Moduli spaces of Gushel–Mukai varieties

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Gushel–Mukai varieties

Definition

A Gushel–Mukai (GM) variety is a smooth Fano variety X such that

- $\mathrm{Pic}(X) = \mathbb{Z}H$ (H is the ample generator);
- $\mathrm{coindex}(X) = 3$, i.e., $K_X = -(n-2)H$, where $n = \dim(X)$;
- $\deg(X) := H^n = 10$.

- The above definition implies $n \geq 3$.
- One can include the case $n = 2$ (GM surfaces), i.e.
  - polarized K3 surfaces of degree 10
  - can replace $\mathrm{Pic}(X) = \mathbb{Z}H$ by *Brill–Noether generality*: (i.e., $h^0(D)h^0(H-D) < h^0(H)$ for $0 < D < H$).
- One can include mildly singular varieties.

Classification of GM varieties

Theorem (Gushel, Mukai, D-K)

If X is a GM variety then

$$X = \mathrm{CGr}(2, V_5) \cap \mathbb{P}(W_{n+2}) \cap Q,$$

where

- $\mathrm{CGr}(2, V_5) \subset \mathbb{P}(\Bbbk \oplus \wedge^2 V_5)$, *a cone over the Grassmannian;*
- $W_{n+5} \subset \Bbbk \oplus \wedge^2 V_5$, *a vector subspace;*
- $Q \subset \mathbb{P}(W_{n+5})$, *a quadric hypersurface.*

- In particular, $\dim(X) \leq 6$.
- Q is defined ambiguously (modulo *Plücker quadrics*).

Ordinary vs. special GM varieties

1. The vertex of the cone is not in $\mathbb{P}(W)$; then

$$X = \mathrm{Gr}(2, V_5) \cap \mathbb{P}(W_{n+5}) \cap Q.$$

   Such GM varieties are called ordinary.

2. The vertex of the cone is in $\mathbb{P}(W)$ (but not on X); then

$$X \xrightarrow{2:1} \mathrm{Gr}(2, V_5) \cap \mathbb{P}(W'_{n+4}),$$

   branched along $\mathrm{Gr}(2, V_5) \cap \mathbb{P}(W'_{n+4}) \cap Q$ (ordinary GM $(n-1)$-fold).
   Such GM varieties are called special.

- $\left\{ \text{special GM } n\text{-folds} \right\} \xleftrightarrow{1:1} \left\{ \text{ordinary GM } (n-1)\text{-folds} \right\}$
- If $\dim(X) = 6$ then X is special.

GM data

Definition

GM-data is a collection (W, V_6, V_5, μ, q), where

- $W = W_{n+5}$, $V_5 \subset V_6$;
- $\mu \colon W \to \wedge^2 V_5$ $\qquad (- \otimes (V_6/V_5))$;
- $q \colon V_6 \to \operatorname{Sym}^2 W^\vee$ $\qquad (- \otimes \det(V_5) \otimes (V_6/V_5)^{\otimes 2})$;

such that

$$\begin{array}{ccc} V_5 \otimes \operatorname{Sym}^2 W & \hookrightarrow & V_6 \otimes \operatorname{Sym}^2 W \\ \downarrow{\scriptstyle \mu\wedge\mu} & & \downarrow{\scriptstyle q} \\ V_5 \otimes \operatorname{Sym}^2(\wedge^2 V_5) & \xrightarrow{\wedge} & \det(V_5) \end{array} \qquad - \otimes (V_6/V_5)^{\otimes 2}$$

In other words $q(v)(w_1, w_2) = v \wedge \mu(w_1) \wedge \mu(w_2)$ if $v \in V_5$.

GM varieties vs. GM data

From GM varieties to GM data:

- $W = H^0(X, \mathcal{O}_X(H))^\vee$;
- $V_6 = H^0(\mathbb{P}(W), \mathcal{I}_X(2))$;
- $V_5 = H^0(\mathbb{P}(\wedge^2 V_5), \mathcal{I}_{\mathrm{Gr}(2,V_5)}(2)) \subset V_6$;
- $\mu \colon H^0(X, \mathcal{O}_X(H))^\vee \to H^0(\mathrm{Gr}(2,5), \mathcal{O}(1))^\vee$;
- $q \colon H^0(\mathbb{P}(W), \mathcal{I}_X(2)) \to H^0(\mathbb{P}(W), \mathcal{O}(2))$.

From GM data to GM varieties:

$$X(W, V_6, V_5, \mu, q) := \bigcap_{v \in V_6} Q_v \subset \mathbb{P}(W).$$

GM data is smooth if $X(W, V_6, V_5, \mu, q)$ is smooth of dimension n.

Theorem

$$\left\{ \textit{smooth GM varieties} \right\} \xleftrightarrow{1:1} \left\{ \textit{smooth GM data} \right\}.$$

Ordinary vs. special GM data

GM data is ordinary if $\mu \colon W \to \wedge^2 V_5$ is injective; otherwise it is special.
Let (W, V_6, V_5, μ, q) be a special smooth GM data:

- $W_1 := \mathrm{Ker}(\mu)$ (then $\dim(W_1) = 1$ and $\mathbb{P}(W_1) = \bigcap_{v \in V_5} \mathrm{Sing}(Q_v)$);
- $W_0 = W_1^{\perp_{Q_{v_0}}}$, $v_0 \in V_6 \setminus V_5$;
- $W = W_0 \oplus W_1$, $\mu = (\mu_0, 0)$, $q = (q_0 \oplus q_1)$;
- $(W_0, V_6, V_5, \mu_0, q_0)$ is an ordinary smooth GM data.
- $X(W_0, V_6, V_5, \mu_0, q_0) = X(W, V_6, V_5, \mu, q)_{\mathrm{ord}}$.

Let $(W_0, V_6, V_5, \mu_0, q_0)$ be an ordinary smooth GM data:

- choose an isomorphism $q_1 \cong V_6/V_5 \xrightarrow{\sim} \mathrm{Sym}^2 W_1^\vee$;
- set $W = W_0 \oplus W_1$, $\mu = (\mu_0, 0)$, $q = (q_0, q_1)$;
- (W, V_6, V_5, μ, q) is a special smooth GM data.
- $X(W, V_6, V_5, \mu, q) = X(W_0, V_6, V_5, \mu_0, q_0)_{\mathrm{spe}}$.

This gives a bijection of isomorphism classes if $\sqrt{\Bbbk} = \Bbbk$.

Lagrangian data

Definition

Lagrangian data is a triple (V_6, V_5, A), where

- $V_5 \subset V_6$;
- $A \subset \wedge^3 V_6$ is *Lagrangian* w.r.to $\wedge^3 V_6 \otimes \wedge^3 V_6 \xrightarrow{\wedge} \det(V_6)$.

In particular, $\dim(A) = 10$.

Theorem

$$\left\{ \textit{ordinary GM data} \right\} \xleftrightarrow{1:1} \left\{ \textit{Lagrangian data} \right\}.$$

From Lagrangian data to GM data

Let $\lambda\colon V_6 \to V_6/V_5$ be the natural map.

- Leibniz rule: $\lambda_p\colon \wedge^p V_6 \to \wedge^{p-1} V_5 \otimes (V_6/V_5)$;
- $W := \mathrm{Im}\left(A \hookrightarrow \wedge^3 V_6 \xrightarrow{\lambda_3} \wedge^2 V_5 \otimes (V_6/V_5)\right)$, $\mu\colon W \hookrightarrow \wedge^2 V_5$ (note that $\mathrm{Ker}(A \twoheadrightarrow W) = \wedge^3 V_5 \cap A$);
- $\tilde{q}\colon V_6 \to \mathrm{Sym}^2 A^\vee$, $\tilde{q}(v)(a_1, a_2) = -\lambda_4(v \wedge a_1) \wedge \lambda_3(a_2)$:
  - Symmetry: $\tilde{q}(v)(a_1, a_2) = -\lambda(v) a_1 \wedge \lambda_3(a_2) + v \wedge \lambda_3(a_1) \wedge \lambda_3(a_2)$,
    $0 = \lambda_6(a_1 \wedge a_2) = \lambda_3(a_1) \wedge a_2 - a_1 \wedge \lambda_3(a_2)$;
  - Kernel: if $a_2 \in \wedge^3 V_5 \cap A$ then $\tilde{q}(v)(a_1, a_2) = 0$ for any $v \in V_6$, $a_1 \in A$;
  - Plücker: if $v \in V_5$ then $\tilde{q}(v)(a_1, a_2) = v \wedge \lambda_3(a_1) \wedge \lambda_3(a_2)$.

Therefore, $\tilde{q}$ descends to $q\colon V_6 \to \mathrm{Sym}^2 W^\vee$.

Lemma

(W, V_6, V_5, μ, q) *is ordinary GM data.*

From GM data to Lagrangian data

Consider the diagram

$$\begin{array}{ccccc} V_5 \otimes W & \xrightarrow{f_1} & \wedge^3 V_5 \oplus V_6 \otimes W & \xrightarrow{f_2} & W^\vee \\ & & \downarrow{\scriptstyle f_3} & & \\ & & \wedge^3 V_6 & & \end{array}$$

- $f_1(v \otimes w) = (-v \wedge \mu(w), v \otimes w)$;
- $f_2(\xi, v \otimes w)(w') = \xi \wedge \mu(w') + q(v)(w, w')$;
- $f_3(\xi, v \otimes w) = \xi + v \wedge \mu(w)$.
- $f_2 \circ f_1 = 0$, $f_3 \circ f_1 = 0$, f_1 mono, f_2 epi, $\mathrm{Ker}(f_3) = \mathrm{Im}(f_1)$.
- $A := f_3(\mathrm{Ker}(f_2)/\,\mathrm{Im}(f_1)) \subset \wedge^3 V_6$.

Lemma

(V_6, V_5, A) *is Lagrangian data.*

Properties of GM variety from Lagrangian data

From Lagrangian data (V_6, V_5, A) we obtain

- ordinary GM variety $X_{\mathrm{ord}}(V_6, V_5, A)$,
- special GM variety $X_{\mathrm{spe}}(V_6, V_5, A)$.

Theorem

$X_{\mathrm{ord}}(V_6, V_5, A)$ (*resp.* $X_{\mathrm{spe}}(V_6, V_5, A)$) *is smooth of dimension* $n \iff$

1. $\dim(\wedge^3 V_5 \cap A) = 5 - n$ (*resp.* $\dim(\wedge^3 V_5 \cap A) = 6 - n$);
2. $\mathbb{P}(A) \cap \mathrm{Gr}(3, V_6) = \varnothing$ (A *has no decomposable vectors*).

Corollary

If $\sqrt{\Bbbk} = \Bbbk$ *isomorphism classes of GM* n*-folds are in bijection with*

$$\left\{(V_6, V_5, A) \;\middle|\; \dim(\wedge^3 V_5 \cap A) \in \{5-n, 6-n\},\ \mathbb{P}(A) \cap \mathrm{Gr}(3, V_6) = \varnothing\right\}.$$

Moduli stack of GM varieties

Let $\mathfrak{M}_n^{\mathrm{GM}}$ be the moduli stack of smooth GM n-folds (smooth GM data):

- $\mathfrak{M}_n^{\mathrm{GM}}(S) = \{\mathscr{X} \to S\}$, $\mathscr{X}_s$ is a smooth GM n-fold for any $s \in S$.
- $\mathfrak{M}_n^{\mathrm{GM}}(S) = \{(\mathcal{V}_6, \mathcal{V}_5, \mathcal{W}, \mu, q) \mid \dots\}$.

Lemma

In the diagram μ is surjective, hence $(\wedge^{n+5} q) = (\wedge^{n+5} \mu^\vee)$.

$$\mathcal{V}_5 \otimes \mathcal{W} \xrightarrow{\ \mu\ } \wedge^2 \mathcal{V}_5^\vee, \qquad \mathcal{V}_5 \otimes \mathcal{W} \xrightarrow{\ q\ } \mathcal{W}^\vee \xleftarrow{\ \mu^\vee\ } \wedge^2 \mathcal{V}_5^\vee$$

Definition

- $S_{\mathrm{spe}} = (\wedge^{n+5} q) = (\wedge^{n+5} \mu^\vee)$;
- $\mathfrak{M}_{n,\mathrm{ord}}^{\mathrm{GM}}(S) := \{\mathscr{X}/S \mid S_{\mathrm{spe}} = \varnothing\}$,
- $\mathfrak{M}_{n,\mathrm{spe}}^{\mathrm{GM}}(S) := \{\mathscr{X}/S \mid S_{\mathrm{spe}} = S\}$.

Moduli stack of Lagrangian data

Let $\mathfrak{M}_n^{\mathrm{Lag}}$ be the moduli stack of Lagrangian data:

- $\mathfrak{M}_n^{\mathrm{Lag}}(S) = \left\{ (\mathcal{V}_6, \mathcal{V}_5, \mathcal{A}) \;\middle|\; \begin{array}{l} \mathbb{P}(\mathcal{A}_s) \cap \mathrm{Gr}(3, \mathcal{V}_{6,s}) = \varnothing, \\ \dim(\wedge^3 \mathcal{V}_{5,s} \cap \mathcal{A}_s) \in \{5-n, 6-n\} \end{array} \right\}$.
- For $(\mathcal{V}_6, \mathcal{V}_5, \mathcal{A}) \in \mathfrak{M}_n^{\mathrm{Lag}}(S)$ set

$$\varphi\colon \mathcal{A} \hookrightarrow \wedge^3 \mathcal{V}_6 \xrightarrow{\lambda_3} \wedge^2 \mathcal{V}_5 \otimes (\mathcal{V}_6/\mathcal{V}_5).$$

Then $(\wedge^{n+6}\varphi) = 0$, $(\wedge^{n+4}\varphi) = 1$.

Definition

- $S_{\mathrm{spe}} = (\wedge^{n+5}\varphi)$;
- $\mathfrak{M}_{n,\mathrm{ord}}^{\mathrm{Lag}}(S) := \{(\mathcal{V}_6, \mathcal{V}_5, \mathcal{A}) \mid S_{\mathrm{spe}} = \varnothing\}$;
- $\mathfrak{M}_{n,\mathrm{spe}}^{\mathrm{Lag}}(S) := \{(\mathcal{V}_6, \mathcal{V}_5, \mathcal{A}) \mid S_{\mathrm{spe}} = S\}$.

Morphism of stacks

As before

$$\begin{array}{ccccc} \mathcal{V}_5 \otimes \mathcal{W} & \xrightarrow{f_1} & \wedge^3\mathcal{V}_5 \oplus \mathcal{V}_6 \otimes \mathcal{W} & \xrightarrow{f_2} & \mathcal{W}^\vee \\ & & \downarrow{\scriptstyle f_3} & & \\ & & \wedge^3\mathcal{V}_6 & & \end{array}$$

Set $\mathcal{A} := f_3(\operatorname{Ker}(f_2)/\operatorname{Im}(f_1)) \subset \wedge^3\mathcal{V}_6$.

Definition

Define a morphism of stacks $\mathfrak{a}\colon \mathfrak{M}_n^{\mathrm{GM}} \to \mathfrak{M}_n^{\mathrm{Lag}}$ by

$$\mathfrak{a}(\mathcal{V}_6, \mathcal{V}_5, \mathcal{W}, \mu, q) = (\mathcal{V}_6, \mathcal{V}_5, \mathcal{A}).$$

Morphism of stacks: special loci

Lemma

Let $(\mathcal{V}_6, \mathcal{V}_5, \mathcal{A}) = \mathfrak{a}(\mathcal{V}_6, \mathcal{V}_5, \mathcal{W}, \mu, q)$.
The ideals of $S_{\mathrm{Lag,spe}} \subset S$ *and* $S_{\mathrm{Lag,spe}} \subset S$ *satisfy* $\mathcal{J}_{\mathrm{Lag,spe}} = \mathcal{J}^2_{\mathrm{GM,spe}}$.

Proof.

$$\begin{array}{ccccc}
\mathrm{Ker}(f_2) & \hookrightarrow & \wedge^3\mathcal{V}_5 \oplus \mathcal{V}_6 \otimes \mathcal{W} & \xrightarrow{f_3} & \wedge^3\mathcal{V}_6 \\
\downarrow & & \downarrow{\scriptstyle (0,\lambda)} & & \downarrow{\scriptstyle \lambda_3} \\
\mathcal{A} & \overset{\nu}{\dashrightarrow} & \mathcal{W} & \xrightarrow{\mu} & \wedge^2\mathcal{V}_5
\end{array}
\qquad \mathcal{A} \xrightarrow{\varphi} \wedge^2\mathcal{V}_5$$

$\mathcal{J}_{\mathrm{Lag,spe}} = (\wedge^{n+5}\varphi) = (\wedge^{n+5}\nu) \cdot (\wedge^{n+5}\mu) = \mathcal{J}^2_{\mathrm{GM,spe}}$. □

Upshot: $\mathfrak{a}$ is not an isomorphism of stacks!

Inverse construction: Step 1

Theorem

Let $(\mathcal{V}_6, \mathcal{V}_5, \mathcal{A}) \in \mathfrak{M}_n^{\mathrm{Lag}}(S)$*. Assume* $S_{\mathrm{Lag,spe}} = 2E$ *for a Cartier divisor* E*. Then there is a unique* $(\mathcal{V}_6, \mathcal{V}_5, \mathcal{W}, \mu, q) \in \mathfrak{M}_n^{\mathrm{GM}}(S)$ *such that*

$$(\mathcal{V}_6, \mathcal{V}_5, \mathcal{A}) = \mathfrak{a}(\mathcal{V}_6, \mathcal{V}_5, \mathcal{W}, \mu, q).$$

- Away from E the pointwise construction works nicely.
- Need to extend it to a neighbrhood of E.
  - Have a morphism $\varphi\colon \mathcal{A} \to \wedge^2\mathcal{V}_5$ of rank $n+5$.
  - Need to find its factorization

$$\mathcal{A} \xrightarrow{\nu} \mathcal{W} \xrightarrow{\mu} \wedge^2\mathcal{V}_5.$$

through a vector bundle $\mathcal{W}$ of rank $n+5$.

Factorization lemma

Lemma

Let $\varphi\colon \mathcal{E} \to \mathcal{F}$ be a morphism of vector bundles on S such that

- $(\wedge^{r_0+1}\varphi) = 0$ (*i.e.*, $\mathrm{rk}(\varphi) \le r_0$);
- $(\wedge^{r_0-1}\varphi) = 1$ (*i.e.*, $\mathrm{rk}(\varphi) \ge r_0 - 1$);
- $(\wedge^{r_0}\varphi) = D$, *a Cartier divisor.*

Then there is a unique factorization of φ

$$\mathcal{E} \twoheadrightarrow \mathcal{E}_1 \xrightarrow{\varphi_1} \mathcal{F}_1 \hookrightarrow \mathcal{F},$$

where

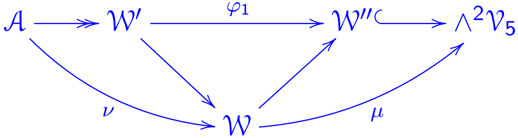
- $\mathcal{E}_1$ *and* $\mathcal{F}_1$ *are vector bundles of rank* r_0;
- $\mathcal{E} \twoheadrightarrow \mathcal{E}_1$ *epi*, $\mathcal{F}_1 \hookrightarrow \mathcal{F}$ *fiberwise mono;*
- φ_1 *is mono*, $\mathrm{Coker}(\varphi_1)$ *is a line bundle on* D.

Inverse construction: Step 1 continued

- Apply factorization lemma to $\varphi \colon \mathcal{A} \to \wedge^2 \mathcal{V}_5$.
- Obtain the factorization

$$\mathcal{A} \twoheadrightarrow \mathcal{W}' \xrightarrow{\varphi_1} \mathcal{W}'' \hookrightarrow \wedge^2 \mathcal{V}_5.$$

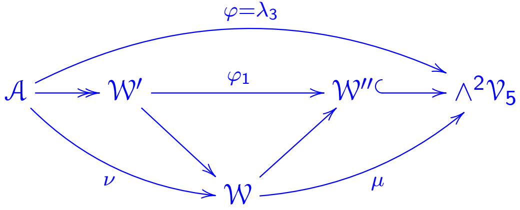
- $\mathrm{Coker}(\varphi_1)$ is a line bundle on $D = 2E$.
- Represent it (uniquely) as an extension of line bundles on E.
- Obtain the diagram



- $\mathrm{Coker}(\mathcal{W}' \to \mathcal{W})$ and $\mathrm{Coker}(\mathcal{W} \to \mathcal{W}'')$ are line bundles on E.

Inverse construction: Step 2

Recall



- As before, $\tilde{q} \colon \mathcal{V}_6 \to \operatorname{Sym}^2 \mathcal{A}^\vee$, $\tilde{q}(v)(a_1, a_2) = -\lambda_4(v \wedge a_1) \wedge \lambda_3(a_2)$.
- $\operatorname{Ker}(\mathcal{A} \to \mathcal{W}') \subset \operatorname{Ker}(\mathcal{A} \to \wedge^2 \mathcal{V}_5) \subset \operatorname{Ker}(\tilde{q})$, hence q factors as
- $\mathcal{V}_6 \xrightarrow{q'} \operatorname{Sym}^2 (\mathcal{W}')^\vee \hookrightarrow \operatorname{Sym}^2 \mathcal{A}^\vee$.
- Need $\mathcal{V}_6 \xrightarrow{q} \operatorname{Sym}^2 \mathcal{W}^\vee \hookrightarrow \operatorname{Sym}^2 (\mathcal{W}')^\vee \hookrightarrow \operatorname{Sym}^2 \mathcal{A}^\vee$.

Hecke transform for families of quadratic forms

Lemma

- $q' \colon \mathcal{V} \to \mathrm{Sym}^2(\mathcal{E}')^\vee$, *a family of quadratic forms,*
- $D = 2E$, *a Cartier divisor,*
- $\mathcal{K} \subset \mathcal{E}'_D$, *a line subbundle contained in* $\operatorname{Ker} q'_D$.

Define the bundle $\mathcal{E}$ *from the exact sequences*

$$0 \to \mathcal{E}^\vee \to (\mathcal{E}')^\vee \to \mathcal{K}_E^\vee \to 0,$$
$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{K}_E(E) \to 0.$$

Then

1. $\exists!$ *factorization* $\mathcal{V} \xrightarrow{q} \mathrm{Sym}^2 \mathcal{E}^\vee \hookrightarrow \mathrm{Sym}^2(\mathcal{E}')^\vee$ *of* q';
2. $\mathcal{E}_E = (\mathcal{E}'_E/\mathcal{K}_E) \oplus \mathcal{K}_E(E)$, *orthogonal with respect to* q;
3. *the restriction of* q *to the first summand is induced by* q'.

Inverse construction: Step 2 continued

- We have $q' \colon \mathcal{V}_6 \to \mathrm{Sym}^2(\mathcal{W}')^\vee$, set $\mathcal{K} = \mathrm{Ker}(\mathcal{W}'_D \xrightarrow{\varphi_1} \mathcal{W}''_D)$.
- Hecke transform gives $q \colon \mathcal{V}_6 \to \mathrm{Sym}^2 \mathcal{W}^\vee$,
- $\mathcal{W}_E = (\mathcal{W}'_E/\mathcal{K}_E) \oplus \mathcal{K}_E(E)$, orthogonal with respect to q, the restriction of q to the first summand is induced by q'.
- $(\mathcal{V}_6, \mathcal{V}_5, \mathcal{W}, \mu, q)$ is a family of GM data, hence gives $\mathscr{X} \to S$, a family of GM intersections; need to prove smoothness.
- Away from E we have $\mathscr{X}_s = X_{\mathrm{ord}}(\mathcal{V}_{6,s}, \mathcal{V}_{5,s}, \mathcal{A}_s)$, hence is smooth.
- On E if $\mathcal{W}_0 := \mathcal{W}'_E/\mathcal{K}_E$ then $(\mathcal{V}_6, \mathcal{V}_5, \mathcal{W}_0, \mu, q)$ is a family of smooth (ordinary) GM data of dimension $n-1$.
- The form $q(v)$ on $\mathcal{W}_1 = \mathcal{K}_E(E)$ is non-degenerate for $v \in \mathcal{V}_{6,s} \setminus \mathcal{V}_{5,s}$, hence $\mathscr{X}_s = X_{\mathrm{spe}}(\mathcal{V}_{6,s}, \mathcal{V}_{5,s}, \mathcal{A}_s)$.

□

Moduli stack $\mathfrak{M}_n^{\mathrm{GM}}$ as a root stack

Upshot:

- There is a morphism of stacks $\mathfrak{a}\colon \mathfrak{M}_n^{\mathrm{GM}} \to \mathfrak{M}_n^{\mathrm{Lag}}$,
- $\mathfrak{a}\colon \mathfrak{M}_n^{\mathrm{GM}}(S,E) \to \mathfrak{M}_n^{\mathrm{Lag}}(S,2E)$ is an isomorphism, where
  - $\mathfrak{M}_n^{\mathrm{GM}}(S,E) = \{\mathscr{X}/S \mid S_{\mathrm{GM,spe}} = E\}$,
  - $\mathfrak{M}_n^{\mathrm{Lag}}(S,D) = \{(\mathcal{V}_6,\mathcal{V}_5,\mathcal{A}) \mid S_{\mathrm{Lag,spe}} = D\}$.

Theorem

The morphism $\mathfrak{a}\colon \mathfrak{M}_n^{\mathrm{GM}} \to \mathfrak{M}_n^{\mathrm{Lag}}$ is a root stack w.r.t. $\mathfrak{M}_{n,\mathrm{spe}}^{\mathrm{Lag}} \subset \mathfrak{M}_n^{\mathrm{Lag}}$.

This is an exotic root stack!

Exotic root stacks

- Assume $Z \subset Y$, $Y \setminus Z$ is smooth, and étale locally along Z

$$Y \cong Z \times (\mathbb{A}^m/\{\pm 1\}).$$

- Locally consider the quotient stack

$$\hat{Y} = Z \times [\mathbb{A}^m/\{\pm 1\}]$$

and glue globally.

Remark

When $m = 1$ this is the usual root stack construction.

Definition

$\hat{Y} \to Y$ is the root stack w.r.t $Z \subset Y$.

Theorem

The morphism $\mathfrak{a} \colon \mathfrak{M}_n^{\mathrm{GM}} \to \mathfrak{M}_n^{\mathrm{Lag}}$ is a root stack w.r.t. $\mathfrak{M}_{n,\mathrm{spe}}^{\mathrm{Lag}} \subset \mathfrak{M}_n^{\mathrm{Lag}}$.

Global quotient stack constructions

Fix V_6, set

$$\mathrm{S}_n = \{(V_5, A) \mid \dim(\wedge^2 V_5 \cap A) = 5 - n,\ \mathbb{P}(A) \cap \mathrm{Gr}(3, V_6) = \varnothing\},$$
$$\bar{\mathrm{S}}_n = \{(V_5, A) \mid \dim(\wedge^2 V_5 \cap A) \in \{5 - n, 6 - n\},\ \mathbb{P}(A) \cap \mathrm{Gr}(3, V_6) = \varnothing\}.$$

Then $\bar{\mathrm{S}}_n = \mathrm{S}_n \sqcup \mathrm{S}_{n-1}$, $\mathrm{codim}_{\bar{\mathrm{S}}_n}(\mathrm{S}_{n-1}) = 6 - n$, locally around S_{n-1}

$$\bar{\mathrm{S}}_n \cong \mathrm{S}_{n-1} \times (\mathbb{A}^{6-n}/\{\pm 1\}).$$

Let $\hat{\mathrm{S}}_n \to \bar{\mathrm{S}}_n$ be the root stack with respect to $\mathrm{S}_{n-1} \subset \bar{\mathrm{S}}_n$.

Theorem

$\mathfrak{M}_n^{\mathrm{Lag}} = [\bar{\mathrm{S}}_n/\mathrm{PGL}(V_6)]$, $\mathfrak{M}_n^{\mathrm{GM}} = [\hat{\mathrm{S}}_n/\mathrm{PGL}(V_6)]$.
These stacks are separated DM stacks of finite presentation over $\mathbb{Q}$.

Remark

$\mathfrak{M}_n^{\mathrm{GM}}$ can be realized as a quotient stack of a scheme by $\mathrm{GL}(V_6)/\mu_{3(5-n)}$.

Coarse moduli spaces

Recall

$$\bar{\mathrm{S}}_n = \{(V_5, A) \mid \dim(\wedge^2 V_5 \cap A) \in \{5-n, 6-n\},\ \mathbb{P}(A) \cap \mathrm{Gr}(3, V_6) = \varnothing\}.$$

Note that $\bar{\mathrm{S}}_n \subset \mathbb{P}(V_6^\vee) \times \mathrm{LGr}(\wedge^3 V_6)$.

Theorem

The coarse moduli spaces $\mathbf{M}_n^{\mathrm{GM}}$ and $\mathbf{M}_n^{\mathrm{Lag}}$ are isomorphic to

$$\mathbf{M}_n^{\mathrm{GM}} \cong \mathbf{M}_n^{\mathrm{Lag}} \cong \bar{\mathrm{S}}_n /\!\!/ \mathrm{PGL}(V_6),$$

the GIT quotients w.r.t. linearizations on $\mathcal{O}(6, 2m)$ for $m \gg 0$.

Applications: period maps

Recall $\bar{\mathrm{S}}_n \subset \mathbb{P}(V_6^\vee) \times \mathsf{LGr}(\wedge^3 V_6)$.

Theorem

The morphism

$$\mathbf{M}_n^{\mathrm{GM}} = \bar{\mathrm{S}}_n /\!\!/ \mathsf{PGL}(V_6) \to \mathsf{LGr}(\wedge^3 V_6) /\!\!/ \mathsf{PGL}(V_6)$$

is the period map for GM n-folds for $n \in \{4, 6\}$.

Remark

For $n \in \{3, 5\}$ the period factors through $\mathbf{M}_n^{\mathrm{GM}} \to \mathsf{LGr}(\wedge^3 V_6) /\!\!/ \mathsf{PGL}(V_6)$.

Applications: complete families of smooth GM varieties

The "inverse construction" can be applied to produce

- smooth families $\mathscr{X} \to \mathbb{P}^1$ of GM 5-folds;
- smooth families $\mathscr{X} \to Y$ of smooth GM 4-folds over double EPW sextics Y, hence also families over $\mathbb{P}^1$;
- smooth families $\mathscr{X} \to Y_2$ of smooth GM 3-folds over EPW surfaces Y_2 (of general type).

Remark

Any complete family of GM varieties is contained in the fiber of the period map $\mathfrak{M}_n^{\mathrm{GM}} \to \mathsf{LGr}(\wedge^3 V_6)/\!\!/\mathsf{PGL}(V_6)$, because $\mathsf{LGr}(\wedge^3 V_6)/\!\!/\mathsf{PGL}(V_6)$ is affine.

Thanks for attention!