Moduli Space of Curves

(1) Some references (see webpage)

- Zvonkine — Introduction to the moduli of curves
- Harris-Morrison — Moduli of curves — Start here
- P — A calculus for the moduli of curves

all expository

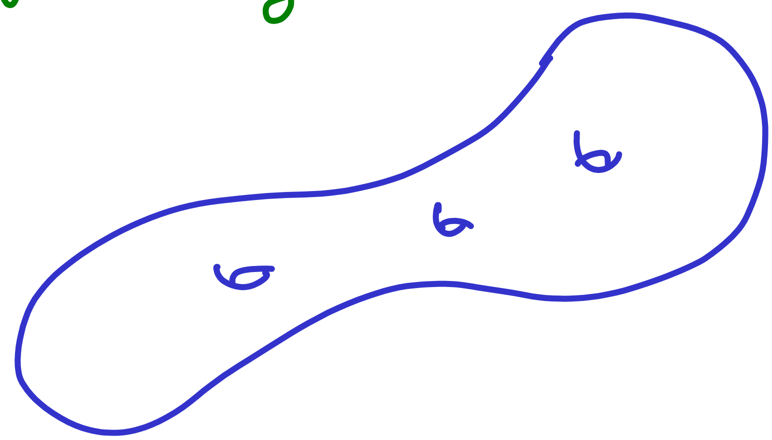
More technical:

- D. Gieseker — Lectures on Moduli of Curves

- video of my ICM lecture in Rio

(2) We start with a curve ← nonsingular complete connected

of genus g



C

Since $g = 0, 1$ are special,

(have continuous parameters of Automorphism)

let $g \geq 2$ to start.

We want deform the complex structure of C.

How can we do this?

We can vary the coefficients of the defining equation.

Example: genus 3

A nonsingular plane curve

$$C \subset \mathbb{P}^2$$

of degree 4 has genus 3

[degree-genus formula for nonsingular plane curves

$$g = \binom{d-1}{2}$$]

degree 4

We can write quartic polynomials in $\mathbb{P}^2$ as

$$a x_0^4 + b x_1^4 + c x_2^4 + d x_0^3 x_1 + \cdots$$

variables x_0, x_1, x_2

$\mathbb{C}$-coefficients which we can vary to change the complex structure

How many coefficients?

$$15 = \binom{4+2}{2}$$

How many parameters of moduli?

Since the zero set does not depend upon the scale $\Rightarrow$

14 parameters

$\mathcal{U} \subset \mathbb{P}^{14}$

$\uparrow$ open locus of degree 4 equations in x_0, x_1, x_2 which are nonsingular

But we can also change coordinates on $\mathbb{P}^2$:

$PGL_3 \leftarrow \dim = 3^2 - 1 = 8$

We have constructed (some) moduli in genus 3 by

$\mathcal{U}/\mathbb{P}GL_3$ $\leftarrow$ $\dim = 14 - 8 = 6.$

Three Main questions:

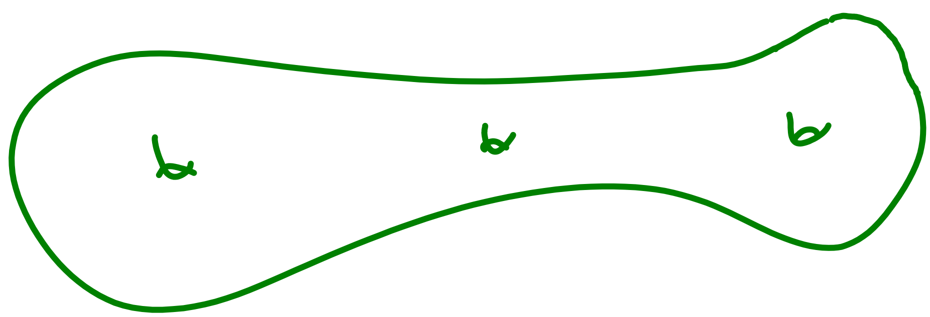
(i) does the quotient $\mathcal{U}/\mathbb{P}GL_3$ make sense in algebraic geometry? YES, GIT

A beautiful theory of quotients

Geometric Invariant Theory

(ii) for $p \in \mathcal{U}/_{\mathbb{P}GL_3}$,

let C_p be the corresponding genus 3 curve

C_p = 

If $p \neq q \in \mathcal{U}/_{\mathbb{P}GL_3}$,

is it true that

$C_p \not\cong C_q$. [Yes, canonical curves]

(iii) If C is curve of genus 3, does there always exist a $p \in \mathcal{U}/\mathbb{P}GL_3$ such that

$$C \cong C_p \;?$$

[Sadly no, we miss all hyperelliptic curves]

We have

$$\underset{\text{open set}}{\mathcal{U}/\mathbb{P}GL_3 \subset} \underset{\text{full moduli}}{\overset{\dim 6}{\mathcal{M}_3}}$$

Moreover :

$$\underset{\text{dim } 6}{\mathcal{M}_3} \setminus \underset{\text{dim } 6}{\mathcal{U}/_{\mathbb{P}GL_3}} = \underset{\text{dim } 5}{\mathcal{H}_3}$$

all genus 3

plane quartics

hyperelliptic genus 3

[A hyperelliptic curve of genus g

$C \xrightarrow[2-1]{} \mathbb{P}^1$ has $2g+2$ branchings

H_g has dim $2g+2-3 = 2g-1$]

(3) How can we construct moduli in general?

- Start with a genus $g \geq 2$
- Let C be a curve of genus g. nonsingular complete connected
- Embed $C \hookrightarrow \mathbb{P}^N$ via a high multiple of the canonical bundle: K_C^M,

$\deg(K_C) = 2g-2$

By RR: $N+1 = M(2g-2) - g + 1$

• The role of $\mathcal{U}$ in genus 3 is played by the Hilbert Scheme of M-canonical curves in $\mathbb{P}^N$:

$\mathcal{U} \subset$ HilbScheme of all curves

$\mathcal{U}$ is nonsingular of dimension

$$H^0(C, Nor_{C/\mathbb{P}^N}) - g$$

deformation theory of the Hilbert scheme

all deformations in $\mathbb{P}^N$

$H^1(C, Nor) = 0$

Since M-Canonical

Let us calculate

Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^N} \to \bigoplus_{0}^{N} \mathcal{O}_{\mathbb{P}^N}(1) \to T_{\mathbb{P}^N} \to 0$$

$$\chi(C, T_{\mathbb{P}^N}|_C)$$

$$= (N+1)\,\chi(C, \mathcal{O}_{\mathbb{P}}(1)|_C) - \chi(C, \mathcal{O}_{\mathbb{P}^N}|_C)$$

$$= (N+1)\,\chi(C, K_C^M) - \chi(C, \mathcal{O}_C)$$

RR

$$= (N+1)\big(M(2g-2)-g+1\big) + g - 1$$

Normal bundle sequence

$$0 \to T_C \to T_{\mathbb{P}^N}\big|_C \to Nor_{C|\mathbb{P}^N} \to 0$$

$$\chi(C, Nor)$$

$$= \chi(C, T_{\mathbb{P}^N}|_C) - \chi(C, T_C)$$

$$= (N+1)\left(\mathcal{M}(2g-2) - g + 1\right) + g - 1$$

$$- (2 - 2g - g + 1)$$

$$\dim \mathcal{U} = (N+1)\left(\mathcal{M}(2g-2) - g + 1\right) - 1$$

$$+ 3g - 3$$

- The role of PGL_3 is now taken by $\mathbb{P}GL_{N+1}$

$$\mathcal{M}_g \cong \mathcal{U} / \mathbb{P}GL_{N+1}$$

$$\dim \mathcal{M}_g = \dim \mathcal{U} - \dim \mathbb{P}GL_{N+1}$$

$$= (N+1)\big(\mu(2g-2)-g+1\big)-1+3g-3 - \big((N+1)^2-1\big)$$

$$= 3g-3$$ (Known already to Riemann)

- $\mathcal{M}_g \cong \mathcal{U}/\mathbb{P}GL_{N+1}$

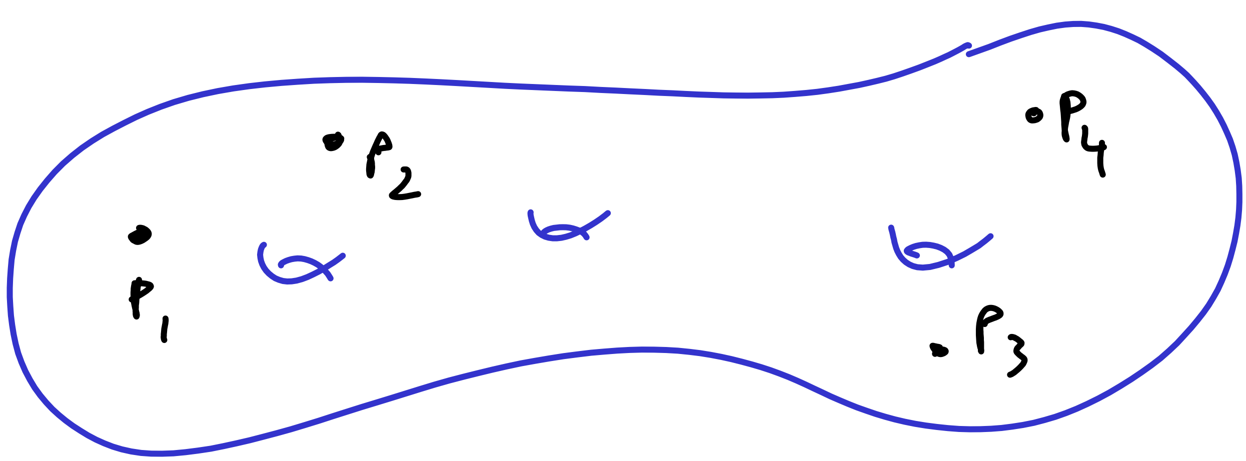
GIT quotient $\Rightarrow$ $\mathcal{M}_g$ is a quasi projective variety with quotient singularities

As a Deligne-Mumford stack (orbifold) $\Rightarrow$ $\mathcal{M}_g$ is <u>non</u>sing<u>ular</u> of dim $3g-3$

(4) Marked points

$\mathcal{M}_{g,n}$ = moduli space of nonsingular algebraic curves with n distinct (ordered) points

$$[C, p_1, \ldots, p_n] \in \mathcal{M}_{g,n}$$



$\mathcal{M}_{g,1}$ plays a special role:

$$\begin{tikzcd} \mathcal{M}_{g,1} \arrow[d, "\pi"'] & \ni & C \arrow[d] \\ \mathcal{M}_g & \ni & [C] \end{tikzcd}$$

$\mathcal{M}_{g,1}$ is the universal curve

Sometimes we write

$$\begin{tikzcd} \mathcal{C}_g \arrow[d, "\pi"'] \\ \mathcal{M}_g \end{tikzcd}$$

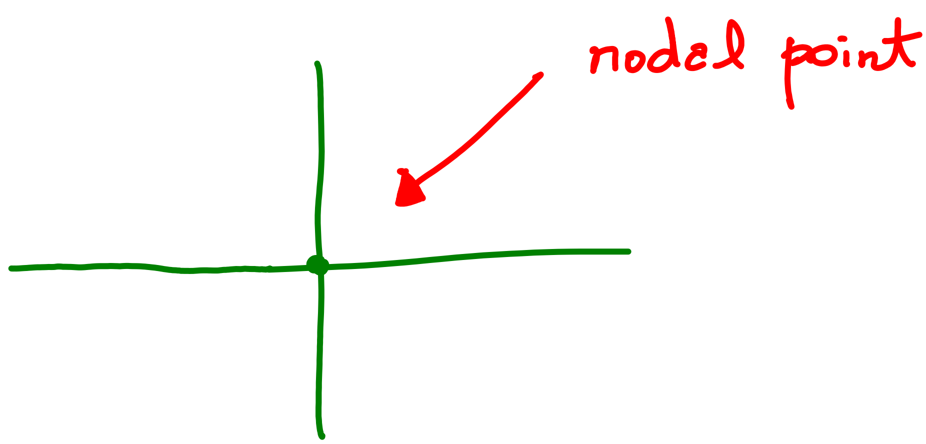
$\mathcal{C}_g \cong \mathcal{M}_{g,1}$

(5) Deligne-Mumford stability

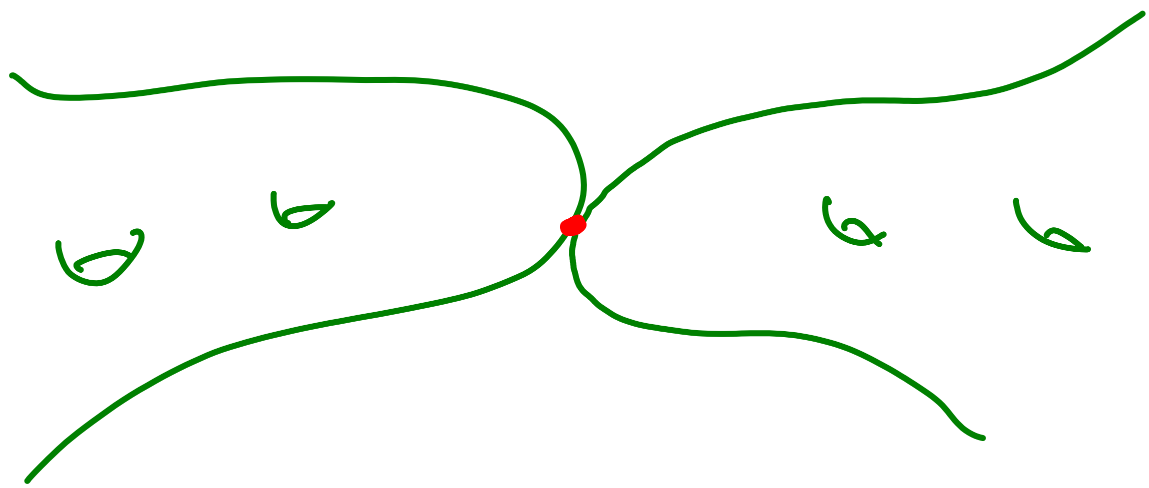
For the first time, we consider singular curves. But only the mildest singularities: nodes

A node is locally (analytically):

$$xy = 0 \subset \mathbb{C}^2$$



The picture topologically :

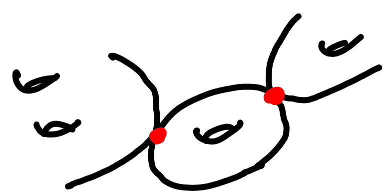


A Deligne-Mumford stable curve :

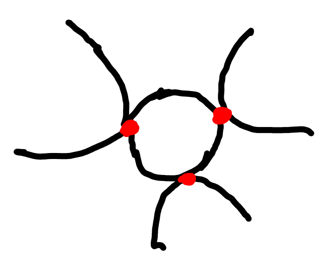
- C is complete connected curve with (at worst) nodal singularities
- C has a finite Aut group

The Finite Aut Condition is equivalent to :

- Every component $E \subset C$ which is nonsingular of genus 1 must contain a nodal point of C

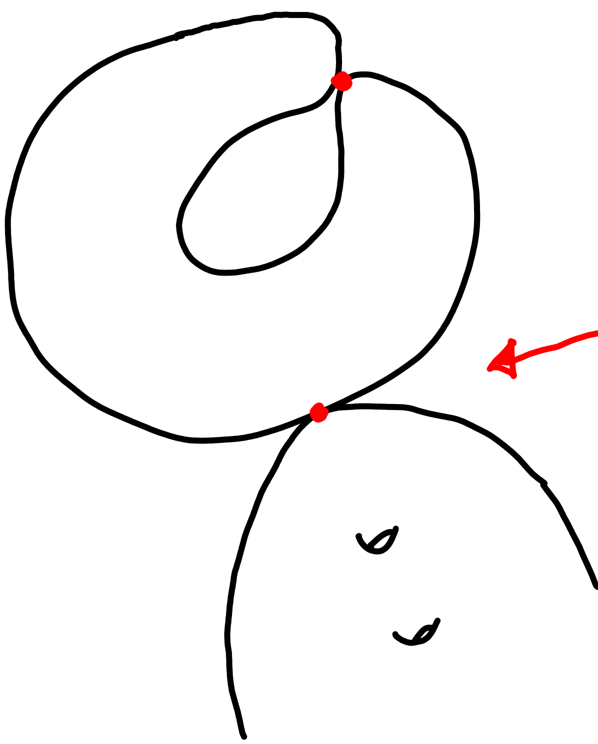


- Every component $P \subset C$ which is nonsingular of genus 0 must contain 3 nodal points of C



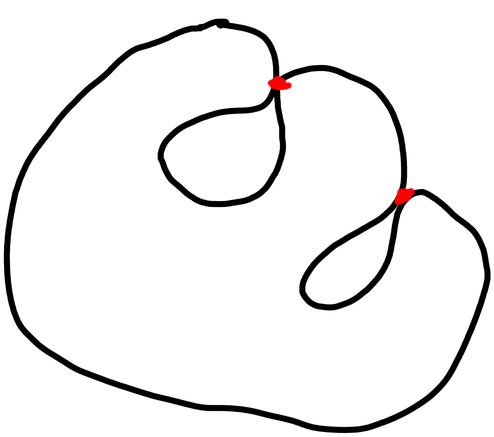
• Nodal rational components:

1-Nodal component

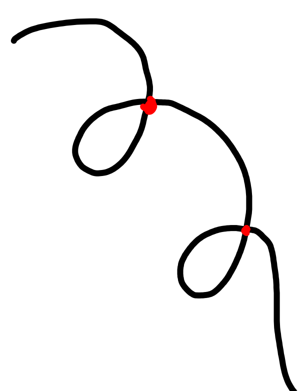


must meet an additional node of C

≥ 2-Nodal components have no further conditions

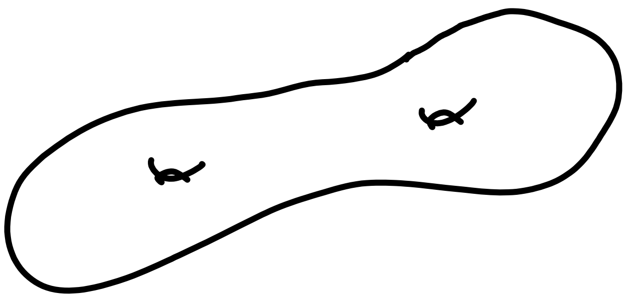


Topological Picture

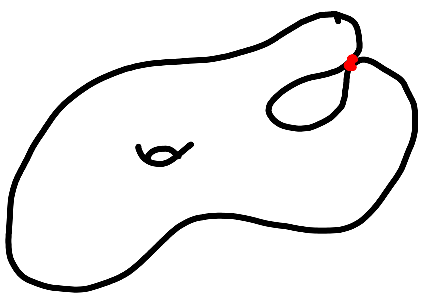


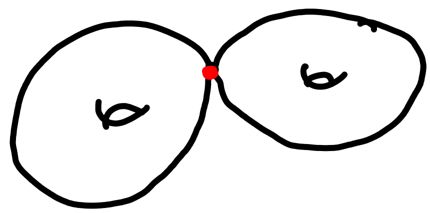
algebro-geometric Picture

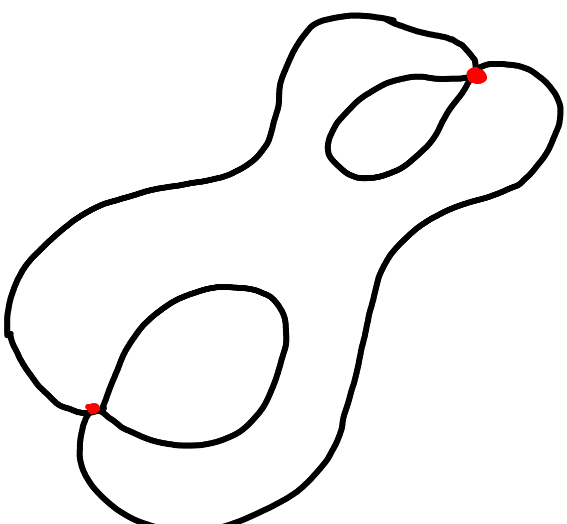
Examples in genus 2

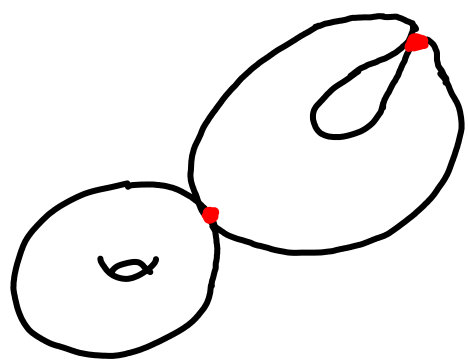


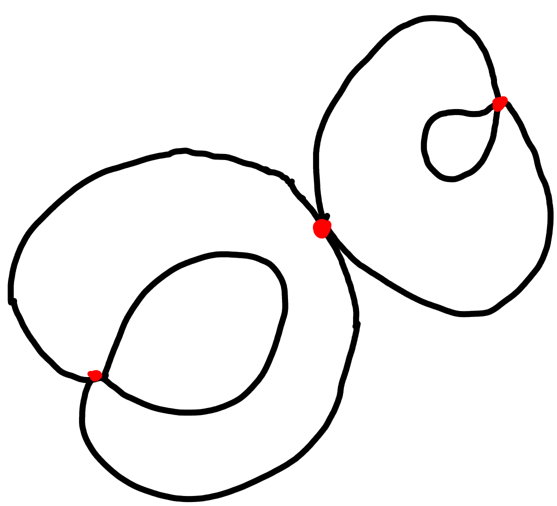
Every nonsingular curve of $g \geq 2$ has finite Aut











Are these all possibilities?

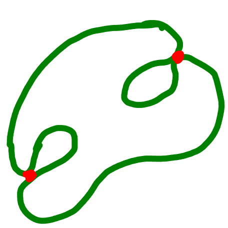
What is the genus of a Deligne-Mumford stable curve C?

$$g(C) \overset{\text{def}}{=} h^1(\mathcal{O}_C)$$

We have $\chi(\mathcal{O}_C) = 1 - g(C)$

Exercise: Show all genus 2 examples in fact have genus 2.

normalization

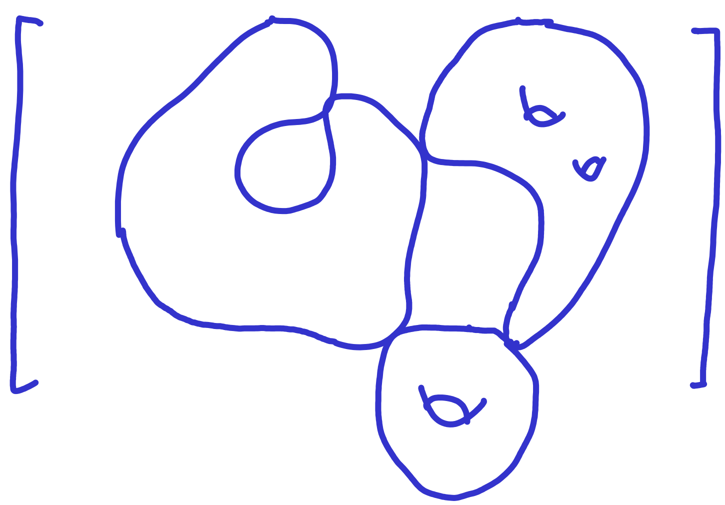
[$R =$  Then $\mu : \mathbb{P}^1 \to R$

use $0 \to \pi_* \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_R \to \mathbb{C}^2 \to 0$]

Theorem (Deligne-Mumford 1969):

The moduli space $\overline{M}_g$ of DM stable curves of genus $g \geq 2$ is a complete connected nonsingular DM stack of dim $3g-3$.

compact

$\overline{M}_5 \ni$ 

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Advice: dont worry about the stack issues now.

(6) Stability with marked points:

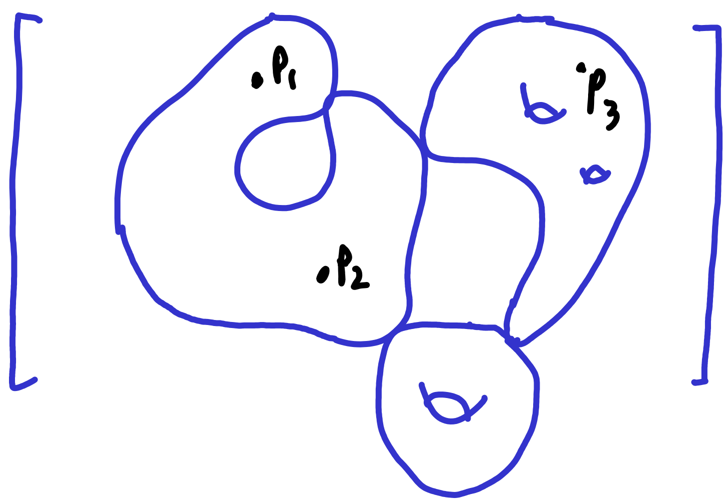
$[C, p_1, \dots, p_n]$ is a Deligne-Mumford stable n-pointed curve if

- C is complete connected curve with (at worst) nodal singularities
- The markings $p_1, \dots, p_n \in C$ are distinct and lie in the nonsingular locus of C
- There are only finitely many Automorphisms of C which fix $p_1, \dots, p_n \in C$.

The moduli space $\overline{\mathcal{M}}_{g,n}$ of DM stable genus g, n pointed curves is a complete connected nonsingular DM stack of dim $3g-3+n$.

compact

$$\boxed{\text{stability} \Rightarrow 2g-2+n>0}$$

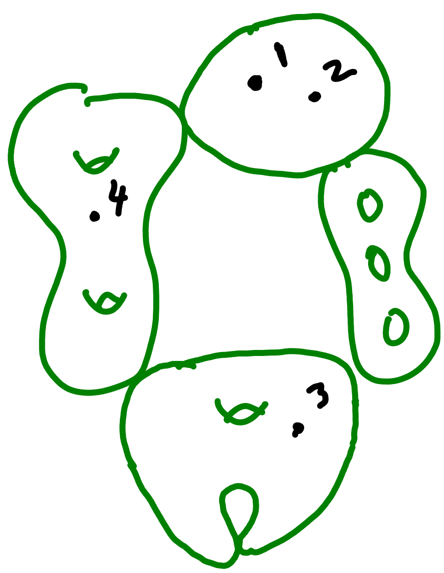
$\overline{\mathcal{M}}_{\underset{g}{5},\underset{n}{3}} \ni$ 

Remark: The same path for the construction of $\mathcal{M}_g$ can be followed for $\overline{\mathcal{M}}_g$, $\overline{\mathcal{M}}_{g,n}$

(7) Stratification by topology

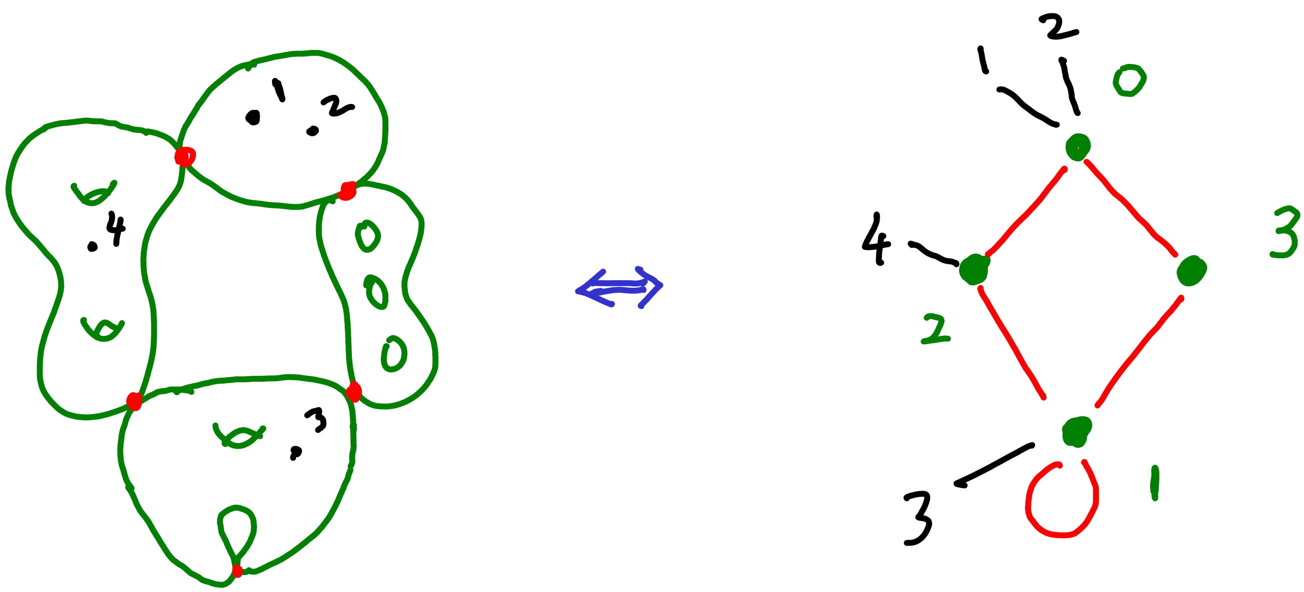
$\overline{M}_{g,n}$ admits a quasi projective stratification by the topological type of the pointed curve.

Example:



Topological type determines a locus of $\overline{M}_{8,4}$

We record the data of the topological type much more efficiently via the dual graph:

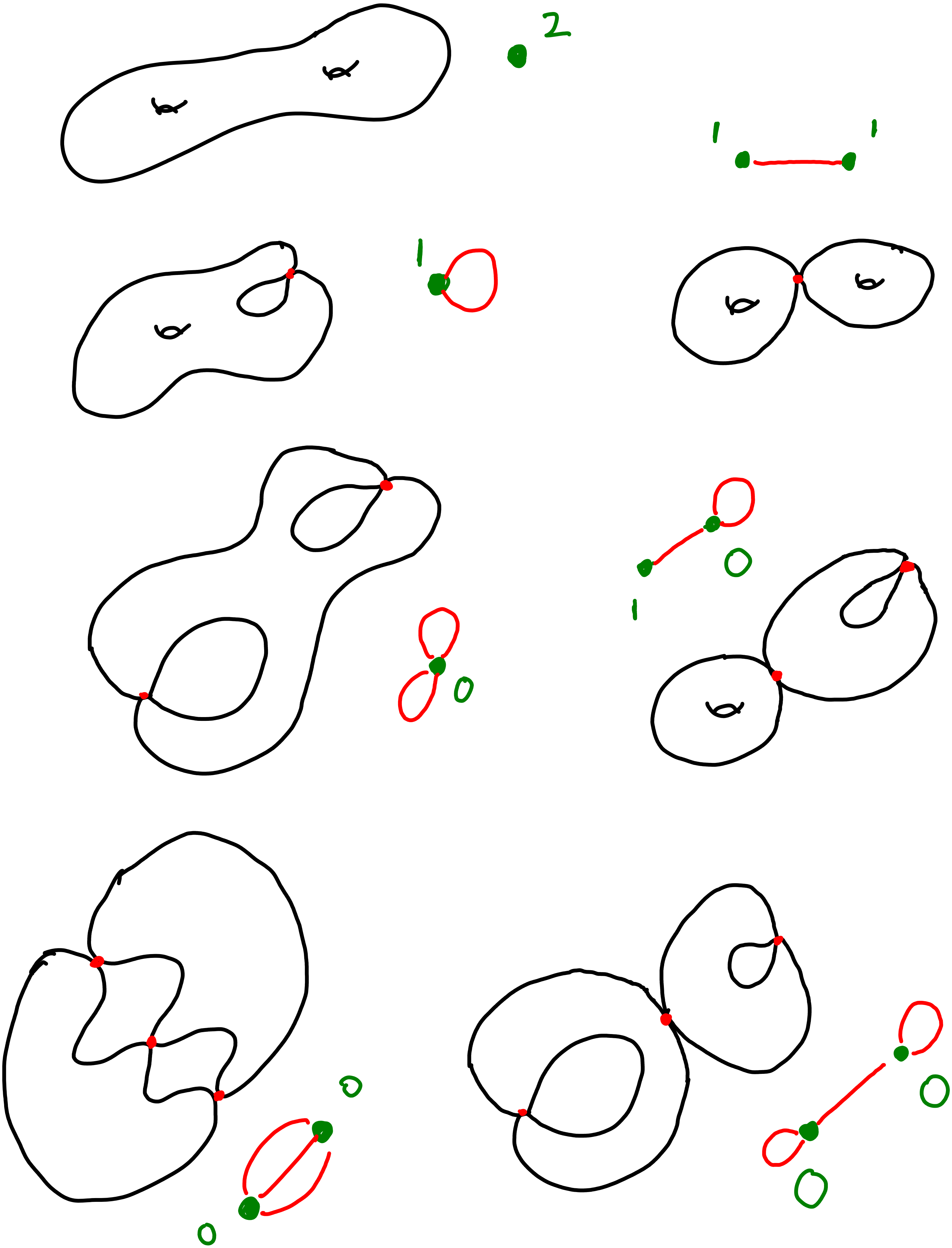


Topological Type

Dual graph Γ

Exercise: $g(C) = g(\Gamma_C)$

$$\overset{\text{def}}{=} h^1(\Gamma_C) + \sum_{v \in \mathrm{Vert}} g(v)$$



2
1
1
1
1
0
1
0
0
0
0
0
0

(8) Strata classes

The simplest classes in the cohomology of $\overline{M}_{g,n}$ correspond to the closures of strata.

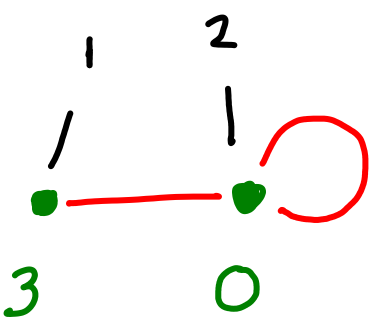
Let $G_{g,n}$ be the set of stable dual graphs corresponding to strata of $\overline{M}_{g,n}$.

Let $\Gamma \in G_{g,n}$ be a dual graph.

We associate to Γ a product of moduli spaces of curves:

$$\Gamma \rightsquigarrow \overline{M}_\Gamma = \prod_{v \in \mathrm{Vert}(\Gamma)} \overline{M}_{g(v),\, \mathrm{Val}(v)}$$

Example:

$\Gamma =$  $\rightsquigarrow \overline{M}_{3,2} \times \overline{M}_{0,4}$

Then we have a canonical glueing morphism

$$\xi_\Gamma : \overline{M}_\Gamma \longrightarrow \overline{M}_{g,n}$$

We define a cohomology class via ξ_Γ.

More precisely,

$$[\Gamma] \in H^{*}(\overline{M}_{g,n})$$

Fundamental class

$$[\Gamma] = \frac{1}{|\mathrm{Aut}(\Gamma)|} \xi_{\Gamma *} [\overline{M}_{\Gamma}]$$

Automorphism group of a stable graph (finite)

Cohomological push-forward via Poincaré duality

Exercise: $[\Gamma] \in H^{2E}(\overline{M}_{g,n})$

where $E = |\mathrm{Edge}(\Gamma)|$

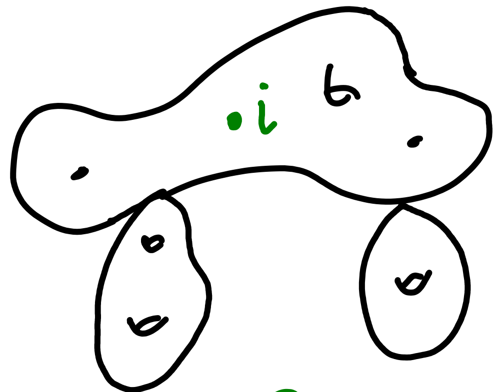
(g) ψ and κ classes

In addition to the strata classes $[\Gamma] \in H^*(\overline{M}_{g,n})$, there are two more basic constructions

- Cotangent line classes ψ

Let $\begin{array}{c} \mathbb{L}_i \\ \downarrow \\ \overline{M}_{g,n} \end{array}$ be the cotangent line associated to the i^{th} marking

$$\begin{array}{ccc} \mathbb{L}_i & \supset & T^*_{p_i}C \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{g,n} & \ni & [C] \end{array}$$



C

Here we are describing a $\mathbb{C}$-bundle by canonically defining the fiber over every moduli point.

$$\psi_i \overset{\text{def}}{=} c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n})$$

• Kappa classes

$$\begin{array}{c} \overline{M}_{g,n+1} \\ \downarrow \pi \\ \overline{M}_{g,n} \end{array}$$

forget marking $n+1$

Define by push-forward along π.

$$\kappa_a \overset{\text{Def}}{=} \pi_* \left(\psi_{n+1}^{a+1} \right) \in H^{2a}(\overline{M}_{g,n})$$

$$RH^*(\overline{M}_{g,n}) \subset H^*(\overline{M}_{g,n})$$

Subring of tautological classes

Complex grading

$$RH^*(\overline{M}_{g,n}) \overset{\text{Def}}{\subset} H^{2*}(\overline{M}_{g,n})$$

is the linear span of

$$[\Gamma, \alpha] \in H^*(\overline{M}_{g,n})$$

Stable dual graph

monomial in all ψ and κ classes on vertices of Γ

$$[\Gamma, \alpha] \overset{\text{Def}}{=} \frac{1}{|\mathrm{Aut}(\Gamma)|} \xi_*\left(\alpha \cap [\overline{M}_\Gamma]\right) \in H^*(\overline{M}_{g,n})$$

While $H^*(\bar{M}_{g,n})$ is mysterious,

$RH^*(\bar{M}_{g,n})$ is less so

but still not completely understood

[Pixton's Conjecture]

Our point of view:

We have generators and a product rule for $RH^*(\bar{M}_{g,n})$

So we can really work in the ring.

(10) Moduli of Elliptic curves $\overline{M}_{1,1}$

The general method of construction is simple for $\overline{M}_{1,1}$:

quasi projective

$$W \subset \mathbb{P}^2 \times \mathbb{P}^9$$

$$W = \left\{ (p, c) \;\middle|\; \begin{array}{l} C \text{ is cubic} \\ \text{which is nonsingular} \\ \text{or 1-nodal,} \\ p \in C^{ns} \text{ is a flex} \end{array} \right\}$$

Theorem: As a quotient stack

$$\overline{M}_{1,1} \cong W / \mathbb{P}GL_3$$

See Notes from my MSRI Lecture

Integration $\displaystyle\int_{\overline{M}_{1,1}} \psi_1$ in $H^2(\overline{M}_{1,1})$

Complex dimension = 1

Real dimension = 2

$$\int_{\overline{M}_{1,1}} \psi_1 \overset{\text{Def}}{=} \text{degree}\left(\psi_1 \cap [\overline{M}_{1,1}]\right)$$

in $H_0(\overline{M}_{1,1})$

$$\in \mathbb{Q}$$

Since $\overline{M}_{1,1}$ is a

Deligne-Mumford stack.

How can we compute $\int_{\overline{M}_{1,1}} \psi_1$?

Idea is to find a nice family of stable 1-pointed elliptic curves:

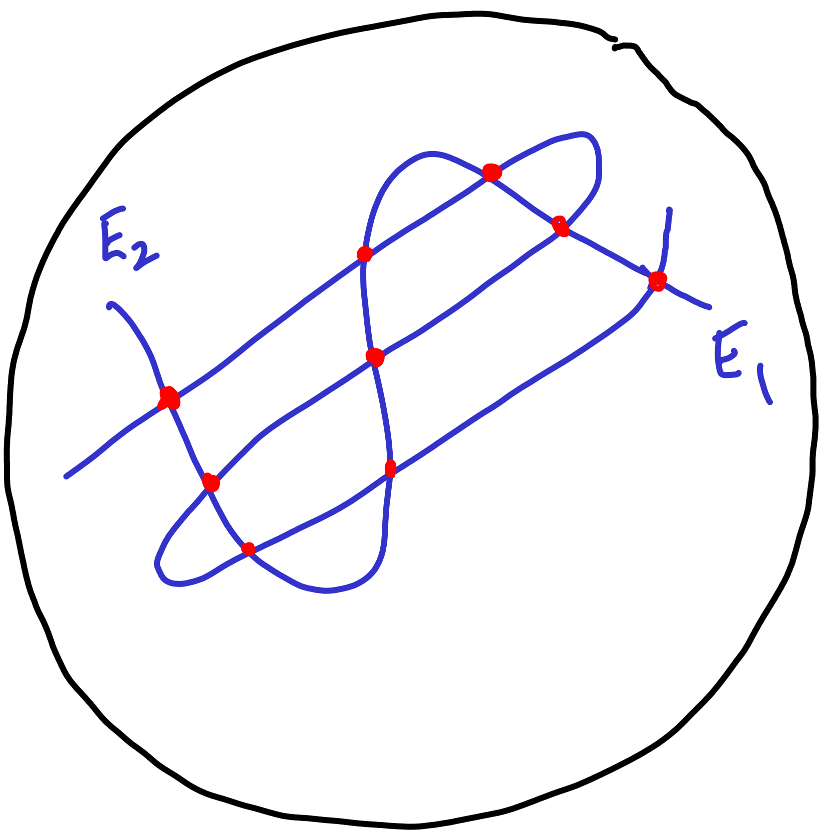
$$\begin{tikzcd} \mathcal{C} \arrow[d, "\pi"'] \\ B \arrow[u, bend left, "p"] \end{tikzcd}$$

fibers are stable elliptic curves with marking p

nonsingular of dimension 1

There is a beautiful family of cubics:

Let E_1 and E_2 be general nonsingular cubics in $\mathbb{P}^2$



There are exactly 9 intersection points

Bezout

Let $\mathcal{E}$ be the blow-up

of $\mathbb{P}^2$ at the nine

intersection points

Then $\mathcal{E} \xrightarrow{\pi} \mathbb{P}^1 = \text{Base}$

by the linear series determined

by E_1, E_2.

Exercise: What is $\chi_{top}(\mathcal{E})$?

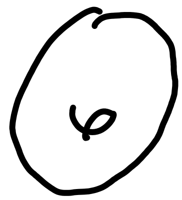
$$\left[\ 3 - 9 + 9 \cdot 2 = 12\ \right]$$

ANSWER

While most fibers of

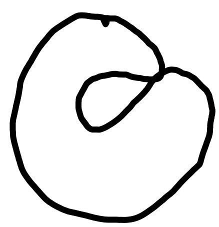
$$\mathcal{C} \xrightarrow{\pi} \mathbb{P}^1$$

$\chi_{top} = 0$

are nonsingular  ,

finitely many are singular

$\chi_{top} = 1$

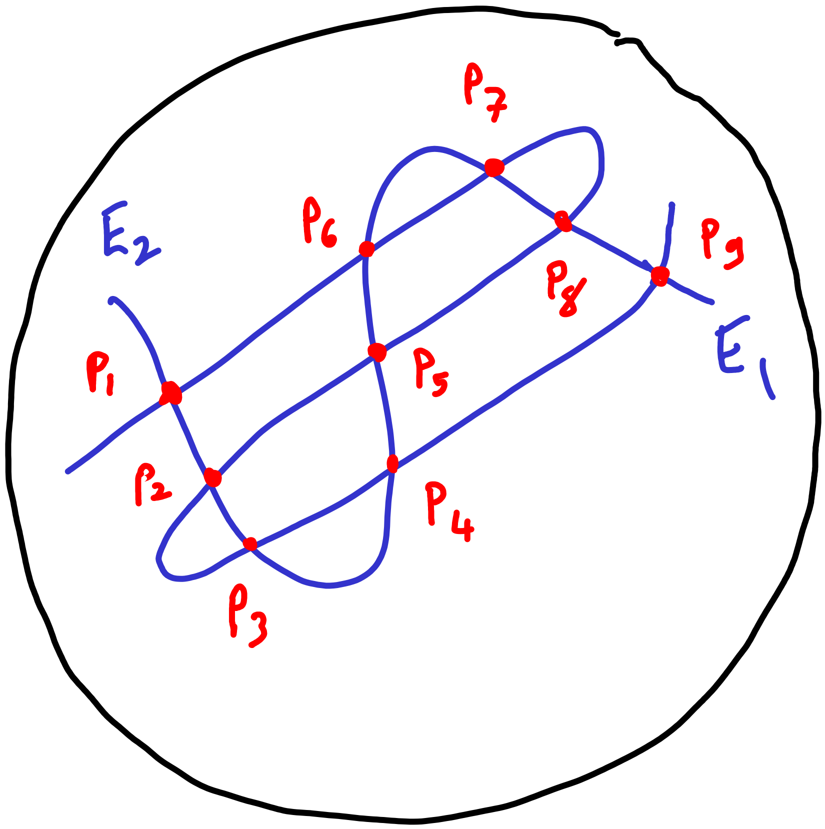
with one node 

How many nodal fibers of π ?

$$\chi_{top}(\mathcal{C}) = 12$$

Since nonsingular fibers do not contribute

$\Longrightarrow$ number of nodal fibers $= 12$



The family $\mathcal{C} \xrightarrow{\pi} \mathbb{P}^1$ has nine sections corresponding to $P_1, P_2, \ldots, P_9 \in \mathbb{P}^2$

Summary:
$$\begin{tikzcd} \mathcal{C} \arrow[d, "\pi"'] \\ \mathbb{P}^1 \arrow[u, bend right, "P_1"'] \end{tikzcd}$$
is a family of stable 1-pointed genus 1 curves

By the universal property of the moduli space of curves, the family

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\pi"'] \\ \mathbb{P}^1 \arrow[u, bend right, "P_1"'] \end{tikzcd}$$

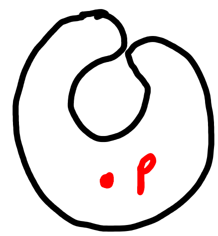
Base

canonically induces a morphism

$$\mathcal{E}_\pi : \underset{\text{Base}}{\overset{\text{dim } 1}{\mathbb{P}^1}} \longrightarrow \overset{\text{dim } 1}{\overline{M}_{1,1}}$$

What is the degree of $\mathcal{E}_\pi$?

We can calculate degree(ε_π) by careful analysis of any fiber of ε_π.

We will analyse the fiber of ε_π over [

] $\in \overline{M}_{1,1}$.

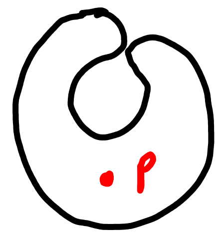
We have already seen that

ε_π^{-1}([

]) $\subset \mathbb{P}^1$

Consists of 12 points of $\mathbb{P}^1$.

Deformation theory shows $\mathcal{E}_\pi$

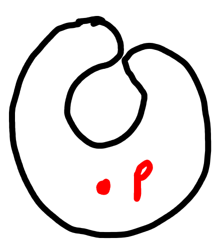
is unramified over $\Big[$ 

$\Big]$.

We might therefore conclude that

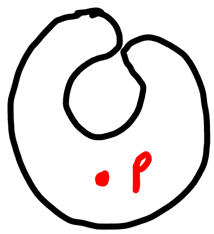
the degree of $\mathcal{E}_\pi$ is 12,

but 12 is wrong.

Why? Because 

is a

stacky point: $\mathrm{Aut} \cong \mathbb{Z}/2\mathbb{Z}$.

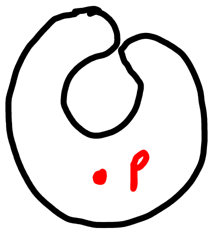
As a conquence, the push-forward to $\mathrm{Spec}(\mathbb{C})$ of $\Big[$  $\Big] \in \overline{\mathcal{M}}_{1,1}$ under

$$\overline{\mathcal{M}}_{1,1} \to \mathrm{Spec}(\mathbb{C})$$

is $\frac{1}{2}$ (in general for proper DM stacks $\frac{1}{|\mathrm{Aut}|}$)

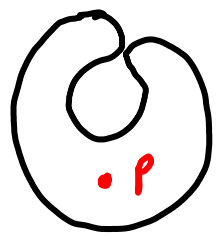
On the other hand, every point of the base $\mathbb{P}^1$ pushes forward to 1 under the map to $\mathrm{Spec}(\mathbb{C})$

In order the satisfy the degree formula:

$\int_{\mathbb{P}^1} \varepsilon_\pi^{-1}\Big(\Big[$ 

$\Big]\Big)$

push forward to Spec($\mathbb{C}$)

$= \deg(\varepsilon_\pi) \int_{\overline{M}_{1,1}} \Big[$ 

$\Big]$,

We must have

$$\text{degree}(\varepsilon_\pi) = 24.$$

We can now finally compute

$$\int_{\overline{M}_{1,1}} \psi_1 \overset{\text{Def}}{=} \operatorname{degree}\left(\psi_1 \cap [\overline{M}_{1,1}]\right)$$

push forward to Spec($\mathbb{C}$)

Use the formula

$$\int_{\mathbb{P}^1} \varepsilon_\pi^*(\psi_1) = \operatorname{degree}(\varepsilon_\pi) \cdot \int_{\overline{M}_{1,1}} \psi_1$$

$$= 24 \cdot \int_{\overline{M}_{1,1}} \psi_1$$

What is $\varepsilon_\pi^*(\psi_1)$ on $\mathbb{P}^1$?

By definition, ψ_1 is Chern class of the cotangent line.

The tangent line at p of the family

$$\begin{tikzcd} C \arrow[d, "\pi"'] \\ \mathbb{P}^1 \arrow[u, bend right, "p"'] \end{tikzcd}$$

is easy to see.

It is the normal bundle to the exceptional curve:

$\mathcal{C}$ = Blow-up of $\mathbb{P}^2$ at the nine intersection points of E_1 and E_2

The image of the section

$$p : \mathbb{P}^1 \to \mathcal{C}$$

is exactly the exception curve of $\mathcal{C}$ corresponding to p.

$$\begin{matrix} \mathbb{L}_1 & & \mathrm{Normal}(\mathcal{C}/\mathbb{P}^1)^* \\ \downarrow & \text{is} & \downarrow \\ \mathbb{P}^1 & & \mathbb{P}^1 \end{matrix}$$

From the geometry of a blow-up

$$\text{degree } \mathrm{Normal}(\mathcal{E}/\mathbb{P}^1) = -1$$

Self intersection of exceptional curve

So we have

$$24 \cdot \int_{\overline{M}_{1,1}} \psi_1 = \int_{\mathbb{P}^1} \varepsilon_\pi^*(\psi_1)$$

$$\int_{\mathbb{P}^1} \varepsilon_\pi^*(\psi_1) = -(-1) = 1$$

We conclude

$$\int_{\overline{M}_{1,1}} \psi_1 = \frac{1}{24}.$$

(11) Witten's Conjecture (1991)

The computation $\int_{\overline{M}_{1,1}} \psi_1 = \frac{1}{24}$

was by explicit geometric study

which is hard to imagine pursuing

for the general integral

$$\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_{g,n} = \int_{\overline{M}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n}$$

n redundant, often dropped

dim constraint: $\sum k_i = 3g-3+n$

$\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_{g,n} = 0$ if constraint fails

Also 0 if $2g-2+n \leq 0$ by stability

To state Witten's Conjecture,

we form a generating series

$$\mathcal{F}_g(t_0, t_1, t_2, \dots) \overset{\text{def}}{=} \sum_{\{n_i\}} \prod_{i=1}^{\infty} \frac{t_i^{n_i}}{n_i!} \langle \tau_0^{n_0} \tau_1^{n_1} \tau_2^{n_2} \cdots \rangle_g$$

$g \geq 0$ genus

Sum over all sequences of non-neg integers with only finitely many nonzero terms

$$\int_{\overline{M}_{g,\Sigma n_i}} \psi_1^0 \cdots \psi_{n_0}^0 \, \psi_{n_0+1}^1 \cdots \psi_{n_0+n_1}^1 \cdots$$

A more compact way:

$$\mathcal{F}_g = \sum_{n=0}^{\infty} \frac{\langle \gamma^n \rangle_g}{n!}$$

where $\gamma = \sum_{i=0}^{\infty} t_i \tau_i$

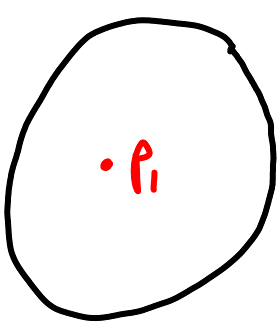
$\mathcal{F}_g$ contains the data of all integrals of ψ-classes over all $\overline{\mathcal{M}}_{g,n}$.

Called descendent integrals

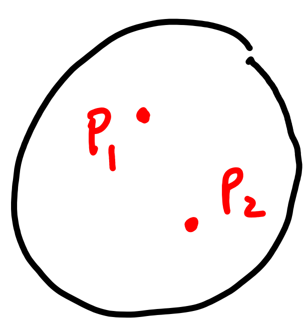
We can put all the data for all genera together

$$\mathcal{F}(\lambda, t) = \sum_{g=0}^{\infty} \lambda^{2g-2} \mathcal{F}_g$$

[For $\mathcal{F}_0$, 1- and 2-point integrals are 0, since



and



are unstable]

Notation for partial derivatives:

$$\langle\langle \tau_{k_1} \cdots \tau_{k_n} \rangle\rangle = \frac{\partial}{\partial t_{k_1}} \frac{\partial}{\partial t_{k_2}} \cdots \frac{\partial}{\partial t_{k_n}} F$$

Witten's Conjecture (KdV):

For $n \geq 1$, we have:

$$(2n+1)\, \lambda^{-2} \langle\langle \tau_n \tau_0^2 \rangle\rangle =$$

$$\langle\langle \tau_{n-1} \tau_0 \rangle\rangle \langle\langle \tau_0^3 \rangle\rangle + 2 \langle\langle \tau_{n-1} \tau_0^2 \rangle\rangle \langle\langle \tau_0^2 \rangle\rangle$$

$$+ \frac{1}{4} \langle\langle \tau_{n-1} \tau_0^4 \rangle\rangle$$

Let $u = \dfrac{\partial^2 F}{\partial t_0^2}$

Then the $n=1$ equation $\Rightarrow$

$$\frac{\partial u}{\partial t_1} = u \frac{\partial u}{\partial t_0} + \frac{1}{12} \frac{\partial^3 u}{\partial t_0^3}$$ (set $\lambda = 1$)

Korteweg-de Vries equation

$t_0 \rightsquigarrow x$ space

$t_1 \rightsquigarrow t$ time

u height of wave

First written in the late 19^{th} century to model shallow water waves

What do water waves have to do with $\overline{\mathcal{M}}_{g,n}$?

Long and interesting story

Witten, 2d quantum gravity and intersection theory on moduli space

$$\text{KdV equations} \xleftrightarrow{\text{integrable systems}} \text{Matrix integrals } \int_{\text{matrices}} \xleftrightarrow{\text{2d Gravity}} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n}$$

Proven by Kontsevich 1992, others

Okounkov-P, GW theory, Hurwitz numbers and Matrix models

What about $\langle \tau_1 \rangle_1 = \frac{1}{24}$?

Take equation for $n=3$:

$$7 \langle \tau_3 \tau_0^2 \rangle_1 = \langle \tau_2 \tau_0 \rangle_1 \langle \tau_0^3 \rangle_0 + \frac{1}{4} \langle \tau_2 \tau_0^4 \rangle_0$$

Set $\lambda = 1$ and all $t_i = 0$

After applying the string equation:

$$\langle \tau_3 \tau_0^2 \rangle_1 = \langle \tau_2 \tau_0 \rangle_1 = \langle \tau_1 \rangle_1$$

$$\langle \tau_2 \tau_0^4 \rangle_0 = \langle \tau_0^3 \rangle = 1$$

$\int_{\overline{M}_{0,3}} 1 = 1$

We find

$$7\langle \tau_1 \rangle_1 = \langle \tau_1 \rangle_1 \cdot 1 + \frac{1}{4} \cdot 1$$

So $\quad 6\langle \tau_1 \rangle_1 = \frac{1}{4}$

Finally $\quad \langle \tau_1 \rangle_1 = \frac{1}{24}$

Exercise: Prove the string equation

$$\left\langle \tau_0 \prod_{i=1}^{n} \tau_{k_i} \right\rangle_{g,n+1} = \sum_{j=1}^{n} \left\langle \tau_{k_j-1} \prod_{i\neq j} \tau_{k_i} \right\rangle_{g,n}$$

Convention: $\tau_k = 0$ for $k < 0$.

Exercise: Use the string equation to prove $\langle \tau_{k_1} \cdots \tau_{k_n} \rangle_{0,n} = \binom{n-3}{k_1, \ldots, k_n}$.

Exercise: Prove the dilaton equation

$$\left\langle \tau_1 \prod_{i=1}^{n} \tau_{k_i} \right\rangle_{g,n+1} = (2g-2+n) \left\langle \prod_{i=1}^{n} \tau_{k_i} \right\rangle_{g,n}$$

What about removing τ_k for $k>1$?

Answer: Virasoro Constraints

See my Notes (linked on the Course Webpage).

(12) genus 0

I havent spoken much about genus 0 because the main Abel-Jacobi questions are all trivial in genus 0.

There is, however, a lot of beautiful geometry in genus 0.

The moduli space $\overline{M}_{0,n}$ is a nonsingular variety of dimension $n-3$.

$\overline{\mathcal{M}}_{0,n}$ is much easier to construct:

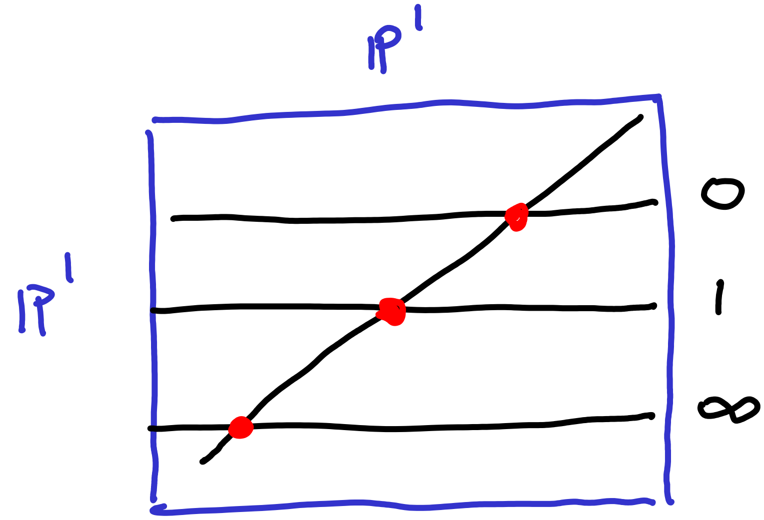
Let $P = \mathbb{P}^1 \setminus \{0,1,\infty\}$

$$\mathcal{M}_{0,n} = \underbrace{P \times \cdots \times P}_{n-3} \setminus \Delta$$

where Δ is the set of all diagonals.

$\overline{\mathcal{M}}_{0,n}$ can be constructed explicitly as a blow up of $(\mathbb{P}^1)^{n-3}$.

Exercise: Prove $\overline{M}_{0,5}$ is obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by blowing up 3 points: $(0,0)$, $(1,1)$, (∞,∞)



Exercise: Identify the five cotangent line bundles

$\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3, \mathbb{L}_4, \mathbb{L}_5$

on $\overline{M}_{0,5} \cong \text{Blow-up}_3(\mathbb{P}^1 \times \mathbb{P}^1)$

After completing these exercises, the general formula

$$\int_{\overline{M}_{0,n}} \psi_1^{k_1} \cdots \psi_1^{k_n} = \binom{n-3}{k_1, \ldots, k_n}$$

Consequence of the String equation

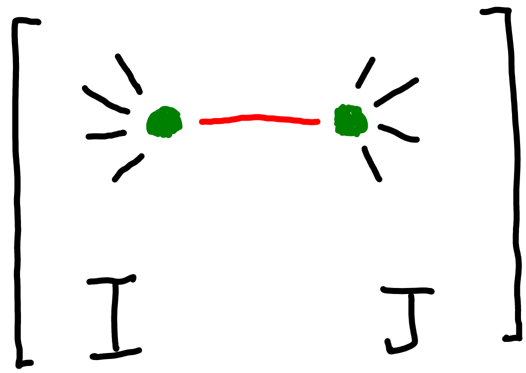
is easy to check for $n=5$.

About tautological classes in genus 0, we know a lot (almost everything):

- $RH^*(\overline{M}_{0,n}) = H^*(\overline{M}_{0,n})$

- $RH^*(\overline{M}_{0,n})$ is generated by boundary divisors

$I \cup J = \{1, \dots, n\}$
$|I| \geq 2$
$|J| \geq 2$,



- Complete ideal of relations is generated by

(i) all empty intersections

(ii) all cross ratio relations

Cross ratio Relations (WDVV relations)

Choose 4 markings

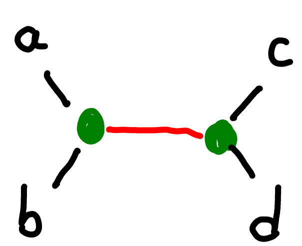
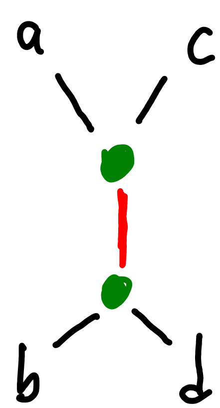
$$\{a, b, c, d\} \subset \{1, 2, 3, \dots, n\}$$

Then there is a forgetful map

$$\varepsilon : \overline{M}_{0,n} \to \overline{M}_{0,4}$$

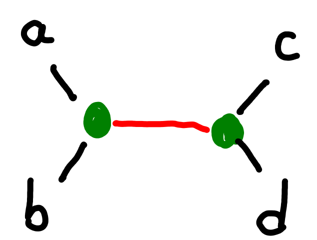
$\{a,b,c,d\}$

Since $\overline{M}_{0,4} \cong \mathbb{P}^1$

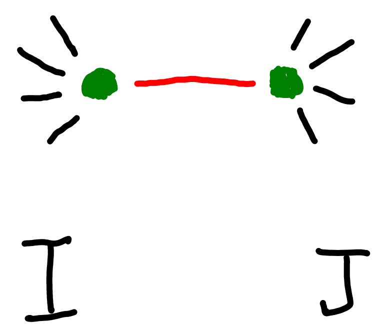
[] = [] $\in H^2(\overline{M}_{0,4})$

After pull-back, we obtain relations among boundary divisors of $\overline{M}_{0,n}$.

Exercise: Prove

ε^* [

] =

$\sum$ [

] $\in H^2(\overline{M}_{0,n})$

$I \cup J = \{1,2,3,\dots,n\}$

$\{a,b\} \subset I$

$\{c,d\} \subset J$

(13) Open questions in $g=0$

Question: What is the cone of effective curve classes in $H_2(\overline{M}_{0,n})$?

Is the effective cone generated by the classes of 1-dimensional strata?

The affirmative assertion is called the F-conjecture.

Warning: F-conjecture could well be false.

Fulton asked: is every effective cycle class on $\overline{M}_{0,n}$ equal to a non-negative linear combination of strata classes $[\Gamma]$?

Answer: No for divisors

Keel-Vermeire 2002

Counter example on $\overline{M}_{0,6}$

Maybe we need positivity:

Is every NEF divisor on $\overline{M}_{0,n}$ equal to non-negative linear combination of boundary strata?

Answer: No for $\overline{M}_{0,12}$

Pixton 2013 constructs an interesting effective and NEF divisor on $\overline{M}_{0,12}$ providing a counterexample.

(14) Kapranov's Construction ~1990

We start with a moduli point

$$[C, p_1, \dots, p_n] \in \mathcal{M}_{0,n}$$

$$C \cong \mathbb{P}^1$$

Consider $T_C^* \leftarrow \omega_C$ dualizing sheaf

Construct the line bundle

$$\omega_C(p_1 + p_2 + \cdots + p_n)$$

on C.

No choices made, degree is $-2+n>0$

Let $V = H^0\left(C, \omega_C\left(\sum_i p_i\right)\right)$

$\dim_{\mathbb{C}} V = n-1$

There is a canonical map

$$f : C \to \mathbb{P}(V^\vee)$$

and this is an embedding,

f embedds C as a rational normal curve in $\mathbb{P}(V^\vee)$

No choices have been made

Exercise: Prove the points

$$f(p_1), f(p_2), \ldots, f(p_n) \in \mathbb{P}(V^\vee)$$

$\dim_{\mathbb{C}} = n-2$

are in linear general position.

no 2 are equal

no 3 lie on a line

no 4 lie on a plane

$\vdots$

no $n-1$ lie on a hyplane

Hint: Easy consequence of the Vandermonde determinant formula

Consider next the fixed projective projective space $\mathbb{P}^{n-2}$ with n points

$$\left.\begin{aligned} q_1 &= [1, 0, 0, \dots, 0] \\ q_2 &= [0, 1, 0, \dots, 0] \\ &\vdots \\ q_{n-1} &= [0, 0, 0, \dots, 0, 1] \\ q_n &= [1, 1, 1, 1, \dots, 1] \end{aligned}\right\} \text{points are in general linear position in } \mathbb{P}^{n-2}$$

By projective linear algebra:

There is a unique linear isomorphism

$$\phi : \mathbb{P}(V^\vee) \xrightarrow{\sim} \mathbb{P}^{n-2}$$

which satisfies

$$\phi(f(p_i)) = q_i$$

Conclusion: to each moduli point

$$[C, p_1, \ldots, p_n] \in \mathcal{M}_{0,n},$$

we associate a rational normal curve $C \subset \mathbb{P}^{n-2}$ through $q_1, \ldots, q_n$

The inverse construction is immediate:

To each rational normal curve

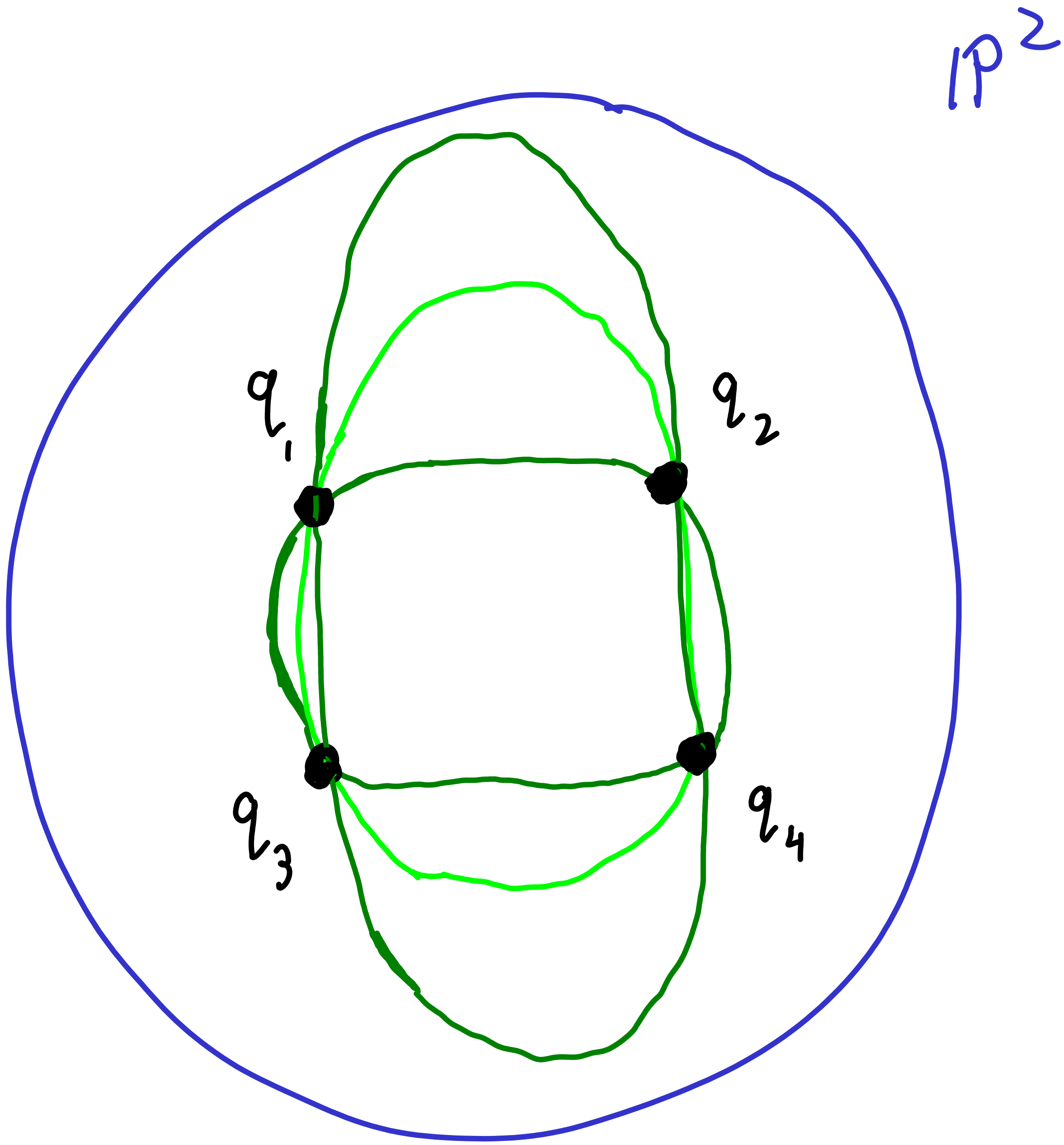
$C \subset \mathbb{P}^{n-2}$ through $q_1, \dots, q_n$,

we associate the moduli point

$$[C, q_1, \dots, q_n] \in \mathcal{M}_{0,n}.$$

These correspondences are inverse to eachother!

Example ($n=4$): RNCs in $\mathbb{P}^2$ are just conics.



We have via Kapranov's Construction:

$$\overline{M}_{0,n} \cong \text{Hilbert Scheme of RNCs in } \mathbb{P}^{n-2} \text{ passing through } q_1, \ldots, q_n$$

Hilbert Scheme of RNCs in $\mathbb{P}^{n-2}$ passing through $q_1, \dots, q_n$ $\overset{\text{quasiprojective}}{\subset}$ Hilbert Scheme

$$\text{Hilbert Scheme of RNCs in } \mathbb{P}^{n-2} \text{ passing through } q_1,\dots,q_n \overset{\text{def}}{=} \mathcal{H}_{RNC}$$

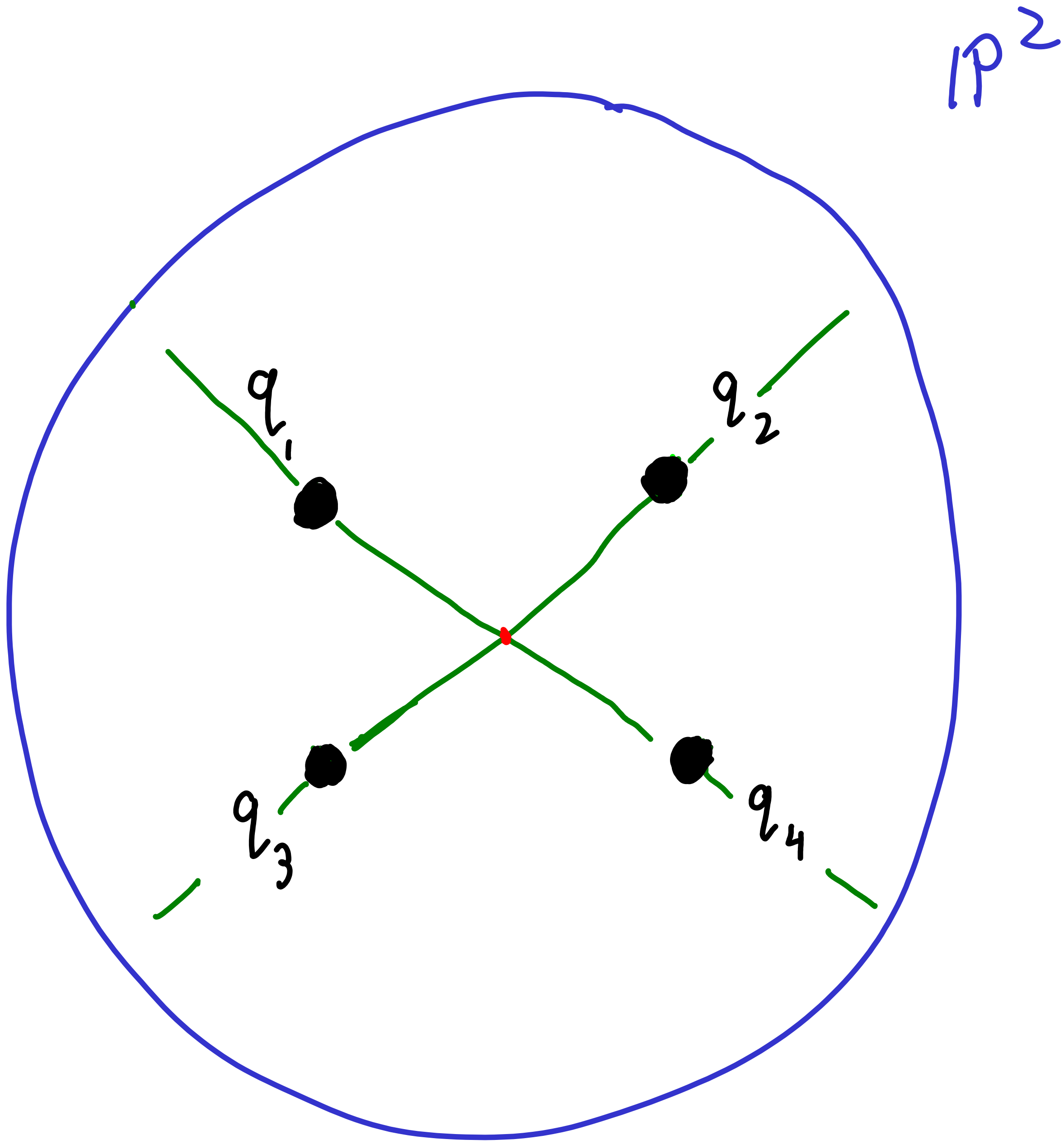
$$\text{Hilbert Scheme} \overset{\text{def}}{=} \mathcal{H}$$

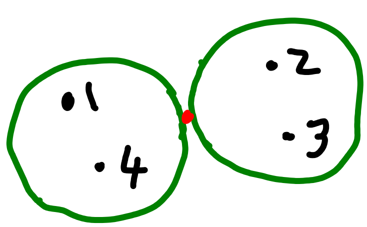
Kapranov's Theorem:

$$\overline{\mathcal{M}}_{0,n} \cong \text{Closure of } \mathcal{H}_{RNC} \text{ in } \mathcal{H}$$

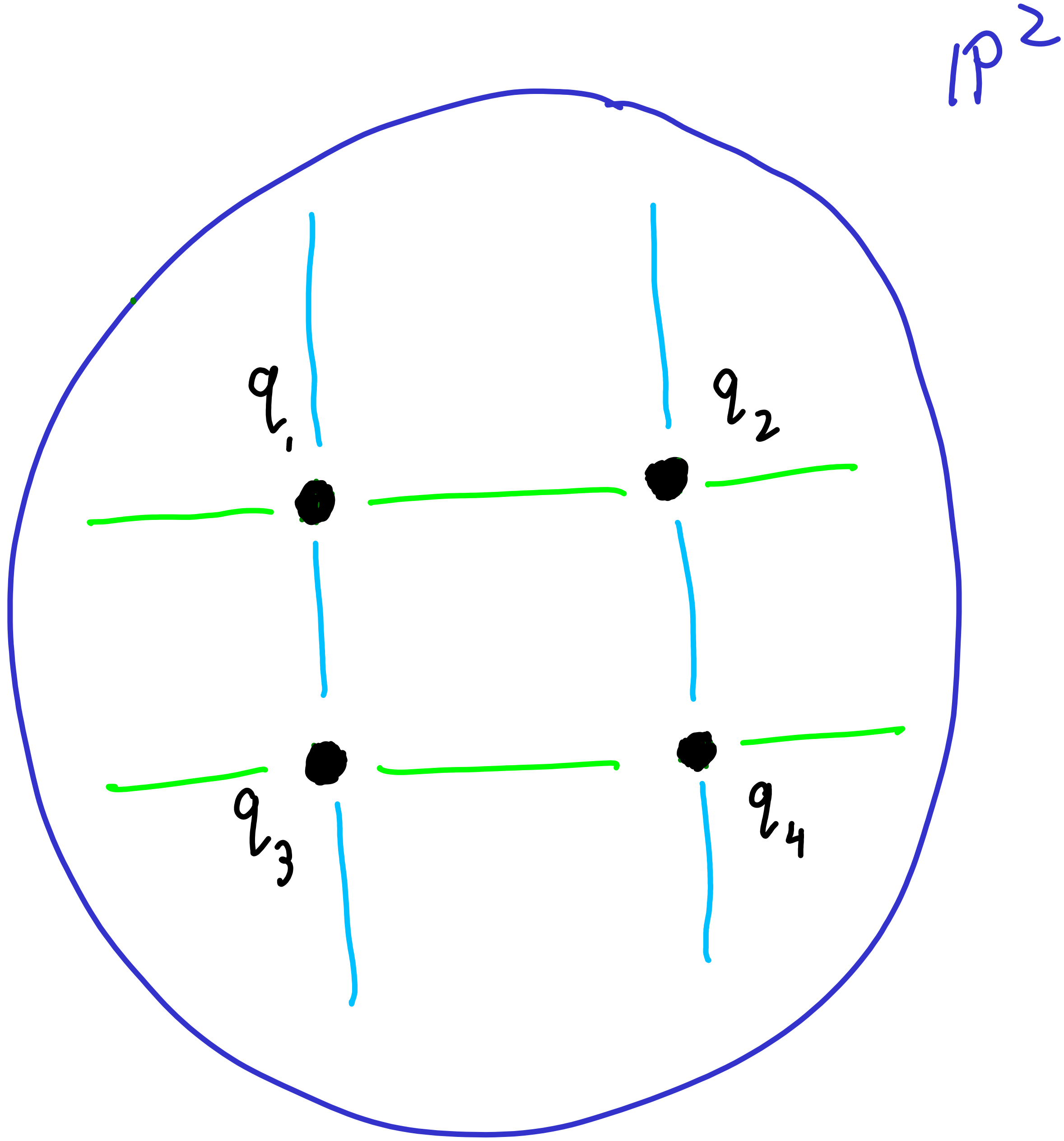
How to find $\overline{\mathcal{M}}_{0,n}$ in nature

Example ($n = 4$ revisited):

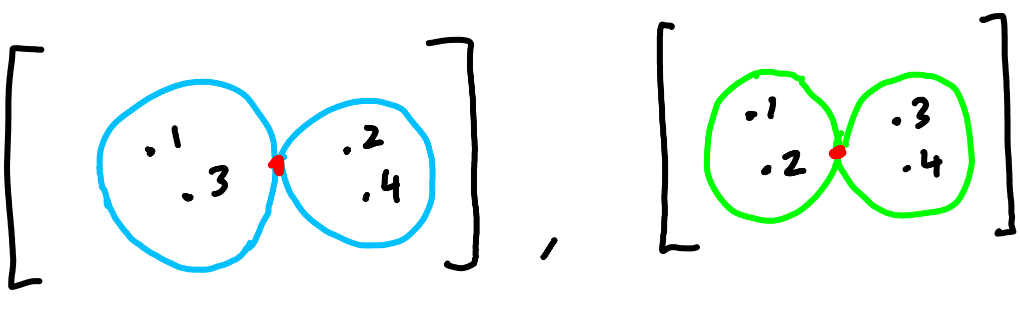


$\left[\right.$ 

$\left.\right] \in \overline{M}_{0,4}$



Two more boundary points:



$\in \overline{\mathcal{M}}_{0,4}$

Proof of Kapranov's Theorem:

Given the existence of $\bar{M}_{0,n}$,

the proof is easy:

We must just check that

the morphism

$$\mathcal{M}_{0,n} \longrightarrow \mathcal{H}_{RNC}$$

extends to

$$\bar{\mathcal{M}}_{0,n} \longrightarrow \bar{\mathcal{H}}_{RNC} \subset \mathcal{H}$$

by the same construction.

We start with a moduli point

$$[C, p_1, \ldots, p_n] \in \overline{\mathcal{M}}_{0,n}$$

C is a nodal genus 0 curve

The dualizing sheaf is still defined

$$\omega_C(p_1 + p_2 + \cdots + p_n)$$

on C.

No choices made, degree is $-2+n>0$

There is a canonical map

$$f : C \to \mathbb{P}(V^\vee)$$

where $V \cong H^0(C, \omega_C(\sum_i p_i))$

f is an embedding, and

$$f(p_1), f(p_2), \dots, f(p_n) \in \mathbb{P}(V^\vee)$$

are in linear general position.

A good exercise in Riemann-Roch and Serre duality for nodal curves

Conclusion: to each

$$[C, p_1, \dots, p_n] \in \overline{M}_{0,n},$$

we associate a genus 0 curve

$C \subset \mathbb{P}^{n-2}$ through $q_1, \dots, q_n$

We obtain the desired extension

$$\overline{\mathcal{M}}_{0,n} \longrightarrow \overline{\mathcal{H}}_{RNC} \subset \mathcal{H}$$

Some details left to check $\square$

(15) Universal Abel-Jacobi theory over M_g

Let us return now to AJ theory:

To a nonsingular curve C of genus g, we constructed $\mathrm{Jac}(C)$

$$\mathrm{Jac}(C) \cong H^0(C, K_C)^\vee \big/ H_1(C, \mathbb{Z})$$

We can consider the family of Jacobians over $\mathcal{M}_g$

$$\begin{tikzcd}
\mathcal{C} \arrow[dr, "\pi"'] & & \mathrm{Jac}(\pi) \arrow[dl, "\varepsilon"] \\
& \mathcal{M}_g &
\end{tikzcd}$$

universal curve

universal Jacobian

$$\begin{tikzcd}
\pi^{-1}([C]) = C \arrow[dr] & & \varepsilon^{-1}([C]) = \mathrm{Jac}(C) \arrow[dl] \\
& {[C]} &
\end{tikzcd}$$

To construct $\mathrm{Jac}(\pi)$ in algebraic geometry, we can construct the universal Jacobian over the Hilbert Scheme:

$$
\begin{tikzcd}
\mathcal{C}^{\mathcal{U}} \arrow[dr, "\pi"'] & & \mathrm{Jac}^{\mathcal{U}}(\pi) \arrow[dl, "\varepsilon"] \\
& \mathcal{U} \subset \text{HilbScheme of all curves} &
\end{tikzcd}
$$

m-canonical curves

And then quotient $\mathrm{Jac}(\pi) = \mathrm{Jac}^{\mathcal{U}}(\pi) \big/ PGL$.

Caporaso: A compactification of the universal Picard variety 1994

P: Universal moduli space of slope-semistable bundles 1996

Let C be a fixed nonsingular curve of genus g, and let

$$A = (a_1, a_2, \ldots, a_n)$$

be a vector of integer which satisfy $\sum_{i=1}^{n} a_i = 0$.

We have a canonical AJ map:

$$AJ : C^n \to Jac(C)$$

$$AJ(p_1, \dots, p_n) = \sum_{i=1}^{n} a_i p_i$$

degree 0

- Previous perspective:

Given $(p_1, \dots, p_n) \in C^n$,

How can we tell if

$$\mathcal{O}_C\left(\sum_{i=1}^{n} a_i p_i\right) \overset{?}{\cong} \mathcal{O}_C$$

Answer: Calculate the AJ map, period integrals

- New perspective:

What is the geometry of the locus of solutions:

$$\left\{ (p_1, \ldots, p_n) \in C^n \;\middle|\; \mathcal{O}_C\left(\sum_{i=1}^{n} a_i p_i\right) \cong \mathcal{O}_C \right\}$$

First Answer: The locus is simply $AJ^{-1}(0) \subset C^n$.

See Part I.18 for $g=1$ example.

- More precise question:

What is the class of the solution set:

$$[AJ^{-1}(0)] \in H^*(C^n) \,?$$

- Universal perspective

$$\begin{tikzcd}
C^n \arrow[rr, "AJ"] \arrow[dr, "\pi^n"'] & & \mathrm{Jac}(\pi) \arrow[dl, "\varepsilon"] \\
& M_g &
\end{tikzcd}$$

universal product

universal Jacobian

Here $\quad \mathcal{C}^n \xrightarrow{AJ} \mathrm{Jac}(\pi)$

is the universal Abel-Jacobi map

$$AJ([C, p_1, \dots, p_n]) = \sum_{i=1}^{n} a_i p_i \in \mathrm{Jac}(C).$$

What is the class of the solution set:

$$[AJ^{-1}(0)] \in H^*(\mathcal{C}^n)\ ?$$

transversality is an issue

Universal product

depends upon $A=(a_1,\dots,a_n)$

universal Jacobian

$$
\begin{tikzcd}
\mathcal{C}^n \arrow[rr, bend left, "AJ"] \arrow[dr, "\pi^n"'] & & \mathrm{Jac}(\pi) \arrow[dl, shift right, "\varepsilon"'] \\
& \mathcal{M}_g \arrow[ur, shift right, "o"'] &
\end{tikzcd}
$$

universal product

universal 0-section

Since the image of the 0-section

is of Codim g in $\mathrm{Jac}(\pi)$,

we expect $AJ^{-1}(0) \subset \mathcal{C}^n$

to also be of Codim g.

But this is not always true.

Example: $A = (1,-1)$ and $g \geq 2$.

Then $AJ^{-1}(0) \subset C^2$ is

exactly the diagonal.

$$AJ^{-1}(0) = C \xhookrightarrow[\text{diagonal}]{} C^2$$

which is of codim 1.

Excess intersection theory $\Rightarrow$

$$[AJ^*(0)] \in H^{2g}(C^n)$$

(16) Universal Abel-Jacobi theory over $\overline{M}_{g,n}$

Let $A = (a_1, a_2, \ldots, a_n)$ be a vector of integers which satisfy

$$\sum_{i=1}^{n} a_i = 0 .$$

Goal: define a cycle which is like $AJ^{-1}(0)$ even though AJ and $Jac(\pi)$ are problematic over $\overline{M}_{g,n}$

The result is the

Double ramification cycle

$$DR_{g,A} \in H^{2g}(\overline{\mathcal{M}}_{g,n})$$

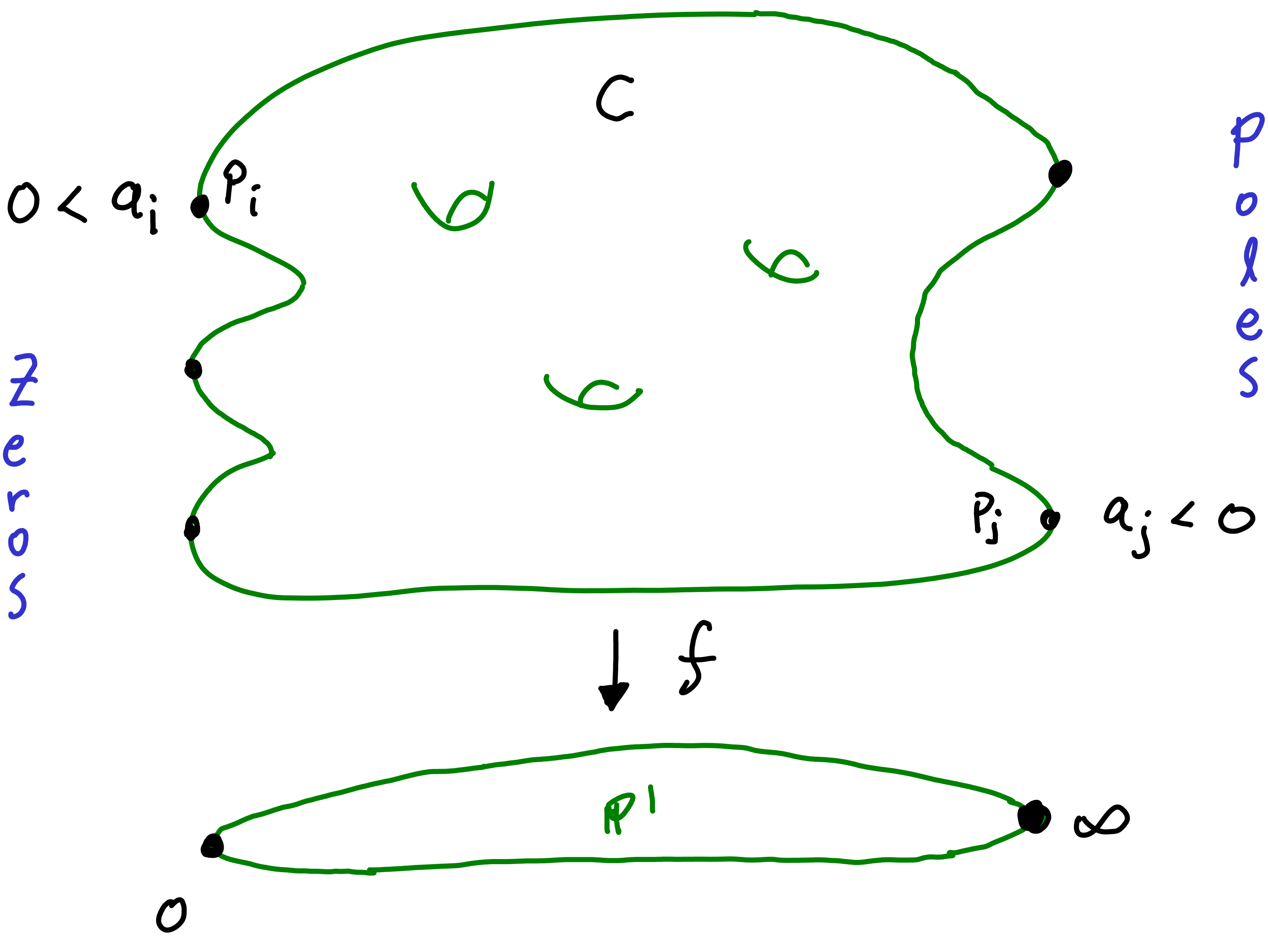
Why the name?

When $\mathcal{O}_C\left(\sum_{i=1}^{n} a_i p_i\right) \cong \mathcal{O}_C$

on a nonsingular curve C,

we obtain a rational function f

with zeros and poles specified by the vector A:



We will define and calculate

$$DR_{g,A} \in RH^{g}(\overline{M}_{g,n})$$

(17) A solution on
$$\begin{tikzcd} \mathcal{C}^n \arrow[d, "\pi^n"'] \\ M_g \end{tikzcd}$$

Let $A = (a_1, \ldots, a_n)$, $\sum_{i=1}^{n} a_i = 0$

Consider
$$\begin{tikzcd} \mathcal{C} \times \mathcal{C}^n \arrow[d, "\pi"'] \\ \mathcal{C}^n \arrow[u, bend right, "p_1"'] \arrow[u, bend right=60, "p_n"'] \end{tikzcd}$$
universal curve with sections

Let $\mathcal{L} \cong \mathcal{O}_{\mathcal{C}}\left(\sum_{i=1}^{n} a_i p_i\right)$

We are interested in $AJ^{-1}(0) \subset \mathcal{C}^n$ which we can write as a set as

$$\left\{ [C, p_1, \ldots, p_n] \in \mathcal{C}^n \;\middle|\; h^0(C, \mathcal{O}_C(\textstyle\sum a_i p_i)) \geq 1 \right\}$$

If we take a 2-term resolution

$$[E^0 \xrightarrow{\varphi} E^1] = R\pi_* \mathcal{L} \in D^b(\mathcal{C}^n)$$

rank e_0 rank e_1

$$e_0 - e_1 = 1 - g$$

The locus is precisely where

$$\operatorname{rank}(\varphi) < e_0 - 1$$

We obtain a natural scheme structure

$\mathcal{S} \subset \mathcal{C}^n$ as a degeneracy locus

Expected codim of $\mathcal{S} \subset \mathcal{C}^n$ is

$$\left(e_0 - (e_0 - 1)\right) \cdot \left(e_1 - (e_0 - 1)\right) = g$$

By the Porteus formula, the class of the degeneracy locus is

$$[\mathcal{S}]^{vir} = c_g\left(E^1 - E^0\right)$$

$$= c_g\left(-R\pi_* \mathcal{L}\right) \in H^{2g}(\mathcal{C}^n)$$

if $\mathcal{S} \subset \mathcal{C}^n$ pure codim g, then $[\mathcal{S}]^{vir} = [\mathcal{S}]$

We have agreement with the calculation in I.18

We have put two natural scheme structures on the locus

$$\left\{ [C, p_1, \dots, p_n] \in \mathcal{C}^n \;\middle|\; h^0(C, \mathcal{O}_C(\textstyle\sum a_i p_i)) \geq 1 \right\}$$

Scheme I: $AJ^{-1}(0) \subset \mathcal{C}^n$

Scheme II: $\mathcal{S} \subset \mathcal{C}^n$

Exercise*: Prove $AJ^{-1}(0) = \mathcal{S}$ as schemes.

Exercise*: Prove $AJ^*(0) = [\mathcal{S}]^{vir}$

* → challenging exercise.

(18)

Tautological classes

$\kappa, \psi, [\Gamma]$

Witten's Conjecture

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n}$$

Pixton's Relations

Birational geometry

Harris-Mumford

$\overline{\mathcal{M}}_g$ irrational for high g

$\overline{\mathcal{M}}_{1,11}$ irrational

Gromov-Witten Theory

Stable maps

$\overline{\mathcal{M}}_{g,n}(X,\beta)$

Double ramification

$DR_{g,A}$

moduli of curves

$\mathcal{M}_g \subset \overline{\mathcal{M}}_g$

$\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$

Geometric Invariant Theory

Deformation theory

Hilbert schemes

Deligne-Mumford Stacks

End of Part II

of the course