^r \hat{H}_{2,\mathbb{C}} \hookrightarrow \Lambda^r \hat{H}_{1,\mathbb{C}}$, where $r := \mathrm{rank}(\underline{\hat{H}}_2)$. Since the Hodge filtration commutes with Λ^r, we may replace the objects $\underline{\hat{H}}_2 \subset \underline{\hat{H}}_1$ by $\Lambda^r \underline{\hat{H}}_2 \subset \Lambda^r \underline{\hat{H}}_1$. Thus without loss of generality we may assume $\mathrm{rank}(\underline{\hat{H}}_2) = 1$. Next the desired assertion is invariant under tensoring with objects of rank 1. Thus after tensoring with $\underline{\hat{H}}_2^\vee$ we may assume $\underline{\hat{H}}_2 = \underline{\mathbb{1}}_{\hat{K}}$. Now $\underline{\hat{H}}_1$ corresponds to a representation of the reductive group $G_{\underline{\hat{H}}}$ and $\underline{\hat{H}}_2$ to a line of $G_{\underline{\hat{H}}}$-invariants. Thus by geometric reductivity (see Mumford-Fogarty [22] Appendix 1) there exists an integer $m \geq 1$ so that $\mathrm{Sym}^m \underline{\mathbb{1}}_{\hat{K}} \subset \mathrm{Sym}^m \underline{\hat{H}}_1$ is a direct factor. It follows that the associated embedding $\mathrm{Sym}^m \mathbb{C} \hookrightarrow \mathrm{Sym}^m \hat{H}_{1,\mathbb{C}}$ is strictly compatible with the Hodge filtrations. Therefore the original embedding $\mathbb{C} \hookrightarrow \hat{H}_{1,\mathbb{C}}$ is strictly compatible with the Hodge filtrations, as desired. **q.e.d.**

Corollary 7.4 *Assume that* $\mathrm{char}(\hat{K}) = 0$. *Then the properties "Hodge additive" and "strongly Hodge additive" are equivalent.*

Proof. Since $\underline{\hat{H}}^{\mathrm{ss}}$ corresponds to a faithful semisimple representation of $G_{\underline{\hat{H}}^{\mathrm{ss}}}$ in characteristic zero, this group is reductive. By Proposition 7.3 (or just linear reductivity) $\underline{\hat{H}}^{\mathrm{ss}}$ is always strongly Hodge additive. The desired equivalence now follows from Proposition 7.2. **q.e.d.**

Arbitrary characteristic: In positive characteristic we cannot argue directly like this, because the Hodge group of a semisimple object is not necessarily reductive. We shall reduce ourselves to a reductive Hodge group via Frobenius pullback.

Proposition 7.5 *Let* $q \geq 1$ *be a power of* $\mathrm{char}(\hat{K})$. *Then* $\underline{\hat{H}}$ *is strongly Hodge additive if and only if* $\mathrm{Frob}_q^* \underline{\hat{H}}$ *is strongly Hodge additive.*

Proof. Consider the map

$$\mathrm{Frob}_q^* \hat{H} = \hat{H} \otimes_{\hat{K},\mathrm{Frob}_q} \hat{K} \longrightarrow \mathrm{Sym}^q \hat{H}, \;\; h \otimes x \mapsto x \cdot h^q \,.$$

From Definition 5.1 one easily checks that it defines an embedding $\mathrm{Frob}_q^* \underline{\hat{H}} \hookrightarrow \mathrm{Sym}^q \underline{\hat{H}}$. It follows that $\langle\!\langle \mathrm{Frob}_q^* \underline{\hat{H}} \rangle\!\rangle \subset \langle\!\langle \underline{\hat{H}} \rangle\!\rangle$, which implies the "only if" part. For the "if" part suppose that $\mathrm{Frob}_q^* \underline{\hat{H}}$ is strongly Hodge additive and consider any object $\underline{\hat{H}}_1$ of $\langle\!\langle \underline{\hat{H}} \rangle\!\rangle$. Then $\mathrm{Frob}_q^* \underline{\hat{H}}_1$ is an object of $\langle\!\langle \mathrm{Frob}_q^* \underline{\hat{H}} \rangle\!\rangle$ and hence Hodge additive. Definition 5.1 implies that the Hodge polygon of $\mathrm{Frob}_q^* \underline{\hat{H}}_1$ is obtained from the corresponding Hodge polygon of $\underline{\hat{H}}_1$ by scaling in the vertical direction by the factor q. In particular, these Hodge polygons determine each other. Since $\mathrm{Frob}_q^* (\underline{\hat{H}}_1^{\mathrm{ss}})$ is a partial semisimplification of $\mathrm{Frob}_q^* \underline{\hat{H}}_1$ and the latter is Hodge additive, their Hodge polygons coincide. It follows that the Hodge polygons of $\underline{\hat{H}}_1^{\mathrm{ss}}$ and $\underline{\hat{H}}_1$ coincide, as desired. **q.e.d.**

Next note the following lemma:

Lemma 7.6 *Let $q \geq 1$ be any power of* $\mathrm{char}(\hat{K})$ *and* $\underline{\hat{H}}$ *arbitrary. Then the following assertions are equivalent:*

(a) The Hodge polygons of $\mathrm{Frob}_q^*\mathrm{Frob}_{q,*}\underline{\hat{H}}$ *and* $\underline{\hat{H}}^{\oplus q}$ *coincide.*

(b) All slopes of the Hodge polygon of $\underline{\hat{H}}$ *are multiples of* q.

Proof. In the preceding proof we have seen that all slopes of the Hodge polygon of $\mathrm{Frob}_q^*\mathrm{Frob}_{q,*}\underline{\hat{H}}$ are multiples of q. By Proposition 5.3 (b) this implies the direction (a)⇒(b). For the converse note that by the elementary divisor theorem there exists a $\mathbb{C}[[z-\zeta]]$-basis $(h_i)_{1\leq i\leq r}$ of $\mathfrak{p}_{\hat{H}}$ such that $\mathfrak{q}_{\hat{H}}$ is generated by the elements $(z-\zeta)^{e_i}\cdot h_i$. Here the e_i are the slopes of the Hodge polygon of $\underline{\hat{H}}$, so we assume that they are multiples of q. Now Definitions 5.1 and 5.2 show that the lattices associated to $\mathrm{Frob}_q^*\mathrm{Frob}_{q,*}\underline{\hat{H}}$ are obtained from those of $\underline{\hat{H}}$ by applying $(\ )\otimes_{\mathbb{C}[[(z-\zeta)^q]]}\mathbb{C}[[z-\zeta]]$, where the new $\mathbb{C}[[z-\zeta]]$-module structure comes from the second factor. Since q divides e_i, the calculation

$$\Big((z-\zeta)^{e_i}\cdot\mathbb{C}[[z-\zeta]]\Big)\underset{\mathbb{C}[[(z-\zeta)^q]]}{\otimes}\mathbb{C}[[z-\zeta]] = \mathbb{C}[[z-\zeta]]\underset{\mathbb{C}[[(z-\zeta)^q]]}{\otimes}\Big((z-\zeta)^{e_i}\cdot\mathbb{C}[[z-\zeta]]\Big)$$

shows that each slope e_i for $\underline{\hat{H}}$ yields q copies of slope e_i for $\mathrm{Frob}_q^*\mathrm{Frob}_{q,*}\underline{\hat{H}}$. Thus the Hodge polygons of $\mathrm{Frob}_q^*\mathrm{Frob}_{q,*}\underline{\hat{H}}$ and $\underline{\hat{H}}^{\oplus q}$ coincide, as desired.

q.e.d.

As an example let us consider the special object $\underline{\hat{H}}$ from 5.16. It is isomorphic to $\mathrm{Frob}_{2,*}\underline{\hat{H}}'$, where $\underline{\hat{H}}'$ is of rank 1, pure of weight 1, and has $\mathfrak{q}_{\hat{H}'} := (z-\zeta)^{-1}\cdot\mathfrak{p}_{\hat{H}'}$. Its single Hodge slope is 1, so $\mathrm{Frob}_2^*\underline{\hat{H}}$ is not Hodge additive by Lemma 7.6. In particular we see that the property "Hodge additive" is not invariant under Frobenius pullback.

Let us now have a closer look at an arbitrary simple object $\underline{\hat{H}}_1$. Its endomorphism ring $\mathrm{End}(\underline{\hat{H}}_1)$ is a division algebra of finite dimension over $\hat{K}$. Such an algebra is called *separable over* $\hat{K}$ if and only if its center is a separable extension of $\hat{K}$. The property "semisimple and separable $\hat{K}$-algebra" is invariant under base extension to any overfield of $\hat{K}$.

Let C_1 denote the center of $\mathrm{End}(\underline{\hat{H}}_1)$, and let q_1 be the degree of the purely inseparable part of the extension $C_1/\hat{K}$. Identifying $\hat{K}$ with a subextension of C_1 via the Frobenius map Frob_{q_1}, we can write $\underline{\hat{H}}_1 = \mathrm{Frob}_{q_1,*}\underline{\hat{H}}_1'$ for some simple object $\underline{\hat{H}}_1'$ whose endomorphism ring is automatically separable over $\hat{K}$.

Definition 7.7 *Let* $\underline{\hat{H}}_1$, q_1, *and* $\underline{\hat{H}}_1'$ *be as above. We call* $\underline{\hat{H}}_1$ quasi-separable *if and only if all slopes of the Hodge polygon of* $\underline{\hat{H}}_1'$ *are multiples of* q_1.

Theorem 7.8 *The following assertions are equivalent:*

(a) $\underline{\hat{H}}$ *is strongly Hodge additive.*

(b) $\mathrm{Frob}_q^*\underline{\hat{H}}$ *is Hodge additive for every $q \geq 1$ that is a power of* $\mathrm{char}(\hat{K})$.

(c) $\underline{\hat{H}}$ *is Hodge additive and every simple subfactor is quasi-separable.*

Proof. First we prove the equivalence (b)⇔(c). Since the semisimplifications of $\mathrm{Frob}_q^*\underline{\hat{H}}$ and $\mathrm{Frob}_q^*(\underline{\hat{H}}^{\mathrm{ss}})$ coincide, this reduces at once to the case that $\underline{\hat{H}}$ is simple. Write $\underline{\hat{H}} = \underline{\hat{H}}_1$ with q_1 and $\underline{\hat{H}}_1'$ as in 7.7. The implication (b)⇒(c) then follows at once from Lemma 7.6, applied to $\underline{\hat{H}} := \underline{\hat{H}}_1'$ and $q := q_1$. Conversely, if (c) holds, then $\mathrm{Frob}_{q_1}^*\underline{\hat{H}}_1$ is Hodge additive and a successive extension of copies of $\underline{\hat{H}}_1'$. Since $\mathrm{End}(\underline{\hat{H}}_1')$ is separable over $\hat{K}$, the Hodge group of $\underline{\hat{H}}_1'$ is reductive, so Proposition 7.3 implies that $\underline{\hat{H}}_1'$ is strongly Hodge additive. It follows that $\mathrm{Frob}_{q_1}^*\underline{\hat{H}}_1$ is strongly Hodge additive. Now Proposition7.5 implies that $\mathrm{Frob}_q^*\underline{\hat{H}}_1$ is strongly Hodge additive for every q, whence (b).

The implication (a)⇒(b) is part of Proposition 7.5. Conversely assume (b) and let q be a sufficiently large power of $\mathrm{char}(\hat{K})$, so that the endomorphism ring of every simple subfactor of $\mathrm{Frob}_q^*\underline{\hat{H}}$ is separable over $\hat{K}$. Then the Hodge group of $(\mathrm{Frob}_q^*\underline{\hat{H}})^{\mathrm{ss}}$ is reductive, so this object is strongly Hodge additive by Proposition 7.3. Since $\mathrm{Frob}_q^*\underline{\hat{H}}$ is Hodge additive by assumption, Proposition 7.2 implies that $\mathrm{Frob}_q^*\underline{\hat{H}}$ is strongly Hodge additive. Using Proposition 7.5 this shows that $\underline{\hat{H}}$ is strongly Hodge additive, as desired. **q.e.d.**

The tannakian category: Now we can prove:

Theorem 7.9 *The strongly Hodge additive objects form a strictly full tannakian subcategory of $\mathscr{H}\!odge_{\hat{K}}$, that is, it is abelian and invariant under subquotients, tensor product, and dual. This category is denoted $\mathscr{H}\!odge_{\hat{K}}^{\mathrm{sha}}$.*

Proof. The definition of "strongly Hodge additive" implies that the subcategory $\mathscr{H}\!odge_{\hat{K}}^{\mathrm{sha}}$ is invariant under subquotients, tensor powers, and dual. Theorem 7.8 implies that it is also abelian; in particular, for any $\underline{\hat{H}}_1$, $\underline{\hat{H}}_2$ in $\mathscr{H}\!odge_{\hat{K}}^{\mathrm{sha}}$ we have $\underline{\hat{H}}_1 \oplus \underline{\hat{H}}_2$ in $\mathscr{H}\!odge_{\hat{K}}^{\mathrm{sha}}$. Since $\underline{\hat{H}}_1 \otimes \underline{\hat{H}}_2$ is a subquotient of $(\underline{\hat{H}}_1 \oplus \underline{\hat{H}}_2)^{\otimes 2}$, it is again in $\mathscr{H}\!odge_{\hat{K}}^{\mathrm{sha}}$. Thus $\mathscr{H}\!odge_{\hat{K}}^{\mathrm{sha}}$ is invariant under tensor product, and everything is proved. **q.e.d.**

Hodge cocharacters: Now consider an object $\underline{\hat{H}}$ of $\mathscr{H}\!odge_{\hat{K}}^{\mathrm{sha}}$. Proposition 6.11 implies that we have a faithful exact tensor functor Gr_F from $\langle\!\langle \underline{\hat{H}} \rangle\!\rangle$ to the category of $\mathbb{Z}$-graded vector spaces over $\mathbb{C}$. Let $G_{\underline{\hat{H}}}^F$ denote the automorphism group of the underlying fiber functor $\langle\!\langle \underline{\hat{H}} \rangle\!\rangle \to \mathscr{V}\!ec_{\mathbb{C}}$. By general tannakian theory (see Deligne-Milne [4] Th. 3.2) the group $G_{\underline{\hat{H}}}^F$ is in a canonical way an inner form of $G_{\underline{\hat{H}},\mathbb{C}} := G_{\underline{\hat{H}}} \times_{\hat{K},\iota} \mathbb{C}$. As $\mathbb{C}$ is algebraically closed, this amounts to an isomorphism $G_{\underline{\hat{H}}}^F \cong G_{\underline{\hat{H}},\mathbb{C}}$ which is canonical up to conjugation.

The grading means that Gr_F is actually a tensor functor $\langle\!\langle \underline{\hat{H}} \rangle\!\rangle \to \mathscr{R}\!ep_{\mathbb{G}_{m,\mathbb{C}}}$, so it corresponds to a unique cocharacter of $G_{\underline{\hat{H}}}^F$. Via the above isomorphism it corresponds to a unique conjugacy class of cocharacters of $G_{\underline{\hat{H}},\mathbb{C}}$. As a whole this conjugacy class is defined over the separable closure of $\iota(\hat{K})$ in $\mathbb{C}$. Thus if $\hat{K}^{\mathrm{sep}}$ denotes any abstractly given separable closure of $\hat{K}$, we obtain a unique $G_{\underline{\hat{H}}}(\hat{K}^{\mathrm{sep}}) \rtimes \mathrm{Gal}(\hat{K}^{\mathrm{sep}}/\hat{K})$-conjugacy class of cocharacters of $G_{\underline{\hat{H}},\hat{K}^{\mathrm{sep}}}$.

Definition 7.10 *Any cocharacter in this $G_{\underline{\hat{H}}}(\hat{K}^{\mathrm{sep}}) \rtimes \mathrm{Gal}(\hat{K}^{\mathrm{sep}}/\hat{K})$-conjugacy class is called a* Hodge cocharacter of $G_{\underline{\hat{H}}}$.

The Hodge cocharacters yield numerical restrictions on the Hodge group, in view of the following result:

Theorem 7.11 *For any object $\underline{\hat{H}}$ in $\mathscr{H}\!odge_K^{\mathrm{sha}}$ the group $G_{\underline{\hat{H}},\hat{K}^{\mathrm{sep}}}$ is generated by the images of all $G_{\underline{\hat{H}}}(\hat{K}^{\mathrm{sep}}) \rtimes \mathrm{Gal}(\hat{K}^{\mathrm{sep}}/\hat{K})$-conjugates of Hodge cocharacters.*

Proof. The subgroup of $G_{\underline{\hat{H}},\hat{K}^{\mathrm{sep}}}$ which is generated by these images is normal and defined over $\hat{K}$. Consider a finite dimensional representation of $G_{\underline{\hat{H}}}$ whose kernel is precisely this subgroup, and let $\underline{\hat{H}}_1$ be its corresponding object of $\langle\!\langle\underline{\hat{H}}\rangle\!\rangle$. Then all Hodge cocharacters of $G_{\underline{\hat{H}}}$ are trivial, and we must prove $G_{\underline{\hat{H}}_1} = 1$. Now the Hodge filtration associated to $\underline{\hat{H}}_1$ is trivial, hence we have $\mathfrak{q}_{\hat{H}_1} = \mathfrak{p}_{\hat{H}_1}$. Since this property is inherited by any subobject of $\underline{\hat{H}}_1$, the semistability implies that $\underline{\hat{H}}_1$ is pure of weight 0. Thus $\underline{\hat{H}}_1$ is a direct sum of copies of $\mathbb{1}_{\hat{K}}$, and hence $G_{\underline{\hat{H}}_1} = 1$, as desired. **q.e.d.**

Possibilities for the Hodge group: A direct consequence of Theorem 7.11 is the following negative result:

Corollary 7.12 *The Hodge group of any object $\underline{\hat{H}}$ in $\mathscr{H}\!odge_K^{\mathrm{sha}}$ does not possess a non-trivial unipotent factor group.*

On the positive side, we can now show that there are no restrictions on the semisimple part of a Hodge group. In particular, any root system can occur:

Proposition 7.13 *For any connected semisimple group G over $\hat{K}$ there exists an object $\underline{\hat{H}}$ of $\mathscr{H}\!odge_K^{\mathrm{sha}}$ with $G_{\underline{\hat{H}}} \cong G$.*

Proof. Fix a conjugacy class C of regular cocharacters of G. It is an algebraic variety of dimension $d := \dim(G) - \mathrm{rank}(G)$. Recall that since $\mathbb{C}$ contains the completion of an algebraic closure of $\iota(\hat{K})$, it has infinite transcendence degree over $\iota(\hat{K})$. Thus we may find a point $\lambda \in C\big(\mathbb{C}[\![z-\zeta]\!]\big)$ whose coordinates are maximally transcendent over $\hat{K}$, i.e. such that for every $n \geq 1$ the coordinates of $\lambda \mod (z-\zeta)^n \in C\big(\mathbb{C}[z-\zeta]/(z-\zeta)^n\big)$ generate an extension of $\hat{K}$ of transcendence degree nd.

For every finite dimensional representation $\hat{H}$ of G define the weight filtration W to be pure of weight 0, let $\mathfrak{q}_{\hat{H}} := \lambda(z-\zeta)\cdot\mathfrak{p}_{\hat{H}}$, and consider the triple $\underline{\hat{H}} := (\hat{H}, W, \mathfrak{q}_{\hat{H}})$. This defines a functor from $\mathscr{R}\!ep_G$ to the category of pure $\hat{K}$-pre-Hodge structures of weight 0. By construction it commutes with tensor products. The main point is the following lemma.

Lemma 7.14 *Let $\underline{\hat{H}}' \subset \underline{\hat{H}}$ be a strict subobject satisfying $\deg_{\mathfrak{q}}(\underline{\hat{H}}') \geq 0$. Then this inequality is an equality, and $\underline{\hat{H}}'$ comes from a G-invariant subspace of $\hat{H}$.*

Proof. Recall that $\deg_{\mathfrak{q}}(\underline{\hat{H}}') = \deg_{\mathfrak{q}}(\Lambda^r\underline{\hat{H}}')$ where $r := \mathrm{rank}(\underline{\hat{H}})$. Thus for both assertions we may replace $\underline{\hat{H}}' \subset \underline{\hat{H}}$ by $\Lambda^r\underline{\hat{H}}' \subset \Lambda^r\underline{\hat{H}}$, after which we may assume $r = 1$. In this case the assumption $\deg_{\mathfrak{q}}(\underline{\hat{H}}') \geq 0$ is equivalent to $\hat{H}' \subset \mathfrak{q}_{\hat{H}}$. Now the construction of $\mathfrak{q}_{\hat{H}}$ implies that $\hat{H} \cap \mathfrak{q}_{\hat{H}}$ coincides with the space of G-invariants in $\hat{H}$. Thus $\underline{\hat{H}}'$ comes from a copy of the trivial one-dimensional representation inside $\hat{H}$. In particular it is isomorphic to $\mathbb{1}_{\hat{K}}$, and hence $\deg_{\mathfrak{q}}(\underline{\hat{H}}') = 0$, as desired. **q.e.d.**

Since G is semisimple, the construction implies $\deg_{\mathfrak{q}}(\underline{\hat{H}}) = 0$ for every representation $\hat{H}$. Together with the first part of Lemma 7.14 this implies the semistability of $\underline{\hat{H}}$. Thus we have obtained a tensor functor $\mathscr{R}ep_G \longrightarrow \mathscr{H}odge_{\hat{K}}$, which is clearly faithful. By the second part of Lemma 7.14 its essential image is invariant under taking subquotients, and looking at graphs of homomorphisms shows that the functor is fully faithful. If $\underline{\hat{H}}$ comes from a faithful representation of G, it follows that we have an equivalence of categories $\mathscr{R}ep_G \xrightarrow{\sim} \langle\langle \underline{\hat{H}} \rangle\rangle$. This implies $G_{\underline{\hat{H}}} \cong G$, as desired. Finally, Proposition 7.3 shows that $\underline{\hat{H}}$ is automatically in $\mathscr{H}odge^{\mathrm{sha}}_{\hat{K}}$. **q.e.d.**

8 Parametrization and extensions

In this section we make some qualitative remarks on parameter spaces of mixed $\hat{K}$-Hodge structures, on extensions, and on infinitesimal deformations.

Period spaces: In order to classify all possible mixed $\hat{K}$-Hodge structures up to isomorphism it is practical to first fix all discrete numerical invariants and then try to endow the set of isomorphy classes with a suitable analytic structure. So let us fix a finite dimensional $\hat{K}$-vector space $\hat{H}$ together with a weight filtration W. This pair is determined up to isomorphism by the rank and the weight polygon. The only remaining numerical invariant is the Hodge polygon, i.e. the collection of elementary divisors relating the lattices $\mathfrak{p}_{\hat{H}}$ and $\mathfrak{q}_{\hat{H}}$. Fixing these, we have:

Proposition 8.1 *The set of lattices $\mathfrak{q}_{\hat{H}} \subset \hat{H} \otimes_{\hat{K}} \mathbb{C}(\!(z-\zeta)\!)$ with the given elementary divisors is in natural one-to-one correspondence with the $\mathbb{C}$-valued points of some irreducible smooth quasi-projective algebraic variety X over $\hat{K}$.*

Proof. Choose a basis $h_1, \ldots, h_r$ of $\hat{H}$, and let $\mathfrak{q}_0$ be the lattice generated by the elements $(z-\zeta)^{-e_i} \cdot h_i$ for $1 \leq i \leq r$, where e_i are the slopes of the Hodge polygon. By the elementary divisor theorem any other lattice with the given invariants is conjugate to $\mathfrak{q}_0$ under $\mathrm{GL}_r(\mathbb{C}[\![z-\zeta]\!])$. Thus the set of all such lattices can be identified with the factor space $\mathrm{GL}_r(\mathbb{C}[\![z-\zeta]\!]) \,/\, \mathrm{Stab}_{\mathrm{GL}_r(\mathbb{C}[\![z-\zeta]\!])}(\mathfrak{q}_0)$. If all $|e_i| \leq e$, the stabilizer contains all invertible matrices which are congruent to the identity modulo $(z-\zeta)^{2e}$. Therefore the set can also be viewed as a homogeneous space over $\mathrm{GL}_r(\mathbb{C}[z-\zeta]/(z-\zeta)^{2e})$. Now this group can be identified with the group of $\mathbb{C}$-valued points of a connected algebraic group over $\hat{K}$, defined as the Weil restriction $\mathcal{R}_{(\hat{K}[z-\zeta]/(z-\zeta)^{2e})/\hat{K}}\mathrm{GL}_r$. The stabilizer of $\mathfrak{q}_0$ corresponds to an algebraic subgroup defined over $\hat{K}$, so the factor space can be endowed with a natural structure of irreducible smooth algebraic variety X over $\hat{K}$. Finally note that $(z-\zeta)^e \mathfrak{p}_{\hat{H}} \subset \mathfrak{q}_{\hat{H}} \subset (z-\zeta)^{-e}\mathfrak{p}_{\hat{H}}$, and giving $\mathfrak{q}_{\hat{H}}$ is equivalent to giving the subspace $\mathfrak{q}_{\hat{H}}/(z-\zeta)^e\mathfrak{p}_{\hat{H}} \subset (z-\zeta)^{-e}\mathfrak{p}_{\hat{H}}/(z-\zeta)^e\mathfrak{p}_{\hat{H}}$. This defines an embedding of X into some Grassmannian, hence X is quasi-projective, as desired. **q.e.d.**

The lattice $\mathfrak{q}_{\hat{H}}$ determines the Hodge filtration, which itself is parametrized by a certain Grassmannian variety. Thus X is fibered over a Grassmannian, and the points in a fiber correspond to some kind of infinitesimal deformations of the associated filtration. In other words X is a kind of jet bundle over a Grassmannian.

Next, the *period space* $\mathcal{X}^{\mathrm{ss}}$ is the set of lattices $\mathfrak{q}_{\hat{H}}$ for which $\underline{\hat{H}} = (\hat{H}, W, \mathfrak{q}_{\hat{H}})$ is semistable. It may, of course, happen that the period space is empty because the numerical invariants are wrong.

Proposition 8.2 *$\mathcal{X}^{\mathrm{ss}}$ is open in $X(\mathbb{C})$, and dense if non-empty.*

Proof. To obtain $\mathcal{X}^{\mathrm{ss}}$ from $X(\mathbb{C})$ we must, for every subspace $\hat{H}' \subset \hat{H}$, remove all those lattices $\mathfrak{q}_{\hat{H}}$ which are in some sense too near to $\hat{H}'$. Note that $\hat{H}'$ corresponds to a $\hat{K}$-valued point on a projective algebraic variety Y over $\hat{K}$, namely a finite union of Grassmannians. By semicontinuity the condition to remove $(\hat{H}', \mathfrak{q}_{\hat{H}})$ is Zariski closed, so the set of offending pairs is $\mathcal{Z} := Z(\mathbb{C}) \cap \big(Y(\hat{K}) \times X(\mathbb{C})\big)$, where Z is a certain closed subvariety of $Y \times X$. Now the properness of Y implies that $\mathrm{pr}_2(\mathcal{Z}) \subset X(\mathbb{C})$ is closed in the analytic topology, hence the complement $\mathcal{X}^{\mathrm{ss}}$ is open.

To prove the density note first that we may replace X by any Zariski open dense subvariety and shrink $\mathcal{X}^{\mathrm{ss}}$ and Z accordingly, since by openness the non-emptyness is preserved. After shrinking X we may assume that the morphism $Z \to X$ is flat. This family of subvarieties of Y then corresponds to a morphism from X to the Hilbert scheme of Y. Let T be the Zariski closure of the image of X. After shrinking X further we may assume that $X \to T$ is smooth and its image lies in the smooth part of T. Then the map $X(\mathbb{C}) \to T(\mathbb{C})$ looks locally like a coordinate projection $\mathbb{C}^{\oplus d} \to \mathbb{C}^{\oplus e}$ with $d \geq e \geq 0$. All points to be removed lie in the inverse image of $W(\hat{K})$, i.e. of $\hat{K}^{\oplus e}$. Since $\hat{K}$ is nowhere dense in $\mathbb{C}$, the desired assertion follows unless $e = 0$. In that case the map $X \to T$ is constant, hence Z is a product $Y_1 \times X$, and $\mathcal{X}^{\mathrm{ss}} = \emptyset$. **q.e.d.**

A similar argument shows:

Proposition 8.3 *The subset $\mathcal{X}^{\mathrm{s}} \subset \mathcal{X}^{\mathrm{ss}}$ parametrizing all simple semistable objects is open and dense.*

For example, an object of Drinfeld module type of rank r (cf. Section 10) is determined by the subspace $F^0\hat{H}_{\mathbb{C}} \subset \hat{H}_{\mathbb{C}}$ of codimension 1. Thus in this case we have $X \cong \mathbb{P}^{r-1}_{\hat{K}}$, and $\mathcal{X}^{\mathrm{s}} = \mathcal{X}^{\mathrm{ss}} \subset \mathbb{P}^{r-1}(\mathbb{C})$ is the complement of all $\hat{K}$-rational hyperplanes.

Extensions: As the second topic of this section we classify the possible extensions in $\mathscr{H}\!odge_{\hat{K}}$ and compare them with those in $\mathscr{H}\!odge^{\mathrm{sha}}_{\hat{K}}$. In either case they will be parametrized by a vector space over $\mathbb{C}$, whose dimension is typically infinite in the former case, but always finite in the second case. We are interested in short exact sequences

$$0 \longrightarrow \underline{\hat{H}}_1 \longrightarrow \underline{\hat{H}} \longrightarrow \underline{\hat{H}}_2 \longrightarrow 0 \tag{8.4}$$

in $\mathscr{H}odge_{\hat{K}}$, where $\underline{\hat{H}}_1$ and $\underline{\hat{H}}_2$ are fixed. If some weight of $\underline{\hat{H}}_2$ is strictly smaller than some weight of $\underline{\hat{H}}_1$, the weight filtration of $\underline{\hat{H}}$ determines a partial splitting. Thus we restrict our discussion to the following basic case:

Assumption 8.5 *Every weight of $\underline{\hat{H}}_2$ is greater than or equal to every weight of $\underline{\hat{H}}_1$.*

Let $\widetilde{\mathrm{Ext}}(\underline{\hat{H}}_2, \underline{\hat{H}}_1)$ denote the set of all extensions 8.4 in $\mathscr{H}odge_{\hat{K}}$ together with a splitting for the underlying $\hat{K}$-vector spaces, up to isomorphy. We have already seen such an extension in Example 6.14.

Proposition 8.6 *There is a canonical isomorphism*

$$\widetilde{\mathrm{Ext}}(\underline{\hat{H}}_2, \underline{\hat{H}}_1) \cong \mathrm{Hom}_{\mathbb{C}[[z-\zeta]]}\left(\mathfrak{q}_{\hat{H}_2}, \frac{\hat{H}_1 \otimes_{\hat{K}} \mathbb{C}((z-\zeta))}{\mathfrak{q}_{\hat{H}_1}}\right).$$

Note that this has infinite dimension over $\mathbb{C}$ unless both $\underline{\hat{H}}_1$ and $\underline{\hat{H}}_2$ are zero.

Proof. Identify the underlying $\hat{K}$-vector space $\hat{H}$ with $\hat{H}_1 \oplus \hat{H}_2$ via the given splitting. Then Assumption 8.5 implies that the weight filtration of $\hat{H}$ is determined uniquely by the rest of the data. Next by 8.5 and Proposition 4.11 any strict exact sequence 8.4 of mixed $\hat{K}$-pre Hodge structures is automatically semistable, i.e. $\underline{\hat{H}}$ is in $\mathscr{H}odge_{\hat{K}}$. Thus giving the extension 8.4 is equivalent to giving the extension of lattices

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \hat{H}_1 \otimes_{\hat{K}} \mathbb{C}((z-\zeta)) & \longrightarrow & \hat{H} \otimes_{\hat{K}} \mathbb{C}((z-\zeta)) & \longrightarrow & \hat{H}_2 \otimes_{\hat{K}} \mathbb{C}((z-\zeta)) & \longrightarrow & 0 \\
 & & \cup & & \cup & & \cup & & \\
0 & \longrightarrow & \mathfrak{q}_{\hat{H}_1} & \longrightarrow & \mathfrak{q}_{\hat{H}} & \longrightarrow & \mathfrak{q}_{\hat{H}_2} & \longrightarrow & 0
\end{array}$$

Here the curved arrow in the upper row indicates the given splitting. Since the lower row consists of free $\mathbb{C}[[z-\zeta]]$-modules, it possesses its own splitting, indicated by the dotted curved arrow. This splitting differs from that in the upper row by a homomorphism $\mathfrak{q}_{\hat{H}_2} \longrightarrow \hat{H}_1 \otimes_{\hat{K}} \mathbb{C}((z-\zeta))$. Since the lower splitting is unique up to a homomorphism $\mathfrak{q}_{\hat{H}_2} \longrightarrow \mathfrak{q}_{\hat{H}_1}$, and $\mathfrak{q}_{\hat{H}_2}$ is a free $\mathbb{C}[[z-\zeta]]$-module, giving the lattice $\mathfrak{q}_{\hat{H}}$ is equivalent to giving the induced homomorphism $\mathfrak{q}_{\hat{H}_2} \longrightarrow \big(\hat{H}_1 \otimes_{\hat{K}} \mathbb{C}((z-\zeta))\big)/\mathfrak{q}_{\hat{H}_1}$. **q.e.d.**

Let $\widetilde{\mathrm{Ext}}^{\mathrm{ha}}(\underline{\hat{H}}_2, \underline{\hat{H}}_1) \subset \widetilde{\mathrm{Ext}}(\underline{\hat{H}}_2, \underline{\hat{H}}_1)$ denote the subset of all extensions for which the Hodge polygons of $\underline{\hat{H}}$ and $\underline{\hat{H}}_1 \oplus \underline{\hat{H}}_2$ coincide. Here the superscript stands for "Hodge additive". If $\underline{\hat{H}}_1$ and $\underline{\hat{H}}_2$ are in $\mathscr{H}odge_{\hat{K}}^{\mathrm{sha}}$, it follows from Proposition 7.2 that $\underline{\hat{H}}$ lies in $\mathscr{H}odge_{\hat{K}}^{\mathrm{sha}}$ if and only if it corresponds to an element of $\widetilde{\mathrm{Ext}}^{\mathrm{ha}}(\underline{\hat{H}}_2, \underline{\hat{H}}_1)$. For the following description of this Ext group note that both $\mathrm{Hom}_{\mathbb{C}[[z-\zeta]]}(\mathfrak{q}_{\hat{H}_2}, \mathfrak{q}_{\hat{H}_1})$ and $\mathrm{Hom}_{\mathbb{C}[[z-\zeta]]}(\mathfrak{p}_{\hat{H}_2}, \mathfrak{p}_{\hat{H}_1})$ can be viewed as lattices in the $\mathbb{C}((z-\zeta))$-vector space $\mathrm{Hom}_{\hat{K}}(\hat{H}_2, \hat{H}_1) \otimes_{\hat{K}} \mathbb{C}((z-\zeta))$.

Proposition 8.7 *There is a canonical isomorphism*

$$\widetilde{\operatorname{Ext}}^{\mathrm{ha}}(\underline{\hat{H}}_2, \underline{\hat{H}}_1) \cong \frac{\operatorname{Hom}_{\mathbb{C}[\![z-\zeta]\!]}\big(\mathfrak{p}_{\hat{H}_2}, \mathfrak{p}_{\hat{H}_1}\big)}{\operatorname{Hom}_{\mathbb{C}[\![z-\zeta]\!]}\big(\mathfrak{p}_{\hat{H}_2}, \mathfrak{p}_{\hat{H}_1}\big) \cap \operatorname{Hom}_{\mathbb{C}[\![z-\zeta]\!]}\big(\mathfrak{q}_{\hat{H}_2}, \mathfrak{q}_{\hat{H}_1}\big)} \,.$$

Note that this Ext group has finite dimension over $\mathbb{C}$. One easily finds that it is zero if and only if $\underline{\hat{H}}_1$ and $\underline{\hat{H}}_2$ are direct sums of copies of the same object of rank 1. In all other cases we deduce that there exist non-trivial Hodge additive extensions.

Proof. For any extension 8.4 there is an integer $e \gg 0$ such that $(z-\zeta)^e \mathfrak{p}_{\hat{H}} \subset \mathfrak{q}_{\hat{H}}$. Thus we may consider the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (z-\zeta)^e \mathfrak{p}_{\hat{H}_1} & \longrightarrow & (z-\zeta)^e \mathfrak{p}_{\hat{H}} & \underset{\longrightarrow}{\overset{\cdots\cdots}{\longleftarrow}} & (z-\zeta)^e \mathfrak{p}_{\hat{H}_2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{q}_{\hat{H}_1} & \longrightarrow & \mathfrak{q}_{\hat{H}} & \underset{\longrightarrow}{\overset{\cdots\cdots}{\longleftarrow}} & \mathfrak{q}_{\hat{H}_2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \dfrac{\mathfrak{q}_{\hat{H}_1}}{(z-\zeta)^e \mathfrak{p}_{\hat{H}_1}} & \longrightarrow & \dfrac{\mathfrak{q}_{\hat{H}}}{(z-\zeta)^e \mathfrak{p}_{\hat{H}}} & \underset{\longrightarrow}{\overset{\cdots\cdots}{\longleftarrow}} & \dfrac{\mathfrak{q}_{\hat{H}_2}}{(z-\zeta)^e \mathfrak{p}_{\hat{H}_2}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{8.8}$$

The elementary divisors of the torsion $\mathbb{C}[\![z-\zeta]\!]$-modules in the bottom row are of the form $e+i$ where i runs through the slopes of the corresponding Hodge polygons. Thus the desired Hodge additivity is equivalent to the splitting of the bottom row. Any splitting of the bottom row can be lifted to a splitting of the middle row, so it gives rise to compatible splittings of all rows, as indicated in Diagram 8.8 by dotted arrows. Looking at the upper row and comparing this splitting with the given one as in the proof of Proposition 8.6, we see that the extension of lattices is given by an element of $\operatorname{Hom}_{\mathbb{C}[\![z-\zeta]\!]}\big(\mathfrak{p}_{\hat{H}_2}, \mathfrak{p}_{\hat{H}_1}\big)$. Again the non-uniqueness amounts to an element of $\operatorname{Hom}_{\mathbb{C}[\![z-\zeta]\!]}\big(\mathfrak{q}_{\hat{H}_2}, \mathfrak{q}_{\hat{H}_1}\big)$, yielding the desired parametrization. **q.e.d.**

Obstructions: Using the above explicit descriptions one can easily show that the extension problems in both $\mathscr{Hodge}_{\hat{K}}$ and $\mathscr{Hodge}_{\hat{K}}^{\mathrm{sha}}$ are unobstructed. The proof is left to the reader:

Proposition 8.9 *Consider objects* $\underline{\hat{H}}_i$ *in* $\mathscr{Hodge}_{\hat{K}}$ *for* $1 \le i \le 3$ *such that for all weights* μ_i *of* $\underline{\hat{H}}_i$ *we have* $\mu_1 \le \mu_2 \le \mu_3$. *Consider extensions*

$$0 \longrightarrow \underline{\hat{H}}_1 \longrightarrow \underline{\hat{H}}_{12} \longrightarrow \underline{\hat{H}}_2 \longrightarrow 0\,,$$
$$0 \longrightarrow \underline{\hat{H}}_2 \longrightarrow \underline{\hat{H}}_{23} \longrightarrow \underline{\hat{H}}_3 \longrightarrow 0$$

in $\mathscr{H}odge_{\hat{K}}$. Then there exists an object $\underline{\hat{H}}_{123}$ in $\mathscr{H}odge_{\hat{K}}$ fitting into a commutative diagram with exact columns and rows

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \underline{\hat{H}}_1 & \longrightarrow & \underline{\hat{H}}_{12} & \longrightarrow & \underline{\hat{H}}_2 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \underline{\hat{H}}_1 & \longrightarrow & \underline{\hat{H}}_{123} & \longrightarrow & \underline{\hat{H}}_{23} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \underline{\hat{H}}_3 & = & \underline{\hat{H}}_3 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

If moreover $\underline{\hat{H}}_{12}$ has the same Hodge polygon as $\underline{\hat{H}}_1 \oplus \underline{\hat{H}}_2$, and $\underline{\hat{H}}_{23}$ the same as $\underline{\hat{H}}_2 \oplus \underline{\hat{H}}_3$, then $\underline{\hat{H}}_{123}$ can be chosen to have the same Hodge polygon as $\underline{\hat{H}}_1 \oplus \underline{\hat{H}}_2 \oplus \underline{\hat{H}}_3$.

Infinitesimal deformations: Finally, the first order infinitesimal deformations of a pure object $\underline{\hat{H}}$ are parametrized by the group $\widetilde{\mathrm{Ext}}(\underline{\hat{H}}, \underline{\hat{H}})$. The subgroup $\widetilde{\mathrm{Ext}}^{\mathrm{ha}}(\underline{\hat{H}}, \underline{\hat{H}})$ corresponds to infinitesimal deformations with constant Hodge polygon. This is the tangent space of the period space $\mathcal{X}^{\mathrm{ss}}$ described above. Note that the unobstructedness result of Proposition 8.9 corresponds to assertion that the parameter space is smooth.

9 Hodge structures over global function fields

Let K be a global function field in one variable over a perfect field $\mathbb{F}$, and let $\hat{K}$ be the completion of K at a fixed place ∞. If k and $z \in K$ denote, respectively, the residue field and a local parameter at ∞, there is an isomorphism $\hat{K} \cong k(\!(z)\!)$. Thus we may use this field as basis for the concepts and results of the preceding sections. The aim of this section is to globalize everything by reducing coefficients from $\hat{K}$ to K. All the notations and assumptions of Sections 3–8 remain in force.

Pre-Hodge structures: Recall the inclusions $K \subset \hat{K} \subset \mathbb{C}[\![z-\zeta]\!]$. The basic definition reads (compare Definition 3.2):

Definition 9.1 *A* mixed K-∞-pre-Hodge structure *is a triple* $\underline{H} = (H, W, \mathfrak{q}_H)$ *where*

(a) H is a finite dimensional K-vector space,

(b) $W = (W_\mu H)_{\mu \in \mathbb{Q}}$ is an increasing filtration by K-subspaces of H, called the weight filtration, *and*

(c) $\mathfrak{q}_H$ *is a lattice in* $H \otimes_K \mathbb{C}(\!(z-\zeta)\!)$.

Any mixed K-∞-pre-Hodge structure $\underline{H}$ determines a mixed $\hat{K}$-pre-Hodge structure $\underline{\hat{H}} = (H \otimes_K \hat{K}, W \otimes_K \hat{K}, \mathfrak{q}_H)$. Naturally $\underline{H}$ is called *pure of weight* μ if and only if $\underline{\hat{H}}$ is so (compare Definition 3.3). A *morphism* of mixed K-∞-pre-Hodge structures is a homomorphism of the underlying K-vector spaces that is compatible with the rest of the data (compare Definition 3.7). *Strict morphisms, (strict) subquotients, tensor products, the unit object* $\underline{\mathbb{1}}_K$*, inner hom*, etc. are likewise defined as in Section 3.

Hodge structures: We rigidify our global objects by the same local condition as in 2.3:

Definition 9.2 *A mixed* K*-*∞*-pre-Hodge structure* $\underline{H}$ *is called* locally semistable *if and only if its associated mixed* $\hat{K}$*-pre-Hodge structure* $\underline{\hat{H}}$ *is semistable. A locally semistable mixed (or pure)* K*-*∞*-pre-Hodge structure is called a* mixed *(resp.* pure) K-∞-Hodge structure.

From Proposition 4.13 it follows that $\underline{H}$ is locally semistable if and only if $\mathrm{Gr}_\mu^W \underline{H}$ is locally semistable for every $\mu \in \mathbb{Q}$. The category of all mixed K-∞-Hodge structures is denoted $\mathscr{H}\!odge_{K,\infty}$. It is equipped with a natural fiber functor

$$\omega : \mathscr{H}\!odge_{K,\infty} \to \mathscr{V}\!ec_K,\ \underline{H} \mapsto H\ ,$$

which obviously commutes with tensor product.

Theorem 9.3 $\mathscr{H}\!odge_{K,\infty}$ *and* ω *form a neutral tannakian category over* K.

Proof. The fact that $\mathscr{H}\!odge_{K,\infty}$ is abelian follows from Propositions 4.7 and 4.14, as in the local case. The rest is deduced from Corollary 5.7. **q.e.d.**

The Hodge group: Let $\langle\!\langle \underline{H} \rangle\!\rangle \subset \mathscr{H}\!odge_{K,\infty}$ denote the strictly full tannakian subcategory generated by an object $\underline{H}$, and $\omega_{\underline{H}} : \langle\!\langle \underline{H} \rangle\!\rangle \to \mathscr{V}\!ec_K$ the associated fiber functor. The *Hodge group of* $\underline{H}$ is defined as $G_{\underline{H}} := \underline{\mathrm{Aut}}^{\otimes}(\omega_{\underline{H}})$. This is an algebraic subgroup of the general linear group $\mathrm{Aut}_K(H)$. The obvious tensor functor $\langle\!\langle \underline{H} \rangle\!\rangle \longrightarrow \langle\!\langle \underline{\hat{H}} \rangle\!\rangle$ shows that the local and global Hodge groups are related by a canonical embedding $G_{\underline{\hat{H}}} \hookrightarrow G_{\underline{H},\hat{K}} := G_{\underline{H}} \times_K \hat{K}$. The global Hodge group satisfies the analogues of Proposition 6.2 and 6.3:

Proposition 9.4 $G_{\underline{H}}$ *is connected and reduced.*

Proposition 9.5 *Consider any object* $\underline{H}$ *of* $\mathscr{H}\!odge_{K,\infty}$ *and let* ρ *denote the representation of* $G_{\underline{H}}$ *on the underlying vector space* H.

(a) There exists an element $\gamma \in G_{\underline{H}}(\mathbb{C}(\!(z-\zeta)\!))$ *such that* $\mathfrak{q}_H = \rho(\gamma)\mathfrak{p}_H$.

(b) Consider any γ *as in (a) and any object* $\underline{H}_1$ *of* $\langle\!\langle \underline{H} \rangle\!\rangle$. *Let* ρ_1 *denote the associated representation of* $G_{\underline{H}}$ *on the underlying vector space* H_1. *Then we have* $\mathfrak{q}_{H_1} = \rho_1(\gamma)\mathfrak{p}_{H_1}$.

Frobenius functoriality: The pullback $\mathrm{Frob}_q^*\underline{H}$ and the pushforward $\mathrm{Frob}_{q,*}\underline{H}$ are constructed as in Definitions 5.1 and 5.2. Both functors preserve local semistability by Proposition 5.5. By the same arguments as in Proposition 6.4 we have:

Proposition 9.6 *If $\underline{H}$ is locally semistable, there is a canonical isomorphism* $G_{\mathrm{Frob}_q^*\underline{H}} \cong \mathrm{Frob}_q^* G_{\underline{H}} := G \times_{K,\mathrm{Frob}_q} K$.

Weight (quasi-)cocharacter: It is constructed as in Proposition 6.5. One obtains a unique $G_{\underline{H}}(K)$-conjugacy class of quasi-cocharacters of $G_{\underline{H}}$, whose image in the reductive part of $G_{\underline{H}}$ is unique and lands in the center. Various properties of polygons from Section 6 translate directly to the global case.

Global Hodge additivity: As in the local case the functors F^i and Gr_F^i on the category $\mathscr{H}\!odge_{K,\infty}$ are not exact. To obtain a satisfactory theory of Hodge cocharacters we repeat the discussion of Section 7 in the global setting, but listing only the main points. Let $\underline{H}$ be an object of $\mathscr{H}\!odge_{K,\infty}$.

Definition 9.7 *(a) $\underline{H}$ is called* (globally) Hodge additive *if and only if its Hodge polygon coincides with that of its semisimplification $\underline{H}^{\mathrm{ss}}$.*

(b) $\underline{H}$ is called (globally) strongly Hodge additive *if and only if every object $\underline{H}_1$ of $\langle\!\langle \underline{H} \rangle\!\rangle$ is Hodge additive.*

Clearly $\underline{H}$ is Hodge additive (resp. strongly Hodge additive) whenever $\hat{\underline{H}}$ is Hodge additive (resp. strongly Hodge additive), but not vice versa. The same proof as in 7.3 shows:

Proposition 9.8 *If $G_{\underline{H}}$ is reductive, then $\underline{H}$ is strongly Hodge additive.*

More generally, consider a simple object $\underline{H}_1$ and let q_1 be the degree over K of the purely inseparable part of the center of $\mathrm{End}(\underline{H}_1)$. Then we can write $\underline{H}_1 = \mathrm{Frob}_{q_1,*}\underline{H}_1'$ for some simple object $\underline{H}_1'$. We say that $\underline{H}_1$ is *(globally) quasi-separable* if and only if all slopes of the Hodge polygon of $\underline{H}_1'$ are multiples of q_1. The same arguments as in Section 7 imply:

Theorem 9.9 *The following assertions are equivalent:*

(a) $\underline{H}$ is strongly Hodge additive.

(b) $\mathrm{Frob}_q^*\underline{H}$ is Hodge additive for every $q \geq 1$ that is a power of $\mathrm{char}(K)$.

(c) $\underline{H}$ is Hodge additive and every simple subfactor is quasi-separable.

Theorem 9.10 *The strongly Hodge additive objects form a strictly full tannakian subcategory $\mathscr{H}\!odge_{K,\infty}^{\mathrm{sha}} \subset \mathscr{H}\!odge_{K,\infty}$.*

Hodge cocharacters: Now let K^{sep} denote a separable closure of K. For any object $\underline{H}$ of $\mathscr{H}\!odge^{\mathrm{sha}}_{K,\infty}$ the analogue of the construction preceding Definition 7.10 yields a unique $G_{\underline{H}}(K^{\mathrm{sep}}) \rtimes \mathrm{Gal}(K^{\mathrm{sep}}/K)$-conjugacy class of cocharacters of $G_{\underline{H},K^{\mathrm{sep}}}$. Any such cocharacter is called a *Hodge cocharacter of* $G_{\underline{H}}$. As in 7.11 we obtain:

Theorem 9.11 *For any object $\underline{H}$ of $\mathscr{H}\!odge^{\mathrm{sha}}_{K,\infty}$ the group $G_{\underline{H},K^{\mathrm{sep}}}$ is generated by the images of all $G_{\underline{H}}(K^{\mathrm{sep}}) \rtimes \mathrm{Gal}(K^{\mathrm{sep}}/K)$-conjugates of Hodge cocharacters.*

The same construction as in Proposition 7.13 yields:

Proposition 9.12 *For any connected semisimple group G over K there exists an object $\underline{H}$ of $\mathscr{H}\!odge_{K,\infty}$ with $G_{\underline{H}} \cong G$.*

Parametrization and extensions: Finally, recall that everywhere in Section 8 we have fixed the underlying $\hat{K}$-vector spaces and their weight filtrations. If we want to deal with global objects, we just have to specify an additional K-structure on all of these. Since the constructions concerned only the lattices $\mathfrak{q}$, the results remain literally the same as in the local case.

10 Hodge structures of Drinfeld module type

The aim of this section is to calculate the Hodge group for the following kind of local or global objects. Consider a $\hat{K}$-Hodge structure $\underline{\hat{H}}$ of rank $r \geq 1$ which may or may not come from a K-∞-Hodge structure $\underline{H}$.

Definition 10.1 *$\underline{\hat{H}}$ resp. $\underline{H}$ is called* of Drinfeld module type *if and only if it is pure and has $\mathfrak{q}_{\hat{H}} \subset \mathfrak{p}_{\hat{H}}$ of $\mathbb{C}$-codimension 1.*

The purity and semistability conditions imply that the weight is $-\frac{1}{r}$. By the lattice condition we must have

$$\dim(F^i\hat{H}_{\mathbb{C}}) = \begin{cases} 0 & \text{if } i > 0, \\ r-1 & \text{if } i = 0, \\ r & \text{if } i \leq -1, \end{cases}$$

and $\mathfrak{q}_{\hat{H}}$ is determined completely by the Hodge filtration. Thus all the information is contained in the subspace $F^0\hat{H}_{\mathbb{C}} \subset \hat{H}_{\mathbb{C}}$ of codimension 1. The semistability condition amounts to the assertion $\hat{H} \cap F^0\hat{H}_{\mathbb{C}} = 0$. This is equivalent to saying that the composite map $\psi : \hat{H} \hookrightarrow \hat{H}_{\mathbb{C}} \twoheadrightarrow \hat{H}_{\mathbb{C}}/F^0\hat{H}_{\mathbb{C}} \cong \mathbb{C}$ is injective. Therefore giving an object of Drinfeld module type $\underline{\hat{H}}$ up to isomorphism is equivalent to giving the $\iota(\hat{K})$-subspace $\psi(\hat{H}) \subset \mathbb{C}$ of dimension r, up to scaling by $\mathbb{C}^{\times}$. Giving $\underline{H}$ involves the additional $\iota(K)$-structure $\psi(H)$ on $\psi(\hat{H})$. Such subspaces arise from the analytic uniformization of Drinfeld modules over $\mathbb{C}$, depending only on the isogeny class. (See Drinfeld [6] §3 or Goss [15] §4.6. The Hodge filtration in this case was also constructed by Gekeler [10], [11] in the framework of a certain kind of de Rham cohomology.)

Proposition 10.2 *Any object $\underline{\hat{H}}$ or $\underline{H}$ of Drinfeld module type is simple and its endomorphism ring is commutative.*

Proof. The proof is the same in both cases, so we stick to $\underline{H}$. Any subobject is again pure of weight $-\frac{1}{r}$, and by Proposition 6.8 the endpoint of its weight polygon has integral coordinates, so its dimension is divisible by r. This proves that $\underline{H}$ is simple. From the above reinterpretation of $\underline{H}$ it is clear that

$$\mathrm{End}(\underline{H}) \cong \{x \in \mathbb{C} \mid x \cdot \psi(H) \subset \psi(H)\} .$$

This is a subring of $\mathbb{C}$, so it is commutative, as desired. **q.e.d.**

The Hodge group: Choosing a basis of $\hat{H}$ we may view $\mathrm{End}(\underline{\hat{H}})$ as a subalgebra of the matrix ring $\mathcal{M}_{r\times r}(\hat{K})$; so in particular it is a finite extension of $\hat{K}$. Also, the Hodge group is then an algebraic subgroup $G_{\underline{\hat{H}}} \subset \mathrm{GL}_{r,\hat{K}}$. In the same way we have $G_{\underline{H}} \subset \mathrm{GL}_{r,K}$. The following result solves a problem raised in [23] Guess 0.5.

Theorem 10.3 *The Hodge group of an object $\underline{\hat{H}}$ of Drinfeld module type of rank r is*

$$G_{\underline{\hat{H}}} = \mathrm{Cent}_{\mathrm{GL}_{r,\hat{K}}}(\mathrm{End}(\underline{\hat{H}})) .$$

The Hodge group of an object $\underline{H}$ of Drinfeld module type of rank r is

$$G_{\underline{H}} = \mathrm{Cent}_{\mathrm{GL}_{r,K}}(\mathrm{End}(\underline{H})) .$$

The proof of this theorem covers the rest of this section. It is exactly the same in the local and the global case, so for ease of notation we restrict ourselves to $\underline{H}$. Abbreviate $E := \mathrm{End}(\underline{H})$ and let d denote its degree over K. Then H is an E-vector space of dimension $s := \frac{r}{d}$, and the right hand side in Theorem 10.3 is isomorphic to the Weil restriction $G_{\mathrm{amb}} := \mathcal{R}_{E/K}\mathrm{GL}_{s,E}$. We know already that $G_{\underline{H}}$ is contained in this ambient group, and it remains to prove equality.

Proof in the separable case: Assume first that E/K is separable. Then G_{amb} will be reductive, so we can use the theory of Section 7. Write $\Sigma := \mathrm{Hom}_K(E, K^{\mathrm{sep}})$. Then we have a decomposition

$$H \otimes_K K^{\mathrm{sep}} \;\cong\; \bigoplus_{\sigma\in\Sigma} H \otimes_{E,\sigma} K^{\mathrm{sep}} \tag{10.4}$$

and correspondingly an inclusion

$$G_{\underline{H},K^{\mathrm{sep}}} \;\subset\; G_{\mathrm{amb}} \times_K K^{\mathrm{sep}} \;\cong\; \prod_{\sigma\in\Sigma} \mathrm{GL}_{s,K^{\mathrm{sep}}} . \tag{10.5}$$

We must prove that this is an equality.

Since $\underline{H}$ is simple, the representation of $G_{\underline{H}}$ on H is irreducible, and its centralizer is E. It follows that $G_{\underline{H},K^{\mathrm{sep}}}$ acts absolutely irreducibly on each summand in 10.4, hence this group is reductive. Now Proposition 7.3 implies that $\underline{H}$ is strongly Hodge additive, so by Theorem 7.11 the group $G_{\underline{H},K^{\mathrm{sep}}}$ is generated by the images of Hodge cocharacters. Fix a Hodge cocharacter λ, and note that its weights are -1 and 0 with respective multiplicities 1 and $r-1$.

The weight -1 must lie in precisely one summand of the decomposition 10.4, say that associated to $\sigma_0 \in \Sigma$. Let us quote the following result:

Proposition 10.6 *(Pink [23] Proposition A.3) Consider a field F, and integer $n \geq 1$, and a connected linear algebraic subgroup $G \subset \mathrm{GL}_{n,F}$. Assume that G acts absolutely irreducibly in the tautological representation and that it possesses a cocharacter over the algebraic closure whose weights are 1 with multiplicity 1, and 0 with multiplicity $n-1$. Then $G = \mathrm{GL}_{n,F}$.*

Applying this to the summand at σ_0 and the cocharacter λ^{-1} we deduce that the projection from $G_{\underline{H},K^{\mathrm{sep}}}$ to the corresponding factor $\mathrm{GL}_{s,K^{\mathrm{sep}}}$ is surjective. On the other hand the image of λ lands only in this factor, and its $\mathrm{GL}_{s,K^{\mathrm{sep}}}$-conjugates generate $\mathrm{GL}_{s,K^{\mathrm{sep}}}$. Therefore the conjugates of λ generate this factor $\mathrm{GL}_{s,K^{\mathrm{sep}}}$ as a subgroup of $G_{\underline{H},K^{\mathrm{sep}}}$. Finally, the Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ transitively permutes the factors on the right hand side of 10.5 and stabilizes $G_{\underline{H},K^{\mathrm{sep}}}$. Thus $G_{\underline{H},K^{\mathrm{sep}}}$ contains all factors as subgroups, hence we have equality in 10.5, as desired. **q.e.d.**

The inseparable case: In general G_{amb} possesses a complicated unipotent part, which makes the proof quite elaborate. Let q denote the degree of the purely inseparable part of E/K. Then we can write $\underline{H} = \mathrm{Frob}_{q,*}\underline{H}'$ for some simple object $\underline{H}'$, which is itself of Drinfeld module type and of rank $r' := \frac{r}{q}$. Its endomorphism ring is $E' := E^q \subset E$, which is now separable over K. By Proposition 6.4 we may work with $\underline{\tilde{H}} := \mathrm{Frob}_q^*\underline{H} \cong \mathrm{Frob}_q^*\mathrm{Frob}_{q,*}\underline{H}'$ in place of $\underline{H}$.

Without loss of generality we may assume that the uniformizer $z \in \hat{K}$ lies already in K. Abbreviating $t := z \otimes 1 - 1 \otimes z$ we then have

$$A := E^q \otimes_{K^q} K = E'[t]/(t^q)\ . \tag{10.7}$$

The ambient group becomes

$$\tilde{G}_{\mathrm{amb}} := \mathrm{Frob}_q^* G_{\mathrm{amb}} \cong \mathcal{R}_{A/K}\mathrm{GL}_{s,A}\ . \tag{10.8}$$

Its unipotent part is composed of two different kinds of pieces which occur, respectively, in the left and right hand sides of the short exact sequence

$$1 \longrightarrow \mathcal{R}_{A/K}\mathrm{SL}_{s,A} \longrightarrow \tilde{G}_{\mathrm{amb}} \xrightarrow{\det} \mathcal{R}_{A/K}\mathbb{G}_{m,A} \longrightarrow 1. \tag{10.9}$$

Our first aim is to prove that the image of $G_{\underline{\tilde{H}}}$ in the maximal abelian quotient $\mathcal{R}_{A/K}\mathbb{G}_{m,A}$ is the whole group. Recall that the Hodge group is defined by representation theoretic information. Thus in order to prove this equality we must take suitable representations of $\mathcal{R}_{A/K}\mathbb{G}_{m,A}$ and show that the corresponding K-∞-Hodge structures are non-trivial. The tautological representation of $\mathcal{R}_{A/K}\mathbb{G}_{m,A}$ on A is not so useful for this purpose, because its extension structure is too complicated. In some sense we would like to go to the Lie algebra, thereby linearizing the formulas. The following calculation performs enough of this for our aims.

Formal logarithm: The power series

$$f := 1 + u_1 t + u_2 t^2 + \ldots \in \mathbb{Z}[u_1, u_2, \ldots][[t]]$$

has constant term 1, so we can speak of its formal logarithm

$$\log f := \sum_{k \geq 1} \frac{(-1)^{k-1}}{k} \cdot (f-1)^k \in \mathbb{Q}[u_1, u_2, \ldots][[t]] \,. \tag{10.10}$$

Let us write its t-expansion in the form

$$\log f =: \sum_{i \geq 1} L_i \cdot \frac{(-1)^{i-1}}{i} \cdot t^i \,. \tag{10.11}$$

Lemma 10.12 *For every $i \geq 1$ we have*

(a) $L_i \in \mathbb{Z}[u_1, \ldots, u_i]$, *of degree* $\leq i$.

(b) $L_i \equiv u_1^i \mod (u_2, \ldots, u_i)$.

(c) $L_i \equiv (-1)^{i-1} \cdot i \cdot u_i \mod (u_1, \ldots, u_{i-1})$.

Proof. Everything follows directly from the definition except for the assertion that L_i has integral coefficients. To prove this assertion note that a monomial $u_1^{k_1} u_2^{k_2} \cdots$ occurs in the k^{th} term of the series 10.10 if and only if $k = \sum_{j \geq 1} k_j$, and it comes to lie in L_i if and only if $\sum_{j \geq 1} j k_j = i$. Its coefficient in L_i can be written in terms of a multiple binomal coefficient as

$$\pm \frac{i}{k} \cdot \binom{k}{k_1, k_2, \ldots} \,. \tag{10.13}$$

We shall prove that its order at every prime p is ≥ 0. Note the following facts, whose easy proof is left to the reader:

Sublemma 10.14 *For any integers $k \geq \ell \geq 0$ and any prime p we have:*

(a) If $p \nmid \ell$, then $\operatorname{ord}_p(k) \leq \operatorname{ord}_p(\binom{k}{\ell})$.

(b) $\operatorname{ord}_p(\binom{pk}{p\ell}) = \operatorname{ord}_p(\binom{k}{\ell})$.

Now assume first that $p \nmid k_j$ for some j. Then the coefficient 10.13 is an integral multiple of $\frac{1}{k}\binom{k}{k_j}$, which is a p-adic integer by Sublemma 10.14 (a). The general case is proved by induction. It suffices to show that whenever the term 10.13 is a p-adic integer, then the same is true after all k_j are multiplied by p. Note that in the process i and k are also multiplied by p, so the desired assertion follows by repeated application of Sublemma 10.14 (b). **q.e.d.**

Next let R be a commutative ring with identity and put $A := R[t]/(t^{n+1})$, where $n \geq 0$ is fixed. Its group of "1-units"

$$U := \{\ u = 1 + u_1 t + u_2 t^2 + \ldots \in A^\times\ \}$$

can be viewed as a linear algebraic group over R, namely as a subgroup of the Weil restriction $\mathcal{R}_{A/R}\mathbb{G}_{m,A}$. It is a successive extension of n copies of the additive group $\mathbb{G}_{a,R}$.

Lemma 10.15 *For every $1 \leq i \leq n$ the map*

$$L_i : \; U \to \mathbb{G}_{a,R}, \;\; u = 1 + u_1 t + u_2 t^2 + \ldots \mapsto L_i(u_1, \ldots, u_i)$$

is a homomorphism of algebraic groups.

Proof. This amounts to a formal identity in a polynomial ring, all of whose coefficients are integers. Thus it suffices to prove it universally over $\mathbb{Z}$, for which in turn we may work over $\mathbb{Q}$. Here the assertion follows directly from the functional equation of the logarithm. **q.e.d.**

Linearization of U/U^p: Now we specialize the preceding discussion to the case that R is a field F, say of characteristic $p \geq 0$. Let I be the set of all integers $1 \leq i \leq n$ which are prime to p, and put $n' := \mathrm{card}(I)$. Let $U^p \subset U$ denote the image of the endomorphism $u \mapsto u^p$.

Lemma 10.16 *The homomorphism*

$$L' : \; U \longrightarrow \mathbb{G}_{a,F}^{\oplus n'}, \;\; u \mapsto \big(L_i(u_1, \ldots, u_i)\big)_{i \in I}$$

induces an isomorphism of algebraic groups $U/U^p \xrightarrow{\sim} \mathbb{G}_{a,F}^{\oplus n'}$.

Proof. Clearly we have

$$U^p = \Big\{ \; u = 1 + \sum_{i=1}^{n} u_i t^i \in U \;\Big|\; u_i = 0 \text{ whenever } p \nmid i \; \Big\} \, ,$$

and the subvariety

$$X := \Big\{ \; u = 1 + \sum_{i=1}^{n} u_i t^i \in U \;\Big|\; u_i = 0 \text{ whenever } p | i \; \Big\}$$

is a direct complement of U^p, although not as a subgroup. By Lemma 10.12 (c) the resulting morphism $\mathbb{A}_F^{n'} \cong X \longrightarrow \mathbb{G}_{a,F}^{\oplus n'} = \mathbb{A}_F^{n'}$ maps each coordinate to a non-zero multiple plus a polynomial in the earlier coordinates. Thus it defines an isomorphism of algebraic varieties. On the other hand p annihilates $\mathbb{G}_{a,F}^{\oplus n'}$, so our homomorphism factors through U/U^p. The desired result follows. **q.e.d.**

Proof in the case $s = 1$: Now we return to the situation of Theorem 10.3, assuming $s = 1$. As a preparation we give explicit descriptions of $\underline{H}'$ and $\underline{\tilde{H}}$. Fix an embedding $K^{\mathrm{sep}} \hookrightarrow \mathbb{C}[\![z - \zeta]\!]$ and put $\Sigma := \mathrm{Hom}_K(E', K^{\mathrm{sep}})$. Since E'/K is separable, we obtain a decomposition

$$\Lambda := E' \otimes_K \mathbb{C}[\![z - \zeta]\!] \cong \bigoplus_{\sigma \in \Sigma} \mathbb{C}[\![z - \zeta]\!] \, . \tag{10.17}$$

The vector space underlying $\underline{H}'$ has dimension 1 over E', so we may identify it with E'. Then we have $\mathfrak{p}_{H'} = \Lambda$, and by assumption $\mathfrak{q}_{H'}$ is a Λ-submodule of $\mathbb{C}$-codimension 1. Thus we must have $\mathfrak{q}_{H'} = \pi \cdot \Lambda$ with $\pi = (\pi_\sigma) \in \Lambda$ such that

$$\pi_\sigma = \begin{cases} z - \zeta & \text{if } \sigma = \sigma_0, \\ 1 & \text{otherwise,} \end{cases} \tag{10.18}$$

where $\sigma_0 \in \Sigma$ is fixed. Next Definitions 5.1 and 5.2 imply

$$\begin{aligned} \underline{\tilde{H}} &\cong \mathrm{Frob}_q^* \mathrm{Frob}_{q,*} \underline{H}' \\ &= \Big(E' \otimes_{K^q} K,\ W \otimes_{K^q} K,\ (\pi \cdot \Lambda) \otimes_{\mathbb{C}[[(z-\zeta)^q]]} \mathbb{C}[[z-\zeta]] \Big)\ , \end{aligned}$$

where the action of K, resp. $\mathbb{C}[[z-\zeta]]$, on each tensor product goes into the second factor. We may write the underlying vector space as $A = E'[t]/(t^q)$, where $t = z \otimes 1 - 1 \otimes z$ as in 10.7. In the same way we obtain an identification

$$\Lambda \otimes_{\mathbb{C}[[(z-\zeta)^q]]} \mathbb{C}[[z-\zeta]] \cong \Lambda[t]/(t^q)\ . \tag{10.19}$$

Thus we have $\mathfrak{p}_{\underline{\tilde{H}}} = \Lambda[t]/(t^q)$, and $\mathfrak{q}_{\underline{\tilde{H}}}$ is an ideal in this ring. To write down a generator note that

$$\begin{aligned} \tilde{G}_{\mathrm{amb}}\big(\mathbb{C}((z-\zeta))\big) &\overset{10.8}{\cong} (\mathcal{R}_{A/K}\mathbb{G}_{m,A})\big(\mathbb{C}((z-\zeta))\big) \\ &= \mathbb{G}_m\big(A \otimes_K \mathbb{C}((z-\zeta))\big) \\ &= \big(\Lambda[t]/(t^q) \otimes_{\mathbb{C}[[z-\zeta]]} \mathbb{C}((z-\zeta))\big)^\times\ . \end{aligned}$$

Consider the idempotent $e = (e_\sigma) \in \Lambda$ with

$$e_\sigma = \begin{cases} 1 & \text{if } \sigma = \sigma_0, \\ 0 & \text{otherwise,} \end{cases}$$

and put $\gamma := \pi + et \in \tilde{G}_{\mathrm{amb}}\big(\mathbb{C}((z-\zeta))\big)$.

Lemma 10.20 *We have* $\mathfrak{q}_{\underline{\tilde{H}}} = \gamma \cdot \mathfrak{p}_{\underline{\tilde{H}}}$.

Proof. It suffices to prove the equality $\pi \otimes 1 = 1 \otimes \pi + et$ in the left hand side of 10.19. For this we use the decomposition 10.17. In the summand associated to σ_0 the equality reads $(z-\zeta) \otimes 1 = t + 1 \otimes (z-\zeta)$, which follows from the definition of t. In the other summands both sides are equal to $1 \otimes 1$. **q.e.d.**

Now we take a closer look at $\tilde{G}_{\mathrm{amb}}$. We shall apply the earlier abstract discussion to the case $F = E'$ and $n = q-1$. Then we have a unique decomposition $\mathcal{R}_{A/E'}\mathbb{G}_{m,A} = \mathbb{G}_{m,E'} \times U$, whence

$$\tilde{G}_{\mathrm{amb}} \cong (\mathcal{R}_{E'/K}\mathbb{G}_{m,E'}) \times (\mathcal{R}_{E'/K}U) \tag{10.21}$$

Here the second factor is unipotent and, since E'/K is separable, the first factor is a torus. Thus any connected algebraic subgroup of $\tilde{G}_{\mathrm{amb}}$ decomposes accordingly.

Note that the first factor in 10.21 is precisely the image of $\tilde{G}_{\mathrm{amb}}$ in $\mathrm{Aut}_K H'$, so the corresponding factor of $G_{\underline{\tilde{H}}}$ is the Hodge group $G_{\underline{H}'}$. Since $\underline{H}'$ is of Drinfeld module type with separable endomorphism ring, we already know that $G_{\underline{H}'} = \mathcal{R}_{E'/K}\mathbb{G}_{m,E'}$. Thus to prove that $G_{\underline{\tilde{H}}}$ equals $\tilde{G}_{\mathrm{amb}}$ it suffices to show that it surjects to the second factor in 10.21. Let $p := \mathrm{char}(K)$, and consider the projection homomorphism

$$\psi:\ \tilde{G}_{\mathrm{amb}} \longrightarrow (\mathcal{R}_{E'/K}U)\big/(\mathcal{R}_{E'/K}U)^p\ .$$

It is enough to prove:

Proposition 10.22 *The restriction of* ψ *to* $G_{\underline{\tilde{H}}}$ *is surjective.*

Proof. Taking Weil restrictions Lemma 10.16 yields an isomorphism

$$(10.23) \quad V := (\mathcal{R}_{E'/K}U)\,/\,(\mathcal{R}_{E'/K}U)^p \cong \mathcal{R}_{E'/K}(U/U^p) \cong \mathcal{R}_{E'/K}\mathbb{G}_{a,E'}^{\oplus n'}\,.$$

If the homomorphism in question is not surjective, its image is contained in the kernel of some non-zero homomorphism of algebraic groups $\varphi : V \to \mathbb{G}_{a,K}$. Identifying the additive group with the subgroup $\left(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right) \subset \mathrm{GL}_{2,K}$, this homomorphism defines a representation of V and hence of the groups $G_{\underline{\tilde{H}}} \subset \tilde{G}_{\mathrm{amb}}$ on $H_\varphi := K^2$. Let $\underline{H}_\varphi$ denote the corresponding K-∞-Hodge structure. If the image of $G_{\underline{\tilde{H}}}$ is contained in the kernel of φ, we must have $\underline{H}_\varphi \cong \mathbb{1}_K^{\oplus 2}$, hence in particular $\mathfrak{q}_{H_\varphi} = \mathfrak{q}_{H_\varphi}$. Thus to prove the proposition it suffices to show that this equality is false whenever $\varphi \neq 0$.

Lemma 10.24 *With γ as in Lemma 10.20 we have*

$$\mathfrak{q}_{H_\varphi} = \left\langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}\varphi(\psi(\gamma))\\1\end{pmatrix} \right\rangle_{\mathbb{C}[\![z-\zeta]\!]}.$$

Proof. More generally, for every representation ρ_1 of $\tilde{G}_{\mathrm{amb}}$ on a finite dimensional K-vector space H_1 the lattice of the associated object $\underline{H}_1$ is given by $\mathfrak{q}_{H_1} = \rho_1(\gamma)\mathfrak{p}_{H_1}$. This follows as in the proof of Proposition 6.3 (b). **q.e.d.**

By Lemma 10.24 it is enough to prove that $\varphi(\psi(\gamma)) \notin \mathbb{C}[\![z-\zeta]\!]$ whenever $\varphi \neq 0$. For this we first determine $\psi(\gamma)$ in

$$V\big(\mathbb{C}(\!(z-\zeta)\!)\big) \;\cong\; \big(\Lambda \otimes_{\mathbb{C}[\![z-\zeta]\!]} \mathbb{C}(\!(z-\zeta)\!)\big)^{\oplus n'}.$$

Lemma 10.25 *We have $\psi(\gamma) = \big(\pi^{-i}e\big)_{i\in I}$.*

Proof. The component of $\gamma = \pi + et$ in the second factor of 10.21 is $1+\pi^{-1}et$. Recall that the identification 10.23 was induced by the homomorphism L' of Lemma 10.16. Thus we have

$$\begin{aligned} \psi(\gamma) &= L'(1+\pi^{-1}et) \\ &\overset{10.16}{=} \big(L_i(\pi^{-1}e, 0, \ldots, 0)\big)_{i\in I} \\ &\overset{10.12\,(b)}{=} \big(\pi^{-i}e\big)_{i\in I}\,. \end{aligned}$$

q.e.d.

Next we must describe φ. Note that V is isomorphic to a direct sum of copies of the additive group $\mathbb{G}_{a,K}$, and recall that the endomorphism ring of $\mathbb{G}_{a,K}$ is generated by K and Frob_p. Note also that every K-linear form on $(E')^{\oplus n'}$ is the composite of some E'-linear form and the trace map $E' \to K$. Thus φ is given by

$$x = (x_i) \mapsto \sum_{i\in I}\sum_{j\geq 0} \mathrm{trace}_{E'/K}(\varphi_{i,j}\cdot x_i^{p^j})$$

with certain coefficients $\varphi_{i,j} \in E'$. It follows that

$$\begin{aligned} \varphi(\psi(\gamma)) &= \sum_{i\in I}\sum_{j\geq 0} \mathrm{trace}_{E'/K}(\varphi_{i,j}\cdot \pi^{-ip^j}e) \\ &\overset{10.18}{=} \sum_{i\in I}\sum_{j\geq 0} \sigma_0(\varphi_{i,j})\cdot (z-\zeta)^{-ip^j}\,. \end{aligned}$$

Now the key observation is that no exponent $-ip^j$ occurs twice, since $p \nmid i$. Choose (i,j) such that $\varphi_{i,j} \neq 0$ and ip^j is maximal. It follows that

$$\mathrm{ord}_{z-\zeta}\big(\varphi(\psi(\gamma))\big) = -ip^j < 0 ,$$

as desired. This finishes the proof of Proposition 10.22 and hence of Theorem 10.3 in the case $s = 1$. **q.e.d.**

The maximal abelian quotient: Now we come back to the general case. From the case $s = 1$ we can deduce:

Proposition 10.26 *The determinant map in the exact sequence 10.9 induces a surjective homomorphism* $G_{\underline{\tilde{H}}} \twoheadrightarrow \mathcal{R}_{A/K}\mathbb{G}_{m,A}$.

Proof. The proof uses a construction corresponding to the "determinant" of a Drinfeld module, which turns out to be another Drinfeld module: see Anderson [1]. Consider the highest exterior power $H_1 := \Lambda_E^s H$ of the vector space underlying $\underline{H}$. It carries a natural representation of the ambient group G_{amb} and hence corresponds to an object $\underline{H}_1$ of $\mathscr{Hodge}_{K,\infty}$. One easily finds that this is again of Drinfeld module type, this time with $\dim_K \mathrm{End}(\underline{H}_1) = \mathrm{rank}(\underline{H}_1)$. Thus Theorem 10.3 is already proved for it. The Frobenius pullback $\underline{\tilde{H}}_1 := \mathrm{Frob}_q^* \underline{H}_1$ comes from the representation of $\tilde{G}_{\mathrm{amb}}$ on $\Lambda_A^s \tilde{H} \cong A$, which is simply the determinant map in 10.9. Since $G_{\underline{\tilde{H}}}$ surjects to $G_{\underline{\tilde{H}}_1} = \mathcal{R}_{A/K}\mathbb{G}_{m,A}$, we are done. **q.e.d.**

The adjoint representation: In order to deal with the case $s > 1$ we study the adjoint representation of $\tilde{G}_{\mathrm{amb}}$ and the K-∞-Hodge structures deduced from it. For every $0 \leq i \leq q$ abbreviate $A_i := E'[t]/(t^i)$, and note that we have natural short exact sequences $0 \to A_i \xrightarrow{t^j} A_{i+j} \to A_j \to 0$. Let H_i denote the space of $r \times r$-matrices over A_i, and $H_i^0 \subset H_i$ the subspace of matrices of trace zero. Let H_i^{00} denote the image of H_i^0 in the space of matrices modulo scalars H_i/A_i. Note that $H_i^{00} \cong H_i^0/A_i$ when $p := \mathrm{char}(K)$ divides r, and $H_i^{00} \cong H_i^0$ otherwise. All these spaces are subquotient representations of the adjoint representation $\mathrm{Lie}\,\tilde{G}_{\mathrm{amb}} \cong H_q$, and multiplication with powers of t induces various equivariant maps between them. The most interesting part is H_q^{00}, which is a successive extension of copies of H_1^{00}. The latter is a non-trivial irreducible representation (cf. for instance [18]).

All these representations restrict to representations of the Hodge group $G_{\underline{\tilde{H}}}$, so they correspond to K-∞-Hodge structures $\underline{H}_i \supset \underline{H}_i^0 \twoheadrightarrow \underline{H}_i^{00}$. We want to show that these objects do not decompose more than under $\tilde{G}_{\mathrm{amb}}$. As a first step consider the projection to the reductive part

$$\tilde{G}_{\mathrm{amb}} \;=\; \mathcal{R}_{A/K}\mathrm{GL}_{s,A} \;\twoheadrightarrow\; \mathcal{R}_{E'/K}\mathrm{GL}_{s,E'} \;.$$

Since Theorem 10.3 is already proved for $\underline{H}'$, this induces an epimorphism

$$\pi:\; G_{\underline{\tilde{H}}} \twoheadrightarrow G_{\underline{H}'} \cong \mathcal{R}_{E'/K}\mathrm{GL}_{s,E'} \tag{10.27}$$

It follows, for instance, that the object $\underline{H}_1^{00}$ is simple. Next we want to determine the subobjects of $\underline{H}_q^{00}$. As preparation we need:

Lemma 10.28 *For every $1 \leq i \leq q$ the Hodge polygon of $\underline{H}_i^{00}$ has slopes $\pm i$ with multiplicity $s-1$ each, all the remaining slopes being* 0.

Proof. We begin with the case $i = 1$. From the separable case recall that the Hodge slopes of $\underline{H}'$ are -1 with multiplicity 1, the remaining slopes being 0, and that the slope -1 corresponds to a certain embedding $\sigma_0 : E' \hookrightarrow K^{\mathrm{sep}}$. Since $\dim_{E'}(H') = s$ and $H_1 \cong (H')^\vee \otimes_{E'} H'$, we deduce that $\underline{H}_1$ has slopes ± 1 with multiplicity $s-1$ each and remaining slopes 0. Now $\underline{H}_1$ is an extension of $\underline{H}_1^{00}$ with one or more copies of the unit object. Thus the desired assertion follows for $\underline{H}_1^{00}$ provided that $\underline{H}_1$ is Hodge additive. But this follows from Proposition 9.8, since $\underline{H}_1 \in \langle\!\langle \underline{H}' \rangle\!\rangle$, whose Hodge group is reductive.

For the general case note that $\underline{H}_q^{00} \cong \mathrm{Frob}_q^* \mathrm{Frob}_{q,*} \underline{H}_1^{00}$, and that $\underline{H}_i^{00}$ is obtained by dividing by t^i, where $t = z \otimes 1 - 1 \otimes z$ as in 10.7. To determine the resulting Hodge slopes it suffices to look at what happens to the lattices $\mathfrak{p}$ and $\mathfrak{q}$. As in the proof of Lemma 7.6 we may choose a $\mathbb{C}[\![z-\zeta]\!]$-basis $\{h_\nu\}$ of $\mathfrak{p}_{H_1^{00}}$ such that $\mathfrak{q}_{H_1^{00}}$ is generated by the elements $(z-\zeta)^{e_\nu} \cdot h_\nu$. Here e_ν runs through all Hodge slopes of $\underline{H}_1^{00}$. Now Definitions 5.1 and 5.2 show that the effect of $\mathrm{Frob}_q^* \mathrm{Frob}_{q,*}$ on the lattices is $(\ ) \otimes_{\mathbb{C}[\![(z-\zeta)^q]\!]} \mathbb{C}[\![z-\zeta]\!]$, where the new $\mathbb{C}[\![z-\zeta]\!]$-module structure comes from the second factor. The chosen basis breaks up everything into summands of rank 1. From each summand in $\mathfrak{p}_{H_1^{00}}$ we obtain a copy of the module

$$\mathbb{C}[\![z-\zeta]\!] \otimes_{\mathbb{C}[\![(z-\zeta)^q]\!]} \mathbb{C}[\![z-\zeta]\!] \cong \mathbb{C}[\![z-\zeta]\!][t]/(t^q) .$$

For the corresponding summand in $\mathfrak{q}_{H_1^{00}}$ we calculate

$$(z-\zeta)^{e_\nu} \otimes 1 = \big((z-\zeta)\otimes 1\big)^{e_\nu} = \big(t + 1 \otimes (z-\zeta)\big)^{e_\nu} ,$$

so we obtain the module

$$\big(t + (z-\zeta)\big)^{e_\nu} \cdot \mathbb{C}[\![z-\zeta]\!][t]/(t^q) .$$

Calculating modulo t^i we must determine the elementary divisors relating the module $\mathbb{C}[\![z-\zeta]\!][t]/(t^i)$ with its fractional ideal $\big(t+(z-\zeta)\big)^{e_\nu}$.

It is not difficult to give a general answer, but in our situation only $e_\nu = 0$ and ± 1 occur. In the former case we obviously obtain i copies of the slope 0. In the case $e_\nu = 1$ we easily find that

$$\big(\mathbb{C}[\![z-\zeta]\!][t]/(t^i)\big)/\big(t+(z-\zeta)\big) \cong \mathbb{C}[\![z-\zeta]\!]/(z-\zeta)^i$$

as $\mathbb{C}[\![z-\zeta]\!]$-module, hence we get one copy of the slope i and $i-1$ copies of the slope 0. The dual argument applies to $e_\nu = -1$, and the lemma follows. **q.e.d.**

Lemma 10.29 *The subobjects $t^{i-j}\underline{H}_j^{00} \subset \underline{H}_i^{00}$ for all $0 \leq j \leq i$ are the only subobjects of $\underline{H}_i^{00}$.*

Proof. Consider a counterexample $\underline{H}_i' \subset \underline{H}_i^{00}$ with i as small as possible. Then we must have $\underline{H}_i' \cap t\underline{H}_{i-1}^{00} = t^{i-j}\underline{H}_j^{00}$ for some $0 \leq j < i$. If $j > 0$, we can divide everything by this, thereby decreasing i. Thus we must have $j = 0$, which means that the composite map $\underline{H}_i' \to \underline{H}_i^{00} \to \underline{H}_1^{00}$ is a monomorphism. Since $\underline{H}_1^{00}$ is simple, our counterexample must induce an isomorphism $\underline{H}_i' \xrightarrow{\sim} \underline{H}_1^{00}$

and hence a decomposition $\underline{H}_i^{00} \cong \underline{H}_{i-1}^{00} \oplus \underline{H}_1^{00}$. As the Hodge polygon is additive in direct sums, this contradicts Lemma 10.28. **q.e.d.**

End of the proof for $s > 1$: We shall show that the Lie algebra of the derived group $G_{\underline{H}}^{\mathrm{der}}$ is sufficiently big. Put

$$\begin{aligned} L &:= \operatorname{Lie} G_{\underline{\tilde{H}}} \subset \operatorname{Lie} \tilde{G}_{\mathrm{amb}} = H_q , \\ L^0 &:= \operatorname{Lie} G_{\underline{\tilde{H}}}^{\mathrm{der}} \subset \operatorname{Lie} \tilde{G}_{\mathrm{amb}}^{\mathrm{der}} = H_q^0 . \end{aligned}$$

We start from the top:

Lemma 10.30 *The composite map $L \hookrightarrow H_q \twoheadrightarrow H_1$ is surjective.*

Proof. The map in question is the derivative of the homomorphism π of 10.27. To prove the surjectivity of $d\pi$ we may work over an algebraic closure $\bar{K}$. Since the kernel of π is unipotent, its restriction to any maximal torus is a monomorphism. Varying the torus it follows that the image of $d\pi$ contains all semisimple elements of the Lie algebra H_1, hence is equal to H_1, as desired. **q.e.d.**

Lemma 10.31 *The composite map $L^0 \hookrightarrow H_q^0 \twoheadrightarrow H_1^0$ is surjective.*

Proof. The map in question is the derivative of the homomorphism $\pi^0 : G_{\underline{\tilde{H}}}^{\mathrm{der}} \longrightarrow \mathcal{R}_{E'/K}\mathrm{SL}_{s,E'}$ (cf. 10.27). Taking derivatives of the commutator maps we obtain a commutative diagram

$$\begin{array}{ccc} L^0 & \xrightarrow{d\pi^0} & H_1^0 \\ \cup & & \parallel \\ [L,L] & \xrightarrow{[d\pi, d\pi]} & [H_1, H_1] . \end{array}$$

The bottom map is surjective by Lemma 10.30, and the equality on the right hand side holds by direct calculation. The desired assertion follows. **q.e.d.**

Lemma 10.32 *The composite map $L^0 \longrightarrow H_q^{00}$ is surjective.*

Proof. The image of L^0 is a $G_{\underline{\tilde{H}}}$-invariant subspace, hence it corresponds to a subobject of $\underline{H}_q^{00}$. By Lemma 10.31 this subobject maps onto $\underline{H}_1^{00}$. By Lemma 10.29 the only such subobject is $\underline{H}_q^{00}$. **q.e.d.**

Lemma 10.33 $L^0 = H_q^0$.

Proof. If $p \nmid r$, this is equivalent to Lemma 10.32. If $p|r$, the same lemma still implies that all non-trivial simple constituents of H_q^0 as representation of $G_{\underline{\tilde{H}}}$ also occur in L^0. It follows that for every $0 \leq i < q$ the subspace

$$\frac{L^0 \cap t^i H_q^0}{L^0 \cap t^{i+1} H_q^0} \hookrightarrow \frac{t^i H_q^0}{t^{i+1} H_q^0} \cong H_1^0$$

surjects onto H_1^{00}. Since $G_{\underline{\tilde{H}}}$ acts on H_1^0 through the epimorphism 10.27, this subspace is invariant under $\mathcal{R}_{E'/K}\mathrm{GL}_{s,E'}$. It is known that any such subspace is equal to H_1^0 (cf. Hiss [18]). By descending induction on i one can now prove $t^i H_q^0 \subset L^0$, whence the lemma. **q.e.d.**

At last, Lemma 10.33 implies that $G_{\underline{H}}^{\mathrm{der}} \subset \tilde{G}_{\mathrm{amb}}^{\mathrm{der}}$ is a subgroup of equal dimension. Since they are connected, we have equality. On the other hand, Proposition 10.26 says that $G_{\underline{\tilde{H}}}$ surjects to $\tilde{G}_{\mathrm{amb}}/\tilde{G}_{\mathrm{amb}}^{\mathrm{der}}$. We deduce that $G_{\underline{\tilde{H}}} = \tilde{G}_{\mathrm{amb}}$, as desired. This finishes the proof of Theorem 10.3. **q.e.d.**

References

[1] Anderson, G., t-motives, *Duke Math. J.* **53** (1986), 457–502.

[2] Borel, A., *Linear algebraic groups*, GTM 126, New York etc.: Springer (1991).

[3] Deligne, P., Théorie de Hodge II, *Publ. Math. IHES* **40** (1971), 5–57.

[4] Deligne, P., Milne, J. S., Tannakian Categories, in: *Hodge Cycles, Motives, and Shimura Varieties*, Deligne, P., et al. (Eds.), ch. VI, LNM **900**, Berlin etc.: Springer (1982), 101–228.

[5] Deligne, P., Catégories tannakiennes, in: *Grothendieck Festschrift*, Boston: Birkhäuser (1990), 111–195.

[6] Drinfeld, V. G., Elliptic Modules (Russian), *Math. Sbornik* **94** (1974), 594–627, = *Math. USSR-Sb.* **23** (1974), 561–592.

[7] Faltings, G., Mumford-Stabilität in der algebraischen Geometrie, in: *Proceedings of the International Congress of Mathematicians, Zürich 1994*, Basel: Birkhäuser (1995), 648–655.

[8] Fontaine, J.-M., Modules galoisiens, modules filtrés et anneaux de Barsotti-Tate, in: *Journées de Géométrie Algébrique de Rennes*, *Astérisque* **65**, Paris: Soc. Math. France (1979), 3–80.

[9] Fontaine, J.-M., Cohomologie de de Rham, cohomologie cristalline et représentations p-adiques, in: *Algebraic Geometry Tokyo-Kyoto*, LNM **1016**, New York etc.: Springer (1983), 86–108.

[10] Gekeler, E.-U., On the de Rham isomorphism for Drinfeld modules, *J. reine angew. Math.* **401** (1989), 189–208.

[11] Gekeler, E.-U., De Rham Cohomology for Drinfeld Modules, *Sém. Théorie des Nombres, Paris 1988–89*, Boston etc.: Birkhäuser (1990), 57–85.

[12] Gieseker, D., Stable Vector Bundles and the Frobenius Morphism, *Ann. scient. Éc. Norm. Sup.*, 4.^{e} série, t. **6** (1973), 95–101.

[13] Gieseker, D., On a theorem of Bogomolov on Chern Classes of Stable Bundles, *Amer. J. Math.* **101** (1979), 77–85.

[14] Goss, D., L-series of t-motives and Drinfeld Modules, in: *The Arithmetic of Function Fields*, Goss, D., et al. (Eds.) Proceedings of the workshop at Ohio State University 1991. Berlin: Walter de Gruyter (1992), 313–402.

[15] Goss, D., *Basic Structures of Function Field Arithmetic*, Ergebnisse **35**, Berlin etc.: Springer (1996).

[16] Hartshorne, R., Ample Vector Bundles on Curves, *Nagoya Math. J.* **43** (1971), 73–89.

[17] Hartshorne, R., Ample Vector Bundles, *Publ. Math. IHES* **29** (1966), 63–94.

[18] Hiss, G., Die adjungierten Darstellungen der Chevalley-Gruppen, *Arch. Math.* **42** (1984), 408–416.

[19] Katz, N. M., Slope filtration of F-crystals, *Astérisque* **63**, Paris: Soc. Math. France (1979), 113–164.

[20] Laffaille, G., Groupes p-divisibles et modules filtrés: le cas peu ramifié, *Bull. Soc. Math. France* **108** (1980), 187–206.

[21] Maruyama, M., The Theorem of Grauert-Mülich-Spindler, *Math. Ann.* **255** (1981), 317–333.

[22] Mumford, D., Fogarty, J., *Geometric Invariant Theory*, 2nd enlarged edition, Berlin etc.: Springer (1982).

[23] Pink, R., The Mumford-Tate conjecture for Drinfeld modules, Preprint Mannheim, (Sept. 1996), 26 S., to be published in *Publ. RIMS, Kyoto University* **33**, 3.

[24] Pink, R., ℓ-adic algebraic monodromy groups, cocharacters, and the Mumford-Tate conjecture, *Preprint Mannheim* (March 1997), 54p.

[25] Pink, R., Motives and Hodge structures over function fields, Summary of a talk at the Arbeitstagung in Bonn, (June 1997), 6p.

[26] Pink, R., Analytic uniformization of motives over function fields, in preparation.

[27] Ramanan, S., Ramanathan, R., Some Remarks on the Instability Flag, *Tôhoku Math. J.* **36** (1984), 269–291.

[28] Totaro, B., Tensor Products in p-adic Hodge Theory, *Duke Math. J.* **83** (1996), 79–104.