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Vector bundles with a Frobenius structure on the punctured unit disc

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Abstract

Let $\mathbb{C}$ be a complete non-archimedean-valued algebraically closed field of characteristic $p > 0$ and consider the punctured unit disc $\dot{D} \subset \mathbb{C}$. Let q be a power of p and consider the arithmetic Frobenius automorphism $\sigma_{\dot{D}} : x \mapsto x^{q^{-1}}$. A *σ-bundle* is a vector bundle $\mathcal{F}$ on $\dot{D}$ together with an isomorphism $\tau_{\mathcal{F}} : \sigma_{\dot{D}}^{*}\mathcal{F} \xrightarrow{\sim} \mathcal{F}$. The aim of this article is to develop the basic theory of these objects and to classify them. It is shown that every σ-bundle is isomorphic to a direct sum of indecomposable σ-bundles $\mathcal{F}_{d,r}$ which depend only on rational numbers d/r. This result has close analogies with the classification of rational Dieudonné modules and of vector bundles on the projective line or on an elliptic curve. It has interesting consequences concerning the uniformizability of Anderson's t-motives that will be treated in a future paper.

Introduction

Let $\mathbb{C}$ be an algebraically closed field of characteristic $p > 0$ which is complete with respect to a non-archimedean absolute value $|\cdot|$ and consider the punctured unit disc

$$\dot{D} := \{x \in \mathbb{C} : 0 < |x| < 1\}.$$

Let q be a power of p and consider the map

$$\sigma_{\dot{D}} : \dot{D} \xrightarrow{\sim} \dot{D}, \quad \sigma_{\dot{D}}(x) := x^{q^{-1}}.$$

The pull-back of a holomorphic function $f(z) = \sum_i a_i z^i$ on $\dot{D}$ is defined as

$$\sigma_{\dot{D}}^{*} f(z) := \sum_i a_i^q z^i,$$

which makes $\sigma_{\dot{D}}$ an automorphism of rigid-analytic spaces relative to the arithmetic Frobenius of $\mathbb{C}$. By definition a *σ-bundle* (*on* $\dot{D}$) consists of a vector bundle $\mathcal{F}$ on $\dot{D}$ together with an isomorphism $\tau_{\mathcal{F}} : \sigma_{\dot{D}}^{*}\mathcal{F} \xrightarrow{\sim} \mathcal{F}$. The σ-bundles are the 'vector bundles with a Frobenius structure' from the title. The main aim of this article is to classify all σ-bundles up to isomorphism.

The building blocks for this classification are constructed as follows. For every integer n the σ-bundle $\mathcal{O}(n)$ is the structure sheaf $\mathcal{O}_{\dot{D}}$, where $\tau_{\mathcal{O}(n)}$ is the above isomorphism $\sigma_{\dot{D}}^{*}\mathcal{O}_{\dot{D}} \xrightarrow{\sim} \mathcal{O}_{\dot{D}}$ followed by multiplication by z^{-n}. For more examples take a positive integer r and consider the morphism of rigid-analytic spaces over $\mathbb{C}$

$$[r] : \dot{D} \to \dot{D}, \quad x \mapsto x^r.$$

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For every integer d that is relatively prime to r we set $\mathcal{F}_{d,r} := [r]_*\mathcal{O}(d)$ together with the induced isomorphism $\tau_{\mathcal{F}_{d,r}} := [r]_*\tau_{\mathcal{O}(d)}$. This defines a σ-bundle of rank r, having $\mathcal{F}_{n,1} = \mathcal{O}(n)$ as a special case.

The Main Theorem 11.1 states that every σ-bundle is isomorphic to one of the form $\bigoplus_{i=1}^{k} \mathcal{F}_{d_i,r_i}$, where the pairs (d_i, r_i) are uniquely determined up to a permutation by Corollary 11.8. In particular, every σ-bundle of rank one is isomorphic to $\mathcal{O}(n)$ for a unique integer n, called its degree; see Theorem 5.4. Moreover, the $\mathcal{F}_{d,r}$ are precisely the indecomposable σ-bundles up to isomorphism, and with a natural definition of stability they are also precisely the stable ones; see Corollary 11.6.

These results are reminiscent of two other well-known classifications. On the one hand, they resemble the facts about rational Dieudonné modules; see Dieudonné [Die57] or Manin [Man63, Theorem 2.1]. This has to do with the presence of a Frobenius map as a common feature in both situations. In other aspects the results remind one of Grothendieck's classification [Gro57] of vector bundles on the projective line. Indeed, the σ-bundle $\mathcal{O}(1)$ enjoys many of the properties of ample twisting sheaves from algebraic and analytic geometry: see §§ 3 and 4.

Furthermore, there are parallels to recent work of Kedlaya [Ked01], who proves the analogous classification theorem for vector bundles with a Frobenius structure in mixed characteristic [Ked01, Theorem 4.16]. An intermediate result [Ked01, Proposition 4.8] corresponds to our Theorem 4.1 and provided the inspiration for its proof. Although the rest of our work was done independently, another intermediate result [Ked01, Proposition 4.15] is a close analogue of our Proposition 9.1. It is interesting to note that the main technical complications of both articles arise in similar places.

The relation with geometry is explained further by the following interpretation. The group $\sigma_{\dot{D}}^{\mathbb{Z}}$ acts properly discontinuously on $\dot{D}$ and we can consider the quotient $\dot{D}/\sigma_{\dot{D}}^{\mathbb{Z}}$. Since $\sigma_{\dot{D}}$ acts non-trivially on the field of coefficients $\mathbb{C}$, this quotient does not carry a natural structure of rigid-analytic space over $\mathbb{C}$. Nevertheless, most likely it can be endowed with a suitable Grothendieck topology so that giving a σ-bundle is equivalent to giving a vector bundle on $\dot{D}/\sigma_{\dot{D}}^{\mathbb{Z}}$. Our results can thus be viewed as the classification of vector bundles on a certain 'twisted rigid-analytic space'. Note that the situation resembles the non-archimedean uniformization of an elliptic curve with non-integral j-invariant.

The notion of σ-bundle was introduced by the second author to investigate the nature of uniformizability of Anderson's t-motives [And86]. In brief, to any t-motive M of rank r over $\mathbb{C}$ one can associate a natural σ-bundle $\mathcal{Q}_M$ of rank r, such that M is uniformizable if and only if $\mathcal{Q}_M \cong \mathcal{O}(0)^{\oplus r}$. The Main Theorem 11.1 of this article thus tells us precisely what happens instead, when M is not uniformizable. Its use lies in the fact that a non-existence statement is transformed into another existence statement. Conversely, the concept of σ-bundles allows one to construct new uniformizable t-motives out of local data, much like abelian varieties are constructed from their Hodge structure. This may play an important role in the study of moduli spaces of t-motives. The respective details will be explained in a future paper. For related results see also Gardeyn [Gar01, ch. 5].

1. The punctured unit disc

Throughout this article we fix a complete non-archimedean valued algebraically closed field of characteristic $p > 0$. By analogy with the field of complex numbers we denote it by $\mathbb{C}$. The main example we have in mind is the completion of the algebraic closure of the field $\mathbb{F}_p((\xi))$ of Laurent series in one variable over the finite field of p elements $\mathbb{F}_p$. The absolute value on $\mathbb{C}$ is denoted by $|\cdot|$. Inside $\mathbb{C}$ we consider the punctured unit disc

$$\dot{D} := \{x \in \mathbb{C} : 0 < |x| < 1\}.$$

We view it as a rigid-analytic space over $\mathbb{C}$ in the usual way. (We do not require the full theory of rigid-analytic geometry here. For an overview of what we need see Lazard [Laz62] or Fresnel and van der Put [FP81]. A general introduction would be that of Bosch *et al.* [BGR84].) The ring R of *holomorphic functions on* $\dot{D}$ consists of all Laurent series $\sum_i a_i z^i$ with coefficients $a_i \in \mathbb{C}$, possibly infinite in both directions, that converge on $\dot{D}$. The following proposition is straightforward to prove and therefore left as an exercise.

Proposition 1.1. *A Laurent series $\sum_i a_i z^i$ with $a_i \in \mathbb{C}$ lies in R if and only if*

$$\limsup_{i\to\infty} \frac{\log|a_i|}{i} \leqslant 0 \quad \textit{and} \quad \limsup_{i\to\infty} \frac{\log|a_{-i}|}{i} = -\infty.$$

We are interested in *locally free coherent sheaves on* $\dot{D}$. By a common abuse of terminology we call them *vector bundles* for short. It is known (see Gruson [Gru68, ch. V, Theorem 1]) that taking global sections defines an equivalence between the category of vector bundles on $\dot{D}$ and the category of finitely generated projective R-modules. If $\mathbb{C}$ is maximally complete, then every vector bundle on $\dot{D}$ is free (see Lazard [Laz62, § 7, Theorem 2]), but otherwise there is no guarantee for that. Nevertheless, we note the following useful fact (see Bartenwerfer [Bar81]).

Theorem 1.2. *A vector bundle on $\dot{D}$ is free if and only if its highest exterior power is free.*

2. σ-Bundles

Once and for all we fix a power q of p and consider the field automorphism

$$\sigma : \mathbb{C} \to \mathbb{C}, \quad a \mapsto \sigma(a) := a^q.$$

The elements of $\mathbb{C}$ that are fixed by σ form the unique subfield $\mathbb{F}_q$ of q elements. Next we let σ act on the coefficients of a Laurent series, obtaining a map

$$R \to R, \quad f(z) = \sum_i a_i z^i \mapsto \sigma(f) := \sum_i a_i^q z^i,$$

denoted again by σ. By Proposition 1.1 this clearly defines an automorphism of R. The corresponding automorphism of $\dot{D}$ is

$$\sigma_{\dot{D}} : \dot{D} \to \dot{D}, \quad x \mapsto \sigma_{\dot{D}}(x) := x^{q^{-1}}.$$

The reader should not confuse the automorphisms σ of $\mathbb{C}$ and R with the automorphism $\sigma_{\dot{D}}$ of $\dot{D}$. Actually they are related by the equation $\sigma(f)(x) = f(\sigma_{\dot{D}}(x))^q$ for all $f \in R$ and $x \in \dot{D}$. In this sense $\sigma_{\dot{D}}$ defines what must be called the *arithmetic Frobenius of* $\dot{D}$.

For any vector bundle $\mathcal{F}$ on $\dot{D}$ with space of global sections M, the pull-back $\sigma_{\dot{D}}^*\mathcal{F}$ is the vector bundle with space of global sections $R \otimes_{\sigma,R} M$.

Definition 2.1. A vector bundle $\mathcal{F}$ on $\dot{D}$ together with an isomorphism $\tau_{\mathcal{F}} : \sigma_{\dot{D}}^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$ is called a *σ-bundle* (on $\dot{D}$).

Giving a σ-bundle $\mathcal{F}$ is equivalent to giving its space of global sections over $\dot{D}$ together with the automorphism induced by $\tau_{\mathcal{F}}$. By the preceding section this data amounts to a finitely generated projective R-module M together with a σ-linear automorphism $\tau_M : M \xrightarrow{\sim} M$, that is, an additive automorphism satisfying $\tau_M(fm) = \sigma(f) \cdot \tau_M(m)$ for all $f \in R$ and all $m \in M$. To be precise τ_M is obtained as follows. By adjunction between $\sigma_{\dot{D}}^*$ and $(\sigma_{\dot{D}})_*$ we obtain from $\tau_{\mathcal{F}}$ the morphism $(\sigma_{\dot{D}})_*\tau_{\mathcal{F}} : \mathcal{F} \xrightarrow{\sim} (\sigma_{\dot{D}})_*\mathcal{F}$, and the corresponding isomorphism of global sections is

$$\tau_M = \Gamma(\dot{D}, (\sigma_{\dot{D}})_*\tau_{\mathcal{F}}) : M = \Gamma(\dot{D}, \mathcal{F}) \xrightarrow{\sim} \Gamma(\dot{D}, (\sigma_{\dot{D}})_*\mathcal{F}) = M.$$

Remark 2.2. For better geometric intuition, note that the group $\sigma_{\dot{D}}^{\mathbb{Z}}$ acts properly discontinuously on $\dot{D}$, because any annulus

$$\{x \in \mathbb{C} : \rho_1 \leqslant |x| \leqslant \rho_2\}$$

with $0 < \rho_2^q < \rho_1 \leqslant \rho_2 < 1$ is disjoint from all its translates. However, $\sigma_{\dot{D}}$ is not an automorphism of $\dot{D}$ as an analytic space *over* $\mathbb{C}$, because it acts non-trivially on the field of coefficients $\mathbb{C}$. Nevertheless, we can imagine the quotient $\dot{D}/\sigma_{\dot{D}}^{\mathbb{Z}}$ as being obtained from an annulus

$$\{x \in \mathbb{C} : \rho^q \leqslant |x| \leqslant \rho\}$$

for $\rho \in |\mathbb{C}|$ with $0 < \rho < 1$ by gluing its two 'edges' via

$$\sigma_{\dot{D}} : \{x \in \mathbb{C} : |x| = \rho^q\} \xrightarrow{\sim} \{x \in \mathbb{C} : |x| = \rho\}.$$

Heuristically speaking, giving a σ-bundle is then equivalent to giving a vector bundle on the quotient $\dot{D}/\sigma_{\dot{D}}^{\mathbb{Z}}$.

The *tensor product* $\mathcal{F} \otimes \mathcal{G}$ of two σ-bundles is defined in the obvious way as the tensor product of the underlying vector bundles together with the isomorphism $\tau_{\mathcal{F}\otimes\mathcal{G}} := \tau_{\mathcal{F}} \otimes \tau_{\mathcal{G}}$. The σ-bundle $\mathcal{O}$ together with $\tau_{\mathcal{O}} := \sigma$ is a unit object for the tensor product. Symmetric and alternating powers of σ-bundles are defined in the obvious way.

Similarly, the *inner hom* $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ of two σ-bundles is defined as the inner hom of the underlying vector bundles together with its own natural τ deduced from $\tau_{\mathcal{F}}$ and $\tau_{\mathcal{G}}$. In particular, the *dual* of a σ-bundle is defined as $\mathcal{F}^\vee := \mathcal{H}om(\mathcal{F}, \mathcal{O})$. Clearly we have $\mathcal{H}om(\mathcal{F}, \mathcal{G}) \cong \mathcal{F}^\vee \otimes \mathcal{G}$ and other compatibilities.

Next, a *global section of* $\mathcal{F}$ is a global section of the underlying vector bundle that is invariant under $\tau_{\mathcal{F}}$. The set of all global sections of $\mathcal{F}$ is denoted as $H^0(\mathcal{F})$. It is a module over the ring $H^0(\mathcal{O}) = \{f \in R : \sigma(f) = f\}$, which we denote by F.

Proposition 2.3. $F = \mathbb{F}_q((z))$.

Proof. By definition $H^0(\mathcal{O})$ consists of all Laurent series $f(z) = \sum_i a_i z^i \in R$ with $a_i^q = a_i$, that is, with $a_i \in \mathbb{F}_q$. Note that this implies that $|a_i| = 1$ whenever $a_i \neq 0$. Thus, by Proposition 1.1 the series converges on $\dot{D}$ if and only if its principal part is finite, that is, if $f(z) \in \mathbb{F}_q((z))$. ☐

A *homomorphism* $\varphi : \mathcal{F} \to \mathcal{G}$ of σ-bundles is a homomorphism of the underlying vector bundles which satisfies $\tau_{\mathcal{G}} \circ \sigma_{\dot{D}}^* \varphi = \varphi \circ \tau_{\mathcal{F}}$. The set of all homomorphisms $\mathcal{F} \to \mathcal{G}$ is denoted $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$, and with these we obtain an F-linear category of σ-bundles. If we included arbitrary coherent sheaves instead of just locally free ones, the category would be abelian. Note that we have a natural isomorphism $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \cong H^0(\mathcal{H}om(\mathcal{F}, \mathcal{G}))$.

Next observe that $H^0(\mathcal{F})$ is the kernel of the F-linear map $\mathrm{id} - \tau_{\mathcal{F}}$ on the space of global sections of $\mathcal{F}$ over $\dot{D}$. We define the *first cohomology group* $H^1(\mathcal{F})$ to be the cokernel of this map. The higher cohomology groups $H^i(\mathcal{F})$ for $i \geqslant 2$ are set to zero. In other words, if M is the R-module associated to $\mathcal{F}$, then the different $H^i(\mathcal{F})$ are the homology groups of the complex

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{\mathrm{id} - \tau_M} M \longrightarrow 0 \longrightarrow \cdots.$$

By the snake lemma every short exact sequence of σ-bundles yields an obvious long exact cohomology sequence. Finally, we set $\mathrm{Ext}(\mathcal{F}, \mathcal{G}) := H^1(\mathcal{H}om(\mathcal{F}, \mathcal{G}))$, and the higher Ext groups are set to zero.

Proposition 2.4. *The group* $\mathrm{Ext}(\mathcal{F}, \mathcal{G})$ *classifies classes of extensions of* σ*-bundles*

$$0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0$$

up to isomorphisms of short exact sequences that are the identity on $\mathcal{G}$ *and* $\mathcal{F}$.

Proof. This follows by the usual arguments in homological algebra; cf. MacLane [Mac75, ch. III]. We want to make this explicit. Let $\mathcal{F}$ and $\mathcal{G}$ correspond to the R-modules M and N with their respective τ_M and τ_N. Then $\mathcal{H}om(\mathcal{F},\mathcal{G})$ corresponds to the R-module $H := \mathrm{Hom}_R(M,N)$ with $\tau_H(h) := \tau_N \circ h \circ \tau_M^{-1}$. For any element $h \in H$ we set $E_h := N \oplus M$ with the σ-linear automorphism

$$\tau_{E_h} := \begin{pmatrix} \tau_N & h \circ \tau_M \\ 0 & \tau_M \end{pmatrix}.$$

The obvious inclusion and projection maps yield a short exact sequence $0 \to N \to E_h \to M \to 0$ and therefore an extension of σ-bundles

$$0 \to \mathcal{G} \to \mathcal{E}_h \to \mathcal{F} \to 0.$$

Now, since any short exact sequence of projective R-modules splits, every extension of $\mathcal{F}$ by $\mathcal{G}$ is isomorphic to one of this form. On the other hand, the extensions associated to h, $h' \in H$ are isomorphic if and only if

$$\begin{pmatrix} \mathrm{id} & k \\ 0 & \mathrm{id} \end{pmatrix} \cdot \begin{pmatrix} \tau_N & h \circ \tau_M \\ 0 & \tau_M \end{pmatrix} \cdot \begin{pmatrix} \mathrm{id} & k \\ 0 & \mathrm{id} \end{pmatrix}^{-1} = \begin{pmatrix} \tau_N & h' \circ \tau_M \\ 0 & \tau_M \end{pmatrix}$$

for some $k \in H$. This equation amounts to

$$\begin{aligned} & h \circ \tau_M + k \circ \tau_M - \tau_N \circ k = h' \circ \tau_M \\ \Longleftrightarrow\ & h' - h = k - \tau_N \circ k \circ \tau_M^{-1} = (\mathrm{id} - \tau_H)(k). \end{aligned}$$

Thus the extensions of $\mathcal{F}$ by $\mathcal{G}$ are classified by the cokernel of the homomorphism $\mathrm{id} - \tau_H : H \to H$, as desired. □

3. Twisting sheaves

For every integer n we let $\mathcal{O}(n)$ denote the following σ-bundle of rank one: the underlying coherent sheaf is simply the structure sheaf $\mathcal{O}_{\dot{D}}$ of $\dot{D}$, and $\tau_{\mathcal{O}(n)}$ is the isomorphism $\sigma_{\dot{D}}^*\mathcal{O}_{\dot{D}} \xrightarrow{\sim} \mathcal{O}_{\dot{D}}$ furnished by σ followed by multiplication by z^{-n}. The corresponding R-module is simply R itself together with the σ-linear automorphism $f(z) \mapsto z^{-n} \cdot \sigma(f)(z)$. We will see that $\mathcal{O}(1)$ enjoys many of the properties of ample twisting sheaves from algebraic and analytic geometry.

The tensor product of a σ-bundle with $\mathcal{O}(n)$ is abbreviated by $\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}(n)$ and called a *twist of* $\mathcal{F}$. Clearly we have $\mathcal{F}(n)(m) \cong \mathcal{F}(n+m)$ and $\mathcal{F}(n)^\vee \cong \mathcal{F}^\vee(-n)$ and various other compatibilities.

Proposition 3.1. *$H^0(\mathcal{O}(n))$ is an F-vector space of dimension*

$$\begin{cases} 0 & \text{if } n < 0, \\ 1 & \text{if } n = 0, \\ \infty & \text{if } n > 0. \end{cases}$$

Proof. By definition $H^0(\mathcal{O}(n))$ consists of all Laurent series $f(z) = \sum_i a_i z^i \in R$ with

$$\sum a_i^q z^{i-n} = z^{-n} \cdot \sum a_i^q z^i = z^{-n} \cdot \sigma(f)(z) = f(z) = \sum a_i z^i.$$

This equation amounts to $a_{i+nj} = a_i^{q^{-j}}$ for all i and j. Suppose first that $n < 0$. Then by Proposition 1.1 we need for any i that

$$\frac{\log |a_{i+nj}|}{|i+nj|} = \frac{q^{-j}}{|i+nj|} \cdot \log |a_i|$$

tends to $-\infty$ as $j \to \infty$. Since the first factor tends to zero, this can be only if $a_i = 0$ for all i, that is, if $f(z)$ vanishes identically. This finishes the case $n < 0$. The case $n = 0$ is contained in Proposition 2.3.

Suppose now that $n > 0$. Then there is no convergence problem for $j \to \infty$, because by Proposition 1.1 it suffices that the lim sup is less than or equal to zero. For $j \to -\infty$ the factor $q^{-j}/|i+nj|$ tends to infinity, so by Proposition 1.1 we have convergence if and only if $\log|a_i| < 0$. All in all we find that the functions in $H^0(\mathcal{O}(n))$ correspond to the tuples $(a_1, \ldots, a_n)$ in $\mathbb{C}$ satisfying $|a_i| < 1$ for all i. It remains to show that the dimension of this space over $F = \mathbb{F}_q((z))$ is infinite. Since F is a finite extension of $\mathbb{F}_q((z^n))$, it suffices to prove the same over this subfield. Now

$$\Big(\sum_j b_j z^{nj}\Big) \cdot \Big(\sum_i a_i z^i\Big) = \sum_k \Big(\sum_j b_j a_{k-nj}\Big) z^k = \sum_k \Big(\sum_j b_j a_k^{q^j}\Big) z^k;$$

hence $g(z) = \sum b_j z^{nj} \in \mathbb{F}_q((z^n))$ maps each coefficient a_i to $\sum_j b_j a_i^{q^j}$. Thus we must prove that $\mathfrak{m}_{\mathbb{C}} := \{a \in \mathbb{C} : |a| < 1\}$ has infinite dimension as vector space over $\mathbb{F}_q((z^n))$ via the action $\big(\sum b_j z^{nj}\big)a := \sum_j b_j a^{q^j}$. For this, note that

$$\begin{aligned}
\log\Big|\sum b_j a^{q^j}\Big| &= \sup\{\log|b_j| + q^j \log|a| : j \in \mathbb{Z}\} \\
&= \sup\{q^j \log|a| : j \in \mathbb{Z} \text{ with } b_j \neq 0\} \\
&= \inf\{q^j : j \in \mathbb{Z} \text{ with } b_j \neq 0\} \cdot \log|a| \\
&\in q^{\mathbb{Z}} \cdot \log|a|.
\end{aligned}$$

Thus in any non-trivial finite linear combination of elements $a_\nu \in \mathfrak{m}_{\mathbb{C}}$, whose $\log|a_\nu|$ are pairwise inequivalent multiplicatively modulo $q^{\mathbb{Z}}$, no two non-zero summands have the same norm, and so the total sum is non-zero. Since $\mathbb{C}$ is algebraically closed, its value group is $\mathbb{Q}$-divisible. We can therefore find infinitely many elements in $\mathfrak{m}_{\mathbb{C}}$ whose logarithmic norms are pairwise inequivalent modulo $q^{\mathbb{Z}}$. Thus the dimension in question is infinite, as desired. □

Combining the isomorphism $\mathrm{Hom}(\mathcal{O}(n), \mathcal{O}(n')) \cong H^0(\mathcal{O}(n'-n))$ with Proposition 3.1 we obtain the following.

Proposition 3.2. *$\mathrm{Hom}(\mathcal{O}(n), \mathcal{O}(n'))$ is an F-vector space of dimension*

$$\begin{cases} 0 & \text{if } n > n', \\ 1 & \text{if } n = n', \\ \infty & \text{if } n < n'. \end{cases}$$

In particular, $\mathcal{O}(n)$ and $\mathcal{O}(n')$ are isomorphic if and only if $n = n'$.

Next we determine the size of H^1.

Proposition 3.3. *$H^1(\mathcal{O}(n))$ is an F-vector space of dimension*

$$\begin{cases} \infty & \text{if } n < 0, \\ 0 & \text{if } n \geqslant 0. \end{cases}$$

Proof. By definition $H^1(\mathcal{O}(n))$ is the cokernel of the homomorphism

$$R \to R, \quad \sum_i a_i z^i \mapsto \sum_i a_i z^i - \sum_i a_i^q z^{i-n} = \sum_i (a_i - a_{i+n}^q) z^i.$$

So for $n \geqslant 0$ we must show that this homomorphism is surjective. Consider a Laurent series $\sum b_i z^i \in R$ and the resulting equations $a_i - a_{i+n}^q = b_i$. Assume first that $n = 0$; then these are independent Artin–Schreier equations. Moreover, any solution $a_i \in \mathbb{C}$ satisfies $|a_i| = |b_i|^{1/q}$ if $|b_i| \geqslant 1$, and for $|b_i| < 1$ there exists a solution satisfying $|a_i| = |b_i|$, namely $a_i = \sum_{j\geqslant 0} b_i^{q^j}$. In both cases we have $|a_i| \leqslant |b_i|$, so the convergence of $\sum_i a_i z^i$ follows from the convergence of $\sum b_i z^i$; hence the former series lies in R. This proves the surjectivity in the case $n = 0$.

For $n > 0$ the equation $a_i - a_{i+n}^q = b_i$ by induction yields the formulas

$$a_{i+jn} = a_i^{q^{-j}} - b_i^{q^{-j}} - b_{i+n}^{q^{1-j}} - \cdots - b_{i+(j-1)n}^{q^{-1}} \tag{3.4}$$

and

$$a_{i-jn} = b_{i-jn} + b_{i-(j-1)n}^{q} + \cdots + b_{i-n}^{q^{j-1}} + a_i^{q^j} \tag{3.5}$$

for all $j > 0$ and all i. Since $\lim_{i\to-\infty} b_i = 0$ by Proposition 1.1, we may select $i \leqslant -n$ in any residue class modulo n such that $|b_{i'}| < 1$ for all $i' \leqslant i$. We set $a_i := 0$ and define the $a_{i\pm jn}$ according to the above formulas, and we will show that the resulting series $\sum_j a_j z^j$ lies in R. First, formula (3.5) shows that

$$\frac{\log|a_{i-jn}|}{|i-jn|} \leqslant \sup\left\{\frac{q^{j-k}\cdot\log|b_{i-kn}|}{|i-jn|} : 1 \leqslant k \leqslant j\right\}.$$

Fix an $N > 0$. The convergence condition in Proposition 1.1 then guarantees that $\log|b_{i-kn}| \leqslant -N\cdot|i-kn|$ for, say, all $k > k_0$. The terms for $1 \leqslant k \leqslant k_0$ in the above supremum are bounded above by $-\varepsilon\cdot q^j/|i-jn|$ for some fixed $\varepsilon > 0$, and this value tends to $-\infty$ as $j \to \infty$. The terms for $k_0 < k \leqslant j$ are bounded above by

$$-N\cdot\frac{q^j}{|i-jn|}\cdot\frac{|i-kn|}{q^k}.$$

Since $k \mapsto q^k/|i-kn|$ is a monotone increasing function for $k > 0$, this value is bounded above by $-N$. It follows that $\log|a_{i-jn}|/|i-jn| \leqslant -N$ for all $j \gg 0$. As N was arbitrary, this shows that

$$\limsup_{j\to\infty}\frac{\log|a_{i-jn}|}{|i-jn|} = -\infty,$$

proving one half of the conditions in Proposition 1.1. For the other half, formula (3.4) shows that

$$\frac{\log|a_{i+jn}|}{|i+jn|} \leqslant \sup\left\{\frac{q^{k-j}\cdot\log|b_{i+kn}|}{|i+jn|} : 0 \leqslant k < j\right\}.$$

Fix an $\varepsilon > 0$. The convergence condition in Proposition 1.1 then guarantees that $\log|b_{i+kn}| \leqslant \varepsilon\cdot|i+kn|$ for, say, all $k \geqslant k_0$. The terms for $0 \leqslant k < k_0$ in the maximum are bounded above by $C/(q^j|i+jn|)$ for some fixed $C > 0$, and this value tends to zero as $j \to \infty$. The terms for $k_0 \leqslant k < j$ are bounded above by

$$\varepsilon\cdot\frac{q^k|i+kn|}{q^j|i+jn|} \leqslant \varepsilon.$$

It follows that $\log|a_{i+jn}|/|i+jn| \leqslant \varepsilon$ for all $j \gg 0$. As $\varepsilon > 0$ was arbitrary, this shows that

$$\limsup_{j\to\infty}\frac{\log|a_{i+jn}|}{|i+jn|} \leqslant 0,$$

proving the other half of the conditions in Proposition 1.1. Thus $\sum_j a_j z^j$ lies in R, proving the surjectivity in the case $n > 0$.

It remains to show that $\dim_F H^1(\mathcal{O}(-n)) = \infty$ for all $n > 0$. For this we use the following fact.

Lemma 3.6. *For any $n > 0$ there exists a short exact sequence of σ-bundles*

$$0 \longrightarrow \mathcal{O}(-n) \longrightarrow \mathcal{O}(0)^{\oplus 2} \longrightarrow \mathcal{O}(n) \longrightarrow 0.$$

Proof. Fix two points $a, b \in \dot{D}$ that are inequivalent under $\sigma_{\dot{D}}^{\mathbb{Z}}$. By Proposition 5.1 below (whose proof does not depend on Lemma 3.6), there exist non-zero functions f_a, $f_b \in H^0(\mathcal{O}(1))$ which possess a zero of exact order one at a^{q^i}, respectively at b^{q^i}, for all $i \in \mathbb{Z}$ and no other zeroes. Thus the vector bundle underlying $\mathcal{O}(n)$ is generated everywhere by the two global sections f_a^n and $f_b^n \in H^0(\mathcal{O}(n))$. The homomorphism of σ-bundles $(f_a^n, f_b^n) : \mathcal{O}(0)^{\oplus 2} \to \mathcal{O}(n)$ is therefore surjective, and its kernel $\mathcal{F}$ is a σ-bundle of rank one. The formula $\mathcal{O}(0) \cong \bigwedge^2(\mathcal{O}(0)^{\oplus 2}) \cong \mathcal{F} \otimes \mathcal{O}(n)$ now implies that $\mathcal{F} \cong \mathcal{O}(-n)$, as desired. □

To finish the proof of Proposition 3.3 we consider the long exact cohomology sequence associated to the short exact sequence from Lemma 3.6. We obtain an exact sequence

$$F^2 = H^0(\mathcal{O}(0)^{\oplus 2}) \longrightarrow H^0(\mathcal{O}(n)) \longrightarrow H^1(\mathcal{O}(-n)).$$

As the dimension of $H^0(\mathcal{O}(n))$ is infinite by Proposition 3.1, the same follows for $H^1(\mathcal{O}(-n))$, as desired. □

Combining the isomorphism $\operatorname{Ext}(\mathcal{O}(n), \mathcal{O}(n')) \cong H^1(\mathcal{O}(n'-n))$ with Proposition 3.3 yields the following.

Proposition 3.7. $\operatorname{Ext}(\mathcal{O}(n), \mathcal{O}(n'))$ *is an F-vector space of dimension*

$$\begin{cases} \infty & \text{if } n > n', \\ 0 & \text{if } n \leqslant n'. \end{cases}$$

4. Upper and lower bounds

In this section we prove the following results.

Theorem 4.1. *For any σ-bundle $\mathcal{F}$ of rank r there exists an integer n_0 such that $\mathcal{F}$ contains a σ-subbundle isomorphic to $\mathcal{O}(-n)^{\oplus r}$ for every $n \geqslant n_0$.*

Theorem 4.2. *For every σ-bundle $\mathcal{F}$ of rank r there exists an integer n_0 such that $\mathcal{F}$ can be embedded as a σ-subbundle into $\mathcal{O}(n)^{\oplus r}$ for every $n \geqslant n_0$.*

Proof of Theorem 4.2. Theorem 4.2 follows by applying Theorem 4.1 to the dual σ-bundle $\mathcal{F}^\vee$. Indeed, there exists an $n_0 \in \mathbb{Z}$ such that for every $n \geqslant n_0$ there is a σ-subbundle $\mathcal{O}(-n)^{\oplus r} \subset \mathcal{F}^\vee$, and therefore $\mathcal{F} \subset (\mathcal{O}(-n)^{\oplus r})^\vee \cong \mathcal{O}(n)^{\oplus r}$. □

Before proving Theorem 4.1 we note also the following consequence. Its proof is left to the interested reader, because in any case it results from the classification Theorem 11.1 together with Propositions 8.4 and 8.7, in whose proofs it is not used.

Theorem 4.3.

a) *For any σ-bundle $\mathcal{F}$ there exists an integer n_0 such that $\mathcal{F}(n)$ is generated by global sections for every $n \geqslant n_0$.*
b) *For any σ-bundle $\mathcal{F}$ there exists an integer n_0 such that $H^1(\mathcal{F}(n))$ vanishes for every $n \geqslant n_0$.*

Remark. One standard way of proving such a result in algebraic or analytic geometry is to first show that all higher cohomology groups $H^i(\mathcal{F})$ are finitely generated and then to make them vanish by explicit construction after a sufficiently high twist. In our case we cannot follow this path, because $H^1(\mathcal{F})$ may be infinite dimensional over F by Proposition 3.3.

Proof of Theorem 4.1. We fix a radius $\rho \in |\mathbb{C}|$ with $0 < \rho < 1$ and consider the following annuli in $\dot{D}$ and their affinoid $\mathbb{C}$-algebras:

$$\begin{aligned} A &:= \{x \in \mathbb{C} : |x| = \rho\}, & R_\rho &:= \mathcal{O}_{\dot{D}}(A),\\ A_+ &:= \{x \in \mathbb{C} : \rho \leqslant |x| \leqslant \rho^{1/q}\}, & R_{\rho+} &:= \mathcal{O}_{\dot{D}}(A_+),\\ A_- &:= \{x \in \mathbb{C} : \rho^q \leqslant |x| \leqslant \rho\}, & R_{\rho-} &:= \mathcal{O}_{\dot{D}}(A_-),\\ A_\pm &:= \{x \in \mathbb{C} : \rho^q \leqslant |x| \leqslant \rho^{1/q}\}, & R_{\rho\pm} &:= \mathcal{O}_{\dot{D}}(A_\pm). \end{aligned}$$

Any vector bundle on a closed annulus is free. We may therefore choose an isomorphism $\varphi : \mathcal{F}|_{A_\pm} \xrightarrow{\sim} \mathcal{O}_{A_\pm}^{\oplus r}$. Then there is a matrix $T \in \mathrm{GL}_r(R_{\rho-})$ such that $\varphi \circ \tau_{\mathcal{F}} = (T \cdot \sigma) \circ \varphi$ as a map $\mathcal{F}(A_+) \to \mathcal{F}(A_-)$.

We denote by $|\cdot|_\rho$ the supremum norm on R_ρ. For every $(r \times r)$-matrix $W = (w_{\mu\nu}) \in M_r(R_\rho)$ we set $|W|_\rho := \sup\{|w_{\mu\nu}|_\rho : \text{all } \mu, \nu\}$. Now let $C := \sup\{|T|_\rho, |\sigma^{-1}(T^{-1})|_\rho\}$, which is greater than or equal to one. Fix a constant $\varepsilon \in |\mathbb{C}|$ with $0 < \varepsilon < 1$. Since $0 < \rho < 1$, we may fix $n_0 \in \mathbb{N}$ such that $(\varepsilon/C)^{q+1} \geqslant \rho^{(q-1)n_0}$. We claim that Theorem 4.1 holds with this choice of n_0. To show this consider any $n \geqslant n_0$ and choose a constant $d \in \mathbb{C}$ with $|d| = \rho^{-n}\varepsilon/C$. Then the monomial $\lambda := dz^n \in R$ satisfies $|\lambda|_\rho C = |d|\rho^n C = \varepsilon$ and

$$|\sigma(\lambda^{-1})|_\rho C = |d|^{-q}\rho^{-n}C = \frac{C^{q+1}}{\varepsilon^q}\rho^{(q-1)n} \leqslant \varepsilon.$$

We are going to describe an iteration process which produces the desired σ-subbundle. The idea for this is based on a p-adic argument of Kedlaya [Ked01, Prop. 4.8]. For every Laurent series $w = \sum_{i \in \mathbb{Z}} a_i z^i \in R_\rho$ we define

$$g(w) := \sum_{i \text{ with } |a_i| > 1} a_i z^i \in R_{\rho-} \quad \text{and} \quad h(w) := \sum_{i \text{ with } |a_i| \leqslant 1} a_i z^i \in R_{\rho+}.$$

Obviously we have $w = g(w) + h(w)$. The use of this decomposition lies in the fact that σ has better approximation properties on $h(w)$, while its inverse σ^{-1} has better approximation properties on $g(w)$. By applying g and h to the entries of matrices we extend them to maps $g : M_r(R_\rho) \to M_r(R_{\rho-})$ and $h : M_r(R_\rho) \to M_r(R_{\rho+})$. Consider the map

$$f : M_r(R_\rho) \to M_r(R_{\rho+}), \quad f(W) := \lambda^{-1}h(W) - \sigma^{-1}(T^{-1}g(W)).$$

This is well defined, because $\sigma^{-1}(R_{\rho-}) = R_{\rho+}$. Now we define sequences (W_l) in $M_r(R_\rho)$ and (V_l) in $M_r(R_{\rho+})$ by

$$\begin{aligned} V_0 &:= \lambda^{-1}\mathrm{Id}_r + \sigma^{-1}(T^{-1}),\\ W_l &:= T\sigma(V_l) - \lambda V_l,\\ V_{l+1} &:= V_l + f(W_l), \end{aligned}$$

for all $l \geqslant 0$. We claim that $W_l \to 0$ in the supremum norm $|\cdot|_\rho$ and that V_l converges in $M_r(R_\rho)$. First we need some estimates.

LEMMA 4.4. *For every $W \in M_r(R_\rho)$ we have*

a) $|\lambda\sigma^{-1}(T^{-1}g(W))|_\rho \leqslant \varepsilon|g(W)|_\rho$,

b) $|\sigma(\lambda^{-1})T\sigma(h(W))|_\rho \leqslant \varepsilon|h(W)|_\rho$,

c) $|f(W)|_\rho \leqslant |\lambda|_\rho^{-1}|W|_\rho$, *and*

d) $|T\sigma(f(W)) - \lambda f(W) + W|_\rho \leqslant \varepsilon|W|_\rho$.

Proof. For $w = \sum_i a_i z^i \in R_\rho$ we observe that $|w|_\rho = \sup\{|a_i|\rho^i : i \in \mathbb{Z}\}$, and so

$$\begin{aligned} |\sigma^{-1}(g(w))|_\rho &= \sup\{|a_i|^{1/q}\rho^i : i \in \mathbb{Z} \text{ with } |a_i| > 1\} \\ &\leqslant \sup\{|a_i|\rho^i : i \in \mathbb{Z} \text{ with } |a_i| > 1\} \\ &= |g(w)|_\rho \end{aligned}$$

and

$$\begin{aligned} |\sigma(h(w))|_\rho &= \sup\{|a_i|^q\rho^i : i \in \mathbb{Z} \text{ with } |a_i| \leqslant 1\} \\ &\leqslant \sup\{|a_i|\rho^i : i \in \mathbb{Z} \text{ with } |a_i| \leqslant 1\} \\ &= |h(w)|_\rho. \end{aligned}$$

Thus we find

$$|\lambda\sigma^{-1}(T^{-1}g(W))|_\rho \leqslant |\lambda|_\rho C|\sigma^{-1}(g(W))|_\rho \leqslant \varepsilon|g(W)|_\rho$$

and

$$|\sigma(\lambda^{-1})T\sigma(h(W))|_\rho \leqslant |\sigma(\lambda^{-1})|_\rho C|\sigma(h(W))|_\rho \leqslant \varepsilon|h(W)|_\rho,$$

proving items a and b. Furthermore,

$$\begin{aligned} |f(W)|_\rho &= |\lambda^{-1}h(W) - \sigma^{-1}(T^{-1}g(W))|_\rho \\ &\leqslant \sup\{|\lambda^{-1}h(W)|_\rho, |\lambda^{-1}|_\rho|\lambda\sigma^{-1}(T^{-1}g(W))|_\rho\} \\ &\leqslant |\lambda|_\rho^{-1}\sup\{|h(W)|_\rho, \varepsilon|g(W)|_\rho\} \\ &\leqslant |\lambda|_\rho^{-1}|W|_\rho \end{aligned}$$

shows item c and

$$\begin{aligned} &|T\sigma(f(W)) - \lambda f(W) + W|_\rho \\ &\quad= |\sigma(\lambda^{-1})T\sigma(h(W)) - g(W) - h(W) + \lambda\sigma^{-1}(T^{-1}g(W)) + W|_\rho \\ &\quad= |\sigma(\lambda^{-1})T\sigma(h(W)) + \lambda\sigma^{-1}(T^{-1}g(W))|_\rho \\ &\quad\leqslant \sup\{|\sigma(\lambda^{-1})T\sigma(h(W))|_\rho, |\lambda\sigma^{-1}(T^{-1}g(W))|_\rho\} \\ &\quad\leqslant \varepsilon\sup\{|h(W)|_\rho, |g(W)|_\rho\} \\ &\quad= \varepsilon|W|_\rho \end{aligned}$$

shows item d. □

Continuing with the proof of Theorem 4.1, we see that Lemma 4.4 item d implies

$$\begin{aligned} |W_{l+1}|_\rho &= |T\sigma(V_{l+1}) - \lambda V_{l+1}|_\rho \\ &= |T\sigma(V_l) + T\sigma(f(W_l)) - \lambda V_l - \lambda f(W_l)|_\rho \\ &= |T\sigma(f(W_l)) - \lambda f(W_l) + W_l|_\rho \\ &\leqslant \varepsilon|W_l|_\rho. \end{aligned}$$

Therefore, W_l converges to zero in the supremum norm $|\cdot|_\rho$, and so by Lemma 4.4 item c the same holds for $V_{l+1} - V_l = f(W_l)$. Thus the sequence (V_l) converges to a matrix $V \in M_r(R_\rho)$. Using Lemma 4.4 item c again we also deduce that

$$\begin{aligned} |V_{l+1} - V_l|_\rho &= |f(W_l)|_\rho \\ &\leqslant |\lambda|_\rho^{-1}|W_l|_\rho \\ &\leqslant |\lambda|_\rho^{-1}|W_0|_\rho \\ &= |\lambda|_\rho^{-1}|\sigma(\lambda^{-1})T + \mathrm{Id}_r - \mathrm{Id}_r - \lambda\sigma^{-1}(T^{-1})|_\rho \end{aligned}$$

$$\begin{aligned}&\leqslant |\lambda|_\rho^{-1}\sup\{|\sigma(\lambda^{-1})T|_\rho, |\lambda\sigma^{-1}(T^{-1})|_\rho\}\\&\leqslant |\lambda|_\rho^{-1}\sup\{|\sigma(\lambda^{-1})|_\rho C, |\lambda|_\rho C\}\\&= \varepsilon|\lambda|_\rho^{-1}\end{aligned}$$

for all $l \geqslant 0$. Since, on the other hand,

$$|V_0 - \lambda^{-1}\mathrm{Id}_r|_\rho = |\sigma^{-1}(T^{-1})|_\rho \leqslant C = \varepsilon|\lambda|_\rho^{-1},$$

we deduce that

$$|V - \lambda^{-1}\mathrm{Id}_r|_\rho \leqslant \sup\{\varepsilon|\lambda|_\rho^{-1}, |V_0 - \lambda^{-1}\mathrm{Id}_r|_\rho\} = \varepsilon|\lambda|_\rho^{-1} < |\lambda|_\rho^{-1}$$

and therefore $V \in \mathrm{GL}_r(R_\rho)$. Next consider the equation

$$T\sigma V_l = \lambda V + W_l + \lambda(V_l - V)$$

in $M_r(R_\rho)$ for $l \to \infty$. The second and the third terms on the right-hand side converge to zero in the norm $|\cdot|_\rho$; and hence also coefficientwise. The left-hand side lies in $M_r(R_{\rho-})$ and converges to $T\sigma(V)$ in the supremum norm on the annulus $\{x \in \mathbb{C} : |x| = \rho^q\}$, and thus again coefficientwise. Thus in the limit we obtain the Laurent series identity $T\sigma(V) = \lambda V$. This identity implies that $V = \lambda^{-1}T\sigma(V)$ converges on the annulus $\{x \in \mathbb{C} : |x| = \rho^q\}$ as well as on A; and so we see that actually $V \in M_r(R_{\rho-})$.

Finally, choose $e \in \mathbb{C}$ such that $e^{q-1} = d$. Then

$$T\sigma(e^{-1}V) = e^{-q}\lambda V = e^{-q}\, dz^n V = z^n \cdot e^{-1}V.$$

Thus $U_0 := \varphi^{-1}(e^{-1}V)$ is an r-tuple of sections in $\mathcal{F}(A_-)$ which over A generates $\mathcal{F}$ and satisfies $\tau_{\mathcal{F}}U_0 = z^nU_0$. If we define $U_k := (z^{-n}\tau_{\mathcal{F}})^k(U_0) \in \mathcal{F}(\sigma_{\dot{D}}^k A_-)$ for all $k \in \mathbb{Z}$, the U_k glue to give a linearly independent r-tuple U of global sections in $\mathcal{F}(\dot{D})$ that satisfies $\tau_{\mathcal{F}}U = z^nU$. Thus U defines the desired injection $\mathcal{O}(-n)^{\oplus r} \hookrightarrow \mathcal{F}$. □

5. σ-Bundles of rank one

To classify σ-bundles of rank one we will need to construct functions with prescribed divisors.

Proposition 5.1. *For any $a \in \dot{D}$ there exists a non-zero function $f_a \in H^0(\mathcal{O}(1))$ which possesses a zero of exact order one at a^{q^i} for all $i \in \mathbb{Z}$ and no other zeroes.*

Proof. Set

$$g_a := \prod_{i\geqslant 0}\left(1 - \frac{a^{q^i}}{z}\right).$$

As a^{q^i} converges to zero at exponential speed, this infinite product converges to a function in R which has a zero of exact order one at a^{q^i} for all $i \geqslant 0$ and no other zeroes. By construction we also have

$$\sigma(g_a)(z) = \left(1 - \frac{a}{z}\right)^{-1} \cdot g_a(z). \tag{5.2}$$

On the other hand, we will construct a function $h_a(z) = \sum_{i\geqslant 0} b_i z^i \in R$ with non-zero constant coefficient and which satisfies

$$\sigma(h_a)(z) = (z - a) \cdot h_a(z). \tag{5.3}$$

This equation amounts to the equations $b_0^q = -ab_0$ and $b_i^q = b_{i-1} - ab_i$ for all $i > 0$. These equations can be solved inductively for all i, by letting b_0 be any $(q-1)$th root of $-a$ and solving an Artin–Schreier equation for every remaining coefficient, using the fact that $\mathbb{C}$ is algebraically closed.

By induction on i one easily proves $|b_i| = |a|^{q^{-i}/(q-1)}$. In particular, the coefficients are bounded, and so h_a indeed defines a holomorphic function on $\dot{D}$, which is also holomorphic at zero. Now consider the divisor $\Delta := \operatorname{div}(h_a) - \sum_{i<0}(a^{q^i})$ on $\dot{D}$. Equation (5.3) implies

$$\sigma_{\dot{D}}^*\Delta - \Delta = \operatorname{div}\left(\frac{\sigma(h_a)}{h_a}\right) - \sum_{i<0}(a^{q^{i+1}}) + \sum_{i<0}(a^{q^i}) = (a) - (a) = 0.$$

Thus if $\operatorname{supp}(\Delta)$ contains a point $c \in \dot{D}$, it contains c^{q^i} for every $i \in \mathbb{Z}$. But this is impossible, because h_a is holomorphic and non-zero at 0 and so 0 is not an accumulation point of $\operatorname{supp}(\Delta)$. Therefore, $\Delta = 0$ and h_a has a zero of exact order one at a^{q^i} for all $i < 0$ and no other zeroes.

Combining all this information, the function $f_a := g_a \cdot h_a \in R$ now has a zero of exact order one at a^{q^i} for all $i \in \mathbb{Z}$ and no other zeroes, and by Equations (5.2) and (5.3) it satisfies $\sigma(f_a)(z) = z \cdot f_a(z)$, that is, it is an element of $H^0(\mathcal{O}(1))$. □

Theorem 5.4. *Every σ-bundle of rank one is isomorphic to $\mathcal{O}(n)$ for a unique integer n.*

Proof. The uniqueness of n is contained in Proposition 3.2. For the existence let us fix a σ-bundle $\mathcal{F}$ of rank one. By Theorem 4.2 we can identify $\mathcal{F}$ with a σ-subbundle of $\mathcal{O}(m)$ for some integer m. Then $\mathcal{F}(-m) \subset \mathcal{O}$ is the ideal sheaf of a σ-invariant divisor Δ on $\dot{D}$. Since the support of any divisor on $\dot{D}$ contains only finitely many points of any closed annulus $\{x \in \mathbb{C} : \rho^q \leqslant |x| \leqslant \rho\}$, the set $\operatorname{supp}(\Delta)/\sigma_{\dot{D}}^{\mathbb{Z}}$ is finite. Let $a_i \in \dot{D}$ be representatives with multiplicities ℓ_i for $1 \leqslant i \leqslant k$. Letting f_{a_i} be the associated functions from Proposition 5.1, we find that Δ is also the divisor of the function $f := \prod_{i=1}^k f_{a_i}^{\ell_i}$. Therefore, multiplication by f induces an isomorphism of the underlying vector bundles $\mathcal{O} \xrightarrow{\sim} \mathcal{F}(-m) \subset \mathcal{O}$. Since f is a section in $H^0(\mathcal{O}(\ell))$ for $\ell := \sum_{i=1}^k \ell_i$, this defines an isomorphism of σ-bundles $\mathcal{O}(0) \xrightarrow{\sim} \mathcal{F}(-m+\ell) \subset \mathcal{O}(\ell)$. Therefore, $\mathcal{F} \cong \mathcal{O}(m-\ell)$, as desired. □

Corollary 5.5. *The vector bundle underlying any σ-bundle is free.*

Proof. Let $\mathcal{F}$ be a σ-bundle of rank r. Then by Theorem 5.4 and the definition of $\mathcal{O}(n)$ the line bundle underlying $\bigwedge^r \mathcal{F}$ is free. From Theorem 1.2 it now follows that the vector bundle underlying $\mathcal{F}$ is free. □

6. Semi-stability

The *rank of* a σ-bundle $\mathcal{F}$ is the rank of the underlying vector bundle and is denoted $\operatorname{rank}\mathcal{F}$. By Theorem 5.4 the highest exterior power $\bigwedge^{\operatorname{rank}\mathcal{F}} \mathcal{F}$ is isomorphic to $\mathcal{O}(d)$ for a unique integer d. This integer is called the *degree of* $\mathcal{F}$ and denoted $\deg \mathcal{F}$.

Proposition 6.1. *The degree is additive in short exact sequences.*

Proof. Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of σ-bundles, of respective ranks r', r and r''. Then there is a natural isomorphism $\bigwedge^r \mathcal{F} \cong \bigwedge^{r'} \mathcal{F}' \otimes \bigwedge^{r''} \mathcal{F}''$, and hence an isomorphism

$$\mathcal{O}(\deg \mathcal{F}) \cong \mathcal{O}(\deg \mathcal{F}') \otimes \mathcal{O}(\deg \mathcal{F}'') \cong \mathcal{O}(\deg \mathcal{F}' + \deg \mathcal{F}'').$$

The additivity thus results from the uniqueness in Theorem 5.4. □

Proposition 6.2. *Let $\mathcal{F}$ be a σ-bundle and $\mathcal{G} \subset \mathcal{F}$ a σ-subbundle with* $\operatorname{rank}\mathcal{G} = \operatorname{rank}\mathcal{F}$. *Then* $\deg \mathcal{G} \leqslant \deg \mathcal{F}$, *and equality holds if and only if* $\mathcal{G} = \mathcal{F}$.

Proof. Let $r := \operatorname{rank}\mathcal{F}$. Then $\mathcal{O}(\deg \mathcal{G}) \cong \bigwedge^r \mathcal{G} \subset \bigwedge^r \mathcal{F} \cong \mathcal{O}(\deg \mathcal{F})$ is a non-zero σ-subbundle. By Proposition 3.2 it follows that $\deg \mathcal{G} \leqslant \deg \mathcal{F}$. If the degrees are equal, the determinant of the inclusion morphism $\mathcal{G} \subset \mathcal{F}$ is an isomorphism; hence the inclusion itself is an isomorphism. □

Next, if $\operatorname{rank}\mathcal{F} > 0$, the *weight of* $\mathcal{F}$ is defined as

$$\mu(\mathcal{F}) := \frac{\deg \mathcal{F}}{\operatorname{rank}\mathcal{F}}.$$

A non-zero σ-bundle $\mathcal{F}$ is said to be *semi-stable* if $\mu(\mathcal{G}) \leqslant \mu(\mathcal{F})$ for all non-zero σ-subbundles $\mathcal{G}$ of $\mathcal{F}$. It is said to be *stable* if $\mu(\mathcal{G}) < \mu(\mathcal{F})$ for all non-zero proper σ-subbundles $\mathcal{G}$ of $\mathcal{F}$. These notions make sense because of the following fact.

Proposition 6.3.

a) *The weights of all non-zero σ-subbundles of any non-zero σ-bundle are bounded above. Moreover, there exists a non-zero σ-subbundle of maximal weight, and any such σ-subbundle is semi-stable.*

b) *$\mathcal{O}(n)^{\oplus r}$ is semi-stable of weight n.*

Proof. First we prove item b. Consider a non-zero σ-subbundle $\mathcal{G} \subset \mathcal{O}(n)^{\oplus r}$ of rank s. Then we have

$$\mathcal{O}(\deg \mathcal{G}) \cong \bigwedge^{s} \mathcal{G} \subset \bigwedge^{s}(\mathcal{O}(n)^{\oplus r}) \cong \mathcal{O}(sn)^{\oplus\binom{r}{s}}.$$

At least one of the coefficients therefore yields a non-zero homomorphism of σ-bundles $\mathcal{O}(\deg \mathcal{G}) \to \mathcal{O}(sn)$. From Proposition 3.2 it thus follows that $\deg \mathcal{G} \leqslant sn$, and hence $\mu(\mathcal{G}) \leqslant n$, as desired.

To prove item a we may assume by Theorem 4.2 that $\mathcal{F} \subset \mathcal{O}(n)^{\oplus r}$ for some r and n. Then every σ-subbundle of $\mathcal{F}$ is also a σ-subbundle of $\mathcal{O}(n)^{\oplus r}$ and therefore of weight less than or equal to n. Next the weights of all non-zero σ-subbundles are rational numbers with denominator less than or equal to r. Thus they form a discrete subset of $\mathbb{R}$ which is non-empty and bounded above; hence it contains a maximum. By construction, any non-zero σ-subbundle of maximal weight is semi-stable. $\square$

Using Proposition 6.3 one can easily show that every σ-bundle possesses a unique Harder–Narasimhan filtration (compare with Harder and Narasimhan [HN75, § 1.3]). Indeed, for non-zero $\mathcal{F}$ let $\mu \in \mathbb{Q}$ be the largest possible weight of a non-zero σ-subbundle; then the first non-zero step of this filtration is the unique largest σ-subbundle of weight μ. The remaining filtration steps are obtained by induction, repeating the procedure with the quotient. See also Corollary 11.7 below.

7. Finite maps

For any positive integer n let us now consider the morphism $[n] : \dot{D} \to \dot{D}$, $x \mapsto x^n$ which on the underlying ring is defined as

$$[n]^* : \ R \to R, \quad f(z) = \sum_i a_i z^i \mapsto \sum_i a_i z^{ni}.$$

Note that this map is given by substituting z^n for z and is therefore a homomorphism of $\mathbb{C}$-algebras, while σ was given by its action on the coefficients. This map induces the following two operations. The *pull-back* $[n]^*\mathcal{F}$ of a σ-bundle $\mathcal{F}$ is the pull-back of the associated vector bundle together with the induced isomorphism

$$\sigma_{\dot{D}}^*[n]^*\mathcal{F} = [n]^*\sigma_{\dot{D}}^*\mathcal{F} \xrightarrow{[n]^*\tau_{\mathcal{F}}} [n]^*\mathcal{F}.$$

On R-modules this operation maps M to $R \otimes_{[n],R} M$ with the σ-linear automorphism $\tau_{[n]^*M} := \sigma \otimes \tau_M$. The *push-forward* $[n]_*\mathcal{F}$ is the push-forward of the associated vector bundle together with the induced isomorphism

$$\sigma_{\dot{D}}^*[n]_*\mathcal{F} = [n]_*\sigma_{\dot{D}}^*\mathcal{F} \xrightarrow{[n]_*\tau_{\mathcal{F}}} [n]_*\mathcal{F}.$$

On R-modules this operation maps M to itself, viewed as a new R-module via $[n] : R \to R$, and with $\tau_{[n]_*M} := \tau_M$. Clearly both operations define functors from the category of σ-bundles to itself, and $[n]_*$ is a right adjoint of $[n]^*$.

PROPOSITION 7.1.

a) $[n]^*\mathcal{O}(m) \cong \mathcal{O}(nm)$.
b) $\deg[n]^*\mathcal{F} = n \cdot \deg \mathcal{F}$.
c) $\operatorname{rank}[n]^*\mathcal{F} = \operatorname{rank} \mathcal{F}$.
d) $\mu([n]^*\mathcal{F}) = n \cdot \mu(\mathcal{F})$.

Proof. Assertions a and c follow directly from the definitions of $[n]^*$ and $\mathcal{O}(m)$. Assertion b follows from a and the fact that $[n]^*$ commutes with exterior powers. Finally, assertions b and c together imply assertion d. □

PROPOSITION 7.2. $\operatorname{Hom}([n]^*\mathcal{F}, [n]^*\mathcal{G}) \cong F \otimes_{[n],F} \operatorname{Hom}(\mathcal{F}, \mathcal{G})$.

Proof. This follows directly from the definitions and the isomorphism $R \otimes_{[n],R} M \cong F \otimes_{[n],F} M$. □

PROPOSITION 7.3. *$\mathcal{F} \cong \mathcal{G}$ if and only if $[n]^*\mathcal{F} \cong [n]^*\mathcal{G}$.*

Proof. The 'only if' part follows from the fact that $[n]^*$ is a functor. For the 'if' part suppose that there exists an isomorphism $\varphi : [n]^*\mathcal{F} \xrightarrow{\sim} [n]^*\mathcal{G}$. Then Proposition 7.1 shows that $r := \operatorname{rank} \mathcal{F} = \operatorname{rank} \mathcal{G}$ and $\deg \mathcal{F} = \deg \mathcal{G}$. By Proposition 7.2 we can select a finite-dimensional F-subspace $V \subset \operatorname{Hom}(\mathcal{F}, \mathcal{G})$ such that $\varphi \in F \otimes_{[n],F} V$. Now by Proposition 6.2 any injective homomorphism $\psi \in V$ is an isomorphism. Thus $\psi \in V$ is an isomorphism if and only if its determinant $\det \psi \in \operatorname{Hom}(\bigwedge^r \mathcal{F}, \bigwedge^r \mathcal{G}) \cong F$ is non-zero. This determinant is given by a homogeneous polynomial map $V \to F$, which after base change via $F \otimes_{[n],F} (\cdot)$ induces the corresponding determinant map for homomorphisms $[n]^*\mathcal{F} \to [n]^*\mathcal{G}$. The fact that φ is an isomorphism means that $\det \varphi \neq 0$, and so the map $\det : V \to F$ is not identically zero. Therefore, there exists $\psi \in V$ with $\det \psi \neq 0$, and this ψ is the desired isomorphism $\mathcal{F} \to \mathcal{G}$. □

PROPOSITION 7.4. $[n]_*[n]^*\mathcal{F} \cong \mathcal{F}^{\oplus n}$.

Proof. $[n]_*[n]^*M$ is obtained from M by adjoining $z^{1/n}$, with the obvious module structure over R. Thus

$$[n]_*[n]^*M \cong \bigoplus_{i=1}^{n} z^{i/n} \otimes M \cong M^{\oplus n},$$

as desired. □

PROPOSITION 7.5. $[n]^*[n]_*\mathcal{O}(m) \cong \mathcal{O}(m)^{\oplus n}$.

Proof. (For the generalization to arbitrary σ-bundles see Corollary 11.5.) By definition $\mathcal{O}(m)$ corresponds to the R-module R together with the map $\tau = z^{-m} \cdot \sigma$. The operation $[n]^*[n]_*$ amounts to tensoring over the subring $R_n := [n](R)$ with a new copy of R. Thus the module corresponding to $[n]^*[n]_*\mathcal{O}(m)$ is isomorphic to $R[w]/(w^n - z^n)$ together with $\tau = w^{-m} \cdot \sigma$. Since z is invertible in R, we may rewrite this in terms of the new variable $u := w/z$ as $R[u]/(u^n - 1)$ together with $\tau = u^{-m}z^{-m} \cdot \sigma$. Thus $\mathcal{H}om(\mathcal{O}(m), [n]^*[n]_*\mathcal{O}(m))$ corresponds to the module $R[u]/(u^n - 1)$ together with $\tau = u^{-m} \cdot \sigma$. This is the base change via $R \otimes_{\mathbb{C}}$ of the finite-dimensional $\mathbb{C}$-vector space $\mathbb{C}[u]/(u^n - 1)$ together with the σ-linear automorphism $\tau = u^{-m} \cdot \sigma$. By Artin–Schreier theory (see Katz [Gro72, exp. XXII, Proposition 1.1]), or by an easy explicit calculation, this vector space possesses a τ-invariant basis. This basis induces the desired isomorphism. □

Proposition 7.6. *$[n]_*\mathcal{O}(m)$ is semi-stable of rank n, degree m, and weight m/n.*

Proof. Propositions 6.3 (item b) and 7.5 show that $[n]^*[n]_*\mathcal{O}(m)$ is semi-stable of rank n and weight m. Consider a non-zero σ-subbundle $\mathcal{G} \subset [n]_*\mathcal{O}(m)$. Then $[n]^*\mathcal{G}$ is a non-zero σ-subbundle of $[n]^*[n]_*\mathcal{O}(m)$, so by the latter's semi-stability its weight satisfies $\mu([n]^*\mathcal{G}) \leqslant m = \mu([n]^*[n]_*\mathcal{O}(m))$. Proposition 7.1 item d now shows that $\mu(\mathcal{G}) \leqslant m/n = \mu([n]_*\mathcal{O}(m))$, as desired. $\square$

8. Building blocks

For any pair of relatively prime integers d, r with $r > 0$, and only for those, we now set $\mathcal{F}_{d,r} := [r]_*\mathcal{O}(d)$. By Proposition 7.6 this is a σ-bundle of rank r and weight d/r. The assumption on d and r thus implies that the rank is precisely the denominator of the weight. Of course we have $\mathcal{F}_{n,1} \cong \mathcal{O}(n)$ for every integer n. In the general case consider the $(r \times r)$-matrix

$$A_{d,r} := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ z^{-d} & 0 & \cdots & \cdots & 0 \end{pmatrix}. \tag{8.1}$$

Then the vector bundle underlying $\mathcal{F}_{d,r}$ can be identified with $\mathcal{O}^{\oplus r}$ such that $\tau_{\mathcal{F}_{d,r}}$ is the isomorphism $\sigma : \sigma_D^*\mathcal{O}^{\oplus r} \xrightarrow{\sim} \mathcal{O}^{\oplus r}$ followed by multiplication by $A_{d,r}$. The corresponding R-module is then $R^{\oplus r}$ together with the σ-linear automorphism

$$A_{d,r} \cdot \sigma : \; R^{\oplus r} \longrightarrow R^{\oplus r}, \quad m \mapsto A_{d,r} \cdot \sigma(m),$$

where σ acts on m componentwise.

Proposition 8.2. *$\mathcal{F}_{d,r}$ is stable.*

Proof. By Proposition 7.6 it is semi-stable. To show that it is stable, consider a non-zero σ-subbundle $\mathcal{G} \subset \mathcal{F}_{d,r}$ with $\mu(\mathcal{G}) = \mu(\mathcal{F}_{d,r}) = d/r$. Since d and r are relatively prime, the denominator $\operatorname{rank}\mathcal{G}$ of $\mu(\mathcal{G})$ must be divisible by r. Thus $\mathcal{G}$ is a σ-subbundle of equal rank r and degree d. With Proposition 6.2 we conclude that $\mathcal{G} = \mathcal{F}_{d,r}$, as desired. $\square$

Proposition 8.3.

a) $\mathcal{F}_{d,r} \otimes \mathcal{F}_{d',r'} \cong \mathcal{F}_{d'',r''}^{\oplus rr'/r''}$*, where* $d/r + d'/r' = d''/r''$ *with each fraction in lowest terms.*
b) $\mathcal{F}_{d,r}(n) \cong \mathcal{F}_{d+rn,r}$.
c) $\mathcal{F}_{d,r}^{\vee} \cong \mathcal{F}_{-d,r}$.
d) $\mathcal{H}om(\mathcal{F}_{d,r}, \mathcal{F}_{d',r'}) \cong \mathcal{F}_{d'',r''}^{\oplus rr'/r''}$*, where* $d'/r' - d/r = d''/r''$ *with each fraction in lowest terms.*

Proof. To prove assertion a we note that by Proposition 7.3 it suffices to prove the isomorphy after applying $[rr']^*$. Using Propositions 7.5 and 7.1 (item a) the left-hand side then becomes

$$\begin{aligned} [rr']^*(\mathcal{F}_{d,r} \otimes \mathcal{F}_{d',r'}) &\cong [r']^*[r]^*\mathcal{F}_{d,r} \otimes [r]^*[r']^*\mathcal{F}_{d',r'} \\ &\cong [r']^*\mathcal{O}(d)^{\oplus r} \otimes [r]^*\mathcal{O}(d')^{\oplus r'} \\ &\cong \mathcal{O}(r'd)^{\oplus r} \otimes \mathcal{O}(rd')^{\oplus r'} \\ &\cong \mathcal{O}(r'd + rd')^{\oplus rr'}. \end{aligned}$$

Similarly, since $r''|rr'$ the right-hand side becomes

$$\begin{aligned}[rr']^*\mathcal{F}_{d'',r''}^{\oplus rr'/r''} &\cong [rr'/r'']^*\mathcal{O}(d'')^{\oplus rr'} \\ &\cong \mathcal{O}(rr'd''/r'')^{\oplus rr'} \\ &= \mathcal{O}(r'd+rd')^{\oplus rr'},\end{aligned}$$

as desired. Assertion b is the special case of assertion a with $(d',r')=(n,1)$. Assertion c can be seen easily from the explicit description of $\mathcal{F}_{d,r}$ above. Finally, assertion d results from assertions a and c. □

Proposition 8.4. *$H^0(\mathcal{F}_{d,r})$ is an F-vector space of dimension*

$$\begin{cases} 0 & \text{if } d/r<0, \\ 1 & \text{if } d/r=0 \text{ and hence } r=1, \\ \infty & \text{if } d/r>0. \end{cases}$$

Proof. This follows from Proposition 3.1 together with the fact that $H^0(\mathcal{F}_{d,r}) = H^0([r]_*\mathcal{O}(d)) \cong H^0(\mathcal{O}(d))$ with the new F-vector space structure via $[r]: F\to F$, $\sum a_i z^i \mapsto \sum a_i z^{ri}$. □

Proposition 8.5. *$\operatorname{Hom}(\mathcal{F}_{d,r},\mathcal{F}_{d',r'})$ is an F-vector space of dimension*

$$\begin{cases} 0 & \text{if } d/r>d'/r', \\ r^2 & \text{if } d/r=d'/r', \\ \infty & \text{if } d/r<d'/r'. \end{cases}$$

In particular, $\mathcal{F}_{d,r}$ and $\mathcal{F}_{d',r'}$ are isomorphic if and only if $(d,r)=(d',r')$.

Proof. This results from Propositions 8.3 (item d) and 8.4 by taking H^0. □

For the sake of completeness we determine the endomorphism ring precisely.

Proposition 8.6. *$\operatorname{End}(\mathcal{F}_{d,r}) := \operatorname{Hom}(\mathcal{F}_{d,r},\mathcal{F}_{d,r})$ is a central division algebra over F of dimension r^2 and invariant $-d/r \bmod \mathbb{Z}$.*

Proof. The σ-bundle $\mathcal{F}_{d,r}$ can be described by the R-module $R^{\oplus r}$ together with the automorphism $\tau = A_{d,r}\cdot\sigma$, where $A_{d,r}$ is the matrix (8.1). Thus $\operatorname{End}(\mathcal{F}_{d,r})$ corresponds to the ring of matrices $B=(b_{i,j})\in M_r(R)$ satisfying $BA_{d,r}=A_{d,r}\sigma(B)$. One easily checks that on the coefficients this means that

$$b_{i,r} = \sigma^r(b_{i,r}) \quad \text{for all } i,$$

and

$$b_{i,j} = \begin{cases} \sigma^{r-j}(b_{i+r-j,r}) & \text{for } i\leqslant j, \\ z^{-d}\sigma^{r-j}(b_{i-j,r}) & \text{for } i>j. \end{cases}$$

The first equation means that $b_{i,r}\in\mathbb{F}_{q^r}((z))$; hence the map $(b_{i,j})_{i,j=1}^r \mapsto (b_{i,r})_{i=1}^r$ induces an isomorphism of additive groups $\operatorname{End}(\mathcal{F}_{d,r}) \xrightarrow{\sim} \mathbb{F}_{q^r}((z))^{\oplus r}$. The second equation means that the product on $\operatorname{End}(\mathcal{F}_{d,r})$ corresponds to

$$(b_{i,r})_{i=1}^r\cdot(c_{i,r})_{i=1}^r = \Bigg(\sum_{j=i}^r c_{j,r}\sigma^{r-j}(b_{r+i-j,r}) \;+\; z^{-d}\sum_{j=1}^{i-1} c_{j,r}\sigma^{r-j}(b_{i-j,r})\Bigg)_{i=1}^r.$$

These equations describe the cyclic central simple F-algebra of degree r associated to the element z^{-d} according to Reiner [Rei75, § 30]. By definition it has invariant $-d/r \bmod \mathbb{Z}$; see [Rei75, p. 266]. □

Proposition 8.7. *$H^1(\mathcal{F}_{d,r})$ is an F-vector space of dimension*

$$\begin{cases} \infty & \text{if } d/r < 0, \\ 0 & \text{if } d/r \geqslant 0. \end{cases}$$

Proof. This follows from Proposition 3.3 as in the proof of Proposition 8.4. ☐

Proposition 8.8. *$\operatorname{Ext}(\mathcal{F}_{d,r}, \mathcal{F}_{d',r'})$ is an F-vector space of dimension*

$$\begin{cases} \infty & \text{if } d/r > d'/r', \\ 0 & \text{if } d/r \leqslant d'/r'. \end{cases}$$

Proof. This results from Propositions 8.3 (item d) and 8.7 by taking H^1. ☐

9. Global sections

As a preliminary step for the classification of σ-bundles in § 11 we prove the following.

Proposition 9.1. *For every extension $0 \to \mathcal{O}(-1) \to \mathcal{F} \to \mathcal{F}_{1,r} \to 0$ of σ-bundles we have $H^0(\mathcal{F}) \neq 0$.*

First we derive a corollary from this.

Corollary 9.2. *For every extension $0 \to \mathcal{O}(-n) \to \mathcal{F} \to \mathcal{O}(n) \to 0$ of σ-bundles we have $H^0(\mathcal{F}) \neq 0$.*

Proof of Corollary 9.2. For $n \leqslant 0$ this follows from Proposition 3.1, and for $n = 1$ from Proposition 9.1. So let $n > 1$. We proceed by induction on n. Choose any non-zero homomorphism $\mathcal{O}(n-1) \to \mathcal{O}(n+1)$ using Proposition 3.2 and consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(-n+1) & \longrightarrow & \mathcal{F}(1) & \longrightarrow & \mathcal{O}(n+1) & \longrightarrow & 0 \\ & & \| & & \bigcup & \square & \bigcup & & \\ 0 & \longrightarrow & \mathcal{O}(-n+1) & \longrightarrow & \widetilde{\mathcal{F}} & \longrightarrow & \mathcal{O}(n-1) & \longrightarrow & 0 \end{array}$$

where the second row is constructed via pull-back from the first row. The induction hypothesis implies that $H^0(\widetilde{\mathcal{F}}) \neq 0$; hence $\operatorname{Hom}(\mathcal{O}(-1), \mathcal{F}) = H^0(\mathcal{F}(1)) \neq 0$. So there exists a subbundle $\mathcal{O}(-1) \cong \mathcal{F}' \subset \mathcal{F}$. If $\mathcal{F}'$ is not saturated, its saturation has degree greater than or equal to zero by Proposition 6.2 and thus possesses non-trivial global sections by Proposition 3.1; hence $H^0(\mathcal{F}) \neq 0$, as desired. If $\mathcal{F}'$ is saturated, Proposition 6.1 shows that $\deg(\mathcal{F}/\mathcal{F}') = 1$; hence $\mathcal{F}$ fits into an exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}(1) \longrightarrow 0,$$

and again Proposition 9.1 shows that $H^0(\mathcal{F}) \neq 0$. ☐

Proof of Proposition 9.1. The σ-bundle $\mathcal{F}$ can be given by the free R-module $M = R^{\oplus(r+1)}$ together with

$$\tau_M = \begin{pmatrix} z & b \\ 0 & A \end{pmatrix} \cdot \sigma,$$

where $A = A_{1,r}$ is the matrix (8.1) and $b = (b_t)_{t=1}^r \in R^r$ is a row vector. We must find an element $u \in R$ and a column vector $v = (v_t)_{t=1}^r \in R^r$, not both zero, such that

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} z & b \\ 0 & A \end{pmatrix} \cdot \sigma \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} z\sigma(u) + b\sigma(v) \\ A\sigma(v) \end{pmatrix}.$$

If v were zero, then u would be a non-zero section of $\mathcal{O}(-1)$, contradicting Proposition 3.1. Thus $v \neq 0$. The equation $v = A\sigma(v)$ amounts to $v_t = \sigma^{r-t}(v_r)$ for all $1 \leqslant t < r$ and $v_r = z^{-1}\sigma^r(v_r)$. As in the proof of Proposition 3.1 we see that the non-zero solutions of these equations are given precisely by

$$v_t = \sum_{i \in \mathbb{Z}} z^{-i} w^{q^{ir-t}}$$

for some $w \in \mathbb{C}$ with $0 < |w| < 1$. For any such choice of w we have $f := b\sigma(v) \in R$, and $u \in R$ must satisfy $u = z\sigma(u) + f$. By induction on m this is equivalent to

$$u = z^m \sigma^m(u) + \sum_{j=0}^{m-1} z^j \sigma^j(f)$$

for all $m \geqslant 1$. Writing $f = \sum_i f_i z^i$ and $u = \sum_i u_i z^i$ this means that

$$u_i = u_{i-m}^{q^m} + \sum_{j=0}^{m-1} f_{i-j}^{q^j}$$

for all $i \in \mathbb{Z}$. By Proposition 1.1 we may take the limit as $m \to \infty$, and so the unique choice for u_i is given by the convergent series

$$u_i = \sum_{j \geqslant 0} f_{i-j}^{q^j}.$$

The only remaining problem is to find $w \in \mathbb{C}$ with $0 < |w| < 1$ such that $u \in R$. This problem is difficult to solve for the following reason. Since $\deg \mathcal{F} = 0$, the classification Theorem 11.1 which we are going to prove implies that for typical b we have $\mathcal{F} \cong \mathcal{O}(0)^{\oplus(r+1)}$. Thus the F-vector space $H^0(\mathcal{F})$ will tend to have finite dimension, whereas $\{w \in \mathbb{C} : |w| < 1\}$ is infinite-dimensional. Finding a non-zero element in that finite-dimensional subspace is somewhat tricky. We study the two sides of the convergence problem separately.

Lemma 9.3. *The part of u with negative powers of z always converges on $\dot{D}$.*

Proof. The fact that $f \in R$ implies that $(1/i)\log|f_{-i}| \to -\infty$ for $i \to \infty$ by Proposition 1.1. If $N \in \mathbb{N}$ is chosen such that $\log|f_{-i}| \leqslant 0$ for all $i \geqslant N$, we deduce for $i \geqslant N$ that

$$\begin{aligned} \frac{\log|u_{-i}|}{i} &\leqslant \sup\left\{ \frac{q^j \log|f_{-i-j}|}{i} : j \geqslant 0 \right\} \\ &\leqslant \sup\left\{ \frac{\log|f_{-i-j}|}{i+j} : j \geqslant 0 \right\} \\ &\to -\infty \quad \text{for } i \to \infty. \end{aligned}$$

By Proposition 1.1 this proves the claim. ∎

Lemma 9.4. *The part of u with positive powers of z converges on $\dot{D}$ if and only if*

$$\limsup_{i \to \infty} \frac{\log|u_i|}{q^i} \leqslant 0.$$

Proof. The respective condition in Proposition 1.1 requires that

$$\limsup_{i\to\infty} \frac{\log|u_i|}{i} \leqslant 0.$$

Both conditions depend only on the terms with $\log|u_i| > 0$ and, since $q^i \geqslant i$ for all i, the condition in the lemma is clearly necessary. To show that it is also sufficient note that for all i and $k \geqslant 1$ we have $u_i = u_{i+k}^{q^{-k}} - \sum_{j=1}^k f_{i+j}^{q^{-j}}$. Therefore,

$$\frac{\log|u_i|}{q^i} \leqslant \sup\left\{\frac{\log|u_{i+k}|}{q^{i+k}}, \frac{\log|f_{i+j}|}{q^{i+j}} : 1 \leqslant j \leqslant k\right\}.$$

Fixing i and letting $k \to \infty$, the condition in the lemma implies that

$$\frac{\log|u_i|}{q^i} \leqslant \sup\left\{0, \frac{\log|f_{i+j}|}{q^{i+j}} : j \geqslant 1\right\}.$$

Therefore,

$$\begin{aligned}\limsup_{i\to\infty} \frac{\log|u_i|}{i} &\leqslant \limsup_{i\to\infty}\left(\sup\left\{0, \frac{\log|f_{i+j}|}{iq^j} : j \geqslant 1\right\}\right)\\ &\leqslant \limsup_{i\to\infty}\left(\sup\left\{0, \frac{\log|f_{i+j}|}{i+j} : j \geqslant 1\right\}\right)\\ &= \sup\left\{0, \limsup_{k\to\infty}\frac{\log|f_k|}{k}\right\}.\end{aligned}$$

Since $f \in R$, this is less than or equal to zero by Proposition 1.1, as desired. □

Now we expand u_i explicitly in terms of w. From

$$\begin{aligned}f = b\sigma(v) &= \sum_{t=1}^r b_t\sigma(v_t)\\ &= \sum_{t=1}^r \left(\sum_{j\in\mathbb{Z}} b_{t,j}z^j\right)\cdot\left(\sum_{i\in\mathbb{Z}} z^{-i}w^{q^{ir-t+1}}\right)\\ &= \sum_{i,j\in\mathbb{Z}}\sum_{t=1}^r b_{t,j}w^{q^{ir-t+1}}z^{j-i}\end{aligned}$$

we obtain

$$f_k = \sum_{j\in\mathbb{Z}}\sum_{t=1}^r b_{t,j}w^{q^{(j-k)r-t+1}}.$$

Therefore,

$$u_i^{q^{-i}} = \sum_{j\geqslant 0} f_{i-j}^{q^{j-i}} = \sum_{k\leqslant i} f_k^{q^{-k}} = \sum_{k\leqslant i}\sum_{j\in\mathbb{Z}}\sum_{t=1}^r b_{t,j}^{q^{-k}} w^{q^{(j-k)r-k-t+1}}.$$

We abbreviate $n := r+1$ and $m_{rj-t+1} := b_{t,j}$. Then $m := \sum_\ell m_\ell z^\ell \in R$ and

$$u_i^{q^{-i}} = \sum_{k\leqslant i}\sum_{\ell\in\mathbb{Z}} m_\ell^{q^{-k}} w^{q^{\ell-nk}}.$$

So by the preceding lemmas Proposition 9.1 is reduced to the following assertion, whose proof occupies the next section. □

Proposition 9.5. *For every $n \geqslant 2$ and every $m = \sum_\ell m_\ell z^\ell \in R$ there exists $w \in \mathbb{C}$ with $0 < |w| < 1$ such that*

$$\limsup_{i\to\infty} \log \Big| \sum_{k\leqslant i} \sum_{\ell\in\mathbb{Z}} m_\ell^{q^{-k}} w^{q^{\ell-nk}} \Big| \leqslant 0.$$

10. The fundamental estimate

In this section we prove Proposition 9.5.

10.1 The convergence of the double series in Proposition 9.5 was shown implicitly in the preceding section. Note that the terms with $|m_\ell| \leqslant 1$ do not affect the desired inequality. Thus after removing them we may assume that the principal part of m is finite. Moreover, the lim sup does not change if m is replaced by $z^{nr}\sigma^r(m)$ for any $r \gg 0$. Therefore, we may assume that

$$m = \sum_{\ell>0} m_\ell z^\ell.$$

10.2 Since now m converges in a neighborhood of 0 and vanishes at 0, there exists a constant $c > 0$ such that $\log|m_\ell| \leqslant c\ell$ for all ℓ. Since $x \leqslant q^x$ for all $x \in \mathbb{R}$, this implies that for all ℓ we have

$$\log|m_\ell| \leqslant cnq^{\ell/n}.$$

10.3 We will choose w with $\log|w| \leqslant -cn$. This has the effect that

$$\begin{aligned}\log|m_\ell^{q^{-k}} w^{q^{\ell-nk}}| &= q^{-k}\log|m_\ell| + q^{\ell-nk}\log|w| \\ &\leqslant q^{-k}q^{\ell/n}cn + q^{\ell-nk}(-cn) \\ &= cn(q^{(\ell-nk)/n} - q^{\ell-nk}),\end{aligned}$$

which is less than or equal to zero if $\ell - nk \geqslant 0$. So only the terms with $\ell < nk$ contribute to our condition.

10.4 To deal with the remaining terms we set $j := nk - \ell$ and write

$$\begin{aligned}F_i(z) &:= \sum_{k\leqslant i} \sum_{0<\ell<nk} m_\ell^{q^{-k}} z^{q^{\ell-nk}} \\ &= \sum_{0<k\leqslant i} \sum_{0<j<nk} m_{nk-j}^{q^{-k}} z^{q^{-j}} \\ &= \sum_{0<j<ni} F_{i,j} z^{q^{-j}}\end{aligned}$$

with

$$F_{i,j} := \sum_{j/n<k\leqslant i} m_{nk-j}^{q^{-k}} \in \mathbb{C}.$$

We need to find w with $\limsup_{i\to\infty}\log|F_i(w)| \leqslant 0$. We begin with some estimates.

10.5 For all $0 < j < ni$ we set

$$\varphi_{i,j} := \log|F_{i,j}|,$$

and for all $j > 0$ we set

$$\varphi_j := \limsup_{i\to\infty} \varphi_{i,j}.$$

A direct calculation shows that $F_{i+\ell,j+n\ell} = F_{i,j}^{q^{-\ell}}$ for all ℓ. Therefore,

$$\varphi_{i+\ell,j+n\ell} = q^{-\ell}\varphi_{i,j}$$

and hence

$$\varphi_{j+n\ell} = q^{-\ell}\varphi_j.$$

10.6 Also, the estimate § 10.2 implies

$$\varphi_{i,j} \leqslant \sup\{q^{-k}\log|m_{nk-j}| : j/n < k \leqslant i\} \leqslant q^{-j/n}cn$$

and hence

$$\varphi_j \leqslant q^{-j/n}cn.$$

10.7 Next we estimate the difference $F_{i'} - F_i$. Take $0 < j < ni \leqslant ni'$. Then by § 10.2 we have

$$\begin{aligned}\log|F_{i',j} - F_{i,j}| &\leqslant \sup\{q^{-k}\log|m_{nk-j}| : i < k \leqslant i'\}\\ &\leqslant \sup\{q^{-k}c(nk-j) : i < k \leqslant i'\}\\ &\leqslant cn\sup\{q^{-k}k : i < k \leqslant i'\}.\end{aligned}$$

The function $x \mapsto q^{-x}x$ has derivative $q^{-x}(1 - x\log q)$; hence, it is monotone decreasing for $x \geqslant 2 > 1/\log q$. Thus we deduce that for $i' \geqslant i \geqslant 2$ we have

$$\log|F_{i',j} - F_{i,j}| \leqslant cniq^{-i}.$$

10.8 Now we distinguish two cases. Suppose first that $\varphi_j \leqslant 0$ for all j. Then the estimate in § 10.7 implies for all $i \geqslant 2$ and all $0 < j < ni \leqslant ni'$ that

$$\begin{aligned}\varphi_{i,j} &= \log|F_{i',j} + (F_{i,j} - F_{i',j})|\\ &\leqslant \sup\{\varphi_{i',j}, cniq^{-i}\}\\ &\leqslant cniq^{-i}\end{aligned}$$

by taking $i' \gg 0$. So for all $i \geqslant 2$ and all $w \in \mathbb{C}$ with $|w| < 1$ we deduce that

$$\log|F_i(w)| < \sup\{\varphi_{i,j} : 0 < j < ni\} \leqslant cniq^{-i}.$$

This goes to zero as $i \to \infty$. Therefore, every w as above with $\log|w| \leqslant -cn$ satisfies the condition in Proposition 9.5. (One can show that in this case the extension in Proposition 9.1 splits, and the only other possibility is that $\mathcal{F} \cong \mathcal{O}(0)^{\oplus(r+1)}$; cf. Theorem 11.1.)

10.9 In the rest of the proof we assume that there exists $j > 0$ with $\varphi_j > 0$. From § 10.5 we see that this must then be so for some $0 < j \leqslant n$. Using § 10.7 and the fact that $cniq^{-i}$ becomes arbitrarily small as $i \to \infty$ we deduce that $\varphi_{i,j} = \varphi_j$ for all sufficiently large $i > j/n$. Let $i_0 \geqslant 2$ be such that $\varphi_{i,j} = \varphi_j$ for all $i \geqslant i_0$ and for all $0 < j \leqslant n$ with $\varphi_j > 0$. Applying § 10.5 and the transformation $(i,j) \to (i+\ell, j+n\ell)$ implies

$$\forall j > 0,\ \forall i \geqslant i_0 + \frac{j}{n} : \varphi_j > 0 \Longrightarrow \varphi_{i,j} = \varphi_j.$$

10.10 Observe now that every $F_i(z)$ is a fractional q-polynomial, meaning that the exponents of z are of the form q^j for $j \in \mathbb{Z}$. We cannot simply go to the limit of the F_i. Instead, we will construct a sequence w_i of zeroes of the respective F_i which converges to the desired w as $i \to \infty$. The first step is to find zeroes of a single F_i of the right absolute value. This depends on the slopes of the Newton polygon of F_i.

10.11 By the theory of Newton polygons F_i possesses a zero of absolute value α if and only if the function $j \mapsto |F_{i,j}|\alpha^{q^{-j}}$ attains its maximum at two different points $j_1 < j_2$. Thus it possesses a zero of logarithmic absolute value $-\rho < 0$ if and only if the function $j \mapsto \varphi_{i,j} - q^{-j}\rho$ attains its maximum at two different points. In view of § 10.9 we first determine the maximum of the function $j \mapsto \varphi_j - q^{-j}\rho$.

Lemma 10.12. *There exist $\rho \geqslant cn$ and $j_2 > j_1 > 0$ such that $\varphi_j - q^{-j}\rho$ attains its maximum among all $j > 0$ at both $j = j_1$ and $j = j_2$ and that maximum is positive.*

Proof. First we determine the maximum within the residue class $\bmod\, n$ of a given j_0 with $\varphi_{j_0} > 0$. This maximum is achieved at $j > n$ if and only if for all $k > -j/n$ we have

$$\varphi_j - q^{-j}\rho \geqslant \varphi_{j+nk} - q^{-j-nk}\rho = q^{-k}\varphi_j - q^{-nk}q^{-j}\rho,$$

the last equality coming from § 10.5. This means that for all $k > -j/n$ we have

$$(1 - q^{-k})\varphi_j \geqslant (1 - q^{-nk})q^{-j}\rho,$$

or equivalently for all integers $k > 0$ and $0 < k' < j/n$ we have

$$\frac{1 - q^{-nk}}{1 - q^{-k}} \leqslant \frac{\varphi_j}{q^{-j}\rho} \leqslant \frac{q^{nk'} - 1}{q^{k'} - 1}.$$

It is enough to have this for $k = k' = 1$, that is, to have

$$q^{1-n} \cdot \frac{q^n - 1}{q - 1} = \frac{1 - q^{-n}}{1 - q^{-1}} \leqslant \frac{\varphi_j}{q^{-j}\rho} \leqslant \frac{q^n - 1}{q - 1}.$$

Now within the given residue class we have $j = j_0 + n\ell$, and so by § 10.5

$$\frac{\varphi_j}{q^{-j}\rho} = \frac{q^{-\ell}\varphi_{j_0}}{q^{-j_0-n\ell}\rho} = q^{(n-1)\ell} \cdot \frac{\varphi_{j_0}}{q^{-j_0}\rho}.$$

Since $n \geqslant 2$, for any $\rho \gg cn$ we deduce that within the residue class $j_0 \bmod n$ the maximum is achieved either at precisely one point $j > n$ or at precisely two points $j + n$, $j > n$. Furthermore, the value of that maximum is

$$\varphi_j - q^{-j}\rho \geqslant q^{-j}\rho\left(\frac{1 - q^{-n}}{1 - q^{-1}} - 1\right) = q^{-j}\rho\frac{q^{-1} - q^{-n}}{1 - q^{-1}} > 0.$$

So combining everything we conclude that for every $\rho \gg cn$ the function $j \mapsto \varphi_j - q^{-j}\rho$ attains a positive maximum for at least one and at most finitely many points $j > 0$. As soon as that maximum is attained at more than one point, we are done. So consider $\rho \gg cn$ for which a positive maximum is attained at some $j > 0$. Then the above formulas show that for $q^{n-1}\rho$ in place of ρ the maximum is attained at $j + n$ instead of j. Thus as ρ increases to $q^{n-1}\rho$, it must reach a point where the maximum is attained at two places. This proves the lemma. ☐

Lemma 10.13. *Let ρ be as in Lemma 10.12. Then there exists $i_1 \geqslant 2$ such that for all $i \geqslant i_1$ the function F_i has a zero $w_i \in \mathbb{C}$ with $\log|w_i| = -\rho$.*

Proof. Let $j_2 > j_1 > 0$ be as in Lemma 10.12. We must determine $i_1 \geqslant 2$ such that for all $i \geqslant i_1$ we have $j_2 < in$ and the function $j \mapsto \varphi_{i,j} - q^{-j}\rho$ attains its maximum at j_1 and j_2. For this we first require that $i_1 \geqslant i_0 + j_2/n$. Then by § 10.9 we have $\varphi_{i,j} = \varphi_j$ for $j = j_1$, j_2 and all $i \geqslant i_1$. It remains to show that

$$\varphi_{i,j} - q^{-j}\rho \leqslant \varphi_{j_1} - q^{-j_1}\rho$$

for all $i \geqslant i_1$ and all $0 < j < in$. Here the right-hand side is positive by Lemma 10.12. By § 10.6 the left-hand side is always less than or equal to $q^{-j/n}cn$; hence the inequality holds whenever $j \gg 0$, independent of i. Fix one of the finitely many remaining j. If $\varphi_j > 0$, we require that $i_1 \geqslant i_0 + j/n$. Then by § 10.9 and Lemma 10.12 for all $i \geqslant i_1$ we have

$$\varphi_{i,j} - q^{-j}\rho = \varphi_j - q^{-j}\rho \leqslant \varphi_{j_1} - q^{-j_1}\rho,$$

as desired. If $\varphi_j \leqslant 0$, by the definition of φ_j for all $i \gg 0$ we have

$$\varphi_{i,j} - q^{-j}\rho \leqslant 0 < \varphi_{i,j_1} - q^{-j_1}\rho.$$

In this case we adjust i_1 so that this holds for all $i \geqslant i_1$. Altogether we have found finitely many lower bounds on i_1 which imply the desired assertion. □

Lemma 10.14. *Consider a non-zero fractional q-polynomial $G(z) = \sum_j G_j z^{q^j}$, indexed by finitely many $j \in \mathbb{Z}$. For every $\lambda \in \mathbb{R}$ consider the ball*

$$B_\lambda := \{\zeta \in \mathbb{C} : \log|\zeta| \leqslant \lambda\}$$

and set

$$\psi(\lambda) := \sup\{\log|G_j| + q^j\lambda : \textit{all } j\}.$$

Then

$$G(B_\lambda) = B_{\psi(\lambda)}.$$

Proof. For $\zeta \in \mathbb{C}$ with $\log|\zeta| \leqslant \lambda$ we have

$$\log|G(\zeta)| \leqslant \sup\{\log|G_j\zeta^{q^j}| : \text{all } j\} \leqslant \psi(\lambda),$$

proving the inclusion '$\subset$'. To prove '$\supset$' consider $\xi \in \mathbb{C}$ with $\log|\xi| \leqslant \psi(\lambda)$. We must show that the equation $G(z) = \xi$ has a solution ζ with $\log|\zeta| \leqslant \lambda$. But by the theory of the Newton polygon the smallest zero ζ of $G(z) - \xi$ satisfies

$$\log|\xi| = \sup\{\log|G_j\zeta^{q^j}| : \text{all } j\} = \psi(\log|\zeta|).$$

As ψ is a strictly monotone increasing function, the inequality $\psi(\log|\zeta|) \leqslant \psi(\lambda)$ now implies $\log|\zeta| \leqslant \lambda$, as desired. □

Lemma 10.15. *Let ρ and i_1 be as in Lemma 10.13. Then there exist $i_2 \geqslant i_1$ and $c' > 0$ with the following property. Consider $i \geqslant i_2$ and a zero w_i of F_i with $\log|w_i| = -\rho$. Then there exists a zero w_{i+1} of F_{i+1} with*

$$\log|w_{i+1} - w_i| \leqslant -c'q^{(n-1)i/2}.$$

Proof. Write $w_{i+1} = w_i + \Delta w_i$. Since $F_{i+1}(z)$ is additive, we have

$$F_{i+1}(w_{i+1}) = F_{i+1}(w_i) + F_{i+1}(\Delta w_i) = (F_{i+1} - F_i)(w_i) + F_{i+1}(\Delta w_i).$$

So we must solve the equation

$$F_{i+1}(\Delta w_i) = -(F_{i+1} - F_i)(w_i).$$

We estimate the right-hand side using §§ 10.6 and 10.7:

$$\begin{aligned}\log|(F_{i+1} - F_i)(w_i)| &\leqslant \sup\left\{\begin{matrix}\log|F_{i+1,j} - F_{i,j}| - q^{-j}\rho & : & 0 < j < ni \\ \log|F_{i+1,j}| - q^{-j}\rho & : & ni \leqslant j < n(i+1)\end{matrix}\right\} \\ &\leqslant \sup\left\{\begin{matrix} cniq^{-i} - q^{-j}\rho & : & 0 < j < ni \\ cnq^{-j/n} - q^{-j}\rho & : & ni \leqslant j < n(i+1)\end{matrix}\right\} \\ &\leqslant cniq^{-i}.\end{aligned}$$

Set

$$\lambda := \min\{q^j(cniq^{-i} - \varphi_{i+1,j}) : 0 < j < n(i+1)\}.$$

Then

$$\sup\{\varphi_{i+1,j} + q^{-j}\lambda : 0 < j < n(i+1)\} = cniq^{-i},$$

so by Lemma 10.14 we can find Δw_i solving the above equation with $\log|\Delta w_i| \leqslant \lambda$. To obtain an explicit bound fix $0 < j_0 \leqslant n$ with $\varphi_{j_0} > 0$. Consider any $0 \leqslant k \leqslant i - i_0$ and set $j := j_0 + nk$. Then §§ 10.5 and 10.9 imply

$$\varphi_{i+1,j} = q^{-k}\varphi_{i+1-k,j_0} = q^{-k}\varphi_{j_0}.$$

Therefore,

$$\log|\Delta w_i| \leqslant \lambda \leqslant q^{j_0+nk}(cniq^{-i} - q^{-k}\varphi_{j_0}) = -q^{(n-1)k}q^{j_0}(\varphi_{j_0} - cniq^{k-i}).$$

Now choose $k := \lceil i/2 \rceil$, the least integer greater than or equal to $i/2$, which is permitted if $i \geqslant 2i_0$. Then

$$\varphi_{j_0} - cniq^{k-i} \geqslant \varphi_{j_0} - cniq^{-(i-1)/2}.$$

There exists $i_2 \geqslant \sup\{i_1, 2i_0\}$ such that this is greater than or equal to $\varphi_{j_0}/2 > 0$ for all $i \geqslant i_2$. Under this condition it follows that

$$\log|\Delta w_i| \leqslant -q^{(n-1)k}q^{j_0}\varphi_{j_0}/2 \leqslant -q^{(n-1)i/2}q^{j_0}\varphi_{j_0}/2,$$

so the lemma holds with $c' := q^{j_0}\varphi_{j_0}/2$. □

10.16 Now we construct the desired sequence of zeroes w_i of F_i with $\log|w_i| = -\rho$ by induction. Let i_2 and c' be as in Lemma 10.15 and choose $i_3 \geqslant i_2$ such that $c'q^{(n-1)i_3/2} > \rho$. Let w_{i_3} be any zero of F_{i_3} with $\log|w_{i_3}| = -\rho$, using Lemma 10.13. If w_i has been constructed for $i \geqslant i_3$, we choose w_{i+1} as in Lemma 10.15. The inequalities

$$\log|w_{i+1} - w_i| \leqslant -c'q^{(n-1)i/2} \leqslant -c'q^{(n-1)i_3/2} < -\rho$$

then imply that $\log|w_{i+1}| = -\rho$. We may therefore repeat the construction indefinitely. Since $-c'q^{(n-1)i/2} \to -\infty$ as $i \to \infty$, the sequence w_i converges to an element $w \in \mathbb{C}$ with $\log|w| = -\rho \leqslant cn$.

Lemma 10.17. $\limsup_{i\to\infty} \log|F_i(w)| \leqslant 0$.

Proof. Note that $F_i(w) = F_i(w - w_i)$, and that Lemma 10.15 implies

$$|w - w_i| \leqslant -c'q^{(n-1)i/2}.$$

Thus, using § 10.6, we can estimate

$$\begin{aligned}\log|F_i(w)| &\leqslant \sup\{\varphi_{i,j} + q^{-j}\log|w - w_i| : 0 < j < ni\} \\ &\leqslant \sup\{q^{-j/n}cn - q^{-j+(n-1)i/2}c' : 0 < j < ni\}.\end{aligned}$$

The term $q^{-j/n}cn - q^{-j+(n-1)i/2}c'$ is less than or equal to zero if and only if

$$\frac{c'}{cn} \geqslant q^{j-j/n-(n-1)i/2} = q^{(n-1)(j-ni/2)/n},$$

that is, if $j \leqslant ni/2 + c''$ for a constant c''. For the remaining j this term is less than

$$q^{-j/n}cn \leqslant q^{-i/2-c''/n}cn,$$

which goes to zero as $i \to \infty$. Therefore,

$$\limsup_{i\to\infty} \log |F_i(w)| \leqslant \limsup_{i\to\infty} (\sup\{0, q^{-i/2-c''/n} cn\}) = 0,$$

as desired. This proves Lemma 10.17 and thereby finishes the proof of Propositions 9.5 and 9.1. ☐

11. Classification

After all these preparations we can now prove the main result of this article.

Theorem 11.1. *Every σ-bundle is isomorphic to one of the form $\bigoplus_{i=1}^k \mathcal{F}_{d_i,r_i}$ where the d_i and r_i are integers with $r_i > 0$ and $\gcd(d_i, r_i) = 1$.*

Proof. We prove the following assertions by induction on r.

A_r) For any σ-bundle $\mathcal{F}$ of rank r and degree zero we have $H^0(\mathcal{F}) \neq 0$.

B_r) Any σ-bundle of rank $\leqslant r$ is isomorphic to a direct sum $\bigoplus_i \mathcal{F}_{d_i,r_i}$.

C_r) For any non-zero σ-bundle $\mathcal{F}$ of rank $\leqslant r$ and degree $\geqslant 0$ we have $H^0(\mathcal{F}) \neq 0$.

For $r = 1$ these assertions hold by Theorem 5.4 and Proposition 3.1. So let $r \geqslant 2$.

Lemma 11.2. *Assertions B_{r-1} and C_{r-1} imply A_r.*

Proof. Let $\mathcal{F}$ be a σ-bundle of rank r and degree zero. By Theorem 4.1 there exists a σ-subbundle $\mathcal{O}(m) \cong \mathcal{F}' \subset \mathcal{F}$. The possible m for this are bounded above by Proposition 6.3 item a. Thus we may choose m maximal. Proposition 6.2 then implies that $\mathcal{F}'$ is saturated, and so we obtain an exact sequence of σ-bundles

$$0 \longrightarrow \mathcal{O}(m) \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0.$$

By assertion B_{r-1} we have $\mathcal{F}'' \cong \bigoplus_i \mathcal{F}_{d_i,r_i}$. Note that $0 = \deg \mathcal{F} = m + \deg \mathcal{F}'' = m + \sum_i d_i$ by Propositions 6.1 and 7.6.

If $r = 2$ we are done by Theorem 5.4 and Corollary 9.2, so we assume $r \geqslant 3$. If $m \geqslant 0$ we are done by Proposition 3.1, so we assume $m \leqslant -1$. Then we may select i such that $d_i \geqslant 1$. By Proposition 8.5 there exists an embedding $\mathcal{O}(0) \hookrightarrow \mathcal{F}_{d_i,r_i} \hookrightarrow \mathcal{F}''$. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(m) & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}'' & \longrightarrow & 0 \\ & & \| & & \bigcup & \square & \bigcup & & \\ 0 & \longrightarrow & \mathcal{O}(m) & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{O}(0) & \longrightarrow & 0 \end{array}$$

obtained via pull-back. By assertion B_{r-1} the σ-bundle $\mathcal{G}$ is isomorphic to $\mathcal{O}(m_1) \oplus \mathcal{O}(m_2)$ or to $\mathcal{F}_{m',2}$ for suitable m_1, m_2, or m'. In the first case we have $m = \deg \mathcal{G} = m_1 + m_2$ by Proposition 6.1. From the maximality of m we deduce that $m \geqslant \sup\{m_1, m_2\} \geqslant m/2$ and thus $m \geqslant 0$, contradicting the assumption $m \leqslant -1$. Therefore, $\mathcal{G} \cong \mathcal{F}_{m',2}$ for some odd m'. By Propositions 6.1 and 7.6 we then have $m = \deg \mathcal{G} = m'$; in particular, m is odd. By Proposition 8.5 there then exists a σ-subbundle $\mathcal{O}((m-1)/2) \hookrightarrow \mathcal{G} \subset \mathcal{F}$; hence the maximality of m implies $m \geqslant (m-1)/2$ and so $m \geqslant -1$. Therefore, $m = -1$.

Now if $\mathcal{F}''$ consists of just one summand $\mathcal{F}_{d_1,r_1}$, we have $d_1 = 1$ and the desired assertion is the content of Proposition 9.1. Otherwise select i such that $d_i \geqslant 1$ and consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(-1) & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}'' & \longrightarrow & 0 \\ & & \| & & \bigcup & \square & \bigcup & & \\ 0 & \longrightarrow & \mathcal{O}(-1) & \longrightarrow & \widetilde{\mathcal{G}} & \longrightarrow & \mathcal{F}_{d_i,r_i} & \longrightarrow & 0 \end{array}$$

obtained via pull-back. By the additivity of degrees we have $\deg \widetilde{\mathcal{G}} \geqslant 0$; hence $H^0(\widetilde{\mathcal{G}}) \neq 0$ by assertion C_{r-1}. Therefore, $H^0(\mathcal{F}) \neq 0$, as desired. ☐

Lemma 11.3. *Assertions B_{r-1} and A_r imply B_r.*

Proof. Let $\mathcal{F}$ be a σ-bundle of rank r and degree d. Then $\mathcal{G} := ([r]^*\mathcal{F})(-d)$ has rank r and degree zero, so by assertion A_r we have $\mathrm{Hom}(\mathcal{O}(d),[r]^*\mathcal{F}) \cong H^0(\mathcal{G}) \neq 0$. Choose a σ-subbundle $\mathcal{O}(d) \hookrightarrow [r]^*\mathcal{F}$ and consider the induced homomorphism $[r]_*\mathcal{O}(d) \hookrightarrow [r]_*[r]^*\mathcal{F} \cong \mathcal{F}^{\oplus r}$, using Proposition 7.4. Some component of it must be non-zero; hence there exists a non-zero homomorphism $\varphi : [r]_*\mathcal{O}(d) \to \mathcal{F}$. We distinguish two cases.

If φ is injective, its image is a σ-subbundle of $\mathcal{F}$ of the same rank and the same degree, which by Proposition 6.2 must be equal to $\mathcal{F}$. Setting $t := \gcd(d,r)$ this implies that

$$\mathcal{F} \cong [r]_*\mathcal{O}(d) \cong [r/t]_*([t]_*[t]^*\mathcal{O}(d/t)) \cong \mathcal{F}_{d/t,r/t}^{\oplus t}$$

using Proposition 7.4, as desired.

If φ is not injective, let $\mathcal{F}'$ be the saturation of $\mathrm{Im}(\varphi)$ in $\mathcal{F}$ and $r' < r$ be the common rank of $\mathrm{Im}(\varphi)$ and $\mathcal{F}'$. By assertion B_{r-1} we may write $\mathcal{F}' \cong \bigoplus_i \mathcal{F}_{d_i,r_i}$. Since $[r]_*\mathcal{O}(d)$ is semi-stable of weight d/r by Proposition 7.6, its quotient $\mathrm{Im}(\varphi)$ has weight $\geqslant d/r$. Thus by Proposition 6.2 we have $\mu(\mathcal{F}') \geqslant d/r$, and so we may fix an i with $d_i/r_i \geqslant d/r$. Consider the exact sequence of σ-bundles

$$0 \longrightarrow \mathcal{F}_{d_i,r_i} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{F}_{d_i,r_i} \longrightarrow 0.$$

By assertion B_{r-1} we may write $\mathcal{F}/\mathcal{F}_{d_i,r_i} \cong \bigoplus_j \mathcal{F}_{d'_j,r'_j}$. By the additivity of degrees we have $\mu(\mathcal{F}/\mathcal{F}_{d_i,r_i}) \leqslant d/r$; hence there is a j with $d'_j/r'_j \leqslant d/r \leqslant d_i/r_i$. Now Proposition 8.8 implies that the extension of $\mathcal{F}_{d'_j,r'_j}$ by $\mathcal{F}_{d_i,r_i}$ splits; hence $\mathcal{F} \cong \mathcal{F}_{d'_j,r'_j} \oplus \mathcal{G}$ for some σ-bundle $\mathcal{G}$ of rank $r - r'_j < r$. Applying assertion B_{r-1} to $\mathcal{G}$ yields B_r, as desired. ☐

Lemma 11.4. *Assertion B_r implies C_r.*

Proof. For $\mathcal{F} \cong \bigoplus_i \mathcal{F}_{d_i,r_i}$ we have $\deg \mathcal{F} = \sum_i d_i$ by Propositions 6.1 and 7.6. Thus if $\mathcal{F} \neq 0$ and $\deg \mathcal{F} \geqslant 0$, at least one of the d_i must be greater than or equal to zero. Therefore, Proposition 8.4 implies $H^0(\mathcal{F}) \neq 0$, as desired. ☐

The three lemmas above complete the proof of Theorem 11.1. ☐

Theorem 11.1 now permits us to generalize Proposition 7.5.

Corollary 11.5. *$[n]^*[n]_*\mathcal{F} \cong \mathcal{F}^{\oplus n}$ for any σ-bundle $\mathcal{F}$ and any positive integer n.*

Proof. By Theorem 11.1 it suffices to prove this for $\mathcal{F} = \mathcal{F}_{d,r}$. In this case Proposition 7.5 implies

$$[r]^*([n]^*[n]_*\mathcal{F}_{d,r}) \cong [rn]^*[rn]_*\mathcal{O}(d) \cong \mathcal{O}(d)^{\oplus rn} \cong [r]^*[r]_*\mathcal{O}(d)^{\oplus n} \cong [r]^*\mathcal{F}_{d,r}^{\oplus n}.$$

From this the corollary follows by Proposition 7.3. ☐

Corollary 11.6.

a) *The semi-stable σ-bundles are, up to isomorphism, exactly the direct sums $\mathcal{F}_{d,r}^{\oplus t}$.*
b) *The stable σ-bundles are, up to isomorphism, exactly the $\mathcal{F}_{d,r}$.*

Proof. Let $\mathcal{F}$ be a semi-stable σ-bundle of weight d/r with $\gcd(d,r) = 1$. Then $\mathcal{F} \cong \bigoplus_i \mathcal{F}_{d_i,r_i}$ by Theorem 11.1, and $d_i/r_i \leqslant d/r$ by semi-stability. This is possible only if all $(d_i,r_i) = (d,r)$; hence $\mathcal{F} \cong \mathcal{F}_{d,r}^{\oplus t}$. If $\mathcal{F}$ is stable, there can be only one summand; hence $\mathcal{F} \cong \mathcal{F}_{d,r}$. Conversely $\mathcal{F}_{d,r}$ is stable by Proposition 8.2, and

$$\mathcal{F}_{d,r}^{\oplus t} \cong [r]_*([t]_*[t]^*\mathcal{O}(d)) \cong [rt]_*\mathcal{O}(td)$$

is semi-stable by Proposition 7.6. ☐

Corollary 11.7. *Let $\mathcal{F} = \bigoplus_i \mathcal{F}_{d_i,r_i}$. Then the Harder–Narasimhan filtration of $\mathcal{F}$ is given by the descending chain of σ-subbundles*

$$\mathcal{F}^{(\mu)} := \bigoplus_{i:d_i/r_i \geqslant \mu} \mathcal{F}_{d_i,r_i}.$$

Corollary 11.8. *The pairs (d_i, r_i) in Theorem 11.1 are uniquely determined up to a permutation.*

Remark 11.9. To every σ-bundle we can therefore associate its *Harder–Narasimhan polygon* (cf. Shatz [Sha77]) in the coordinate plane, which begins at the origin and is composed of a line segment of horizontal width r_i and slope d_i/r_i for every pair (d_i, r_i), arranged in ascending order of slopes. By construction all break points and end points of this polygon have integral coordinates, and clearly all such upper convex finite polygons occur for some σ-bundle. Conversely, Theorem 11.1 states that the isomorphy class of a σ-bundle is uniquely determined by its Harder–Narasimhan polygon.

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Vector bundles with a Frobenius structure

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