Moduli Space of Stable maps

(1) Some references (see webpage)

- Fulton - P   Notes on stable maps
- Koch - Vainsencher   Invitation to Quantum Cohomology
- P - Thomas $13/2$ ways of Counting Curves

(2) Definition

Let X be a variety

nonsingular projective is the best case

We will consider maps

$$f: C \to X$$

the target

algebraic morphism

complete connected nodal curve of genus g

$$f_*[C] = \beta \in H_2(X, \mathbb{Z})$$

curve class

- $\overline{\mathcal{M}}_g(X,\beta)$ is the moduli space of stable maps of genus g curves to X representing the class $\beta \in H_2(X,\mathbb{Z})$

- $[f : C \to X] \in \overline{\mathcal{M}}_g(X,\beta)$ is stable if and only if

$$|\mathrm{Aut}(f)| < \infty .$$

- An automorphism of f is an automorphism of C which commutes with f:

$$\begin{tikzcd} C \arrow[rd, "f"] \arrow[d, "\phi"'] \arrow[rd, phantom, "G", shift right=6ex] & \\ C \arrow[r, "f"'] & X \end{tikzcd}$$

$$\mathrm{Aut}(f) \subset \mathrm{Aut}(C)$$

C stable curve $\Rightarrow$

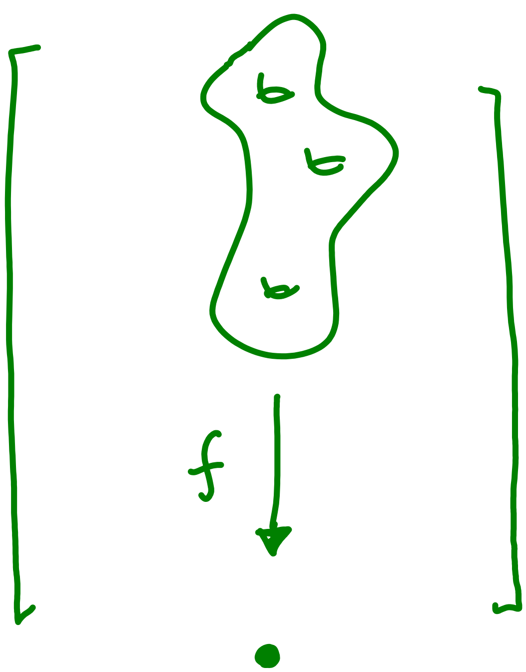
f is a stable map

Example: Maps to a point

Consider $\overline{M}_g(\bullet, 0)$

Target X is the point

$H_2(\bullet, \mathbb{Z}) = 0$



$\in \overline{M}_g(\bullet, 0)$

The morphism f carries no data for point targets.

We have an isomorphism $(g \geq 2)$

$$\overline{\mathcal{M}}_g(\bullet, 0) \cong \overline{\mathcal{M}}_g$$

For $g = 0, 1$, the space is empty

$$\overline{\mathcal{M}}_0(\bullet, 0) \cong \emptyset$$

$$\overline{\mathcal{M}}_1(\bullet, 0) \cong \emptyset$$

because of the stability condition.

We have discussed Automorphisms, but what is an isomorphism of stable maps?

The maps

$$f : C \to X$$

$$\hat{f} : \hat{C} \to X$$

are isomorphic if and only if

$\exists\ \phi : C \underset{\text{isom}}{\xrightarrow{\sim}} \hat{C}$ which commutes with $f, \hat{f}$:

$$\begin{tikzcd} C \arrow[dr, "f"] \arrow[d, "\phi"', "\wr"] & \\ \hat{C} \arrow[r, "\hat{f}"'] \arrow[ur, phantom, "\circlearrowright"] & X \end{tikzcd}$$

Example: genus 0 with target $\mathbb{P}^1$

Since $H_2(\mathbb{P}^1, \mathbb{Z}) \cong \mathbb{Z} \cdot [\mathbb{P}^1]$

We can represent β by an integer.

What is $\overline{\mathcal{M}}_0(\mathbb{P}^1, 1)$?

$$f : C \cong \mathbb{P}^1 \xrightarrow{\text{deg } 1} \mathbb{P}^1$$

All differ by reparameterization of the domain $\mathbb{P}^1$.

[Check that nothing else is stable

We see

$$\overline{\mathcal{M}}_0(\mathbb{P}^1, 1) \cong \text{point}$$

What about $\overline{\mathcal{M}}_0(\mathbb{P}^1, 2)$?

A much more interesting case.

First approach as coarse moduli

(forgetting the stack structure)

Suppose we have a map

$$f: C \cong \mathbb{P}^1 \xrightarrow{\deg 2} \mathbb{P}^1,$$

How can we classify f up to isomorphism of stable maps?

Riemann-Hurwitz:

$$br(f) + 2(-2) = -2$$

degree of K_C

Number of branch points of f

degree of f

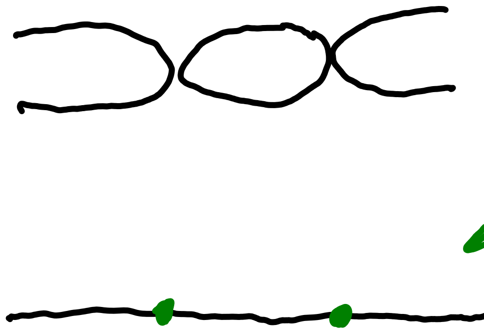
degree of $K_{\mathbb{P}^1}$

We see $br(f) = 2$.

C

$\downarrow f$

$\mathbb{P}^1$



2 branch points

If $f : C \cong \mathbb{P}^1 \to \mathbb{P}^1$

and $\hat{f} : \hat{C} \cong \mathbb{P}^1 \to \mathbb{P}^1$

are isomorphic then the two branch points in the target $\mathbb{P}^1$ must be the same.

Exercise: f and $\hat{f}$ are isomorphic if and only if the two branch points in the target $\mathbb{P}^1$ are the same.

Hint: if the two branch points are $0, \infty \in \mathbb{P}^1$, then the map $f : \mathbb{P}^1 \to \mathbb{P}^1$ must be of the form

$$[x, y] \xrightarrow{f} [x^2, \lambda y^2] \qquad \lambda \in \mathbb{C}^*$$

The locus $\mathcal{M}_0(\mathbb{P}^1, 2) \subset \overline{\mathcal{M}}_0(\mathbb{P}^1, 2)$

is therefore (coarsely) isomorphic to

$$\left(\mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta \right) / \mathbb{Z}_2 \;\cong\; \mathbb{P}^2 \setminus \text{Conic}$$

diagonal discriminant

We must then examine the broken domains

$$f : \mathbb{P}^1 \cup \mathbb{P}^1 \to \mathbb{P}^1$$

only invariant is image of the node

These yield a $\mathbb{P}^1$ parameterizing the image of the node.

All together

$$\overline{M}_0(\mathbb{P}^1, 2) \underset{\text{coarse}}{\cong} \mathbb{P}^2$$

(3) Marked points

- $\overline{M}_{g,n}(X, \beta)$ is the moduli space of stable maps to X of genus g curves with n markings (distinct, nonsingular) representing the class $\beta \in H_2(X, \mathbb{Z})$

- $[f:(C,p_1,\dots,p_n)\to X]\in\overline{\mathcal{M}}_{g,n}(X,\beta)$ is stable if and only if
$$|\mathrm{Aut}(f)|<\infty.$$

- An automorphism of f is an automorphism of C which commutes with f:

$$\begin{tikzcd} C \arrow[dr, "f"] \arrow[d, "\phi"'] & \\ C \arrow[r, phantom, "G"] \arrow[r, "f"'] & X \end{tikzcd}$$

and fixes the markings $f(p_i)=p_i$

We have an isomorphism $(2g-2+n>0)$

$$\overline{\mathcal{M}}_{g,n}(\bullet, 0) \cong \overline{\mathcal{M}}_{g,n}$$

Stable maps are a significant generalization of the theory of stable curves.

Theorem: If X is a projective variety,

$$\overline{\mathcal{M}}_{g,n}(X, \beta)$$

is a proper Deligne-Mumford stack.

(4) Genus 0 maps to $\mathbb{P}^m$

The moduli space $\overline{M}_{0,n}(\mathbb{P}^m, d)$ is a nonsingular Deligne-Mumford stack (we will discuss the deformation theory later).

What is the dimension?

$$f : (C, p_1, \ldots p_n) \to \mathbb{P}^m$$

generically $C \cong \mathbb{P}^1$

and the map f is determined

by $m+1$ polynomials

$$[x,y] \xmapsto{f} [P_0(x,y), \dots, P_m(x,y)]$$

homogeneous of degree d

Dimension count:

- Vary $P_0, \dots, P_m \Rightarrow (m+1)(d+1)$
- projective equivalence $\Rightarrow -1$
- Reparametrization of domain $\Rightarrow -3$
- markings $\Rightarrow n$

Total: $(m+1)(d+1) - 4 + n$

Exercise: Use the idea of expressing maps

$$f : (C, p_1, \dots, p_n) \to \mathbb{P}^r$$

explicitly in terms of polynomials to prove $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r, d)$ is a nonsingular irreducible Deligne Mumford stack of dim

$$(m+1)(d+1) - 4 + n$$

[Fulton-P]

We can write the dimension formula as .

$$\dim \overline{M}_{0,n}(\mathbb{P}^m, d) =$$

genus 0

$$\chi(f^*T_{\mathbb{P}^m}) + 3\cdot 0 - 3 + n$$

$\Big\downarrow$ Riemann-Roch

genus 0

$$\underbrace{\operatorname{degree}(f^*T_{\mathbb{P}^m})}_{= \, m(d+1)} + \underbrace{\operatorname{rank}(f^*T_{\mathbb{P}^m})}_{= \, m} \cdot (1-0)$$

use the Euler sequence

$$0 \to \mathcal{O} \to \bigoplus_0^m \mathcal{O}(1) \to T_{\mathbb{P}^m} \to 0$$

Definition: The virtual dimension of

$\overline{\mathcal{M}}_{g,n}(X, \beta)$ is

$$\text{vir dim} = \int_\beta c_1(T_X) + \dim_{\mathbb{C}}(X)(1-g) + 3g-3+n$$

Theorem: Every irreducible component of $\overline{\mathcal{M}}_{g,n}(X, \beta)$ has dimension $\geq$ vir dim.

(5) Moduli of genus 0 maps to $\mathbb{P}^2$

$\overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d)$ nonsingular DM stack of dimension $3d-1+n$

n markings yield n evaluation maps

$$ev_i : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^2, d) \to \mathbb{P}^2$$

$$ev_i \left[f : (C, p_1, \ldots, p_i, \ldots, p_n) \to \mathbb{P}^2 \right] \overset{\text{def}}{=} f(p_i)$$

Gromov-Witten invariants:

$$\langle \gamma_1, \dots, \gamma_n \rangle_{0,d}^{\mathbb{P}^2} = \int_{\overline{M}_{0,n}(\mathbb{P}^2, d)} \prod_{i=1}^{n} ev_i^*(\gamma_i)$$

$\gamma_i \in H^*(\mathbb{P}^2)$

genus

degree

cohomology class

S_n symmetric construction

Meaning of the integral is:

$$\begin{array}{c} \displaystyle\prod_{i=1}^{n} ev_i^*(\gamma_i) \cap \left[\overline{M}_{0,n}(\mathbb{P}^2, d)\right] \in H_*\left(\overline{M}_{0,n}(\mathbb{P}^2, d)\right) \\ \Big\downarrow \text{Project} \\ H_0\left(\overline{M}_{0,n}(\mathbb{P}^2, d)\right) \\ \cong \\ \mathbb{Q} \end{array}$$

Cohomology

fundamental class in Homology

value of the integral $\int$

- Dimension constraint:

$$\langle \gamma_1, \dots, \gamma_n \rangle_{0,d}^{\mathbb{P}^2} = 0$$

unless $\quad 3d-1+n = \sum_{i=1}^{n} \text{degree}_{\mathbb{C}}(\gamma_i)$.

- Symmetry:

$$\langle \gamma_1, \dots, \gamma_n \rangle_{0,d}^{\mathbb{P}^2} = \langle \gamma_{\sigma(1)}, \dots, \gamma_{\sigma(n)} \rangle_{0,d}^{\mathbb{P}^2}$$

$\sigma \in S_n$ symmetric group

- Fundamental class:

If $\gamma_i = 1 \in H^*(\mathbb{P}^2)$

Then $\langle \gamma_1, \dots, \gamma_n \rangle_{0,d}^{\mathbb{P}^2} = 0$

Unless $n=3$ and $d=0$,

$$\langle \gamma_1, \gamma_2, 1 \rangle = \int_{\mathbb{P}^2} \gamma_1 \cup \gamma_2$$

- **Divisor equation**

Cohomology of $\mathbb{P}^2$

$H^0 \cong \mathbb{Q} \cdot 1$

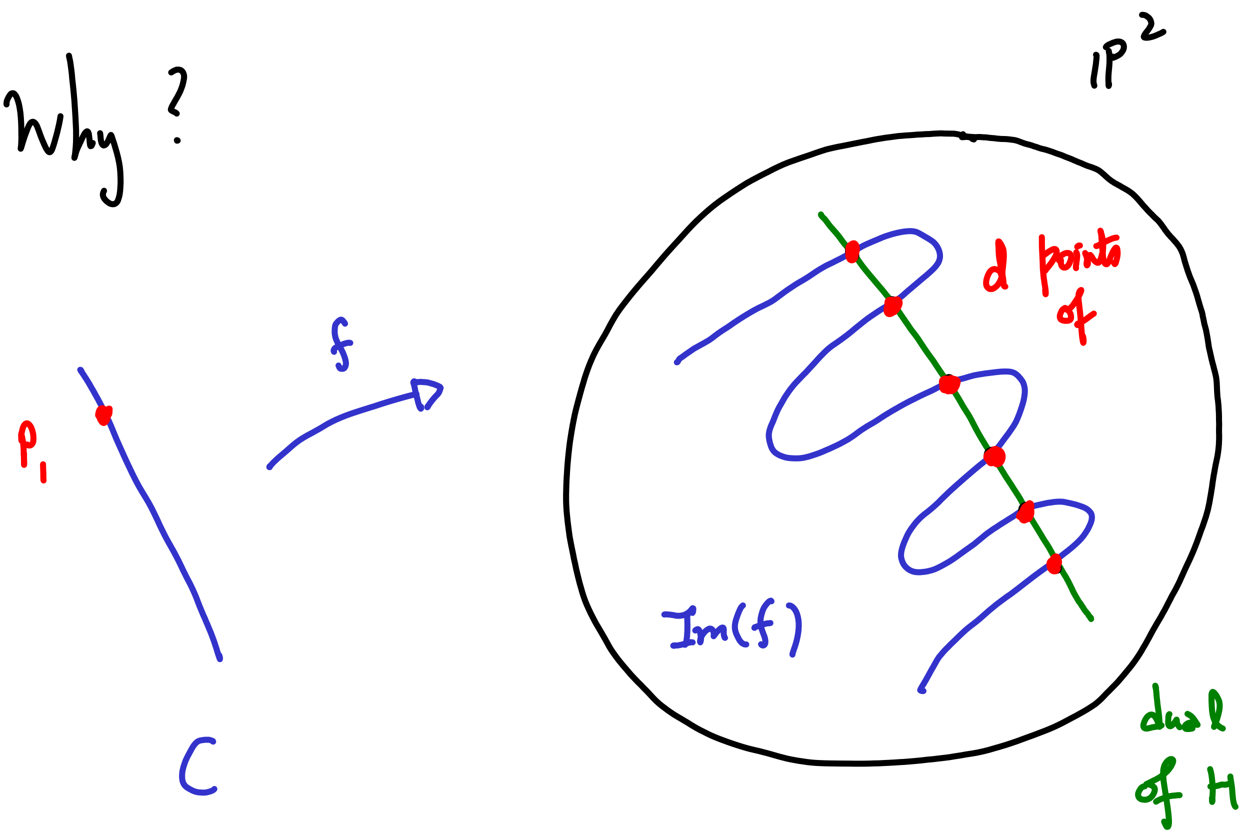
$H^2 \cong \mathbb{Q} \cdot H$

$H^4 \cong \mathbb{Q} \cdot p$

If $\gamma_1 = H$ then

$$\langle H, \gamma_2, \dots, \gamma_n \rangle_{0,d}^{\mathbb{P}^2} = d \, \langle \gamma_2, \dots, \gamma_n \rangle_{0,d}^{\mathbb{P}^2}$$

Why ?



[exceptional case $\langle H, \gamma_2, \gamma_3 \rangle_{0,3}^{\mathbb{P}^2} = \int_{\mathbb{P}^2} H \cup \gamma_2 \cup \gamma_3$]

In fact, since $\overline{M}_{0,3}(\mathbb{P}^2, 0) \cong \overline{M}_{0,3} \times \mathbb{P}^2$

$\cong \mathbb{P}^2$

We easily see:

$$\langle \gamma_1, \gamma_2, \gamma_3 \rangle_{0,3}^{\mathbb{P}^2} = \int_{\mathbb{P}^2} \gamma_1 \cup \gamma_2 \cup \gamma_3$$

What is left?

$$\langle \underbrace{p, \dots, p}_{3d-1} \rangle_{0,d}^{\mathbb{P}^2} \overset{\text{def}}{=} N_d$$

Can we compute N_d?

Yes! But we require an additional idea.

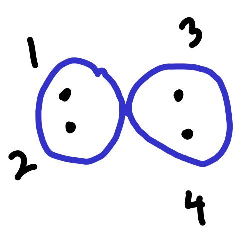
Consider the forgetful morphism

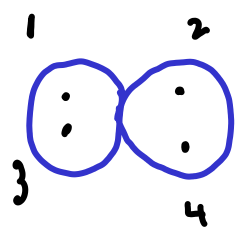
$$\varepsilon : \overline{M}_{0,n}(\mathbb{P}^2, d) \to \overline{M}_{0,4}$$

[Require $n \geq 4$]

which forgets everything except the first 4 markings.

We have the cross ratio relation

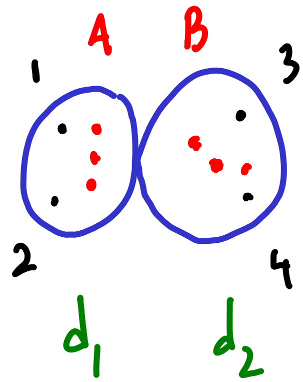
$\Big[$ 

$\Big] = \Big[$ 

$\Big] \in H^2(\overline{M}_{0,4})$

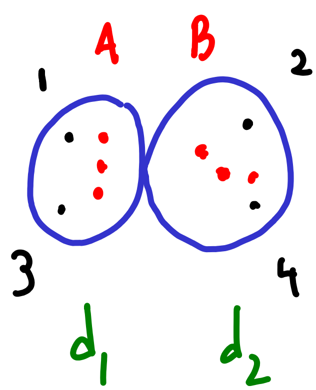
We simply pull back the relation by ε.

We obtain

$\sum_{\substack{A \cup B = [5,6,\dots,n] \\ d_1 + d_2 = d}} \Big[$  $\Big]$

WDVV relation

$\|$

$\sum_{\substack{A \cup B = [5,6,\dots,n] \\ d_1 + d_2 = d}} \Big[$  $\Big]$

in $H^2(\overline{M}_{0,n}(\mathbb{P}^2, d))$

After applying $\prod_{i=1}^{n} ev_i^*(\gamma_i)$ and

integrating $\int_{\overline{M}_{0,n}(\mathbb{P}^2,d)}$ we obtain

$$\sum_{\substack{A \cup B \\ d_1, d_2}} \langle \gamma_1, \gamma_2, \gamma_{i\in A}, \bullet \rangle_{d_1} \langle \bullet, \gamma_{i\in B}, \gamma_3, \gamma_4 \rangle_{d_2}$$

diagonAL

$$\|$$

$$\sum_{\substack{A \cup B \\ d_1, d_2}} \langle \gamma_1, \gamma_3, \gamma_{i\in A}, \bullet \rangle_{d_1} \langle \bullet, \gamma_{i\in B}, \gamma_2, \gamma_4 \rangle_{d_2}$$

diagonAL

The diagonal has Künneth components

$$1 \otimes P + H \otimes H + P \otimes 1$$

So the left side is

$$\sum_{\substack{A \cup B \\ d_1, d_2}} \langle \gamma_1, \gamma_2, \gamma_{i \in A}, 1 \rangle_{d_1} \langle P, \gamma_{i \in B}, \gamma_3, \gamma_4 \rangle_{d_2}$$

$$+$$

$$\sum_{\substack{A \cup B \\ d_1, d_2}} \langle \gamma_1, \gamma_2, \gamma_{i \in A}, H \rangle_{d_1} \langle H, \gamma_{i \in B}, \gamma_3, \gamma_4 \rangle_{d_2}$$

$$+$$

$$\sum_{\substack{A \cup B \\ d_1, d_2}} \langle \gamma_1, \gamma_2, \gamma_{i \in A}, P \rangle_{d_1} \langle 1, \gamma_{i \in B}, \gamma_3, \gamma_4 \rangle_{d_2}$$

and the right side is

$$\sum_{\substack{A \cup B \\ d_1, d_2}} \langle \gamma_1, \gamma_3, \gamma_{i \in A}, 1 \rangle_{d_1} \langle P, \gamma_{i \in B}, \gamma_2, \gamma_4 \rangle_{d_2}$$

$$+$$

$$\sum_{\substack{A \cup B \\ d_1, d_2}} \langle \gamma_1, \gamma_3, \gamma_{i \in A}, H \rangle_{d_1} \langle H, \gamma_{i \in B}, \gamma_2, \gamma_4 \rangle_{d_2}$$

$$+$$

$$\sum_{\substack{A \cup B \\ d_1, d_2}} \langle \gamma_1, \gamma_3, \gamma_{i \in A}, P \rangle_{d_1} \langle 1, \gamma_{i \in B}, \gamma_2, \gamma_4 \rangle_{d_2}$$

The equality is the WDVV equation in Gromov-Witten theory.

We apply the above WDVV equation with

$d \geq 2$

$\gamma_1 = H,\ \gamma_2 = H,\ \gamma_3 = p,\ \gamma_4 = p$

$\gamma_5 = \cdots = \gamma_{3d} = p$

Total of $3d-2$ point conditions

Left side

$$N_d + \sum_{\substack{d_1+d_2=d \\ d_i>0}} d_1^3 d_2 \binom{3d-4}{3d_1-1} N_{d_1} N_{d_2}$$

Right side

$$\sum_{\substack{d_1+d_2=d \\ d_i>0}} d_1^2 d_2^2 \binom{3d-4}{3d_1-2} N_{d_1} N_{d_2}$$

Together, for $d \geq 2$:

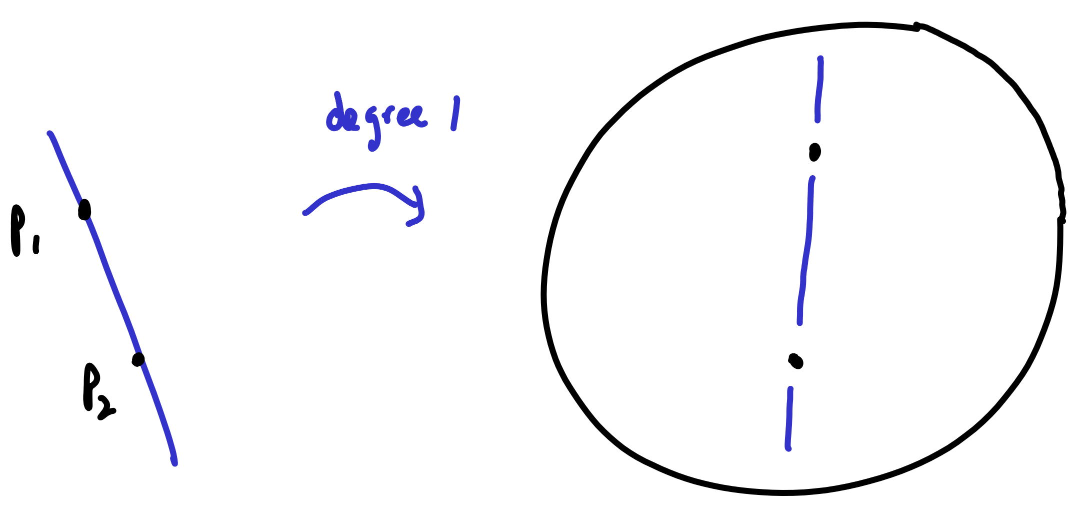
Kontsevich 94

$$N_d + \sum_{\substack{d_1+d_2=d \\ d_i>0}} d_1^3 d_2 \binom{3d-4}{3d_1-1} N_{d_1} N_{d_2}$$

$$= \sum_{\substack{d_1+d_2=d \\ d_i>0}} d_1^2 d_2^2 \binom{3d-4}{3d_1-2} N_{d_1} N_{d_2}$$

Exercise: $N_1 = 1$

Unique line through 2 points [Euclid]



WDVV equation determines all N_d from N_1

$$N_2 + 1^3 \cdot 1 \cdot \binom{2}{2} \underset{\substack{\uparrow \\ 1}}{N_1} \underset{\substack{\uparrow \\ 1}}{N_1} = 1^2 \cdot 1^2 \cdot \binom{2}{1} \underset{\substack{\uparrow \\ 1}}{N_1} \underset{\substack{\uparrow \\ 1}}{N_1}$$

so $N_2 = 1$

Exercise: Calculate

$$N_1 = 1 \qquad N_2 = 1 \qquad N_3 = 12 \qquad N_4 = 620$$

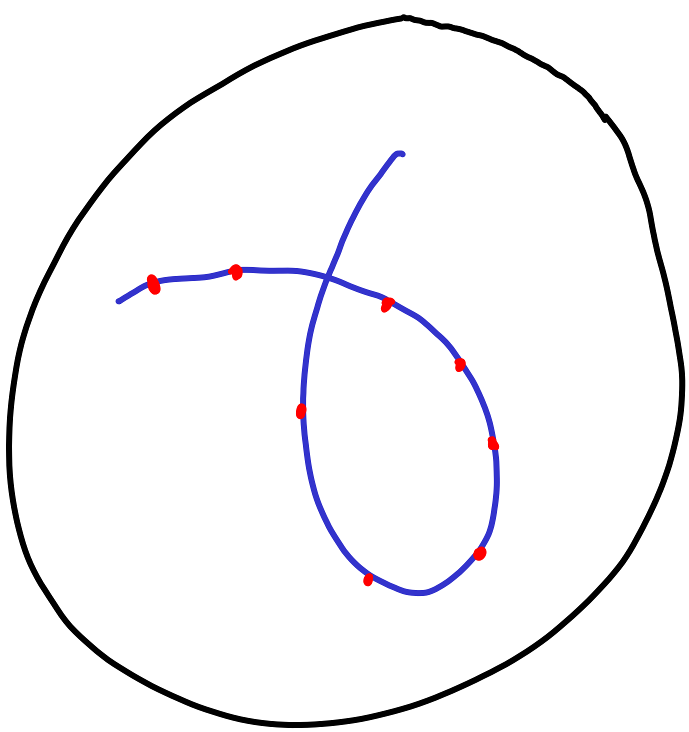
$$N_5 = 87304 \qquad N_6 = 26312976$$

Do the numbers N_d have geometric meaning?

Yes: N_d is the number of genus 0 curves in $\mathbb{P}^2$ of degree d passing through $3d-1$ generic points.

Proof uses Bertini's Theorem

Example: $N_3 = 12$



12 nodal cubic passing through 8 general points

(6) Quantum Cohomology for $\mathbb{P}^2$

Ordinary cohomology : $H^*(\mathbb{P}^2)$ is $\mathbb{Q}$-algebra

Let t_0, t_1, t_2 be coordinates

$$H^*(\mathbb{P}^2) = \left\{ t_0 \cdot 1 + t_1 \cdot H + t_2 \cdot p \;\middle|\; t_1, t_2, t_3 \in \mathbb{Q} \right\}$$

Quantum cohomology : $QH^*(\mathbb{P}^2)$ is a

$\mathbb{Q}[[t_0, t_1, t_2]]$ - algebra

$QH^*(\mathbb{P}^2)$ is free $\mathbb{Q}[[t_0, t_1, t_2]]$ - module

with basis $1, H, p$

and a $\mathbb{Q}[[t_0, t_1, t_2]]$ - linear

quantum product $\star$

Quantum product is determined by

$$\langle \gamma_1 * \gamma_2, \gamma_3 \rangle$$

$\gamma_i \in H^*(\mathbb{P}^2)$ quantum product Poincaré pairing

$$\langle \mu, \nu \rangle = \int_{\mathbb{P}^2} \mu \cup \nu$$

Formula for the Quantum product:

$$\langle \gamma_1 * \gamma_2, \gamma_3 \rangle = \sum_{d=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{n!} \langle \gamma_1, \gamma_2, \gamma_3, T^n \rangle_{0,d}^{\mathbb{P}^2}$$

$$T = t_0 \cdot 1 + t_1 \cdot H + t_2 \cdot p$$

Exercise: $QH^*(\mathbb{P}^2)$ with $\star$ is a commutative, associative ring with unit $1 \in H^*(\mathbb{P}^2)$.

Requires WDVV equations

Let X be a nonsingular projective variety. Then the quantum cohomology

$$\left(QH^*(X), \star, 1 \right)$$

is defined in the same way

- $\mathbb{Z}_2$-grading, super commutative
- $\overline{\mathcal{M}}_{0,n}(X, \beta)$ may not be of the expected dimension. Need virtual fundamental class

(7)

QH^*, QK^*

Less well Known

[YP Lee PhD]

All computable

Connection to enumerative geometry

Cohomology

Poincaré poly

All tautological

[Getzler-P, Oprea]

Birational geometry:

Rational

[B. Kim-P]

$\overline{M}_{0,n}(G/P, \beta)$

A lot known

(ask Andrew Kresch)

$\overline{M}_{0,n}(\mathbb{P}^m, d)$

Proper

Nonsingular

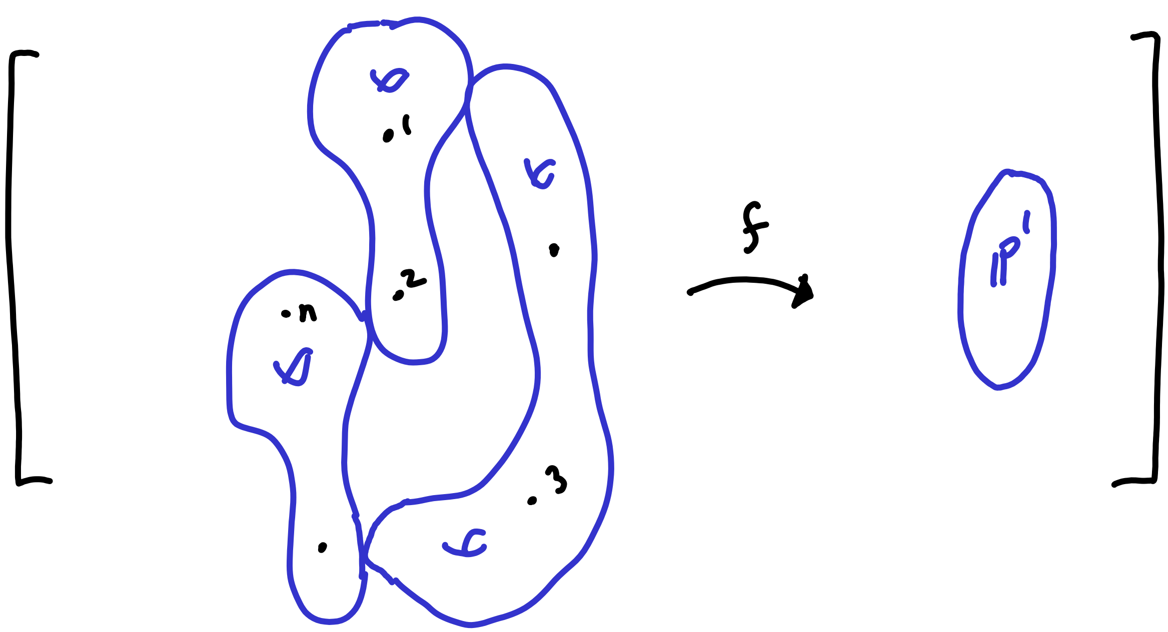
Deligne-Mumford Stack

(8) Maps to $\mathbb{P}^1$ and the Abel-Jacobi problem

We have already defined and discussed

$$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$$

$$\cup$$



But to address the questions of universal Abel-Jacobi theory, we need a new construction. Why?

Let $A = (a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$

Assume for simplicity that $\forall i \; a_i \neq 0$

satisfy $\sum_{i=1}^{n} a_i = 0$

Let $(C, p_1, \dots, p_n) \in \mathcal{M}_{g,n}$

Abel-Jacobi condition

satisfy $\mathcal{O}_C(\sum a_i p_i) \cong \mathcal{O}_C$

Then $\exists f$ meromorphic function

$f : C \to \mathbb{P}^1.$

with zero and poles determined

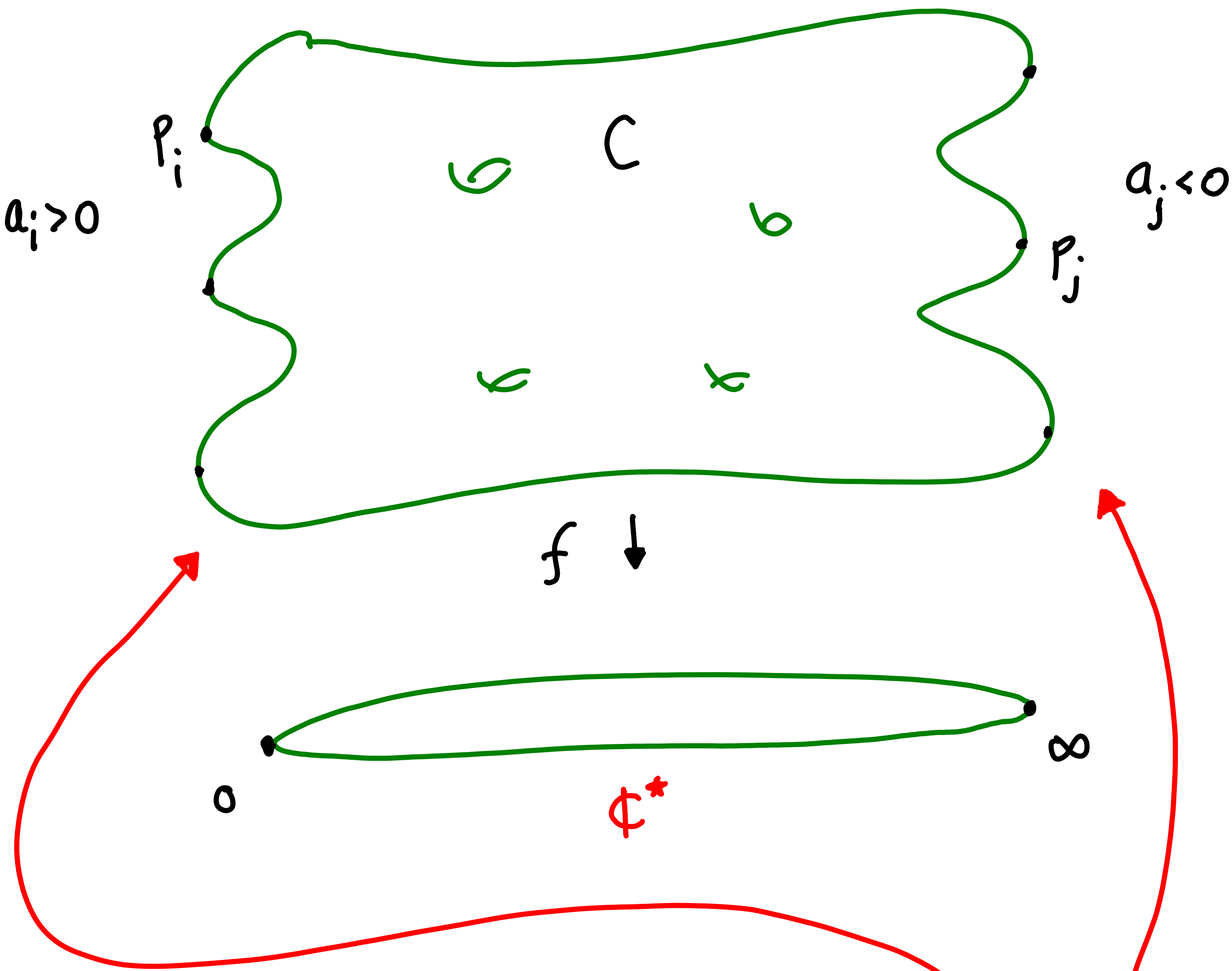
by A. So what is the problem?

f is only determined up to $\mathbb{C}^*$ scale.

Geometrically: $\mathcal{Z} \subset \mathcal{M}_{g,n}$

locus of solutions to the Abel-Jacobi problem for A.

Then, $\mathcal{Z}$ is almost isomorphic to a moduli space of maps to $\mathbb{P}^1$ (up to the $\mathbb{C}^*$-scale).



Besides the $\mathbb{C}^*$-scale issue,
we also have fixed profiles
over $0, \infty \in \mathbb{P}^1$

Moduli space of stable relative maps
solves both problems!

(g) Stable relative maps

Let $g \geq 0$ be the genus

Let μ, ν be two ordered partions of $d \geq 0$

$\mu = (\mu_1, \dots, \mu_{\ell(\mu)})$

$\nu = (\nu_1, \dots, \nu_{\ell(\nu)})$

[Think of these as coming from the positive and negative parts of A]

We will define a moduli space

$$\overline{M}_g(\mathbb{P}^1/0,\infty)^{\sim}_{\mu,\nu}$$

$\mathbb{C}^*$ equivalence

Relative to the points $0, \infty \in \mathbb{P}^1$

determine profiles over 0 and ∞

No bar

$$\mathcal{M}_g(\mathbb{P}^1/0,\infty)^{\sim}_{\mu,\nu} \cong Z \subset \mathcal{M}_{g,n}.$$

Abel-Jacobi solutions

$$A = (a_1, \ldots, a_n)$$

$a_i \neq 0$

$\mu = (\mu_1, \ldots)$ $\nu = (\nu_1, \ldots)$

positive parts of A

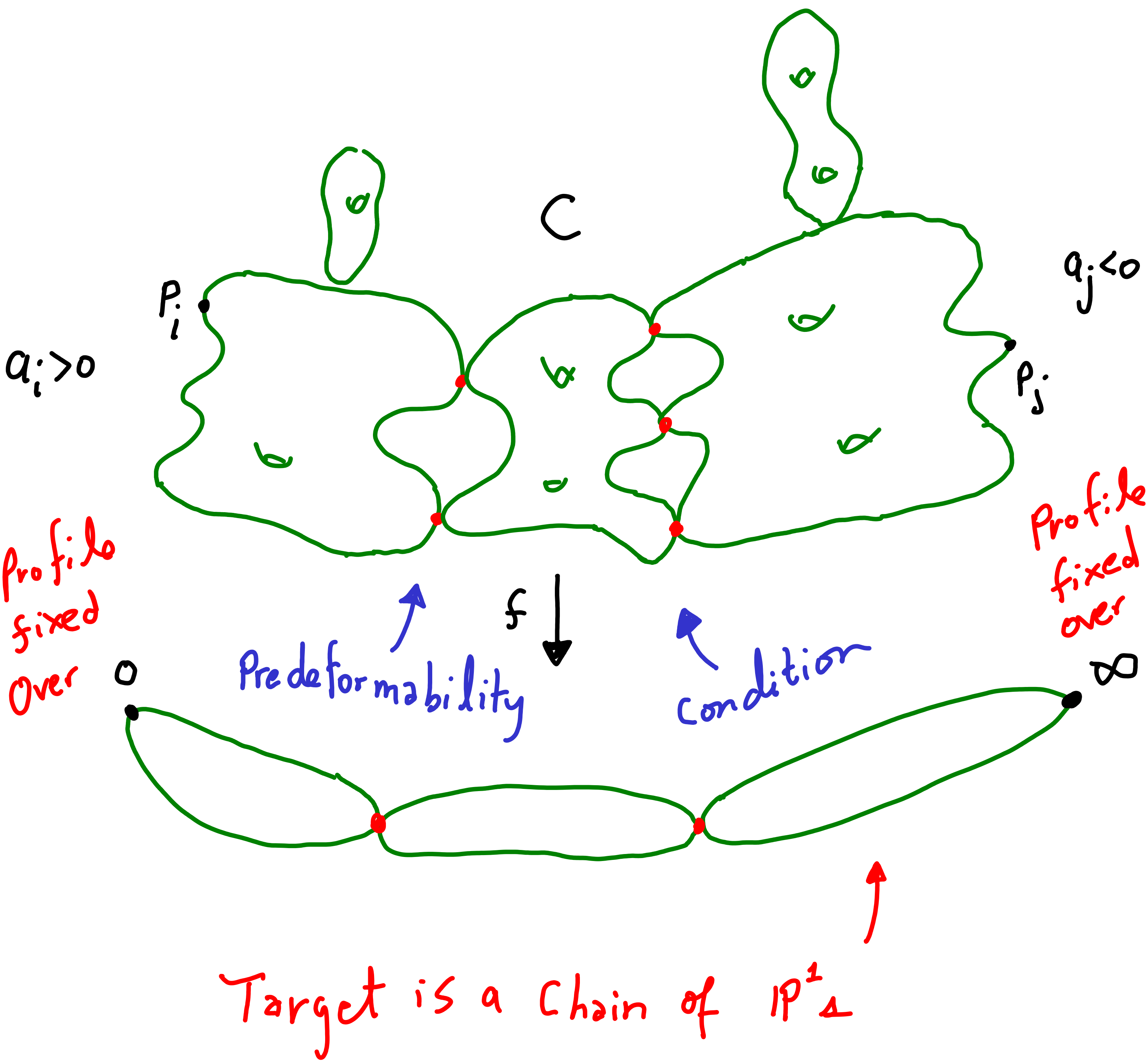
absolute value of the negative parts of A

The geometric matching is perfect on the open set

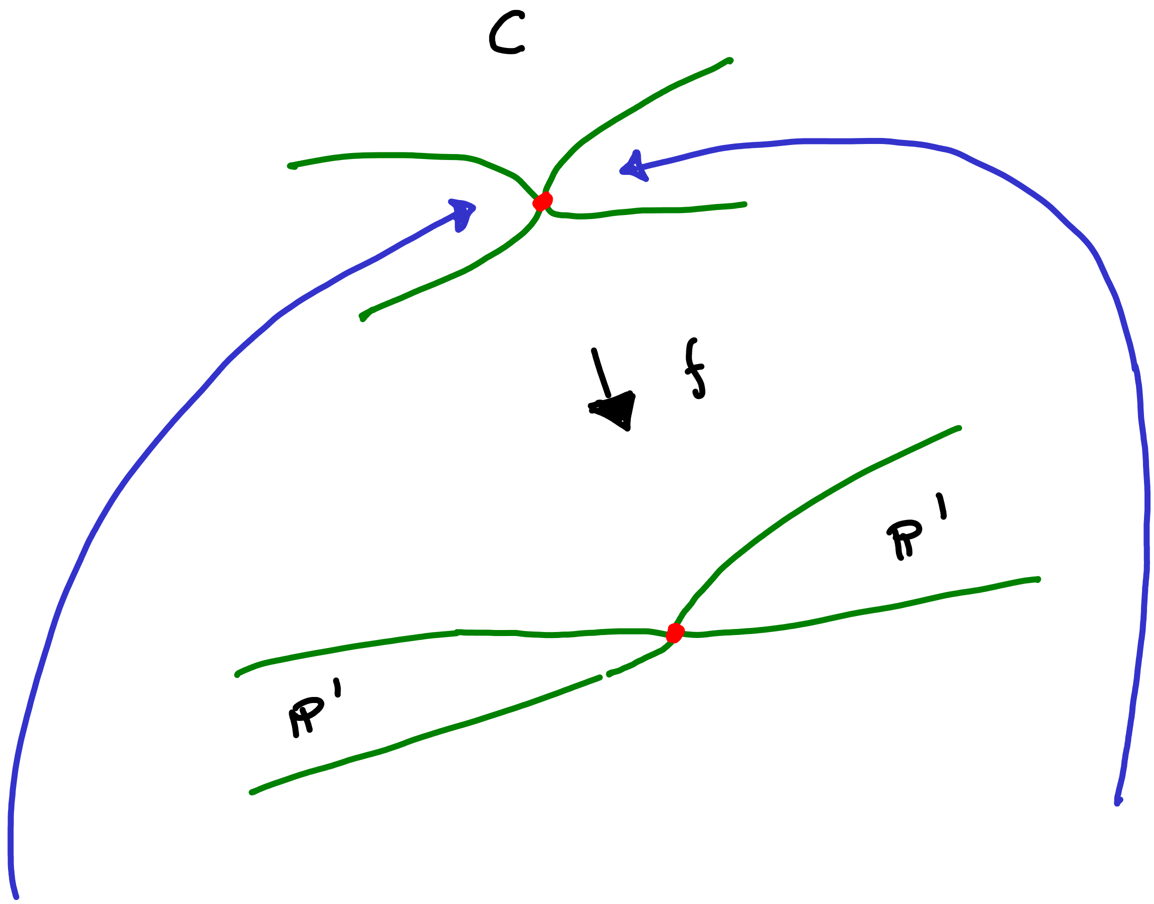
$$\mathcal{M}_g(\mathbb{P}^1/0,\infty)^{\sim}_{\mu,\nu} \subset \overline{\mathcal{M}}_g(\mathbb{P}^1/0,\infty)^{\sim}_{\mu,\nu}$$

what happens here?

A Stable relative map is of the following form:



- Predeformability



Each side has a ramification number.

Predeformability: The two numbers are equal.

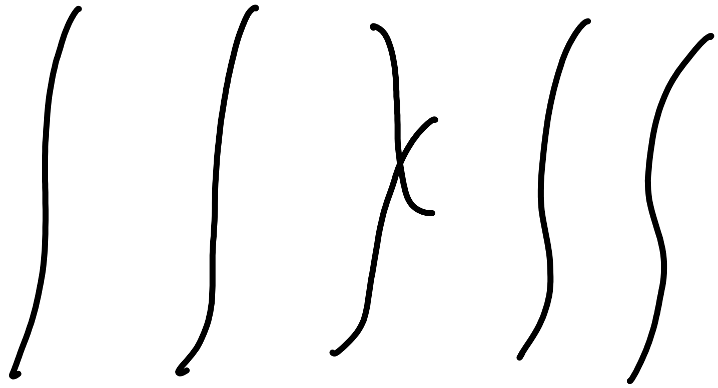
Why is this related to deformability?

Exercise: Let

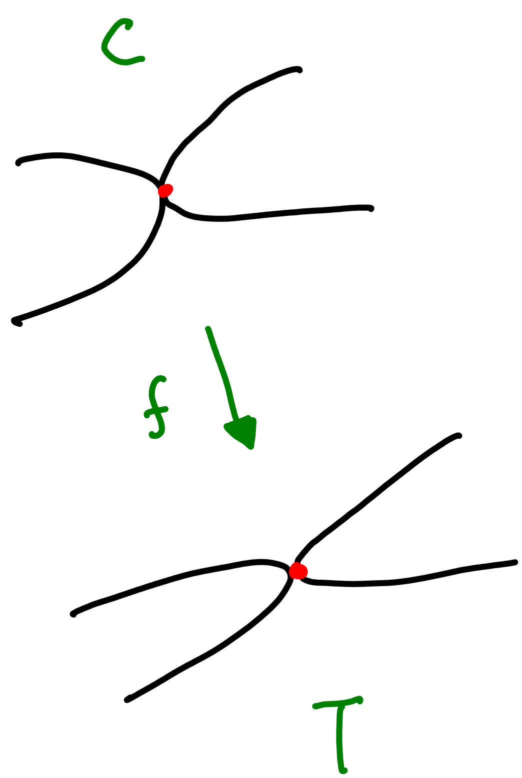
$$\begin{array}{c} \mathcal{C} \\ f \downarrow \\ T \end{array}$$

be a flat family of maps

where T is a 1-parameter deformation of a nodal curve

T 

and C looks like

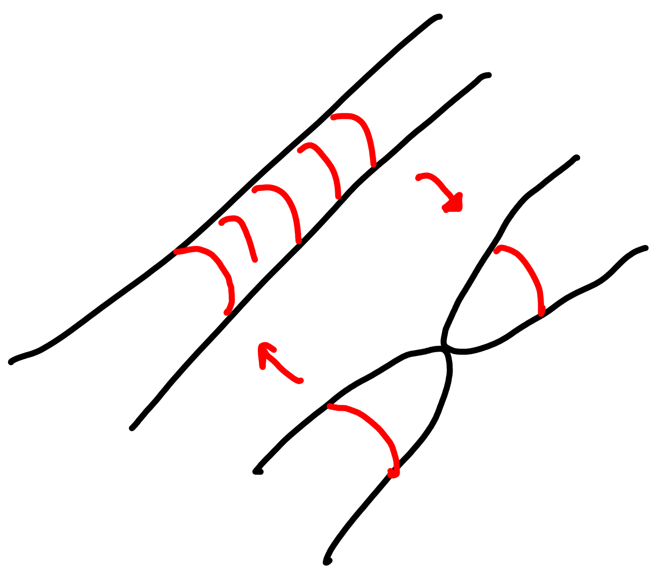


over the node of T and étale elsewhere

Prove the predeformability condition holds.

Hint: Look what happens on C to the preimages of the red loops.

T



- Stability

$$f : C \to T$$

predef condition holds over internal nodes of T

Connected nodal curve with fixed profiles over $0, \infty \in T$

Chain of $\mathbb{P}^1$'s with $0, \infty \in T$ on extremal components

Stability condition:

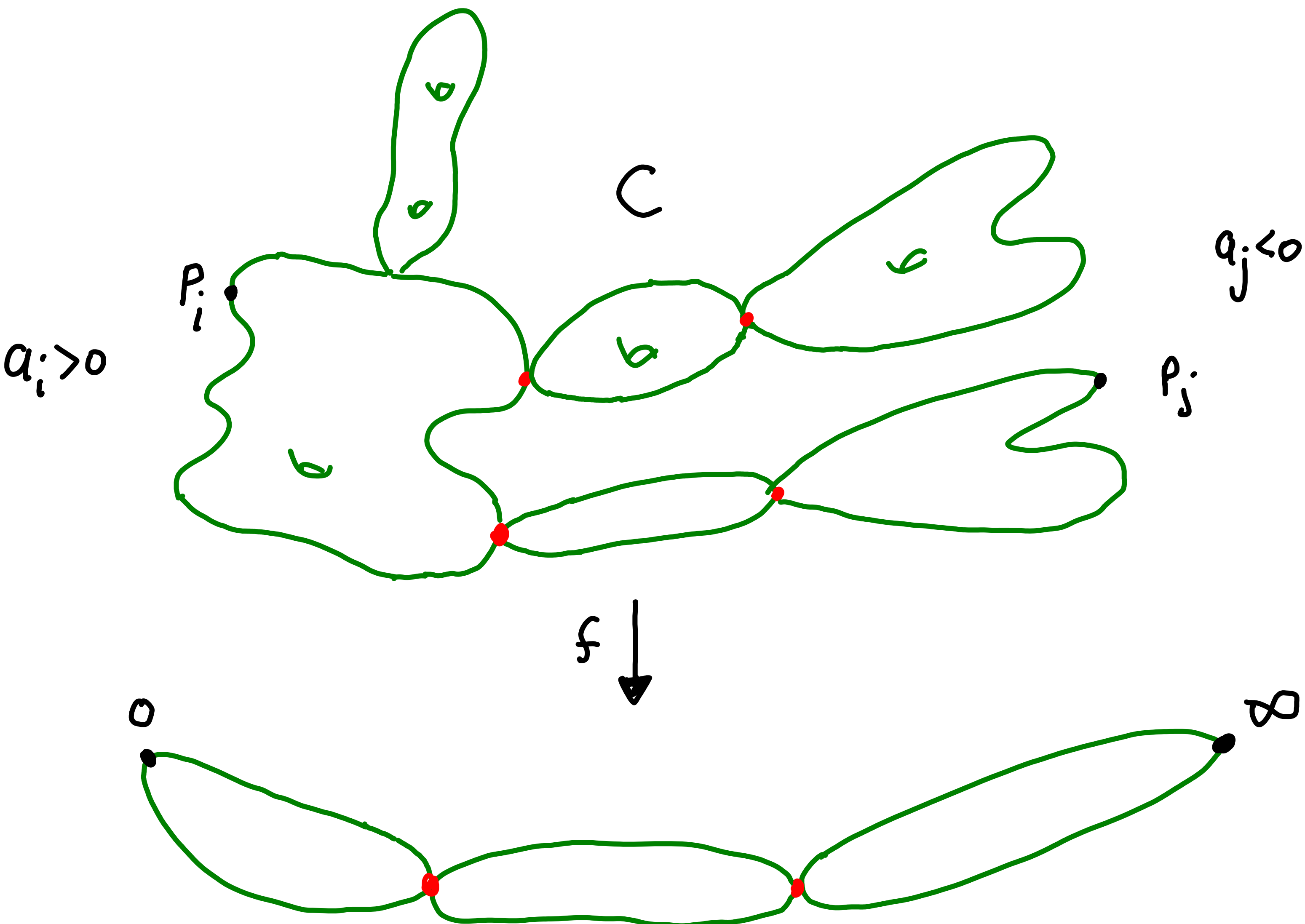
f has finite Automorphisms.

- An automorphism of f is an automorphism of C and a scaling automorphism of T which commutes with f:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\phi"'] \arrow[dr, phantom, "G"] & T \arrow[d, "\gamma"] \\ C \arrow[r, "f"] & T \end{tikzcd}$$

A scaling Aut of T is $\mathbb{C}^*$ scale on each component.

Another possible picture of a stable relative map:



Target is a chain of $\mathbb{P}^1$s

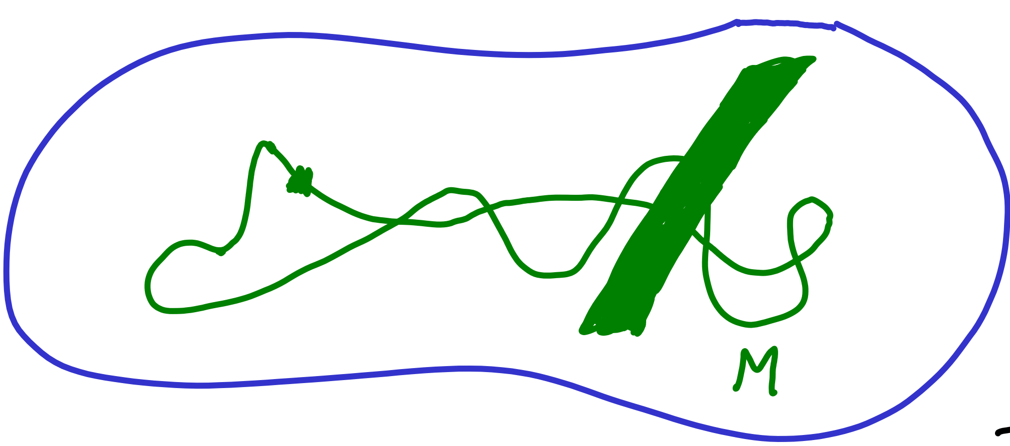
(9) Deformation theory and the virtual fundamental class

General theory (Behrend-Fantechi)

Let M be scheme (or DM stack).

The cotangent complex (2-term cut off):

$$L^\bullet_M = \left[\, {}^{I_M}/_{I_M^2} \to \Omega_Y \big|_M \right] \in D^b_{Coh}(M)$$



Y nonsingular

$I_M \subset \mathcal{O}_Y$ ideal sheaf defining M

$M \subset Y$ not canonical, but $L^\bullet_M \in D^b_{Coh}(M)$ is canonical.

$$h^0(L^\bullet_M) = \mathrm{Coker}\left(I_M/I_M^2 \to \Omega_Y|_M \right)$$

$$\cong \Omega_M$$ sheaf of differentials

A 2 term obstruction theory for M is

$$E^\bullet \xrightarrow{\phi} L^\bullet_M$$

where $E^\bullet = [E^{-1} \to E^0] \in D^b_{Coh}(M)$

is represented by 2 vector bundles and

ϕ is a morphism in $D^b_{Coh}(M)$ satisfying:

- ϕ^0 is an isomorphism on h^0
- ϕ^{-1} is surjective on h^1

By the first condition,

$$h^0(E^\bullet) \overset{\phi^0}{\cong} \Omega_M$$

Information about the Zariski Tangent space of M.

Let $p \in M$, then we have

$$0 \to \mathrm{Ker} \to E^{-1}_p \to E^0_p \to \Omega_{M,p} \to 0$$

$E_i = (E^{-i})^\vee$

We dualize ($\mathbb{C}$- vector spaces)

$$0 \to T_{M,p} \to E_{0,p} \to E_{1,p} \to \mathrm{Obs} \to 0$$

Tangent space

obstruction space

We can assume we have maps

$$\begin{tikzcd} E^{-1} \arrow[r] \arrow[d, "\phi"'] & E^{0} \arrow[d] \\ I_M/I_M^2 \arrow[r] & \Omega_Y|_M \end{tikzcd}$$

The condition of an obstruction theory $\Rightarrow$

$$E^{-1} \to E^{0} \oplus I_M/I_M^2 \xrightarrow{\gamma} \Omega_Y|_M \to 0$$

Then we obtain an exact sequence of ablian cones:

$$0 \to T_Y \to E_0 \times_M C(I_M/I_M^2) \to C(Q) \to 0$$

$C(Q)$ is the cone associated to $\mathrm{Ker}(\gamma)$.

The Normal cone of M in Y is

$$C(M/Y) = \mathrm{Spec}\left(\bigoplus_{k=0}^{\infty} I_M^k / I_M^{k+1} \right)$$

So $$C(M/Y) \subset C\left(I_M / I_M^2 \right)$$

pure dim Y

We have

$$E_0 \times_M C(M/Y) \subset E_0 \times_M C\left(I_M / I_M^2 \right)$$

Behrend-Fantechi : T_Y - cone

Let $$D = \frac{E_0 \times_M C(M/Y)}{T_Y} \subset C(Q)$$

Since $C(Q) \subset E_1$

We have
$$\begin{tikzcd} D \arrow[r, phantom, "\subset"] & E_1 \arrow[d, shift right] \\ & M \arrow[u, shift right, "s"'] \end{tikzcd}$$

$e_1 = \text{rank } E_1$

$s = 0$-section

$\dim = e_0 + \dim Y - \dim Y = e_0$

Definition: $[M]^{vir} \in A_{e_0 - e_1}(M)$

virtual dim $= e_0 - e_1$

$$[M]^{vir} \overset{\text{def}}{=} s^*([D])$$

(10) Virtual class of the moduli of maps

There are three stages of the analysis

- Moduli of maps from a fixed curve C to X ← Nonsingular projective variety

- $\overline{\mathcal{M}}_{g,n}(X,\beta)$

  involves varying complex structure on the domain

- $\overline{\mathcal{M}}_g(\mathbb{P}^1/0,\infty)^{\sim}_{\mu,\nu}$

  most relevant for Abel-Jacobi Theory

Start with the simplest:

C fixed nonsingular curve of genus g

X fixed nonsingular projective target variety

$\mathcal{M}_C(X, \beta)$ moduli of maps representing $\beta \in H_2(X, \mathbb{Z})$

universal map

$$\begin{tikzcd} \mathcal{M} \times C \arrow[r, "f"] \arrow[d, "\pi"'] & X \\ \mathcal{M} = \mathcal{M}_C(X, \beta) & \end{tikzcd}$$

projection

We have

$$f^*\Omega_X \xrightarrow{df} L^{\bullet}_{\mathcal{M}\times C} \cong \pi^* L^{\bullet}_{\mathcal{M}} \oplus L^{\bullet}_{C}$$

Then $$f^*\Omega_X \to \pi^* L^{\bullet}_{\mathcal{M}}$$ via projection

So $$(\pi^* L^{\bullet}_{\mathcal{M}})^{\vee} \to f^* T_X$$

Then $$R\pi_*\left((\pi^* L^{\bullet}_{\mathcal{M}})^{\vee}\right) \to R\pi_* f^* T_X$$

So $$\left(R\pi_* f^* T_X\right)^{\vee} \to R\pi_*\left((\pi^* L^{\bullet}_{\mathcal{M}})^{\vee}\right)^{\vee}$$

$$R\pi_*\left(\left(\pi^* L^\bullet_M\right)^\vee\right)^\vee =$$

Then $R\pi_*\left(\pi^*\left(L^{\bullet\vee}_M\right)\right)^\vee = \left(L^{\bullet\vee}_M \otimes R\pi_* \mathcal{O}_{M\times C}\right)^\vee$

$$= L^\bullet_M \otimes \left(R\pi_* \mathcal{O}_{M\times C}\right)^\vee$$

Then $\mathcal{O}_M \to R\pi_* \mathcal{O}_{M\times C}$

So $\left(R\pi_* \mathcal{O}_{M\times C}\right)^\vee \to \mathcal{O}_M$

So $L^\bullet_M \otimes \left(R\pi_* \mathcal{O}_{M\times C}\right)^\vee \to L^\bullet_M$

Finally we have: $\left(R\pi_* f^* T_X\right)^\vee \to L^\bullet_M$

We have a 2 term obstruction theory of virtual dimension

Illusie

$$\text{Vir dim } M_C(X, \beta) = \chi(C, f^*T_X)$$

$$= \int_\beta c_1(X) + \dim_{\mathbb{C}} X(1-g)$$

Next step is $\bar{M}_g(X, \beta)$

which is the essentially the same as for fixed C but studied relatively over the varying moduli of curves

Since the moduli of curves is nonsingular,

we obtain

$$[\overline{M}_{g,n}(X,\beta)]^{vir} \in A_{exp}(\overline{M}_{g,n}(X,\beta))$$

$$\text{vir dim} = \text{vir dim } M_C(X,\beta) + 3g-3+n$$

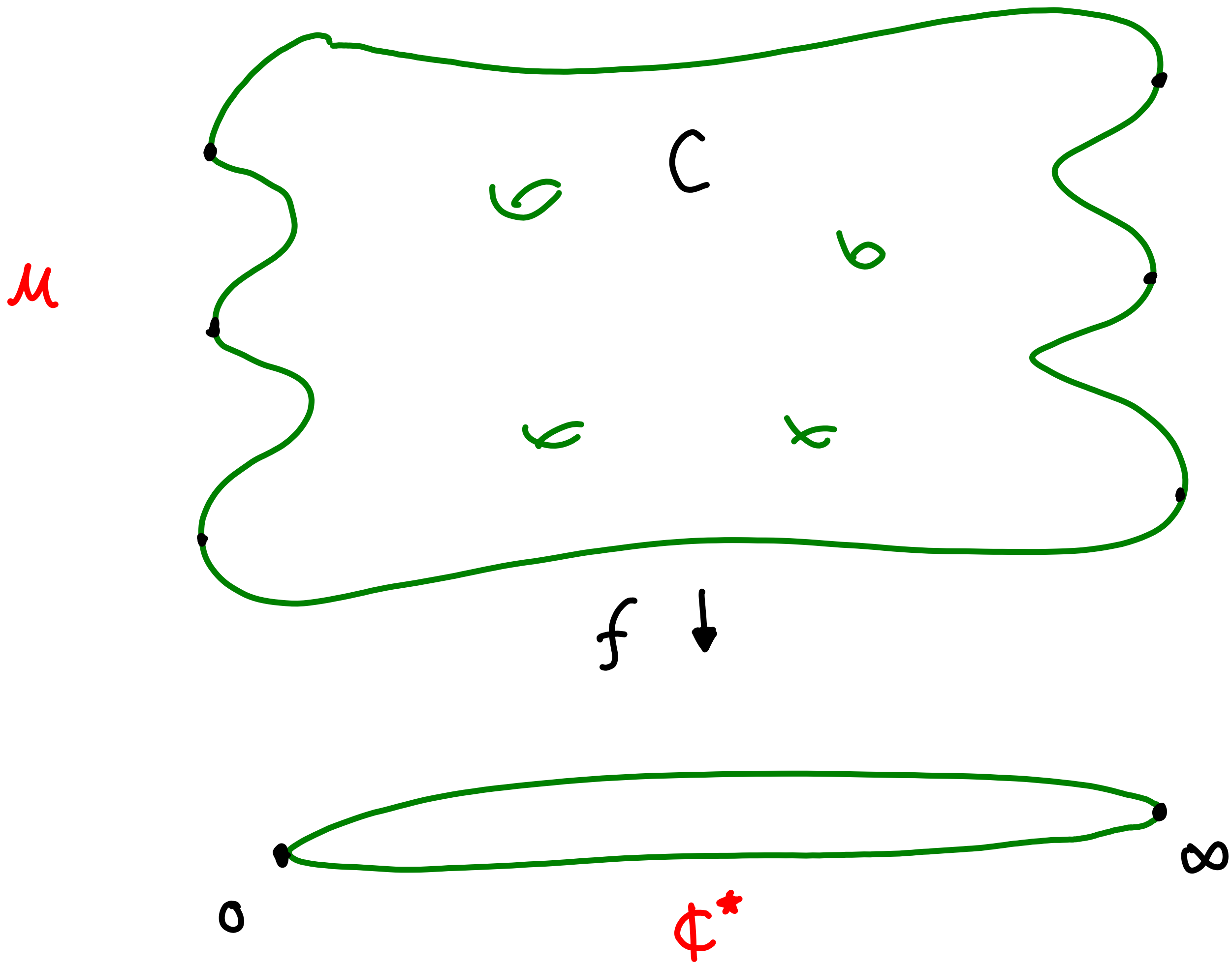
Third step is the relative case

which is technically more complicated.

$$[\overline{M}_g(\mathbb{P}^1/0,\infty)_{\mu,\nu}]^{vir}$$

what is the expected dimension?

When the domain and the target are fixed, the theory is similar :



$$\text{vir dim } \overline{M}_g(\mathbb{P}^1/0,\infty)^{\sim}_{\mu,\nu} = \underset{d=|\mu|=|\nu|}{2d} + 1 - g + 3g - 3 \underset{\mathbb{C}^*}{-1}$$

$$- (|\mu| - \ell(\mu)) - (|\nu| - \ell(\nu))$$

Log geometry $\quad = \ell(\mu) + \ell(\nu) + 2g - 3$

(11) The double ramification cycle

For the open set $\mathcal{M}_g(\mathbb{P}^1/0,\infty)^{\sim}_{\mu,\nu} \subset \overline{\mathcal{M}}_g(\mathbb{P}^1/0,\infty)^{\sim}_{\mu,\nu}$

we have

Abel-Jacobi solutions

$$\mathcal{M}_g(\mathbb{P}^1/0,\infty)^{\sim}_{\mu,\nu} \cong Z \subset \mathcal{M}_{g,n}.$$

$$A = (a_1, \dots, a_n)$$

$a_i \neq 0$

$$\mu = (\mu_1, \dots) \qquad \nu = (\nu_1, \dots)$$

positive parts of A

absolute value of the negative parts of A

We have $\varepsilon : \overline{\mathcal{M}}_g(\mathbb{P}^1/0,\infty)_{\mu,\nu} \to \overline{\mathcal{M}}_{g,n}$

We can define

$$\varepsilon\left(\overline{M}_g(\mathbb{P}^1/0,\infty)^{\sim}_{\mu,\nu}\right) \overset{\text{def}}{=} Z_{g,A} \subset \overline{M}_{g,n}$$

Many components of different dimensions

defined as the image of ε

To define a cycle class for universal Abel-Jacobi theory, we use the virtual class:

$$DR_{g,A} = \varepsilon_*\left[\overline{M}_g(\mathbb{P}^1/0,\infty)^{\sim}_{\mu,\nu}\right]^{vir}$$

$$DR_{g,A} \in H^{2g}(\overline{M}_{g,n})$$

(12) Pixton's Formula

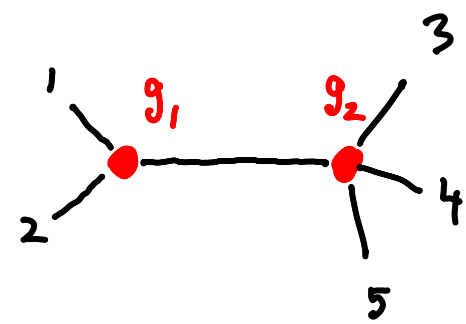
Let $G_{g,n}$ = set of stable graphs of genus g with n markings

finite set

for $\Gamma \in G_{g,n} \rightsquigarrow \overline{M}_\Gamma \xrightarrow{\xi_\Gamma} \overline{M}_{g,n}$

product of moduli spaces determinded by the vertices

for $\Gamma =$



$$\overline{M}_\Gamma = \overline{M}_{g_1,3} \times \overline{M}_{g_2,4}$$

Tautological classes are given by

$$\xi_{\Gamma *}\Big(\underset{\text{markings}}{\prod} \psi_i^{m_i} \underset{\text{halves of edges}}{\prod} \psi_j^{n_j} \underset{\text{vertices}}{\prod} \text{kappas} \Big) \in RH^*(\overline{M}_{g,n})$$

$$\overline{M}_\Gamma \xrightarrow{\xi_\Gamma} \overline{M}_{g,n}$$

The linear span of all such classes defines the tautological ring

$$RH^*(\overline{M}_{g,n}) \subset H^*(\overline{M}_{g,n})$$

We can also define the tautological ring

$$R^*(\overline{M}_{g,n}) \subset CH^*(\overline{M}_{g,n}).$$

Let $\Gamma \in G_{g,n}$ be a stable graph.

Let r be a positive integer

A Weighting mod r of Γ is

$$w : \mathcal{H}(\Gamma) \to \{0, 1, \dots, r-1\}$$

half edges

Remember

$A = (a_1, \dots, a_n)$

$\sum a_i = 0$

(I) $i \in$ Marking, $w(i) = a_i \mod r$

(II) $e = (h, h') \in$ Edge, $w(h) + w(h') = 0 \mod r$

(III) $v \in$ Vertex, $\sum_{h \vdash v} w(h) = 0 \mod r$

$W_{\Gamma,r}$ is set of weightings mod r of Γ

$$|W_{\Gamma,r}| = r^{h^1(\Gamma)}$$

Definition (Pixton)

Let $P_g^{d,r}(A) \in R^d(\overline{M}_{g,n})$ be the degree d component of

$$\sum_{\Gamma \in G_{g,n}} \sum_{w \in W_{\Gamma,r}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \frac{1}{r^{h'(\Gamma)}} \cdot$$

$$\xi_{\Gamma *}\left[\prod_{1}^{n} \exp\left(\frac{a_i^2}{2}\psi_i\right) \cdot \prod_{e=(h,h')} \frac{1-\exp\left(-\frac{w(h)w(h')}{2}\cdot(\psi_h+\psi_{h'})\right)}{\psi_h+\psi_{h'}}\right]$$

Various motivations: Compact type restriction,
Givental - Teleman theory

Claim (Pixton):

$P_g^{d,r}(A) \in R^d(\overline{\mathcal{M}}_{g,n})$ is polynomial in r for all $r >> 0$.

Definition (Pixton):

$P_g^d(A) \in R^d(\overline{\mathcal{M}}_{g,n})$ is the constant term of $P_g^{d,r}(A)$

$r=0$

Theorem (Conjectured by Pixton, proven in JPPZ)

$$DR_{g,A} = P_g^g(A) \in R^g(\overline{\mathcal{M}}_{g,n}).$$

(13) formula for $\lambda_g = c_g(\mathbb{E}_g)$

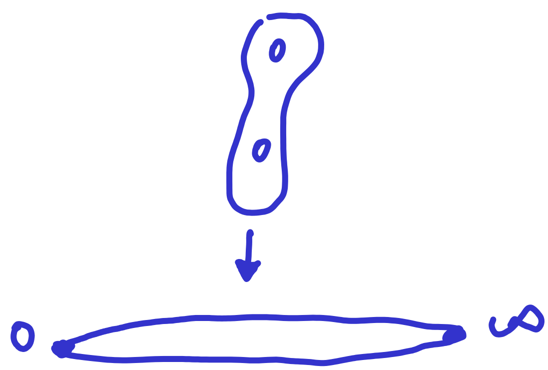
supported on $\Delta_0 \subset \overline{M}_g$

Answer: We view $\overline{M}_g = \overline{M}_{g,0}$

$n=0$

Let $A = \emptyset$

Geometry $\Rightarrow$ $\overline{M}_g(\mathbb{P}^1, \emptyset)^\sim \cong \overline{M}_g$



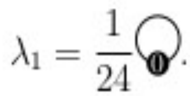
Moreover $[\overline{M}_g(\mathbb{P}^1, \emptyset)^\sim]^{vir} = (-1)^g \lambda_g$

$$DR_g(\emptyset) = (-1)^g \lambda_g$$

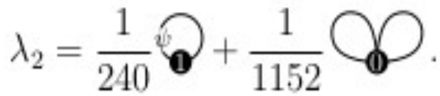
So we can apply the DR cycle Formula:

Diagrams from JPPZ

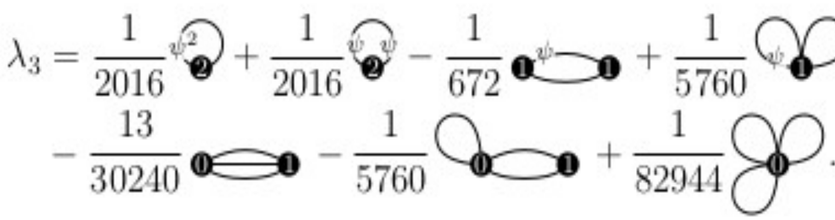
Genus 1.



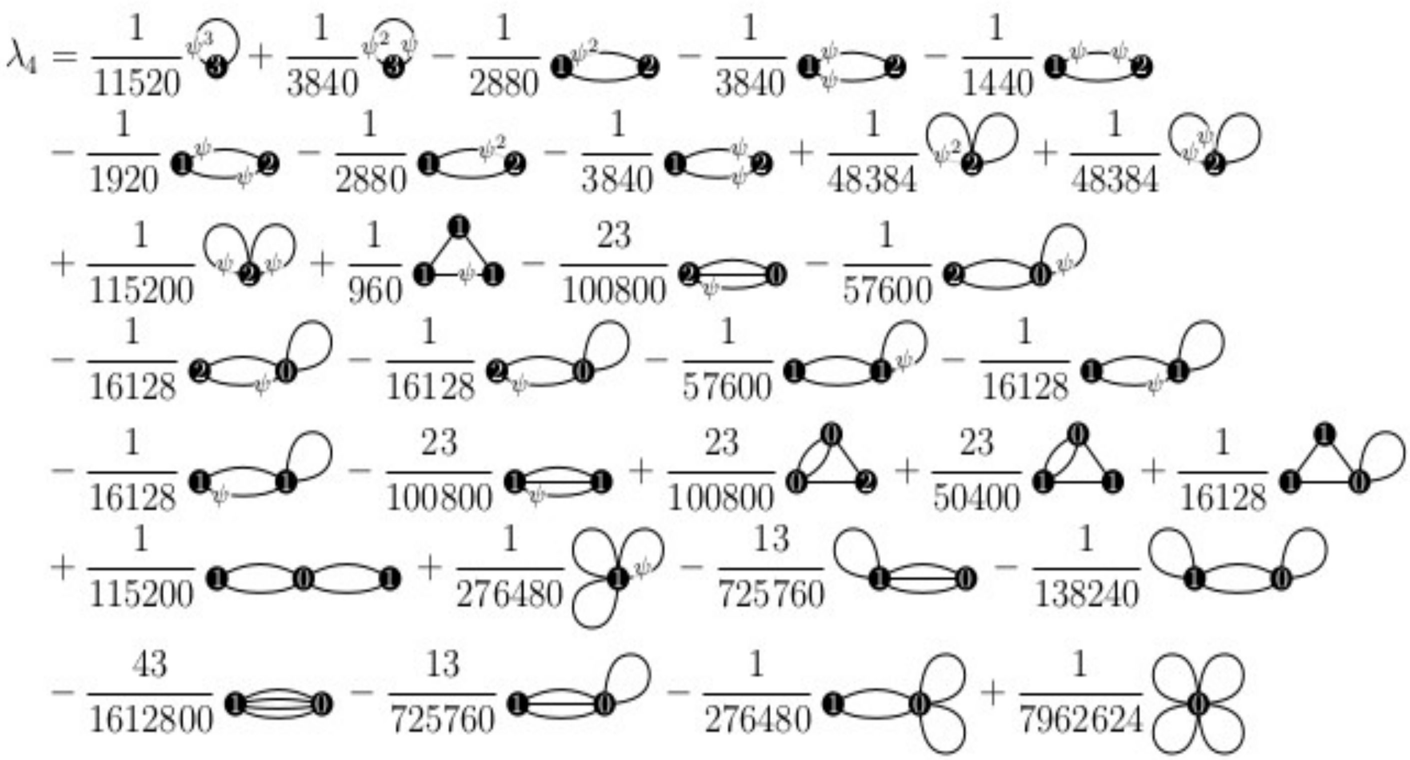
Genus 2.



Genus 3.



Genus 4.



Why only graphs with loops?

If Γ is an unpointed tree,

then $w = 0$.

$$\sum_{\Gamma \in G_{g,n}} \sum_{w \in W_{\Gamma,r}} \frac{1}{\mathrm{Aut}(\Gamma)} \underbrace{\frac{1}{r^{h'(\Gamma)}}}_{1} \cdot$$

$$\xi_{\Gamma *} \left[\prod_{e=(h,h')} \frac{1-\exp\left(-\frac{w(h)w(h')}{2} \cdot (\psi_h + \psi_{h'})\right)}{\psi_h + \psi_{h'}} \right]$$

0 0

The formula has no contributions in degree ≥ 1!

Can have no nodes

For $g>0 \Rightarrow$ Tree contributions vanish

(14) Map:

Pixton's Formula

$\Uparrow$ limit $r \to \infty$, set $r=0$

$\boxed{\mathbb{C}^* \text{ action/Localization}}$

$\boxed{\text{Chiodo's formula}}$ $\quad$ $\boxed{\text{Rubber geometry}}$

$\boxed{\text{Orbi-GW theory}}$

$$\overline{\mathcal{M}}_{g,\mu}(\mathbb{P}^1[0/r],\infty)_\nu$$

relative

Orbifold conditions

orbifold point

$$\cup$$

$$\overline{\mathcal{M}}_{g,\mu}(\mathbb{P}^1/0,\infty)_\nu$$

$$\Big\updownarrow \text{def}$$

$$DR_{g,A}$$

Double ramification cycle