The Proof of Pixton's formula

(1) References

- JPPZ Double ramification cycles on moduli spaces of curves
- BHPSS Pixton's formula and Abel-Jacobi theory on the Picard stack

(2) Stable relative maps (revisited)

We have discussed

- $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$ ← Stable maps

- $\overline{\mathcal{M}}_g(\mathbb{P}^1/0,\infty)^{\sim}_{\mu,\nu}$ ← Stable relative maps to rubber

We will need an intermediate moduli space

- $\overline{\mathcal{M}}_g(\mathbb{P}^1/\infty)_\nu$

Moduli of stable maps relative to ∞

No rubber

$$\overline{\mathcal{M}}_g(\mathbb{P}^1/\infty)_\nu$$

genus

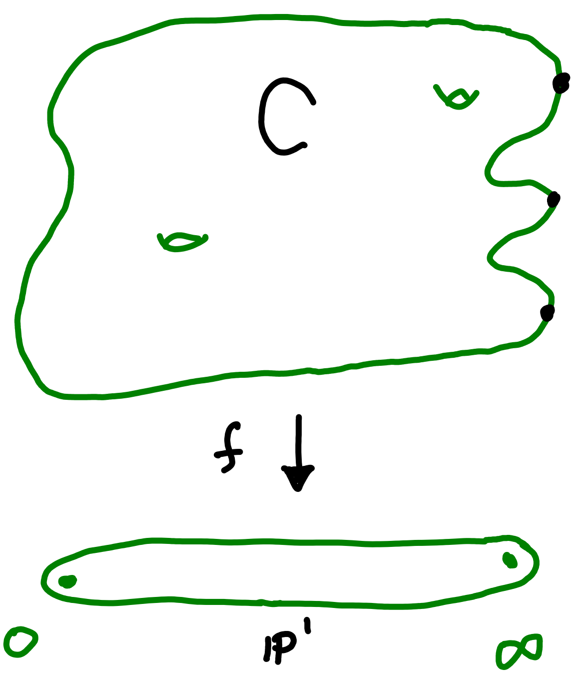
profile over $\infty \in \mathbb{P}^1$

degree $d = |\nu|$

Stable maps relative to $\infty \in \mathbb{P}^1$ are of the form

markings

No conditions over $0 \in \mathbb{P}^1$



Profile ν over ∞

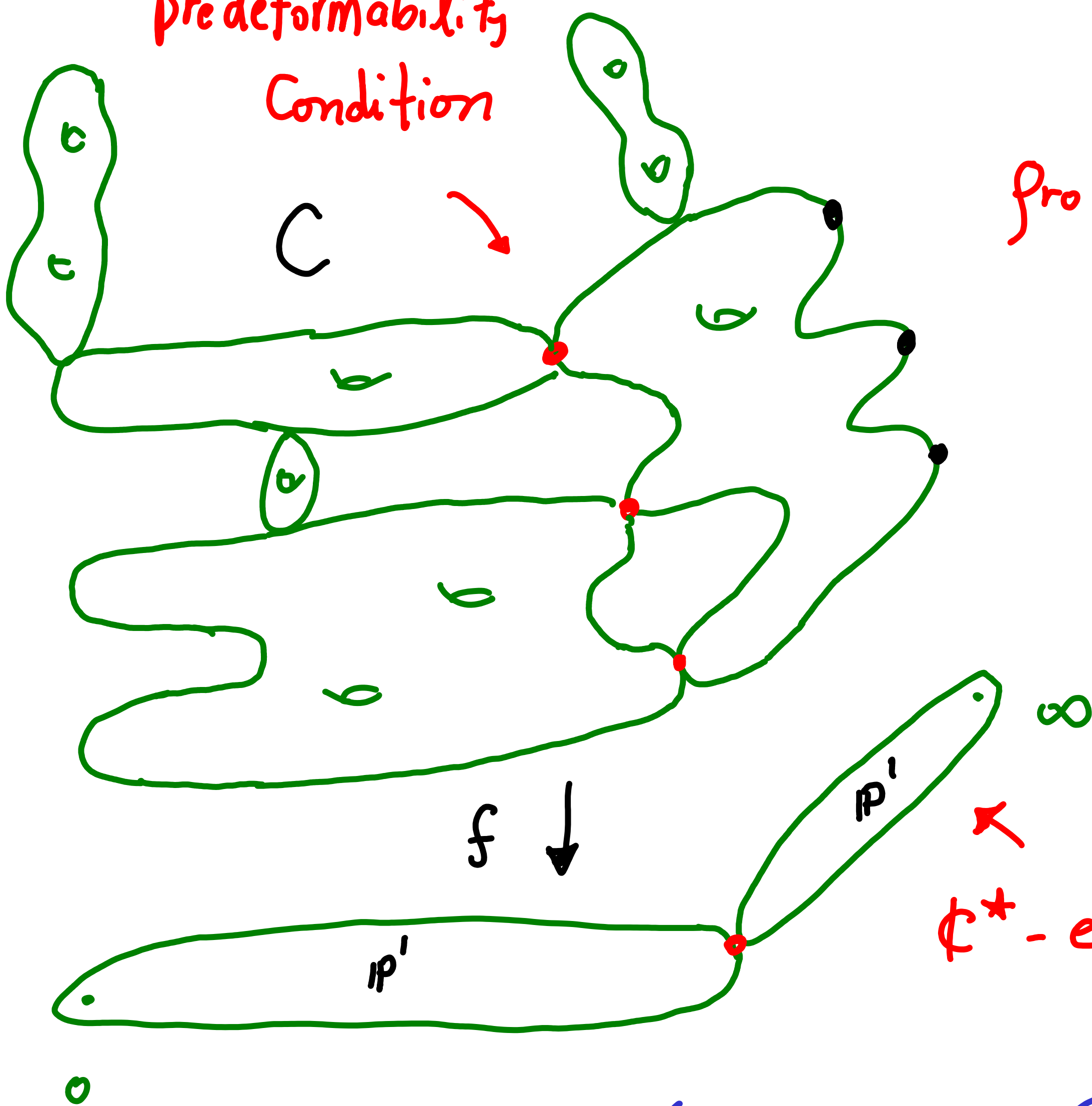
No $\mathbb{C}^*$-equivalence

What happens over ∞?

Allow degenerations of rubber type over ∞

predeformability condition

profile ν



$\mathbb{C}^*$-equivalence

usual stable map

Stable relative maps to rubber

Stability for $\overline{\mathcal{M}}_g(\mathbb{P}^1/\infty)_\nu$

is defined then as before

by finiteness of automorphisms.

$$\operatorname{virdim} \overline{\mathcal{M}}_g(\mathbb{P}^1/\infty)_\nu$$

$$= 2g-2+d+l_\nu$$

length of ν

So we have now three

variations:

$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$

Standard space for GW theory:

Axioms, Toda equations

Virasoro Constraints

$\mathbb{C}^*$-action

[Okounkov - P]

$\overline{\mathcal{M}}_g(\mathbb{P}^1/0,\infty)^{\sim}_{\mu,\nu}$

Connection to AJ theory

Definition of $DR_{g,A}$

no $\mathbb{C}^*$-action

$\overline{\mathcal{M}}_g(\mathbb{P}^1/\infty)_{\nu}$

Lives in both worlds,

Over 0: Stable maps

Over ∞: rubber

$\mathbb{C}^*$-action

(3) Bott residue formula

Let M be a nonsingular algebraic variety with a $\mathbb{C}^*$-action:

$$\mathbb{C}^* \times M \xrightarrow{\phi} M.$$

Example: $\mathbb{P}^n$ with the action

$$\phi(z, [x_0, \ldots, x_N]) = [z^{c_0} x_0, \ldots, z^{c_N} x_N]$$

for a vector $(c_0, \ldots, c_N) \in \mathbb{Z}^{N+1}$

$\mathbb{C}^*$-actions have been intensively studied. Most results hold also for Torus actions,

$$T = (\mathbb{C}^*)^r$$

$$\phi : T \times M \to M$$

Two basic facts:

(i) $M^T \subset M$, M^T is nonsingular

T-fixed locus

(but may be disconnected of varying dimensions)

(ii) Let $\mathcal{M}^T = \bigcup_j \mathcal{M}_j^T$

disjoint union
of the connected components
of the T-fixed locus.

on $\mathcal{M}_j^T$ we have

$$0 \to T_{\mathcal{M}_j^T} \to T_{\mathcal{M}}\big|_{\mathcal{M}_j^T} \to Nor_j \to 0$$

Exact sequence
of bundles with
T-actions

Normal bundle
of $\mathcal{M}_i^T$ in $\mathcal{M}$

$$T_\mu|_{M_j^T}$$ decomposes as a direct sum of T-representions

$$T_\mu|_{M_j^T} = T^{\text{moving}}_{\mu|_{M_j^T}} \oplus T^{\text{fixed}}_{\mu|_{M_j^T}}$$

all other weights

0-weight

Claim: $T_{M_j^T} = T^{\text{fixed}}_{\mu|_{M_j^T}}$

$$Nor_j = T^{\text{moving}}_{\mu|_{M_j^T}}$$

Bott residue / Atiyah-Bott localization

$$\sum_j i_* \frac{[M_j^T]}{e(Nor_j)} = [M]$$

in localized T-equivariant cohomology

CHow results by Edidin Graham

$$\mathcal{H}_T^*(M) \otimes \mathbb{Q}(t_1, \dots, t_r)$$

What does this mean?

What is T-equivariant cohomology?

Let ET be a contractible space on which T acts freely, and let BT be the quotient $ET/T = BT$

Algebraic Theory: Edidin-Graham, Equivariant Chow groups

in algebraic geometry, $E\mathbb{C}^*$ is approximated by $\mathbb{C}^N - 0$

$$E\mathbb{C}^* = \lim_{N\to\infty} \mathbb{C}^N - 0$$

$$B\mathbb{C}^* = \lim_{N\to\infty} \mathbb{P}^{N-1} = \mathbb{P}^\infty$$

Def: $H_T^*(M) = H^*(M \times_T ET)$

usual cohomology

we have
$$\begin{array}{c} M \times_T ET \\ \downarrow \\ BT \end{array}$$

so $H_T^*(M)$ is a module over the algebra

$$H^*(BT) = H_T^*(\cdot)$$

$\mathbb{Q}[t_1, \ldots, t_r]$

$t_i =$ first Chern class of the standard representation of i^{th} factor

Bott residues / Atiyah-Bott localization

fundamental class

holds after localization $\mathbb{Q}(t_1,\dots,t_r)$

Components of T-fixed locus

$$\sum_j i_* \frac{[M_j^T]}{e(Nor_j)} = [M]$$

T-equivariant push forward

fundamental class in $H_T^*(M)$

Most interesting aspect

$$T \text{ acts } \begin{array}{ll} \text{nontrivially} & Nor_j \\ & \downarrow \\ \text{trivially} & M_j^T \end{array}$$

$\mathcal{N}or_j$ decomposes into summands associated to T-representations

$$\mathcal{N}or_j \cong \bigoplus_k \omega_k \otimes \mathcal{N}or_{j,k}$$

$$e(\mathcal{N}or_j) = \prod e(\omega_k \otimes \mathcal{N}or_{j,k})$$

top Chern class

invertible in

$$\mathcal{H}^*_T(\mathcal{M}_j)$$

For the proof of the localization formula:

Atiyah-Bott, Edidin-Graham

Example: Calculate $i_j^*[M]$.

Here $i_j : M_j^T \hookrightarrow M$

using the localization formula

$$[M] = \sum_j i_* \frac{[M_j^T]}{e(Nor_j)}$$

Then,

$$i_j^*[M] = i_j^* i_{j*} \frac{[M_j^T]}{e(Nor_j)}$$

$$= \frac{1}{e(Nor_j)} i_j^* i_{j*} [M_j^T]$$

By excess intersection,

$$i_j^* \, i_{j*} [M_j^T] = e(N_{or_j}) \cdot [M_j^T]$$

so $$i_j^* [M] = [M_j^T]$$

which is of course correct.

Example: $\mathbb{P}^n$ with the action

$$\phi(z, [x_0, \dots, x_N]) = [z^{c_0} x_0, \dots, z^{c_N} x_N]$$

for a vector $(c_0, \dots, c_N) \in \mathbb{Z}^{N+1}$

Let us assume $C_i \neq C_j$ for $i \neq j$

Then the T-fixed locus

Consists of the $n+1$ coordinate

points $p_0 = [1, 0, 0 \dots, 0]$

$$p_1 = [0, 1, 0, \dots, 0]$$

$$\vdots$$

$$p_n = [0, \dots, 0, 1]$$

What is $e(Nor_j)$?

at p_j

Answer: $e(Nor_j) = \prod_{k \neq j} c_k t - c_j t$

Localization:

$$\sum_j \frac{i_* [p_j]}{\prod_{k \neq j} c_k t - c_j t} = [\mathbb{P}^n]$$

in $H^*_{\mathbb{C}^*}(\mathbb{P}^n) \otimes \mathbb{Q}(t)$

need only invert t

Example: Equivariant push-forward

$$\begin{tikzcd}
\{p_0, p_1, \ldots, p_n\} \arrow[r, phantom, "\subset"] \arrow[dr, "\varepsilon"'] & \mathbb{P}^n \arrow[d, "\varepsilon"] \\
& \cdot \text{ point}
\end{tikzcd}$$

$\mathbb{C}^*$-fixed locus

$$\varepsilon_* : H^*_{\mathbb{C}^*}(\mathbb{P}^n) \to H^*_{\mathbb{C}^*}(\cdot) \cong \mathbb{Q}[t]$$

We often denote ε_* by $\int$. ← equivariant integration

(i) $\varepsilon_* [\mathbb{P}^n] = 0$ (positive fiber dimension)

so
$$\sum_j \frac{i_* [p_j]}{\prod_{k \neq j} c_k t - c_j t} = 0$$

Identity in $\mathbb{Q}(t)$.

Check for $N=2$:

$$\frac{1}{(c_2t-c_0t)(c_1t-c_0t)}$$

$$+\quad\frac{1}{(c_2t-c_1t)(c_0t-c_1t)}$$

$$+\quad\frac{1}{(c_1t-c_2t)(c_0t-c_2t)}$$

$$=0$$

(ii) Let $\gamma \in \mathcal{H}^{*}_{\mathbb{C}^*}(\mathbb{P}^n)$ be any class. There is a restriction map

$$\begin{aligned} \mathcal{H}^{*}_{\mathbb{C}^*}(\mathbb{P}^n) &\longrightarrow H^*(\mathbb{P}^n) \\ \gamma &\longmapsto \hat{\gamma} \end{aligned}$$

Non Equivariant limit

Relationship with push-forward:

$$\widehat{\varepsilon_{\star}\gamma} = \int_{\mathbb{P}^n} \hat{\gamma} \in \mathbb{Q}$$

(iii) Let $h = c_1(\mathcal{O}_{\mathbb{P}}(1)) \in \mathcal{H}^*_{\mathbb{C}^*}(\mathbb{P}^n)$

Then for $0 \le k < n$,

$$\varepsilon_* \, h^k \cap [\mathbb{P}^n] = 0$$

so

$$\varepsilon_* \sum_j \frac{i^* h^k \cap i_* [p_j]}{\prod_{k \neq j} c_k t - c_j t} = 0$$

$$\Updownarrow$$

$$\sum_j \frac{(-c_j t)^k}{\prod_{k \neq j} c_k t - c_j t} = 0$$

(iv) $\varepsilon_\star \, h^n \cap [\mathbb{P}^n] = 1$

so

$$\varepsilon_\star \sum_j \frac{i^* h^n \cap i_* [p_j]}{\prod_{k \neq j} c_k t - c_j t} = 1$$

$$\Updownarrow$$

$$\sum_j \frac{(-c_j t)^n}{\prod_{k \neq j} c_k t - c_j t} = 1$$

There is some magic in the localization formula.

(v) Let $\delta \in H^{2n-2}_{\mathbb{C}^*}(\mathbb{P}^n)$

Consider $c_1(t \otimes \mathcal{O}_{\mathbb{P}^n}) \in H^2_{\mathbb{C}^*}(\mathbb{P}^n)$

trivial line with weight t

Let $\gamma = \delta \cdot c_1(t \otimes \mathcal{O}_{\mathbb{P}^n})$

Certainly $\hat{\gamma} = 0$ because of

We obtain

$$\varepsilon_\star\left(\delta \cdot c_1(t \otimes \mathcal{O}_{\mathbb{P}^n})\right) = 0$$

By localization:

$$\varepsilon_* \sum_j \frac{t\, i^*(\delta) \cap i_* [p_j]}{\prod_{k \neq j} c_k t - c_j t} = 0$$

$$\Updownarrow$$

$$\sum_j \frac{t\, i^*(\delta)}{\prod_{k \neq j} c_k t - c_j t} = 0$$

Geometric method for generating identies in $\mathbb{Q}(t)$

(4) Localization of the virtual fundamental class

Let $\mathcal{M}$ be a scheme with

- T-action, $T \times \mathcal{M} \to \mathcal{M}$
- a T-equivariant 2-term Obstruction theory

$$E^\bullet = [E^{-1} \to E^0]$$

$$\phi : E^\bullet \to L^\bullet_{\mathcal{M}}$$ cotangent complex

Then there is localization formula for $[\mathcal{M}]^{vir} \in H^T_*(\mathcal{M})$

Localization of virtual classes

Graber - P
1998

$$[M]^{vir} = \sum_j i_* \frac{[M_j^T]^{vir}}{e(Nor_j^{vir})}$$

fundamental class in $H_*^T(M)$

Component of the T- fixed locus

holds after localization $\mathbb{Q}(t_1,\ldots,t_r)$

New Aspects

New: Each T-fixed components

$\mathcal{M}_j^T$ carries a canonical

T-fixed 2-term obstruction

theory.

$$\left(E^\bullet|_{\mathcal{M}_j^T}\right)^{\text{fixed}} = \left[\left(E^{-1}|_{\mathcal{M}_j^T}\right)^{\text{fixed}} \to \left(E^0|_{\mathcal{M}_j^T}\right)^{\text{fixed}}\right]$$

$$\phi^{\text{fixed}} \searrow$$

$$\left(L^\bullet_{\mathcal{M}}|_{\mathcal{M}_j^T}\right)^{\text{fixed}} = L^\bullet_{\mathcal{M}_j^T}$$

2 term obstruction theory $\Rightarrow$ $[\mathcal{M}_j^T]^{vir}$

We define $e(N_{or_j}^{vir})$ using the moving parts:

$$e(N_{or_j}^{vir}) = \frac{e\left(E_0^{moving}\big|_{M_j^T}\right)}{e\left(E_1^{moving}\big|_{M_j^T}\right)}$$

where $E_0 = (E^0)^\vee$

$E_1 = (E^{-1})^\vee$

Exercise: Apply the virtual localization formula to the moduli space $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$

(5) Idea to calculate $DR_{g,A}$:

As usual, we form partitions

μ, ν from $A = (a_1, \dots, a_n)$

length l_μ $\quad l_\nu$

$n = l_\mu + l_\nu$

$|\mu| = |\nu| = d$

Consider the moduli space

$$\overline{\mathcal{M}}_{g, l_\mu}(\mathbb{P}^1/\infty)_\nu$$

There is a $\mathbb{C}^*$-action by

rotating $\mathbb{P}^1$ with $0, \infty \in \mathbb{P}^1$ fixed forgetful

$$\varepsilon : \overline{\mathcal{M}}_{g,l_\mu}(\mathbb{P}^1/\infty)_\nu \longrightarrow \overline{\mathcal{M}}_{g,n}$$

nontrivial $\mathbb{C}^*$-action

$\mathbb{C}^*$-equivariant

trivial $\mathbb{C}^*$-action

$$\operatorname{vir\,dim}_{\mathbb{C}} \overline{\mathcal{M}}_{g,l_\mu}(\mathbb{P}^1/\infty)_\nu$$

$$= 2g-2+d+l_\mu+l_\nu$$

Idea: use equivariant push-forward and the localization formula

$c_1(t \otimes \theta)$ cotangent lines $\mathbb{C}^*$- equivariant virtual fundamental class

$$\varepsilon_*\left(t \cdot \prod_{i=1}^{l_\mu} \psi_i^{\mu_i} \cdot \left[\overline{\mathcal{M}}_{g,l_\mu}(\mathbb{P}^1/\infty)_\nu \right]^{vir} \right)$$

$$\shortparallel$$

$$0 \in H^{2g}(\overline{\mathcal{M}}_{g,n})$$

$\dim_{\mathbb{C}}$ calculation

$$-1 - \sum_{i=1}^{l_\mu} \mu_i + 2g-2 + d + l_\mu + l_\nu$$

$$= 2g-3 + l_\mu + l_\nu$$

exactly $\operatorname{codim}_{\mathbb{C}} g$ in $\overline{\mathcal{M}}_{g,n}$

Now use $\mathbb{C}^*$-equivariant localization

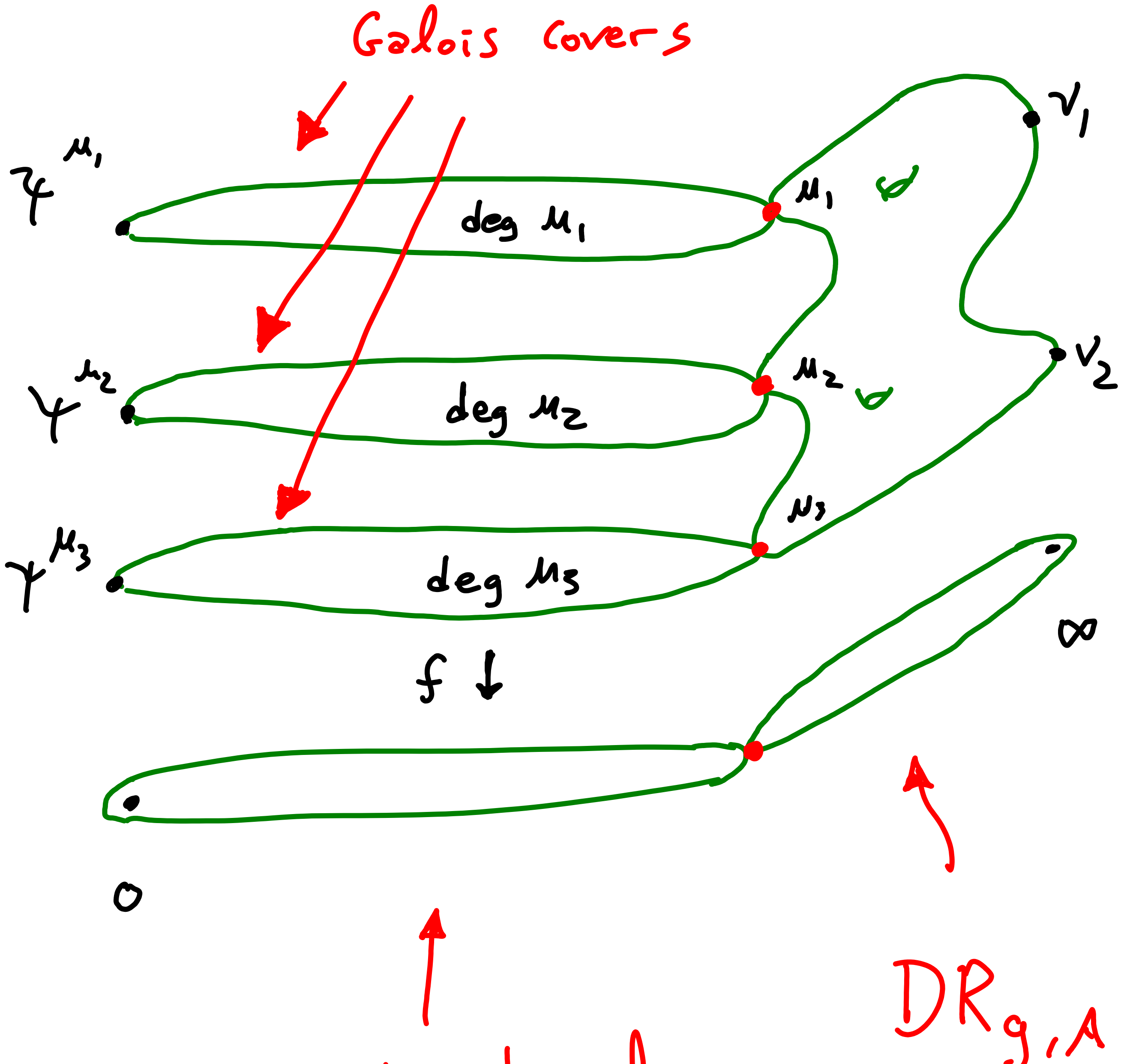
to analyze

$$\varepsilon_*\left(t \cdot \prod_{i=1}^{l_\mu} \psi_i^{\mu_i} \cdot [\overline{M}_{g,l_\mu}(\mathbb{P}^1/\infty)_\nu]^{vir} \right)$$

Remarks

- Must find all $\mathbb{C}^*$-fixed points
- Need localization for the virtual fundamental class

$D_{g,A}$ does arise as a localization contribution



[Faber-P 2004]

- Sadly there are a huge number of other $\mathbb{C}^*$-fixed loci which complicate the relation.

No way to prove Pixton's formula

(6) Another geometric idea:

Orbifold Gromov-Witten theory

Related to Pixton's r

What is an orbifold?

- Special case M/G

Nonsingular algebraic variety

finite group

- More generally: define locally as such a quotient

Development in topology, see book by Chen-Ruan

- or define as a nonsingular DM stack

Main examples for us:

point

- $B\mathbb{Z}_r = [\bullet / \mathbb{Z}_r]$ indicate orbifold/DM stack

trivial action

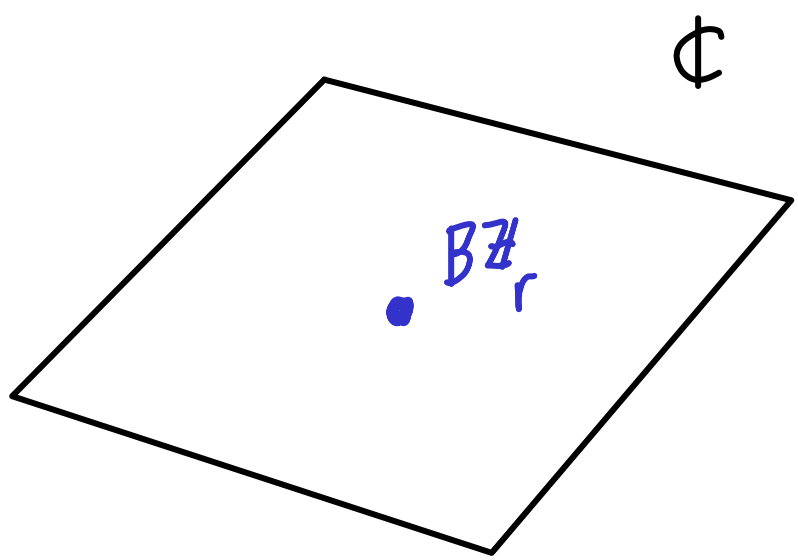
- $[\mathbb{C}/W_r] \ni B\mathbb{Z}_r$ stacky point

group of r^{th} roots of unity

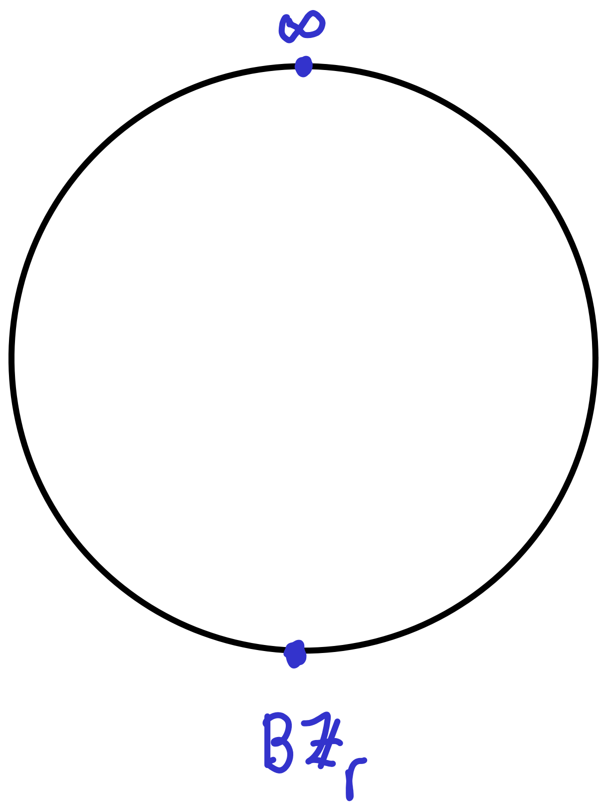
Away from $B\mathbb{Z}_r$,

$$[\mathbb{C}/W_r] \cong \mathbb{C}^*$$

Picture:



- $\mathbb{P}^1[r]$ ⟵ $\mathbb{P}^1$ with $B\mathbb{Z}_r$ at 0 and standard structure everywhere else



Two charts

$$B\mathbb{Z}_r \in \mathbb{C}/\mu_r \quad \cup \quad \mathbb{C} \ni \infty$$

gluing $\quad z \longmapsto \dfrac{1}{z^r}$

Torus action

$$\mathbb{C}^* \times \mathbb{P}^1[r] \to \mathbb{P}^1[r]$$

in charts

- $\mathbb{C}^* \times \mathbb{C}/_{\mu_r} \to \mathbb{C}/_{\mu_r}$

$$(\lambda, z) \mapsto \lambda^{\frac{1}{r}} z$$

- $\mathbb{C}^* \times \mathbb{C} \to \mathbb{C}$

$$(\lambda, z) \mapsto \lambda^{-1} z$$

Tangent representation:

weight $\frac{1}{r}$ at $B\mathbb{Z}_r$

-1 at ∞

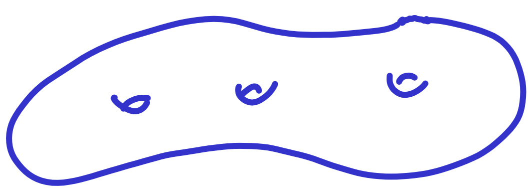
A Strange world

Orbifold stable maps

$$\overline{\mathcal{M}}_{g,n}(\mathcal{X},\beta)$$

orbifold target

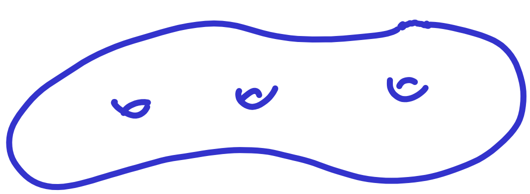
What is a map from a space to an orbifold?

C :  $\xrightarrow{f}$ M/G

$\Updownarrow$ equivalent

G-bundle $\quad P \xrightarrow[G\text{-equivariant}]{F} M$

$\pi \downarrow$

C 

Main example for us :

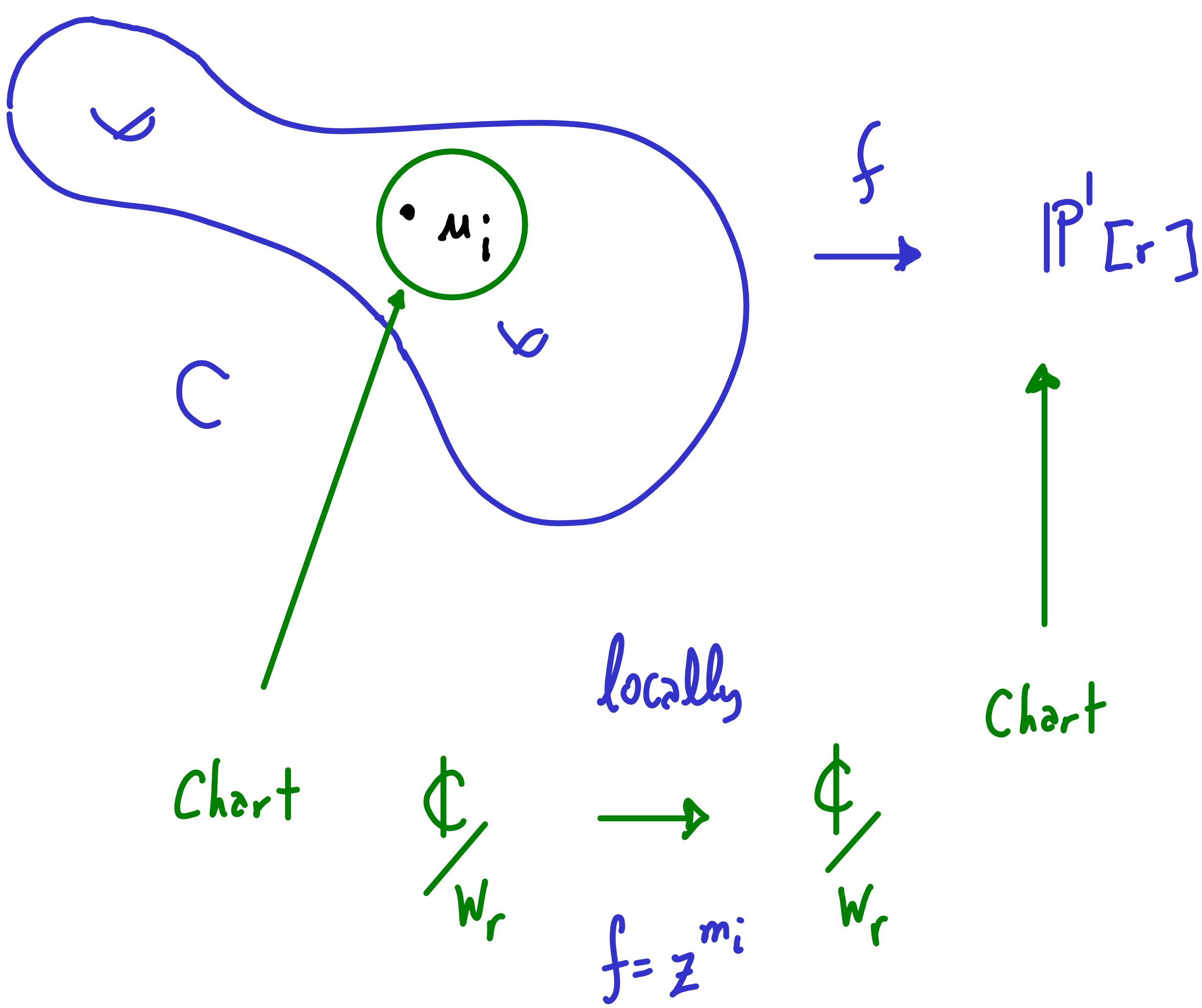
$$\overline{\mathcal{M}}_{g,\mu}\left(\mathbb{P}^1[r]/\infty\right)_\nu$$

relative conditions as before

Orbifold markings

Orbifold point $B\mathbb{Z}_r = [0/\mathbb{Z}_r]$

relative point $\infty \in \mathbb{P}^1$



(7) Steps of Proof of Pixton's formula

forgetful

$$\varepsilon : \overline{\mathcal{M}}_{g,\mu}(\mathbb{P}^1[r]/\infty)_\nu \longrightarrow \overline{\mathcal{M}}_{g,n}$$

nontrivial $\mathbb{C}^*$-action

$\mathbb{C}^*$-equivariant

trivial $\mathbb{C}^*$-action

$$\operatorname{vir\,dim}_{\mathbb{C}} \overline{\mathcal{M}}_{g,\mu}(\mathbb{P}^1[r]/\infty)_\nu$$

$$= 2g-2+l_\mu+l_\nu$$

$$= 2g-2+n$$

$c_1(t \otimes \mathcal{O})$

$\mathbb{C}^*$- equivariant virtual fundamental class

$$\varepsilon_* \left(t \cdot [\overline{M}_{g,\mu} (\mathbb{P}^1/\infty)_\nu]^{vir} \right)$$

$$\parallel$$

$$0 \in H^{2g}(\overline{M}_{g,n})$$

STEP I : Calculate using the virtual localization formula.

- Find $\mathbb{C}^*$-fixed points
- Find the localization contributions and write equation

Over $B\mathbb{Z}_r$ $\hookrightarrow$ orbifold contributions

Over ∞ $\hookrightarrow$ double ramification cycle contributions

Step II: Study the $r \gg 0$ limit, polynomiality, set $r=0$

Step III: Almost all terms in the equation are killed in step II

The remaining terms are exactly Pixton's formula.

(8) Localization example

$$\langle \tau_2(p) \rangle^{\mathbb{P}^1}_{0,\, d=2} = \int_{\overline{M}_{0,1}(\mathbb{P}^1,2)} \psi_1^2 \, ev_1^*(p)$$

genus 0

degree = 2

Nonsingular stack of dim 3

$\mathbb{C}^*$ acts on $\mathbb{P}^1 = \mathbb{P}(\mathbb{C} \oplus \mathbb{C})$

weight 0 weight 1.

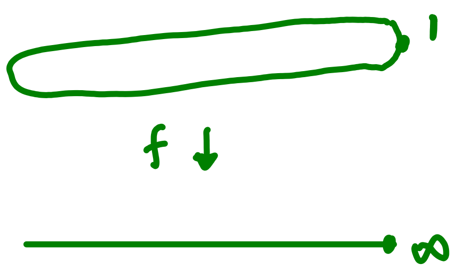
action formula $\lambda \cdot [x, y] \mapsto [x, \lambda y]$

$\mathbb{C}^*$ $\mathbb{P}^1$ $\mathbb{P}^1$

Equivariant lift : $$\int_{\overline{M}_{0,1}(\mathbb{P}^1,2)} \psi_1^2 \, ev_1^*(\infty)$$

$\mathbb{C}^*$- fixed loci (which contribute) :

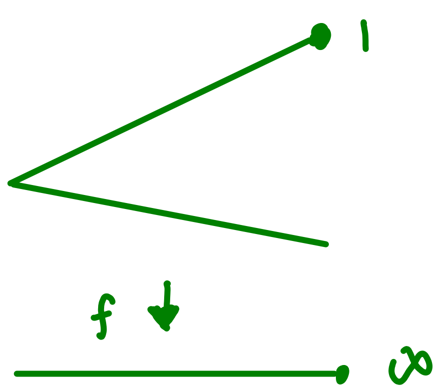
(a)



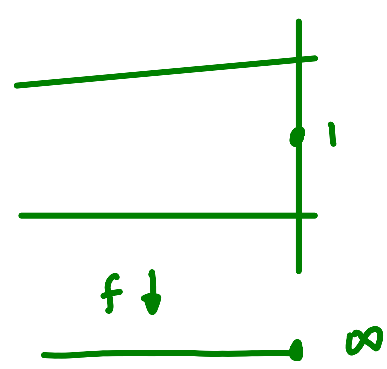
Galois cover

$[x,y] \to [x^2, y^2]$

(b)



(c)



(a) We must calculate 3 Normal weights

Remember: deformations of f are given

by $H^0(f^* T_{\mathbb{P}^1}) - H^0(T_{\mathbb{P}^1}(-p))$

dim 5 dim 2

$-t \quad -\frac{t}{2} \quad 0 \quad \frac{t}{2} \quad t \qquad -\frac{t}{2} \quad 0$

$\rightsquigarrow$ Normal weights are

$-t, \frac{t}{2}, t$

Stack Automorphism

Contribution of (a): $$\frac{(t)\left(-\frac{t}{2}\right)^2}{(-t)\left(\frac{t}{2}\right)(t)}\left(\frac{1}{2}\right)$$

(b) We must calculate 3 Normal weights

$-2t, \; -t, \; t$

Smoothing the node

Contribution of (b) : $\dfrac{(t)\,(-t)^2}{(-2t)\,(-t)\,(t)}$

Contribution of (c) vanishes because of the cotangent line ψ_1

Answer: $-\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$

$$\langle \tau_2(p) \rangle_{0,d=2}^{\mathbb{P}^1} = \frac{1}{4} = \frac{1}{(2!)^2}$$

More generally $\langle \tau_{2d-2}(p) \rangle_{0,d}^{\mathbb{P}^1} = \dfrac{1}{(d!)^2}$

How can we prove such a formula?

Use structures in $g=0$ GW theory:

QDE, fundamental solution

Topological recursion

Reference: $\mathcal{P}$ Rational curves on hypersurfaces
SEM Bourbaki (after A. Givental)

Consider $$\int_{\overline{M}_{0,1}(\mathbb{P}^1, 2)} t \, \psi_1 \, ev_1^*(\infty) = 0$$

As before, only (a) and (b) contributions

(a):

$$\frac{(t)^2 \left(-\frac{t}{2}\right)}{(-t)\left(\frac{t}{2}\right)(t)} \left(\frac{1}{2}\right) = \frac{1}{2}$$

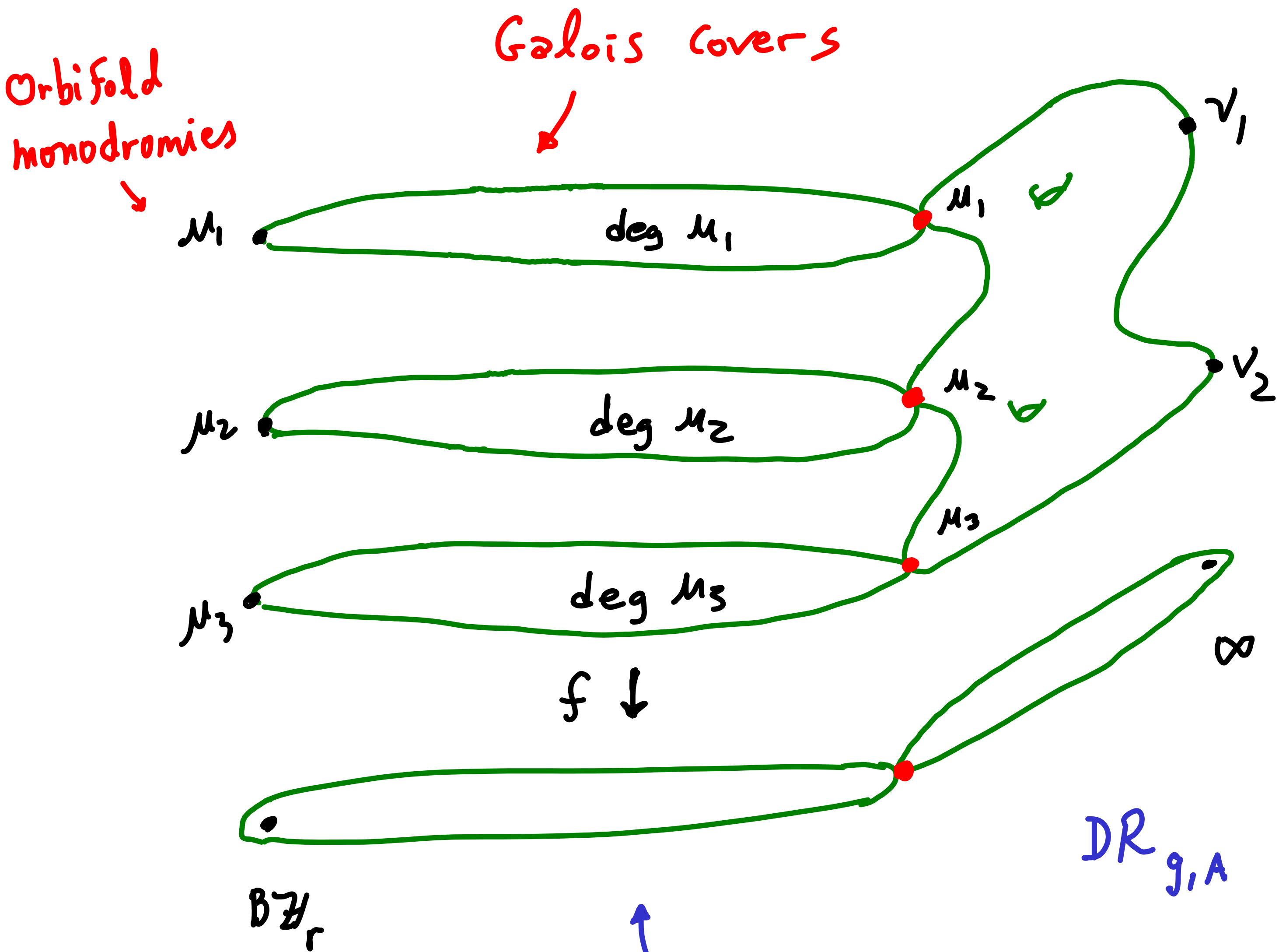
(b):

$$\frac{(t)^2 (-t)}{(-2t)(-t)(t)} = -\frac{1}{2}$$

(g) Return to the proof of Pixton's formula (I)

- Find $\mathbb{C}^*$-fixed points:

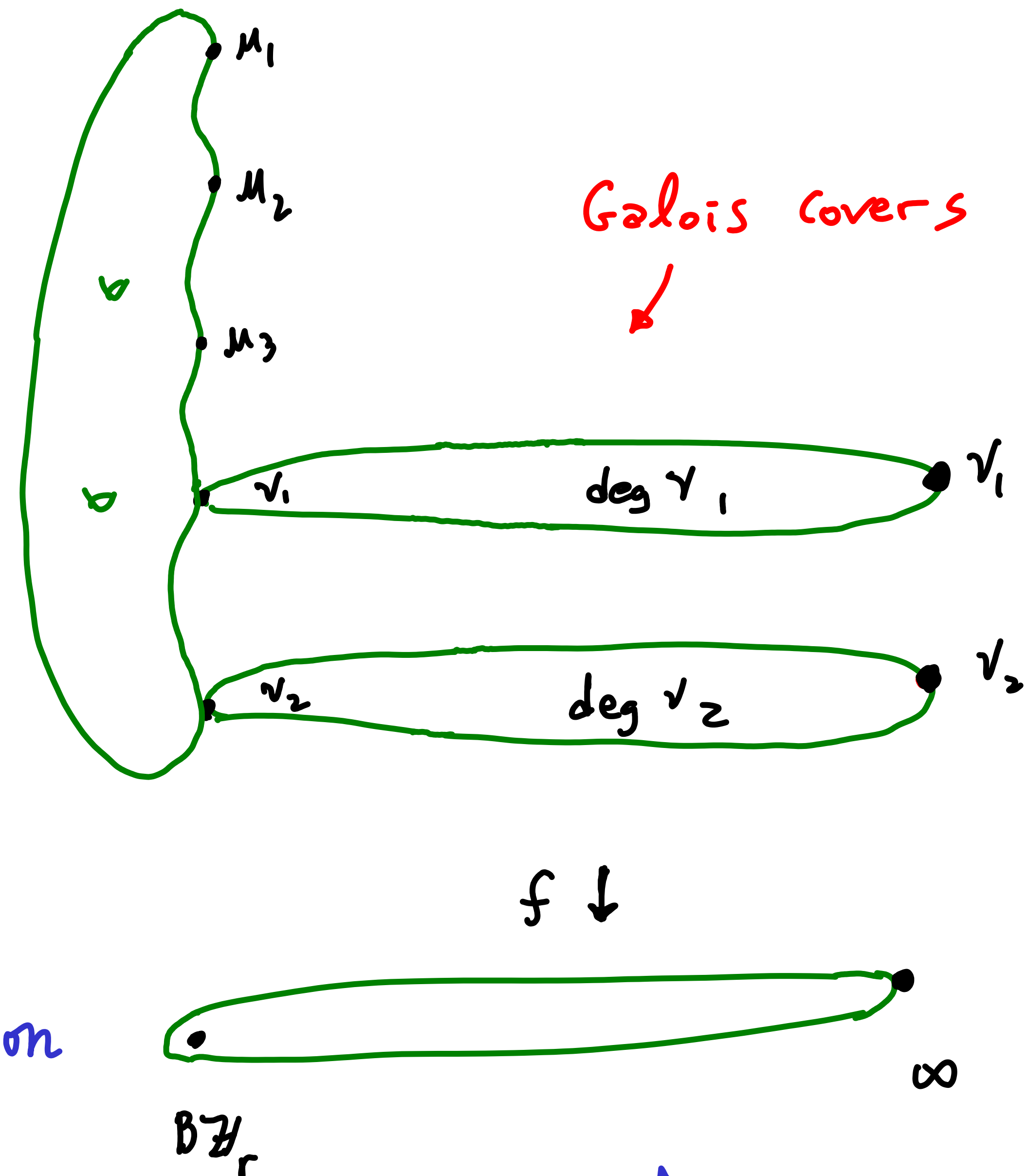
$\boxed{\text{Type } \infty}$



$r >> d = |\mu|$

Combinatorial factor

$\boxed{\text{Type } 0}$



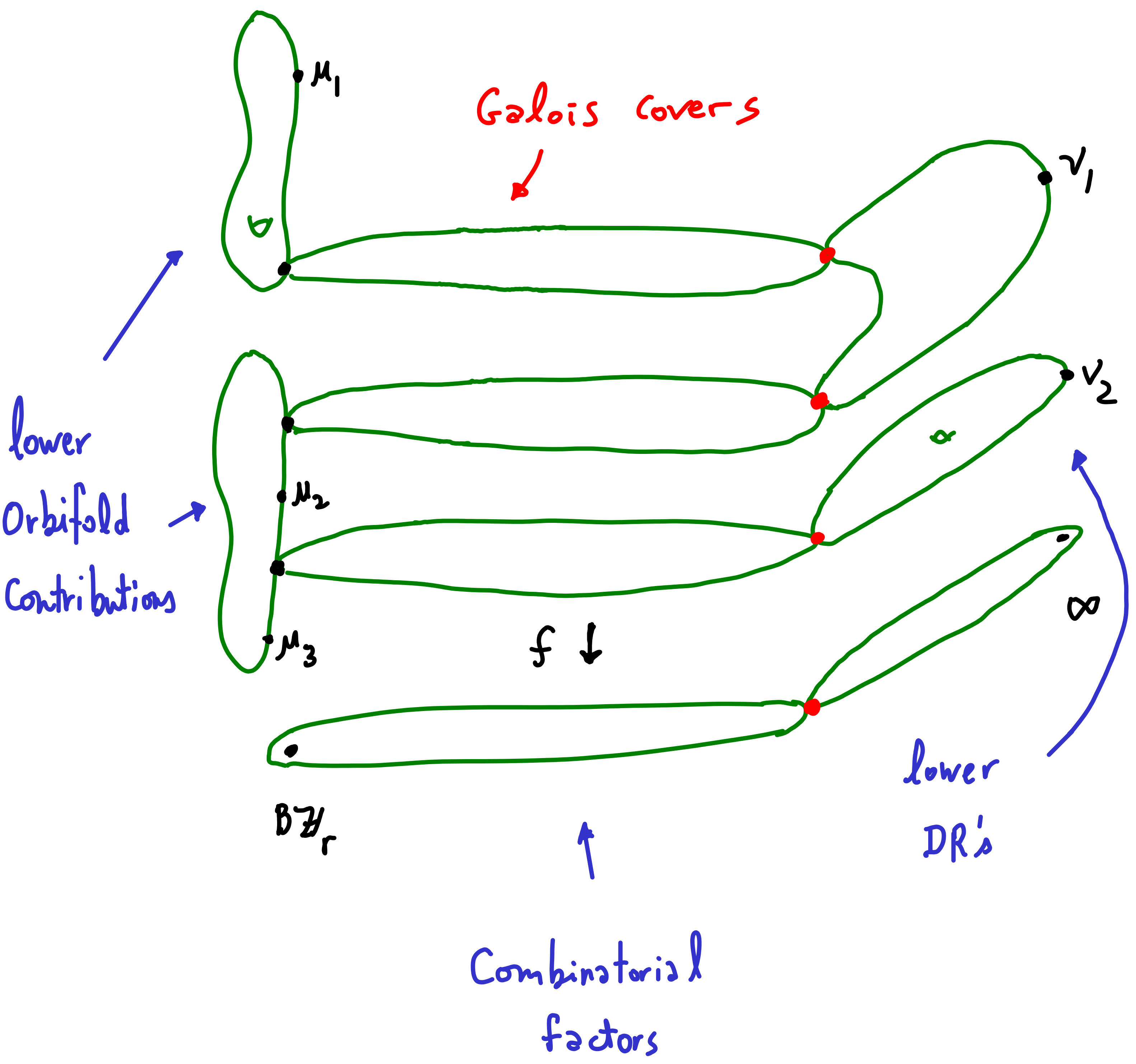
Pure Orbifold Contribution

$r >> d = |\mu|$

Combinatorial factor

$\boxed{\text{The rest}}$

Huge number of graphs



$\boxed{\text{We have not yet used } r}$

- Find the localization contributions and Write equation

$$\varepsilon : \overline{\mathcal{M}}_{g,\mu}(\mathbb{P}^1[r]/\infty)_\nu \longrightarrow \overline{\mathcal{M}}_{g,n}$$

$\mathbb{C}^*$-fixed loci

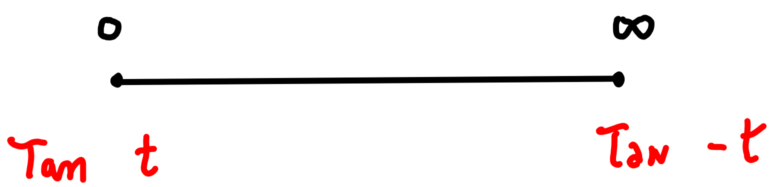
We will calculate

$$\varepsilon_*\left(t \cdot [\overline{\mathcal{M}}_{g,\mu}(\mathbb{P}^1/\infty)_\nu]^{vir} \right) = 0$$

in $H^{2g}(\overline{\mathcal{M}}_{g,n})$

Contributions: Type ∞, Type 0, Rest

$\mathbb{C}^*$ acts on $\mathbb{P}^1 = \mathbb{P}(\underset{\text{weight } 0}{\mathbb{C}} \oplus \underset{\text{weight } 1.}{\mathbb{C}})$

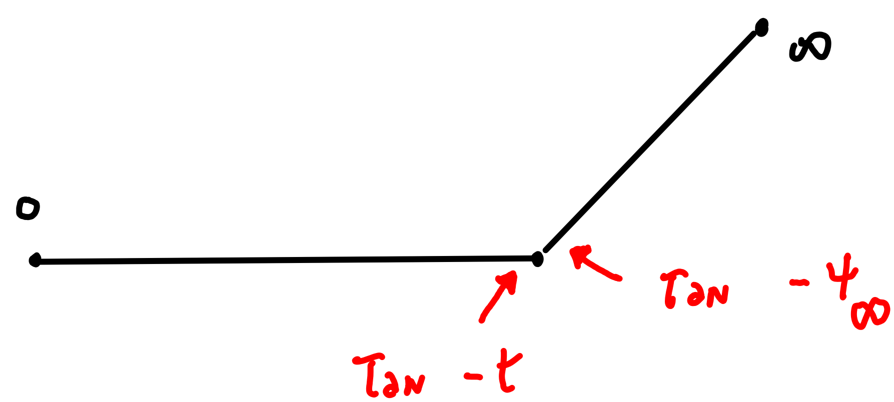


Type ∞ and Type 0 are simple.

The Rest is complicated

Contribution of Type ∞

$$\boxed{-\frac{t}{t+\psi_\infty}\cdot DR_{g,A}}$$



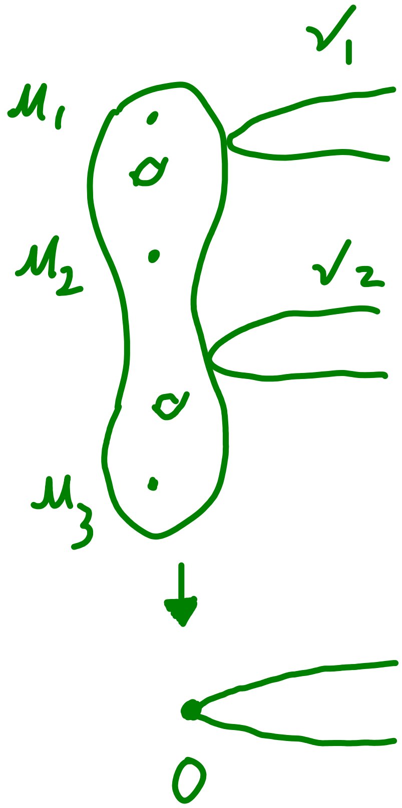
$$\frac{1}{e(\mathrm{Nor}^{\mathrm{vir}})}=\frac{1}{-t-\psi_\infty}$$

Contribution of Type 0

$$\boxed{\; t \prod_{i=1}^{l_\nu} \frac{r}{t - \nu_i \psi_i} \sum_{\delta \geq 0} c_\delta \left(-R\pi_* \mathcal{L}\right) \left(\frac{t}{r}\right)^{g-1+l_\nu-\delta} \;}$$

Where does this come from?

Constant map contributions in standard Gromov-Witten theory should be considered first



The $\mathbb{C}^*$-fixed locus here is

$$\overline{\mathcal{M}}_{g,\ell_\mu+\ell_\nu}$$

The $\mathbb{C}^*$-moving part of the obstruction theory is

$\mathbb{C}^*$-weight is t on Tan_0

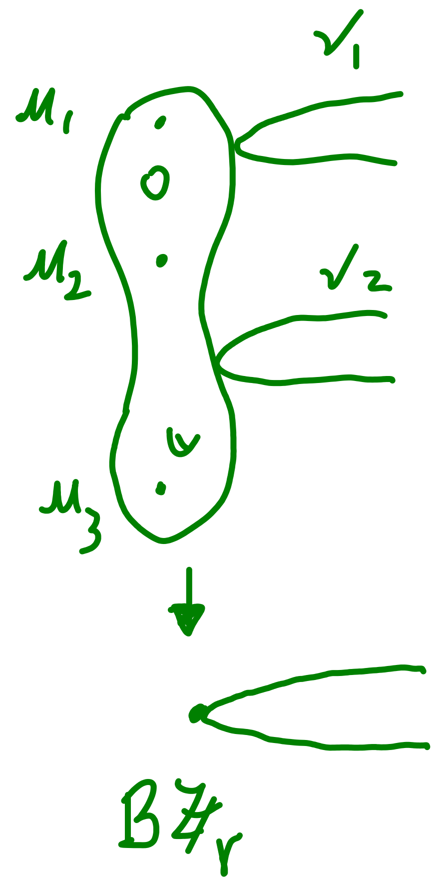
$$\sum_{\delta\geq 0} c_\delta\left(-R\pi_* f^*\Theta\right)(t)^{g-1-\delta}$$

together with

$$\prod_{i=1}^{\ell_\nu} \frac{1}{\frac{t_i}{\nu_i}-\psi_i}$$

suppressed factors of t and ν_i

But we are in orbifold Gromov-Witten theory



the moving part from the nodes is essentially the same

$$\prod_{i=1}^{l_\nu} \frac{1}{\frac{t}{r\nu_i} + \frac{\psi_i}{r}}$$

suppressed factors of t and ν_i

for the other $\mathbb{C}^*$-moving part, we have

$$\sum_{\delta \geq 0} c_\delta \left(-R\pi_* \mathcal{L}\right) \left(\frac{t}{r}\right)^{g-1+l_v-\delta}$$

$\mathcal{L}$ is the universal rth root.

$\mathcal{L}$ is orbifold tangent space at $B\mathbb{Z}_r$

Contribution of the Rest is complicated

But we write an equation

$$- \frac{t}{t+\psi_\infty} \cdot DR_{g,A}$$

$$+ \ t \prod_{i=1}^{l_\nu} \frac{r}{t - \nu_i \psi_i} \sum_{\delta \geq 0} c_\delta \left(-R\pi_* \mathcal{L}\right) \left(\frac{t}{r}\right)^{g-1+l_\nu-\delta}$$

$$+ \quad \text{Rest}$$

$$= 0 \quad \text{in } H^{2g}(\overline{\mathcal{M}}_{g,n})$$

All terms understood as taken after push forward to $\overline{\mathcal{M}}_{g,n}$ and extraction of t^0-coefficient

(10) Proof of Pixton's formula (II) and (III)

Step II : Study the $r \gg 0$ limit,

polynomiality, set $r = 0$

- Type ∞ We have

$$- \frac{t}{t + \psi_\infty} \cdot DR_{g,A} =$$

$$- DR_{g,A} + \frac{1}{t} \psi_\infty DR_{g,A} - \dots$$

No r here
only t^0-term

So $\quad -\dfrac{t}{t+\psi_\infty}\cdot DR_{g,A}$

contributes $\quad -DR_{g,A}$

- Type 0 $\quad$ We have

$$t\prod_{i=1}^{l_\nu}\frac{r}{t-\nu_i\psi_i}\sum_{\delta\geq 0} c_\delta\left(-R\pi_*\mathcal{L}\right)\left(\frac{t}{r}\right)^{g-1+l_\nu-\delta}$$

on $\quad \overline{\mathcal{M}}^{\frac{1}{r}}_{g,A}$

We must study $r\gg 0$,

push forward

$$\varepsilon : \overline{\mathcal{M}}^{\frac{1}{r}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

and take the r constant term.

We write

(i) $$\hat{c}_\delta = r^{2\delta-2g+1}\, \varepsilon_\star c_g(-R\pi_* \mathcal{L})$$

in $H^{2\delta}(\overline{\mathcal{M}}_{g,n})$

(ii) $$s = tr$$

[want $t^0 r^0$ coefficient, equivalently $s^0 r^0$ coefficient]

We rewrite

$$\varepsilon_\star \left[t \prod_{i=1}^{l_\nu} \frac{r}{t - \nu_i \psi_i} \sum_{\delta \geq 0} c_\delta(-R\pi_\star \mathcal{L}) \left(\frac{t}{r}\right)^{g-1+l_\nu-\delta} \right]$$

in the new conventions as

$$\prod_{i=1}^{l_\nu} \frac{1}{1 - \frac{r}{s}\nu_i \psi_i} \sum_{\delta \geq 0} \hat{c}_\delta \, s^{g-\delta}$$

Analysis of

$$\hat{c}_\delta = r^{2\delta - 2g + 1} \varepsilon_\star c_\delta(-R\pi_\star \mathcal{L})$$

By GRR (Chiodo's formula) $\Rightarrow$

$\hat{C}_g$ is polynomial in r for all $r >> 0$.

4 steps:

- Chern class $\Rightarrow$ Chern Characters
- Chern characters by GRR Chiodo
- Expansion in the tautological ring of $\overline{\mathcal{M}}_{g,n}$
- Ehrhart theory as before

Now we can take $r^0 s^0$ coefficient:

Type 0 contributes $\hat{C}_g \big|_{r=0}$

- Rest Complicated, but

  - polynomial in r with no $r=0$ terms!

Miracle upon which the whole proof rests

STEP $\underline{\mathrm{III}}$ We have

$$DR_{g,A} = \hat{C}_g \Big|_{r=0} \in H^{2g}(\bar{M}_{g,n})$$

The only remaining task is to explicitly calculate $\hat{C}_g \Big|_{r=0}$

(11) Final calculation of $\hat{C}_g \big|_{r=0}$

Pure orbifold geometry

$$\varepsilon : \overline{\mathcal{M}}^{\frac{1}{r}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

Strata here indexed by (Γ, w)

mod r weighting

as in Pixtons formula

$$\begin{tikzcd} & \mathcal{L} \arrow[dl] \\ \mathcal{C} \arrow[d, "\pi"'] & \\ \overline{\mathcal{M}}^{\frac{1}{r}}_{g,n} & \end{tikzcd}$$

universal r^{th} root

Chiodo's GRR calculation

Bernoulli polynomials

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{t e^{xt}}{e^t - 1}$$

$$c(-R\pi_* \mathcal{L}) =$$

$$\sum_{\Gamma \in G_{g,n}} \sum_{w \in W_{\Gamma, r}} \frac{r^{|E(\Gamma)|}}{|\mathrm{Aut}\, \Gamma|} \cdot$$

$$(\zeta_{\Gamma, r})_* \Bigg[\prod_{i=1}^{n} \exp\left(\sum_{m \geq 1} (-1)^{m-1} \frac{B_{m+1}(a_i/r)}{m(m+1)} \psi_i^m \right) \cdot$$

$$\prod_{\substack{e=(h,h') \\ \text{in } E(\Gamma)}} \frac{1 - \exp\left(\sum_{m \geq 1} (-1)^{m-1} \frac{B_{m+1}(w(h)/r)}{m(m+1)} [\psi_h^m - (-\psi_{h'})^m] \right)}{\psi_h + \psi_{h'}} \Bigg]$$

It is then straightforward to prove polynomiality of $\hat{C}_g$ and derive $\hat{C}_g \big|_{r=0}$ = Pixton's Formula $\square$

(12) Pixton's formula on the Picard stack

What role is stability playing?

Answer: none (except in the proof)

Let $\mathfrak{M}_{g,n}$ be the Artin stack of connected, nodal, pointed curves

markings are not nodes and are distinct

Let $\mathfrak{P}_{g,n}$ be the Artin stack of curves together with a line bundle

Picard Stack

$$\begin{array}{c} \mathfrak{P}_{g,n} \\ \downarrow \\ \mathfrak{M}_{g,n} \end{array}$$

See BHPSS for an introduction

Let
$$\begin{array}{c} \mathcal{L} \\ \swarrow \\ C \\ \downarrow \;\; \uparrow p_1 \cdots p_n \\ B \end{array}$$
pure dim = b

be a flat family of genus g curves with n markings and a line bundle $\mathcal{L}$

Let $A = (a_1, \ldots, a_n)$, $\sum a_i = \deg(\mathcal{L})$ fiber degree

Abel-Jacobi Cycle in $CH_{b-g}(B)$

defined by where $\mathcal{L}|_b \cong \mathcal{O}_C(\Sigma a_i p_i)|_b$

uses some theory to define,

termed $DR_{g,A}[B]$, see BHPSS

Theorem [BHPSS 2020]

Universal Pixton formula

$$DR_{g,A} = P_{g,A} \quad \text{in} \quad CH^g_{op}(\mathfrak{P}_{g,n})$$

Universal DR cycle

cycle theory of the Picard stack

$$f_B : B \to \mathfrak{P}_{g,n}$$

Then $$DR_{g,A}[B] = f_B^* P_{g,A} [B]$$

Most general statement that we know.

$A = (a_1, \dots, a_n)$

$\sum a_i = d$

$P_{g,A} \in CH^g_{op}(\mathcal{M}_{g,n})$ is the

r constant, degree g component of

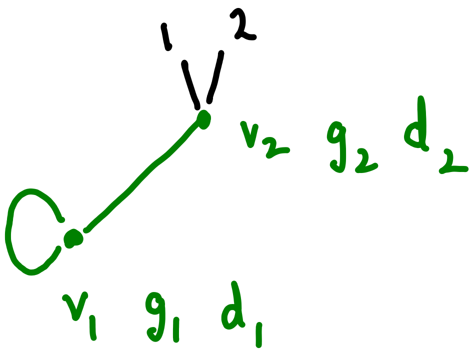
$$\sum_{\Gamma \in G^d_{g,n}} \sum_{w \in W^d_{\Gamma,r}} \frac{1}{|\operatorname{Aut}\Gamma|} \frac{1}{r^{h^1(\Gamma)}} \cdot$$

$$\xi_{\Gamma *}\left[\prod_{i=1}^{n} \exp\left(\frac{a_i^2}{2} \psi_i + a_i \xi_i \right) \cdot \prod_{v} \exp\left(-\frac{1}{2}\eta(v)\right) \right.$$

$$\left. \cdot \prod_{e=(h,h')} \frac{1 - \exp\left(-\frac{w(h)w(h')}{2} \cdot (\psi_h + \psi_{h'})\right)}{\psi_h + \psi_{h'}} \right]$$

What is different?

$\Gamma \in G^d_{g,n}$ Graphs with no stability and a degree assignment on vertices (sum to degree d)



∞ sum!

$w \in W^d_{\Gamma,r}$ Same as before except

$$\sum_{h \vdash v} w(h) = d(v) \mod r$$

Finally, what are the classes ξ_i, η?

These are defined using the universal line bundle:

$$\begin{tikzcd} \mathcal{L} \arrow[d] \\ \mathcal{C} \arrow[d, "\pi"] \\ \mathcal{M}_{g,n} \arrow[u, bend left, "p_1 \cdots p_n"] \end{tikzcd}$$

$$\xi_i = p_i^* c_1(\mathcal{L}), \quad \eta = \pi_* c_1(\mathcal{L})^2$$

Theta

Map of papers to read:

Farkas-P 1508.07940

JPPZ DR 1602.0470

Holmes-Schmitt 1909.11981

Schmitt 1607.08429

JPPZ DR(x) 1812.10136

BHPSS Pic 2004.08676

holomorphic k-differentials

DR cycle development

The End