Thursday, January 3, 2019 11:16 AM

1) What do we want?

Given a cyclic A_∞ algebra A, let $V = HH_*(A)$ (e.g., if A is something like Fukaya category, V is like de Rham cohomology).

Fix a splitting of HdR:

$$r: HH_*(A)((u)) \xrightarrow{\sim} HP_*(A) \qquad (u = \text{formal var of degree } -2)$$

Given this data want to construct a categorical GW potential

$$F^{cat} \in \mathrm{Sym}(V^-)[[\lambda]]((\hbar))$$

where $V^- := u^{-1}V[u^{-1}]$.

λ encodes the Euler characteristic $\lambda^{-\chi} = \lambda^{2g-2+n}$
$\hbar$ encodes the genus $\hbar^{g-1}$
u^{-i-1} corresponds to τ_i (doing a ψ^i descendant)
use Mukai pairing to do insertions.

2). How do we want to construct F^{cat}?

First idea: it should solve a certain differential eqn (Quantum Master Eqn)

$$dF^{cat} + \hbar \Delta F^{cat} + \frac{1}{2}\{F^{cat}, F^{cat}\} = 0.$$

which models the combinatorics of the boundary strata of $\overline{\mathcal{M}}_{g,n}$.

For this the space $\mathrm{Sym}(V^-)[[\lambda]]((\hbar))$ should have a (dg) BV algebra structure:

- commutative (dg) algebra
- $\Delta^2 = 0$, $d\Delta + \Delta d = 0$, Δ of order 2
- $\{x, y\} = \Delta(xy) - x\Delta y - y\Delta x$ is of order 1 in each argument

Problems:

1). This QME only defined at chain level: replace $V = HH_*(A)$ by $V_A = C_*(A)$ (Hochschild chains)
$d = b + uB$, Δ and $\{-,-\}$ come from Deligne conjecture.

2). The solution to the QME in $\mathcal{F}_A := \mathrm{Sym}(V_A^-)[[\lambda]]((\hbar))$ is <u>not</u> unique if we only impose the obvious boundary condition (analogue of saying that the 3-point function induces the product on Hochschild cohomology).

3). A solution of the QME gives a cohomology class for $d + \hbar\Delta$:

$$\text{QME} \iff (d + \hbar\Delta) e^{\frac{F^{cat}}{\hbar}} = 0.$$

What we really wanted was a cohomology class for just $d = b + uB$ (this would give us a potential in

$$\mathrm{Sym}(HC^-)[[\lambda]]((\hbar))$$

and we could use H-dR splitting to get what we want).

3). Next idea: first solve the QME in a different BV algebra where it <u>has</u> a unique solution, and which maps to our BV algebra.

<u>Theorem 1</u>: (Zwiebach-Sen, Costello)

a) There is a BV algebra structure on

$$\mathcal{F}_C := \mathrm{Sym}(V_C^-)[[\lambda]]((\hbar))$$

where

$$V_C = \bigoplus_{g,n} C_*(\mathcal{M}_{g,n}^{fr}/\Sigma_n)$$

$d = b + uB$

$\bullet$ = symmetric algebra product (= disjoint union)

Δ = "twist sew. (explain)

b). The QME has a unique solution S ("the string vertex") in $\mathcal{F}_C$ if we impose the boundary condition

$$S_{0,3} = [\mathcal{M}_{0,3}/S_3]$$

c) There "should be" a map of BV algebras

$$\mathcal{F}_C \longrightarrow \mathcal{F}_A$$

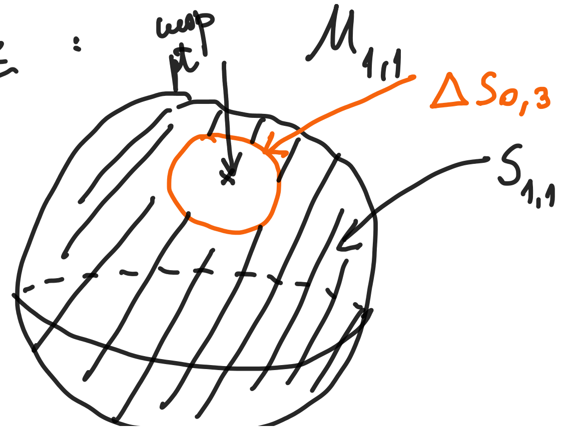
Define $F^{abs} \in \mathcal{F}_A$ to be the image of S. This is a distinguished solution of the QME in $\mathcal{F}_A$.

4). Some explanations for the above theorem:

a) What are the string vertices: they are non-closed singular chains that are the fundamental classes of

$$\overline{\mathcal{M}}_{g,n} \setminus \{\text{open } \varepsilon\text{-nbhd of } \partial\overline{\mathcal{M}}_{g,n}\}$$

Example: $\overline{\mathcal{M}}_{1,1}$



<u>QME:</u> $dS_{1,1} = \Delta S_{0,3}$

b) Existence and uniqueness of S (i.e. uniqueness of the $(d+\hbar\Delta)$ class $e^{S/\hbar}$) follows from the fact that

$$H_{6g-7-2n}(\mathcal{M}_{g,n}) = 0.$$

For example $\Delta S_{0,3} \in C_1(\mathcal{M}_{1,1})$. From BV algebra properties

$$d(\Delta S_{0,3}) = \Delta(dS_{0,3}) = 0$$

($S_{0,3} = [\mathcal{M}_{0,3}/\Sigma_3]$ is closed). $\Rightarrow$ $\Delta S_{0,3}$ gives class in $H_1(\mathcal{M}_{1,1}) = 0$ $\Rightarrow$ $\Delta S_{0,3} = d(S_{1,1})$ for some $S_{1,1} \in C_2(\mathcal{M}_{1,1})$

c) The "should be" part of the theorem would follow from the following result of Kontsevich - Soibelman / Costello:

<u>Theorem</u>: There is an action of the PROP of $C_*(\mathcal{M}_{g,n}^{fr})$ on $C_*(A)$ for a cyclic A_∞-algebra A.

boundary of chains = b
circle rotation = B (Connes)
Δ, $\{-,-\}$ = corresponding operators.

(Proof of theorem based on identifying $C_*(\mathcal{M}_{g,n}^{fr})$ with a complex of ribbon graphs and describing an action of these)

<u>Problem</u>: In this PROP the n-points are marked, and at least one must be an <u>input</u>. We can only sew inputs to outputs (or, in fact, with some work, also two outputs) but not two inputs.

So we can not really define Δ which is defined as sewing any pair of markings.

We'll fix this in next lecture.

5) How do we address the problem that we have only constructed a $(d+\hbar\Delta)$-closed class and not a d-closed one?

We can see that the problem is in the picture of $S_{1,1}$: we should think of taking the image of S as "integrating". But we have not integrated along all of $\overline{\mathcal{M}}_{1,1}$ (which would be closed for d) but only along $S_{1,1}$ which is not closed.

What we need to do is add the contribution of , a little disk in $\overline{\mathcal{M}}_{1,1}$ centered at the cusp and whose boundary is $\Delta S_{0,3}$. In other words we need to choose an operator H whose boundary is Δ.

This is <u>not</u> possible in $\mathcal{F}_c$ (obviously, since to fill in the disk we would need the nodal curve at the cusp) but it <u>can</u> be done in $\mathcal{F}_A$, with a choice of splitting

of H-dR:
let

$$\tilde{R} : (C_*(A), b) \longrightarrow (C_*(A)[[u]], b + uB)$$

be a chain level lift of r. such that

$$\tilde{R}(x) = x + \tilde{R}_1(x)u + \tilde{R}_2(x)u^2 + \dots$$

Then we claim that the map $\Delta^{\tilde{R}} : \mathrm{Sym}^2(C_*(A)u^{-1}) \to \mathbb{K}$ given by

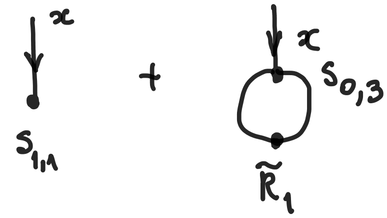
$$(xu^{-1}) \cdot (yu^{-1}) \xmapsto{\Delta^{\tilde{R}}} \frac{1}{2}\left[\langle \tilde{R}_1(x), y \rangle + \langle x, \tilde{R}_1(y) \rangle \right]$$

satisfies

$$\left((b + uB)\,\Delta^{\tilde{R}}\right)\left[(xu^{-1}) \cdot (yu^{-1})\right] = \langle Bx, y \rangle$$

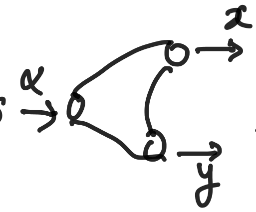
so $\Delta^{\tilde{R}}$ bounds the twist sewing.

So if we define

$F^{cat}_{1,1}(x) =$ 

we should get a $(b+uB)$ closed operation, accounting for all of $\overline{\mathcal{M}}_{1,1}$.

Explicitly to compute the 1,1 invariant with insertion a certain Hochschild homology class α we should insert it in

- $S_{1,1}$
- also insert it in $S_{0,3}$ thought of as 

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