The Isogeny Conjecture for A-Motives

Richard Pink*

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Dedicated to Günter Harder on the occasion of his 70th birthday

Abstract

We prove the isogeny conjecture for A-motives over finitely generated fields K of transcendence degree $\leqslant 1$. This conjecture says that for any semisimple A-motive M over K, there exist only finitely many isomorphism classes of A-motives M' over K for which there exists a separable isogeny $M' \to M$. The result is in precise analogy to known results for abelian varieties and for Drinfeld modules and will have strong consequences for the $\mathfrak{p}$-adic and adelic Galois representations associated to M. The method makes essential use of the Harder-Narasimhan filtration for locally free coherent sheaves on an algebraic curve.

*Dept. of Mathematics, ETH Zurich, 8092 Zurich, Switzerland, pink@math.ethz.ch

1 Introduction

The aim of this article is to prove the following result, called *the isogeny conjecture for A-motives* (in the case of transcendence degree $\leqslant 1$):

Theorem 1.1 *Let K be a field which is finitely generated of transcendence degree $\leqslant 1$ over a finite field $\mathbb{F}_q$. Let M be a semisimple A-motive over K. Then there exist only finitely many isomorphism classes of A-motives M' over K for which there exists a separable isogeny $M' \to M$.*

For the meaning of A and the other concepts involved see below. Caution: The direction of the isogeny $M' \to M$ must not be reversed: see Counterexample 1.6 below.

The concept of A-motives were invented by Anderson [1] in the case $A = \mathbb{F}_q[t]$ and under the name of t-motives. They can be viewed as analogues of abelian varieties or more general Grothendieck motives, with the essential difference that both the field of definition and the ring of coefficients of an A-motive have positive characteristic. Many related concepts, theorems, and conjectures for abelian varieties possess natural analogues for A-motives, and vice versa. The isogeny conjecture is an analogue of a result for abelian varieties proved by Faltings [3] resp. Zarhin [25].

A special class of A-motives arises from Drinfeld modules. The isogeny conjecture for these translates directly into the isogeny conjecture for Drinfeld modules, which was proved by Taguchi in [17], [21]. The isogeny conjecture for direct sums thereof was proved by the present author with Traulsen in [12], resp. with Rütsche in [13].

As in the case of abelian varieties, the isogeny conjecture can be used to deduce the Tate conjecture for endomorphisms and the semisimplicity conjecture, proved previously by Taguchi [17], [18], [19], [20], Tamagawa [22], [23], [24], resp. Stalder [16]. The isogeny conjecture also has consequences for the $\mathfrak{p}$-adic and adelic Galois representations associated to A-motives beyond the results in [4], [8], [9], [10], [11], [12], [13], [14]. We plan to discuss these, and possibly the generalization to finitely generated fields K of arbitrary transcendence degree, in a later article.

In the rest of the introduction we define the concepts involved in the isogeny conjecture, explain why the assumptions in the conjecture are necessary, and describe the strategy of proof. For more of the theory of A-motives see Anderson [1], Goss [5].

Let $\mathbb{F}_q$ be a finite field with q elements. Throughout the article, tensor products of rings and modules and fiber products of schemes over $\mathbb{F}_q$ are taken over $\mathbb{F}_q$ except where indicated otherwise. Let C be an irreducible smooth projective curve over $\mathbb{F}_q$. Fix a closed point $\infty \in C$ and set $C^\circ := C \smallsetminus \{\infty\}$. Let $A := \Gamma(C^\circ, \mathcal{O}_{C^\circ})$ denote the ring of regular functions on C°. Consider a field K together with a ring homomorphism $\gamma : A \to K$. Then $\mathfrak{p}_0 := \ker(\gamma)$ is either zero or a maximal ideal of A; we allow both possibilities.

Let σ denote the Frobenius endomorphism $s \mapsto s^q$ of K. As σ is the identity on $\mathbb{F}_q$, it induces an endomorphism $\mathrm{id}\otimes\sigma$ of the ring $A\otimes K := A\otimes_{\mathbb{F}_q} K$. For any $A\otimes K$-module M, an

$\mathrm{id}\otimes\sigma$*-linear map* $\tau : M \to M$ is an additive map which satisfies $\tau((a\otimes u)\cdot m) = (a\otimes u^q)\cdot\tau(m)$ for all $(a, u, m) \in A\times K\times M$. Setting $(\mathrm{id}\otimes\sigma)_*M := M \otimes_{K,\sigma} K$, giving an $\mathrm{id}\otimes\sigma$-linear map $\tau : M \to M$ is equivalent to giving an $A\otimes K$-linear map $\tau^{\mathrm{lin}} :\ (\mathrm{id}\otimes\sigma)_*M \to M$, called the *linearization of* τ. Note that τ^{lin} is injective if and only if its cokernel is $A\otimes K$-torsion.

Definition 1.2 *An* A-motive of characteristic γ over K *is a finitely generated projective* $A\otimes K$*-module* M *together with an* $\mathrm{id}\otimes\sigma$*-linear map* $\tau : M \to M$*, such that* $a\otimes 1 - 1\otimes\gamma(a)$ *is nilpotent on* $\mathrm{coker}(\tau^{\mathrm{lin}})$ *for every* $a \in A$*.*

Definition 1.3 *Let* M *and* N *be* A*-motives of characteristic* γ *over* K*. An* $A\otimes K$*-linear map* $f : M \to N$ *that commutes with* τ *is called a* homomorphism*. An injective homomorphism whose cokernel is* $A\otimes K$*-torsion is called an* isogeny*. If an isogeny* $M \to N$ *exists, then* M *and* N *are called* isogenous*. An isogeny* f *is called* separable *if* τ^{lin} *induces an isomorphism* $(\mathrm{id}\times\sigma)_*\,\mathrm{coker}(f) \to \mathrm{coker}(f)$*.*

Basic facts on isogenies (not used in this paper) include the following: Any composite of isogenies is an isogeny. Any element $a \in A\smallsetminus\{0\}$ defines an isogeny $a_M : M \to M$, $m \mapsto am$. A homomorphism $f : M \to N$ is an isogeny if and only if there exists a homomorphism $g : N \to M$ such that $gf = a_M$, or equivalently $fg = a_N$, for some $a \in A \smallsetminus \{0\}$. In particular g is then an isogeny, and being isogenous is an equivalence relation.

If $\mathfrak{p}_0 = 0$, every isogeny is separable. In general a composite of isogenies is separable if and only if its constituents are separable. If $\mathfrak{p}_0 \neq 0$ and $M \neq 0$, the isogeny $a_M : M \to M$ is separable if and only $a \in A \smallsetminus \mathfrak{p}_0$. If $\mathfrak{p}_0 \neq 0$ and $f : M \to N$ is a separable isogeny, it may or may not be possible to choose the 'dual' isogeny $g : N \to M$ above separable as well. Thus in general the existence of a separable isogeny $M \to N$ is not an equivalence relation.

Definition 1.4 *An* A*-motive* M *over* K *is called* simple up to isogeny*, or just* simple*, if it is non-zero and every non-zero injective homomorphism of* A*-motives* $N \hookrightarrow M$ *is an isogeny. An* A*-motive is called* semisimple up to isogeny*, or just* semisimple*, if it is isogenous to a direct sum of simple* A*-motives.*

Now we discuss the different assumptions in Theorem 1.1.

The assumption that K is finitely generated appears for the same reason as in the Tate conjecture for endomorphisms. Indeed—as for abelian varieties—the isogeny conjecture for M over K implies the Tate conjecture for M over K, i.e., the isomorphy

$$\mathrm{End}_K(M) \otimes_A A_{\mathfrak{p}} \xrightarrow{\sim} \mathrm{End}_{A_{\mathfrak{p}}[\mathrm{Gal}(K^{\mathrm{sep}}/K)]}\big(T_{\mathfrak{p}}(M)\big), \tag{1.5}$$

where $T_{\mathfrak{p}}(M)$ is the $\mathfrak{p}$-adic Tate module of M for any prime $\mathfrak{p} \neq \mathfrak{p}_0$ of A. This statement gives a lower bound on the image of Galois in terms of $\mathrm{End}_K(M)$. Since this endomorphism ring can be small, even when $K = K^{\mathrm{sep}}$, the isomorphy cannot hold without strong restrictions on K.

Next, the assumption that the isogeny $M' \to M$ be separable is vacuous if $\mathfrak{p}_0 = 0$. But in the case $\mathfrak{p}_0 \neq 0$ it is really necessary, as the following example shows. The example also shows that Theorem 1.1 becomes false if instead of a separable isogeny $M' \to M$ one requires a separable isogeny $M \to M'$.

Counterexample 1.6 Take $A := \mathbb{F}_q[t]$ and $K := \mathbb{F}_q(x)$ with $\gamma : A \to K$, $\sum \alpha_i t^i \mapsto \alpha_0$. Then $\mathfrak{p}_0 = (t) \neq 0$. We first do the construction with Drinfeld modules, where everything is dual, and then translate it into A-motives. For any $n \geqslant 0$ consider the Drinfeld A-module $\varphi_n : A \to K[\tau]$ sending t to $x^{q^n}\tau + \tau^2$, which is of rank 2 and characteristic γ. The calculations

$$\begin{aligned} \tau \cdot (x^{q^n}\tau + \tau^2) &= (x^{q^{n+1}}\tau + \tau^2) \cdot \tau, \\ (x^{q^n} + \tau) \cdot (x^{q^{n+1}}\tau + \tau^2) &= (x^{q^n}\tau + \tau^2) \cdot (x^{q^n} + \tau) \end{aligned}$$

show that we have an inseparable isogeny $\tau : \varphi_n \to \varphi_{n+1}$ and a separable isogeny $x^{q^n} + \tau : \varphi_{n+1} \to \varphi_n$. Taking composites we find an inseparable isogeny $\varphi_0 \to \varphi_n$ and a separable isogeny $\varphi_n \to \varphi_0$. Moreover, we claim that all φ_n are pairwise non-isomorphic. Indeed, an isomorphism $\varphi_n \to \varphi_{n'}$ is an element $u \in K^\times$ with

$$ux^{q^n}\tau + u\tau^2 \;=\; u \cdot (x^{q^n}\tau + \tau^2) \;\stackrel{!}{=}\; (x^{q^{n'}}\tau + \tau^2) \cdot u \;=\; x^{q^{n'}} u^q \tau + u^{q^2}\tau^2.$$

This means that $u^{q-1} = x^{q^n - q^{n'}}$ and $u^{q^2-1} = 1$. Since x is transcendental over $\mathbb{F}_q$, these equations cannot be simultaneously fulfilled unless $n = n'$, proving the claim.

Finally, by Anderson [1] there is a fully faithful contravariant functor $\varphi \mapsto M_\varphi$ from the category of Drinfeld A-modules over K to the category of A-motives over K. Moreover M_φ is always simple. Thus $M := M_{\varphi_0}$ is a simple A-motive over K, for which there exist infinitely many pairwise non-isomorphic A-motives M_{φ_n} over K with inseparable isogenies $M_{\varphi_n} \to M$ and separable isogenies $M \to M_{\varphi_n}$.

Counterexample 1.7 The statement in Theorem 1.1 also becomes false when M is not semisimple. Suppose for instance that we have a short exact sequence of A-motives $0 \to M' \to M \to M'' \to 0$ where M' and M'' are simple, but M not semisimple. Fix a maximal ideal $\mathfrak{p} \neq \mathfrak{p}_0$ of A, and for every integer $n \geqslant 0$ consider the A-submotive $M_n := M' + \mathfrak{p}^n M \subset M$. Then the inclusion $M_n \hookrightarrow M$ is a separable isogeny, because so is the composite isogeny $a^n : M \to a^n M \subset M_n \subset M$ for any $a \in \mathfrak{p} \setminus \mathfrak{p}_0$. We claim that no infinite set of M_n can be pairwise isomorphic. Therefore the M_n form infinitely many isomorphism classes.

Proof. For any n consider the short exact sequence

$$0 \longrightarrow M' \longrightarrow M_n \longrightarrow \mathfrak{p}^n M'' \longrightarrow 0.$$

Taken modulo $\mathfrak{p}^n$ the construction provides a splitting

$$\begin{tikzcd} 0 \arrow[r] & M'/\mathfrak{p}^n M' \arrow[r] & M_n/\mathfrak{p}^n M_n \arrow[r] & \mathfrak{p}^n M''/\mathfrak{p}^{2n} M'' \arrow[r] & 0. \\ & & \mathfrak{p}^n M/\mathfrak{p}^n M_n \arrow[u, phantom, "\cup"] \arrow[ur, "\cong"'] & & \end{tikzcd}$$

It follows that the short exact sequence of Tate modules

$$(S_n) \qquad\qquad 0 \longrightarrow T_{\mathfrak{p}}(M') \longrightarrow T_{\mathfrak{p}}(M_n) \longrightarrow T_{\mathfrak{p}}(\mathfrak{p}^n M'') \longrightarrow 0$$

possesses a $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-equivariant splitting modulo $\mathfrak{p}^n$.

Suppose now that some M_n is isomorphic to infinitely many other M_{n_i}. Then any isomorphism $f : M_n \stackrel{\sim}{\to} M_{n_i}$ must map $M' \subset M_n$ to itself, because otherwise it would induce an isogeny $(\mathrm{id}, f|M') : M' \oplus M' \hookrightarrow M_{n_i} \hookrightarrow M$ and show that M is semisimple, contrary to the assumption. In the resulting commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d] & M_n \arrow[r] \arrow[d, "f"', "\cong"] & \mathfrak{p}^n M'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r] & M_{n_i} \arrow[r] & \mathfrak{p}^{n_i} M'' \arrow[r] & 0,
\end{tikzcd}$$

the right hand vertical map is surjective, hence an isomorphism, and therefore all vertical maps are isomorphisms. Thus it induces an isomorphism between the exact sequences (S_n) and (S_{n_i}), and so the splitting of (S_{n_i}) modulo $\mathfrak{p}^{n_i}$ yields a $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-equivariant splitting of (S_n) modulo $\mathfrak{p}^{n_i}$. This being the case for infinitely many n_i, a compactness argument shows that such a splitting exists already for the sequence (S_n) itself. In other words, there exists a $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-equivariant $A_{\mathfrak{p}}$-linear map $T_{\mathfrak{p}}(M_n) \to T_{\mathfrak{p}}(M')$ whose restriction to $T_{\mathfrak{p}}(M')$ is the identity. By the Tate conjecture for homomorphisms ([19], [20], [22], [23], [24]) this map can be expressed as an $A_{\mathfrak{p}}$-linear combination of homomorphisms of A-motives $M_n \to M'$. Then for at least one of these homomorphisms the restriction to M' is non-zero. If N denotes its kernel, we obtain isogenies $M' \oplus N \to M_n \to M$, again contradicting the assumption that M is not semisimple. **q.e.d.**

Now we will sketch the proof of Theorem 1.1, while disregarding several technical difficulties that are addressed in the body of this article. Abbreviate $C_K := C \times \operatorname{Spec} K$ and $C_K^\circ := C^\circ \times \operatorname{Spec} K = \operatorname{Spec}(A \otimes K)$, where the fiber product is taken over $\operatorname{Spec} \mathbb{F}_q$. Every finitely generated projective $A \otimes K$-module M is the group of global sections of a locally free coherent sheaf on C_K°. Let $\mathcal{G}$ be the dual sheaf thereof. Then giving an $A \otimes K$-linear map $\tau^{\mathrm{lin}} : (\mathrm{id} \otimes \sigma)_* M \to M$ is equivalent to giving a homomorphism of coherent sheaves $\kappa : \mathcal{G} \to (\mathrm{id} \times \sigma)^* \mathcal{G}$. Moreover M is an A-motive of characteristic γ if and only if κ is an isomorphism outside the closed point $\theta \in C_K^\circ$ corresponding to γ (see Proposition 8.3). We call the pair $(\mathcal{G}, \kappa)$ a *κ-sheaf of characteristic θ on C_K°*. In a natural way, isogenies of A-motives $M' \hookrightarrow M$ correspond to inclusions of κ-sheaves of equal rank $\mathcal{G} \hookrightarrow \mathcal{G}'$.

In order to use finiteness results in algebraic geometry, we must compactify the situation. To this end we extend $\mathcal{G}$ to a locally free coherent sheaf $\overline{\mathcal{G}}$ on C_K. Set $\infty_K := \infty \times \operatorname{Spec} K$. Then κ extends to a homomorphism $\kappa : \overline{\mathcal{G}} \to (\mathrm{id} \times \sigma)^* \overline{\mathcal{G}}(d\infty_K)$ for some integer d. We call the pair $(\overline{\mathcal{G}}, \kappa)$ a *κ-sheaf of pole order $\leqslant d$ on C_K*.

This extension plays a role similar to that of a polarization of an abelian variety. In fact, following Faltings's proof for abelian varieties we should define a height for A-motives, prove that this height remains bounded under separable isogenies, and prove that for any r and h there are only finitely many isomorphism classes of A-motives over K of rank r and height $\leqslant h$. But the definition of a height requires the extra structure of a polarization, which is somehow related to the infinite prime. This makes it natural to look for some data at ∞ as an analogue of a polarization. Of course, the analogy is not complete, because our data has nothing to do with a symplectic pairing.

From a different point of view, only the extension to ∞ allows us to define numerical invariants of $\overline{\mathcal{G}}$. A natural analogue of the degree of a polarization consists of the pole order d together with the slopes in the Harder-Narasimhan filtration of $\overline{\mathcal{G}}$. Bounding these invariants should be necessary and sufficient for our objects to be parametrized by a moduli stack of finite type. Once these algebro-geometric numerical invariants are bounded, the remaining arithmetic problem can be interpreted as bounding the number of K-rational points of height $\leqslant h$ on this moduli stack. Our method is guided by these principles, although we do not formally speak of moduli stacks or heights.

A crucial result in our case is that any κ-sheaf that is isogenous to a given semisimple κ-sheaf $\mathcal{G}$ possesses an extension whose numerical invariants are bounded only in terms of $\mathcal{G}$. This is proved in Proposition 8.21 by the following argument. Set $r := \mathrm{rank}(\mathcal{G})$ and fix an extension $\overline{\mathcal{G}}$ of $\mathcal{G}$ of pole order $\leqslant d$. Then for any inclusion of κ-sheaves of equal rank $\mathcal{G} \hookrightarrow \mathcal{G}'$, there exists an extension $\overline{\mathcal{G}}'$ of $\mathcal{G}'$ that coincides with $\overline{\mathcal{G}}$ at ∞, and which therefore also is of pole order $\leqslant d$. If $\mathcal{G}$ is simple, in a sense analogous to 1.4, so is $\mathcal{G}'$, and in this case we prove that the slopes in the Harder-Narasimhan filtration of $\overline{\mathcal{G}}'$ lie in an interval of length $< rd\ell_C$, where ℓ_C denotes the degree of ∞ over $\mathbb{F}_q$. Then a suitable twist $\overline{\mathcal{G}}'(n\infty_K)$ is another extension of $\mathcal{G}'$ of pole order $\leqslant d$, all of whose slopes lie in a fixed bounded interval. If $\mathcal{G}$ is only semisimple, i.e., isogenous to a direct sum of simple κ-sheaves, we extend this argument by allowing different twists of $\overline{\mathcal{G}}'$ in the directions corresponding to different simple summands of $\mathcal{G}$. The semisimplicity assumption allows us to construct independent twists in all directions, and this freedom suffices to obtain the same bound on the slopes. This is the only place in the argument where the semisimplicity assumption comes in.

The result just sketched already implies Theorem 1.1 when K is finite, even for all isogenies instead of just separable ones: see Theorem 8.23. Indeed, when K is finite it is standard knowledge that there are only finitely many isomorphism classes of locally free coherent sheaves $\overline{\mathcal{G}}'$ on C_K of given rank and slopes. For any such $\overline{\mathcal{G}}'$, the associated homomorphism κ lies in the group $\mathrm{Hom}\big(\overline{\mathcal{G}}', (\mathrm{id}{\times}\sigma)^*\overline{\mathcal{G}}'(d\infty_K)\big)$, which is a finite dimensional vector space over K. Thus there are at most finitely many possibilities for κ. Forgetting the extension to ∞ it follows that there are only finitely many possibilities for the isomorphism class of the κ-sheaf $\mathcal{G}'$ and hence for the A-motive M', as desired.

Assume now that K has transcendence degree 1 over $\mathbb{F}_q$. Let X be the irreducible smooth projective curve over $\mathbb{F}_q$ with function field K. Over its generic point η_X we do essentially the same as above. Next we define a κ-sheaf of pole order $\leqslant d$ on the surface $C{\times}X$ as a locally free sheaf $\mathcal{F}$ on $C{\times}X$ together with an injective homomorphism $\kappa : \mathcal{F} \hookrightarrow$

$(\mathrm{id}\times\sigma)^*\mathcal{F}(d,0)$, where $(d,0)$ indicates a twist of $d\infty$ in the direction of C and no twist in the direction of X. We show that every κ-sheaf of pole order $\leqslant d$ on $C\times\eta_X$ possesses a unique minimal extension to $C\times X$ that is contained in all other extensions. This minimal extension is an analogue of the Néron model of an abelian variety. It can be viewed as containing information on good reduction and degeneration and thus on the height of the original κ-sheaf over K.

We regard this height as being encoded in the slopes of the Harder-Narasimhan filtration of $\mathcal{F}$ along the fibers $c\times X$ for all points $c\in C$. We show that these slopes remain bounded under separable isogenies. This is the only place where the separability assumption comes in.

It remains to prove that there are at most finitely many isomorphism classes of κ-sheaves $\mathcal{F}$ of rank r and pole order $\leqslant d$ on $C\times X$ that satisfy the indicated bounds along $C\times\eta_X$ and $c\times X$ for all $c\in C$. This is done in Section 7. Actually, the result depends on a further minimality condition 7.1 (f) which requires some additional effort to achieve.

The method involves the sheaves $\mathcal{G}_n := \mathrm{pr}_{1*}(\mathcal{F}(0,n))$ on C for suitable twists $(0,n)$ in the direction of X. The bounds along $c\times X$ for all $c\in C$ imply that the homomorphism $\mathrm{pr}_1^*\,\mathcal{G}_n \to \mathcal{F}(0,n)$ obtained by adjunction is surjective whenever n is greater than some explicit bound. The rank of $\mathcal{G}_n$ can also be determined explicitly. In Section 5 we show that for the desired finiteness it suffices to bound the slopes in the Harder-Narasimhan filtration of $\mathcal{G}_n$ from above and below.

The main problem here is to control the gaps between successive slopes of $\mathcal{G}_n$. For this fix a suitable n and let μ be the largest of the slopes of $\mathcal{G}_n$ with the property that $\mathcal{G}_n$ has no slope in the interval $[\mu - d\ell_C, \mu)$ for some explicit constant $d\ell_C$. Then all slopes of $\mathcal{G}_n$ are $\leqslant \mu + \mathrm{rank}(\mathcal{G}_n)\cdot d\ell_C$, and a priori we have no control over the smaller slopes. But let $\mathcal{G}_n^\mu$ be the corresponding step in the Harder-Narasimhan filtration of $\mathcal{G}_n$. Using the relation between the Harder-Narasimhan filtration and the homomorphism $\kappa : \mathcal{F} \to (\mathrm{id}\times\sigma)^*\mathcal{F}(d,0)$ we show in Lemma 7.18 that the image of the homomorphism $\mathrm{pr}_1^*\,\mathcal{G}_n^\mu \to \mathcal{F}(0,n)$ obtained by adjunction coincides generically with a κ-invariant subsheaf of $\mathcal{F}$.

Suppose for the moment that $\mathcal{F}$ is simple, i.e., that any non-zero κ-invariant subsheaf has equal rank. Then the κ-invariant subsheaf obtained is equal to $\mathcal{F}$, and so the homomorphism $\mathrm{pr}_1^*\,\mathcal{G}_n^\mu \to \mathcal{F}(0,n)$ is generically surjective. Ideally, we would like to deduce from this that $\mathcal{G}_n$ has in fact no slopes $<\mu$, but we are unable to do so. Instead, using standard methods for coherent sheaves, in Lemma 7.22 we prove that all slopes of $\mathcal{G}_{n'}$ are $\geqslant \mu$ whenever n' is greater than some explicit bound in terms of n.

This leaves us with the new problem of bounding the slopes of $\mathcal{G}_{n'}$ from above. Here the homomorphism $\kappa : \mathcal{F} \to (\mathrm{id}\times\sigma)^*\mathcal{F}(d,0)$ comes to our aid, because it induces homomorphisms between the sheaves $\mathcal{G}_{n''}$ for different indices n''. More precisely, it induces injective homomorphisms $\mathcal{G}_{qn''} \hookrightarrow \mathcal{G}_{n''+a}(d\infty_K)^{\oplus N}$ for all n'', where a and N are fixed: see Lemma 7.12. Suppose for ease of presentation that $X = \mathbb{P}^1$, in which case we can take $a=0$. Then by iteration the slopes of $\mathcal{G}_{q^j n}$ are bounded above in terms of the slopes of $\mathcal{G}_n$ up to adding a linear multiple of j: see Lemma 7.24. It then becomes crucial that $q^j n$ grows exponentially with j, while the bound itself grows only linearly with j. This seems to be a manifestation of the strong contracting properties of Frobenius. Combining these arguments we can find a sequence of explicit numbers $n' > n$ such that all slopes of $\mathcal{G}_{n'}$ lie in the interval $[\mu, \mu+\text{some explicit constant}]$.

From this we can also deduce upper and lower bounds for μ. Indeed, for any two numbers $n'' > n'$ with the above property, the difference $\deg(\mathcal{G}_{n''}) - \deg(\mathcal{G}_{n'})$ is on the one hand a certain multiple of $(n'' - n')\mu$ plus a bounded number. On the other hand the Riemann-Roch formula expresses $\deg(\mathcal{G}_{n''})$ and $\deg(\mathcal{G}_{n'})$ as $\chi(C\times X, \mathcal{F})$ plus something linear in n'', resp. n', where the coefficients depend only on the given numerical invariants of $\mathcal{F}$: see Proposition 5.3 (i). The unknown value $\chi(C\times X, \mathcal{F})$ vanishes in the difference, and solving the resulting equation for μ yields the desired upper and lower bounds.

As explained above, these bounds for the sheaf $\mathcal{G}_{n'}$ in place of $\mathcal{G}_n$ imply the desired finiteness for $\mathcal{F}$, which finishes the proof if $\mathcal{F}$ is simple. In the general case we need to carry out the above arguments for different values of μ and obtain upper and lower bounds for slopes related to a filtration of $\mathcal{F}$ by κ-invariant subsheaves; for details see Section 7.

One further point which calls for an explanation is the passage from the A-motive M to its dual. Its immediate effect is that the associated sheaf on $C\times X$ has a homomorphism $\kappa : \mathcal{F} \hookrightarrow (\mathrm{id}\times\sigma)^*\mathcal{F}$ instead of the other way around. During the development of the proof we have found this more convenient in some ways, although not in others; it can possibly be avoided. We have not determined whether there is a relation with the dualization in [2].

Finally, we review the content of the individual sections. Section 1 is the present introduction. The next three sections collect known preparatory information on different topics: Section 2 on locally free coherent sheaves on regular schemes of dimension $\leqslant 2$, Section 3 on the Harder-Narasimhan filtration for locally free coherent sheaves on a smooth projective curve, and Section 4 on Frobenius. In Section 5 we use standard methods to prove a finiteness result for locally free coherent sheaves $\mathcal{F}$ on a product of two curves $C\times X$ under suitable assumptions on the Harder-Narasimhan slopes of $\mathcal{F}$ and of $\mathcal{G}_n := \mathrm{pr}_{1*}(\mathcal{F}(0,n))$.

In Section 6 we explain the basic notion of κ-sheaves over different base schemes. The remaining three sections contain the hard work. In Section 7 we use the finiteness result from Section 5 to derive the much more subtle finiteness result for κ-sheaves on $C\times X$. In some sense it is analogous to the Shafarevich conjecture for abelian varieties, proved by Faltings [3], which asserts that there are only finitely many isomorphism classes of abelian varieties of fixed dimension over a global field K which possess a polarization of a given degree and good reduction outside a given finite set of places of K. Section 8 deals with the relation between A-motives and κ-sheaves over a field from different angles and discusses various technical constructions. It also proves Theorem 1.1 in the case that K is finite. The final Section 9 combines everything over $C\times X$ and proves Theorem 1.1 in the case of transcendence degree 1.

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2 Locally free sheaves

In this section we recall some basic properties of locally free sheaves. First note that any torsion free coherent sheaf on a regular noetherian scheme of dimension 1 is locally free. In dimension 2 we have:

Proposition 2.1 *Let Z be a regular noetherian scheme of equidimension 2 and $j : U \hookrightarrow Z$ an open embedding with finite complement. Then:*

(a) For any locally free coherent sheaf $\mathcal{F}$ on Z, the adjunction homomorphism $\mathcal{F} \to j_*j^*\mathcal{F}$ is an isomorphism.

(b) For any locally free coherent sheaf $\mathcal{G}$ on U, the direct image $j_*\mathcal{G}$ is a locally free coherent sheaf on Z.

Proof. The assertion being local on Z, we may assume that $Z = \operatorname{Spec} R$ for a regular noetherian local ring R of Krull dimension 2 and that U is the complement of the closed point. Fix local parameters u and v which generate the maximal ideal of R and consider the closed embedding $i : Y = \operatorname{Spec} R/(u) \hookrightarrow Z$. The proof of Langton [7, §3 Prop. 6], adapted almost verbatim to the present situation, implies (a) and shows that in (b), the sheaf $j_*\mathcal{G}$ is coherent and its pullback $i^*j_*\mathcal{G}$ is torsion free. But since Y is regular of dimension 1, it follows that $i^*j_*\mathcal{G}$ is locally free. Its rank is then the rank of $\mathcal{G}$, and so by the Nakayama lemma the stalk of $j_*\mathcal{G}$ at the closed point has the same number of generators as the stalk at the generic point of Z. Thus $j_*\mathcal{G}$ is locally free, as desired. **q.e.d.**

Proposition 2.2 *Let Z be a regular noetherian scheme of equidimension 2. Then for any homomorphism $f : \mathcal{G} \to \mathcal{F}$ of locally free coherent sheaves on Z and any locally free coherent subsheaf $\mathcal{F}' \subset \mathcal{F}$, the sheaf $f^{-1}(\mathcal{F}')$ is locally free.*

Proof. As $f^{-1}(\mathcal{F}')$ is a torsion free coherent sheaf on a regular noetherian scheme, it is locally free at all points of codimension 1. Thus the set of points $U \subset Z$ where $f^{-1}(\mathcal{F}')$ is locally free is open and its complement has codimension 2. Consider the commutative diagram obtained by combining the definition of $f^{-1}(\mathcal{F}')$ with the adjunction homomorphism $\mathrm{id} \to j_*j^*$ for the open embedding $j : U \hookrightarrow Z$:

$$\begin{tikzcd}
\mathcal{G} \arrow[rr] \arrow[dr, equal] & & \mathcal{F} \arrow[dr, equal] & \\
& j_*j^*\mathcal{G} \arrow[rr] & & j_*j^*\mathcal{F} \\
f^{-1}(\mathcal{F}') \arrow[uu, hook] \arrow[rr] \arrow[dr, hook] & & \mathcal{F}' \arrow[uu, hook] \arrow[dr, equal] & \\
& j_*j^*f^{-1}(\mathcal{F}') \arrow[uu, hook] \arrow[rr] & & j_*j^*\mathcal{F}' \arrow[uu]
\end{tikzcd}$$

Here the three indicated oblique equalities result from Proposition 2.1 (a). The definition of $f^{-1}(\mathcal{F}')$ thus implies that the fourth oblique arrow $f^{-1}(\mathcal{F}') \hookrightarrow j_*j^*f^{-1}(\mathcal{F}')$ is also an equality. Since the latter is locally free by Proposition 2.1 (b), we are done. **q.e.d.**

Proposition 2.3 *Let Z be a regular noetherian scheme of equidimension 2 and $j : U \hookrightarrow Z$ an open dense embedding. Let $z_1, \ldots, z_n$ be the generic points of $Z \smallsetminus U$ of codimension 1 in Z, and abbreviate $Z_i := \operatorname{Spec} \mathcal{O}_{Z,z_i}$.*

(a) For any locally free coherent sheaves $\mathcal{F}_U$ on U and $\mathcal{F}_i$ on Z_i for all i, which coincide at all generic points of Z, there exists a unique locally free coherent sheaf $\mathcal{F}$ on Z whose restrictions to U and Z_i are $\mathcal{F}_U$ and $\mathcal{F}_i$, respectively.

(b) For any locally free coherent sheaves $\mathcal{F}'$ and $\mathcal{F}$ on Z and any homomorphisms $f_U : \mathcal{F}'|U \to \mathcal{F}|U$ and $f_i : \mathcal{F}'|Z_i \to \mathcal{F}|Z_i$ for all i, which agree at all generic points of Z, there exists a unique homomorphism $f : \mathcal{F}' \to \mathcal{F}$ extending f_U and all f_i.

Proof. By induction on n it suffices to prove this in the case $n = 1$.

For (a) choose any extension of $\mathcal{F}_1$ to a locally free coherent sheaf $\widetilde{\mathcal{F}}_1$ on some irreducible open neighborhood $U_1 \subset Z$ of Z_1. Then $\widetilde{\mathcal{F}}_1$ coincides with $\mathcal{F}_U$ outside some proper closed subset $T \subset U_1 \cap U$. For dimension reasons z_1 is not contained in the closure $\overline{T}$ of T in Z. Thus after replacing U_1 by $U_1 \smallsetminus \overline{T}$, the sheaves $\widetilde{\mathcal{F}}_1$ and $\mathcal{F}_U$ coincide on $U_1 \cap U$ and are therefore the restrictions of a locally free coherent sheaf on $U_1 \cup U$. But this sheaf extends to a locally free coherent sheaf on Z by Proposition 2.1 (b), proving the existence part of (a). The uniqueness part of (a) follows from (b) applied to the identity maps f_U and f_1 for two extensions.

In (b) the homomorphism f_1 extends to a homomorphism $\tilde{f}_1 : \mathcal{F}'|U_1 \to \mathcal{F}|U_1$ for some irreducible open neighborhood $U_1 \subset Z$ of Z_1. Since $\mathcal{F}'$, $\mathcal{F}$ are locally free and f_U, $\tilde{f}_1$ coincide at the generic point of the integral scheme $U_1 \cap U$, the restrictions of f_U, $\tilde{f}_1$ to $U_1 \cap U$ must coincide. They therefore induce a homomorphism $\mathcal{F}'|U_1 \cap U \to \mathcal{F}|U_1 \cap U$. Proposition 2.1 (a) implies that this homomorphism extends to a homomorphism $f : \mathcal{F}' \to \mathcal{F}$, proving the existence part of (b). The uniqueness of f follows from the fact that Z is regular and $\mathcal{F}'$, $\mathcal{F}$ are locally free. **q.e.d.**

Now let C and X be irreducible smooth curves over a field k with generic points η_C and η_X. Consider the natural inclusions

$$\begin{tikzcd} \eta_C \times \eta_X \arrow[r, hook] \arrow[d, hook] & C \times \eta_X \arrow[d, hook] \\ \eta_C \times X \arrow[r, hook] & C \times X, \end{tikzcd}$$

where all fiber products are taken over $\operatorname{Spec} k$. Here $\eta_C \times \eta_X$ is simultaneously a subscheme of the curve $C \times \eta_X$ over η_X and a subscheme of the curve $\eta_C \times X$ over η_C. Viewed as a subscheme of the surface $C \times X$, it consists of the generic points of $C \times X$ and the generic points of all irreducible curves in $C \times X$ which map surjectively to both C and X.

Proposition 2.4 *(a) For any locally free coherent sheaves $\mathcal{G}$ on $C \times \eta_X$ and $\mathcal{H}$ on $\eta_C \times X$ which coincide over $\eta_C \times \eta_X$, there exists a unique locally free coherent sheaf $\mathcal{F}$ on $C \times X$ extending both $\mathcal{G}$ and $\mathcal{H}$.*

(b) *For any locally free coherent sheaves $\mathcal{F}'$ and $\mathcal{F}$ on $C\times X$ and any homomorphisms $g : \mathcal{F}'|\, C\times\eta_X \to \mathcal{F}|\, C\times\eta_X$ and $h : \mathcal{F}'|\, \eta_C\times X \to \mathcal{F}|\, \eta_C\times X$ which agree over $\eta_C\times\eta_X$, there exists a unique homomorphism $f : \mathcal{F}' \to \mathcal{F}$ extending g and h.*

Proof. In (a) choose any locally free coherent sheaf $\mathcal{F}_1$ on an open dense subscheme $U \subset C\times X$ which coincides with $\mathcal{G}$ on $U \cap (C\times\eta_X)$. Then the restrictions of $\mathcal{F}_1$ and $\mathcal{H}$ to $U \cap (\eta_C\times X)$ coincide outside a nowhere dense closed subset $T \subset \eta_C\times X$. After replacing U by $U \setminus T$ we may thus assume that $\mathcal{F}_1$ and $\mathcal{H}$ coincide over $U \cap (\eta_C\times X)$. Since the points of codimension 1 in $C\times X$ are precisely the points of codimension 1 in $C\times\eta_X$ and in $\eta_C\times X$, Proposition 2.3 (a) yields a locally free coherent sheaf $\mathcal{F}$ on $C\times X$ which simultaneously extends $\mathcal{G}$ and $\mathcal{H}$. Any other locally free extension with this property coincides with $\mathcal{F}$ on an open dense subscheme $U \subset C\times X$. Since it also coincides with it at all points of codimension 1, it coincides everywhere by the uniqueness in Proposition 2.3 (a). This proves (a).

For (b) note that g extends to some open neighborhood $U \subset C\times X$ of $C\times\eta_X$. Since $C\times X$ is regular and $\mathcal{F}'$ and $\mathcal{F}$ are locally free, this extension must coincide with h over $U \cap (\eta_C\times X)$. Thus by Proposition 2.3 (b) it extends to a homomorphism $f : \mathcal{F}' \to \mathcal{F}$. Again by regularity, this extension is unique. **q.e.d.**

3 Harder-Narasimhan filtration

In this section we recall some basic facts concerning the Harder-Narasimhan filtration of a locally free coherent sheaf on a curve. For a reference see [6], [15]. We generalize the formulas slightly to curves that are not necessarily geometrically irreducible, and normalize degrees and slopes in a way that behaves well under base change.

Let C be an irreducible smooth projective curve of genus g over a field k. We do not assume that C is geometrically irreducible; thus its field of constants may be an arbitrary finite separable extension k' of k, say of degree e. Consider a locally free coherent sheaf $\mathcal{G}$ on C, and set $h^i(C,\mathcal{G}) := \dim_k H^i(C,\mathcal{G})$ for $i = 0, 1$. By the Riemann-Roch theorem we have

$$\chi(C,\mathcal{G}) \;:=\; h^0(C,\mathcal{G}) - h^1(C,\mathcal{G}) \;=\; \deg(\mathcal{G}) + (1-g)\cdot e\cdot \operatorname{rank}(\mathcal{G}) \tag{3.1}$$

for an integer $\deg(\mathcal{G})$ called the *degree of $\mathcal{G}$ (over k)*. If $\mathcal{G}$ is non-zero, the rational number

$$\mu(\mathcal{G}) \;:=\; \frac{\deg(\mathcal{G})}{e\cdot \operatorname{rank}(\mathcal{G})} \tag{3.2}$$

is called the *weight of $\mathcal{G}$*. A non-zero $\mathcal{G}$ is called *semistable* if $\mu(\mathcal{G}') \leqslant \mu(\mathcal{G})$ for all non-zero coherent subsheaves $\mathcal{G}' \subset \mathcal{G}$. The *Harder-Narasimhan filtration of $\mathcal{G}$* is a decreasing filtration by coherent subsheaves $\mathcal{G}^\mu$ indexed by rational numbers μ, which is separated, exhaustive, and left continuous, such that $\mathcal{G}^\mu / \bigcup_{\mu'>\mu} \mathcal{G}^{\mu'}$ is locally free and semistable of weight μ whenever this subquotient is non-zero. Such a filtration always exists and is unique. The numbers μ whose associated subquotient is non-zero are called the *slopes of* $\mathcal{G}$, with *multiplicities* $e \cdot \operatorname{rank}(\mathcal{G}^\mu / \bigcup_{\mu'>\mu} \mathcal{G}^{\mu'})$. If $\mathcal{G}$ is non-zero, we denote its smallest slope by $\mu^{\min}(\mathcal{G})$ and its largest slope by $\mu^{\max}(\mathcal{G})$. If $\mathcal{G} = 0$ we set $\mu^{\min}(\mathcal{G}) := \infty$ and $\mu^{\max}(\mathcal{G}) := -\infty$. Basic properties are:

(3.3) $\deg(\mathcal{G})$ is the sum of all slopes of $\mathcal{G}$ counted with multiplicities.

(3.4) $e \cdot \operatorname{rank}(\mathcal{G}) \cdot \mu^{\min}(\mathcal{G}) \leqslant \deg(\mathcal{G}) \leqslant e \cdot \operatorname{rank}(\mathcal{G}) \cdot \mu^{\max}(\mathcal{G})$.

(3.5) $\mu^{\min}(\mathcal{G}) \geqslant \deg(\mathcal{G}) - (e \cdot \operatorname{rank}(\mathcal{G}) - 1) \cdot \mu^{\max}(\mathcal{G})$.

(3.6) $\mu^{\max}(\mathcal{G}) \leqslant \deg(\mathcal{G}) - (e \cdot \operatorname{rank}(\mathcal{G}) - 1) \cdot \mu^{\min}(\mathcal{G})$.

(3.7) The slopes of $\mathcal{G}^{\oplus N}$ are the slopes of $\mathcal{G}$.

(3.8) The slopes of the dual sheaf $\mathcal{G}^\vee$ are minus the slopes of $\mathcal{G}$.

(3.9) If $\mu^{\max}(\mathcal{G}) < 0$, then $H^0(C, \mathcal{G}) = 0$.

(3.10) If $\mathcal{G}$ is generated by global sections, then $\mu^{\min}(\mathcal{G}) \geqslant 0$.

(3.11) If $\mu^{\min}(\mathcal{G}) > 2g - 2$, then $H^1(C, \mathcal{G}) = 0$.

(3.12) If $\mu^{\min}(\mathcal{G}) > 2g - 1$, then $\mathcal{G}$ is generated by global sections.

Also, for any homomorphism of non-zero locally free coherent sheaves $f : \mathcal{F} \to \mathcal{G}$ we have:

(3.13) $f(\mathcal{F}^\mu) \subset \mathcal{G}^\mu$ for every $\mu \in \mathbb{Q}$.

(3.14) $\mu^{\min}(\mathcal{F}) \leqslant \mu^{\max}(\mathcal{G})$ if f is non-zero.

(3.15) $\mu^{\max}(\mathcal{F}) \leqslant \mu^{\max}(\mathcal{G})$ if f is injective.

(3.16) $\mu^{\min}(\mathcal{F}) \leqslant \mu^{\min}(\mathcal{G})$ if f has torsion cokernel.

(3.17) $\deg(\mathcal{F}) \leqslant \deg(\mathcal{G})$ if f is injective with torsion cokernel.

Furthermore, for any short exact sequence $0 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$ of locally free coherent sheaves we have:

(3.18) $\mu^{\min}(\mathcal{G}) \geqslant \min\{\mu^{\min}(\mathcal{G}'), \mu^{\min}(\mathcal{G}'')\}$.

(3.19) $\mu^{\max}(\mathcal{G}) \leqslant \max\{\mu^{\max}(\mathcal{G}'), \mu^{\max}(\mathcal{G}'')\}$.

Next let $\mathcal{L}$ be an ample invertible sheaf of weight $\ell := \mu(\mathcal{L})$ on C. For any coherent sheaf $\mathcal{G}$ on C and any integer n we define $\mathcal{G}(n) := \mathcal{G} \otimes \mathcal{L}^{\otimes n}$. Then:

(3.20) The slopes of $\mathcal{G}(n)$ are the slopes of $\mathcal{G}$ plus $n\ell$.

(3.21) $\deg(\mathcal{G}(n)) = \deg(\mathcal{G}) + n\ell e \cdot \operatorname{rank}(\mathcal{G})$.

Now consider an arbitrary field extension $k \hookrightarrow L$ and let C_L denote the curve over L obtained from C by base change. Then C_L is a finite disjoint union of at most e irreducible smooth projective curves $C_{L,i}$ over L. For any locally free coherent sheaf $\mathcal{G}_L$ on C_L all the above concepts and properties apply to $\mathcal{G}_L|C_{L,i}$ for every i. Thus the direct sum of the Harder-Narasimhan filtrations for these constituents yields a *Harder-Narasimhan filtration of* $\mathcal{G}_L$. The *slopes of* $\mathcal{G}_L$ are those of all constituents combined, each counted with the sum of the respective multiplicities. Furthermore $\mu^{\min}(\mathcal{G}_L)$ and $\mu^{\max}(\mathcal{G}_L)$ are defined exactly as before, and one sets $\deg(\mathcal{G}_L) := \sum_i \deg(\mathcal{G}_L|C_{L,i})$. Then all the above properties hold verbatim over C_L, except that in the formulas involving $\mathrm{rank}(\mathcal{G}_L)$ one must assume that $\mathcal{G}_L$ has constant rank.

Finally, we revert to a locally free coherent sheaf $\mathcal{G}$ on C and consider its pullback $\pi^*\mathcal{G}$ via the morphism $\pi : C_L \to C$. This is a locally free coherent sheaf of constant rank on C_L, whose rank, degree, and weight all coincide with those of $\mathcal{G}$.

(3.22) The Harder-Narasimhan filtration of $\pi^*\mathcal{G}$ is the pullback of the Harder-Narasimhan filtration of $\mathcal{G}$. In particular the degree and all slopes and multiplicities of $\pi^*\mathcal{G}$ are equal to those of $\mathcal{G}$.

For the existence and uniqueness of the Harder-Narasimhan filtration see [15, 1ère Partie, I, Th. 4], where the assumption that k be algebraically closed is irrelevant. Assertions (3.3)–(3.8) and (3.17)–(3.21) are straightforward consequences of the definition. Assertion (3.13) follows from [15, ibid., Prop. 6]); this in turn implies (3.14) and (3.15), and by duality the latter yields (3.16). The special case $\mathcal{F} = \mathcal{O}_C$ of (3.14) and (3.16) shows (3.9) and (3.10); for (3.11) and (3.12) see [15, ibid., Lemma 20]). Finally (3.22) follows from [15, 3ème Partie, III, Prop. 17].

4 Frobenius

From now on we let $\mathbb{F}_q$ be a finite field with q elements. Let X be an irreducible smooth projective curve of genus g over $\mathbb{F}_q$. Let $\sigma : X \to X$ denote its Frobenius endomorphism over $\operatorname{Spec} \mathbb{F}_q$ which is the identity on the underlying topological space and the map $s \mapsto s^q$ on the structure sheaf. For any field extension $\mathbb{F}_q \hookrightarrow L$ we let σ again denote the endomorphism of X_L over L deduced from $\sigma : X \to X$ by base change. Let $\mathcal{L}$ be an ample invertible sheaf of weight ℓ on X. For any coherent sheaf $\mathcal{F}$ on X_L we let $\mathcal{F}(n)$ denote the tensor product of $\mathcal{F}$ with the pullback of $\mathcal{L}^{\otimes n}$.

Proposition 4.1 *For any coherent sheaf $\mathcal{F}$ on X_L we have:*

(a) $\deg(\sigma^*\mathcal{F}) = q \cdot \deg(\mathcal{F})$.

(b) $\sigma^*(\mathcal{F}(n)) \cong (\sigma^*\mathcal{F})(qn)$ *for any integer n.*

Proof. Assertion (a) follows from the fact that $\sigma : X_L \to X_L$ is finite of constant degree q. Assertion (b) reduces to the isomorphy $\sigma^*\mathcal{L} \cong \mathcal{L}^{\otimes q}$ on X. But this follows from the fact that the cocycles defining both sides of the equation are obtained from the cocycle defining $\mathcal{L}$ by applying the same map $s \mapsto s^q$. **q.e.d.**

Proposition 4.2 *There exists a_0 such that for all $a \geqslant a_0$ there exists $N > 0$ and an injective homomorphism $\sigma_*\mathcal{O}_X \hookrightarrow \mathcal{O}_X(a)^{\oplus N}$ whose image is locally a direct summand.*

Proof. By dualizing the assertion is equivalent to the existence of a locally split surjection $\mathcal{O}_X(-a)^{\oplus N} \twoheadrightarrow (\sigma_*\mathcal{O}_X)^\vee$. Since the sheaf $(\sigma_*\mathcal{O}_X)^\vee$ is locally free, any surjection is already locally split. But whenever

$$\mu^{\min}\big((\sigma_*\mathcal{O}_X)^\vee(a)\big) \overset{(3.20)}{=} \mu^{\min}\big((\sigma_*\mathcal{O}_X)^\vee\big) + a\ell \;>\; 2g-1,$$

(3.12) asserts that $(\sigma_*\mathcal{O}_X)^\vee(a)$ is generated by global sections; hence there exists the desired surjection. Thus the proposition holds with $a_0 := \big(2g - \mu^{\min}((\sigma_*\mathcal{O}_X)^\vee)\big)/\ell$. **q.e.d.**

5 Finiteness for locally free coherent sheaves

As a warm-up, we recall the proof of a well-known finiteness result over a curve, which is implicit in both [6] and [15]. Let C, g, e, $\mathcal{L}$, and ℓ be as in Section 3, with $k = \mathbb{F}_q$.

Theorem 5.1 *Fix constants $r > 0$, d, and μ. Then up to isomorphism, there exist at most finitely many locally free coherent sheaves $\mathcal{F}$ on C with the following properties:*

(a) $\mathcal{F}$ has constant rank r.

(b) $\deg(\mathcal{F}) = d$.

(c) $\mu^{\min}(\mathcal{F}) \geqslant \mu$.

Proof. Fix any integer $m > (2g-1-\mu)/\ell$. Then for any $\mathcal{F}$ with the given properties we have

$$\mu^{\min}(\mathcal{F}(m)) \overset{(3.20)}{=} \mu^{\min}(\mathcal{F}) + m\ell \overset{(c)}{\geqslant} \mu + m\ell \;>\; 2g-1.$$

By (3.11) and (3.12) this implies that $H^1(\mathcal{F}(m)) = 0$ and that $\mathcal{F}(m)$ is generated by global sections. Also, we calculate

$$\begin{aligned} N \;:=\; h^0(\mathcal{F}(m)) \;&=\; \chi(\mathcal{F}(m)) \\ &\overset{(3.1)}{=} \deg(\mathcal{F}(m)) + (1-g)\cdot e\cdot \mathrm{rank}(\mathcal{F}(m)) \\ &\overset{(3.21)}{=} \deg(\mathcal{F}) + (m\ell + 1 - g)\cdot e\cdot \mathrm{rank}(\mathcal{F}) \\ &\overset{(a),\,(b)}{=} d + (m\ell+1-g)\cdot e\cdot r, \end{aligned}$$

which is independent of $\mathcal{F}$. Together we find that there exists a surjection $\mathcal{O}_C^{\oplus N} \twoheadrightarrow \mathcal{F}(m)$. Let $\mathcal{F}'$ denote its kernel. Then $\mathcal{F}'$ is locally free of rank $r' := N - r$. Next,

$$d' := \deg(\mathcal{F}') = -\deg(\mathcal{F}(m)) \overset{(3.21)}{=} -\deg(\mathcal{F}) - m\ell e \cdot \mathrm{rank}(\mathcal{F}) = -d - m\ell er$$

is also independent of $\mathcal{F}$. Furthermore, by (3.15) we have $\mu^{\max}(\mathcal{F}') \leqslant \mu^{\max}(\mathcal{O}_C^{\oplus N}) = 0$ and hence, by (3.5), $\mu^{\min}(\mathcal{F}') \geqslant \deg(\mathcal{F}') = d' =: \mu'$. Thus $\mathcal{F}'$ satisfies the same kind of conditions as $\mathcal{F}$ with (r', d', μ') in place of (r, d, μ). In particular, for any integer $m' > (2g-1-\mu')/\ell$ there exists a surjection $\mathcal{O}_C^{\oplus N'} \twoheadrightarrow \mathcal{F}'(m')$ with $N' := d' + (m'\ell + 1 - g) \cdot e \cdot r$. Combining this with the earlier surjection and twisting back we obtain an exact sequence

$$\mathcal{O}_C^{\oplus N'}(-m-m') \xrightarrow{h} \mathcal{O}_C^{\oplus N}(-m) \longrightarrow \mathcal{F} \longrightarrow 0.$$

Here the numbers m, N, m', N' depend only on the invariants fixed in Theorem 5.1, but not otherwise on $\mathcal{F}$. As the homomorphism h lies in the finite dimensional $\mathbb{F}_q$-vector space

$$\mathrm{Hom}\big(\mathcal{O}_C^{\oplus N'}(-m-m'), \mathcal{O}_C^{\oplus N}(-m)\big),$$

there are only finitely many possibilities for it, and hence for the isomorphism class of $\mathcal{F}$, as desired. **q.e.d.**

From now on and throughout the rest of this article we consider two irreducible smooth projective curves C and X over $\mathbb{F}_q$. We let η_C denote the generic point of C and η_X the generic point of X. All fiber products are taken over $\mathrm{Spec}\,\mathbb{F}_q$.

For any locally free coherent sheaf $\mathcal{F}$ on $C \times X$ and any point $c \in C$ we let $\mathcal{F}_c$ denote the pullback of $\mathcal{F}$ to the fiber $c \times X$. Likewise, for any point $x \in X$ we let $\mathcal{F}_x$ denote the pullback of $\mathcal{F}$ to the fiber $C \times x$. In both situations we will apply the conventions of Section 3. Note that by flatness the number $\deg(\mathcal{F}_c)$ is independent of $c \in C$, and the number $\deg(\mathcal{F}_x)$ is independent of $x \in X$.

Fix ample invertible sheaves $\mathcal{L}_C$ on C and $\mathcal{L}_X$ on X. For any coherent sheaf $\mathcal{F}$ on $C \times X$ and any two integers m and n we set

$$\mathcal{F}(m,n) := \mathcal{F} \otimes \mathrm{pr}_1^* \mathcal{L}_C^{\otimes m} \otimes \mathrm{pr}_2^* \mathcal{L}_X^{\otimes n} \tag{5.2}$$

Let g_C denote the genus and e_C the degree over $\mathbb{F}_q$ of the constant field of C, and ℓ_C the weight $\mu(\mathcal{L}_C)$. Let g_X, e_X, and ℓ_X denote the corresponding invariants for the curve X.

Proposition 5.3 *Let $\mathcal{F}$ be a locally free coherent sheaf on $C \times X$. Assume that:*

(a) $\mathcal{F}$ has constant rank r.

(b) $\deg(\mathcal{F}_c) = d_X$ for all $c \in C$.

(c) $\mu^{\min}(\mathcal{F}_c) \geqslant \mu_X$ for all $c \in C$.

(d) $\deg(\mathcal{F}_{\eta_X}) = d_C$.

Then for any integer $n > (2g_X - 1 - \mu_X)/\ell_X$ we have:

(e) $\mathcal{G}_n := \mathrm{pr}_{1*}(\mathcal{F}(0,n))$ *is a locally free coherent sheaf on C.*

(f) $R^1 \,\mathrm{pr}_{1*}(\mathcal{F}(0,n)) = 0$.

(g) *The adjunction homomorphism $\mathrm{pr}_1^* \mathcal{G}_n \to \mathcal{F}(0,n)$ is surjective.*

(h) $\mathrm{rank}(\mathcal{G}_n) = d_X + (n\ell_X + 1 - g_X)e_X r$.

(i) $\deg(\mathcal{G}_n) = \chi(C{\times}X, \mathcal{F}) - (1-g_C)e_C\big(d_X + (1-g_X)e_X r\big) + n\ell_X e_X d_C$.

Proof. As a torsion free coherent sheaf on a smooth curve, $\mathcal{G}_n$ is locally free, proving (e). The assumption on n implies that

$$\mu^{\min}(\mathcal{F}_c(n)) \overset{(3.20)}{=} \mu^{\min}(\mathcal{F}_c) + n\ell_X \overset{(c)}{\geqslant} \mu_X + n\ell_X > 2g_X - 1$$

for any point $c \in C$. By (3.11) and (3.12) this implies that $H^1(\mathcal{F}_c(n)) = 0$ and that $\mathcal{F}_c(n)$ is generated by global sections. Using base change the first of these facts implies (f). This in turn implies that base change also holds in degree 0; in other words, that the natural map $\mathcal{G}_n \otimes k(c) \to H^0(\mathcal{F}_c(n))$ is an isomorphism. That $\mathcal{F}_c(n)$ is generated by global sections then implies that $\mathrm{pr}_1^* \mathcal{G}_n \to \mathcal{F}(0,n)$ is surjective in all fibers over C, and hence everywhere, proving (g). Also we find that

$$\begin{aligned}
\mathrm{rank}(\mathcal{G}_n) \;=\; h^0(\mathcal{F}_c(n)) \;&=\; \chi(\mathcal{F}_c(n)) \\
&\overset{(3.1)}{=} \deg(\mathcal{F}_c(n)) + (1-g_X)\cdot e_X \cdot \mathrm{rank}(\mathcal{F}_c(n)) \\
&\overset{(3.21)}{=} \deg(\mathcal{F}_c) + (n\ell_X + 1 - g_X)\cdot e_X \cdot \mathrm{rank}(\mathcal{F}_c) \\
&= d_X + (n\ell_X + 1 - g_X)e_X r,
\end{aligned}$$

proving (h). To show (i) we calculate $\chi(C{\times}X, \mathcal{F}(0,n))$ in two ways. First observe that

$$\chi(C{\times}X, \mathcal{F}(0,n)) \;=\; \chi(X, R\,\mathrm{pr}_{2*}(\mathcal{F}(0,n)) \;=\; \chi(\mathcal{H}^0(n)) - \chi(\mathcal{H}^1(n)),$$

where $\mathcal{H}^i := R^i\,\mathrm{pr}_{2*}(\mathcal{F})$ is a coherent sheaf on X. Let $\mathcal{H}^i_{\mathrm{tor}}$ denote its torsion subsheaf, so that $\mathcal{H}^i/\mathcal{H}^i_{\mathrm{tor}}$ is locally free. Then

$$\begin{aligned}
\chi(\mathcal{H}^i(n)) \;&=\; \chi((\mathcal{H}^i/\mathcal{H}^i_{\mathrm{tor}})(n)) + \chi(\mathcal{H}^i_{\mathrm{tor}}(n)) \\
&\overset{(3.1)}{=} \deg((\mathcal{H}^i/\mathcal{H}^i_{\mathrm{tor}})(n)) + (\text{some value independent of } n) \\
&\overset{(3.21)}{=} n\ell_X e_X \cdot \mathrm{rank}(\mathcal{H}^i/\mathcal{H}^i_{\mathrm{tor}}) + (\text{some value independent of } n).
\end{aligned}$$

Furthermore, by base change we have

$$\begin{aligned}
\operatorname{rank}(\mathcal{H}^0/\mathcal{H}^0_{\mathrm{tor}}) - \operatorname{rank}(\mathcal{H}^1/\mathcal{H}^1_{\mathrm{tor}}) &= \dim\bigl(\mathcal{H}^0 \otimes k(\eta_X)) - \dim(\mathcal{H}^1 \otimes k(\eta_X)\bigr) \\
&= h^0(C\times\eta_X, \mathcal{F}_{\eta_X}) - h^1(C\times\eta_X, \mathcal{F}_{\eta_X}) \\
&= \chi(C\times\eta_X, \mathcal{F}_{\eta_X}) \\
&\overset{(3.1)}{=} \deg(\mathcal{F}_{\eta_X}) + (1-g_C)\cdot e_C \cdot \operatorname{rank}(\mathcal{F}_{\eta_X}) \\
&= d_C + (1-g_C)e_C r.
\end{aligned}$$

Putting the last three calculations together we deduce that

$$\chi(C\times X, \mathcal{F}(0,n)) \;=\; n\ell_X e_X\bigl(d_C + (1-g_C)e_C r\bigr) + (\text{some value independent of } n).$$

The case $n=0$ shows that the unknown value in parentheses is $\chi(C\times X, \mathcal{F})$. On the other hand we have

$$\begin{aligned}
\chi(C\times X, \mathcal{F}(0,n)) &\overset{(f)}{=} \chi(C, \mathrm{pr}_{1*}\mathcal{F}(0,n)) \;=\; \chi(C, \mathcal{G}_n) \\
&\overset{(3.1)}{=} \deg(\mathcal{G}_n) + (1-g_C)\cdot e_C \cdot \operatorname{rank}(\mathcal{G}_n) \\
&\overset{(h)}{=} \deg(\mathcal{G}_n) + (1-g_C)\cdot e_C \cdot \bigl(d_X + (n\ell_X + 1 - g_X)e_X r\bigr) \\
&= \deg(\mathcal{G}_n) + (1-g_C)e_C\bigl(d_X + (1-g_X)e_X r\bigr) + (1-g_C)e_C r \cdot n\ell_X e_X.
\end{aligned}$$

Comparing these formulas yields

$$\deg(\mathcal{G}_n) \;=\; \chi(C\times X, \mathcal{F}) - (1-g_C)e_C\bigl(d_X + (1-g_X)e_X r\bigr) + n\ell_X e_X d_C,$$

proving (i). **q.e.d.**

Theorem 5.4 *Fix constants $r>0$, d_X, μ_X, d_C, d, μ, and $n > (2g_X - 1 - \mu_X)/\ell_X$. Then up to isomorphism, there exist at most finitely many locally free coherent sheaves $\mathcal{F}$ on $C\times X$ with the following properties, where $\mathcal{G}_n := \mathrm{pr}_{1*}(\mathcal{F}(0,n))$:*

(a) $\mathcal{F}$ *has constant rank* r.

(b) $\deg(\mathcal{F}_c) = d_X$ *for all* $c \in C$.

(c) $\mu^{\min}(\mathcal{F}_c) \geqslant \mu_X$ *for all* $c \in C$.

(d) $\deg(\mathcal{F}_{\eta_X}) = d_C$.

(e) $\deg(\mathcal{G}_n) = d$.

(f) $\mu^{\min}(\mathcal{G}_n) \geqslant \mu$.

Proof. Note that Proposition 5.3 applies in this case. In particular 5.3 (g) implies that $\mathcal{G}_n$ is non-zero. Fix any integer $m > (2g_C - 1 - \mu)/\ell_C$. Then for any $\mathcal{F}$ we have

$$\mu^{\min}(\mathcal{G}_n(m)) \overset{(3.20)}{=} \mu^{\min}(\mathcal{G}_n) + m\ell_C \overset{(f)}{\geqslant} \mu + m\ell_C > 2g_C - 1.$$

By (3.11) and (3.12) this implies that $H^1(\mathcal{G}_n(m)) = 0$ and that $\mathcal{G}_n(m)$ is generated by global sections. Also, we find that

$$\begin{aligned} N := h^0(\mathcal{G}_n(m)) &= \chi(\mathcal{G}_n(m)) \\ &\overset{(3.1)}{=} \deg(\mathcal{G}_n(m)) + (1-g_C)\cdot e_C \cdot \operatorname{rank}(\mathcal{G}_n(m)) \\ &\overset{(3.21)}{=} \deg(\mathcal{G}_n) + (m\ell_C + 1 - g_C)\cdot e_C \cdot \operatorname{rank}(\mathcal{G}_n) \\ &\overset{5.3\,(h)}{=} d + (m\ell_C + 1 - g_C)\cdot e_C \cdot \big(d_X + (n\ell_X + 1 - g_X)e_X r\big) \end{aligned}$$

depends only on the given invariants. That $\mathcal{G}_n(m)$ is generated by global sections means that there exists a surjection $\mathcal{O}_C^{\oplus N} \twoheadrightarrow \mathcal{G}_n(m)$. Combined with 5.3 (g) this yields a surjection $\mathcal{O}_{C\times X}^{\oplus N} \twoheadrightarrow \mathcal{F}(m,n)$. Let $\mathcal{F}'$ denote its kernel, so that we have a short exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{O}_{C\times X}^{\oplus N} \longrightarrow \mathcal{F}(m,n) \longrightarrow 0. \tag{5.5}$$

We want to repeat the above arguments with $\mathcal{F}'$ in place of $\mathcal{F}$. For this we set

$$\begin{aligned} r' &:= N - r, \\ d'_X &:= -d_X - n\ell_X e_X r, \\ d'_C &:= -d_C - m\ell_C e_C r, \end{aligned}$$

choose an integer $n' > (2g_X - 1 - d'_X)/\ell_X$, and abbreviate

$$d' := -d + n'\ell_X e_X d'_C + m\ell_C e_C d'_X - m\ell_C e_C (1-g_X) e_X r.$$

Lemma 5.6 *With $\mathcal{G}'_{n'} := \mathrm{pr}_{1*}(\mathcal{F}'(0,n'))$ we have:*

(a) $\mathcal{F}'$ is locally free of constant rank r'.

(b) $\deg(\mathcal{F}'_c) = d'_X$ for all $c \in C$.

(c) $\mu^{\min}(\mathcal{F}'_c) \geqslant d'_X$ for all $c \in C$.

(d) $\deg(\mathcal{F}'_{\eta_X}) = d'_C$.

(e) $\deg(\mathcal{G}'_{n'}) = d'$.

(f) $\mu^{\min}(\mathcal{G}'_{n'}) \geqslant d'$.

Proof. Note that $\mathcal{F}(m,n)$ is locally free of constant rank r. Thus the sequence (5.5) locally splits, which implies (a). Next the short exact sequence

$$0 \to \mathcal{F}'_c \to \mathcal{O}^{\oplus N}_{c\times X} \to \mathcal{F}_c(n) \to 0$$

and the resulting calculation

$$\deg(\mathcal{F}'_c) = -\deg(\mathcal{F}_c(n)) \stackrel{(3.21)}{=} -\deg(\mathcal{F}_c) - n\ell_X e_X \cdot \mathrm{rank}(\mathcal{F}_c) = -d_X - n\ell_X e_X r = d'_X$$

imply (b). The exact sequence together with (3.15) also implies that $\mu^{\max}(\mathcal{F}'_c) \leqslant \mu^{\max}(\mathcal{O}^{\oplus N}_{c\times X}) = 0$. Together with (b) and (3.5) this implies (c). Assertion (d) is proved in precisely the same way as (b).

The assertions (a) through (d) which have already been proved show that Proposition 5.3 may be applied to $\mathcal{F}'$ and n'. In particular $\mathcal{G}'_{n'}$ is a locally free coherent sheaf on C. Also $R^1\,\mathrm{pr}_{1*}(\mathcal{F}'(0,n')) = 0$; hence after twisting the sequence (5.5) by $(0,n')$ and applying pr_{1*} we obtain a short exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{pr}_{1*}(\mathcal{F}'(0,n')) & \longrightarrow & \mathrm{pr}_{1*}(\mathcal{O}^{\oplus N}_{C\times X}(0,n')) & \longrightarrow & \mathrm{pr}_{1*}(\mathcal{F}(m,n+n')) & \longrightarrow & 0 \\
 & & \| & & \|\wr & & \|\wr & & \\
0 & \longrightarrow & \mathcal{G}'_{n'} & \longrightarrow & \mathcal{O}^{\oplus N}_{C} \otimes H^0(X,\mathcal{O}_X(n')) & \longrightarrow & \mathcal{G}_{n+n'}(m) & \longrightarrow & 0.
\end{array}$$

From this we deduce that

$$\begin{aligned}
\deg(\mathcal{G}'_{n'}) &= -\deg(\mathcal{G}_{n+n'}(m)) \\
&\stackrel{(3.21)}{=} -\deg(\mathcal{G}_{n+n'}) - m\ell_C e_C \cdot \mathrm{rank}(\mathcal{G}_{n+n'}) \\
&\stackrel{5.3}{=} -\deg(\mathcal{G}_n) - n'\ell_X e_X d_C - m\ell_C e_C\big(d_X + ((n+n')\ell_X + 1 - g_X)e_X r\big) \\
&= -d + n'\ell_X e_X d'_C + m\ell_C e_C d'_X - m\ell_C e_C(1-g_X)e_X r \\
&= d',
\end{aligned}$$

proving (e). Finally, the exact sequence together with (3.15) also implies that $\mu^{\max}(\mathcal{G}'_{n'}) \leqslant \mu^{\max}(\mathcal{O}^{\oplus N}_C) = 0$. Together with (e) and (3.5) this implies (f). **q.e.d.**

Lemma 5.6 shows that $\mathcal{F}'$ satisfies the same assumptions as $\mathcal{F}$, only with other constants. The same arguments as in the first part of the proof thus imply that for any fixed integer $m' > (2g_C - 1 - d')/\ell_C$ and

$$N' \;:=\; d' + (m'\ell_C + 1 - g_C)\cdot e_C \cdot \big(d'_X + (n'\ell_X + 1 - g_X)e_X r'\big)$$

there exists a surjection $\mathcal{O}^{\oplus N'}_{C\times X} \twoheadrightarrow \mathcal{F}'(m',n')$. Combining this with the short exact sequence (5.5) and twisting back we obtain an exact sequence

$$\mathcal{O}^{\oplus N'}_{C\times X}(-m-m',-n-n') \xrightarrow{h} \mathcal{O}^{\oplus N}_{C\times X}(-m,-n) \longrightarrow \mathcal{F} \longrightarrow 0.$$

Here the numbers m, n, N, m', n', N' depend only on the invariants fixed in Theorem 5.4, but not otherwise on $\mathcal{F}$. Moreover, the homomorphism h lies in the finite dimensional $\mathbb{F}_q$-vector space

$$\operatorname{Hom}\big(\mathcal{O}_{C\times X}^{\oplus N'}(-m-m',-n-n'),\mathcal{O}_{C\times X}^{\oplus N}(-m,-n)\big).$$

Thus there are only finitely many possibilities for it, and hence only finitely many possibilities for the isomorphism class of $\mathcal{F}$, as desired. **q.e.d.**

6 κ-Sheaves

As we shall consider κ-sheaves over various base schemes, we introduce the concept in a suitable generality. Let Y and X be quasicompact schemes over $\mathbb{F}_q$. Let σ be the Frobenius endomorphism of X over $\mathbb{F}_q$ as in Section 4, which induces an endomorphism $\mathrm{id}\times\sigma$ of the fiber product $Y\times X$ over $\mathbb{F}_q$. Let $\mathcal{L}_Y$ be an ample invertible sheaf on Y. For any coherent sheaf $\mathcal{F}$ on $Y\times X$ and any integer d we abbreviate $\mathcal{F}(d,0) := \mathcal{F}\otimes\mathrm{pr}_1^*\mathcal{L}_Y^{\otimes d}$.

Definition 6.1 *(a) A* κ-sheaf of pole order $\leqslant d$ on $Y\times X$ *is a locally free coherent sheaf $\mathcal{F}$ on $Y\times X$ together with an injective homomorphism $\kappa:\mathcal{F}\hookrightarrow(\mathrm{id}\times\sigma)^*\mathcal{F}(d,0)$. A κ-sheaf of pole order $\leqslant 0$ is called simply a* κ-sheaf on $Y\times X$.

(b) A coherent subsheaf $\mathcal{F}'$ of $\mathcal{F}$ is called κ-invariant *if κ induces a homomorphism $\mathcal{F}'\hookrightarrow(\mathrm{id}\times\sigma)^*\mathcal{F}'(d,0)$ in a commutative diagram*

$$\begin{tikzcd}
\mathcal{F} \arrow[r, hook] & (\mathrm{id}\times\sigma)^*\mathcal{F}(d,0) \\
\mathcal{F}' \arrow[r, hook] \arrow[u, hook] & (\mathrm{id}\times\sigma)^*\mathcal{F}'(d,0). \arrow[u, hook]
\end{tikzcd}$$

A locally free κ-invariant coherent subsheaf is called a κ-subsheaf.

(c) A homomorphism *of κ-sheaves of pole order $\leqslant d$ is a homomorphism f of the underlying coherent sheaves satisfying $\kappa\circ f=((\mathrm{id}\times\sigma)^*f)\circ\kappa$.*

Lemma 6.2 *If Y and X are irreducible and localizations of schemes of finite type over $\mathbb{F}_q$, then any κ-sheaf of some pole order on $Y\times X$ has constant rank.*

Proof. Since κ is injective, the local rank of $\mathcal{F}$ at any generic point of $Y\times X$ is less than or equal to the local rank of $(\mathrm{id}\times\sigma)^*\mathcal{F}(d,0)$ at the same generic point. But the latter is the local rank of $\mathcal{F}$ at the image of the generic point under $\mathrm{id}\times\sigma$. Since under the assumptions on Y and X the finitely many generic points of $Y\times X$ are permuted transitively by $\mathrm{id}\times\sigma$, it follows that this rank is constant. **q.e.d.**

We will apply the above concepts in the following situations. The scheme X will be either the spectrum of a field or an irreducible smooth projective curve over $\mathbb{F}_q$. The scheme Y will be either an irreducible smooth projective curve C over $\mathbb{F}_q$, or the open curve $C^\circ := C \smallsetminus \{\infty\}$ for some closed point $\infty \in C$, or the generic point η_C of C.

In the case $Y = C$ the notion is not very interesting unless $\mathcal{L}_Y$ is ample, because only then there are enough homomorphisms $\kappa : \mathcal{F} \hookrightarrow (\mathrm{id}{\times}\sigma)^*\mathcal{F}(d,0)$ for $d \gg 0$. In Section 7 we allow $\mathcal{L}_Y = \mathcal{L}_C$ to be any ample invertible sheaf on C; in Sections 8 and 9 we specialize it to $\mathcal{L}_C = \mathcal{O}_C(\infty)$. In the cases $Y = C^\circ$ and $Y = \eta_C$ we only consider κ-sheaves without poles.

Some of our work will consist of comparing κ-sheaves on the base schemes

$$\eta_C{\times}X \hookrightarrow C^\circ{\times}X \hookrightarrow C{\times}X.$$

The pullback under each inclusion maps κ-sheaves to κ-sheaves. An inclusion of κ-sheaves on $C^\circ{\times}X$ becomes an isomorphism on $\eta_C{\times}X$ if and only if it is an isomorphism outside $D \times X$ for some divisor $D \subset C$. Thus we can view κ-sheaves on $\eta_C{\times}X$ as *κ-sheaves on $C^\circ{\times}X$ up to isogeny.* At the end of Section 8 we deal with the problem of extending κ-sheaves on $C^\circ{\times}X$ to $C{\times}X$.

When X is an irreducible smooth curve with generic point η_X, we can consider κ-sheaves on each of the base schemes in the commutative diagram

(6.3)
$$\begin{tikzcd}
\eta_C{\times}\eta_X \arrow[r, hook] \arrow[d, hook] & C^\circ{\times}\eta_X \arrow[r, hook] \arrow[d, hook] & C{\times}\eta_X \arrow[d, hook] \\
\eta_C{\times}X \arrow[r, hook] & C^\circ{\times}X \arrow[r, hook] & C{\times}X.
\end{tikzcd}$$

We can then also study restriction and extension in the direction of X. This is done in Section 9.

7 Finiteness for κ-sheaves

We keep the notations of Section 5, with $\mathbb{F}_q$, C, $\mathcal{L}_C$, X, $\mathcal{L}_X$ and consequently g_C, ℓ_C, e_C and g_X, ℓ_X, e_X all fixed. The aim of this section is to prove the following theorem.

Theorem 7.1 *Fix any constants d, r, d_X, μ_X, d_C, and μ_C. Then up to isomorphism, there exist at most finitely many κ-sheaves $\mathcal{F}$ of pole order $\leqslant d$ on $C{\times}X$ with the following properties:*

(a) $\mathcal{F}$ has constant rank r.

(b) $\deg(\mathcal{F}_c) = d_X$ for all $c \in C$.

(c) $\mu^{\min}(\mathcal{F}_c) \geqslant \mu_X$ for all $c \in C$.

(d) $\deg(\mathcal{F}_{\eta_X}) = d_C$.

(e) $\mu^{\max}(\mathcal{F}_{\eta_X}) \leqslant \mu_C$.

(f) *Every κ-invariant coherent subsheaf of $\mathcal{F}$ of rank r coincides with $\mathcal{F}$ along $\eta_C \times X$.*

Throughout this section we consider a κ-sheaf $\mathcal{F}$ satisfying the above properties. Note that the conditions (a)–(d) here are the same as in Theorem 5.4, while (e) and (f) are different. To reduce Theorem 7.1 to Theorem 5.4 we will show that the remaining numerical invariants in 5.4 (e) and (f) are bounded by constants independent of $\mathcal{F}$, or more precisely: depending only on q, g_C, ℓ_C, e_C, g_X, ℓ_X, e_X, d, r, d_X, μ_X, d_C, and μ_C.

We begin with some preparatory results before embarking on the real work in (7.7). First, by the following lemma we may—and do—assume that $\mu_X \leqslant 0$ and $\mu_C \leqslant 0$:

Lemma 7.2 *Theorem 7.1 in the case $\mu_X \leqslant 0$ and $\mu_C \leqslant 0$ implies Theorem 7.1 in general.*

Proof. We may decrease μ_X, because that can only increase the range of possibilities for $\mathcal{F}$. On the other hand, for any κ-sheaf $\mathcal{F}$ and any integer m we obtain a κ-sheaf $\mathcal{F}(m,0)$ with $\mu^{\max}(\mathcal{F}(m,0)_{\eta_X}) = \mu^{\max}(\mathcal{F}_{\eta_X}) + m\ell_C$ by (3.20) and $\deg(\mathcal{F}(m,0)_{\eta_X}) = \deg(\mathcal{F}_{\eta_X}) + m\ell_C e_C r$ by (3.21), while all other numerical invariants in Theorem 7.1 are the same. This process is reversible, so it allows us to replace μ_C by $\mu_C + m\ell_C$. For suitable $m \ll 0$ we can thus achieve $\mu_C \leqslant 0$. **q.e.d.**

Lemma 7.3 *Consider a point $c \in C$ and a locally free coherent sheaf $\mathcal{H}$ on $c \times X$ together with an injective homomorphism $\kappa : \mathcal{H} \hookrightarrow (\mathrm{id} \times \sigma)^*\mathcal{H}$. Then*

(a) *$\mathcal{H}$ is locally free of constant rank.*

(b) $\mu^{\min}(\mathcal{H}) \geqslant 0$.

(c) $\deg(\mathcal{H}) \geqslant 0$.

Proof. (a) is a special case of Lemma 6.2. For (b) suppose that $\mu := \mu^{\min}(\mathcal{H}) < \infty$, so that $\mathcal{H} \neq 0$. Then $\mathcal{H}$ possesses a non-zero semistable quotient $\overline{\mathcal{H}}$ of weight μ. Since $\kappa : \mathcal{H} \to (\mathrm{id} \times \sigma)^*\mathcal{H}$ is injective by assumption, from (a) we deduce that its cokernel is torsion. We can therefore calculate

$$\begin{aligned}
\mu \;=\; \mu^{\min}(\mathcal{H}) &\overset{(3.16)}{\leqslant} \mu^{\min}((\mathrm{id} \times \sigma)^*\mathcal{H}) \\
&\overset{(3.16)}{\leqslant} \mu^{\min}((\mathrm{id} \times \sigma)^*\overline{\mathcal{H}}) \\
&\overset{(3.4)}{\leqslant} \deg((\mathrm{id} \times \sigma)^*\overline{\mathcal{H}})/e \cdot \mathrm{rank}((\mathrm{id} \times \sigma)^*\overline{\mathcal{H}}) \\
&\overset{4.1(a)}{=} q \cdot \deg(\overline{\mathcal{H}})/e \cdot \mathrm{rank}(\overline{\mathcal{H}}) \\
&\overset{(3.2)}{=} q \cdot \mu^{\min}(\overline{\mathcal{H}}) \;=\; q\mu.
\end{aligned}$$

This implies that $\mu \geqslant 0$, proving (b). Finally, (c) is a direct consequence of (b) and (3.3). **q.e.d.**

Note that part (b) of Lemma 7.3 for $\mathcal{H} = \mathcal{F}_c$ subsumes condition 7.1 (c) whenever κ is an isomorphism at $c \times \eta_X$. But at the remaining finitely many points $c \in C$ the condition 7.1 (c) is still necessary. In actual fact we will need Lemma 7.3 for κ-invariant subsheaves of $\mathcal{F}$, as in the next lemma:

Lemma 7.4 *For any κ-invariant locally free coherent subsheaf $\mathcal{F}' \subset \mathcal{F}$ we have:*

(a) $\mathcal{F}'_{\eta_C}$ is locally free of constant rank $\leqslant r$.

(b) $0 \leqslant \deg(\mathcal{F}'_{\eta_C}) \leqslant d_X$.

Proof. Applying Lemma 7.3 with $c = \eta_C$ proves (a) and the first inequality in (b). For the second inequality recall from (3.3) that $\deg(\mathcal{F}'_{\eta_C})$ is the sum of all slopes of $\mathcal{F}'_{\eta_C}$ counted with multiplicities. Let $r' := \operatorname{rank}(\mathcal{F}'_{\eta_C})$ denote the total number of these slopes; then by (3.13) for the inclusion $\mathcal{F}'_{\eta_C} \hookrightarrow \mathcal{F}_{\eta_C}$ their sum is less than or equal to the sum of the r' largest slopes of $\mathcal{F}_{\eta_C}$. As the remaining slopes of $\mathcal{F}_{\eta_C}$ are $\geqslant 0$ by 7.3 (b), applying (3.3) again we find that $\deg(\mathcal{F}'_{\eta_C}) \leqslant \deg(\mathcal{F}_{\eta_C}) = d_X$, proving (b). **q.e.d.**

The next two lemmas describe two kinds of saturations of subsheaves of $\mathcal{F}$. Consider the iterates of κ, which are injective homomorphisms

$$\kappa^m : \mathcal{F} \longhookrightarrow (\mathrm{id}\times\sigma^m)^*\mathcal{F}(md, 0).$$

Lemma 7.5 *For any κ-invariant coherent subsheaf $\mathcal{F}' \subset \mathcal{F}$ there exists a unique largest κ-invariant coherent subsheaf $\widetilde{\mathcal{F}}' \subset \mathcal{F}$ which is sent into $(\mathrm{id}\times\sigma^m)^*\mathcal{F}'(md, 0)$ by some iterate κ^m. Moreover, if $\mathcal{F}'$ is locally free, then so is $\widetilde{\mathcal{F}}'$, and it has the same rank as $\mathcal{F}'$.*

Proof. Since $\mathcal{F}'$ is κ-invariant, we have an increasing sequence of coherent subsheaves

$$\mathcal{F}' \subset \kappa^{-1}((\mathrm{id}\times\sigma)^*\mathcal{F}'(d,0)) \subset \ldots \subset (\kappa^m)^{-1}((\mathrm{id}\times\sigma^m)^*\mathcal{F}'(md,0)) \subset \ldots\ldots \subset \mathcal{F}.$$

As $C\times X$ is noetherian and $\mathcal{F}$ is coherent, this sequence becomes stationary. Let $\widetilde{\mathcal{F}}'$ be its union. Clearly it is the unique largest coherent subsheaf of $\mathcal{F}$ which is sent into $(\mathrm{id}\times\sigma^m)^*\mathcal{F}'(md, 0)$ by some iterate κ^m. By construction it is κ-invariant. This proves the first assertion of the lemma.

If $\mathcal{F}'$ is locally free, Proposition 2.2 implies that all members of the above sequence are locally free. Since κ is an isomorphism at all generic points of $C\times X$, they all have the same rank as $\mathcal{F}'$. Both assertions follow for $\widetilde{\mathcal{F}}'$, as desired. **q.e.d.**

Lemma 7.6 *For any coherent subsheaf $\mathcal{F}' \subset \mathcal{F}$ there exists a unique largest coherent sheaf $\mathcal{F}' \subset \mathcal{F}'^+ \subset \mathcal{F}$ which coincides with $\mathcal{F}'$ along $\eta_C\times X$. Moreover $\mathcal{F}'^+$ is locally free.*

Proof. The set of subsheaves which coincide with $\mathcal{F}'$ along $\eta_C \times X$ contains $\mathcal{F}'$ and is therefore non-empty. Let $\mathcal{F}'^{+}$ be their sum. Since $C \times X$ is noetherian and $\mathcal{F}$ is coherent, this is already the sum of finitely many thereof; hence it is again coherent and coincides with $\mathcal{F}'$ along $\eta_C \times X$. Clearly it is the unique largest coherent subsheaf with this property. Let $U \subset C \times X$ be the set of points where $\mathcal{F}'^{+}$ is locally free. Then U is open and its complement has codimension $\geqslant 2$. Thus if j denotes the open embedding $U \hookrightarrow C \times X$, Proposition 2.1 (b) shows that $j_* j^* \mathcal{F}'^{+}$ is locally free, and Proposition 2.1 (a) shows that $j_* j^* \mathcal{F}'^{+} \hookrightarrow j_* j^* \mathcal{F} = \mathcal{F}$. By the maximality of $\mathcal{F}'^{+}$ we therefore deduce that $\mathcal{F}'^{+} = j_* j^* \mathcal{F}'^{+}$; hence it is locally free, as desired. **q.e.d.**

Now we begin with the detailed analysis of $\mathcal{F}$. We associate to $\mathcal{F}$ certain other sheaves and subsheaves, as follows. For any integer n we set

$$\mathcal{G}_n \;:=\; \mathrm{pr}_{1*}(\mathcal{F}(0,n)). \tag{7.7}$$

This is a torsion free coherent, and hence locally free, sheaf on C. Let $\mathcal{G}_n^\mu$ denote the subsheaf associated to $\mu \in \mathbb{Q}$ in the Harder-Narasimhan filtration of $\mathcal{G}_n$. We consider the inclusion $\mathcal{G}_n^\mu \hookrightarrow \mathcal{G}_n = \mathrm{pr}_{1*}(\mathcal{F}(0,n))$ and take its adjoint homomorphism $\mathrm{pr}_1^* \mathcal{G}_n^\mu \to \mathcal{F}(0,n)$. We twist it back by $(0,-n)$, take the image sheaf, and apply the saturation procedure from Lemma 7.6 to make the resulting subsheaf

$$\mathcal{F}_n^\mu \;:=\; \mathrm{im}\big((\mathrm{pr}_1^* \mathcal{G}_n^\mu)(0,-n) \to \mathcal{F}\big)^{+} \;\subset\; \mathcal{F} \tag{7.8}$$

locally free. The defining property in Lemma 7.6 implies that $\mathrm{pr}_1^* \mathcal{G}_n^\mu(0,-n) \to \mathcal{F}$ factors through a homomorphism $\mathrm{pr}_1^* \mathcal{G}_n^\mu(0,-n) \to \mathcal{F}_n^\mu$ which is surjective over $\eta_C \times X$. Twisting again by $(0,n)$ and using the adjunction between pr_1^* and pr_{1*} yields inclusions

$$\mathcal{G}_n^\mu \;\subset\; \mathrm{pr}_{1*}(\mathcal{F}_n^\mu(0,n)) \;\subset\; \mathcal{G}_n. \tag{7.9}$$

The sheaves $\mathcal{F}_n^\mu$ and their behavior under κ will enable us to study the slopes of $\mathcal{G}_n$ and in particular to compare them for different n. First observe that for any $\mu' \geqslant \mu$ we have $\mathcal{G}_n^{\mu'} \subset \mathcal{G}_n^\mu$ and hence $\mathcal{F}_n^{\mu'} \subset \mathcal{F}_n^\mu$.

Lemma 7.10 *For all n we have $\mu^{\max}(\mathcal{G}_n) \leqslant \mu_C \leqslant 0$.*

Proof. Suppose that $\mu := \mu^{\max}(\mathcal{G}_n) > -\infty$, so that $\mathcal{G}_n \neq 0$. Then $\mathcal{G}_n$ possesses a non-zero semistable coherent subsheaf $\mathcal{G}'$ of weight μ. Since $\mathcal{G}' \hookrightarrow \mathcal{G}_n = \mathrm{pr}_{1*}(\mathcal{F}(0,n))$ is a non-zero homomorphism, so is its adjoint $\mathrm{pr}_1^* \mathcal{G}' \to \mathcal{F}(0,n)$. Here source and target are locally free sheaves on $C \times X$, hence the induced homomorphism $(\mathrm{pr}_1^* \mathcal{G}') \,|\, C \times \eta_X \to \mathcal{F}_{\eta_X}$ is also non-zero. But (3.22) shows that $(\mathrm{pr}_1^* \mathcal{G}') \,|\, C \times \eta_X$ is semistable of weight μ. Thus (3.14) and 7.1 (e) imply that $\mu \leqslant \mu^{\max}(\mathcal{F}_{\eta_X}) \leqslant \mu_C$. Finally, we have $\mu_C \leqslant 0$ by the reduction in Lemma 7.2. **q.e.d.**

Lemma 7.11 *For any integers n and $n' > (e_X r+1)n+(d_X+2g_X-2)/\ell_X$ and any $\mu \in \mathbb{Q}$ we have*
$$\mathrm{pr}_{1*}(\mathcal{F}_n^\mu(0,n')) \subset \mathcal{G}_{n'}^\mu.$$

Proof. Let $v:\ \mathrm{pr}_1^*\,\mathcal{G}_n^\mu \longrightarrow \mathcal{F}_n^\mu(0,n)$ be the homomorphism that is adjoint to the inclusion (7.9). By the construction of $\mathcal{F}_n^\mu$ its restriction to $\eta_C{\times}X$ is surjective. Thus it gives rise to a short exact sequence
$$0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{G}_{n,\eta_C}^\mu \otimes \mathcal{O}_{\eta_C\times X} \xrightarrow{v} \mathcal{F}_{n,\eta_C}^\mu(n) \longrightarrow 0$$
for some torsion free coherent, and hence locally free, sheaf $\mathcal{H}$ on $\eta_C{\times}X$. After twisting by $n'-n$ and taking cohomology over $\eta_C{\times}X$ we obtain a long exact sequence
$$\ldots \longrightarrow H^0\big(\mathcal{G}_{n,\eta_C}^\mu \otimes \mathcal{O}_{\eta_C\times X}(n'{-}n)\big) \longrightarrow H^0(\mathcal{F}_{n,\eta_C}^\mu(n')) \longrightarrow H^1(\mathcal{H}(n'{-}n)) \longrightarrow \ldots.$$
We claim that $H^1(\mathcal{H}(n'-n))=0$ under the given condition on n'. To see this note first that (3.15) and (3.7) imply that $\mu^{\max}(\mathcal{H}) \leqslant \mu^{\max}(\mathcal{O}_{\eta_C\times X})=0$. By (3.5) this in turn yields $\mu^{\min}(\mathcal{H}) \geqslant \deg(\mathcal{H})$. Using the fact that $\deg(\mathcal{O}_{\eta_C\times X})=0$, we deduce that
$$\begin{aligned}
\mu^{\min}(\mathcal{H}(n'-n)) &\overset{(3.20)}{=} \mu^{\min}(\mathcal{H})+(n'-n)\ell_X \\
&\geqslant \deg(\mathcal{H})+(n'-n)\ell_X \\
&= -\deg(\mathcal{F}_{n,\eta_C}^\mu(n))+(n'-n)\ell_X \\
&\overset{(3.21)}{=} -\deg(\mathcal{F}_{n,\eta_C}^\mu)-n\ell_X e_X\cdot\mathrm{rank}(\mathcal{F}_{n,\eta_C}^\mu)+(n'-n)\ell_X \\
&\overset{7.4\,(b)}{\geqslant} -d_X-n\ell_X e_X r+(n'-n)\ell_X.
\end{aligned}$$
The bound on n' is equivalent to this last value being $> 2g_X-2$. Thus by (3.11) it guarantees that $H^1(\mathcal{H}(n'-n))=0$, as claimed.

The claim implies that the homomorphism
$$\mathcal{G}_n^\mu \otimes_{\mathbb{F}_q} H^0(X,\mathcal{O}_X(n'{-}n)) \;\cong\; \mathrm{pr}_{1*}\big((\mathrm{pr}_1^*\,\mathcal{G}_n^\mu)(0,n'{-}n)\big) \;\longrightarrow\; \mathrm{pr}_{1*}(\mathcal{F}_n^\mu(0,n'))$$
induced by v is surjective at the generic point of C. By (3.16) and (3.7) this implies that
$$\mu \;\leqslant\; \mu^{\min}(\mathcal{G}_n^\mu) \;\leqslant\; \mu^{\min}\big(\mathrm{pr}_{1*}(\mathcal{F}_n^\mu(0,n'))\big).$$
But by (3.13) this implies that $\mathrm{pr}_{1*}(\mathcal{F}_n^\mu(0,n'))$ lies in the filtration step $\mathcal{G}_{n'}^\mu$ of $\mathcal{G}_{n'}$, as desired. **q.e.d.**

Next we fix an integer a_0 as in Proposition 4.2.

Lemma 7.12 *For all n and all $a \geqslant a_0$ the homomorphism κ induces a homomorphism*
$$\kappa:\ \mathcal{F}_{qn}^\mu \hookrightarrow (\mathrm{id}{\times}\sigma)^*\mathcal{F}_{n+a}^{\mu-d\ell_C}(d,0).$$

Proof. Let $u : \sigma_*\mathcal{O}_X \hookrightarrow \mathcal{O}_X(a)^{\oplus N}$ be the locally split monomorphism from Proposition 4.2. Then there is a unique homomorphism $\mathcal{G}_{qn} \longrightarrow \mathcal{G}_{n+a}(d)^{\oplus N}$ making the right hand side of the following diagram commute:

$$
\begin{tikzcd}
\mathcal{G}^{\mu}_{qn} \arrow[r, hook] \arrow[dddd] & \mathcal{G}_{qn} \arrow[r, equal, "(7.7)"] \arrow[dddd] & \mathrm{pr}_{1*}(\mathcal{F}(0,qn)) \arrow[r, equal] & \mathrm{pr}_{1*}(\mathrm{id}\times\sigma)_*\big[\mathcal{F}(0,qn)\big] \arrow[d, hook, "\mathrm{pr}_{1*}(\mathrm{id}\times\sigma)_*(\kappa)"] \\
 & & & \mathrm{pr}_{1*}(\mathrm{id}\times\sigma)_*(\mathrm{id}\times\sigma)^*\big[\mathcal{F}(d,n)\big] \arrow[d, equal, "\wr"'] \\
 & & & \mathrm{pr}_{1*}\big[\mathcal{F}(d,n)\otimes\mathrm{pr}_2^*\,\sigma_*\mathcal{O}_X\big] \arrow[d, hook, "\mathrm{pr}_{1*}(\mathrm{id}\otimes\,\mathrm{pr}_2^*\,u)"] \\
 & & & \mathrm{pr}_{1*}\big[\mathcal{F}(d,n)\otimes\mathrm{pr}_2^*\,\mathcal{O}_X(a)^{\oplus N}\big] \arrow[d, equal, "\wr"'] \\
\mathcal{G}^{\mu-d\ell_C}_{n+a}(d)^{\oplus N} \arrow[r, hook] & \mathcal{G}_{n+a}(d)^{\oplus N} \arrow[rr, equal, "(7.7)"] & & \mathrm{pr}_{1*}\big[\mathcal{F}(d,n+a)^{\oplus N}\big].
\end{tikzcd}
\tag{7.13}
$$

By (3.13) and (3.20) there is a unique homomorphism $\mathcal{G}^{\mu}_{qn} \hookrightarrow \big(\mathcal{G}_{n+a}(d)^{\oplus N}\big)^{\mu} = \mathcal{G}^{\mu-d\ell_C}_{n+a}(d)^{\oplus N}$ making the left hand side commute. Applying adjunction between pr_1^* and pr_{1*} to the outer edge of the preceding diagram yields the outer edge of the following diagram:

$$
\begin{tikzcd}
\mathrm{pr}_1^*\,\mathcal{G}^{\mu}_{qn} \arrow[rr] \arrow[dr, dashed] \arrow[dddd] & & (\mathrm{id}\times\sigma)_*\big[\mathcal{F}(0,qn)\big] \arrow[d, hook, "(\mathrm{id}\times\sigma)_*(\kappa)"] \\
 & (\mathrm{id}\times\sigma)_*(\mathrm{id}\times\sigma)^*\big[\mathcal{F}^{\mu-d\ell_C}_{n+a}(d,n)\big] \arrow[r, hook] \arrow[d, equal, "\wr"'] & (\mathrm{id}\times\sigma)_*(\mathrm{id}\times\sigma)^*\big[\mathcal{F}(d,n)\big] \arrow[d, equal, "\wr"'] \\
 & \mathcal{F}^{\mu-d\ell_C}_{n+a}(d,n)\otimes\mathrm{pr}_2^*\,\sigma_*\mathcal{O}_X \arrow[r, hook] \arrow[d, hook, "\mathrm{id}\otimes\,\mathrm{pr}_2^*\,u"] & \mathcal{F}(d,n)\otimes\mathrm{pr}_2^*\,\sigma_*\mathcal{O}_X \arrow[d, hook, "\mathrm{id}\otimes\,\mathrm{pr}_2^*\,u"] \\
 & \mathcal{F}^{\mu-d\ell_C}_{n+a}(d,n)\otimes\mathrm{pr}_2^*\,\mathcal{O}_X(a)^{\oplus N} \arrow[r, hook] \arrow[d, equal, "\wr"'] & \mathcal{F}(d,n)\otimes\mathrm{pr}_2^*\,\mathcal{O}_X(a)^{\oplus N} \arrow[d, equal, "\wr"'] \\
\mathrm{pr}_1^*\big[\mathcal{G}^{\mu-d\ell_C}_{n+a}(d)^{\oplus N}\big] \arrow[r] \arrow[rr, bend right] & \mathcal{F}^{\mu-d\ell_C}_{n+a}(d,n+a)^{\oplus N} \arrow[r, hook] & \mathcal{F}(d,n+a)^{\oplus N}.
\end{tikzcd}
\tag{7.14}
$$

Here the factorization in the bottom row is the twist by $(d,n+a)$ of the factorization

$$
(\mathrm{pr}_1^*\,\mathcal{G}^{\mu-d\ell_C}_{n+a})(0,-n-a) \longrightarrow \mathcal{F}^{\mu-d\ell_C}_{n+a} \hookrightarrow \mathcal{F}
$$

obtained from the definition of $\mathcal{F}^{\mu-d\ell_C}_{n+a}$. The right half of (7.14) is by construction commutative. It is even cartesian, because the tensor product of any inclusion of coherent sheaves with the locally split monomorphism of locally free coherent sheaves $\mathrm{pr}_2^* u : \mathrm{pr}_2^* \sigma_* \mathcal{O}_X \hookrightarrow \mathrm{pr}_2^* \mathcal{O}_X(a)^{\oplus N}$ yields a cartesian square. Thus there exists a unique dashed arrow making the whole diagram (7.14) commute. Applying adjunction between $(\mathrm{id}{\times}\sigma)^*$ and $(\mathrm{id}{\times}\sigma)^*$ to its two upper rows yields the commutative diagram

$$\begin{tikzcd}
(\mathrm{id}{\times}\sigma)^*\, \mathrm{pr}_1^* \mathcal{G}^\mu_{qn} \arrow[r, equal] \arrow[dr] & \mathrm{pr}_1^* \mathcal{G}^\mu_{qn} \arrow[r] \arrow[d] & \mathcal{F}(0,qn) \arrow[d, hook, "\kappa"] \\
& (\mathrm{id}{\times}\sigma)^*\big[\mathcal{F}^{\mu-d\ell_C}_{n+a}(d,n)\big] \arrow[r, hook] & (\mathrm{id}{\times}\sigma)^*\big[\mathcal{F}(d,n)\big].
\end{tikzcd} \tag{7.15}$$

Twisting back by $(0,-qn)$ this in turn yields the outer edge of the commutative diagram

$$\begin{tikzcd}
(\mathrm{pr}_1^* \mathcal{G}^\mu_{qn})(0,-qn) \arrow[r] \arrow[d] \arrow[rr, bend left] & \mathcal{F}^\mu_{qn} \arrow[r, hook] \arrow[dl, hook, dashed] & \mathcal{F} \arrow[d, hook, "\kappa"] \\
(\mathrm{id}{\times}\sigma)^*\big[\mathcal{F}^{\mu-d\ell_C}_{n+a}(d,0)\big] \arrow[rr, hook] & & (\mathrm{id}{\times}\sigma)^*\big[\mathcal{F}(d,0)\big].
\end{tikzcd} \tag{7.16}$$

By the definition (7.8) of $\mathcal{F}^\mu_{qn}$ the top left horizontal arrow $(\mathrm{pr}_1^* \mathcal{G}^\mu_{qn})(0,-qn) \longrightarrow \mathcal{F}^\mu_{qn}$ is surjective on $\eta_C{\times}X$. Thus the composite homomorphism

$$\begin{tikzcd}
\mathcal{F}^\mu_{qn} \arrow[r, hook] & \mathcal{F} \arrow[d, hook, "\kappa"] & (\mathrm{id}{\times}\sigma)^*\big[\mathcal{F}/\mathcal{F}^{\mu-d\ell_C}_{n+a}\big](d,0) \arrow[d, equal, "\wr"] \\
& (\mathrm{id}{\times}\sigma)^*\big[\mathcal{F}(d,0)\big] \arrow[r, two heads] & (\mathrm{id}{\times}\sigma)^*\big[\mathcal{F}(d,0)\big]/(\mathrm{id}{\times}\sigma)^*\big[\mathcal{F}^{\mu-d\ell_C}_{n+a}(d,0)\big]
\end{tikzcd} \tag{7.17}$$

is zero on $\eta_C{\times}X$, and so its image is $\mathcal{O}_C$-torsion. But Lemma 7.6 and the construction (7.8) imply that $\mathcal{F}/\mathcal{F}^{\mu-d\ell_C}_{n+a}$ is $\mathcal{O}_C$-torsion free, and so the target of (7.17) is $\mathcal{O}_C$-torsion free. Thus the composite homomorphism (7.17) is zero everywhere. This means that there exists a unique dashed arrow making the diagram (7.16) commute, as desired. **q.e.d.**

Lemma 7.18 *For any integer $n \geqslant a_0/(q-1)$ and any $\mu \in \mathbb{Q}$ such that $\mathcal{G}_{qn}$ has no slopes in the interval $[\mu - d\ell_C, \mu)$, the subsheaf $\mathcal{F}^\mu_{qn} \subset \mathcal{F}$ is κ-invariant.*

Proof. The assumption means that $\mathcal{G}^{\mu-d\ell_C}_{qn} = \mathcal{G}^\mu_{qn}$, which implies that $\mathcal{F}^{\mu-d\ell_C}_{qn} = \mathcal{F}^\mu_{qn}$. Thus applying Lemma 7.12 with $a = (q-1)n \geqslant a_0$ shows that κ induces a homomorphism

$$\mathcal{F}^\mu_{qn} \hookrightarrow (\mathrm{id}{\times}\sigma)^*\mathcal{F}^{\mu-d\ell_C}_{qn}(d,0) \;=\; (\mathrm{id}{\times}\sigma)^*\mathcal{F}^\mu_{qn}(d,0),$$

as desired. **q.e.d.**

Now we fix an integer

$$n_0 \;>\; \max\Big\{\frac{a_0}{q-1}, \frac{2g_X - 1 - \mu_X}{q\ell_X}\Big\}, \tag{7.19}$$

independent of $\mathcal{F}$. Then Proposition 5.3 holds for all integers $n \geqslant qn_0$. In particular it implies that $R := \mathrm{rank}(\mathcal{G}_{qn_0})$ is independent of $\mathcal{F}$. We let $\mu_1 > \ldots > \mu_s$ be those among the slopes μ of $\mathcal{G}_{qn_0}$ for which $\mathcal{G}_{qn_0}$ has no slopes in the interval $[\mu - d\ell_C, \mu)$. Then evidently $\mu_s = \mu^{\min}(\mathcal{G}_{qn_0})$. Setting $\mu_0 := \infty$ and $\mathcal{G}_{qn_0}^{\infty} := 0$, we are thus interested in the steps

$$0 = \mathcal{G}_{qn_0}^{\mu_0} \subset \mathcal{G}_{qn_0}^{\mu_1} \subset \ldots \subset \mathcal{G}_{qn_0}^{\mu_s} = \mathcal{G}_{qn_0}$$

of the Harder-Narasimhan filtration of $\mathcal{G}$. For any $1 \leqslant i \leqslant s$ we abbreviate

$$\mathcal{F}^{(i)} := \mathcal{F}_{qn_0}^{\mu_i}, \tag{7.20}$$

which is locally free by the construction (7.8) and κ-invariant by Lemma 7.18. We also set $\mu_0 := \infty$ and $\mathcal{F}^{(0)} := \mathcal{F}_{qn_0}^{\infty} := 0$. We thus have a sequence of κ-subsheaves

$$0 = \mathcal{F}^{(0)} \subset \mathcal{F}^{(1)} \subset \ldots \subset \mathcal{F}^{(s)} = \mathcal{F}.$$

By construction the first inclusion is proper, but any or all of the others may conceivably be inclusions of equal rank or even equalities.

Lemma 7.21 *For any $1 \leqslant i \leqslant s$ we have $\mathcal{F}_{qn_0}^{\mu_i + Rd\ell_C} \subset \mathcal{F}^{(i-1)}$.*

Proof. By construction the slopes of $\mathcal{G}_{qn_0}^{\mu_i}/\mathcal{G}_{qn_0}^{\mu_{i-1}}$ have successive differences $\leqslant d\ell_C$, and the smallest slope is μ_i. Since the rank of this sheaf is $\leqslant \mathrm{rank}(\mathcal{G}_{qn_0}) = R$, it follows that its largest slope is $< \mu_i + Rd\ell_C$. But this means that $\mathcal{G}_{qn_0}^{\mu_i + Rd\ell_C} \subset \mathcal{G}_{qn_0}^{\mu_{i-1}}$, which in turn implies that $\mathcal{F}_{qn_0}^{\mu_i + Rd\ell_C} \subset \mathcal{F}_{qn_0}^{\mu_{i-1}} = \mathcal{F}^{(i-1)}$, as desired. **q.e.d.**

Also, Lemma 7.11 immediately implies:

Lemma 7.22 *For any $n' > (e_X r + 1)qn_0 + (d_X + 2g_X - 2)/\ell_X$ and any $1 \leqslant i \leqslant s$ we have*

$$\mathrm{pr}_{1*}\big(\mathcal{F}^{(i)}(0, n')\big) \subset \mathcal{G}_{n'}^{\mu_i}.$$

Next we define a sequence of integers n_j beginning with n_0 by recursively solving $n_j + a_0 = qn_{j-1}$ for all $j \geqslant 1$. Then for all $j \geqslant 0$ we have

$$n_j \;=\; q^j \cdot \left(n_0 - \frac{a_0}{q-1}\right) + \frac{a_0}{q-1}. \tag{7.23}$$

Since $n_0 - \frac{a_0}{q-1} > 0$ by (7.19), this tends to ∞ for $j \to \infty$. Let $\widetilde{\mathcal{F}}^{(i-1)}$ be the saturation of $\mathcal{F}^{(i-1)}$ constructed in Lemma 7.5.

Lemma 7.24 *For every $j \geqslant 0$ and every $1 \leqslant i \leqslant s$ we have*

$$\mathcal{G}_{qn_j}^{\mu_i + (R+j)d\ell_C} \subset \mathrm{pr}_{1*}\big(\widetilde{\mathcal{F}}^{(i-1)}(0, qn_j)\big).$$

Proof. Lemma 7.12 and the recursive definition of n_j show that κ induces homomorphisms

$$\mathcal{F}^{\mu}_{qn_j} \hookrightarrow (\mathrm{id}\times\sigma)^*\mathcal{F}^{\mu-d\ell_C}_{qn_{j-1}}(d,0)$$

for all j and μ. By iteration we deduce that κ^j induces a homomorphism

$$\mathcal{F}^{\mu_i+(R+j)d\ell_C}_{qn_j} \;\hookrightarrow\; (\mathrm{id}\times\sigma^j)^*\mathcal{F}^{\mu_i+Rd\ell_C}_{qn_0}(jd,0).$$

By Lemma 7.21 the target is contained in $(\mathrm{id}\times\sigma^j)^*\mathcal{F}^{(i-1)}(jd,0)$. Thus by Lemma 7.5 it follows that $\mathcal{F}^{\mu_i+(R+j)d\ell_C}_{qn_j} \subset \widetilde{\mathcal{F}}^{(i-1)}$. This in turn implies that

$$\mathcal{G}^{\mu_i+(R+j)d\ell_C}_{qn_j} \overset{(7.9)}{\subset} \mathrm{pr}_{1*}\big(\mathcal{F}^{\mu_i+(R+j)d\ell_C}_{qn_j}(0,qn_j)\big) \subset \mathrm{pr}_{1*}\big(\widetilde{\mathcal{F}}^{(i-1)}(0,qn_j)\big),$$

as desired. **q.e.d.**

We will use Lemmas 7.22 and 7.24 to estimate $\deg(\mathcal{G}_{n'})$ from below and $\deg(\mathcal{G}_{qn_j})$ from above. For this recall from Lemmas 7.4 (a) and 7.5 that $\mathcal{F}^{(i)}$ and $\widetilde{\mathcal{F}}^{(i)}$ are locally free of constant and equal rank. Abbreviate

$$\begin{aligned} r_i &:= \mathrm{rank}(\mathcal{F}^{(i)}) = \mathrm{rank}\big(\widetilde{\mathcal{F}}^{(i)}\big),\\ d_i &:= \deg(\mathcal{F}^{(i)}_{\eta_C}),\\ \tilde{d}_i &:= \deg\big(\widetilde{\mathcal{F}}^{(i)}_{\eta_C}\big), \end{aligned}$$

and note that Lemma 7.4 (b) implies that

(7.25) $\quad 0 \leqslant d_i \leqslant \tilde{d}_i \leqslant d_X.$

Let s' be the smallest integer $\leqslant s$ such that $r_{s'} = r$. Then for every $s' \leqslant i \leqslant s$ we have $\mathcal{F}^{(i)}\,|\,\eta_C\times X = \mathcal{F}\,|\,\eta_C\times X$ by assumption 7.1 (f). Combined with the fact that $\mathcal{F}^{(i)} = \mathcal{F}^{(i)+}$ and Lemma 7.6 this implies that $\mathcal{F}^{(i)} = \mathcal{F}$ and hence

(7.26) $\quad \mathcal{F}^{(i)} = \widetilde{\mathcal{F}}^{(i)} = \mathcal{F}$ for all $s' \leqslant i \leqslant s$, and

(7.27) $\quad r_i = r_{i-1} = r$ and $d_i = d_{i-1} = \tilde{d}_i = \tilde{d}_{i-1} = d_X$ for all $s' < i \leqslant s$.

Lemma 7.28 *For all $n' \geqslant qn_0$ and all $0 \leqslant i \leqslant s$ we have:*

(a) $D_i(n') := \mathrm{rank}\big(\mathrm{pr}_{1*}\big(\mathcal{F}^{(i)}(0,n')\big)\big) = d_i + (n'\ell_X+1-g_X)e_X r_i.$

(b) $\tilde{D}_i(n') := \mathrm{rank}\big(\mathrm{pr}_{1*}\big(\widetilde{\mathcal{F}}^{(i)}(0,n')\big)\big) = \tilde{d}_i + (n'\ell_X+1-g_X)e_X r_i.$

Proof. Observe that

$$\begin{aligned} \mu^{\min}(\mathcal{F}^{(i)}_{\eta_C}(n')) &\overset{(3.20)}{=} \mu^{\min}(\mathcal{F}^{(i)}_{\eta_C}) + n'\ell_X \\ &\overset{7.3\,(b)}{\geqslant} 0 + n'\ell_X \geqslant qn_0\ell_X \overset{(7.19)}{>} 2g_X - 1 - \mu_X \overset{7.2}{>} 2g_X - 2. \end{aligned}$$

By (3.11) this implies that $h^1(\eta_C{\times}X, \mathcal{F}^{(i)}_{\eta_C}(n'))=0$. With Riemann-Roch we deduce that

$$\begin{aligned}\operatorname{rank}\bigl(\mathrm{pr}_{1*}\bigl(\mathcal{F}^{(i)}(0,n')\bigr)\bigr) &= h^0(\eta_C{\times}X, \mathcal{F}^{(i)}_{\eta_C}(n'))\\ &\overset{(3.1)}{=} \deg(\mathcal{F}^{(i)}_{\eta_C}(n')) + (1-g_X)\cdot e_X \cdot \operatorname{rank}(\mathcal{F}^{(i)}_{\eta_C}(n'))\\ &\overset{(3.21)}{=} d_i + (n'\ell_X + 1 - g_X)\cdot e_X \cdot r_i.\end{aligned}$$

This proves (a), and in the same way one proves (b). **q.e.d.**

Lemma 7.29 *For all n' as in Lemma 7.22 and all $j \geqslant 0$ we have:*

(a) $\deg(\mathcal{G}_{n'}) \;\geqslant\; \sum_{i=1}^{s}\bigl(D_i(n') - D_{i-1}(n')\bigr)\cdot e_C \cdot \mu_i.$

(b) $\deg(\mathcal{G}_{qn_j}) \;\leqslant\; \sum_{i=1}^{s}\bigl(\tilde{D}_i(qn_j) - \tilde{D}_{i-1}(qn_j)\bigr)\cdot e_C \cdot \bigl(\mu_i + (R{+}j)d\ell_C\bigr).$

Proof. Recall from Section 3 that the multiplicity of each slope in the total degree is e_C times the rank of the associated subquotient of the Harder-Narasimhan filtration.
Lemma 7.22 implies that $\operatorname{rank}(\mathcal{G}^{\mu_i}_{n'}) \geqslant D_i(n')$ for every $1 \leqslant i \leqslant s$. Thus the $D_i(n')\cdot e_C$ largest slopes of $\mathcal{G}_{n'}$—counted with multiplicities—are $\geqslant \mu_i$. At the same time $D_{i-1}(n')\cdot e_C$ of these are already $\geqslant \mu_{i-1}$. Thus by bounding $(D_i(n') - D_{i-1}(n'))\cdot e_C$ slopes from below by μ_i for all $1 \leqslant i \leqslant s$ and summing up over all i we obtain (a).
Similarly, Lemma 7.24 implies that $\operatorname{rank}(\mathcal{G}^{\mu_i+(R+j)d\ell_C}_{qn_j}) \leqslant \tilde{D}_{i-1}(qn_j)$ for every $1 \leqslant i \leqslant s$. Thus the $(\operatorname{rank}(\mathcal{G}_{qn_j}) - \tilde{D}_{i-1}(qn_j))\cdot e_C$ smallest slopes of $\mathcal{G}_{qn_j}$ with multiplicities are $< \mu_i + (R{+}j)d\ell_C$. At the same time $(\operatorname{rank}(\mathcal{G}_{qn_j}) - \tilde{D}_i(qn_j))\cdot e_C$ of these are already $< \mu_{i+1} + (R{+}j)d\ell_C$. Thus by bounding $(\tilde{D}_i(qn_j) - \tilde{D}_{i-1}(qn_j))\cdot e_C$ slopes from below by $\mu_i + (R{+}j)d\ell_C$ for all $1 \leqslant i \leqslant s$ and summing up over all i we obtain (b). **q.e.d.**

Lemma 7.30 *For all n' as in Lemma 7.22 and all $j \geqslant 0$ we have:*

$$\begin{aligned}(qn_j{-}n')\cdot \ell_X e_X d_C \;\leqslant\; &\bigl(d_X + (qn_j\ell_X{+}1{-}g_X)e_X r\bigr)\cdot e_C(R{+}j)d\ell_C\\ &+ \bigl((qn_j{-}n')\ell_X e_X(r_{s'}{-}r_{s'-1}) - Rd_X\bigr)\cdot e_C\cdot \mu_{s'}\\ &+ (qn_j{-}n')\cdot \textstyle\sum_{i=1}^{s'-1} \ell_X e_X(r_i{-}r_{i-1})\cdot e_C\cdot \mu_i.\end{aligned}$$

Proof. We calculate

$$\begin{aligned}
(qn_j{-}n')\cdot \ell_X e_X d_C \;&\overset{5.3\,(i)}{=}\; \deg(\mathcal{G}_{qn_j}) - \deg(\mathcal{G}_{n'}) \\
&\overset{7.29}{\leqslant}\; \sum_{i=1}^{s}\Big(\tilde{D}_i(qn_j) - \tilde{D}_{i-1}(qn_j)\Big)\cdot e_C(R{+}j)d\ell_C \\
&\qquad + \sum_{i=1}^{s}\Big(\tilde{D}_i(qn_j) - \tilde{D}_{i-1}(qn_j) - D_i(n') + D_{i-1}(n')\Big)\cdot e_C\cdot\mu_i \\
&\overset{7.28}{=}\; \big(\tilde{D}_s(qn_j) - \tilde{D}_0(qn_j)\big)\cdot e_C(R{+}j)d\ell_C \\
&\qquad + \sum_{i=1}^{s}\big(\tilde{d}_i{-}d_i{-}\tilde{d}_{i-1}{+}d_{i-1}\big)\cdot e_C\cdot\mu_i. \\
&\qquad\quad + \sum_{i=1}^{s}(qn_j{-}n')\ell_X e_X(r_i{-}r_{i-1})\cdot e_C\cdot\mu_i.
\end{aligned}$$

We look at the three terms on the right hand side in turn. Since $\mathcal{F}^{(0)} = 0$ and $\mathcal{F}^{(s)} = \mathcal{F}$, the first term is equal to

$$\operatorname{rank}(\mathcal{G}_{qn_j})\cdot e_C(R{+}j)d\ell_C \;\overset{5.3\,(h)}{=}\; (d_X + (qn_j\ell_X{+}1{-}g_X)e_X r)\cdot e_C(R{+}j)d\ell_C.$$

For the second term note that $\mu_i \leqslant 0$ by Lemma 7.10. Also observe that the summands for $s' < i \leqslant s$ vanish by (7.27). Thus using (7.25) and the fact that $s' \leqslant s \leqslant \operatorname{rank}(\mathcal{G}_{qn_0}) = R$ we find that the second term is

$$\leqslant\; \sum_{i=1}^{s'} d_X\cdot e_C\cdot|\mu_i| \;\leqslant\; s'd_X e_C\cdot|\mu_{s'}| \;\leqslant\; -Rd_X e_C\cdot\mu_{s'}.$$

In the third term again the summands for $s' < i \leqslant s$ vanish by (7.27). Thus by combining the summand for $i = s'$ with the second term the lemma follows. **q.e.d.**

Lemma 7.31 *The slope $\mu_{s'}$ is bounded below by a constant μ that is independent of $\mathcal{F}$.*

Proof. Fix any integer n' as in Lemma 7.22. Thereafter, fix any $j \geqslant 0$ such that

$$(qn_j{-}n')\ell_X e_X(r_{s'}{-}r_{s'-1}) - Rd_X \;>\; 0,$$

which is possible by (7.23) and because $\ell_X e_X(r_{s'}{-}r_{s'-1}) > 0$ by the choice of s'. Then in particular $qn_j{-}n' \geqslant 0$; hence so is the coefficient of each μ_i in the last line of Lemma 7.30. Since $\mu_i \leqslant 0$ by Lemma 7.10, the inequality in Lemma 7.30 remains true after removing that line. Solving for $\mu_{s'}$ then yields a lower bound which is independent of $\mathcal{F}$, as desired. **q.e.d.**

Lemma 7.32 *There exist constants $n' > (2g_X{-}1{-}\mu_X)/\ell_X$ and d_1, d_2, μ, all independent of $\mathcal{F}$, such that $d_1 \leqslant \deg(\mathcal{G}_{n'}) \leqslant d_2$ and $\mu^{\min}(\mathcal{G}_{n'}) \geqslant \mu$.*

Proof. Take any μ as in Lemma 7.31 and any n' as in Lemma 7.22, independent of $\mathcal{F}$. Since $\mathcal{F}^{(s')} = \mathcal{F}$ by (7.26), Lemma 7.22 implies that $\mathcal{G}_{n'} \subset \mathcal{G}_{n'}^{\mu_{s'}}$ and hence $\mu^{\min}(\mathcal{G}_{n'}) \geqslant \mu_{s'} \geqslant \mu$. On the other hand we have $\mu^{\max}(\mathcal{G}_{n'}) \leqslant \mu_C$ by Lemma 7.10. Thus with (3.4) we deduce that

$$\begin{aligned} d_1 \;:=\; e_C \cdot \mathrm{rank}(\mathcal{G}_{n'}) \cdot \mu \;&\leqslant\; e_C \cdot \mathrm{rank}(\mathcal{G}_{n'}) \cdot \mu^{\min}(\mathcal{G}_{n'}) \\ &\leqslant\; \deg(\mathcal{G}_{n'}) \\ &\leqslant\; e_C \cdot \mathrm{rank}(\mathcal{G}_{n'}) \cdot \mu^{\max}(\mathcal{G}_{n'}) \;\leqslant\; e_C \cdot \mathrm{rank}(\mathcal{G}_{n'}) \cdot \mu_C \;=:\; d_2. \end{aligned}$$

Here $\mathrm{rank}(\mathcal{G}_{n'})$ and hence d_1 and d_2 are independent of $\mathcal{F}$ by Proposition 5.3 (h). **q.e.d.**

Proof of Theorem 7.1. Combining Lemma 7.32 with Theorem 5.4 for every integer $d_1 \leqslant d \leqslant d_2$ shows that there exist at most finitely many possibilities for the isomorphism class of the coherent sheaf underlying $\mathcal{F}$. For any fixed $\mathcal{F}$, the homomorphism κ lies in the group $\mathrm{Hom}(\mathcal{F}, (\mathrm{id}\times\sigma)^*\mathcal{F}(d,0))$, which is a finite dimensional vector space over $\mathbb{F}_q$. Thus there are at most finitely many possibilities for κ, which finishes the proof of Theorem 7.1. **q.e.d.**

8 A-motives and κ-sheaves over a field

Let C be an irreducible smooth projective curve over $\mathbb{F}_q$. Fix a closed point $\infty \in C$ and set $C^\circ := C \smallsetminus \{\infty\}$. Let K be a field together with a ring homomorphism $\gamma : A \to K$. We are interested in the curve $C^\circ_K := C^\circ \times \mathrm{Spec}\, K$ over K, where the fiber product is taken over $\mathrm{Spec}\, \mathbb{F}_q$. Let θ denote the closed point of C°_K corresponding to γ. Definition 6.1 introduces the notion of *κ-sheaves on C°_K*.

Definition 8.1 *A κ-sheaf $\mathcal{G}$ on C°_K is called* of characteristic θ *if $\kappa : \mathcal{G} \hookrightarrow (\mathrm{id}\times\sigma)^*\mathcal{G}$ is an isomorphism outside θ.*

Let $A := \Gamma(C^\circ, \mathcal{O}_{C^\circ})$ denote the ring of regular functions on C°. For any κ-sheaf $\mathcal{G}$ on C°_K, the global sections of the dual sheaf $\mathcal{G}^\vee$ form a finitely generated projective $A\otimes K$-module $M := \Gamma(C^\circ_K, \mathcal{G}^\vee)$, and κ corresponds to an injective $A\otimes K$-linear map $\tau^{\mathrm{lin}} : (\mathrm{id}\otimes\sigma)_* M \hookrightarrow M$. Moreover, κ is an isomorphism outside θ if and only if $\mathrm{coker}(\tau^{\mathrm{lin}})$ is annihilated by a power of $a\otimes 1 - 1\otimes\gamma(a)$ for all $a \in A$. Thus any κ-sheaf of characteristic θ on C_K yields an A-motive of characteristic γ over K by Definition 1.2. Clearly this process can be reversed and yields:

Proposition 8.2 *The above construction induces an anti-equivalence of categories between the category of κ-sheaves (resp. those of characteristic θ) on C°_K and the category of A-motives (resp. those of characteristic γ) over K.*

Throughout the following we call an injective homomorphism of κ-sheaves $\mathcal{G} \hookrightarrow \mathcal{G}'$ an *inclusion* and denote its cokernel by $\mathcal{G}'/\mathcal{G}$. Two inclusions $i_1 : \mathcal{G} \hookrightarrow \mathcal{G}_1$ and $i_2 : \mathcal{G} \hookrightarrow \mathcal{G}_2$ for which there exists an isomorphism $f : \mathcal{G}_1 \to \mathcal{G}_2$ such that $i_2 = f \circ i_1$ are called *isomorphic*. An inclusion which is an isomorphism everywhere is sometimes—by abuse of notation—called an equality. We will now study inclusions of κ-sheaves of equal rank from different angles.

Proposition 8.3 *Fix an A-motive M of characteristic γ over K and its associated κ-sheaf $\mathcal{G}$ of characteristic θ on C_K°. Then there is a natural bijection between isomorphism classes of*

(a) isogenies $M' \hookrightarrow M$ of A-motives of characteristic γ over K, and

(b) inclusions $\mathcal{G} \hookrightarrow \mathcal{G}'$ of κ-sheaves of equal rank and characteristic θ on C_K°.

The isogeny $M' \hookrightarrow M$ is separable if and only if the homomorphism $\mathcal{G}'/\mathcal{G} \to (\mathrm{id}\times\sigma)^*(\mathcal{G}'/\mathcal{G})$ induced by κ is an isomorphism. Moreover, the isomorphism class of M' (without the isogeny) is determined uniquely by the isomorphism class of $\mathcal{G}'$.

Proof. Consider a homomorphism of A-motives $f : M' \to M$ corresponding to a homomorphism of κ-sheaves $\varphi : \mathcal{G} \to \mathcal{G}'$. Then f is an isogeny if and only if it becomes an isomorphism over $\mathrm{Quot}(A \otimes K)$, if and only if φ is generically an isomorphism, if and only if φ is injective and $\mathrm{rank}(\mathcal{G}) = \mathrm{rank}(\mathcal{G}')$. The desired bijection is thus a consequence of Proposition 8.2.

It remains to determine when f is separable. By definition it is so if and only if τ^{lin} induces an isomorphism $(\mathrm{id}\times\sigma)_* \operatorname{coker}(f) \to \operatorname{coker}(f)$. By usual diagram arguments one checks that this is equivalent to the exactness of the sequence

$$0 \longrightarrow (\mathrm{id}\otimes\sigma)_* M' \xrightarrow{\left(\begin{smallmatrix}\tau^{\mathrm{lin}} \\ -f\end{smallmatrix}\right)} M' \oplus (\mathrm{id}\otimes\sigma)_* M \xrightarrow{f+\tau^{\mathrm{lin}}} M \longrightarrow 0.$$

Dualizing, this is equivalent to the exactness of the sequence

$$0 \longleftarrow (\mathrm{id}\times\sigma)^*\mathcal{G}' \xleftarrow{\kappa-\varphi} \mathcal{G}' \oplus (\mathrm{id}\times\sigma)^*\mathcal{G} \xleftarrow{\left(\begin{smallmatrix}\varphi \\ \kappa\end{smallmatrix}\right)} \mathcal{G} \longleftarrow 0,$$

which in turn is equivalent to the isomorphy of $\operatorname{coker}(\varphi) \to (\mathrm{id}\times\sigma)^* \operatorname{coker}(\varphi)$, as desired.

q.e.d.

Next we look more closely at the points where an inclusion of κ-sheaves of equal rank is not an equality. Here we drop the assumption on the characteristic. To any coherent torsion sheaf $\mathcal{T}$ on C_K° we associate the effective divisor

(8.4) $$\mathrm{Div}(\mathcal{T}) := \sum_{P \in C_K^\circ} \mathrm{length}(\mathcal{T}_P) \cdot P.$$

Clearly it is additive in short exact sequences. For any κ-sheaf $\mathcal{G}$ on C_K° we abbreviate

$$\mathrm{Char}(\mathcal{G}) := \mathrm{Div}\big(\mathrm{coker}(\kappa|\mathcal{G})\big). \tag{8.5}$$

By definition $\mathcal{G}$ is of characteristic θ if and only if $\mathrm{Char}(\mathcal{G})$ is a multiple of θ. Thus $\mathrm{Char}(\mathcal{G})$ can be viewed as a generalized characteristic of $\mathcal{G}$.

Lemma 8.6 *For any inclusion $\mathcal{G} \hookrightarrow \mathcal{G}'$ of κ-sheaves of equal rank on C_K° we have*

$$\mathrm{Char}(\mathcal{G}) + (\mathrm{id}\times\sigma)^* \,\mathrm{Div}(\mathcal{G}'/\mathcal{G}) = \mathrm{Char}(\mathcal{G}') + \mathrm{Div}(\mathcal{G}'/\mathcal{G}).$$

Moreover, if $\bar{K}$ is an algebraic closure of K, there exist integers $n, i_1, \ldots, i_n \geqslant 0$, closed points $P_1, \ldots, P_n \in C_{\bar{K}}$, and an effective divisor $D = (\mathrm{id}\times\sigma)^* D$ on $C_{\bar{K}}$ such that

$$\begin{aligned}
\mathrm{Char}(\mathcal{G})_{\bar{K}} &= \textstyle\sum_{\nu=1}^n P_\nu, \\
\mathrm{Char}(\mathcal{G}')_{\bar{K}} &= \textstyle\sum_{\nu=1}^n (\mathrm{id}\times\sigma^{i_\nu})^* P_\nu, \\
\mathrm{Div}(\mathcal{G}'/\mathcal{G})_{\bar{K}} &= \textstyle\sum_{\nu=1}^n \sum_{i=0}^{i_\nu-1} (\mathrm{id}\times\sigma^i)^* P_\nu \;+\; D.
\end{aligned}$$

Proof. The snake lemma yields a commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
 & & 0 \arrow[d] & 0 \arrow[d] & \mathcal{T}' \arrow[d] & \\
 & 0 \arrow[r] & \mathcal{G} \arrow[r] \arrow[d] & (\mathrm{id}\times\sigma)^*\mathcal{G} \arrow[r] \arrow[d] & \mathrm{coker}(\kappa|\mathcal{G}) \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \mathcal{G}' \arrow[r] \arrow[d] & (\mathrm{id}\times\sigma)^*\mathcal{G}' \arrow[r] \arrow[d] & \mathrm{coker}(\kappa|\mathcal{G}') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{T}' \arrow[r] & \mathcal{G}'/\mathcal{G} \arrow[r] \arrow[d] & (\mathrm{id}\times\sigma)^*(\mathcal{G}'/\mathcal{G}) \arrow[r] \arrow[d] & \mathcal{T} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & 0 &
\end{tikzcd}$$

whose last row and last column consist of coherent torsion sheaves. Thus the additivity of Div( ) implies that

$$\begin{aligned}
\mathrm{Char}(\mathcal{G}') - \mathrm{Char}(\mathcal{G}) &= \mathrm{Div}(\mathcal{T}') - \mathrm{Div}(\mathcal{T}) \\
&= \mathrm{Div}(\mathcal{G}/\mathcal{G}') - \mathrm{Div}\big((\mathrm{id}\times\sigma)^*(\mathcal{G}/\mathcal{G}')\big) \\
&= \mathrm{Div}(\mathcal{G}/\mathcal{G}') - (\mathrm{id}\times\sigma)^* \,\mathrm{Div}(\mathcal{G}/\mathcal{G}'),
\end{aligned} \tag{8.7}$$

proving the first assertion. We prove the second assertion more generally for any effective divisors E, E', F on $C_{\bar{K}}$ satisfying

$$E + (\mathrm{id}\times\sigma)^*F = E' + F. \tag{8.8}$$

First, let D be the largest $\mathrm{id}\times\sigma$-invariant effective divisor $\leqslant F$. After replacing F by $F-D$ we may assume that F does not contain a full $\mathrm{id}\times\sigma$-orbit of points of $C_{\bar{K}}$. Next suppose that E is non-zero and take any point P_1 occurring in E. Then (8.8) implies that P_1 occurs in $E'+F$. Thus if P_1 does not appear in E', it appears in F and so $(\mathrm{id}\times\sigma)^*P_1$ appears in $E'+F$ by (8.8). We repeat this procedure with $(\mathrm{id}\times\sigma)^*P_1$ and $(\mathrm{id}\times\sigma^2)^*P_1$ and so on in place of P_1, as long as this point does not appear in E'. Since F does not contain a full $\mathrm{id}\times\sigma$-orbit, the procedure must stop for some integer $i_1 \geqslant 0$ such that $(\mathrm{id}\times\sigma^{i_1})^*P_1$ occurs in E' and $(\mathrm{id}\times\sigma^{i})^*P_1$ occurs in F but not in E' for any $0 \leqslant i < i_1$. We can then replace E and E' and F, respectively, by $E-P_1$ and $E' - (\mathrm{id}\times\sigma^{i_1})^*P_1$ and $F - \sum_{i=0}^{i_1-1}(\mathrm{id}\times\sigma^{i})^*P_1$, preserving condition (8.8). By induction this reduces us to the case that $E=0$. Then (8.8) implies that $\deg(E') = \deg(E) = 0$ and hence $E'=0$, too. Now $(\mathrm{id}\times\sigma)^*F = F$, and we are done. **q.e.d.**

Lemma 8.9 *Any effective divisor D on C°_K satisfying $D = (\mathrm{id}\times\sigma)^*D$ is the pullback of an effective divisor on C°.*

Proof. Let $I \subset A\otimes K$ denote the ideal of D and set $I_0 := \{a \in A \mid a\otimes 1 \in I\}$. We must show that $I = I_0 \otimes K$. For this let J denote the image of I in the factor ring $(A/I_0)\otimes K$. If J is non-zero, among all non-zero elements $u = \sum_{i=1}^r b_i \otimes x_i \in J$, choose one for which r is minimal. Then $r>1$ and the b_i, respectively the x_i, are linearly independent over $\mathbb{F}_q$. The assumption implies that

$$\sum_{i=2}^{r} b_i \otimes (x_i^q - x_i x_1^{q-1}) \;=\; (\mathrm{id}\otimes\sigma)(u) - u\cdot x_1^{q-1} \;\in\; J;$$

hence by minimality this element must be zero. As the b_i are linearly independent, we deduce that $x_i^q - x_i x_1^{q-1} = 0$. Since x_1 and x_i are linearly independent over $\mathbb{F}_q$, this yields a contradiction. This proves that $J=0$, and so $I = I_0\otimes K$, as desired. **q.e.d.**

As before we let η_C denote the generic point of C. We abbreviate $\eta_{C,K} := \eta_C \times \operatorname{Spec} K$, which consists of all points of C°_K that lie over η_C instead of a closed point of C°. If K is algebraic over $\mathbb{F}_q$, these are only the generic points of C°_K. Otherwise it also contains infinitely many closed points of C°_K.

Proposition 8.10 *Any inclusion of κ-sheaves of equal rank and characteristic θ on C°_K is an equality over $\eta_{C,K}$.*

Proof. If both $\mathcal{G} \hookrightarrow \mathcal{G}'$ in Lemma 8.6 have characteristic θ, we must have $P_\nu = \theta$ and $\mathrm{Char}(\mathcal{G}) = \mathrm{Char}(\mathcal{G}') = n\theta$, and so $\mathrm{Div}(\mathcal{G}'/\mathcal{G})$ is $\mathrm{id}\times\sigma$-invariant by (8.7). By Lemma 8.9 $\mathrm{Div}(\mathcal{G}'/\mathcal{G})$ is therefore the pullback of a divisor on C; hence it has empty intersection with $\eta_{C,K}$, as desired. **q.e.d.**

Proposition 8.11 *For any κ-sheaf $\mathcal{G}$ on C_K°, there exists up to isomorphism at most one inclusion $\mathcal{G} \hookrightarrow \mathcal{G}'$ of κ-sheaves of equal rank which is an equality outside $\eta_{C,K}$, such that $\mathcal{G}'$ is of characteristic θ.*

Proof. For any two such inclusions $\mathcal{G}_1$, $\mathcal{G}_2$ we can form their sum within $\mathcal{G} \otimes \mathrm{Quot}(\mathcal{O}_{C_K^\circ})$. Then $\mathcal{G} \hookrightarrow \mathcal{G}_1 + \mathcal{G}_2$ is another inclusion of κ-sheaves of equal rank which is an equality outside $\eta_{C,K}$. Moreover, since κ is surjective outside θ for both $\mathcal{G}_1$ and $\mathcal{G}_2$, the same holds for $\mathcal{G}_1 + \mathcal{G}_2$; hence this sum is again of characteristic θ. Now $\mathcal{G}_1 \hookrightarrow \mathcal{G}_1 + \mathcal{G}_2$ is an inclusion of κ-sheaves of equal rank and characteristic θ; hence by Proposition 8.10 it is an equality over $\eta_{C,K}$. Since it is also an equality outside $\eta_{C,K}$, it is an equality everywhere. By symmetry we deduce that $\mathcal{G}_1 = \mathcal{G}_1 + \mathcal{G}_2 = \mathcal{G}_2$, as desired. **q.e.d.**

Definition 8.12 *A κ-sheaf $\mathcal{G}$ on C_K° is called* generically minimal *if every κ-subsheaf of equal rank coincides with $\mathcal{G}$ along $\eta_{C,K}$.*

If K is algebraic over $\mathbb{F}_q$, then $\eta_{C,K}$ contains only the generic points of C_K, and in this case every κ-sheaf is generically minimal. So the following is relevant only if K has transcendence degree $\geqslant 1$. Also, we have:

Proposition 8.13 *If θ lies over a closed point of C, then every κ-sheaf of characteristic θ on C_K° is generically minimal.*

Proof. Let $\mathcal{G} \subset \mathcal{G}'$ be an inclusion of κ-sheaves of equal rank where $\mathcal{G}'$ has characteristic θ. Then by assumption $\mathrm{Char}(\mathcal{G}')$ has empty intersection with $\eta_{C,K}$; hence by Lemma 8.6 the same follows for $\mathrm{Char}(\mathcal{G})$, and so $(\mathrm{id}\times\sigma)^* \mathrm{Div}(\mathcal{G}'/\mathcal{G})$ and $\mathrm{Div}(\mathcal{G}'/\mathcal{G})$ coincide over $\eta_{C,K}$. Using Lemma 8.9 we deduce that $\mathrm{Div}(\mathcal{G}'/\mathcal{G})$ has empty intersection with $\eta_{C,K}$. But this means that the inclusion $\mathcal{G} \subset \mathcal{G}'$ is an equality over $\eta_{C,K}$, as desired. **q.e.d.**

For the next assertion note that the morphism $\sigma\times\mathrm{id} : C_K \to C_K$ is bijective on points.

Lemma 8.14 *Assume that K is finitely generated of transcendence degree $\geqslant 1$ over $\mathbb{F}_q$. Then for any closed point $P \in \eta_{C,K}$, the degree over K of the field of definition of $(\sigma^i\times\mathrm{id})^{-1}P$ goes to ∞ for $i \to \infty$.*

Proof. Let K' denote the field of definition of P, and F the function field of C. Then K' is a finite extension of K that contains F. For any integer $i \geqslant 0$ the field of definition of $(\sigma^i\times\mathrm{id})^{-1}P$ can be identified with the subfield $F^{p^{-i}}K'$ of an algebraic closure of K'. As K' is finitely generated over $\mathbb{F}_q$ and F contains a transcendent element, the degree of $F^{p^{-i}}K'$ over K goes to ∞, as desired. **q.e.d.**

Proposition 8.15 *If K is finitely generated over $\mathbb{F}_q$, any κ-sheaf $\mathcal{G}$ on C_K° possesses a unique generically minimal κ-subsheaf $\mathcal{G}_{\mathrm{gmin}}$ that coincides with $\mathcal{G}$ outside $\eta_{C,K}$. If moreover $\mathcal{G}$ has characteristic θ, then it is determined up to unique isomorphism by $\mathcal{G}_{\mathrm{gmin}}$.*

Proof. We first consider an arbitrary κ-subsheaf of equal rank $\mathcal{G}' \subset \mathcal{G}$. Take any point $P' \in \mathrm{Char}(\mathcal{G}')$. Then Lemma 8.6 with the roles of $\mathcal{G}$ and $\mathcal{G}'$ interchanged shows that $P := (\mathrm{id}\times\sigma^i)^{-1}(P') \in \mathrm{Char}(\mathcal{G})$ for some $i \geqslant 0$. We can rewrite this equality in the form $(\sigma^i\times\mathrm{id})^{-1}(P) = (\sigma^i\times\sigma^i)^{-1}(P') = P'$, because the absolute Frobenius $\sigma^i\times\sigma^i$ is the identity on points. Note that there are only finitely many possibilities for the point $P \in \mathrm{Char}(\mathcal{G})$. Note also that the degree over K of the field of definition of P' is $\leqslant \deg_K(\mathrm{Char}(\mathcal{G}'))$, which is equal to $\deg_K(\mathrm{Char}(\mathcal{G}))$ by Lemma 8.6. Thus if P' and hence P lie in $\eta_{C,K}$—which can happen only if K is not algebraic over $\mathbb{F}_q$—Lemma 8.14 implies that $i \leqslant i_0$ for a constant i_0 that depends only on $\mathrm{Char}(\mathcal{G})$. Using Lemma 8.6 again, we deduce that

$$\mathrm{Char}(\mathcal{G}') + \mathrm{Div}(\mathcal{G}/\mathcal{G}') \;\leqslant\; \sum_{i=0}^{i_0} (\mathrm{id}\times\sigma^i)^* \,\mathrm{Char}(\mathcal{G}) + D'$$

for some divisor D' with $D' \cap \eta_{C,K} = 0$. In particular the degree of $\mathrm{Div}(\mathcal{G}/\mathcal{G}') \cap \eta_{C,K}$ is bounded by a constant depending only on $\mathrm{Char}(\mathcal{G})$.

Thus among all κ-subsheaves of equal rank there exists one for which this degree is maximal. Any such κ-subsheaf is generically minimal. After enlarging it again outside $\eta_{C,K}$ where necessary we obtain the desired $\mathcal{G}_{\mathrm{gmin}}$.

For the uniqueness suppose that $\mathcal{G}'$ and $\mathcal{G}''$ are two generically minimal κ-subsheaves of $\mathcal{G}$ that coincide with $\mathcal{G}$ outside $\eta_{C,K}$. Then $\mathcal{G}' \cap \mathcal{G}''$ is another κ-subsheaf with the same properties. By the generic minimality of $\mathcal{G}'$ and $\mathcal{G}''$ the inclusions $\mathcal{G}' \subset \mathcal{G}' \cap \mathcal{G}'' \supset \mathcal{G}''$ are equalities over $\eta_{C,K}$, and by construction they are also equalities outside $\eta_{C,K}$; hence they are equalities everywhere, as desired.

The last statement is a direct consequence of Proposition 8.11. **q.e.d.**

Proposition 8.16 *Assume that K is finitely generated over $\mathbb{F}_q$. Then any inclusion $\mathcal{G} \hookrightarrow \mathcal{G}'$ of κ-sheaves of equal rank on C_K° induces a cartesian diagram*

$$\begin{tikzcd} \mathcal{G} \arrow[r, hook] & \mathcal{G}' \\ \mathcal{G}_{\mathrm{gmin}} \arrow[u, hook] \arrow[r, hook] & \mathcal{G}'_{\mathrm{gmin}}. \arrow[u, hook] \end{tikzcd}$$

If moreover $\mathcal{G}$ and $\mathcal{G}'$ have characteristic θ, the diagram is also cocartesian, and then the induced κ-equivariant homomorphism $\mathcal{G}'_{\mathrm{gmin}}/\mathcal{G}_{\mathrm{gmin}} \to \mathcal{G}'/\mathcal{G}$ is an isomorphism.

Proof. The defining properties of $\mathcal{G}'_{\mathrm{gmin}}$ from Proposition 8.15 imply that the κ-subsheaf $\mathcal{G} \cap \mathcal{G}'_{\mathrm{gmin}}$ is generically minimal and coincides with $\mathcal{G}$ outside $\eta_{C,K}$; hence it is equal to $\mathcal{G}_{\mathrm{gmin}}$, proving the first assertion. Since the vertical inclusions are equalities outside $\eta_{C,K}$, the diagram is automatically cocartesian there. On the other hand the generic minimality of $\mathcal{G}'_{\mathrm{gmin}}$ implies that the inclusion $\mathcal{G}_{\mathrm{gmin}} \hookrightarrow \mathcal{G}'_{\mathrm{gmin}}$ is an equality over $\eta_{C,K}$. If $\mathcal{G}$ and $\mathcal{G}'$ have characteristic θ, the inclusion $\mathcal{G} \hookrightarrow \mathcal{G}'$ is also an equality over $\eta_{C,K}$ by Proposition 8.10. Thus in that case, the diagram is cocartesian over $\eta_{C,K}$ as well, and hence everywhere. The last statement follows from the fact that the diagram is cartesian and cocartesian. **q.e.d.**

Definition 8.17 *A κ-sheaf $\mathcal{G}$ on C_K° is called* simple *if it is non-zero and every non-zero κ-subsheaf has equal rank with $\mathcal{G}$. A κ-sheaf is called* semisimple *if it possesses a κ-subsheaf of equal rank that is a direct sum of simple κ-sheaves.*

One easily shows that both properties are invariant under inclusions of κ-sheaves of equal rank.

Proposition 8.18 *An A-motive is simple, resp. semisimple, if and only if its associated κ-sheaf is simple, resp. semisimple.*

Proof. Let $\mathcal{G}$ be the κ-sheaf associated to an A-motive M. Assume first that $\mathcal{G}$ is simple. By the anti-equivalence from Proposition 8.2, any non-zero injective homomorphism of A-motives $M' \hookrightarrow M$ corresponds to a non-zero homomorphism of κ-sheaves $\mathcal{G} \to \mathcal{G}'$ which is generically surjective. The kernel of the latter is a κ-subsheaf of $\mathcal{G}$ which is not generically equal to $\mathcal{G}$. By assumption it is therefore zero; hence $\mathcal{G} \to \mathcal{G}'$ is injective and thus generically an isomorphism. It follows that the inclusion $M' \hookrightarrow M$ has torsion cokernel and is therefore an isogeny. Thus M is simple, as desired.

Conversely assume that M is simple and consider a non-zero κ-subsheaf $\mathcal{G}' \subset \mathcal{G}$. Let M be of characteristic γ, so that $\mathcal{G}$ is of characteristic θ. As no such assumption is given for $\mathcal{G}'$, we consider the largest coherent subsheaf $\mathcal{G}^+ \subset \mathcal{G}$ containing $\mathcal{G}'$ whose quotient by $\mathcal{G}'$ is torsion. By construction it is generically equal to $\mathcal{G}'$, and the quotient $\mathcal{G}'' := \mathcal{G}/\mathcal{G}^+$ is a torsion free coherent, hence locally free, sheaf on C_K° with an induced homomorphism $\kappa : \mathcal{G}'' \to (\mathrm{id}\times\sigma)^*\mathcal{G}''$. The cokernel $\mathrm{coker}(\kappa|\mathcal{G}'')$ is a quotient of $\mathrm{coker}(\kappa|\mathcal{G})$ and therefore supported at θ; hence $\mathcal{G}''$ is a κ-sheaf of characteristic θ. Let M'' be the corresponding A-motive of characteristic γ over K. Then the surjection $\mathcal{G} \twoheadrightarrow \mathcal{G}''$ corresponds to an injective homomorphism of A-motives $M'' \hookrightarrow M$. But since $\mathcal{G}'$ and hence $\mathcal{G}^+$ is non-zero, the rank of $\mathcal{G}''$ and M'' is strictly smaller than that of $\mathcal{G}$ and M, so that $M'' \hookrightarrow M$ is not an isogeny. By assumption we therefore have $M'' = 0$ and hence $\mathcal{G}^+ = \mathcal{G}$. Thus $\mathcal{G}'$ is generically equal to $\mathcal{G}$, proving that $\mathcal{G}$ is simple.

This proves the equivalence for the property 'simple'. The equivalence for the property 'semisimple' follows in the same fashion. **q.e.d.**

In the remainder of this section we discuss how to extend a κ-sheaf on C_K° to $C_K := C\times \mathrm{Spec}\, K$ and how to modify such an extension. To construct an extension at all we must allow poles at $\infty_K := \infty \times \mathrm{Spec}\, K$, i.e., use κ-sheaves of some pole order with $\mathcal{L}_C = \mathcal{O}_C(\infty)$. By Definition 6.1 a *$\kappa$-sheaf of pole order $\leqslant d$ on C_K* is a locally free coherent sheaf $\overline{\mathcal{G}}$ on C_K together with an injective homomorphism $\kappa : \overline{\mathcal{G}} \hookrightarrow (\mathrm{id}\times\sigma)^*\overline{\mathcal{G}}(d\infty_K)$. Clearly any such $\overline{\mathcal{G}}$ restricts to a κ-sheaf on C_K°.

Conversely, for any κ-sheaf $\mathcal{G}$ on C_K° one can choose any locally free coherent sheaf $\overline{\mathcal{G}}$ on C_K extending $\mathcal{G}$. Then for every sufficiently large integer d the homomorphism κ on $\mathcal{G}$ extends to a homomorphism $\kappa : \overline{\mathcal{G}} \hookrightarrow (\mathrm{id}\times\sigma)^*\overline{\mathcal{G}}(d\infty_K)$, turning $\overline{\mathcal{G}}$ into a κ-sheaf of pole order $\leqslant d$ on $C\times\eta_X$. The data in such an extension plays a role similar to that of a polarization of an abelian variety.

We call $\overline{\mathcal{G}}$ simple, resp. semisimple, if and only if its restriction to C_K° has that property. Recall that $\mu^{\min}(\overline{\mathcal{G}})$ and $\mu^{\max}(\overline{\mathcal{G}})$ denote the smallest resp. largest slopes in the Harder-Narasimhan filtration of $\overline{\mathcal{G}}$. Let ℓ_C denote the degree over $\mathbb{F}_q$ of the closed point $\infty \in C$, so that $\mathcal{O}(d\infty_K)$ has slope $d\ell_C$.

Proposition 8.19 *For any simple κ-sheaf $\overline{\mathcal{G}}$ of rank r and of pole order $\leqslant d$ on C_K, the difference of any two successive slopes in the Harder-Narasimhan filtration of $\overline{\mathcal{G}}$ is $\leqslant d\ell_C$. Consequently*

$$\mu^{\min}(\overline{\mathcal{G}}) \ \geqslant \ \mu^{\max}(\overline{\mathcal{G}}) - (r-1)d\ell_C.$$

Proof. For any rational number μ let $\overline{\mathcal{G}}^\mu$ denote the subsheaf of slopes $\geqslant \mu$ in the Harder-Narasimhan filtration of $\overline{\mathcal{G}}$. Then $(\mathrm{id}\times\sigma)^*\overline{\mathcal{G}}^{\mu-d\ell_C}(d\infty_K)$ is the subsheaf of slopes $\geqslant \mu$ in the Harder-Narasimhan filtration of $(\mathrm{id}\times\sigma)^*\overline{\mathcal{G}}(d\infty_K)$ by (3.20). Thus the functoriality (3.13) of the Harder-Narasimhan filtration implies that κ induces a homomorphism

$$\overline{\mathcal{G}}^\mu \hookrightarrow (\mathrm{id}\times\sigma)^*\overline{\mathcal{G}}^{\mu-d\ell_C}(d\infty_K).$$

Suppose now that μ is a slope of $\overline{\mathcal{G}}$ such that $\overline{\mathcal{G}}$ has no slopes in the interval $[\mu - d\ell_C, \mu)$. Then $\overline{\mathcal{G}}^\mu$ is equal to $\overline{\mathcal{G}}^{\mu-d\ell_C}$ and hence a non-zero κ-subsheaf of $\overline{\mathcal{G}}$. As $\overline{\mathcal{G}}$ is simple, it is therefore generically equal to $\overline{\mathcal{G}}$. As a step in the Harder-Narasimhan filtration it is also saturated; hence it is equal to $\overline{\mathcal{G}}$; and so μ is the smallest slope of $\overline{\mathcal{G}}$. This proves the first assertion. The second assertion follows directly from the first and the fact that the number of distinct slopes is $\leqslant r$. **q.e.d.**

Construction 8.20 For $1 \leqslant i \leqslant s$ let $\overline{\mathcal{G}}_i$ be a simple κ-sheaf of pole order $\leqslant d$ on C_K. Let $\mathcal{G}_i$ denote its restriction to C_K° and let $\bigoplus_{i=1}^s \mathcal{G}_i \hookrightarrow \mathcal{G}$ be an inclusion of κ-sheaves of equal rank on C_K°. Thus $\mathcal{G}$ is semisimple. For any tuple of integers $\underline{n} = (n_1, \ldots, n_s)$ we let $\overline{\mathcal{G}}(\underline{n})$ denote the locally free coherent sheaf on C_K which coincides with $\mathcal{G}$ over C_K° and with $\bigoplus_{i=1}^s \overline{\mathcal{G}}_i(n_i\infty_K)$ along ∞_K. Since twisting by $(n_i\infty_K)$ and pullback by $\mathrm{id}\times\sigma$ commute, each $\overline{\mathcal{G}}_i(n_i\infty_K)$ and hence $\overline{\mathcal{G}}(\underline{n})$ is again a κ-sheaf of pole order $\leqslant d$ on C_K.

Proposition 8.21 *In Construction 8.20 one can choose the tuple $\underline{n}$ such that*

$$\mu^{\max}\big(\overline{\mathcal{G}}(\underline{n})\big) \leqslant 0 \qquad \textit{and} \qquad \mu^{\min}\big(\overline{\mathcal{G}}(\underline{n})\big) \geqslant -rd\ell_C,$$

where r denotes the rank of $\overline{\mathcal{G}}$.

Proof. We use an auxiliary filtration. For every $0 \leqslant j \leqslant s$ define $\overline{\mathcal{G}}(\underline{n})_{\leqslant j}$ as the largest coherent subsheaf of $\overline{\mathcal{G}}(\underline{n})$ containing $\bigoplus_{i=1}^j \overline{\mathcal{G}}_i(n_i\infty_K)$ whose quotient by $\bigoplus_{i=1}^j \overline{\mathcal{G}}_i(n_i\infty_K)$ is torsion. This defines an increasing filtration by κ-subsheaves of pole order $\leqslant d$ satisfying $\overline{\mathcal{G}}(\underline{n})_{\leqslant 0} = 0$ and $\overline{\mathcal{G}}(\underline{n})_{\leqslant s} = \overline{\mathcal{G}}(\underline{n})$. For every $1 \leqslant j \leqslant s$ the subquotient

$$\overline{\mathcal{G}}(\underline{n})_{[j]} \ := \ \overline{\mathcal{G}}(\underline{n})_{\leqslant j}/\overline{\mathcal{G}}(\underline{n})_{\leqslant j-1}$$

is torsion free and hence again a κ-sheaf of pole order $\leqslant d$ on C_K. Moreover, the construction yields a natural inclusion $\overline{\mathcal{G}}_j(n_j\infty_K) \hookrightarrow \overline{\mathcal{G}}(\underline{n})_{[j]}$, which is an equality along ∞_K. In particular it is generically an equality; hence $\overline{\mathcal{G}}(\underline{n})_{[j]}$ is again simple.

Abbreviate $\overline{\mathcal{G}}_{[j]} := \overline{\mathcal{G}}((0,\ldots,0))_{[j]}$. By what we have just seen we have a natural inclusion $\overline{\mathcal{G}}_j \hookrightarrow \overline{\mathcal{G}}_{[j]}$ which is an equality along ∞_K. Thus for arbitrary $\underline{n}$ we have inclusions

$$\overline{\mathcal{G}}(\underline{n})_{[j]} \hookleftarrow \overline{\mathcal{G}}_j(n_j\infty_K) \hookrightarrow \overline{\mathcal{G}}_{[j]}(n_j\infty_K)$$

that are equalities along ∞_K. But the sheaves $\overline{\mathcal{G}}(\underline{n})_{[j]}$ and $\overline{\mathcal{G}}_{[j]}(n_j\infty_K)$ coincide already over C_K°, because the twist is irrelevant there. Together we obtain a natural isomorphism

$$\overline{\mathcal{G}}(\underline{n})_{[j]} \;\cong\; \overline{\mathcal{G}}_{[j]}(n_j\infty_K).$$

This isomorphism together with (3.20) implies that

$$\mu^{\max}\big(\overline{\mathcal{G}}(\underline{n})_{[j]}\big) = \mu^{\max}\big(\overline{\mathcal{G}}_{[j]}(n_j\infty_K)\big) = \mu^{\max}\big(\overline{\mathcal{G}}_{[j]}\big) + n_j\ell_C.$$

Thus we can choose $\underline{n}$ such that for every j we have

$$-\ell_C \leqslant \mu^{\max}\big(\overline{\mathcal{G}}(\underline{n})_{[j]}\big) \leqslant 0.$$

As $\overline{\mathcal{G}}(\underline{n})_{[j]}$ is a simple κ-sheaf of pole order $\leqslant d$ and of rank $\leqslant r$, Proposition 8.19 implies that

$$\mu^{\min}\big(\overline{\mathcal{G}}(\underline{n})_{[j]}\big) \;\geqslant\; -\ell_C - (r-1)d\ell_C \;\geqslant\; -rd\ell_C.$$

Finally, since $\overline{\mathcal{G}}(\underline{n})$ is a successive extension of all $\overline{\mathcal{G}}(\underline{n})_{[j]}$, the same inequalities follow for the slopes of $\overline{\mathcal{G}}(\underline{n})$ using induction and the formulas (3.18) and (3.19). **q.e.d.**

Remark 8.22 From the point of view that an extension of a κ-sheaf from C_K° to C_K constitutes an analogue of a polarization of an abelian variety, the above facts can be interpreted as follows. First, every κ-sheaf on C_K° possesses a 'polarization' of pole order $\leqslant d$ for some integer $d > 0$. Second, Construction 8.20 is based on the fact that the property of having a 'polarization' of pole order $\leqslant d$ is invariant under isogenies. Next, a second invariant of a 'polarization' besides the pole order is given by the slopes in the Harder-Narasimhan filtration of $\overline{\mathcal{G}}$. Proposition 8.19 states that these lie in an interval of bounded length if $\mathcal{G}$ is simple. Based on this, Proposition 8.21 shows that for semisimple κ-sheaves, the property of possessing a 'polarization' of pole order $\leqslant d$ and slopes in a certain bounded range is also invariant under isogenies. The appearance of a semisimplicity assumption is not so strange, considering that abelian varieties are semisimple, but semiabelian varieties, which do not possess a polarization in the same sense as abelian varieties do, are in general not semisimple.

We now use some of the above facts to prove the isogeny conjecture over finite fields. Note that this result concerns all isogenies, not only separable ones.

Theorem 8.23 *Let M be a semisimple A-motive over a finite field K. Then there exist only finitely many isomorphism classes of A-motives M' over K which are isogenous to M. In particular Theorem 1.1 is true when K is finite.*

Proof. Let $\mathcal{G}$ be the κ-sheaf on C_K° associated to M. By Proposition 8.3 it suffices to show that there are only finitely many isomorphism classes of κ-sheaves $\mathcal{G}'$ on C_K° possessing an inclusion of equal rank $\mathcal{G} \hookrightarrow \mathcal{G}'$.

Proposition 8.18 asserts that $\mathcal{G}$ is again semisimple. Thus we can choose finitely many simple κ-sheaves $\mathcal{G}_i$ on C_K° and an inclusion of equal rank $\bigoplus_{i=1}^s \mathcal{G}_i \hookrightarrow \mathcal{G}$. We can also choose extensions $\overline{\mathcal{G}}_i$ of $\mathcal{G}_i$ to κ-sheaves of some pole order $\leqslant d$ on C_K. Here d depends on the $\mathcal{G}_i$ but will remain fixed. Let r denote the rank of $\mathcal{G}$.

For any inclusion of equal rank $\mathcal{G} \hookrightarrow \mathcal{G}'$, we apply Construction 8.20 to the composite inclusion $\bigoplus_{i=1}^s \mathcal{G}_i \hookrightarrow \mathcal{G}'$, yielding extensions $\overline{\mathcal{G}}'(\underline{n})$ of $\mathcal{G}'$ to κ-sheaves of pole order $\leqslant d$ on C_K. Proposition 8.21 shows that $\underline{n}$ can be chosen such that

$$\mu^{\max}\big(\overline{\mathcal{G}}'(\underline{n})\big) \leqslant 0 \qquad \text{and} \qquad \mu^{\min}\big(\overline{\mathcal{G}}'(\underline{n})\big) \geqslant -rd\ell_C.$$

The irreducible components of C_K are irreducible smooth projective curves over K. Since $\overline{\mathcal{G}}'(\underline{n})$ has all slopes in a bounded range, the same holds for its degree on every irreducible component of C_K; hence there are only finitely many possibilities for this degree. As K is finite, applying Theorem 5.1 over every irreducible component shows that there are only finitely many possibilities for the isomorphism class of the coherent sheaf $\overline{\mathcal{G}}'(\underline{n})$. Moreover the associated κ lies in the group $\mathrm{Hom}\big(\overline{\mathcal{G}}'(\underline{n}), (\mathrm{id}\times\sigma)^*\overline{\mathcal{G}}'(\underline{n})(d\infty_K)\big)$, which is a finite dimensional vector space over K. As K is finite, there are at most finitely many possibilities for it. Forgetting the extension to C_K it follows that there are only finitely many possibilities for the isomorphism class of the κ-sheaf $\mathcal{G}'$, as desired. **q.e.d.**

9 Finiteness for A-motives

In this section we prove Theorem 1.1 in the case that K has transcendence degree 1 over $\mathbb{F}_q$. We keep the notations of the preceding sections. In particular, we let X be the irreducible smooth projective curve over $\mathbb{F}_q$ with function field K and generic point $\eta_X = \operatorname{Spec} K$, and let η_C denote the generic point of C. We are interested in the relations between κ-sheaves on each of the base schemes in the diagram (6.3). In Section 8 we have dealt with the problem of extending κ-sheaves from $C^\circ\times\eta_X$ to $C\times\eta_X$. Here we study extensions in the direction of X.

Proposition 9.1 *(a) Any κ-sheaf $\mathcal{G}$ on $\eta_C\times\eta_X$ possesses a unique extension to $\eta_C\times X$ that is contained in every other extension, called the* minimal extension $\mathcal{G}_{\min}$ *of $\mathcal{G}$.*

(b) The minimal extension is functorial in $\mathcal{G}$. In particular, for any finite collection of κ-sheaves $\mathcal{G}_i$ on $\eta_C\times\eta_X$ we have $(\bigoplus_i \mathcal{G}_i)_{\min} = \bigoplus_i \mathcal{G}_{i,\min}$.

Proof. (Compare the maximal extension in Gardeyn [4, Prop. 2.13]) Let j denote the embedding $\eta_C\times\eta_X \hookrightarrow \eta_C\times X$. Since any torsion free coherent sheaf on $\eta_C\times X$ is locally free, the extensions of $\mathcal{G}$ to $\eta_C\times X$ can be identified with the κ-invariant coherent subsheaves of $j_*\mathcal{G}$.

For (a) we first prove that some extension exists. For this choose any coherent subsheaf $\mathcal{F} \subset j_*\mathcal{G}$ with $j^*\mathcal{F} = \mathcal{G}$. Then the homomorphism κ on $\mathcal{G}$ induces a homomorphism $\mathcal{F} \hookrightarrow ((\mathrm{id}\times\sigma)^*\mathcal{F})(D)$ for some effective divisor $D \subset \eta_C\times X$ that is disjoint from $\eta_C\times\eta_X$. The last property means that $D \subset \eta_C\times E$ for some effective divisor $E \subset X$. Thus after enlarging D we may assume that $D = \eta_C\times E$. Then $(\mathrm{id}\times\sigma)^*D = qD$, and since $q \geqslant 2$, we deduce that κ induces a homomorphism

$$\mathcal{F}(D) \;\hookrightarrow\; ((\mathrm{id}\times\sigma)^*\mathcal{F})(2D) \;\subset\; ((\mathrm{id}\times\sigma)^*\mathcal{F})(qD) \;=\; (\mathrm{id}\times\sigma)^*(\mathcal{F}(D)).$$

Therefore $\mathcal{F}(D)$ is a κ-sheaf on $\eta_C\times X$ extending $\mathcal{G}$.

Next, Lemma 7.3 (c) asserts that $\deg(\mathcal{F}) \geqslant 0$ for every κ-sheaf $\mathcal{F}$ on $\eta_C\times X$. As the degree is always an integer, it follows that among all extensions of $\mathcal{G}$ to $\eta_C\times X$, there exists an extension $\mathcal{F}_0$ for which $\deg(\mathcal{F}_0)$ is minimal. Then for every extension $\mathcal{F}$, the intersection $\mathcal{F}\cap\mathcal{F}_0$ is an extension that satisfies $\deg(\mathcal{F}\cap\mathcal{F}_0) \leqslant \deg(\mathcal{F}_0)$. The minimality then implies that $\deg(\mathcal{F}\cap\mathcal{F}_0) = \deg(\mathcal{F}_0)$. Being an inclusion of locally free sheaves of equal rank and degree on a projective curve, the inclusion $\mathcal{F}\cap\mathcal{F}_0 \subset \mathcal{F}_0$ is therefore an equality, and hence $\mathcal{F}_0 \subset \mathcal{F}$. Evidently, an extension contained in every other extension is unique, proving (a).

For (b) let $\varphi : \mathcal{G} \to \mathcal{G}'$ be any homomorphism of κ-sheaves on $\eta_C\times\eta_X$. Then the pullback of $\mathcal{G}'_{\min} \subset j_*\mathcal{G}'$ under $j_*\varphi : j_*\mathcal{G} \to j_*\mathcal{G}'$ is another extension of $\mathcal{G}$. By the minimality of $\mathcal{G}_{\min}$ that extension contains $\mathcal{G}_{\min}$; hence $\varphi(\mathcal{G}_{\min}) \subset \mathcal{G}'_{\min}$, proving the desired functoriality. The functoriality in turn implies the compatibility with direct sums. **q.e.d.**

Proposition 9.2 *(a) Any κ-sheaf $\mathcal{G}$ on $C^\circ\times\eta_X$ possesses a unique extension to a κ-sheaf on $C^\circ\times X$ that is contained in every other extension, called the* minimal extension $\mathcal{G}_{\min}$ *of $\mathcal{G}$. Its restriction to $\eta_C\times X$ is the minimal extension of the restriction $\mathcal{G}|\,\eta_C\times\eta_X$ from Proposition 9.1.*

(b) The minimal extension is functorial in $\mathcal{G}$. In particular, for any finite collection of κ-sheaves $\mathcal{G}_i$ on $C^\circ\times\eta_X$ we have $(\bigoplus_i \mathcal{G}_i)_{\min} = \bigoplus_i \mathcal{G}_{i,\min}$.

The analogous assertions hold for κ-sheaves of pole order $\leqslant d$ on $C\times\eta_X$ and $C\times X$.

Proof. The argument is the same in both cases. For ease of notation we consider the case of $C^\circ\times\eta_X$. Let $\mathcal{G}$ be a κ-sheaf on $C^\circ\times\eta_X$. Let $\mathcal{H}$ denote the minimal extension of $\mathcal{G}|\eta_C\times\eta_X$ from Proposition 9.1. Then by Proposition 2.4 (a) there exists a unique locally free coherent sheaf $\mathcal{F}$ on $C^\circ\times X$ that extends both $\mathcal{G}$ and $\mathcal{H}$. Since the homomorphisms κ for $\mathcal{G}$ and $\mathcal{H}$ coincide over $\eta_C\times\eta_X$, by Proposition 2.4 (b) they extend to a unique homomorphism $\mathcal{F} \to (\mathrm{id}\times\sigma)^*\mathcal{F}$, turning $\mathcal{F}$ into a κ-sheaf on $C^\circ\times X$. For any other κ-sheaf $\mathcal{F}'$ on $C^\circ\times X$ that extends $\mathcal{G}$, the minimality of $\mathcal{H}$ implies that $\mathcal{H} \subset \mathcal{F}'|\eta_C\times X$. Thus the functoriality in Proposition 2.4 (b) implies that $\mathcal{F} \subset \mathcal{F}'$. This shows that $\mathcal{F}$ possesses the minimality property in (a), and with this property it is evidently unique. The last assertion in (a) follows from the construction. The functoriality of the minimal extension in (a) follows in the same way from that in Propositions 9.1 and Proposition 2.4 (b) **q.e.d.**

Lemma 9.3 *For any inclusion $\mathcal{G} \hookrightarrow \mathcal{G}'$ of κ-sheaves of equal rank on $C^\circ \times \eta_X$, where $\mathcal{G}'$ is generically minimal, the induced inclusion $\mathcal{G}_{\min} \hookrightarrow \mathcal{G}'_{\min}$ is an equality outside $D \times X$ for some divisor $D \subset C^\circ$.*

Proof. Since $\mathcal{G}'$ is generically minimal, the inclusion $\mathcal{G} \hookrightarrow \mathcal{G}'$ is an equality over $\eta_C \times \eta_X$. By constructibility the induced inclusion $\mathcal{G}_{\min} \hookrightarrow \mathcal{G}'_{\min}$ is then an equality outside $(D \times X) \cup (C^\circ \times E)$ for some divisors $D \subset C^\circ$ and $E \subset X$. Let $\mathcal{F}''$ denote the kernel of the homomorphism $\mathcal{G}'_{\min} \twoheadrightarrow (\mathcal{G}'_{\min}/\mathcal{G}_{\min})|(C^\circ \times E)$. By construction its restriction to $C^\circ \times \eta_X$ is $\mathcal{G}'$; hence it underlies another κ-sheaf on $C^\circ \times X$ extending $\mathcal{G}'$. By the minimality of $\mathcal{G}'_{\min}$ it must therefore coincide with $\mathcal{G}'_{\min}$. Thus the inclusion $\mathcal{G}_{\min} \hookrightarrow \mathcal{G}'_{\min}$ is in fact an equality outside $D \times X$, as desired. **q.e.d.**

Now we prepare the setup for the proof of Theorem 1.1. Fix an A-motive M of characteristic γ over K and let $\mathcal{G}$ be the associated κ-sheaf of characteristic θ on $C^\circ \times \eta_X$. Let $\mathcal{G}_{\mathrm{gmin}} \subset \mathcal{G}$ be the generically minimal κ-subsheaf defined by Proposition 8.15, and let $\mathcal{F} := (\mathcal{G}_{\mathrm{gmin}})_{\min}$ be its minimal extension to $C^\circ \times X$ defined by Proposition 9.2. By combining earlier results we obtain:

Proposition 9.4 *In the above situation, any separable isogeny $M' \hookrightarrow M$ of A-motives of characteristic γ over K induces an inclusion of κ-sheaves $\mathcal{F} \hookrightarrow \mathcal{F}'$ on $C^\circ \times X$ which*

(a) is an equality outside $D \times X$ for some divisor $D \subset C^\circ$, such that

(b) the homomorphism $\mathcal{F}'/\mathcal{F} \to (\mathrm{id} \times \sigma)^*(\mathcal{F}'/\mathcal{F})$ induced by κ is an isomorphism over $C^\circ \times \eta_X$.

Moreover, the isomorphism class of M' (without the isogeny) is determined uniquely by the isomorphism class of $\mathcal{F}'$.

Proof. Combining Propositions 8.3 and 8.16, every separable isogeny $M' \hookrightarrow M$ of A-motives of characteristic γ over K gives rise to an inclusion $\mathcal{G}_{\mathrm{gmin}} \hookrightarrow \mathcal{G}'_{\mathrm{gmin}}$ of κ-sheaves on $C^\circ \times \eta_X$ that is an equality over $\eta_C \times \eta_X$, such that the homomorphism $\mathcal{G}'_{\mathrm{gmin}}/\mathcal{G}_{\mathrm{gmin}} \to (\mathrm{id} \times \sigma)^*(\mathcal{G}'_{\mathrm{gmin}}/\mathcal{G}_{\mathrm{gmin}})$ induced by κ is an isomorphism. Moreover, the isomorphism class of M' (without the isogeny) is determined uniquely by the isomorphism class of $\mathcal{G}'_{\mathrm{gmin}}$ by Proposition 8.15. By Proposition 9.2 the inclusion $\mathcal{G}_{\mathrm{gmin}} \hookrightarrow \mathcal{G}'_{\mathrm{gmin}}$ extends to an inclusion of minimal extensions $\mathcal{F} \hookrightarrow \mathcal{F}'$ on $C^\circ \times X$. Then (a) follows from Lemma 9.3. Finally, the property of $\mathcal{G}'_{\mathrm{gmin}}/\mathcal{G}_{\mathrm{gmin}}$ stated above is equivalent to (b). **q.e.d.**

Lemma 9.5 *For any inclusion $\mathcal{F} \hookrightarrow \mathcal{F}'$ as in Proposition 9.4 and any point $c \in C^\circ$ we have*

$$\mu^{\min}(\mathcal{F}'_c) \;\geqslant\; \min\{\mu^{\min}(\mathcal{F}_c), 0\}.$$

Proof. Recall that $(\ )_c$ denotes the pullback of a coherent sheaf to the fiber $c{\times}X$. As this defines a right exact functor, we have an exact sequence $\mathcal{F}_c \to \mathcal{F}'_c \to (\mathcal{F}'/\mathcal{F})_c \to 0$. Let $\mathcal{H}''$ denote the quotient of $(\mathcal{F}'/\mathcal{F})_c$ by its torsion subsheaf, and define $\mathcal{H}'$ by the short exact sequence $0 \to \mathcal{H}' \to \mathcal{F}'_c \to \mathcal{H}'' \to 0$. Then the homomorphism $\mathcal{F}_c \to \mathcal{F}'_c$ induces a homomorphism $\mathcal{F}_c \to \mathcal{H}'$ with torsion cokernel.

The condition 9.4 (b) implies that the homomorphism $(\mathcal{F}'/\mathcal{F})_c \to (\mathrm{id}{\times}\sigma)^*(\mathcal{F}'/\mathcal{F})_c$ induced by κ is an isomorphism at $c{\times}\eta_X$. By the definition of $\mathcal{H}''$, the same follows for the induced homomorphism $\mathcal{H}'' \to (\mathrm{id}{\times}\sigma)^*\mathcal{H}''$. Since $\mathcal{H}''$ is locally free, this last homomorphism is therefore injective, and so Lemma 7.3 (b) implies that $\mu^{\min}(\mathcal{H}'') \geqslant 0$. Therefore

$$\mu^{\min}(\mathcal{F}'_c) \;\overset{(3.18)}{\geqslant}\; \min\bigl\{\mu^{\min}(\mathcal{H}'), \mu^{\min}(\mathcal{H}'')\bigr\} \;\overset{(3.16)}{\geqslant}\; \min\bigl\{\mu^{\min}(\mathcal{F}_c), 0\bigr\},$$

as desired. **q.e.d.**

Now we assume that M is semisimple. Then $\mathcal{G}$ and hence $\mathcal{G}_{\mathrm{gmin}}$ is semisimple by Proposition 8.18. We choose simple κ-sheaves $\mathcal{G}_i$ on $C^\circ{\times}\eta_X$ and an inclusion of equal rank

$$\textstyle\bigoplus_{i=1}^s \mathcal{G}_i \hookrightarrow \mathcal{G}_{\mathrm{gmin}}. \tag{9.6}$$

Let $\mathcal{F}_i := \mathcal{G}_{i,\min}$ denote the minimal extension of $\mathcal{G}_i$ to $C^\circ{\times}X$ defined by Proposition 9.2 (a). Then Proposition 9.2 (b) yields an inclusion of equal rank

$$\textstyle\bigoplus_{i=1}^s \mathcal{F}_i \hookrightarrow \mathcal{F}. \tag{9.7}$$

By the generic minimality of $\mathcal{G}_{\mathrm{gmin}}$, Lemma 9.3 implies that (9.7) is an equality outside $E{\times}X$ for some divisor $E \subset C^\circ$. We also choose extensions $\overline{\mathcal{G}}_i$ of $\mathcal{G}_i$ to κ-sheaves of some pole order $\leqslant d$ on $C{\times}\eta_X$. Here d depends on the $\mathcal{G}_i$ but will remain fixed. We let $\overline{\mathcal{F}}_i$ denote their minimal extensions to $C{\times}X$ from Proposition 9.2, which also extend the $\mathcal{F}_i$. We can then repeat Construction 8.20 over $C{\times}X$. Recall that $(n,0)$ denotes the twist by $\mathrm{pr}_1^*\mathcal{L}_C^{\otimes n} = \mathrm{pr}_1^*\mathcal{O}_C(n\infty)$.

Construction 9.8 Consider any inclusion of κ-sheaves $\mathcal{F} \hookrightarrow \mathcal{F}'$ as in Proposition 9.4. Then the composite inclusion $\bigoplus_{i=1}^s \mathcal{F}_i \hookrightarrow \mathcal{F}'$ is an equality outside $(D \cup E){\times}X$. Thus for any tuple of integers $\underline{n} = (n_1, \ldots, n_s)$ we can define a locally free coherent sheaf $\overline{\mathcal{F}}'(\underline{n})$ on $C{\times}X$ which coincides with $\mathcal{F}'$ over $C^\circ{\times}X$ and with $\bigoplus_{i=1}^s \overline{\mathcal{F}}_i(n_i,0)$ along $\infty{\times}X$. Since twisting by $(n_i,0)$ and pullback by $\mathrm{id}{\times}\sigma$ commute, each $\overline{\mathcal{F}}_i(n_i,0)$ and hence $\overline{\mathcal{F}}'(\underline{n})$ is again a κ-sheaf of pole order $\leqslant d$ on $C{\times}X$.

Lemma 9.9 *There exist constants r, d_X, μ_X, μ_C and a finite set D_C such that for any inclusion of κ-sheaves $\mathcal{F} \hookrightarrow \mathcal{F}'$ as in Proposition 9.4, there exists a tuple $\underline{n}$ such that* *(a)* $\overline{\mathcal{F}}'(\underline{n})$ *has constant rank* r*.*

(b) $\deg\big(\overline{\mathcal{F}}'(\underline{n})_c\big) = d_X$ *for all* $c \in C$*.*

(c) $\mu^{\min}\big(\overline{\mathcal{F}}'(\underline{n})_c\big) \geqslant \mu_X$ *for all* $c \in C$*.*

(d) $\deg\big(\overline{\mathcal{F}}'(\underline{n})_{\eta_X}\big) \in D_C$*.*

(e) $\mu^{\max}\big(\overline{\mathcal{F}}'(\underline{n})_{\eta_X}\big) \leqslant \mu_C$*.*

(f) Every κ*-invariant coherent subsheaf of* $\overline{\mathcal{F}}'(\underline{n})$ *of rank* r *coincides with* $\overline{\mathcal{F}}'(\underline{n})$ *along* $\eta_C{\times}X$*.*

Proof. Condition (a) holds trivially with $r := \mathrm{rank}(\mathcal{F})$. Condition (b) holds for $c = \eta_C$ with $d_X := \deg(\mathcal{F}_{\eta_C})$, because $\overline{\mathcal{F}}'(\underline{n})$ coincides with $\mathcal{F}$ over $\eta_C{\times}X$. By flatness (b) then follows for all $c \in C$.

For (c) note that for almost all points $c \in C^\circ$ the homomorphism $\mathcal{F}_c \to (\mathrm{id}{\times}\sigma)^*\mathcal{F}_c$ induced by κ is an isomorphism at $c{\times}\eta_X$. It is then injective, and so Lemma 7.3 (b) implies that $\mu^{\min}(\mathcal{F}_c) \geqslant 0$. The minimum of 0 and the finitely many remaining values yields a constant $\mu_X \leqslant 0$ such that $\mu^{\min}(\mathcal{F}_c) \geqslant \mu_X$ for all $c \in C^\circ$. By Lemma 9.5 the same inequality then follows for $\mu^{\min}\big(\overline{\mathcal{F}}'(\underline{n})_c\big)$. On the other hand, Construction 9.8 implies that

$$\overline{\mathcal{F}}'(\underline{n})_\infty \;=\; \bigoplus_{i=1}^s \overline{\mathcal{F}}_i(n_i,0)_\infty \;=\; \bigoplus_{i=1}^s \overline{\mathcal{F}}_{i,\infty} \otimes \mathrm{pr}_1^*\,\mathcal{O}_C(n_i\infty)_\infty \;\cong\; \bigoplus_{i=1}^s \overline{\mathcal{F}}_{i,\infty},$$

where the last isomorphism is induced by any local generator of $\mathcal{O}_C(n_i\infty)$ at ∞. Therefore $\mu^{\min}\big(\overline{\mathcal{F}}'(\underline{n})_\infty\big)$ is independent of $\mathcal{F}'$. Thus after decreasing μ_X, if necessary, condition (c) holds for all $c \in C$.

The next two conditions (d) and (e) concern the restriction of $\overline{\mathcal{F}}'(\underline{n})$ to the generic fiber $C{\times}\eta_X$. This restriction is precisely the extension $\overline{\mathcal{G}}(\underline{n})$ defined in Construction 8.20. Thus by Proposition 8.21 one can choose the tuple $\underline{n}$ such that

$$\mu^{\max}\big(\overline{\mathcal{F}}'(\underline{n})_{\eta_X}\big) \leqslant 0 \qquad \text{and} \qquad \mu^{\min}\big(\overline{\mathcal{F}}'(\underline{n})_{\eta_X}\big) \geqslant -rd\ell_C.$$

Then (3.4) and (a) imply that

$$-e_C r^2 d\ell_C \;\leqslant\; \deg\big(\overline{\mathcal{F}}'(\underline{n})_{\eta_X}\big) \;\leqslant\; 0.$$

Thus condition (d) holds with the finite set $D_C := \mathbb{Z} \cap [-e_C r^2 d\ell_C, 0]$, and condition (e) holds with $\mu_C := 0$.

For the last condition (f) consider any κ-invariant coherent subsheaf $\overline{\mathcal{F}}'' \subset \overline{\mathcal{F}}'(\underline{n})$ of rank r. Taking its restriction to $C^\circ{\times}X$ and then the pullback under the inclusion $\mathcal{F} \hookrightarrow \mathcal{F}'$ yields a κ-invariant subsheaf of equal rank $\mathcal{F}'''$ of $\mathcal{F}$. Since $\mathcal{F}|\,C^\circ{\times}\eta_X = \mathcal{G}_{\mathrm{gmin}}$ is generically minimal, we find that $\mathcal{F}'''|\,\eta_C{\times}\eta_X = \mathcal{F}|\,\eta_C{\times}\eta_X$. But the construction of $\mathcal{F}$ and the last sentence in Proposition 9.2 (a) show that $\mathcal{F}|\,\eta_C{\times}X$ is the minimal extension of $\mathcal{F}|\,\eta_C{\times}\eta_X$. It follows that $\mathcal{F}'''|\,\eta_C{\times}X = \mathcal{F}|\,\eta_C{\times}X$. Condition 9.4 (a) implies that the latter is equal to $\overline{\mathcal{F}}'(\underline{n})|\,\eta_C{\times}X$. Thus $\mathcal{F}'''$ and hence $\overline{\mathcal{F}}''$ coincides with $\overline{\mathcal{F}}'(\underline{n})$ along $\eta_C{\times}X$, proving (f).

q.e.d.

Proposition 9.10 *Theorem 1.1 is true when K has transcendence degree 1 over $\mathbb{F}_q$.*

Proof. For any semisimple A-motive M over K, the above constructions associate to any separable isogeny $M' \hookrightarrow M$ a κ-sheaf $\overline{\mathcal{F}}'(\underline{n})$ of pole order $\leqslant d$ on $C{\times}X$, which determines the isomorphism class of M' and satisfies the conditions in Lemma 9.9. Here d and the constants and the finite set D_C in Lemma 9.9 are independent of the isogeny. For each value $d_C \in D_C$, Theorem 7.1 asserts that there are only finitely many possibilities for the isomorphism class of $\overline{\mathcal{F}}'(\underline{n})$. Thus there are only finitely many possibilities for the isomorphism class of M', as desired. **q.e.d.**

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