A quick proof that $K\# -K$ is doubly slice

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Throughout this short note we fix a category CAT where $CAT = Diff$, PL or TOP. An *n-knot* is a morphism $K\colon S^n \hookrightarrow S^{n+2} = \mathbb{R}^{n+2} \cup \{\infty\}$. As usual we denote the oriented manifold $K(S^n)$ by K. We denote by $-K$ the knot which is given by reflecting K in a hyperplane and by reversing the orientation. We say that a knot K is *doubly slice* if K is the intersection of a trivial $(n+1)$-knot in S^{n+3} with the equator sphere $S^{n+2} \subset S^{n+3}$. Note that an $(n+1)$-knot $J \subset S^{n+3}$ is trivial if J is the boundary of an $(n+2)$-ball in S^{n+3}.

Zeeman [Zee65, p. 487] introduced the ± 1-twist spin $S_{\pm 1}(K)$ of a knot $K \subset S^{n+2}$ and showed that it is a trivial knot in S^{n+3}. Sumners [Sum71, Corollary 2.9] observed that this result of Zeeman's almost immediately implies the following theorem.

Theorem 0.1. If $K\colon S^n \hookrightarrow S^{n+2}$ is a knot, then $K\# -K$ is doubly slice.

An unscientific poll among the authors and a wider group of topologists showed that the statement of Theorem 0.1 is well-known but that the proof is less well understood, and is shrouded in mystery. In this short note we present a self-contained proof of this theorem, similar in spirit to Zeeman. In fact, if one looks up the the definition of the ± 1-twist spun of a knot, then one can easily see that along the way we in fact reprove Zeeman's theorem that the ± 1-twist spuns of a knot are trivial. This is not the first reproof of this theorem and in [GK78, Corollary 1.11] a different technique is applied to recover the same result. It has been brought to the authors' attention that a proof very similar to our appears in the proof of [Lev83, Theorem C].

The idea of the proof is quite simple. We will embed the *knot exterior* $X = \overline{S^{n+2} \setminus (K \times D^2)}$ into $S^{n+3} = \mathbb{R}^{n+3} \cup \{\infty\}$ in such a way that a copy of $S^{n+2} = \mathbb{R}^{n+2} \cup \{\infty\}$ cuts it along $K \sqcup -K$. We then show that it is possible to glue a 2-handle along a meridian of the embedded X to form an embedded D^{n+3} that intersects S^{n+2} along $K\# -K$. The technical difficulties arise in showing that the 2-handle can be embedded along with X.

Algebraically, slice knots correspond to metabolic Seifert forms and doubly slice knots correspond to hyperbolic Seifert forms. As a simple algebraic application of Theorem 0.1, when n is odd, we use our proof to write down two complementary lagrangian submodules for a Seifert form of $K\# -K$. These lagrangians were algebraically observed as early as [Ran73, Lemma 1.4] but this is the first direct geometric description of them.

Proof. Given any k we define $\mathbb{R}^k_{\geq 0} = \{(x_0, \dots, x_k) \in \mathbb{R}^k \,|\, x_k \geq 0\}$ and we similarly define $\mathbb{R}^k_{>0}$, $\mathbb{R}^k_{\leq 0}$, $\mathbb{R}^k_{<0}$ and $\mathbb{R}^k_{=0}$. We then consider the map

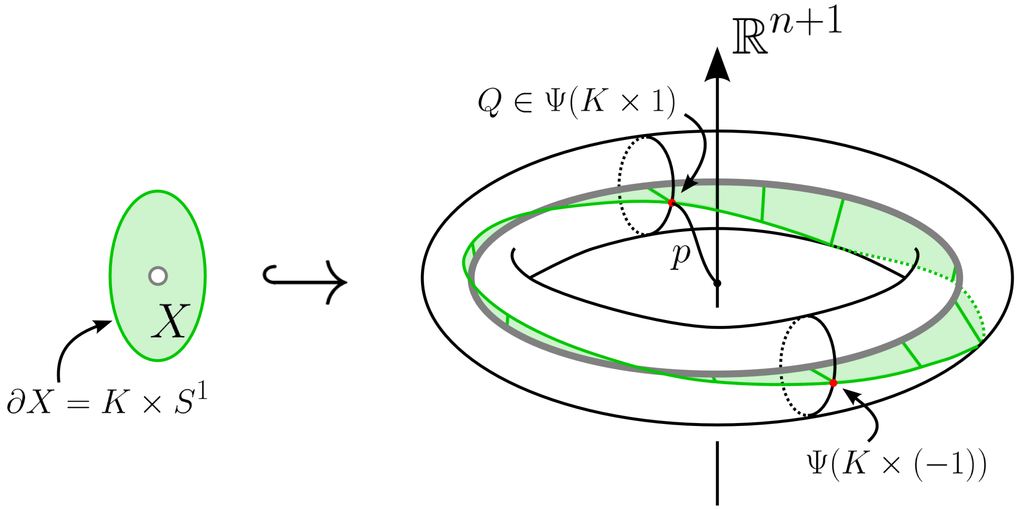
$$\begin{array}{rcl} \Phi\colon \mathbb{R}^{n+2}_{\geq 0} \times S^1 & \to & \mathbb{R}^{n+1} \times \mathbb{C} = \mathbb{R}^{n+3} \\ ((x_1, \dots, x_{n+2}), z) & \mapsto & (x_1, x_2, \dots, z x_{n+2}). \end{array}.$$

Note that the map $x \mapsto \Phi(x, -1)$ is just the reflection of $\mathbb{R}^{n+2}_{\geq 0}$ in the plane $\mathbb{R}^{n+2}_{=0}$.
Now let $K : S^n \hookrightarrow S^{n+2} = \mathbb{R}^{n+2} \cup \{\infty\}$ be a knot. We pick a point Q on K and we pick a tubular neighbourhood $K \times D^2$ of K. Recall the knot exterior $X = \overline{S^{n+2} \setminus (K \times D^2)}$. Note that after a slight isotopy of K and by 'increasing the size of the tubular neighbourhood' we can and will assume that $X \cup (Q \times D^2)$ lies outside the $(n+2)$-ball $\mathbb{R}^{n+2}_{\leq 0} \cup \{\infty\}$, i.e. we can assume that $X \cup (Q \times D^2)$ lies in $\mathbb{R}^{n+2}_{>0}$.
We write $K' = K \times 1$ and define $-K'$ as $\Phi(K \times -1, -1)$ with the reversed orientation. Note that $-K'$ is indeed the inverse of K' up to isotopy. Furthermore note that K' lies in $\mathbb{R}^{n+2}_{>0}$ and $-K'$ lies in $\mathbb{R}^{n+2}_{<0}$.
We now pick an embedded path $p\colon [0, \frac{1}{2}] \to \mathbb{R}^{n+2}_{\geq 0}$ from the origin to the point Q which intersects $X \cup Q \times D^2$ only in the point Q. Given $z \in S^1$, we denote by $q_z\colon [\frac{1}{2}, 1] \to Q \times D^2$ the straight path from Q to $Q \times z$. We then denote by $pq_z\colon [0,1] \to \mathbb{R}^{n+2}_{\geq 0}$ the concatenation of the paths p and q_z. Note that this is a path from the origin to $Q \times z$ with no self-intersections.
We now denote by $\overline{pq_1}$ the path pq_1 with the opposite orientation. The concatenation of the two paths $\overline{pq_1}$ and $\Phi(pq_{-1}, -1)$ is then a path from K' to $-K'$. After a slight perturbation we can assume that this is a CAT-morphism and we use this path to perform the connect sum $K'\# -K'$. In order to prove the theorem it suffices to show that there exists a manifold $Y \cong D^{n+3}$ in $\mathbb{R}^{n+3}$ such that $Y \cap \mathbb{R}^{n+3}_{=0} = K'\# -K'$.
We will first find an embedding $\Psi\colon X \to \mathbb{R}^{n+3}$ such that $\Psi(X) \cap \mathbb{R}^{n+3}_{=0} = K' \sqcup -K'$. We will then show that we can attach a 2-handle to $\Psi(X)$ along the meridian $\Psi(Q \times S^1)$. The resulting manifold Y is then an $(n+2)$-ball and we will see that $Y \cap \mathbb{R}^{n+3}_{=0} = K'\# -K'$.
To carry out this idea, first note that by elementary obstruction theory there exists a map $f\colon X \to S^1$ which has the property that $f|_\partial\colon \partial X = K \times S^1 \to S^1$ is the projection to the

second factor. We then consider the embedding

$$\begin{aligned} \Psi\colon X &\to \mathbb{R}^{n+3} \\ x &\mapsto \Phi(x, f(x)). \end{aligned}$$

Note that $\Psi(x) = \Phi(x, f(x))$ lies in $\mathbb{R}^{n+3}_{=0}$ if and only if $f(x) = 1$ or $f(x) = -1$. We now



see that

$$\partial(\Psi(X)) \cap \mathbb{R}^{n+3}_{=0} = \Psi(\partial X) \cap \mathbb{R}^{n+3}_{=0} = \Psi(K \times S^1) \cap \mathbb{R}^{n+3}_{=0} = \Psi(K \times 1) \sqcup \Psi(K \times -1) = K' \sqcup -K'.$$

We now consider the continuous map

$$\begin{aligned} g\colon D^2 &\to \mathbb{R}^{n+3} \\ rz &\mapsto \Phi((pq_z)(r), z) \end{aligned}$$

where $r \in [0,1]$ and $z \in S^1$. It is straightforward to see that this map is injective and continuous. After a small perturbation we can assume that this map is furthermore a CAT morphism.
Note that

$$g(S^1) = \{\Phi(Q \times z, z) \,|\, z \in S^1\} = \{\Psi(Q \times z) \,|\, z \in S^1\} = \Psi(Q \times S^1).$$

It follows easily from the definitions that $g(D^2)$ intersects $\Psi(X)$ precisely in $g(S^1) = \Psi(Q \times S^1)$. We can therefore thicken up the disk $g(D^2)$ and attach it as a $(n+2)$-dimensional 2-handle to the $(n+2)$-dimensional manifold $\Psi(X)$ along $g(S^1) = \Psi(Q \times S^1) \subset \Psi(\partial X)$. We denote the resulting manifold by Y. The result of attaching a 2-handle along a meridian gives a manifold which is the complement of an open $(n+2)$-ball in S^{n+2}, i.e. Y is a closed $(n+2)$-dimensional ball.

On the other hand, note that we changed the intersection of $\Psi(X)$ with $\mathbb{R}^{n+3}_{=0}$ by attaching a 1-handle to $K' \sqcup -K'$. It follows that $Y \cap \mathbb{R}^{n+3}_{=0} = K' \# - K'$. ■

Remark. In the proof, given a knot K we constructed an $(n+2)$-dimensional manifold Y in $\mathbb{R}^{n+3}$ with $Y \cong D^{n+2}$. It follows easily from the definitions that the boundary of Y is precisely the $+1$-twist spun $S_{+1}(K)$ of Zeeman. We thus gave in particular a proof that $+1$-twist spun knots are trivial. If we replace the map Ψ in the proof by $x \mapsto \Phi(x, f(x)^{-1})$, then the same proof shows that $S_{-1}(K)$ is also a trivial knot.

Some algebra of Seifert forms

We now address the corresponding algebra. Let $\varepsilon = \pm 1$ and A be a ring with involution. Consider a pair (P, ψ) consisting of a finitely generated, projective A-module P, together with an A-module morphism $\psi : P \to P^* = \mathrm{Hom}_A(P, A)$ such that $\psi + \varepsilon\psi^* : P \to P^*$ is an isomorphism. We say that (P, ψ) is a *non-singular ε-symmetric Seifert form* over A, cf. [Ran03]. Define $e = (\psi + \varepsilon\psi^*)^{-1}\psi : P \to P$ and note that this action induces an $A[e]$-module structure on P. A *lagrangian* of (P, ψ) is an $A[e]$-submodule $j : L \hookrightarrow P$ such that the sequence

$$0 \longrightarrow L \xrightarrow{\quad j \quad} P \xrightarrow{j^*(\psi + \varepsilon\psi^*)} L^* \longrightarrow 0$$

is an exact sequence of $A[e]$-modules (in other words, $j(L)^\perp = j(L)$ with respect to $\psi + \varepsilon\psi^*$). If a Seifert form (P, ψ) admits a lagrangian, it is called *metabolic*. If a Seifert form admits two lagrangians $j_\pm : L_\pm \hookrightarrow P$ such that $(j_+ \ j_-) : L_+ \oplus L_- \to P$ is an isomorphism of A-modules, the lagrangians are called *complementary* and the form is called *hyperbolic*.

Lemma 0.2 ([Ran73, Lemma 1.4])**.** For any non-singular ε-symmetric Seifert form (P, ψ), the form $(P \oplus P, \psi \oplus -\psi)$ is hyperbolic.

Proof. The following are lagrangian submodules

$$j_- = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ : \ (P, 0) \to (P \oplus P, \psi \oplus -\psi),$$

$$j_+ = \begin{pmatrix} e \\ e-1 \end{pmatrix} \ : \ (P, 0) \to (P \oplus P, \psi \oplus -\psi).$$

They are complementary as the matrix $(j_+ \quad j_-)$ has invertible determinant. ■

Example 0.3. Given a knot $K : S^n \hookrightarrow S^{n+2}$, a *Seifert manifold* for K is a codimension 1 submanifold $F^{n+1} \subset S^{n+2}$ with boundary $\partial F = K$. If $n = 2q - 1$ and $\varepsilon = (-1)^q$, then we can associate to F a non-singular, ε-symmetric Seifert form $(H^q(F;\mathbb{Z}), \psi)$ such that $\psi + \varepsilon\psi^*$ is the adjoint of the middle-dimensional intersection pairing on F and $\psi : H_q(F) \to H_q(F)^*$ is the adjoint of the Seifert matrix for F.

We will now relate the two lagrangians of Lemma 0.2 to the embedding $\Psi : X \hookrightarrow S^{n+3}$ of Theorem 0.1.

Let $I \subset S^1$ be a closed arc between 1 and -1. We may assume $f : X \to S^1$ from the proof of Theorem 0.1 is transverse regular at some $x \in I$ so that $f^{-1}(x) = F$ is a Seifert manifold for K. We may rescale f so that $f^{-1}(I) = F \times [0,1]$, a tubular neighbourhood of $F \subset X$. Define $X_F = \overline{X \backslash (F \times [-1,1])}$, denote the inclusion of the two ends of X_F by $i_\pm : F_\pm \hookrightarrow X_F$ and recall that the maps $(i_+ - i_-)_*$ and $(i_+ - i_-)^*$ are isomorphisms on reduced homology and cohomology respectively (by Alexander duality). As X_F is a cobordism (rel boundary) from F to F, it determines a lagrangian submodule for the middle dimensional intersection pairing on $F \sqcup -F$ given by

$$j_+ = \begin{pmatrix} (i_+ - i_-)^{-1} i_+ \\ (i_+ - i_-)^{-1} i_- \end{pmatrix} : H_q(F) \to H_q(F) \oplus H_q(F),$$

coming from the composition

$$H_q(F) \cong H^q(F) \xrightarrow{((i_+ - i_-)^*)^{-1}} H^q(X_F) \xrightarrow{\begin{pmatrix} i_+^* \\ i_-^* \end{pmatrix}} H^q(F) \oplus H^q(F) \cong H_q(F) \oplus H_q(F).$$

Let $\psi : H_q(F) \to H_q(F)^*$ be the adjoint map for the Seifert matrix of F. ψ has the property that $\psi(a) = (\psi + \varepsilon\psi^*)((i_+ - i_-)^{-1} i_+(a))$ (see e.g. [Far83, Chapter 1, §1]) so that we may write the lagrangian above as

$$j_+ = \begin{pmatrix} (\psi + \varepsilon\psi^*)^{-1}\psi \\ (\psi + \varepsilon\psi^*)^{-1}\psi^* \end{pmatrix} : H_q(F) \to H_q(F) \oplus H_q(F).$$

Note that the excised $F \times [0,1] \subset X$ is also a cobordism (rel boundary) from F to F and repeating the reasoning above we obtain the trivial lagrangian submodule

$$j_- = \begin{pmatrix} 1 \\ 1 \end{pmatrix} : H_q(F) \to H_q(F) \oplus H_q(F).$$

Now the embedding $\Psi : X \hookrightarrow X \times S^1 \subset \mathbb{R}^{n+3}$ from the proof of Theorem 0.1 has the property that $\Psi(X) \cap \mathbb{R}^{n+3}_{=0} = \Psi(F_-) \sqcup \Psi(F_+)$ so that $\Psi(X_F)$ is an embedded cobordism from $\Psi(F_+)$ to $\Psi(F_-)$ (relative to K) and, similarly, $\Psi(F \times [0,1])$ is an embedded cobordism from $\Psi(F_+)$ to $\Psi(F_-)$ (relative to K). That the lagrangians $j_\pm$ are complementary corresponds to fact that the corresponding slice disks glue together to make $Y \cong D^{n+3}$, a contractible space.

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