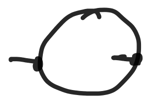
Tuesday, January 8, 2019 10:51 PM

Day 3:

First let me recap what we want from a splitting of H-dR.

The Kontsevich - Soibelman construction gives a map associated to  : $C_*(A) \otimes C_*(A) \longrightarrow k$

(chain level Mukai pairing).
It descends to homology $HH_*(A) \otimes HH_*(A) \longrightarrow k$.

If $\alpha, \beta \in C_*(A)((u))$ define $\langle \alpha, \beta \rangle_{hres} \in k((u))$ by

$$\langle \alpha, \beta \rangle_{hres} = \sum_{i+j} (-1)^i \langle \alpha_i, \beta_j \rangle \, u^{i+j}$$

<u>Def</u>: A good splitting of the Hodge filtration for A is an isomorphism

$$HH_*(A)((u)) \xrightarrow[R]{\sim} HP_*(A)$$

respecting the higher residue pairings.

Equivalently, for each $\alpha \in HH_*(A)$ we want to choose $\tilde{\alpha} \in HC_-(A)$

$$\tilde{\alpha} = \alpha + R_1(\alpha) + \dots$$

s.t. $\langle \tilde{\alpha}, \tilde{\beta} \rangle_{\mathrm{hres}} = \langle \alpha, \beta \rangle_{\mathrm{Mukai}}$

What does this get us?

We have a pairing on $V^{\pm} \simeq HH_*((u))$ given by

$$\langle \alpha u^i, \beta u^j \rangle = (-1)^i \langle \alpha, \beta \rangle \delta_{i+j,-1}.$$

and a similar one on $L^{\pm} = C_*(A)((u))$

Define $W(V^{\pm}) := T(V^{\pm}) \big/ \langle xy - yx = \langle x,y \rangle \rangle$, $W(L^{\pm})$

$\mathcal{F}(V^{\pm}) = W(V^{\pm}) \big/ \langle \alpha u^i \mid i \geq 0 \rangle$, same for $\mathcal{F}(L^{\pm})$

$$\mathcal{F}(V^{\pm}) \simeq \mathrm{Sym}(V^-) \qquad \mathcal{F}(L^{\pm}) \simeq (\mathrm{Sym}(L^-), b + uB + \Delta)$$

The splitting gives us a map

$$\begin{tikzcd}
 & W(V^{\pm}) \arrow[r, "R"] & W(L^{\pm}) \arrow[d] & \\
\mathrm{Sym}(V^-) = & \mathcal{F}(V^{\pm}) \arrow[u] \arrow[r, dashed, "\mathrm{qiso}"] & \mathcal{F}(L^{\pm}) \simeq (\mathrm{Sym}(L^-), b + uB + \Delta, \{-,-\}) &
\end{tikzcd}$$

BV algebra

$\exists!\ F^{dss}$ (up to gauge) $\downarrow$ MC.

$\Updownarrow$

$(b + uB + \Delta) e^{F^{dss}} = 0$

qiso $\downarrow$ dgla map

$$\left(\mathrm{Hom}(\Lambda^{\geq 1} L^+, \mathrm{Sym}^* L^-), b + uB + \iota + \Delta, \{-,-\} \right)$$
$$\cup$$
$$\varphi_* S \quad \text{M.C. elt.}$$

Universal formula for $S_{1,1}$:

Lemma: Assume we have a chain level lift $\tilde{R}$ of R

$$\tilde{R} : C_*(A) \longrightarrow C_*(A)[[u]]$$
$$\alpha \longmapsto \alpha + \tilde{R}_1(\alpha) u + \ldots$$

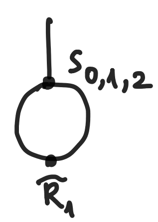
$$\Delta^{\tilde{R}} : \mathrm{Sym}^2(C_*(A) u^{-1}) \longrightarrow k$$

$$(xu^{-1})(yu^{-1}) \longmapsto \left[\langle \tilde{R}_1 x, y \rangle + \langle x, \tilde{R}_1 y \rangle \right] / 2$$

Then $\left((b + uB) \Delta^{\tilde{R}} \right) \left((xu^{-1})(yu^{-1}) \right) = \langle Bx, y \rangle$.

In pictures $d \Big($  $\Big) =$  $\leftarrow$ twist sew.

... discuss $\overline{M}_{1,1}$

Thm:  $+$  is $(b + uB)$ closed.

and ... maps to something gauge equiv. to $(\varphi_* S)_{1,2}$

(Note it is not equal: it only has terms w/ one input!!)

Elliptic curve calculation:

$E_\tau := \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau \qquad D^b(E_\tau) \simeq D^b(\mathcal{A}_\tau\text{-mod})$

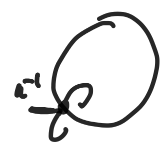
A_τ = 6d cyclic $\mathcal{A}_\infty$ algebra $= \mathrm{Ext}^*_{E_\tau}(\mathcal{O}\oplus\mathcal{O}_{(p)}, \mathcal{O}\oplus\mathcal{O}_{(p)})$

ξ = elt in $\mathcal{A}_\tau$ of degree (-1) homological, generator of $H^1(E_\tau, \mathcal{O})$.

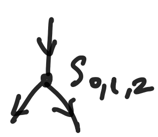
$[\xi] \in HH_{-1}(\mathcal{A}_\tau)$

To compute (1,1) inv't of E_τ: find correct lift to cyclic homology:

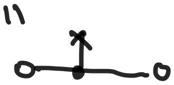
$$[\xi] \xmapsto{R} \xi + R_1(\xi)\,u + \text{h.o.t.}$$

Insert this into $S_{1,1}$:   : 0 (easy)

 insert $R_1(\xi)$: 0 (hard computer calculation) or other argument

$S_{0,1,2}$:  : $[\xi] \longmapsto \frac{1}{2}[\xi]\otimes[\xi]$. easy

$\shortparallel$

$\frac{1}{2}$ 

$$\langle R_1(\xi), \xi \rangle = \frac{1}{2} \langle \alpha, \xi \rangle = -\frac{1}{24} E_2(\tau). \text{ quasi-mod. holo form.}$$

How do we single out the correct lift $\widetilde{R}_1(\xi)$:

Mirror symmetry tells us that the mirror to the class $[pt]^{PD}$ is the class $\underset{[\xi]}{\underset{\parallel}{}} \frac{1}{z-\bar{z}}[d\bar{z}] \in HH_1(E_\tau)$.

$$0 \to H^0(\Omega^1) \to H^1_{dR} \to H^1(\mathcal{O}) \to 0.$$

Lemma: TFAE for a family of lifts $[\xi]^{geom}$ of $[\xi]$:

(1) $[\xi]^{geom}$ is inv't under monodromy around cusp

(2) $[\xi]^{geom}$ is flat w.r.t. G-M connection.

Note (2) can be checked at each point:

If we write $[\xi]^{geom} = \xi + u\alpha + \ldots$

$$\left(\begin{cases} b + uB \text{ closed} \iff b(\alpha) = -B(\xi) = -1|\xi \\ \text{flat} \iff b^{1|1}(\partial_z \mu^* |\alpha) \text{ is } b\text{-exact} \end{cases} \right.$$

solve, fixes α up to b-exact chains

Can also do the same computations w/out computers using another splitting and a gauge-equivalent rep'n of A_τ.