Curves, Jacobians,

and Modern Abel-Jacobi Theory

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https://people.math.ethz.ch/~rahul/AJ.html

Plan of the Course

(i) Basics of algebraic curves

- Jacobian
- Abel-Jacobi map
- Abel's Theorem
- Jacobi Inversion

(ii) Moduli of curves, universal AJ map

(iii) Stable curves, Stable maps, definition of the DR cycle

(iv) Tautological classes, Pixton's formula

(v) Proof of Pixton's formula

(vi) Universal AJ theory on the Picard stack

Niels Henrik Abel 1802-1829

Carl Gustav Jacob Jacobi 1804-1851

Some References:

Books

- Griffiths-Harris Algebraic Geometry
- Mumford Curves and their Jacobians
- Forster Riemann Surfaces GTM 81
- Harris-Morrison Moduli of Curves
- Kock-Vainsencher Quantum Cohomology

All can be found on my webpage

Articles

- Fulton-P Notes on stable maps
- P Calculus for the moduli of curves
- JPPZ DR cycles on the moduli
- BHPSS Universal DR cycles

Lectures

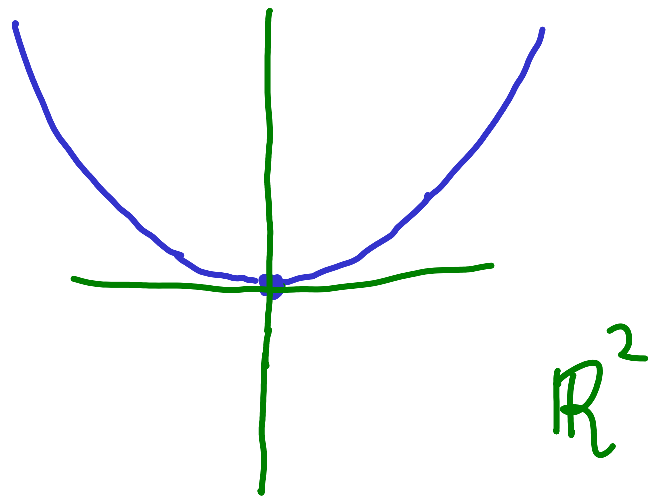
- P Geometry of Moduli of Curves (Rio 2018)
- P AJ maps and DR cycles (ZOOM, MOSCOW 2020)

Our basic object of study is an algebraic curve:

$y = x^2$

$y - x^2 = 0$

$y - x^2$



But we always work over $\mathbb{C}$ not $\mathbb{R}$

To start, we will consider Nonsingular Curves (later nodal)

(1) Three equivalent ways of viewing the definition:

(i) C is a nonsingular, complete, connected, complex algebraic variety of $\dim_{\mathbb{C}} 1$

Example: $(zy - x^2) \subset \mathbb{CP}^2_{[x,y,z]}$

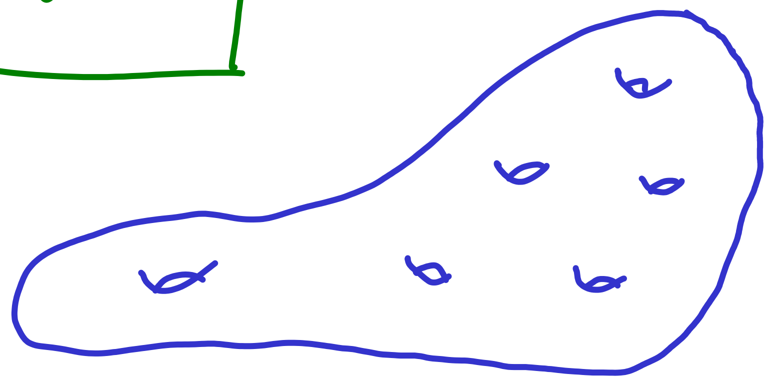
$\boxed{\text{Conic}}$

Example: $(xw - yz,\ xz - y^2,\ yw - z^2) \subset \mathbb{CP}^3_{[x,y,z,w]}$

$\boxed{\text{Twisted Cubic}}$

Example: $(x^5+y^5+z^5) \subset \mathbb{CP}^2_{[x,y,z]}$

$\boxed{\text{Fermat Curve}}$



(ii) C is a compact, connected 1-$\dim_{\mathbb{C}}$ complex manifold.

$\boxed{\text{Riemann Surface}}$

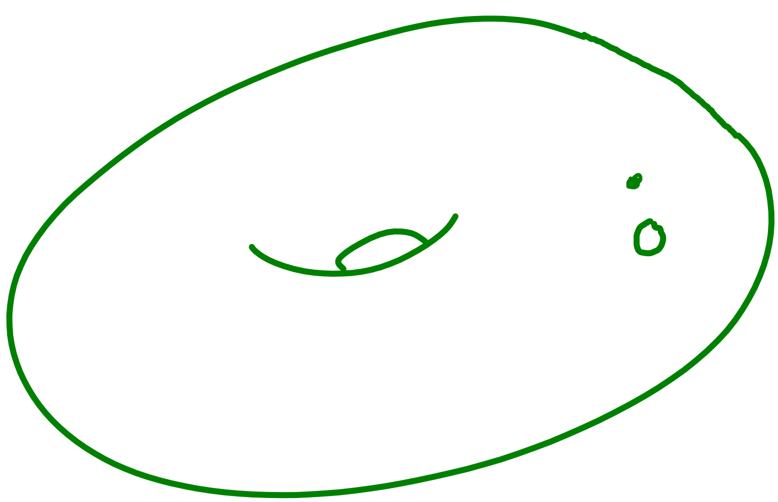
Gaussian integers

Example: $\mathbb{Z}^2 = \mathbb{Z} + i\mathbb{Z} \subset \mathbb{C}$.

Consider $E = \mathbb{C} / \mathbb{Z} + i\mathbb{Z}$

well defined topology and theory of holomorphic functions.

Weierstrass $\wp$ function



Elliptic Curve

(iii) $\begin{array}{c} K \\ | \\ \mathbb{C} \end{array}$ is a finitely generated field extension of transcendence degree 1.

Tr degree 1 means

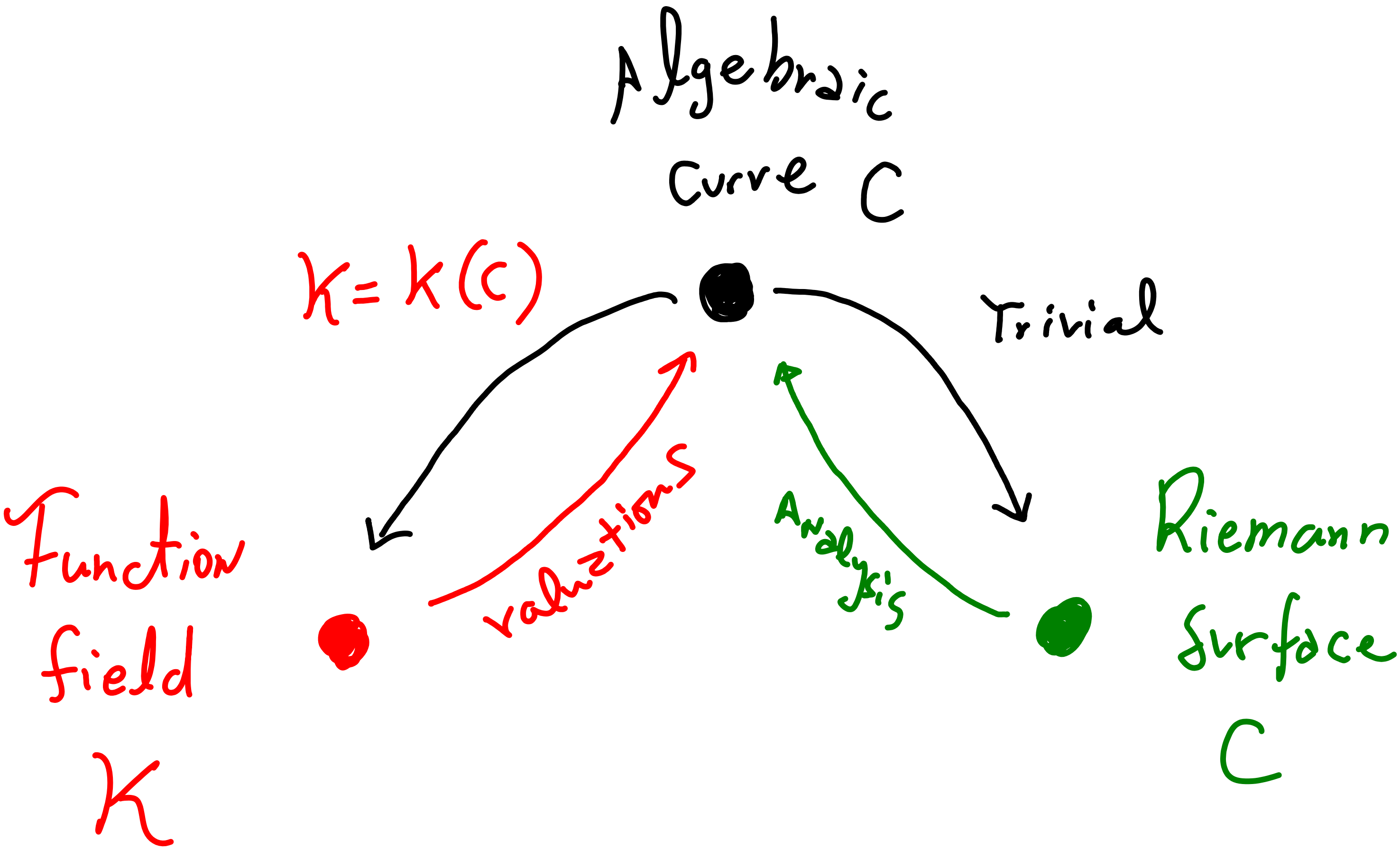
$$\begin{tikzcd} K \arrow[d, dash, "\text{algebraic}"] \\ \mathbb{C}(t) \arrow[d, dash, "\text{pure tr}"] \\ \mathbb{C} \end{tikzcd}$$

Example: $K = \begin{array}{c} \mathbb{C}(t) \\ | \\ \mathbb{C} \end{array}$

Example: K = field of fractions of $\mathbb{C}[x,y]/(f(x,y))$

where $f(x,y) \in \mathbb{C}[x,y]$ is an irreducible polynomial.

The Amazing Fact is that these three definitions are equivalent!



The first step in understanding the theory of curves is to understand this diagram.

(2) We are interested in line bundles

$$\mathcal{L} \to C$$

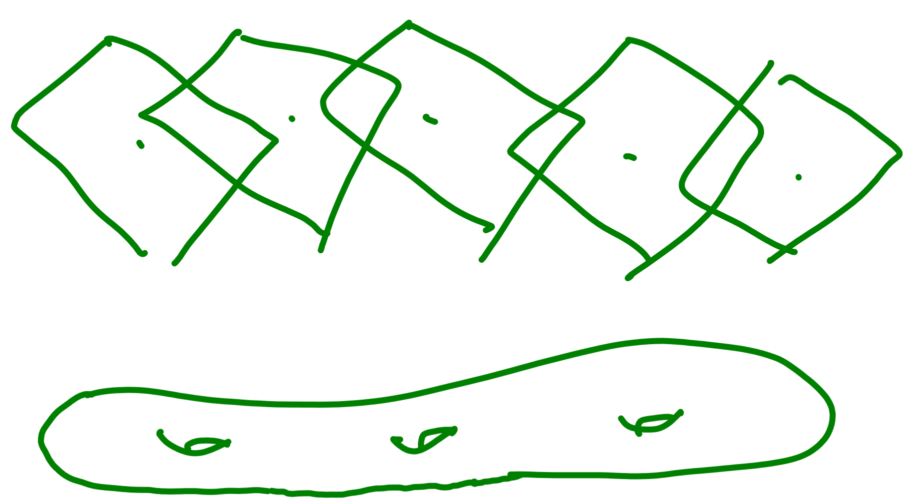
line bundle

Curve

- C is a Riemann Surface

$\mathcal{L}$ is holomorphic $\mathbb{C}$-bundle of rank 1.

$$\begin{array}{c} \mathcal{L} \\ \downarrow \\ C \end{array}$$



$\mathcal{O}_C^{an}$ sheaf of holomorphic functions

$\mathcal{O}_C^{an*}$ invertible holomorphic functions

$\mathcal{L}$ is classified by an element of $H^1(\mathcal{O}_C^{an*})$

$$[\mathcal{L}] \in H^1(\mathcal{O}_C^{an*})$$

Exercise: Check this carefully using the definition of a line bundle and Čech cohomology of sheaves.

The most basic sheaf sequence on a Riemann Surface:

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2\pi i} \mathcal{O}_C^{an} \xrightarrow{\exp} \mathcal{O}_C^{an*} \to 0$$

Constant sheaf

Sheaf with $+$

Sheaf with $\cdot$

Associated cohomology sequence

$$H^1(\mathcal{O}_C^{an*}) \to H^2(C, \mathbb{Z})$$

$$H^1(\mathcal{O}_C^{an*}) \ni [\mathcal{L}] \longmapsto c_1(\mathcal{L}) \in H^2(C)$$

Definition of Chern class

The degree of $\mathcal{L}$ is

$$\deg(\mathcal{L}) = \int_C c_1(\mathcal{L}) \in \mathbb{Z}$$

orientation in $H_2(C)$

$$= c_1(\mathcal{L}) \cap [C]$$

$$H_0(C) \cong \mathbb{Z}$$

Example: $\mathcal{L} = \mathbb{C} \times C \to C$

trivial line bundle (called $\mathcal{O}$)

$[\mathcal{O}] = 1 \in H^1(\mathcal{O}_C^{an*})$, $\deg(\mathcal{O}) = 0$

Example: $\mathcal{L}$ is the complex tangent bundle of C (called T_C)

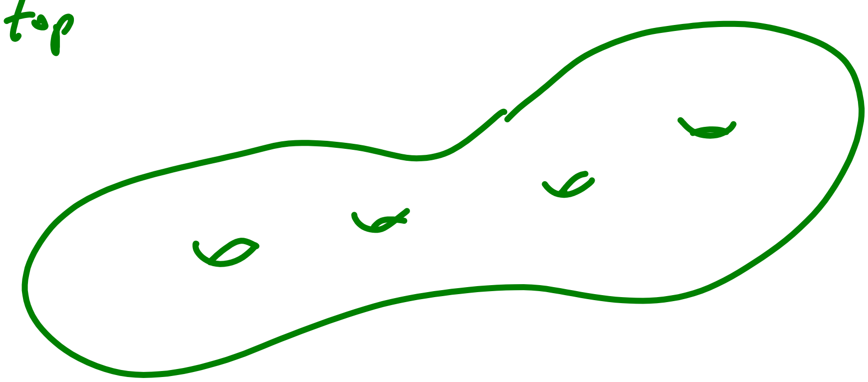
Gauss - Bonnet

$$\deg(T_C) = \int_C c_1(T_C) = \chi_{top}(C)$$

Topological Euler Char

What is $\chi_{top}(C)$?

Suppose



C has g holes, $\quad g = g_{geom}$

Then $g =$ geometric genus of C

singular cohomology

$$H^*(C, \mathbb{Z}) = \begin{array}{ll} \mathbb{Z} & H^0(C) \\ \mathbb{Z}^{2g} & H^1(C) \\ \mathbb{Z} & H^2(C) \end{array}$$

So $\chi_{top}(C) = 2 - 2g_{eom}$

We see $\deg(T_c) = 2 - 2g_{geom}$.

Since $\deg(\mathcal{L}) = -\deg(\mathcal{L}^*)$,

$$\deg(K_c) = 2g_{geom} - 2$$

where $T_c^* = K_c$, the canonical bundle

- C is an algebraic curve

$\mathcal{O}_C^{alg}$ ← sheaf of algebraic functions

Line bundles are associated to locally free sheaves of rank 1.

$\mathcal{L}(\mathcal{U})$ = sheaf of algebraic sections over

$\mathcal{U} \subset C$

↑ Zariski Open

if $s \in H^0(\mathcal{L})$ is a nontrivial global section, then.

$\deg(\mathcal{L}) = \#$ zeros of s (with multiplicity).

if s is a rational section,

$\deg(\mathcal{L}) = \#$ zeros $- \#$ poles of s.

Every algebraic line bundle on an algebraic curve C determines a holomorphic line bundle on the corresponding Riemann Surface.

Exercise: show the two definitions of degree match.

In fact, for a Riemann Surface, the degree of a line bundle can be computed by the zero and poles of any meromorphic section.

Sheaf cohomology

$\mathcal{L} \to C$, $\quad H^0(\mathcal{L})$ and $H^1(\mathcal{L})$

Riemann-Roch :

$$\dim H^0(\mathcal{L}) - \dim H^1(\mathcal{L})$$
$$= \deg(\mathcal{L}) - g_{arith} + 1$$

Example: $\mathcal{L} = \mathcal{O}_C$ trivial sheaf

$$\underset{\underline{1}}{\underset{\shortparallel}{\dim H^0(\mathcal{O}_C)}} - \dim H^1(\mathcal{O}_C) = 0 - g_{arith} + 1$$

so $\dim H^1(\mathcal{O}_C) = g_{arith}$.

Serre Duality:

$$H^0(\mathcal{L}) \cong H^1(\mathcal{L}^\vee \otimes K_C)^\vee$$

Example: Let $\mathcal{L} = K_C$, then

$$H^0(K_C) = H^1(\mathcal{O}_C)^\vee.$$

so $\dim H^0(K_C) = g_{arith}$

(3) Question: is $g_{geom} = g_{arith}$?

Answer: Yes for nonsingular complete curves

How can we prove this?

The statement is nontrivial:

Connects topology (g_{geom})

to algebraic sheaf cohomology (g_{arith})

Standard approach is via

Hodge decomposition:

$$H^1(C,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^{1,0} \oplus H^{0,1}$$

$\uparrow$ rank$_{\mathbb{C}}$ $2g_{geom}$

$H^{1,0}$: $H^0(K_C)$ rank$_{\mathbb{C}}$ g_{arith}

$H^{0,1}$: $H^1(\mathcal{O}_C)$ rank$_{\mathbb{C}}$ g_{arith}

But we can also see

$g_{geom} = g_{arith}$ using

- $\deg(K_C) = 2g_{geom} - 2$ (Gauss Bonnet)

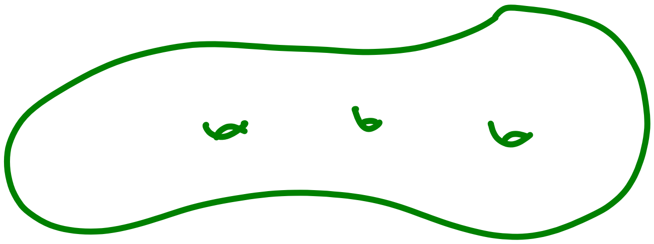
- $H^0(K_C) - H^1(K_C) = \deg(K_C) - g_{arith} + 1$ (Rieman Roch)

- $\dim H^0(K_C) = g_{arith}$ (Serre Duality)

  $\dim H^1(K_C) = 1$

$$\Rightarrow \quad g_{arith} - 1 = 2g_{geom} - 2 - g_{arith} + 1 \quad \blacksquare$$

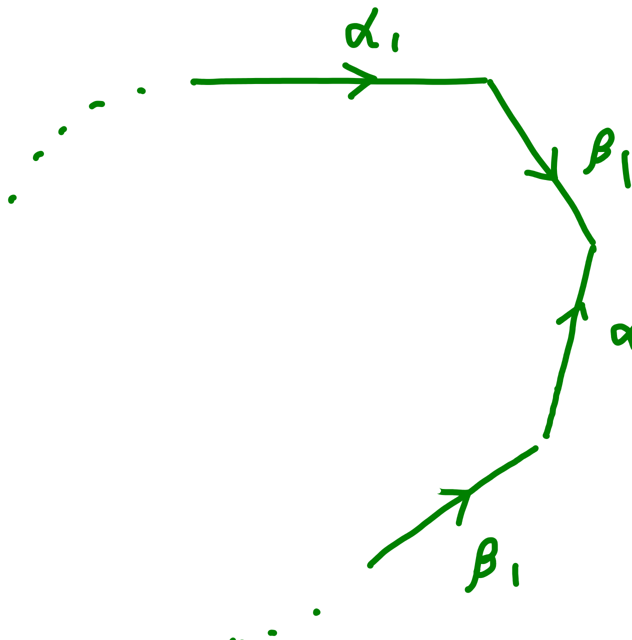
(4) Let C be curve of genus $g > 0$



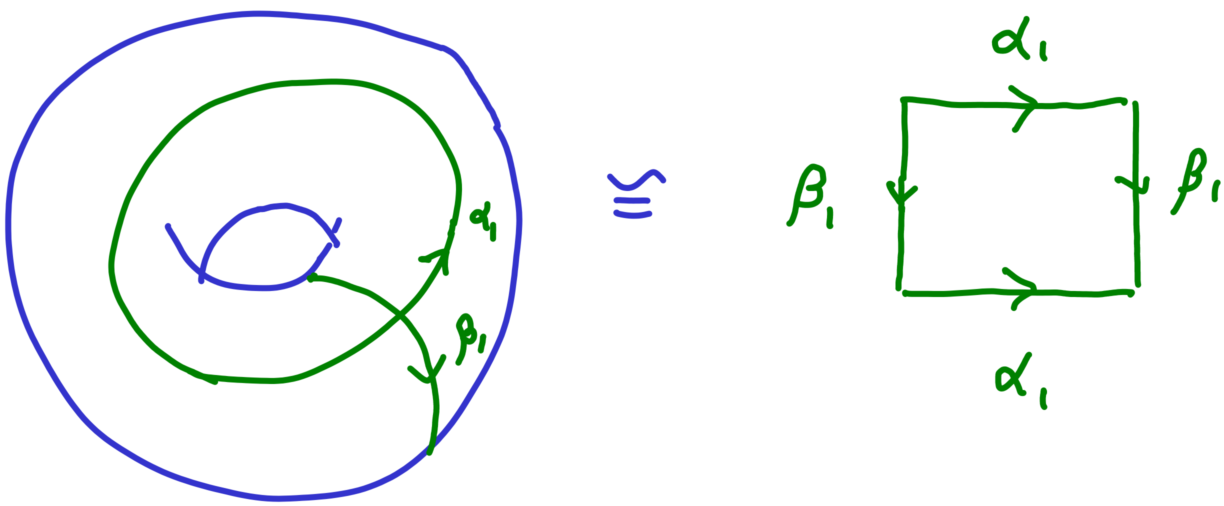
$H_1(C, \mathbb{Z}) = \mathbb{Z}^{2g}$ $\qquad$ $H^0(K_C) = \mathbb{C}^g$

- Basis of $H_1(C, \mathbb{Z})$ is usually written as A and B cycles

Topologically C can be represented as:



Simplest case is when $g=1$



Basis (as a $\mathbb{Z}$-module) of $H_1(C)$:

$$\alpha_1, \alpha_2, \dots, \alpha_g, \beta_1, \dots, \beta_g$$

- Basis (as a $\mathbb{C}$-vector space) of $H^0(K_C)$ is

$$\omega_1, \omega_2, \dots, \omega_g$$

holomorphic differential forms

Example: $E = \mathbb{C}/\mathbb{Z}\oplus i\mathbb{Z}$, $\omega_1 = dz$

(5) There is a canonical map

$$\epsilon : H_1(C,\mathbb{Z}) \longrightarrow H^0(K_C)^\vee$$

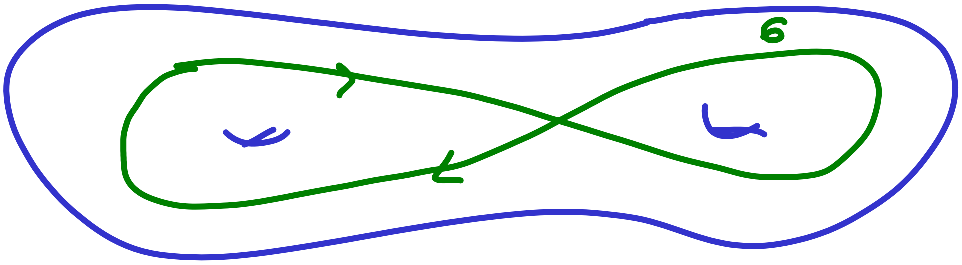
$$\gamma \mapsto \epsilon(\gamma) \in H^0(K_C)^\vee$$

$$\epsilon(\gamma)(\omega) \in \mathbb{C}$$

Definition: $\epsilon(\gamma)(\omega) = \int_\gamma \omega$

Does this make sense?

$\gamma \in H_1(C,\mathbb{Z})$ so γ is represented by a cycle σ, $\gamma = [\sigma]$



Definition:

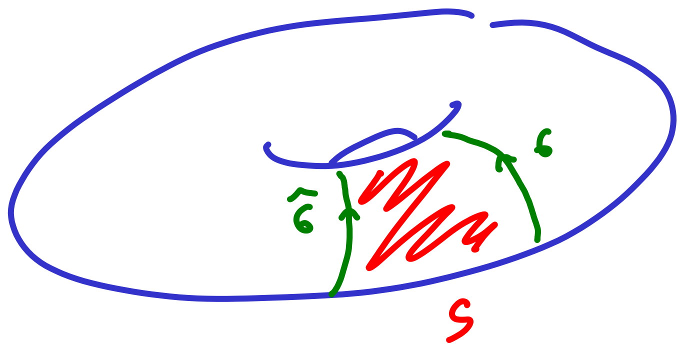
$$\boxed{\int_\gamma \omega = \int_\sigma \omega}$$

But there are other representatives:

$\gamma = [\sigma]$

$\gamma = [\hat{\sigma}]$

Then $\sigma - \hat{\sigma} = \partial S$



Let us check:

$$\int_{\sigma} \omega - \int_{\hat{\sigma}} \omega = \int_{\partial S} \omega \overset{\text{STOKES}}{=} \int_{S} d\omega = 0$$

why?

ω is a holomorphic differential form,

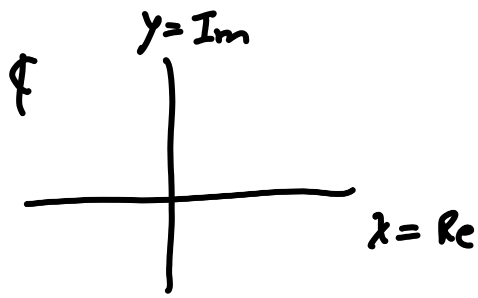
locally $\omega = f(z)\,dz$ where

z is a local holomorphic chart on C

$$df = \frac{\partial f}{\partial \bar z}\, d\bar z + \frac{\partial f}{\partial z}\, dz$$

$$dw = d(f\,dz) = df \wedge dz = \cancelto{0}{\frac{\partial f}{\partial \bar z}}\, d\bar z \wedge dz + \frac{\partial f}{\partial z}\, \cancelto{0}{dz \wedge dz}$$

Reminder:



Wirtinger

$z = x + iy$

$dz = dx + i\,dy$

$d\bar z = dx - i\,dy$

$$dz \wedge d\bar z = -2i\, dx \wedge dy$$

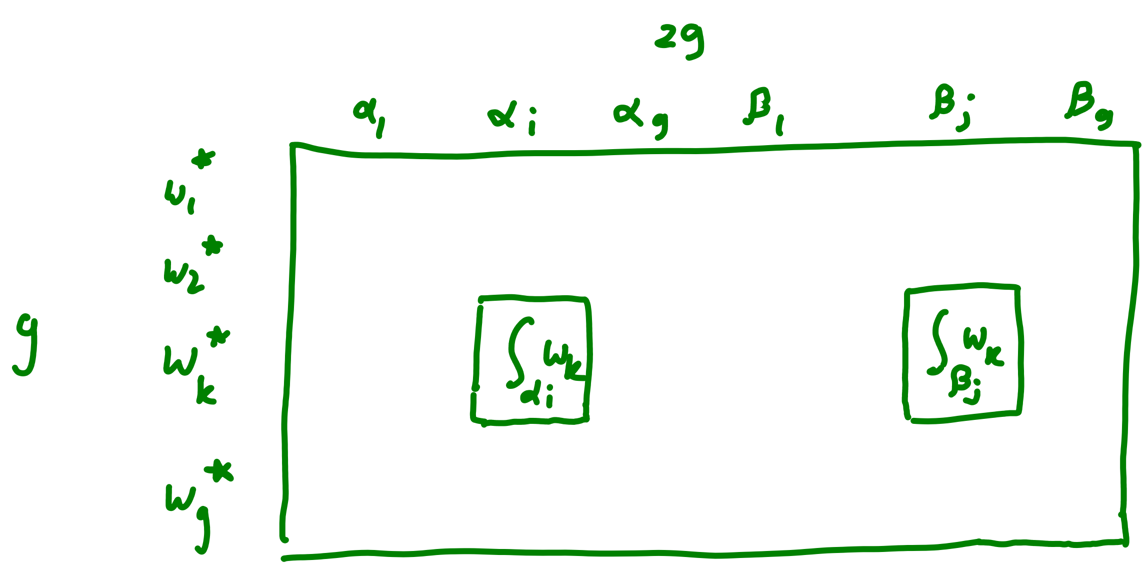
so

$$i\, dz \wedge d\bar z = 2\, dx \wedge dy$$

$$\frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2} i \frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \bar z} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2} i \frac{\partial}{\partial y}$$

We can write the canonical map $\mathcal{E}$ explicitly in terms of the bases.



It is useful to think of this matrix as consisting of $2g$ column vectors

$$\begin{array}{cc} \alpha_i & \beta_j \\ \begin{bmatrix} \int_{\alpha_i}\omega_1 \\ \\ \\ \int_{\alpha_i}\omega_g \end{bmatrix} & \begin{bmatrix} \int_{\beta_j}\omega_1 \\ \\ \\ \int_{\beta_j}\omega_g \end{bmatrix} \end{array}$$

Period vectors along A an B cycles.

In bases, $\mathcal{E} : \mathbb{Z}^{2g} \to \mathbb{C}^g$

<u>Proposition</u>: The $2g$ period vectors are $\mathbb{R}$-linearly independent in $\mathbb{C}^g$.

Proof: Suppose there is an $\mathbb{R}$-dependence.

Then $\exists\, r_1, \dots, r_g, s_1, \dots, s_g \in \mathbb{R}$, not all 0, so that

$$\sum_{i=1}^{g} r_i \int_{\alpha_i} \omega_j + s_i \int_{\beta_i} \omega_j = 0$$

for all $j \in \{1, 2, \dots, g\}$. Then,

$$\sum_{i=1}^{g} r_i \int_{\alpha_i} \overline{\omega_j} + s_i \int_{\beta_i} \overline{\omega_j} = 0$$

We conclude that $\sigma = \sum r_i \alpha_i + s_i \beta_i \neq 0$
$\cap$
$H_1(C, \mathbb{R})$

Has the property: $\int_\sigma \omega = 0$, $\int_\sigma \overline{\omega} = 0$ $\forall \omega \in H^0(K_C)$

But $\omega_1, \dots, \omega_g$ is a basis of $H^{1,0}$

$\bar\omega_1, \dots, \bar\omega_g$ is a basis of $H^{0,1}$

so together $\omega_1, \dots, \omega_g, \bar\omega_1, \dots, \bar\omega_g$ span $H^1(C, \mathbb{C})$.

By Poincaré duality $\Rightarrow \sigma = 0$

Contradiction. $\blacksquare$

Using $\varepsilon : H_1(C) \to H^0(K_C)^\vee$

We can define a complex torus

$$\mathrm{Jac}(C) = H^0(K_C)^\vee \big/ \varepsilon(H_1(C))$$

Jacobian of C

ε often dropped in notation

Jac(C) is topologically $(S^1)^{2g}$

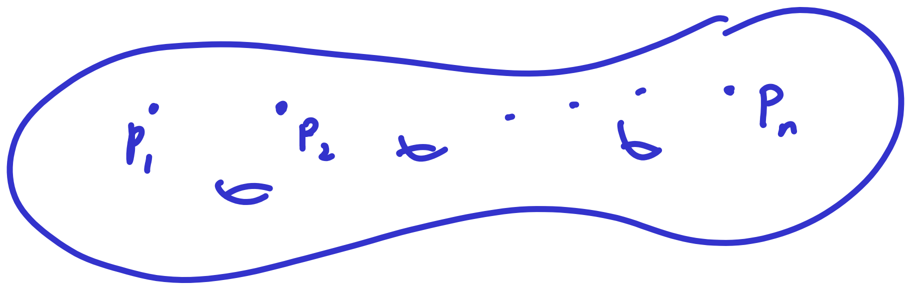
because the periods are $\mathbb{R}$-independent.

Jac(C) is a group under +.

In fact Jac(C) is a complex proj variety [a later discussion].

(6) A divisor on a curve C

is a finite formal linear combination



$$D = \sum_{i=1}^{n} m_i \, p_i$$

points of C

integers

The degree of D is:

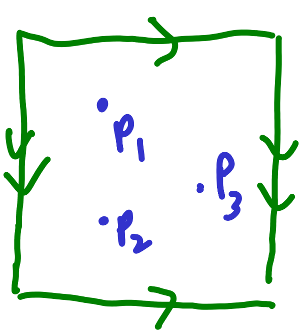
$$|D| = \sum m_i$$

Div(C) is the group (under +) of the divisors on C

$0 \in \mathrm{Div}(C)$ is empty sum.

$\mathrm{Div}_0(C) \subset \mathrm{Div}(C)$ is the subgroup of degree 0 divisors

Ex $\quad E = \mathbb{C}/\mathbb{Z} \oplus i\mathbb{Z}$



$$D = 12\, p_1 - 8\, p_2 - 4\, p_3$$

$$D \in \mathrm{Div}_0(E)$$

(7) Abel-Jacobi map.

We define $AJ : \mathrm{Div}_0(C) \to \mathrm{Jac}(C)$

Let $D \in \mathrm{Div}_0(C)$, $D = \sum_{i=1}^{n} m_i P_i$

Since $\sum_{i=1}^{n} m_i = 0$, $\exists$ 1-chain (path) σ

satisfying: $\partial\sigma = D$

Define: $AJ(D) \in \mathrm{Jac}(C) = H^0(K_C)^\vee / H_1(C,\mathbb{Z})$

Let $\omega \mapsto \int_\sigma \omega \in H^0(K_C)^\vee$

Well defined up to an element of $H_1(C,\mathbb{Z})$,

so $\omega \mapsto \int_\sigma \omega \in H^0(K_C)^\vee / H_1(C,\mathbb{Z})$

We must check that if paths σ and $\hat{\sigma}$ both satisfy

$$\partial\sigma = D, \quad \partial\hat{\sigma} = D$$

then $\omega \mapsto \int_\sigma \omega$ and $\omega \mapsto \int_{\hat{\sigma}} \omega$

differ in $H^0(K_C)^\vee$ by an element of $\varepsilon(H_1(C,\mathbb{Z}))$:

$$\int_\sigma \omega - \int_{\hat{\sigma}} \omega = \int_{\sigma-\hat{\sigma}} \omega$$

since $\partial(\sigma - \hat{\sigma}) = 0$,

$\sigma - \hat{\sigma}$ determines an element of $H_1(C)$.

$AJ(D)$ is defined to be $\omega \mapsto \int_\sigma \omega$ up to $H_1(C, \mathbb{Z})$.

$AJ(0) = 0$, AJ is a homomorphism

(8) Let $x \in C$ be a base point

Then we define

$$AJ : C \longrightarrow Jac(C)$$

by $AJ(p \in C) = AJ(p - x)$

in $Div_0(C)$

Ex. Let $E = \mathbb{C}/\mathbb{Z} \oplus i\mathbb{Z}$

Prove $AJ : E \to Jac(E)$

is an isomorphism

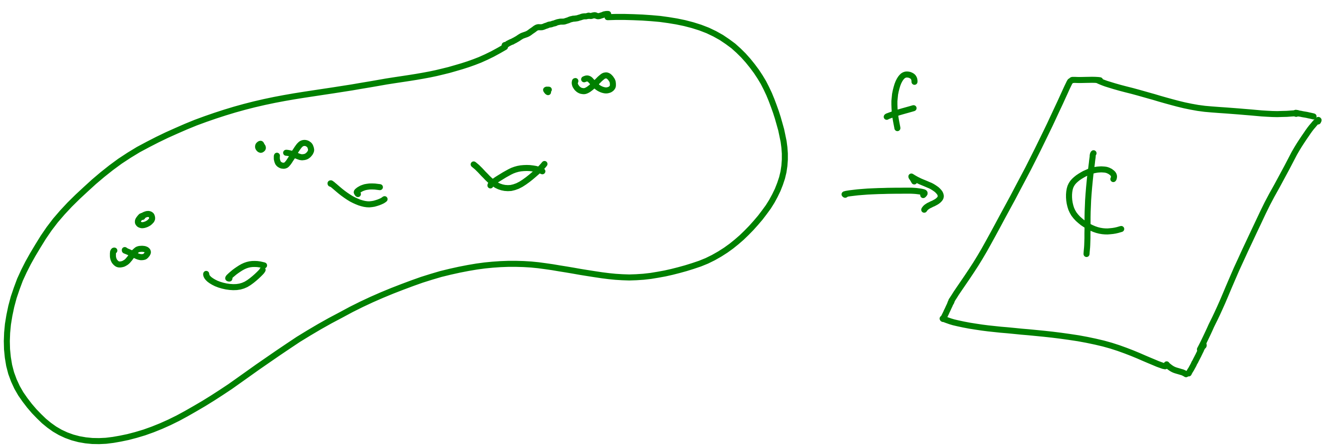
(9) Rational functions / Meromorphic functions

C is an algebraic curve $\rightsquigarrow$ $K(C)$

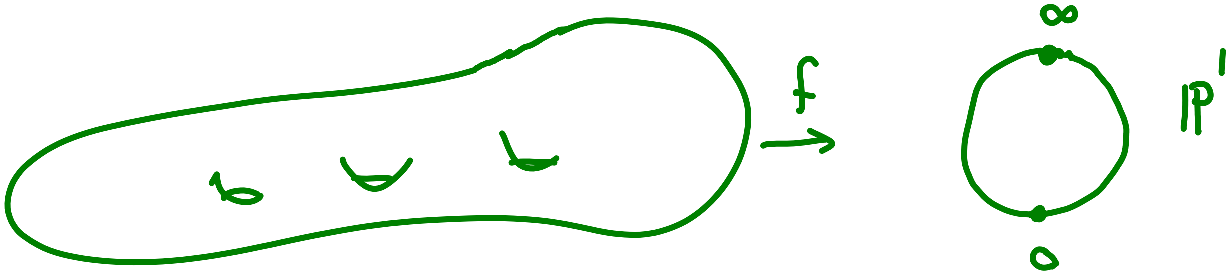
field of rational functions

C is a Riemann Surface $\rightsquigarrow$ $K(C)$

field of meromorphic functions



Better way to view a rational/meromorphic function:



$K(C)$ is a field, abelian group with $+$,

$K(C)^* = K(C) \setminus \{0\}$ abelian group with $\cdot$.

Basic homomorphism of Abelian groups

$$\mathrm{div} : K(C)^* \longrightarrow \mathrm{Div}_0(C)$$

Defined by $\mathrm{div}(f) =$ zeros $-$ poles

More precisely, f has finitely many zeros p_i of multiplicity $a_i > 0$

poles q_j of multiplicity $b_j > 0$,

$$\mathrm{div}(f) = \sum_i a_i p_i - \sum_j b_j q_j .$$

Exercise. $\sum_i a_i - \sum_j b_j = 0$.

There are several solutions:

- degree of a morphism $f: C \to \mathbb{P}^1$

finite dim field extension $\longrightarrow$

$$\begin{array}{c} k(C) \\ \Big| \, f^* \\ k(\mathbb{P}^1) \\ | \\ \mathbb{C} \end{array}$$

$$\deg(f) : \left[k(C) : k(\mathbb{P}^1) \right]$$

Then $\sum_i a_i = \sum_j b_j = \deg(f)$.

- Use Stokes Theorem / Residue Theorem

from $f: C \to \mathbb{P}^1$, construct

a meromorphic differential

form $\frac{df}{f}$ on C

$\frac{df}{f}$ has simple poles at the zeros and poles of f

[if $f = z^k + \dots$ in a local coordinate z,

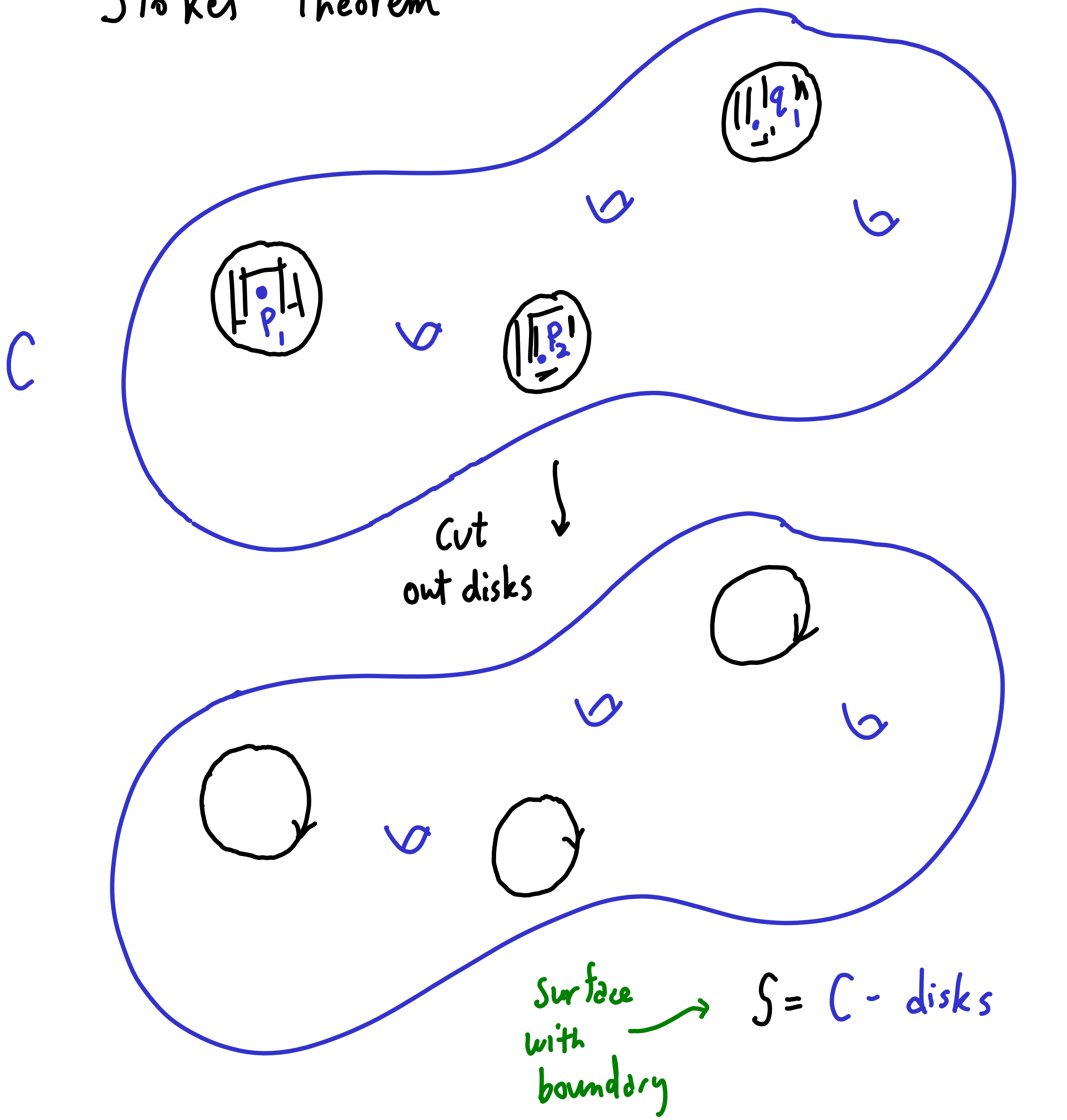
$$\frac{df}{f} = k\,\frac{z^{k-1} + \dots}{z^k + \dots}\,dz = \left(\frac{k}{z} + \dots\right) dz$$]

Moreover, the Residues are

$$\operatorname{Res}_{p_i} \frac{df}{f} = a_i \,, \quad \operatorname{Res}_{q_j} \frac{df}{f} = -b_j$$

The Residue Theorem $\Rightarrow$ $\sum_i a_i - \sum_j b_j = 0$

The Residue Theorem is a consequence of Stokes Theorem:



$$2\pi i \left(\sum b_j - \sum a_i \right) = \int_{\partial S} \frac{df}{f} \overset{\text{Stokes}}{=} \int_S d\left(\frac{df}{f}\right) = \int_S 0 = 0.$$

We have now the basic sequence of Abel-Jacobi Theory

$$K(C)^* \xrightarrow{\text{div}} \mathrm{Div}_0(C) \xrightarrow{AJ} \mathrm{Jac}(C)$$

Theorem (Abel)

Sequence is exact in the middle

<u>or</u> $\quad \mathrm{Im}(\text{div}) = \ker(AJ)$

<u>or</u> A divisor of degree 0

$$\sum a_i p_i - \sum b_j q_j \qquad a_i, b_j > 0$$

is a divisor of a rational function if and only if

$$AJ\left(\sum a_i p_i - \sum b_j q_j\right) = 0 \text{ in } \mathrm{Jac}(C).$$

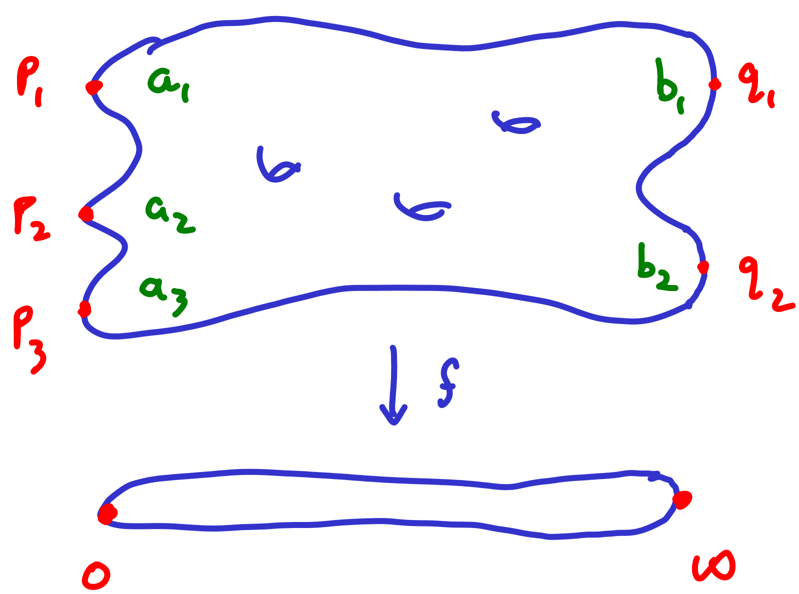
(10) Abel's Theorem Part I (easy):

$$\mathrm{Im}(\mathrm{div}) \subset \ker(AJ)$$

Let $f : C \to \mathbb{P}^1$ be a morphism

Then $\mathrm{div}(f) = f^{-1}(0) - f^{-1}(\infty)$

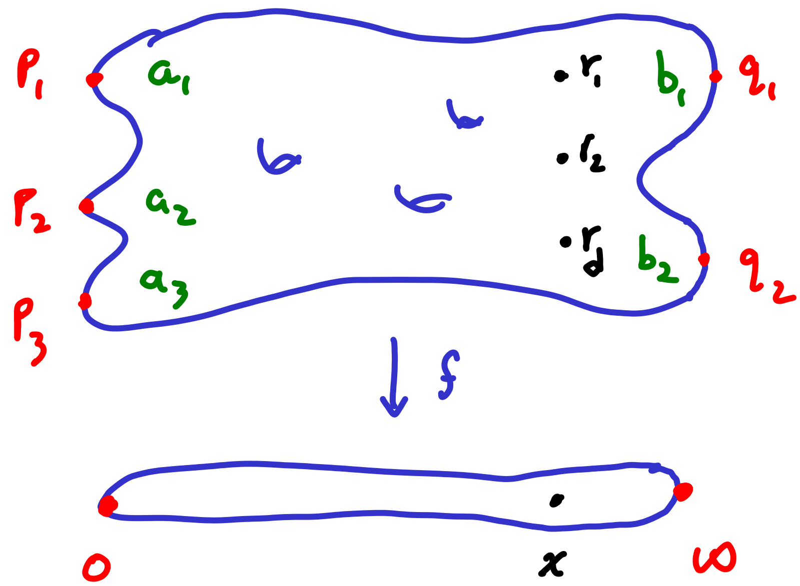
points with multiplicities



$$\mathrm{div}(f) = a_1 p_1 + a_2 p_2 + a_3 p_3 - b_1 q_1 - b_2 q_2$$

We must show $AJ(\mathrm{div}(f)) = 0$.

There is a simple geometric argument:



Let d = degree f = $a_1 + a_2 + a_3 = b_1 + b_2$

Let $x \in \mathbb{P}^1$ be a general point

Consider $f^{-1}(x) = r_1 + r_2 + \cdots + r_d$

Define a map

$$\mu : \mathbb{P}^1 \to \mathrm{Jac}(C)$$

$$\mu(p) = f^{-1}(0) - f^{-1}(p)$$

Claim 1:

(i) μ is well-defined and holomorphic

(ii) $\mu(0) = 0 \in Jac(C)$

(iii) μ is constant

Claim (i) excercise, (ii) definition

for (iii) use lifting property

$$\begin{tikzcd}
 & \mathbb{C}^g \arrow[d, "\text{universal cover}"] \\
\mathbb{P}^1 \arrow[ur, "\exists \hat{\mu}"] \arrow[r, "\mu"] & Jac(C)
\end{tikzcd}$$

simply connected

We have a holomorphic lift $\hat{\mu} : \mathbb{P}^1 \to \mathbb{C}^g$ which must be constant by the max principle.

(11) Abel's Theorem Part II (harder):

$\ker(AJ) \subset \mathrm{Im}(\mathrm{div})$

We require several auxillary results which are important in their own right.

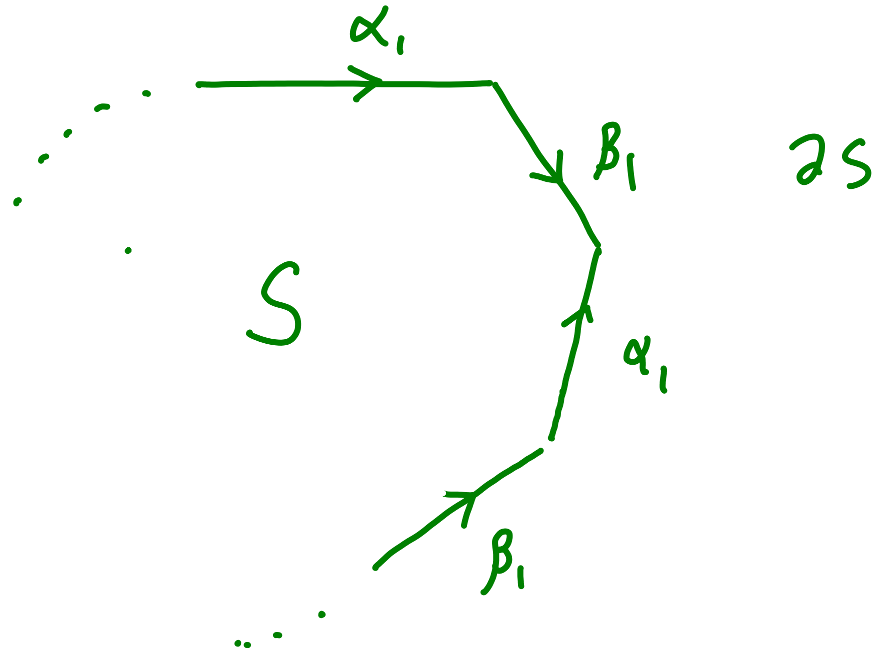
Let C be a curve (Riemann Surface)

→ • A differential of the first kind is a holomorphic differential

Not used here • A differential of the second kind is a meromorphic differential with vanishing residue at each pole.

→ • A differential of the third kind is a meromorphic differential with only simple poles ← order 1.

Topologically C can be represented as:



The first Reciprocity Law:

Let ω, η be differentials of the first and third kind. Then,

$$\sum_{i=1}^{g} \left(\int_{\alpha_i} \omega \int_{\beta_i} \eta - \int_{\alpha_i} \eta \int_{\beta_i} \omega \right)$$

$$= 2\pi i \sum_{k} \operatorname{Res}_{x_k}(\eta) \int_{x_0}^{x_k} \omega .$$

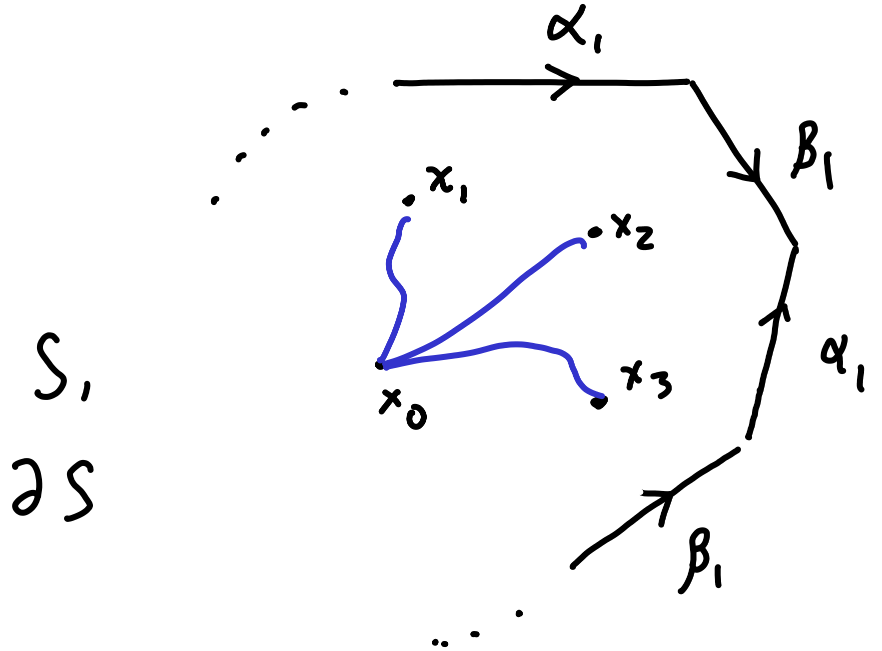
What does this mean?

- poles of η must <u>not</u> lie on the paths $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$.

- $x_1, \ldots, x_k$ are the locations of the poles of η.

interior point

- $X_0 \in C$ is any point (not on the paths $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$).



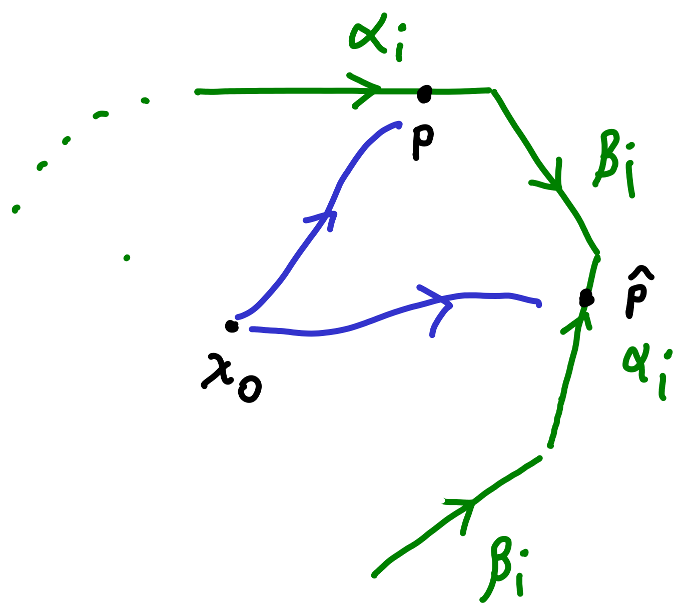
Integrate along any interior path.

RHS will be independent of X_0.

Proof (of the first Reciprocity):

Let $\pi(x) = \int_{x_0}^{x} \omega$ x is interior, path is interior

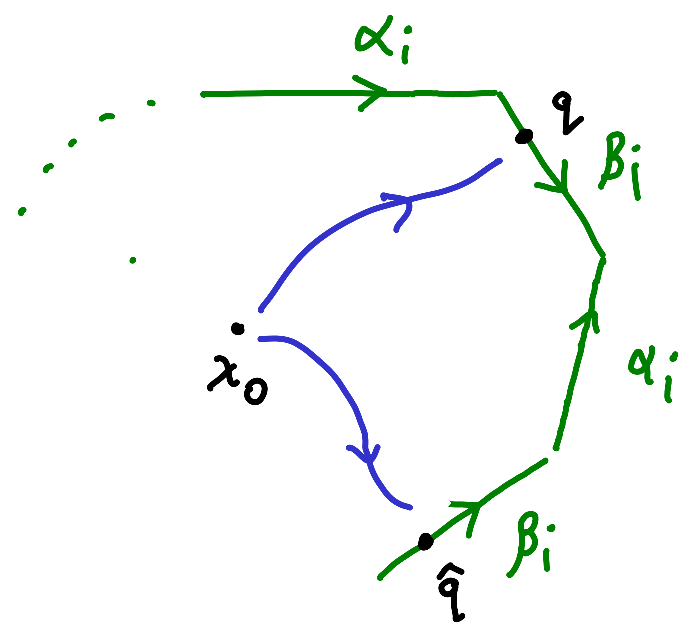
$\pi(x)$ is a holomorphic function on the cut surface S with continuity on the boundary ∂S.



Consider $p \sim \hat{p}$ identified

Then $\pi(\hat{p}) - \pi(p) = \int_{\beta_i} \omega$

uses Cauchy's Theorem of course!



Consider

$q \sim \hat{q}$

identified

Then $\quad \pi(\hat{q}) - \pi(q) = -\int_{\alpha_i} \omega$

Now to the Reciprocity:

$$\sum_{i=1}^{g} \left(\int_{\alpha_i} \omega \int_{\beta_i} \eta - \int_{\alpha_i} \eta \int_{\beta_i} \omega \right) = \; ?$$

Consider

$$\int_{\alpha_i + \alpha_i^{-1}} \pi \eta = \int_{\alpha_i} \pi \eta + \int_{\alpha_i^{-1}} \Big(\overset{\pi(\hat{p})}{\overset{\updownarrow}{\int_{\beta_i} \omega + \pi(p)}} \Big) \eta$$

$$= - \int_{\alpha_i} \eta \int_{\beta_i} \omega$$

$$\int_{\beta_i + \beta_i^{-1}} \pi\eta = \int_{\alpha_i} \omega \int_{\beta_i} \eta$$

Equality uses

- ω holomorphic
- η smooth near ∂S

So we see

$$\int_{\partial S} \pi\eta = \sum_{i=1}^{g} \left(\int_{\alpha_i} \omega \int_{\beta_i} \eta - \int_{\alpha_i} \eta \int_{\beta_i} \omega \right)$$

Now apply the Residue Formula

$$\int_{\partial S} \pi\eta = 2\pi i \sum_k \operatorname{Res}_{x_k}(\eta) \int_{x_0}^{x_k} \omega .$$

$\blacksquare$

A basic Corollary of the proof of Reciprocity is

Positivity: Let $\underset{\neq 0}{\omega} \in H^0(K_c)$, then

$$i \sum_{i=1}^{g} \left(\int_{\alpha_i} \omega \int_{\beta_i} \bar{\omega} - \int_{\alpha_i} \bar{\omega} \int_{\beta_i} \omega \right) > 0$$

Proof (of positivity)

Since ω is a holomorphic differential form, we conclude

$$\int_{\partial S} \underset{\substack{\uparrow \\ \text{smooth}}}{\pi} \bar{\omega} = \sum_{i=1}^{g} \left(\int_{\alpha_i} \omega \int_{\beta_i} \bar{\omega} - \int_{\alpha_i} \bar{\omega} \int_{\beta_i} \omega \right)$$

Exactly as before with $\pi(x) = \int_{x_0}^{x} \omega$.

Now use Stokes:

$$\int_{\partial S} \pi \, \bar{\omega} = \int_S d(\pi \bar{\omega})$$

$$= \int_S d\pi \wedge \bar{\omega} + \cancelto{0}{\pi \wedge d\bar{\omega}}$$

$$= \int_S \omega \wedge \bar{\omega}$$

locally $\omega = f(z)\,dz$, $\bar{\omega} = \overline{f(z)}\,d\bar{z}$

so $\omega \wedge \bar{\omega} = |f(z)|^2 \, dz \wedge d\bar{z}$

$$= |f(z)|^2 (-2i) \, dx \wedge dy$$

Therefore $i\,\omega \wedge \hat{\omega}$ is a positive multiple of the volume form:

$$i \sum_{i=1}^{g} \left(\int_{\alpha_i} \omega \int_{\beta_i} \bar{\omega} - \int_{\alpha_i} \bar{\omega} \int_{\beta_i} \omega \right)$$

$$= i \int_S \omega \wedge \bar{\omega} > 0 \qquad \blacksquare$$

Next, we choose a special basis

$$\omega_1, \omega_2, \dots, \omega_g \in H^0(K_C)$$

which is normalized on the A-cycles:

$$\int_{\alpha_i} \omega_j = \delta_{ij} \quad \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Why is this possible?

Suppose Not: then $\exists\ \omega \neq 0$

Satisfying $\displaystyle\int_{\alpha_i} \omega = 0 \quad \forall\ i \in \{1, .., g\}$.

But then we contradict Positivity

Since the vanishing A-periods imply

$$i \sum_{i=1}^{g} \left(\int_{\alpha_i} \omega \int_{\beta_i} \overline{\omega} - \int_{\alpha_i} \overline{\omega} \int_{\beta_i} \omega \right) = 0$$

Note $\displaystyle\int_{\alpha_i} \overline{\omega} = \overline{\int_{\alpha_i} \omega}$ ∎

Riemann's Bilinear Relations :

I. Let $\omega, \eta \in H^0(K_C)$, then

$$\sum_{i=1}^{g} \int_{\alpha_i} \omega \int_{\beta_i} \eta - \int_{\alpha_i} \eta \int_{\beta_i} \omega = 0$$

Proof: Immediate from Reciprocity.

If $\omega_1, \ldots, \omega_g$ is normalized on A-cycles,

$$0 = \sum_{i=1}^{g} \int_{\alpha_i} \omega_k \int_{\beta_i} \omega_\ell - \int_{\alpha_i} \omega_\ell \int_{\beta_i} \omega_k$$

$$= \sum_{i=1}^{g} \delta_{ik} \int_{\beta_i} \omega_\ell - \delta_{i\ell} \int_{\beta_i} \omega_k$$

$$= \int_{\beta_k} \omega_\ell - \int_{\beta_\ell} \omega_k$$

Hence, the matrix

$$Z_{k\ell} = \int_{\beta_k} w_\ell \quad \text{is symmetric.}$$

We will impose the normalization on A-cycles.

II. $\mathrm{Im}(Z) > 0$ Positive definite

Proof: We have

$$i \sum_{i=1}^{g} \left(\int_{\alpha_i} w \int_{\beta_i} \bar{w} - \int_{\alpha_i} \bar{w} \int_{\beta_i} w \right) > 0$$

If we write

Normalized

$$\underset{\neq 0}{\omega} = \sum_{k=1}^{g} \lambda_k w_k$$

Real numbers

We obtain

$$i \sum_{i=1}^{g} \left(\int_{\alpha_i} \omega \int_{\beta_i} \bar{\omega} - \int_{\alpha_i} \bar{\omega} \int_{\beta_i} \omega \right) =$$

$$i \sum_{k,l} \lambda_k \int_{\beta_k} \bar{\omega}_l \, \lambda_l - \lambda_k \int_{\beta_k} \omega_l \, \lambda_l$$

$$> 0$$

Hence,

$$i \left(\bar{Z} - Z \right) = 2 \operatorname{Im} Z$$

is a positive definite Real matrix. ∎

We now prove the second part of Abel's Theorem:

$$\ker(AJ) \subset \operatorname{Im}(\operatorname{div})$$

Proof: Let $\sum a_i p_i - \sum b_j q_j$ be a divisor:

- all p_i's and q_j's are distinct.
- $a_i > 0$ zeros
- $b_j > 0$ poles
- $\sum a_i - \sum b_j = 0$

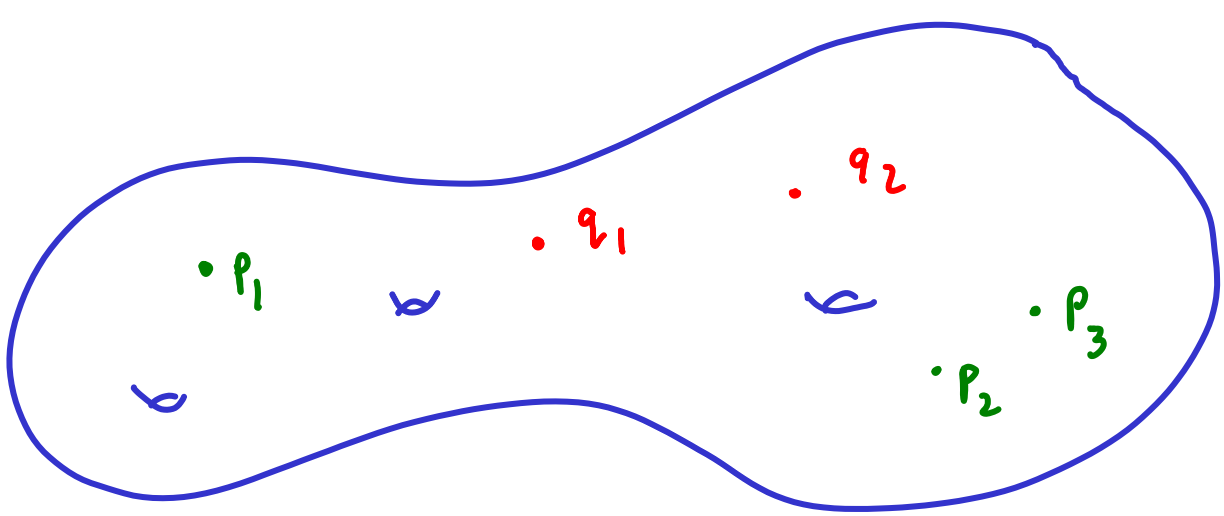
Main hypothesis: $AJ\left(\sum a_i p_i - \sum b_j q_j\right) = 0$

We must find a meromorphic function

$$f: C \longrightarrow \mathbb{P}^1$$

satisfying $\quad \mathrm{div}(f) = \sum a_i p_i - \sum b_j q_j$.

The interesting idea is to look instead for an appropriate meromorphic differential on C



We search for a meromorphic differential η satisfying

(1) η has simple poles exactly at the points

$$p_1, p_2, p_3, \dots, q_1, q_2, \dots \in C$$

and is holomorphic elsewhere.

(2) At p_i, $\mathrm{Res}_{p_i}(\eta) = \dfrac{a_i}{2\pi i}$

At q_j, $\mathrm{Res}_{q_j}(\eta) = -\dfrac{b_j}{2\pi i}$

(3) All the A and B-periods of η are integers:

$$\int_{\alpha_i} \eta \in \mathbb{Z}, \quad \int_{\beta_i} \eta \in \mathbb{Z}$$

If we find such an η,

then choose $x_0 \in C$ and define

$$f(x) = \exp\left(2\pi i \int_{x_0}^{x} \eta\right)$$

Using the residue condition and the integrality of the periods,

$$f : C \to \mathbb{P}^1$$

is well defined.

<u>Claim</u>: $\operatorname{div}(f) = \sum a_i p_i - \sum b_j q_j$

Proof of claim: Need only study p_i, q_j

since f is clearly finite and nonzero away from these points.

Near p_i: η is of the form

$$\eta = \frac{a_i}{2\pi i} \frac{1}{z} + \text{holomorphic terms}$$

local coordinate

$$\int \eta = \int \frac{a_i}{2\pi i} \frac{1}{z} + \text{holomorphic terms}$$

$$= \frac{a_i}{2\pi i} \log z + \text{holomorphic terms}$$

multiple valued

$$f(x) = \exp\left(2\pi i \int \eta\right)$$

single valued

$$= \exp(a_i \log z) \cdot \exp(\text{holomorphic})$$

$$= z^{a_i} \cdot (1 + \text{higher order in } z)$$

So f has a zero of order a_i at $p_i \in C$.

Similarly f has a pole of order b_j at $q_j \in C$. $\blacksquare$

The last step in the proof of Abel's Theorem is the construction of the required η.

Proof of Abel's Theorem Part II

Main hypothesis: $AJ\left(\sum a_i p_i - \sum b_j q_j\right) = 0$

We will construct η.

To find an η satisfying conditions

(1) η has simple poles exactly at the points

$$p_1, p_2, p_3, \ldots, q_1, q_2, \ldots \in C$$

and is holomorphic elsewhere.

(2) At p_i, $\mathrm{Res}_{p_i}(\eta) = \dfrac{a_i}{2\pi i}$

At q_j, $\mathrm{Res}_{q_j}(\eta) = -\dfrac{b_j}{2\pi i}$

follows immediately from Riemann-Roch:

View K_C as a locally free sheaf on C.

Let N be the cardinality of

$$\{ p_1, p_2, p_3, \dots, q_1, q_2, \dots \},$$

the total number of zeros and poles.

Exact sequence of sheaves:

$$0 \to K_C \to K_C\left(\sum p_i + \sum q_j\right) \to \mathbb{C}^N \to 0$$

Sheaf of holomorphic differential form

Sheaf of meromorphic differential forms with at most simple poles

Residue map

Take the associated long exact sequence in cohomology:

$$0 \to \underset{g}{H^0(K_C)} \to \underset{g+N-1}{H^0\big(K_C(\textstyle\sum p_i + \sum q_j)\big)} \to \underset{N}{\mathbb{C}^N}$$

R-R

$$\to \underset{1}{H^1(K_C)} \to \underset{0}{H^1\big(K_C(\textstyle\sum p_i + \sum q_j)\big)} \to 0$$

Serre duality

Exercise: The sum of residues of a meromorphic differential form is 0 (Stokes)

Therefore,

$$\operatorname{Im} H^0\left(K_C\left(\sum p_i + \sum q_j\right)\right) \subset \mathbb{C}^N$$

is exactly the subspace of $\mathbb{C}^N$ where the residues sum to 0.

Conclusion: We can find η satisfying conditions (1) and (2).

A final argument is required to show that we can achieve (3).

(3) All of the A and B-periods of η are integers:

$$\int_{\alpha_i} \eta \in \mathbb{Z}, \quad \int_{\beta_i} \eta \in \mathbb{Z}$$

Recall the basis $\omega_1, \dots, \omega_g$ of holomorphic differential forms is normalized

$$\int_{\alpha_i} \omega_j = \delta_{ij}$$

Let $$\hat{\eta} = \eta - \sum_{i=1}^{g} \int_{\alpha_i} \eta \cdot \omega_i$$

Then $\hat{\eta}$ still satisfies (1) and (2)

Since ω_i are holomorphic (no poles).

Moreover

$$\int_{\alpha_i} \hat{\eta} = 0 \qquad \text{for } i = 1, \dots, g$$

Only the B-periods are left to control.

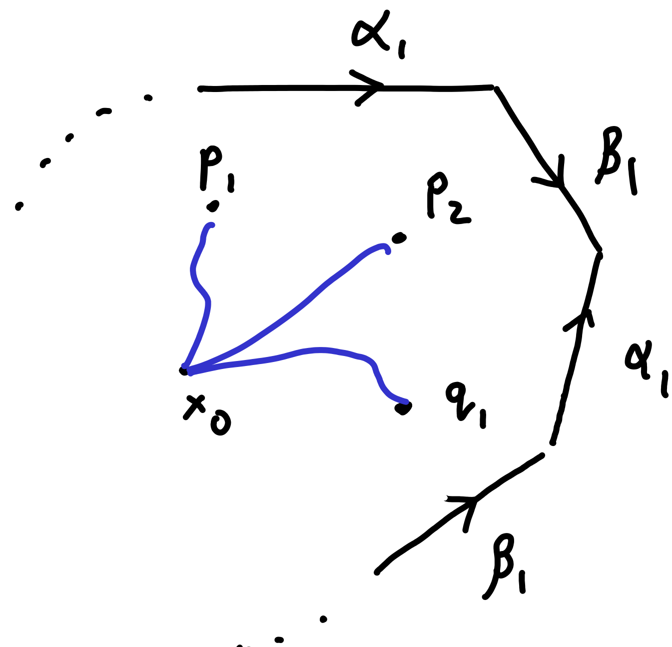
Apply the reciprocity law for $\hat{\eta}$ and ω_j :

$$\sum_{i=1}^{g}\left(\overbrace{\int_{\alpha_i}\omega_j}^{\delta_{ij}}\int_{\beta_i}\hat{\eta}-\overbrace{\int_{\alpha_i}\hat{\eta}}^{0}\int_{\beta_i}\omega_j\right)$$

$$= 2\pi i \sum_{k} \operatorname{Res}_{x_k}(\hat{\eta}) \int_{x_0}^{x_k} \omega_j$$

Rewrite as

$$\int_{\beta_j} \hat{\eta} = \sum_{r} a_r \int_{x_0}^{p_r} \omega_j - \sum_{s} b_s \int_{x_0}^{q_s} \omega_j$$



Let γ be the path $a_1 \overset{p_1}{\underset{x_0}{\uparrow}} + a_2 \; {}_{x_0}\!\nearrow^{p_2} - b_1 \; x_0 \rightarrow q_1$

Then we have

$$\int_{\beta_j} \hat{\eta} = \int_{\gamma} \omega_j , \qquad \partial\gamma = \sum_r a_r p_r - \sum_s b_s q_s$$

By definition

$$AJ\left(\sum_r a_r p_r - \sum_s b_s q_s\right) = \left[\int_\gamma \omega_1, \int_\gamma \omega_2, \ldots, \int_\gamma \omega_g\right]$$

equivalence class in $Jac(C) = \mathbb{C}^g / H_1(C)$

By hypothesis

$$AJ\left(\sum_r a_r p_r - \sum_s b_s q_s\right) = 0 \quad \text{in } Jac(C).$$

Hence, $\exists\, \sigma \in H_1(C)$ satisfying

$$\int_\gamma \omega_1 = \int_\sigma \omega_1 , \quad \int_\gamma \omega_2 = \int_\sigma \omega_2 , \ldots, \int_\gamma \omega_g = \int_\sigma \omega_g$$

We can write $\sigma = \sum_{i=1}^{g} l_i \alpha_i + \sum_{i=1}^{g} m_i \beta_i$

integers

$$\int_{\beta_j} \hat{\eta} = \int_{\gamma} \omega_j = \int_{\sigma} \omega_j$$

$$= l_j + \sum_{i=1}^{g} m_i \int_{\beta_i} \omega_j$$

Finally, define

$$\hat{\hat{\eta}} = \hat{\eta} - \sum_{k=1}^{g} m_k \omega_k$$

A periods : $\displaystyle\int_{\alpha_i} \hat{\hat{\eta}} = -m_i \in \mathbb{Z}.$

B periods:

$$\int_{\beta_j} \hat{\hat{\eta}} = \int_{\beta_j} \hat{\eta} - \sum_{k=1}^{g} m_k \int_{\beta_j} \omega_k$$

$$= l_j + \sum_{k=1}^{g} m_k \int_{\beta_k} \omega_j - \sum_{k=1}^{g} m_k \int_{\beta_j} \omega_k$$

$$= l_j \in \mathbb{Z}$$

We have used the symmetry

$$\int_{\beta_k} \omega_j - \int_{\beta_j} \omega_k = 0$$

The proof of Abel's Theorem is complete!

Let us return to the basic sequence of Abel-Jacobi theory:

$$0 \to \mathbb{C}^* \to K(C)^* \xrightarrow{\text{div}} \text{Div}_0(C) \xrightarrow{AJ} \text{Jac}(C) \to 0$$

Exact here since all holomorphic functions on C constant

Exact here by Abel's Theorem

Exact here by Jacobi inversion

Our next step is to prove Jacobi inversion (much easier than Abel's Theorem).

(12) Jacobi inversion

We will prove more:

Theorem: Let $x_0 \in C$ be a base point. The morphism

$$\mathrm{Sym}^g C \xrightarrow{\phi} \mathrm{Jac}(C)$$

defined by

$$\phi(\underset{\mathrm{Sym}^g C}{\underset{\cap}{\Delta}}) = AJ(\Delta - g \cdot x_0)$$

is surjective and birational.

proof: Let $p_1, \dots, p_g \in C$ be distinct points, and let

$$s = p_1 + p_2 + \cdots + p_g \in \mathrm{Sym}^g(C).$$

We calculate the differential

$$d\phi : \mathrm{Tan}_s(\mathrm{Sym}^g C) \to \mathrm{Tan}_{\phi(s)} \mathrm{Jac}(C)$$

$\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_g}$ local coords $z_1, \dots, z_g$ at $p_1, \dots, p_g \in C$

$\mathbb{C}^g$ since $\mathrm{Jac}(C) = \mathbb{C}^g / H_1(C)$

Writing AJ via the basis

$$\omega_1, \dots, \omega_g \in H^0(K_C),$$

$d\phi$ is the $g \times g$ matrix [columns defined up to $\mathbb{C}^*$]

$$\begin{bmatrix} \omega_1(p_1) & \omega_1(p_2) & \cdots & \omega_1(p_g) \\ \omega_2(p_1) & & & \\ \vdots & & \boxed{\omega_i(p_j)} & \\ \omega_g(p_1) & & & \end{bmatrix}$$

If $d\phi$ is degenerate at

$$s \in \mathrm{Sym}^g(C),$$

Then $\exists\, \underset{0}{\omega^*} \in H^0(K_C)$ which vanishes at all

$$p_1, p_2, \dots, p_g \in C.$$

However, since $\dim H^0(K_C) = g$,

for general $p_1, \dots, p_g \in C$,

there is no nonzero holomorphic

differential ω which

vanishes at all p_i.

We conclude $d\phi$ is

somewhere nondegenerate.

Hence $\operatorname{Im} \phi \subset \operatorname{Jac}(C)$

contains an open subset

by the Implicit Function Theorem.

We conclude that ϕ is surjective using dimension theory of analytic varieties (or algebraic varieties since in fact everything is algebraic here)

Another approach:

$$AJ : \mathrm{Div}_0(C) \to \mathrm{Jac}(C)$$

must be surjective once an open subset of $\mathrm{Jac}(C)$ is captured (using the group law).

But this is weaker than the surjectivity of ϕ.

Finally, we study the fiber of ϕ:

$$\phi^{-1}(\mathcal{L}) = \text{all } s \in \mathrm{Sym}^g C \text{ such that } AJ(s - g x_0) = \mathcal{L}$$

The question is when do we have

$$AJ(\hat{s} - g x_0) = AJ(s - g x_0) \, ?$$

By Abel's Theorem, we must then have

$$\hat{s} - s = \mathrm{div}(f)$$

for some meromorphic $f \in K(C)^*$.

For fixed $s \in \mathrm{Sym}^g(C)$,

such $\hat{s}$ are parametrized

exactly by the projective space

$$\mathbb{P}\left(H^0\left(\mathcal{O}_C(s) \right) \right)$$

[By Riemann-Roch

$\dim H^0(\mathcal{O}_C(s)) - \dim H^1(\mathcal{O}_C(s)) = g - g + 1 = 1$]

By dimension theory, the fibers of

$$\phi : \mathrm{Sym}^g(C) \to \mathrm{Jac}(C)$$

can not all be positive dimensional

Since ϕ is surjective and

$\dim \mathrm{Sym}^g(C) = \dim \mathrm{Jac}(C) = g$.

The generic fiber must be 0-dimensional and therefore must be $\mathbb{P}^0 \cong$ point ■

About the notation $\mathcal{O}_C(s)$:

$\mathcal{O}_C(s)$ is the locally free sheaf with sections on

$\mathcal{U} \subset C$ given by meromorphic functions

$$f : \mathcal{U} \to \mathbb{C}$$

Satisfying the following property

$$\mathrm{div}(f) + s \big|_{\mathcal{U}} \geq 0$$

all coefficients non-negative

This means f has poles bounded by the positive terms of s and zeros forced by the negative terms of s.

The global sections $H^0(\mathcal{O}_C(\Delta))$ therefore correspond to meromorphic functions

$$f : C \to \mathbb{C}$$

with $\operatorname{div}(f) + \Delta \geqslant 0$ on C.

(13) We have completed the proof of the basic exact sequence of Abel-Jacobi theory:

$$0 \to \mathbb{C}^* \to K(C)^* \xrightarrow{\operatorname{div}} \operatorname{Div}_0(C) \xrightarrow{AJ} \operatorname{Jac}(C) \to 0$$

$$\mathrm{Pic}_0(C) \overset{\text{Def}}{=} \mathrm{Div}_0(C) \big/ \mathrm{Im}\, K(C)^*$$

Divisors

modulo linear equivalence

Another formulation of the main result of Abel-Jacobi theory:

$$\mathrm{Pic}_0(C) \cong \mathrm{Jac}(C)$$

via the Abel-Jacobi map.

What about $\mathrm{Div}(C)$? not just $\mathrm{Div}_0(C)$

By definition

$$\mathrm{Coker}\left(K(C)^* \xrightarrow{\mathrm{div}} \mathrm{Div}(C)\right) = A^1(C)$$

Chow group

(also $A_0(C)$)

Exact sequence

$$0 \to \mathrm{Div}_0(C) \to \mathrm{Div}(C) \xrightarrow{\deg} \mathbb{Z} \to 0$$

We have

$$0 \to \mathrm{Jac}(C) \to A^1(C) \xrightarrow{\deg} \mathbb{Z} \to 0$$

(14) Abel-Jacobi theory says nothing interesting in $g=0$:

$$K(\mathbb{P}^1)^* \xrightarrow{\text{div}} \mathrm{Div}_0(\mathbb{P}^1)$$

is surjective and

$$\mathrm{Pic}_0(\mathbb{P}^1) \cong \mathrm{Jac}(\mathbb{P}^1)$$

are both points.

Because there always exists a rational function with specified zeros and poles in $\mathbb{P}^1$ (so long as the sum of orders is 0).

(15) Abel-Jacobi theory for an elliptic curve

$$E = \mathbb{C}/\Lambda$$

lattice $\mathbb{Z}^2 \subset \mathbb{C}$

is interesting.

Basis of holomorphic differential forms is simply

$$dz \in H^0(K_E) = \mathbb{C}^\vee$$

Moreover

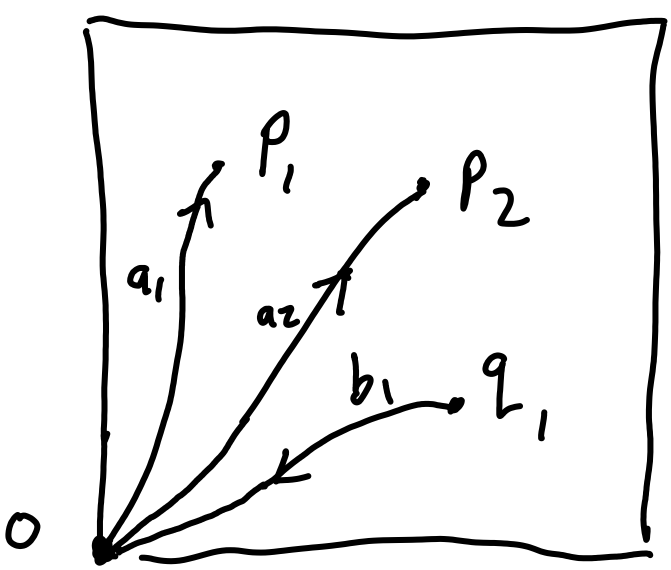
$$Jac(E) = (\mathbb{C}^\vee)^\vee / H_1(E)$$

$$= \mathbb{C}/\Lambda = E$$

What is the Abel-Jacobi map

$$\mathrm{Div}_0(E) \xrightarrow{AJ} \mathrm{Jac}(E) = E \quad ?$$

Let $\sum \underset{>0}{a_i} p_i - \sum \underset{>0}{b_j} q_j \in \mathrm{Div}_0(E)$



Define γ such that

$$\partial\gamma = \sum a_i p_i - \sum b_j q_j$$

Since $\displaystyle\int_{0 \xrightarrow{a_1} p_1} dz = a_1 p_1$, $\displaystyle\int_{q_1 \xrightarrow{b_1} 0} dz = -b_1 q_1$, We find

$$AJ\left(\sum a_i p_i - \sum b_j q_j\right) = \sum a_i p_i - \sum b_j q_j$$

$$\begin{array}{ccc} \in & & \in \\ \mathrm{Div}_0(E) & & \mathrm{Jac}(E) = E \end{array}$$

In particular, there exists a meromorphic function

$$f : E \to \mathbb{P}^1$$

with $\mathrm{div}(f) = \sum a_i p_i - \sum b_j q_j$

if and only if $\sum a_i p_i - \sum b_j q_j = 0 \in E$.

Explicit constructions of f are given by the theory of doubly periodic functions.

(16) Let C be a curve of genus 2. By the proof of Jacobi inversion,

$$\mathrm{Sym}^2 C \xrightarrow{\phi} \mathrm{Jac}(C)$$

is birational and surjective.

Let $s \in \mathrm{Sym}^2 C$. The fiber over s:

$$\phi^{-1}(\phi(s)) = \mathbb{P}(H^0(\mathcal{O}_C(s)))$$

What can these fibers be?

We have

$s = p + q$ with $p, q \in C$.

By Serre Duality,

$$\dim H^1(\mathcal{O}_C(s)) = \dim H^0(K_C(-s))$$

Since $\deg K_C = 2 \cdot 2 - 2 = 2$,

we have $\deg K_C(-s) = 0$

Since a degree line bundle $\mathcal{L}$ has a section if and only if

$$\mathcal{L} \cong \mathcal{O}_C,$$

$$H^0(K_C(-s)) = \begin{cases} 0 & \text{if } \mathcal{O}_C(s) \not\cong K_C \\ \mathbb{C} & \text{if } \mathcal{O}_C(s) \cong K_C \end{cases}$$

By Riemann-Roch

$$H^0(\mathcal{O}_C(s)) = \begin{cases} \mathbb{C} & \text{if } \mathcal{O}_C(s) \not\cong K_C \\ \mathbb{C}^2 & \text{if } \mathcal{O}_C(s) \cong K_C \end{cases}$$

[R-R: $\dim H^0(\mathcal{O}_C(s)) - \dim H^1(\mathcal{O}_C(s)) = 2-2+1$]

so the fibers of

$$\mathrm{Sym}^2 C \xrightarrow{\phi} \mathrm{Jac}(C)$$

are $\mathbb{P}^0 \cong$ point in all cases except when $\mathcal{O}_C(s) \cong K_C$ where the fiber is $\mathbb{P}^1$.

Conclusion:

$$\mathrm{Sym}^2 C \xrightarrow{\phi} \mathrm{Jac}(C)$$

is the blowdown along

$$\mathbb{P}^1 \subset \mathrm{Sym}^2 C$$

Consisting of Divisors

$$\underbrace{\Delta = p+q}_{\text{Called canonical divisors.}} \quad \text{with} \quad \mathcal{O}_C(\Delta) \cong K_C$$

(17) Let C be a curve of genus g, and let $d > 2g-2$.

Let $x_0 \in C$ be a basepoint.

Define $\mathrm{Sym}^d C \xrightarrow{\phi} \mathrm{Jac}(C)$

$$\phi(\underset{\mathrm{Sym}^d C}{\underset{\cap}{\Delta}}) = AJ(\Delta - d \cdot x_0)$$

Exercise: The morphism ϕ expresses $\mathrm{Sym}^d C$ as a projective bundle over $\mathrm{Jac}(C)$ with fiber $\mathbb{P}^{d-g}$.

Hint: Use the analysis of the fiber of ϕ as in the proof of Jacobi Inversion (together with Riemann-Roch and Serre Duality).

Try to visualize the last result

for an elliptic curve E:

- $d=1$ (condition $d > 2g-2 = 0$ satisfied).

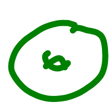
$$E = \mathrm{Sym}^1 E \xrightarrow{\phi} \mathrm{Jac}(E) = E$$

projective bundle with fiber $\mathbb{P}^{d-g} = \mathbb{P}^0$

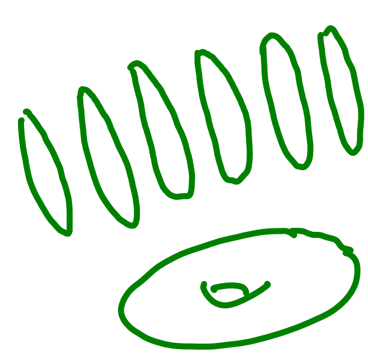
- $d=2$, $\mathrm{Sym}^2 E \xrightarrow{\phi} \mathrm{Jac}(E) = E$

is a $\mathbb{P}^1$-bundle

Even topologically, the geometry is nontrivial.

$E =$ 

$\mathbb{P}^1$ bunde over E

$\mathrm{Sym}^2 E = E \times E / \mathbb{Z}_2 \cong$ 

(18) Let E be an elliptic curve.

Let $E^{n,n} = \underbrace{E \times \cdots \times E}_{n} \times \underbrace{E \times \cdots \times E}_{n}$ (2n factors)

Let $Z \subset E^{n,n}$ be the locus of solutions to the following Abel-Jacobi problem:

$$Z \ni (p_1, \ldots, p_n, q_1, \ldots, q_n)$$

$$\Updownarrow \text{ if and only if}$$

$$\mathcal{O}_E\left(\sum p_i - \sum q_j\right) \cong \mathcal{O}_E$$

$Z \subset E^{n,n}$ is certainly an algebraic subvariety:

$Z = AJ^{-1}(0)$ where

$AJ : E^{n,n} \to Jac(E) \cong E$

$AJ(p_1, \dots, p_n, q_1, \dots, q_n) = \sum p_i - \sum q_j$

Moreover, since AJ is surjective for all $n \geq 1$,

$Z \subset E^{n,n}$ is a divisor (codim = 1)

What more can we ask about Z?

The basic question here:

What is the class of Z:

in cohomology

$$[Z] \in H^2(E^{n,n}, \mathbb{Z})?$$

and in Chow

$$[Z] \in A^1(E^{n,n}, \mathbb{Z})?$$

There are many ways to compute the class of Z.

Let
$$\begin{array}{c} E \times E^{n,n} \\ \downarrow \pi \\ E^{n,n} \end{array}$$
be the universal family with $2n$ sections

$$\begin{tikzcd} E \times E^{n,n} \arrow[d] \\ E^{n,n} \arrow[u, bend left, "p_1, \dots, p_n"] \arrow[u, bend right, "q_1, \dots, q_n"'] \end{tikzcd}$$

On the total space $E \times E^{n,n}$,

we have a universal

line bundle

$$\begin{array}{c} \mathcal{L} \cong \mathcal{O}_{E\times E^{n,n}}(P_1+\cdots+P_n-Q_1-\cdots-Q_n) \\ \downarrow \\ E\times E^{n,n} \end{array}$$

where P_i, Q_j are the divisors corresponding to the sections.

$Z \subset E^{n,n}$ is the locus over which $\mathcal{L}$ is trivial on fibers

More precisely :

$$\xi \in Z \overset{\text{def}}{\iff} \mathcal{L}\big|_{E\times\xi} \cong \mathcal{O}_E$$

$$\overset{\text{degree } 0}{\iff} H^0\left(\mathcal{L}\big|_{E\times\xi}\right) = \mathbb{C}$$

$$\overset{\text{Riemann-Roch}}{\iff} H^1\left(\mathcal{L}\big|_{E\times\xi}\right) = \mathbb{C}$$

Hence :

uses Base change

$$[Z] \in A^1(E^{n,n}) = -\mathrm{ch}_1\left(R\pi_* \mathcal{L}\right).$$

We can calculate using Grothendieck-Riemann-Roch:

$$\mathrm{ch}(R\pi_* \mathcal{L}) = \pi_*(\mathrm{ch}\,\mathcal{L} \cdot \mathrm{Td}_\pi)$$

chern character

derived push forward

relative Todd class

$$\mathrm{Td}_\pi = \frac{\mathrm{Td}(E \times E_{n,n})}{\mathrm{Td}(E_{n,n})} = 1,$$

$$-\mathrm{ch}_1(R\pi_* \mathcal{L}) = -\pi_*(\mathrm{ch}\,\mathcal{L})$$

$$\mathrm{ch}(\mathcal{L}) = \exp(c_1(\mathcal{L}))$$

$$= 1 + c_1(\mathcal{L}) + \frac{c_1(\mathcal{L})^2}{2} + \cdots$$

only term relevant

for $\mathrm{ch}_1(R\pi_* \mathcal{L})$

$$-\mathrm{ch}_1(R\pi_* \mathcal{L}) = -\pi_* \frac{c_1(\mathcal{L})^2}{2}$$

$$c_1(\mathcal{L}) = P_1 + \cdots + P_n - Q_1 \cdots - Q_n$$

what is $c_1(\mathcal{L})^2$?

$$-\frac{1}{2}\pi_* c_1(\mathcal{L})^2 = -\frac{\pi_*}{2}\left(P_1 + \cdots + P_n - Q_1 \cdots - Q_n\right)^2$$

$$= -\sum \Delta^{P}_{ij} + \sum \Delta^{PQ}_{ij} - \sum \Delta^{Q}_{ij}$$

diagonals among p-factors

$1 \leq i \neq j \leq n$ $\quad p_i = p_j$

diagonals between p, q-factors

$1 \leq i \leq n \;,\; 1 \leq j \leq n$

$p_i = q_j$

diagonals among q-factors

$1 \leq i \neq j \leq n$ $\quad q_i = q_j$

Sadly, GRR is only a formula with $\mathbb{Q}$-coefficients:

$$[Z] = -\sum \Delta^{P}_{ij} + \sum \Delta^{PQ}_{ij} - \sum \Delta^{Q}_{ij}$$

in $A^1(E^{n,n}, \mathbb{Q})$.

Exercise: Use the explicit understanding of the AJ map for E to compute the class of $Z \subset E^{n,n}$ in $H^2(E^{n,n}, \mathbb{Z})$ or $A^1(E^{n,n}, \mathbb{Z})$.

Hint: Torsion?

(19) So far we have studied Abel-Jacobi theory for a fixed curve C.

The modern directions concern Abel-Jacobi theory for the full moduli space of curves $\overline{\mathcal{M}}_g$.

The next topic in the course will be the moduli space curves

- Definitions of $\overline{\mathcal{M}}_g$, $\overline{\mathcal{M}}_{g,n}$
- How to think about these spaces
- Ideas about their constructions

(20)

Brill-Noether theory g^d_r

Higher differentials (C, ω_C^k)

Flat structures (C, ω_C)

Hurwitz covers $C \to \mathbb{P}^1$

Canonical curves $C \subset \mathbb{P}^g$

Riemann Mapping, Uniformization

Curves on Abelian varieties

Curves on K3 surfaces

Curves on Del Pezzo Surfaces $\mathbb{P}^1 \times \mathbb{P}^1$

RNC, complete intersections

Curves in $\mathbb{P}^n$ Syzygies

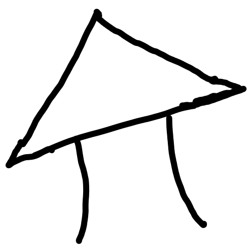
Curves in $\mathbb{P}^3$ Halphen

Curves in $\mathbb{P}^2$

Bezout, planar singularities

Severi varieties

cubics quartics

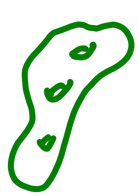


Classical study

$$0 \to \mathbb{C}^* \to K(C)^* \xrightarrow{\text{div}} \mathrm{Div}_0(C) \xrightarrow{AJ} \mathrm{Jac}(C) \to 0$$

Divisors, line bundles, Sheaves, Riemann-Roch, Serre duality

Topology, cohomology $H^*(C, \mathbb{Z})$, Hodge theory



Algebraic Curve

Function field

Riemann Surface

$H^{0,1}$

$H^{1,0}$

Topological String,
Gromov-Witten theory,
Mirror symmetry

Moduli of maps $C \to X$

$\overline{\mathcal{M}}_{g,n}(X,\beta)$

Torelli

$\mathcal{M}_g \to \mathcal{A}_g$

$[C] \to Jac(C)$

Moduli of curves

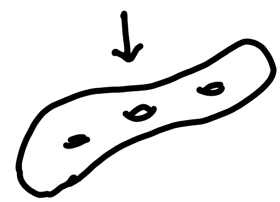
$\overline{\mathcal{M}}_g, \overline{\mathcal{M}}_{g,n}$

Deformation theory

Deligne-Mumford stacks

Moduli of Abelian varieties $\mathcal{A}_g$

E



Teichmüller Theory
mapping class group

Moduli

Moduli of Bundles of higher rank

Stable bundles

Higgs bundles

Representations $\pi_1(C)$

Verlinde formulas

Mumford's Conjecture

Classical Theory

Many other directions

End of Part I

of the Course