Monday, January 7, 2019 9:46 PM

How to deal with the problem that ribbon graphs must have one input?

Idea (due to Costello):

Let V be a vector space on which Σ_n acts.
For $k+l=n$ define

$$V_{k,l} = V \otimes_{\Sigma_k \times \Sigma_l} (\mathrm{Alt}_k \otimes \mathbb{1}_l)$$

Then there exists an exact complex

$$0 \to V_{0,n} \to V_{1,n-1} \to \cdots \to V_{n,0} \to 0$$

(Think of Koszul resolution of k over $k[x_1, \ldots, x_k]$ and keep only the degree n piece.)

Now $C_*(\mathcal{M}^{fr}_{g,n})$ is acted on by Σ_n. We can replace

$$V = C_*(\mathcal{M}^{fr}_{g,n}) = C_*(\mathcal{M}^{fr}_{g,n}) \otimes \mathbb{1}_n$$

by the above Koszul resolution

$$C_*(\mathcal{M}^{fr}_{g,1,n-1}) \xrightarrow{\iota} C_*(\mathcal{M}^{fr}_{g,2,n-2}) \xrightarrow{\iota} \cdots \xrightarrow{\iota} C_*(\mathcal{M}^{fr}_{g,n,0})$$

(each is itself a complex; take the total complex)

- $C_*(\mathcal{M}_{g,k,\ell})$ = chains on moduli space of curves of genus g, with k framed inputs (anti-symmetrized) and ℓ framed outputs (symmetrized)
- ι = turn an output into an input
- Each marking is labeled with a negative power of u
- still have b, uB
- total differential is $b + uB + \iota$.

What have we accomplished?

- $C_*(\mathcal{M}^{fr}_{g,k,\ell})$ can be described combinatorially by means of ribbon graphs
- can sew (combinatorially) any two outputs on the same graph, or an input to an output on different graphs, or an input to an output on same graph as long as they are not overlapping.

<u>Thm</u>: Consider $\mathfrak{g} = \bigoplus_{\substack{g \geq 0 \\ k \geq 1 \\ \ell \geq 0}} C_*(\mathcal{M}^{fr}_{g,k,\ell})[[\hbar]]$ with operators

Δ = twist sew two outputs

$$\{-,-\} = \{-,-\}_0 + \hbar\{-,-\}_1 + \hbar^2\{-,-\}_2 + \dots$$

twist sew one input to one output

two-two ...

Then $(\mathfrak{g},\ b + uB + \iota + \hbar\Delta,\ \{-,-\})$ forms a dgla

and $\uparrow \iota$

$$\left(\bigoplus_{\substack{g\geq 0\\ n\geq 0}} C_*(\mathcal{M}_{g,0,n}),\ b + uB + \hbar\Delta,\ \{-,-\}\right) \text{ is a qiso of}$$

dgla's.

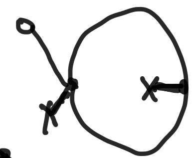
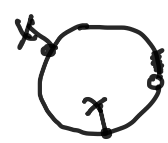
<u>Note</u>: In $\mathfrak{g}$ we can explicitly calculate (with ribbon graphs) and solve the MC eqn explicitly

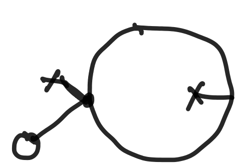
The solution (string vertex) S will be of the form

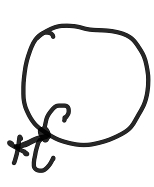
$$S = \sum_{g,k,l} S_{g,k,l}\, \hbar^{g-1} \lambda^{-\chi}$$

where $S_{g,k,l}$ is a linear combination, with rational coeff, of ribbon graphs with k faces (inputs) and l white vertices (outputs).

<u>Ex</u>: $S_{0,1,2} = \frac{1}{2}$  $\left(\frac{1}{2} = 3 \cdot \frac{1}{6}\right)$

$S_{0,2,1} = -\frac{1}{2} \cdot$  $- \frac{1}{4}$ 

$+ \frac{1}{4}$  $- \frac{1}{2}$ 

$S_{1,1} = \frac{1}{4}$  $- \frac{1}{24}$ 

First attempt at defining Costello's invariants:

Notation: if V is a graded vector space, define

$$V^{\pm} := V((u))$$
$$V^{+} := V[[u]]$$
$$V^{-} := V^{\pm}/V^{+} \simeq u^{-1}V[u^{-1}].$$

If V has a pairing, get residue pairing on $V^{\pm}$.

<u>Define</u>: $V := HH_*(A)$ $\quad L := C_*(A)$, both have a pairing.

On L also have $b, B \rightsquigarrow$ action of category of annuli.

Have a sequence of maps once we fix a splitting $\tilde{R} : V \to L^+$ respecting the pairing:

$$\begin{tikzcd}
W(V^{\pm}) \arrow[r, "\tilde{R}"] & W(L^{\pm}) \arrow[d] & \\
\mathrm{Sym}(V^-) \arrow[u] \arrow[r, dashed, "\text{qiso}"] & \mathrm{Sym}(L^-) \arrow[r, "\text{qiso}"', "\iota"] & \mathrm{Hom}(\Lambda^{\geq 1}L^+, \mathrm{Sym} L^-) \\
\rotatebox{90}{$\in$} & b+uB & b+uB+\iota \\
 & & \rotatebox{90}{$\in$}
\end{tikzcd}$$

Find: $\quad F^{cat} \longmapsto \varphi_* S + \text{gauge}$

Could define it abstractly by finding an L_∞ qiso inverse of ι. (Such a thing exists, but requires choices:

e.g. $L^- \xrightarrow{\iota} \mathrm{Hom}(L^+, k) \simeq (L^-)^{\vee\vee}$ need to invert.)