# Introduction to the equivariant vertex.

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## 1 Introduction.

A classical problem in enumerative geometry is:

*How many lines lie on a generic cubic surface?*

Let $S = V(F) \subset \mathbb{P}^3$ be a cubic surface given by the vanishing of a cubic polynomial $F$. The Grassmannian $\mathbb{G}(1,3)$ parameterizes the lines $L \subset \mathbb{P}^3$. The polynomial $F$ restricted to the line $L$ gives a cubic form $F|_L \in H^0(\mathcal{O}_L(3))$. In this way the restriction of $F$ to each line $L$ gives a section $s_F$ of the bundle $H^0(\mathcal{O}_L(3)) \to \mathbb{G}(1,3)$. The zeros of this section are lines $L$ with $F|_L = 0$, i.e. those that lie on the cubic surface.

Since the section $s_F$ takes values in a bundle of rank $h^0(\mathcal{O}_L(3)) = 4$ over a manifold of dimension $\dim(\mathbb{G}(1,3)) = 4$ we expect that, for generic $F$, the zero locus of $s_F$ will consist of a finite number of points.

The number of lines can now be interpreted as the degree of the cycle corresponding to the zero locus of $s_F$. This cycle is know to be rationally equivalent to the top Chern class of the bundle so then

$$\#\{L : L \subset S\} = \int_{\mathbb{G}(1,3)} c_4(H^0(\mathcal{O}_L(3))) \in \mathbb{Z}.$$

One way to do this integral is via the method of localization. We have an action of $(\mathbb{C}^*)^4$ on the Grassmannian $\mathbb{G}(1,3)$. By lifting this integral to equivariant cohomology we can reduce the integral to a sum over simple contributions from each of the six fixed points of $\mathbb{G}(1,3)$.

Equivariant cohomology can be defined as follows [1]. Let $X$ be a smooth manifold with the action of a Lie group $G$. Let $EG \to BG$ be the classifying space of $G$-torsors. The space $X_G = X \times_G EG$ is defined to be the quotient of $X \times EG$ by $G$ given by $(x, v) \sim (x \cdot g^{-1}, g \cdot v)$. The equivariant cohomology is then given by $H^*_G(X) := H^*(X_G)$. Likewise given a $G$-equivariant vector bundle $V \to X$ on $X$ we can define the equivariant Chern classes by $c_i^G(V) := c_i(V_G) \in H^*_G(X)$ where $V_G = V \times_G EG$. Just as in ordinary cohomology other models exist realizing this theory [2],[6].

For example, consider the case when $X = *$ is a point and $G = T_r := (\mathbb{C}^*)^r$. The classifying space of $T_r$ is given by $(\mathbb{C}^\infty - \{0\})^r \to (\mathbb{CP}^\infty)^r$. So $*_G = (\mathbb{CP}^\infty)^r$ and $H^*_{T_r}(*) = \mathbb{Z}[s_1, \ldots, s_r]$. A $G$-equivariant line bundle $L \to *$ is a one dimensional $T_r$ representation $L$ of the form $t \cdot v = t_1^{a_1} \cdots t_r^{a_r} \cdot v$. Hence the line bundle $L_G$ on $(\mathbb{CP}^\infty)^r$ is given by the quotient of the action

$$t \cdot (x, v) = (x_1 t_0^{-1}, \ldots, x_r t_r^{-1}, t_1^{a_1} \cdots t_r^{a_r} v)$$

so that $L_G = \mathcal{O}(-a_1) \boxtimes \cdots \boxtimes \mathcal{O}(-a_r)$ and $c_1^G(L) = -(a_1 s_1 + a_2 s_2 + \cdots + a_r s_r)$.

Let $i_X : X \to X_G$ be the inclusion map of $X$ in a fiber of $X_G$. The pull back $i_X^*$ gives a map from equivariant cohomology to ordinary cohomology. In particular we have a Cartesian square:

$$\begin{array}{ccc} X & \to & p.t. \\ \downarrow & & \downarrow \\ X_G & \to & p.t._G \end{array}$$

so that when $X$ is proper we can do ordinary integrals in equivaraint cohomology

$$i^*_{p.t.} \int_{X_G} P(c_1^G(V), \ldots, c_r^G(V)) = \int_X P(c_1(V), \ldots, c_r(V)) \in \mathbb{Z}$$

since $i^*_{p.t.} \circ \int_{X_G} = \int_X \circ i_X^*$. In equivariant cohomology such integrals often become much simpler after the following famous localization theorem:

**Theorem 1.** *(Atiyah and Bott [1]) Let $(X, H)$ be a complex projective manifold with $T_r$-action. Let $i_j : Z_j \to X$ be the inclusion of the smooth fixed point locus $Z_i$ with $N_i$ its normal bundle. Then there is an isomorphism*

$$H^*_{T_r}(X) \otimes \mathbb{C}(s_1, \ldots, s_r) \to \bigoplus_j H^*_{T_r}(Z_j) \otimes \mathbb{C}(s_1, \ldots, s_r)$$

*given by the invertability of the Euler classes of the normal bundles,*

$$\alpha \mapsto \sum_j \frac{i_j^*(\alpha)}{c_{top}^{T_r}(N_j)}.$$

Applying this we will get a simple count for the number of lines on the cubic surface. Let $T_4$ act on $\mathbb{P}^3$ via $t \cdot [x_0, \ldots, x_3] = [t_0^{-1}x_0, \ldots, t_3^{-1}x_3]$ for which we have four fixed points $p_0, \ldots, p_3$. We can lift this to an action on the Grassmannian of lines on $\mathbb{P}^3$. The fixed lines $L_{i,j}$ are those passing through two of the fixed points $p_i, p_j$. Now we apply the method of localization

$$\int_{\mathbb{G}(1,3)} c_4(H^0(\mathcal{O}_L(3))) = \int_{\mathbb{G}(1,3)_{T_4}} c_4^{T_4}(H^0(\mathcal{O}_L(3))) = \sum_{0\leq i<j\leq 3} \frac{c_4^{T_4}(H^0(\mathcal{O}_{L_{i,j}}(3)))}{c_4^{T_4}(T\mathbb{G}(1,3)_{L_{i,j}})}.$$

The first equality coming from considering the integral in equivariant cohomology and the second from the localization theorem. It suffices to compute the $T_4$ representations, at each fixed point $L_{i,j}$, of the vector spaces $H^0(\mathcal{O}_{L_{i,j}}(3))$ and $T\mathbb{G}(1,3)_{L_{i,j}}$ to get the Chern roots and finish the calculation.

Let us consider the situation at the fixed point $L_{0,1}$. The vector space $H^0(\mathcal{O}_{L_{0,1}}(3))$ splits into one dimensional representations with characters $t_0^3$, $t_0^2t_1$, $t_0t_1^2, t_1^3$ while $T\mathbb{G}(1,3)_{L_{0,1}} = \mathrm{Hom}(\mathbb{C}^2_{0,1}, \mathbb{C}^4/\mathbb{C}^2_{0,1})$ has $T_4$ characters $t_0t_2^{-1}$, $t_0t_3^{-1}, t_1t_2^{-1}, t_1t_3^{-1}$. By identical calculations at the other fixed points we have

$$\#\{L : L \subset S\} = \sum_{0\leq i<j\leq 3} \frac{\prod_{a=0}^3 (as_i + (3-a)s_j)}{\prod_{\substack{a\in\{i,j\} \\ b\notin\{i,j\}}} (s_a - s_b)} = 27.$$

The goal of these notes is to explain a formalism for counting curves on a toric threefold $X$. These numbers will be given as integrals over moduli spaces parameterizing such curves. By using the three dimensional torus action on $X$ these numbers can be computed by the method of localization described above. The equivariant vertex can be understood as a system of formal power series expressing all the virtual counts of curves in terms of a finite amount of topological data on $X$.

# 2 Curves on threefolds.

In this section we describe three alternative ways to parameterize curves lying in a non-singular projective threefold $X$.

**Stable Maps.** The first method is to parametrize curves by maps of abstract curves into $X$. A map $[f, C, p_1 \dots, p_n]$ consist of a regular map $f : C \to X$ from a connected nodal curve $C$ with $p_1, \dots, p_n$ distinct smooth points on the curve $p_i \in C$. An isomorphism between two maps $[f, C, p_1 \dots, p_n]$ and $[f', C', p'_1 \dots, p'_n]$ is an isomorphism of the underlying curves $\tau : C \to C'$ with $\tau(p_i) = p'_i$ and $f = f' \circ \tau$. A map with finitely many automorphisms is called stable. The functor parameterizing flat families of stable maps is represented by a Deligne-Mumford stack $\overline{\mathcal{M}}_{g,n}(X, \beta)$ with projective coarse space $\overline{M}_{g,n}(X, \beta)$ [10]. Here $g = h^1(\mathcal{O}_C)$, $\beta = f_*[C] \in H_2(X, \mathbb{Z})$, and $n \in \mathbb{Z}$, are the genus, degree and number of marked points.

The moduli space $\overline{\mathcal{M}}_{g,n}(X, \beta)$ has a universal curve $\pi_{n+1} : \overline{\mathcal{M}}_{g,n+1}(X, \beta) \to \overline{\mathcal{M}}_{g,n}(X, \beta)$ given by forgetting the last marked point and stabilizing the curve as necessary. Also we have natural line bundles $L_i \to \overline{\mathcal{M}}_{g,n+1}(X, \beta)$ with fibers given by the cotangent line $TC^*_{p_i}$ at the marked point $p_i \in C$.

**Ideal Sheaves.** The second method is to parametrize curves by their associated ideal sheaves $\mathcal{I}_C \subset \mathcal{O}_X$. Abstractly an ideal sheaf of a curve on $X$ is equal to a rank one torsion free sheaf $\mathcal{E}$ equipped with a trivialization $\phi : \det(\mathcal{E}) = \mathcal{O}_X$. The double dual of such an object is a reflexive sheaf with trivial determinant and hence equal to $\mathcal{O}_X$. The sheaf $\mathcal{E}$ is then realized as an ideal sheaf via the inclusion

$$0 \to \mathcal{E} \to \mathcal{E}^{\vee\vee} = \mathcal{O}_X.$$

Then one must show that flat families of such ideal sheaves $\mathcal{I}_C$ are equivalent to flat families of embedded subschemes $C = \mathrm{Spec}_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}_C)$. Hence the moduli space of ideal sheaves of curves in $X$ actually equals the projective Hilbert scheme $\mathrm{Hilb}^P(X)$ of subschemes of $X$ with Hilbert polynomial $P(k) = kH \cdot \beta + n$ where $\beta = c_2(\mathcal{I}_C)$ and $n = \chi(\mathcal{I}_C)$ [8].

There is a universal ideal sheaf over $I_n(X, \beta) \times X$ given by the flat family $\mathbb{I} \to I_n(X, \beta) \times X$.

**Stable Pairs.** The third method of parameterizing curves is to consider pairs consisting of sheaves $F$ supported on a curve with a section $s : \mathcal{O}_X \to F$. Such a pair is stable if the sheaf $F$ is pure and the section $s$ has a zero dimensional cokernel. In particular the support of the sheaf must be a Cohen-Macaulay curve. The stable pairs can be realized as the stable objects in a G.I.T. quotient and as such we have a fine projective moduli space $P_n(X, \beta)$

parameterizing flat families of stable pairs with $\chi(F(k)) = kH \cdot \beta + n$ for $k \gg 0$ where $[supp(F)] = \beta \in H_2(X, \mathbb{Z})$ and $n = \chi(F)$ [22].

There is a universal stable pair over $P_n(X, \beta) \times X$ given by the flat family $\mathbb{I}^\bullet = \{s : \mathcal{O} \to \mathbb{F}\} \to P_n(X, \beta) \times X$.

Considering a stable pair $I^\bullet = [s : \mathcal{O}_X \to F]$ as a complex in the derived category it can be shown that no two distinct stable pairs are quasi-isomorphic [19]. In this way $P_n(X, \beta)$ can be considered as parameterizing complexes in the derived category satisfying a certain stability condition. Likewise $I_n(X, \beta)$ can be considered as parameterizing complexes $I^\bullet = [1 : \mathcal{O}_X \twoheadrightarrow \mathcal{O}_C]$ satisfying another such stability condition. In this way ideal sheaves and stable pairs are naturally two parts of the same story in $D^b(X)$ [24], [26].

**Twisted cubic curves.** We consider how each of the three examples above compactify the space of twisted cubic curves in $\mathbb{P}^3$. A twisted cubic curve in $\mathbb{P}^3$ is given by the image of a degree three map $\mathbb{P}^1 \to \mathbb{P}^3$ such as

$$[x : y] \mapsto [x^3 : x^2y : xy^2 : y^3].$$

Firstly the space of stable maps $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ provides a compactification of the space of such curves. A generic element, representing a smooth curve $C$, is given by the associated degree three stable map $f : \mathbb{P}^1 \to C \subset \mathbb{P}^3$. To see some of the behavior at the boundary consider the family of curves $C_t$ given by the stable maps

$$f_t : [x : y] \mapsto [x^3 : x^2y - x^3 : y^3 - xy^2 : t \cdot y^3].$$

In the limit as $t \to 0$ the image curve tends to $C_0$ a rational planar cubic curve with one node. The associated stable map $f_0 : \mathbb{P}^1 \to C_0 \subset \mathbb{P}^3$ is given by the normalization of $C_0$. In fact $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$ is smooth and projective. Its dimension is 12 given by the sixteen choices of coordinates in the degree three map, minus one for rescaling in projective coordinates, minus three for reparameterization of the domain $\mathbb{P}^1$. Its Poincare polynomial $P(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3), t)$ was computed in [23] and equals

$$1+2t^2+6t^4+10t^6+17t^8+20t^{10}+24t^{12}+20t^{14}+17t^{16}+10t^{18}+6t^{20}+2t^{22}+t^{24}.$$

Unfortunately this simple example of a stable maps space is unrepresentative of the higher genus case where more complications arise. For example $\overline{\mathcal{M}}_{1,0}(\mathbb{P}^3, 3)$ has multiple components one coming from maps $f : E \to C \subset \mathbb{P}^3$ of genus one curves to a planar cubic curve $C$ and another from maps $f : E \cup \mathbb{P}^1 \to C_3 \subset \mathbb{P}^3$ coming from maps of a nodal curve of genus one to a twisted cubic curve that contract the elliptic curve.

Secondly we consider the moduli space of ideal sheaves as a compactification of the space of twisted cubic curves. These curves have Hilbert polynomial $P(k) = 3k+1$. Now the flat family of curves $C_t$ are described by a family of ideals $I_t < \mathbb{C}[X, Y, Z, W]$ given by three quadrics,

$$\left(tX(W-Z) - W^2, XW - (W-tX)(X+Y), tW(X+Y) - (W-tX)^2\right).$$

The associated one dimensional scheme now degenerates to a rational cubic curve in the plane with one node and an embedded point at the node given by the ideal

$$I_0 = \left(W^2, WY, WZ, XZ^2 - Y^3 - XY^2\right) < \mathbb{C}[X, Y, Z, W].$$

These boundary points form a smooth divisor on the 12 dimensional component parameterizing the twisted cubic curves [21]. However such curves can deform in another way as the embedded point is now free to move as a zero dimensional subscheme of $\mathbb{P}^3$ while the nodal cubic can deform to a general planar cubic curve. The choice of a plane, an embedded planar cubic curve, and the location of the embedded point produces a second component of dimension $15 = 3 + 9 + 3$. Even though $I_1(\mathbb{P}^3, 3)$ is singular the main 12 dimensional component parameterizing the twisted cubic curves $I_1(\mathbb{P}^3, 3)^m$ is smooth and birational to $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$. This component has Poincare polynomial $P(I_1(\mathbb{P}^3, 3)^m, t)$ equal to [7],

$$1+2t^2+6t^4+10t^6+16t^8+19t^{10}+22t^{12}+19t^{14}+16t^{16}+10t^{18}+6t^{20}+2t^{22}+t^{24}.$$

Finally the third compactifcation of the space of twisted cubics is given by the space of stable pairs $P_1(\mathbb{P}^3, 3)$. Over the locus of twisted cubic curves the space of stable pairs is isomorphic to the space of ideal sheaves, the pairs being represented by the complexes $1 : \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_C$. However in the limit of the family $C_t$ specializing to $C_0$ the limiting stable pair is given by the canonical section

$$\mathcal{O}_{\mathbb{P}^3} \to f_{0*}\mathcal{O}_{\mathbb{P}^1}$$

with cokernel supported at the node $p$. Like ideal sheaves the pairs lie on the boundary of the 12 dimensional component parameterizing the twisted cubic curves and can deform in another way. In particular the above pair can now deform to $\mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_E(p')$ where $E$ is a plannar cubic containing the point $p'$. The choice of a plane, an embedded cubic curve, and a divisor on the curve gives a second component of $P_1(\mathbb{P}^3, 3)$ of dimension $13 = 3 + 9 + 1$.

In fact the main example of a stable pair is given by a Cohen–Macaulay curve $C$ with a divisor $D$ and the canonical section $s_D : \mathcal{O}_X \to \mathcal{O}_X(D)$ so that the space of stable pairs differs from the space of ideal sheaves in that floating points are constrained to lie on the curve.

# 3 Virtual fundamental classes.

In the introduction we saw how to make a count of curves by integrating Chern classes over a moduli space of lines in $\mathbb{P}^3$. However in the previous section we saw that the various compact moduli spaces parameterizing twisted cubic curves in $\mathbb{P}^3$ are not smooth and have components of different dimensions. Intersection theory on these singular spaces in not well behaved. In particular, we cannot apply the method of localization to compute the degree of a cycle. Moreover evaluating tautological classes against the fundamental class will not necessarily produce deformation invariants of the threefold $X$.

We explain the construction of a virtual fundamental class replacing the badly behaved fundamental class by using the additional data of a perfect obstruction theory. Then integrals against the virtual fundamental class will give deformation invariant (virtual) counts of curves on $X$.

**Physics sketch picture.** In quantum field theory one studies collections of fields $\mathcal{F}$. At a given point on the worldsheet each field has a kinetic energy density $K$ and a potential energy density $V$ measuring how strong the field is at that point in space and time. The Lagrangian density is the difference $L = K - V$ and the action is the function

$$\begin{array}{cccc} S & : & \mathcal{F} & \to & \mathbb{R} \\ & : & \phi & \mapsto & \int L(\phi). \end{array}$$

where we integrate the Lagrangian density over the worldsheet. The dimension of the field theory is the dimension of the worldsheet. For example a particle moving in some manifold has a one dimensional world sheet, given by the timeline parmeterizing its path. Another example is that of gauge fields on $\mathbb{R}^{1,3}$ given by the value of a connection at each point on the four dimensional worldsheet.

In quantum theory the observable quantities such as position and momentum are non-deterministic. Correlation functions give a measure of the expected values of observable quantities $\mathcal{O} : \mathcal{F} \to \mathbb{R}$, such as the energy density $K$ at a point on the worldsheet,

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_r \rangle = \int_{\mathcal{F}} \mathcal{O}_1(\phi) \cdots \mathcal{O}_r(\phi) e^{-S(\phi)} d\mu_{\mathcal{F}}.$$

The space of fields is usually infinite dimensional and the measure $d\mu_{\mathcal{F}}$ rarely has a mathematical definition. However in the presence of a good definition the fields contributing most to the integral should be those minimizing the action functional $\delta S = 0$, these are the Euler-Lagrange equations of motion. The leading order contribution to these correlation functions should equal an integral over a neighborhood of

$$\mathcal{M} = \{\delta S = 0\} \subset \mathcal{F}.$$

Certain correlation functions that are invariant under rescaling and deformations of the action $S$ can even be computed exactly in terms of such integrals over a neighborhood of $\mathcal{M}$.

We can consider the moduli space of stable maps as coming from a twisted non-linear sigma model [28, Chap. 16.4]. Let $\mathcal{F} = C^\infty \mathrm{Maps}(\Sigma_g, X)$ be the space of $C^\infty$-maps $\phi : \Sigma_g \to X$ from a genus $g$ Riemann surface to a symplectic manifold $(X, \omega)$, together with some fermionic fields we will ignore. The action functional is given by

$$S(\phi) = 2 \int_{\Sigma_g} \partial_{\bar{z}} \phi \partial_z \bar{\phi} dz d\bar{z} + \int_{\Sigma_g} \phi^* \omega + \text{ fermionic bit...}$$

If we let $\partial_{\bar{z}} : \mathcal{F} \to \mathcal{E}$ be the section of the vector bundle with fibers $\Gamma(\Omega^{0,1}_{\Sigma_g}(\phi^* TX))$ then the fields minimizing the action are

$$\{\phi : \partial_{\bar{z}}(\phi) = 0\} \subset \mathcal{F}$$

so the space of holomorphic maps is realized as the vanishing of a section of a vector bundle on the space of all maps.

Secondly we can consider the space of ideal sheaves as coming from a holomorphic Chern–Simons theory. Let $X$ be a Calabi–Yau threefold with non-vanishing section $\theta \in \Omega^{3,0}(X)$ and vector bundle $E \to X$. The space of fields $\mathcal{F} = \Omega^{0,1}(\mathrm{ad}(E))$ is given by the space of half connections on $E$. The action is given by a holomorphic version of the Chern–Simons function [28, Chap. 38]

$$S(\phi) = \frac{1}{4\pi^2} \int_X \mathrm{tr} \left( \frac{1}{2} \bar{\partial}_{A_0} \phi \wedge \phi + \frac{1}{3} \phi \wedge \phi \wedge \phi \right) \wedge \theta.$$

If we let $F^{0,2} : \mathcal{F} \to \Omega^{0,2}(\mathrm{ad}(E))$ be the curvature map then the fields minimizing the action are

$$\{\phi : F^{0,2}(\phi) = 0\} \subset \mathcal{F}$$

so the space of holomorphic bundles is realized as the vanishing of a section of a vector bundle on the space of all half connections.

The virtual fundamental class of the moduli space $[\mathcal{M}]^{vir} \in H_*(\mathcal{M})$ is recovered from the extra information of an (infinitesimal) embedding of our moduli space in some bigger space of all fields. In special situations correlation functions like those above can be computed by integrating cohomology classes against this virtual class so that

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_r \rangle := \sum_i e^{-a_i} \int_{[\mathcal{M}]^{vir}} \alpha_1 \cup \cdots \cup \alpha_r$$

where $\mathcal{M}_i$ is the component of $\mathcal{M}$ on which $S$ has minimum value $a_i$.

**Basic toy model.** In this subsection let us assume that our moduli space $\mathcal{M}$ is given as the zero locus of a section $F : A \to E$ of a finite dimensional vector bundle $\pi : E \to A$ over a smooth manifold $A$,

$$\mathcal{M} = \{a : F(a) = 0\} \subset A.$$

We should expect that the dimension of $\mathcal{M}$ equals $ed = \dim(A) - \mathrm{rank}(E)$. Indeed in the simplest case when the graph of $F$ Graph$(F : A \to E)$ is transverse to the zero section then the space $\mathcal{M}$ will be a *smooth manifold of the expected dimension* and we define

$$[\mathcal{M}]^{vir} := [\{F = 0\}] \in A_{ed}(\mathcal{M}).$$

The next slightly harder case is when $E = E' \oplus E''$ and the section $F$ takes values in the sub bundle $E'$ with $\mathrm{Graph}(F : A \to E')$ transverse to the zero section. Then the space $\mathcal{M}$ will be a *smooth manifold not of the expected dimension* having dimension equal to $\dim(A) - \mathrm{rank}(E')$ instead. This may be resolved by deforming the section $F$ to $F_\epsilon = (F, \epsilon)$ where $\epsilon$ is a section of $E''$ whose graph is transverse to the zero section. Then $F_\epsilon$ is transverse with $\{F_\epsilon = 0\} \subset \mathcal{M}$ and we define a class of the expected dimension

$$[\mathcal{M}]^{vir} := [\{F_\epsilon = 0\}] \in A_{ed}(\mathcal{M}).$$

This locus is given by a smooth space cut out from $\mathcal{M}$ by a transverse section of $E''$ so $c_{top}(E'') = [\{\epsilon = 0\}]$ and we have that $[\mathcal{M}]^{vir} = c_{top}(E'') \cdot [M]$. This definition subsumes the previous case.

Finally we need to have a definition that works even when the graph of $F$ cannot be made transverse. In this case the space $\mathcal{M}$ may for example be *singular and not of the expected dimension*. The general such definition comes from the intersection product developed by Fulton and MacPherson [9]. Our moduli space $\mathcal{M}$ is given as the scheme theoretic intersection

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & A \\ \big\downarrow & & \big\downarrow F \\ A & \xrightarrow{0} & E. \end{array}$$

The definition involves replacing the graph of $F$ with the normal cone $C_{A/\mathcal{M}}$. If $\mathcal{M} \subset A$ is defined by the sheaf of ideals $\mathcal{I}_\mathcal{M} \subset \mathcal{O}_A$ then the normal cone $C_{A/\mathcal{M}}$ equals $\mathrm{Spec}\left(\bigoplus_{i \geq 0} \mathcal{I}^i_\mathcal{M}/\mathcal{I}^{i+1}_\mathcal{M}\right)$. It is a cone in $E$ with zero section $\mathcal{M}$ taking the role of a tubular neighborhood of $\mathcal{M}$ inside $\mathrm{Graph}(F)$. We are now in the situation

$$\begin{array}{ccc} C_{A/\mathcal{M}} & \longrightarrow & E \\ \big\downarrow & & \big\downarrow \pi \\ \mathcal{M} & \longrightarrow & A. \end{array}$$

The map $\pi$ here is flat and the associated pullback induces a Thom isomorphism $\pi^* : A_k(A) \to A_{k+r}(E)$ via $\pi^*([Z]) = [E|_Z]$. In particular $[C_{A/\mathcal{M}}] = \sum_i n_i [E|_{Z_i}]$ the refined intersection of the graph of $F$ with the zero section of $E$ is defined to be $0^!([C_{A/\mathcal{M}}]) = \sum_i n_i [Z_i] \in A_{ed}(\mathcal{M})$. This gives the general definition of the virtual fundamental class in the basic toy model

$$[\mathcal{M}]^{vir} = 0^!_E([C_{A/\mathcal{M}}]) \in A_{ed}(\mathcal{M}).$$

**Deformation Theory.** In the last section we assumed we had the additional information of a finite dimensional vector bundle with a section cutting out the moduli space. Generally this is not given and instead we only have an (infinitesimal) neighborhood of our moduli space in the space of all fields. This infinitesimal structure will be given in our cases by a perfect obstruction theory.

A deformation functor is a covariant functor [8, Chap. 6] from the category of Artin local rings to the category of sets

$$D : (Art/k) \to (Sets).$$

The example that will interest us is the functor given by $h_R(A) = \mathrm{Hom}(R, A)$ where $R = \hat{\mathcal{O}}_{\mathcal{M},p}$ is the complete local ring of a moduli space $\mathcal{M}$ at a point $p$. By the moduli theoretic definition of $\mathcal{M}$ we have that $h_{\hat{\mathcal{O}}_{\mathcal{M},p}}(A)$ is equal to the set of flat families of objects over $\mathrm{Spec}(A)$ with reduced fiber given by $p \in \mathcal{M}$, i.e. the infinitesimal deformations of $p \in \mathcal{M}$ over $\mathrm{Spec}(A)$.

Every Artin local ring can be built from the ground field $k$ by a sequence of small extensions

$$0 \to M \to B \to A \to 0$$

where $M \cdot m_B = 0$, i.e. the $B$ module structure on $M$ is induced by its structure as a $k = B/m_B$ vector space. A map between two small extensions is given by a map of the short exact sequences considered as chain complexes.

Every Artin local ring can be produced from a series of such small extensions starting from the ground field $k$. So breaking up deformations into a sequence of these small extensions the basic question in deformation theory is: Given an infinitesimal deformation over $\mathrm{Spec}(A)$ in how many ways can this be extended to an infinitesimal deformation over $\mathrm{Spec}(B)$?

Indeed sometimes it is not possible to extend. A deformation functor $D$ is said to have a tangent-obstruction theory if there exist finite dimensional vector spaces $T_1, T_2$ and maps $ob$ such that for each small extensions $0 \to M \to B \to A \to 0$ we have an exact sequence of sets

$$T_1 \otimes_k M \longrightarrow D(B) \longrightarrow D(A) \xrightarrow{ob} T_2 \otimes_k M.$$

These sequences are required to be functorial with respect to the maps of small extensions in the natural way. Moreover for the simple extension $0 \to \epsilon k \to k[\epsilon]/(\epsilon^2) \to k \to 0$ this sequence has to be exact on the left.

In summary this means that an infinitesimal deformation $\alpha \in D(A)$ extends to one in $D(B)$ if and only if $ob(\alpha) = 0$. In this case the collection of all such extension are contained in $T_1 \otimes_k M$.

The vector space $T_1$ is uniquely determined. For example in the case of the functor $D = h_{\hat{\mathcal{O}}_{\mathcal{M},p}}$ we have $D(k) = p$ and $D(k[\epsilon]/(\epsilon^2)) = T\mathcal{M}_p$ and $T_1$ equals the tangent space of $\mathcal{M}$ at $p$. However the obstruction space is not unique. For example we can take a trivial modification $T_2' = T_2 \oplus k$ and $ob' = (ob, 0)$ to get another theory.

A canonical tangent-obstruction theory can be constructed for the local rings $\hat{\mathcal{O}}_{\mathcal{M},p}$ using the cotangent complex $L^\bullet_{\mathcal{M}} \in D^b(\mathcal{M})$ which has the fundamental property that given a small extension as above and an infinitesimal deformation $f : \mathrm{Spec}(A) \to \mathcal{M}$ of $p \in \mathcal{M}$ there exists a tangent space $T_1 = h^0(f^* L^{\bullet\vee}_{\mathcal{M}})$, and an obstruction space $T_2 = h^1(f^* L^{\bullet\vee}_{\mathcal{M}})$ with obstruction class $ob(B \to A, f) \in T_2 \otimes M$ whose vanishing ensures the existence to an extension of the infinitesimal deformation of $p$ over $\mathrm{Spec}(A)$ to one over $\mathrm{Spec}(B)$.

Given an embedding of $\mathcal{M}$ in a smooth manifold $A$ via the sheaf of ideals $\mathcal{I}$ the cotangent complex can be represented by a complex

$$L^\bullet_{\mathcal{M}} = \{\, \cdots \longrightarrow \mathcal{I}/\mathcal{I}^2 \xrightarrow{\ d\ } \Omega_A|_{\mathcal{M}} \,\} \in D^b(\mathcal{M})$$

concentrated in degrees $\ldots, -1, 0$. In particular in a neighborhood of a point $p \in \mathcal{M}$ where $\mathcal{M}$ is a local complete intersection we have $L^\bullet_{\mathcal{M}} = \mathbb{L}^\bullet_{\mathcal{M}} := \{\, \mathcal{I}/\mathcal{I}^2 \xrightarrow{\ d\ } \Omega_A|_{\mathcal{M}} \,\}$ equals the truncated cotangent complex. If moreover $\mathcal{M}$ is smooth then $\mathbb{L}^\bullet_{\mathcal{M}} = \Omega_{\mathcal{M}}$ so that $h^1(f^* L^{\bullet\vee}_{\mathcal{M}}) = 0$ and hence all infinitesimal deformations can be extended [8, Theorem 6.1.19].

While this obstruction theory is canonical as mentioned above it is not unique. In fact the virtual fundamental class will be constructed from the data of a perfect obstruction theory on $\mathcal{M}$ [4]. A perfect obstruction theory on $\mathcal{M}$ consists of a two term complex of vector bundles $E^\bullet = [E^{-1} \to E^0] \in D^b(\mathcal{M})$ together with a map $\phi : E^\bullet \to \mathbb{L}^\bullet_{\mathcal{M}}$ to the truncated cotangent complex such that $h^\bullet(\phi)$ is isomorphism in degree 0 and is surjective in degree $-1$. A prefect obstruction theory in particular gives, for a small extension as above, a tangent-obstruction theory with tangent space $T_1 = h^0(f^* E^{\bullet\vee})$, and an obstruction space $T_2 = h^1(f^* E^{\bullet\vee})$ with obstruction class $ob(B \to A, f) \in T_2 \otimes M$ [4, Theorem, 4.5]. The perfectness of the tangent-obstruction theory now implies that $\dim(T_1) - \dim(T_2) = \mathrm{rk}(E^0) - \mathrm{rk}(E^{-1})$ is constant over all of $\mathcal{M}$ so the dimension of the obstruction space jumps in the same way as that of the tangent space.

The basic example of such a perfect obstruction theory is given by our toy model considered earlier. If $F : A \to E$ is a section of a vector bundle with $\mathcal{M} = \{F = 0\}$ then the linearization of this section produces a perfect obstruction theory on $\mathcal{M}$

$$\begin{array}{ccc} \{E^\vee|_{\mathcal{M}} & \xrightarrow{dF^\vee} & \Omega_A|_{\mathcal{M}}\} \\ \big\downarrow {\scriptstyle F^\vee} & & \big\| \\ \{\mathcal{I}/\mathcal{I}^2|_{\mathcal{M}} & \xrightarrow{d} & \Omega_A|_{\mathcal{M}}\}. \end{array}$$

From this local data on $\mathcal{M}$ we can construct a virtual fundamental class.

**General definition.** Let us assume we have some embedding of our moduli space $\mathcal{M}$ in some smooth manifold (or DM stack) $A$ given by a sheaf of ideals $\mathcal{I}$ and a perfect obstruction theory given by a map of complexes

$$\phi : E \to \mathbb{L}_{\mathcal{M}}.$$

Taking the cone of this morphism gives an exact sequence of sheaves

$$E^{-1} \xrightarrow{(-d_E, -\phi^{-1})^t} E^0 \oplus \mathcal{I}/\mathcal{I}^2 \xrightarrow{(-\phi^0, d_{\mathbb{L}_{\mathcal{M}}})} \Omega_A|_{\mathcal{M}} \longrightarrow 0\ .$$

Given a sheaf $\mathcal{F}$ define the cone $C(\mathcal{F}) = \mathrm{Spec}_{\mathcal{O}_{\mathcal{M}}}(\oplus_{n\geq 0} Sym^n \mathcal{F})$. For example if $\mathcal{F}$ is the sheaf of sections of a vector bundle $F$ then $C(\mathcal{F}) = F^\vee$. Applying this to the exact sequence above we have cones

$$\begin{array}{ccccccc} 0 & \longrightarrow & TA|_{\mathcal{M}} & \longrightarrow & (E^0)^\vee \times_{\mathcal{M}} C(\mathcal{I}/\mathcal{I}^2) & \longrightarrow & (E^{-1})^\vee \\ & & \big\| & & \uparrow & \nearrow & \\ 0 & \longrightarrow & TA|_{\mathcal{M}} & \longrightarrow & (E^0)^\vee \times_{\mathcal{M}} C_{A|\mathcal{M}}. & & \end{array}$$

The image of the $(E^0)^\vee \times_{\mathcal{M}} C_{A|\mathcal{M}}$ defines a cone $C^{vir}$ in the vector bundle $(E^{-1})^\vee$. Using this local model of the moduli space the virtual fundamental class can be defined [25, c.f. Section 5.3] as before as a refined intersection

$$[\mathcal{M}]^{vir} := 0^!_{(E^{-1})^\vee}([C^{vir}]) \in A_{ed}(\mathcal{M})$$

where here the expected dimension equals

$$ed = \mathrm{rk}(E^0) - \mathrm{rk}(E^{-1}) \in \mathbb{Z}.$$

To see that this definition is independent of all choices made one can work with stacks [4]. The intrinsic normal cone $\mathfrak{C}_{\mathcal{M}}$ is a zero dimension cone stack over $\mathcal{M}$ defined without reference to any embedding of $\mathcal{M}$. Then given a perfect obstruction theory we have a smooth vector bundle stack $\mathfrak{E} = h^1/h^0(E^\vee)$ over $\mathcal{M}$ and an embedding of $\mathfrak{C}_{\mathcal{M}}$ in $\mathfrak{E}$. Then the virtual fundamental class equals the intersection of $\mathfrak{C}_{\mathcal{M}}$ with the zero section of $\mathfrak{E}$ using the intersection theory of cycles on Artin stacks [13]. This construction also removes the need for the global resolution by vector bundles assumed here.

**Maps, Ideals, Pairs.** Here is a quick description of the perfect obstruction theories on the spaces of stable maps, ideal sheaves, and stable pairs used to construct the virtual fundamental class. We also do a calculation of their expected dimensions.

The perfect obstruction theory on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ can be found in [3]. A stable map $f : C \to X$ can be deformed in two directions either by deforming the complex structure on the curve $C$ or by fixing the curve and deforming the map $f$. The second obstruction theory is a relative obstruction theory over the moduli stack of (prestable) curves via the forgetful map

$$p : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \mathfrak{M}_{g,n}.$$

This gives an exact triangle of cotangent complexes on $\overline{\mathcal{M}}_{g,n}(X,\beta)$

$$p^* L_{\mathfrak{M}_{g,n}} \to L_{\overline{\mathcal{M}}_{g,n}(X,\beta)} \to L_p \to p^* L_{\mathfrak{M}_{g,n}}[1].$$

The cotangent complex $p^* L_{\mathfrak{M}_{g,n}}$ has a relsolution by vector bundles $A$ in degrees $0, 1$ since $\mathfrak{M}_{g,n}$ is an Artin stack. The perfect obstruction theory coming from deforming the map is given by

$$R\pi_*(f^*TX)^\vee \to L_p$$

where $\pi : \mathcal{C} \to \overline{\mathcal{M}}_{g,n}(X,\beta)$ is the projection map from the universal curve and $f : \mathcal{C} \to X$ is the map from the universal curve into $X$. This obstruction theory has a resolution $B$ with a natural boundary homomorphism $B \to A[1]$. The cone over this map $E[1]$ fits into a map of exact triangles

$$\begin{array}{ccccccc} B & \longrightarrow & A[1] & \longrightarrow & E[1] & \longrightarrow & B[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ L_p & \longrightarrow & p^* L_{\mathfrak{M}_{g,n}}[1] & \longrightarrow & L_{\overline{\mathcal{M}}_{g,n}(X,\beta)}[1] & \longrightarrow & L_p[1]. \end{array}$$

The complex $E$ is supported in degrees $-1, 0, +1$. By the stability condition the co-homology of $E$ vanishes in degree 1 and so it is derived equivalent to a complex in degrees $-1, 0$. To compute the expected dimension at a point $x \in \overline{\mathcal{M}}_{g,n}(X, \beta)$ we take the cohomology of the above triangle of obstruction theories

$$\begin{array}{ccccccc} & & 0 & \longrightarrow & H^{-1}(p^* L_{\mathfrak{M}_{g,n}}) & \longrightarrow & \\ H^0(C, f^*TX) & \longrightarrow & T_1\overline{\mathcal{M}}_{g,n}(X,\beta)(x) & \longrightarrow & H^0(p^* L_{\mathfrak{M}_{g,n}}) & \longrightarrow & \\ H^1(C, f^*TX) & \longrightarrow & T_2\overline{\mathcal{M}}_{g,n}(X,\beta)(x) & \longrightarrow & 0 & & \end{array}$$

The deformation theory of algebraic curves gives

$$h^0(p^* L_{\mathcal{M}_{g,n}}) - h^{-1}(p^* L_{\mathcal{M}_{g,n}}) = 3g - 3 + n$$

and the Riemann-Roch theorem for curves gives

$$h^0(f^*(TX)) - h^1(f^*(TX)) = \operatorname{rank}(TX)\chi(\mathcal{O}_C) + \int_\beta c_1(TX).$$

So for this perfect obstruction theory the expected dimension of $\overline{\mathcal{M}}_{g,n}(X, \beta)$ equals

$$\int_\beta c_1(X) + (\dim(X) - 3)(1 - g) + n$$

in particular on a threefold $X$ the expected dimension is independent of the genus.

In the case of ideal sheaves of subschemes embedded in a threefold $X$ we have the universal ideal sheaf $\mathbb{I} \to I_n(X,\beta) \times X$ and projection $\pi : I_n(X,\beta) \times X \to X$. There is a natural obstruction theory coming from deformations of subschemes $\mathcal{O}_C = \mathcal{O}_X/\mathcal{I}_C$ given by $R\pi_* RHom(\mathbb{I}, \mathcal{O}/\mathbb{I})$, with tangent-obstruction spaces at $I \in I_n(X,\beta)$ given by $\mathrm{Hom}(I, \mathcal{O}_X/I)$ and $\mathrm{Ext}^1(I, \mathcal{O}_X/I)$. However the dimension $\hom(I, \mathcal{O}_X/I) - \mathrm{ext}^1(I, \mathcal{O}_X/I)$ is not constant over $I_n(X,\beta)$ and this obstruction theory is not perfect. A perfect obstruction theory comes from the model discussed in our section on the physics motivation and holomorphic Chern-Simons theory [27] [14]. There we considered ideal sheaves as holomorphic vector bundles. Deformations of such sheaves have a perfect obstruction theory given by $R\pi_* RHom(\mathbb{I}, \mathbb{I})_0$ here the groups are traceless as we have sheaves of fixed determinant. The tangent and obstruction spaces at $I \in I_n(X,\beta)$ are then given by $\mathrm{Ext}^1(I,I)_0$ and $\mathrm{Ext}^2(I,I)_0$. Then for the collection of ideals in $I_n(X,\beta)$ with $ch(I) = (1, 0, -\beta, -n) \in H^*(X, \mathbb{Z})$ the expected dimension of $I_n(X,\beta)$ can be computed via Hirzebruch–Riemann–Roch

$$\mathrm{ext}^1(I,I)_0 - \mathrm{ext}^2(I,I)_0 = \chi(\mathcal{O}_X, \mathcal{O}_X) - \chi(I,I) = \int_\beta c_1(X)$$

in particular the expected dimension of $I_n(X,\beta)$ is not dependent on $n$.

The case of stable pairs is identical to that of ideal sheaves. Considering the stable pair $I^\bullet = [s : \mathcal{O}_X \to F]$ in the derived category of stable sheaves [19] [14] a perfect obstrution theory comes from $R\pi_* RHom(\mathbb{I}^\bullet, \mathbb{I}^\bullet)$ where $\mathbb{I}^\bullet \to P_n(X,\beta) \times X$ is the universal stable pair and $\pi : P_n(X,\beta) \times X \to X$ is the projection map. The tangent-obstruction spaces at $I^\bullet \in P_n(X,\beta)$ are then given by $\mathrm{Ext}^1(I^\bullet, I^\bullet)_0$ and $\mathrm{Ext}^2(I^\bullet, I^\bullet)_0$ and the expected dimension also equals $\int_\beta c_1(X)$.

# 4 Corresponding virtual counts.

**Calabi–Yau Threefolds.** Let $X$ be a non-singular projective threefold with trivial canonical bundle $K_X = \mathcal{O}_X$. Let $\overline{\mathcal{M}}'_{g,0}(X,\beta)$ be the moduli space of stable maps with possibly disconnected domain curve and no contracted components. Since $c_1(X) = 0$, the following moduli spaces all have expected dimension zero

$$\overline{\mathcal{M}}'_{g,0}(X,\beta), I_n(X,\beta), P_n(X,\beta).$$

We expect that each such moduli space contains a finite number of curves equal to

$$N_\beta^{GW}(g) := \int_{[\overline{\mathcal{M}}'_{g,0}(X,\beta)]^{vir}} 1, N_\beta^{DT}(n) := \int_{[I_n(X,\beta)]^{vir}} 1, N_\beta^{PT}(n) := \int_{[P_n(X,\beta)]^{vir}} 1.$$

These are the Gromov–Witten, Donaldson–Thomas, Pandharipande–Thomas invariants of $X$. Note that since $\overline{\mathcal{M}}'_{g,n}(X,\beta)$ is a DM stack the numbers $N_\beta^{GW}(g)$ may lie in $\mathbb{Q}$ whereas $N_\beta^{DT}(n), N_\beta^{PT}(n) \in \mathbb{Z}$. Following our discussion of the physics we can consider these numbers as summands in the correlation function. Let

$$\begin{aligned} Z_\beta^{GW}(u) &:= \sum_{g\in\mathbb{Z}} N_\beta^{GW}(g)u^{2g-2}, \\ Z_\beta^{DT}(q) &:= \sum_{n\in\mathbb{Z}} N_\beta^{DT}(n)q^n, \\ Z_\beta^{PT}(q) &:= \sum_{n\in\mathbb{Z}} N_\beta^{PT}(n)q^n. \end{aligned}$$

It is a theorem that

$$Z_\beta^{DT}(q) = Z_\beta^{PT}(q) \cdot M(-q)^{\chi(X)}$$

where $M(q) = \prod_{n\geq 1}(1-q^n)^{-n}$ is the generating series for the number of plane partitions. This is proven by considering DT and PT as theories counting stable objects in the derived category for two different choices of stability and then applying wall crossing formulas [5]. There it is also shown that the series $Z_\beta^{PT}(q)$ is the Laurent expansion of a rational function in $q$.

It is then further conjectured [15] (and proven in many cases [18]) that

$$Z_\beta^{GW}(u) = Z^{PT}(q)$$

under the change of variables $q = -e^{iu}$. The rationality of $Z_\beta^{PT}(q)$ puts strong conditions on the invariants $N_\beta^{GW}(g)$. By letting

$$F^{GW} = \log\left(1 + \sum_{\beta\neq 0} Z_\beta^{GW}(u)v^\beta\right)$$

we can define BPS counts $n_\beta^{BPS}(g) \in \mathbb{Q}$ via

$$F^{GW} = \sum_{g\geq 0}\sum_{\beta\neq 0} n_\beta^{BPS}(g)u^{2g-2}\sum_{d>0}\frac{1}{d}\left(\frac{\sin(du/2)}{u/2}\right)^{2g-2} v^{d\beta}.$$

It is conjectured that these counts are integer and for fixed $\beta$ are non-zero for only finitely many $g$. Recently a proof of the integrality was put forward in [12].

**Primary Fields.** In the case that the expected dimension is greater than zero we expect to have infinite dimensional families of curves in $X$. We can introduce incidence conditions to get finite counts of curves. The moduli spaces $\overline{\mathcal{M}}'_{g,k}(X,\beta)$ have evaluation maps

$$e_i : \overline{\mathcal{M}}'_{g,k}(X,\beta) \to X$$

and for $\gamma_i \in H^*(X,\mathbb{Z})$ can define

$$\langle \gamma_1, \dots, \gamma_k \rangle_\beta^{GW}(g) = \int_{[\overline{\mathcal{M}}'_{g,k}(X,\beta)]^{vir}} e_1(\gamma_1) \cup \cdots \cup e_k(\gamma_k) \in \mathbb{Q}.$$

Giving a finite count of genus $g$ curves meeting a collection meeting a collection of cycles dual to $\gamma_i$.

For ideal sheaves we have the universal sheaf $\mathbb{I} \to I_n(X,\beta) \times X$ and projection maps $\pi_X : I_n(X,\beta) \times X \to X$ and $\pi_I : I_n(X,\beta) \times X \to I_n(X,\beta)$. For each $\gamma \in H^l(X,\mathbb{Z})$ we have a map $\gamma : H_*(I_n(X,\beta),\mathbb{Q}) \to H_{*+2-l}(I_n(X,\beta),\mathbb{Q})$ given by

$$\alpha \mapsto -\pi_{I*}(\mathrm{ch}_2(\mathbb{I}) \cdot \pi_X^*(\gamma) \cap \pi_I^*(\alpha))$$

and for $\gamma_i \in H^*(X,\mathbb{Z})$ we can define

$$\langle \gamma_1, \dots, \gamma_k \rangle_\beta^{DT}(n) = \deg\left(\gamma_1 \circ \cdots \circ \gamma_k\left([I_n(X,\beta)]^{vir}\right)\right) \in \mathbb{Z}.$$

Likewise for stable pairs we have the universal pair $\mathbb{I}^\bullet = [\mathcal{O} \to \mathbb{F}] \to P_n(X,\beta) \times X$ and projection maps $\pi_X : P_n(X,\beta) \times X \to X$ and $\pi_P : P_n(X,\beta) \times X \to P_n(X,\beta)$. For each $\gamma \in H^l(X,\mathbb{Z})$ we have a map $\gamma : H_*(P_n(X,\beta),\mathbb{Q}) \to H_{*+2-l}(P_n(X,\beta),\mathbb{Q})$ given by

$$\alpha \mapsto \pi_{P*}(\mathrm{ch}_2(\mathbb{F}) \cdot \pi_X^*(\gamma) \cap \pi_P^*(\alpha))$$

and for $\gamma_i \in H^*(X,\mathbb{Z})$ we can define

$$\langle \gamma_1, \dots, \gamma_k \rangle_\beta^{PT}(n) = \deg\left(\gamma_1 \circ \cdots \circ \gamma_k\left([P_n(X,\beta)]^{vir}]\right)\right) \in \mathbb{Z}.$$

As above we make generating series

$$\begin{aligned}\langle\gamma_1,\dots,\gamma_k\rangle_\beta^{GW} &:= \sum_{g\in\mathbb{Z}}\langle\gamma_1,\dots,\gamma_k\rangle_\beta^{GW}(g)u^{2g-2},\\ \langle\gamma_1,\dots,\gamma_k\rangle_\beta^{DT} &:= \sum_{n\in\mathbb{Z}}\langle\gamma_1,\dots,\gamma_k\rangle_\beta^{DT}(n)q^{n},\\ \langle\gamma_1,\dots,\gamma_k\rangle_\beta^{PT} &:= \sum_{n\in\mathbb{Z}}\langle\gamma_1,\dots,\gamma_k\rangle_\beta^{PT}(n)q^{n}.\end{aligned}$$

It is again conjectured that

$$\langle\gamma_1,\dots,\gamma_k\rangle_\beta^{DT} = \langle\gamma_1,\dots,\gamma_k\rangle_\beta^{PT}\cdot M(-q)^{\int_X c_3(TX\otimes K_X)}$$

and that

$$(-iu)^{d_\beta}\langle\gamma_1,\dots,\gamma_k\rangle_\beta^{GW}(u) = (-q)^{-d_\beta/2}\langle\gamma_1,\dots,\gamma_k\rangle_\beta^{PT}(q)$$

under the change of variables $q=-e^{iu}$ where $d_\beta=\int_\beta c_1(X)$.

**Descendent Fields.** Descendent invariants are defined in GW theory via Chern classes of the cotangent line bundle and in DT/PT theory via higher Chern characters of the universal sheaves.

Let $L_i\to\overline{\mathcal{M}}'_{g,k}(X,\beta)$ be the $i$th cotangent line bundle, let $\psi_i=c_1(L_i)\in H^*(\overline{\mathcal{M}}'_{g,k}(X,\beta),\mathbb{Q})$, and let $\gamma_i\in H^*(X,\mathbb{Z})$ we define

$$\langle\tau_{a_1}(\gamma_1)\dots\tau_{a_k}(\gamma_k)\rangle_\beta^{GW}(g) = \int_{[\overline{\mathcal{M}}'_{g,k}(X,\beta)]^{vir}}\psi^{a_1}e_1(\gamma_1)\cup\dots\cup\psi^{a_k}e_k(\gamma_k)\in\mathbb{Q}.$$

For ideal sheaves we modify the above homology operation by taking a higher Chern character. For each $\gamma\in H^l(X,\mathbb{Z})$ we have a map $\tau_a(\gamma): H_*(I_n(X,\beta),\mathbb{Q})\to H_{*-2a+2-l}(I_n(X,\beta),\mathbb{Q})$ given by

$$\alpha\mapsto(-1)^{a+1}\pi_{I*}(\mathrm{ch}_{a+2}(\mathbb{I})\cdot\pi_X^*(\gamma)\cap\pi_I^*(\alpha))$$

and for $\gamma_i\in H^*(X,\mathbb{Z})$ we can define

$$\langle\tau_{a_1}(\gamma_1)\dots\tau_{a_k}(\gamma_k)\rangle_\beta^{DT}(n) = \deg\left(\tau_{a_1}(\gamma_1)\circ\dots\circ\tau_{a_k}(\gamma_k)\left([I_n(X,\beta)]^{vir}]\right)\right)\in\mathbb{Q}.$$

Likewise a similar modification gives descendent PT invariants

$$\langle\tau_{a_1}(\gamma_1)\dots\tau_{a_k}(\gamma_k)\rangle_\beta^{PT}(n)\in\mathbb{Q}.$$

We form generating series

$$\begin{aligned}
\langle \tau_{a_1}(\gamma_1)\dots\tau_{a_k}(\gamma_k)\rangle_\beta^{GW} &:= \sum_{g\in\mathbb{Z}} \langle \tau_{a_1}(\gamma_1)\dots\tau_{a_k}(\gamma_k)\rangle_\beta^{GW}(g)u^{2g-2},\\
\langle \tau_{a_1}(\gamma_1)\dots\tau_{a_k}(\gamma_k)\rangle_\beta^{DT} &:= \sum_{n\in\mathbb{Z}} \langle \tau_{a_1}(\gamma_1)\dots\tau_{a_k}(\gamma_k)\rangle_\beta^{DT}(n)q^{n},\\
\langle \tau_{a_1}(\gamma_1)\dots\tau_{a_k}(\gamma_k)\rangle_\beta^{PT} &:= \sum_{n\in\mathbb{Z}} \langle \tau_{a_1}(\gamma_1)\dots\tau_{a_k}(\gamma_k)\rangle_\beta^{PT}(n)q^{n}.
\end{aligned}$$

The relationship between the decendents in these theories is more complicated an explicit correspondence is conjectured via a universal correspondence matrix. This conjecture [17] has been proven in many cases [18] expressing the GW descendent series as a finite sum of PT descendent series

$$(-iu)^{d_\beta}\langle \tau_{a_1}(\gamma_1)\dots\tau_{a_k}(\gamma_k)\rangle_\beta^{GW} = (-q)^{-d_\beta/2}\sum \langle \tau_{\hat{a}_1}(\hat{\gamma}_1)\dots\tau_{\hat{a}_j}(\hat{\gamma}_j)\rangle_\beta^{PT}$$

under the change of variables $q = -e^{iu}$ where $d_\beta = \int_\beta c_1(X)$. This descedent conjecture specializes to the primary and Calabi–Yau cases.

# 5 Equivariant theory.

From now on we will work with a three dimensional toric variety $X$. In this section we will lift the integrals of the previous section to equivariant cohomology. When working in equivariant cohomology we can define counting invariants for quasi-projective toric three folds.

**Toric Threefolds.** Let $M \equiv \mathbb{Z}^3$ be a rank 3 lattice and $\Delta \subset M$ a convex integral polytope. Let $\mathcal{V}, \mathcal{E}, \mathcal{F}$ be the sets of vertices, edges and faces of $\Delta$. In order that we have a smooth threefold we assume that each vertex is contained in exactly three edges. Also at each vertex $\alpha \in \mathcal{V}$ if we let $g_1, g_2, g_3$ be the primitive vectors along the edges then we insist that they generate all of $M$ as a $\mathbb{Z}$ module. Given such a polytope we define at each vertex an affine neighborhood $U_\alpha$ isomorphic to $\mathbb{C}^3$ with coordinate ring

$$R_v = \mathbb{C}[x^{g_1}, x^{g_2}, x^{g_3}] = \mathbb{C}[x_1, x_2, x_3].$$

If the two vertices $\alpha, \beta \in \mathcal{V}$ are adjacent with charts having coordinate rings $R_v = \mathbb{C}[x^{g_1}, x^{g_2}, x^{g_3}]$ and $R_w = \mathbb{C}[x^{f_1}, x^{f_2}, x^{f_3}]$ then we have

$$f_1 = -g_1, f_2 = m_{\alpha,\beta}g_1 + g_2, f_3 = m'_{\alpha,\beta}g_1 + g_3$$

therefore $R_\alpha[x^{-g_1}] = R_\beta[x^{-f_1}]$ giving a natural gluing of the two charts. Gluing these affine pieces gives the threefold $X$. Each chart contains the torus $\mathbf{T} = \mathrm{Spec}(\mathbb{C}[x_1^{\pm}, x_2^{\pm}, x_3^{\pm}])$ inducing an action on $X$ via that on each co-ordinate patch.

Form this description each vertex corresponds to a torus fixed point $\alpha$, each edge to a torus fixed line between the fixed points $C_{\alpha,\beta}$, and each face corresponds to a torus fixed divisor. Using the torus action we see that every cycle is rationally equivalent to a sum of these invariant cycles.

The intersection of two divisors is either empty or equals a line given exactly by the edge corresponding to the intersection of the two faces. Therefore the Chow ring is generated by the divisor classes $D_j$ for $j \in \mathcal{F}$. To get the self intersections of divisor classes we can use the relations given by the three one dimensional torus actions. If we let $e_i$ be the $i$th coordinate axis and $r_j$ the primitive vector perpendicular to the face $F_j$ then we have the relations for $i = 1, 2, 3$ given by

$$\sum_{j \in \mathcal{F}} \langle e_i, r_j \rangle D_j = 0.$$

Essentially we have proven that the Chow ring of $X$ equals

$$A^*(X) = \mathbb{Z}[D_j : j \in \mathcal{F}]/I_R$$

Where the ideal of relations $I_R$ is given by the obvious quadratic transverse intersection relations and the additional three linear ones above. A quasi-projective toric threefold is just an invariant open subset of a projective one.

**Equivariant integrals.** As in the introduction we would like to lift to equivariant cohomology the integrals

$$\langle \tau_{a_1}(\gamma_1) \dots \tau_{a_k}(\gamma_k) \rangle_\beta^{GW}(g), \langle \tau_{a_1}(\gamma_1) \dots \tau_{a_k}(\gamma_k) \rangle_\beta^{DT}(n), \langle \tau_{a_1}(\gamma_1) \dots \tau_{a_k}(\gamma_k) \rangle_\beta^{PT}(n).$$

Indeed the action of the three dimensional torus $\mathbf{T}$ on the projective toric threefold $X$ lifts to an action on the moduli spaces $\overline{\mathcal{M}}'_{g,k}(X, \beta), I_n(X, \beta)$, $P_n(X, \beta)$ and corresponding actions on all the universal bundles give equivariant characteristic classes as above. If we let $H^*_{\mathbf{T}}(p.t.) = \mathbb{Q}[s_1, s_2, s_3]$ and $H_*^{\mathbf{T}}(\mathcal{M})$ the equivariant homology of our compact moduli space then for $\alpha \in H_*^{\mathbf{T}}(\mathcal{M})$ and $\beta \in H^*_{\mathbf{T}}(\mathcal{M})$ we can evaluate integrals

$$\int_\alpha \beta \in \mathbb{Q}[s_1, s_2, s_3]$$

via push-forward to a point. If the degree of $\alpha$ equals that of $\beta$ we will get a rational number. In this way we can recover the usual integrals in ordinary cohomology above as equivariant integrals. If the degree of $\alpha$ is less than that of $\beta$ we will get an polynomial in $s_1, s_2, s_3$. If the degree of $\alpha$ is greater than that of $\beta$ we will get zero.

Like in the case of a smooth manifold there exists a localization formula for the (virtual) fundamental class. Given $\mathcal{M}$ a DM stack with a good $\mathbf{T}$ action and $\mathbf{T}$ perfect equivariant obstruction theory $E^\bullet$. The obstruction theory has a fixed part $E^{\bullet,f}$ and a moving part $E^{\bullet,m}$ depending on whether or not $\mathbf{T}$ acts trivially on the vector bundles. The fixed part restricts to a perfect obstruction theory on each of the fixed point loci $\mathcal{M}_i \subset \mathcal{M}^{\mathbf{T}}$ and we have associated virtual fundamental classes $[\mathcal{M}_i]^{vir}$.

**Theorem 2.** *([11], c.f. [13]) Let $\mathcal{M}, E^\bullet$ be a DM stack with perfect equivariant obstruction theory as above then*

$$[\mathcal{M}]^{vir} = \sum_i \frac{[\mathcal{M}_i]^{vir}}{e(N^{vir})} \in H_*^{\mathbf{T}}(\mathcal{M}) \otimes \mathbb{Q}(s_1, s_2, s_3)$$

*where $N^{vir}$ equals the $\mathbf{T}$-varrying $E^{\bullet,m}$ so $e(N^{vir}) = e(E^{0,m})/e(E^{1,m})$.*

As long as the right hand side of this localization formula is compact we can define virtual invariants as follows. If the fixed point locus $\mathcal{M}^{\mathbf{T}}$ is proper we can integrate cohomology classes at these fixed points to get a sum of elements in $\mathbb{Q}(s_1, s_2, s_3)$. This allows us to define virtual counts in quasi projective toric threefolds $X$ generalizing everything above:

$$\langle \tau_{a_1}(\gamma_1) \ldots\ldots \tau_{a_k}(\gamma_k) \rangle_\beta^{GW,\mathbf{T}}(g) \in \mathbb{Q}(s_1, s_2, s_3),$$
$$\langle \tau_{a_1}(\gamma_1) \ldots\ldots \tau_{a_k}(\gamma_k) \rangle_\beta^{DT,\mathbf{T}}(n) \in \mathbb{Q}(s_1, s_2, s_3),$$
$$\langle \tau_{a_1}(\gamma_1) \ldots\ldots \tau_{a_k}(\gamma_k) \rangle_\beta^{PT,\mathbf{T}}(n) \in \mathbb{Q}(s_1, s_2, s_3).$$

We have generating series

$$\begin{aligned}
\langle \tau_{a_1}(\gamma_1) \ldots \tau_{a_k}(\gamma_k) \rangle_\beta^{GW,\mathbf{T}} &:= \sum_{g \in \mathbb{Z}} \langle \tau_{a_1}(\gamma_1) \ldots \tau_{a_k}(\gamma_k) \rangle_\beta^{GW,\mathbf{T}}(g) u^{2g-2}, \\
\langle \tau_{a_1}(\gamma_1) \ldots \tau_{a_k}(\gamma_k) \rangle_\beta^{DT,\mathbf{T}} &:= \sum_{n \in \mathbb{Z}} \langle \tau_{a_1}(\gamma_1) \ldots \tau_{a_k}(\gamma_k) \rangle_\beta^{DT,\mathbf{T}}(n) q^n, \\
\langle \tau_{a_1}(\gamma_1) \ldots \tau_{a_k}(\gamma_k) \rangle_\beta^{PT,\mathbf{T}} &:= \sum_{n \in \mathbb{Z}} \langle \tau_{a_1}(\gamma_1) \ldots \tau_{a_k}(\gamma_k) \rangle_\beta^{PT,\mathbf{T}}(n) q^n.
\end{aligned}$$

Similar correspondences are conjectured to hold between these equivariant series [16], [17].

**Fixed loci in the moduli spaces.** First consider the fixed stable maps. Each torus fixed locus can be described combinatorially in terms of a labeled graph $\Gamma$. Any fixed stable map must have all marked points, nodes, contracted components and ramification points sent to $\mathbf{T}$ fixed points. Each non-contracted component of the stable map mapping to some $\mathbf{T}$ fixed line and corresponds to an edge $e$ of $\Gamma$ labeled with its degree $d_e$. The vertices of $\Gamma$ are given by the connected components that are contracted in the inverse image of the set of fixed points of $X$. Each vertex $i$ is labeled by its genus $g(i)$ and by the fixed point $v(i)$ to which it maps. There are a collection of labeled half edges at each vertex corresponding to the marked points on that component. The non-contracted curves along the edges must be rational and totally ramified over the vertices of order $d_e$ by the Hurwitz formula. Letting $\overline{\mathcal{M}}_\Gamma = \prod_i \overline{\mathcal{M}}_{g(i),val(i)}$ we get a map of fixed curves $\overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}'_{g,k}(X,\beta)$. The fixed locus carries some orbifold structure coming from the automorphisms of stable maps. The automorphism group $\mathbf{A}_\Gamma$ fits into the following sort exact sequence

$$1 \to \prod_e \mathbb{Z}/d_e \to \mathbf{A}_\Gamma \to \mathrm{Aut}(\Gamma) \to 1$$

and is given as the semi-direct product of $\mathrm{Aut}(\Gamma)$ acting on $\prod_e \mathbb{Z}/d_e$. So we have an immersion of closed substacks $\overline{\mathcal{M}}_\Gamma/\mathbf{A}_\Gamma \to \overline{\mathcal{M}}'_{g,k}(X,\beta)$ where each $\mathbb{Z}/d_e$ acts trivially on $\overline{\mathcal{M}}_\Gamma$. Given the labeled graph $\Gamma$ we associate to each fixed curve $C_{\alpha,\beta}$ passing through the fixed points $\alpha, \beta$ a partition $\mu_{\alpha,\beta}$ given by the degrees of the components $d_e$ mapping to the curve.

Secondly in the case of ideal sheaves the associated one dimensional embedded sub-scheme must be supported on the fixed rational curves. The above toric charts $U_\alpha$ of $X$ were indexed by $\alpha \in \mathcal{V}$. The intersection $U_{\alpha,\beta}$ has a coordinate ring of the form $\mathbb{C}[x,y][z^\pm]$ with a three dimensional torus $\mathbf{T}$-action. Consequently a torus fixed sub-scheme must be given by a partition $\mu_{\alpha,\beta} \subset \mathbb{Z}^2_{\geq 0}$

$$I_{\alpha,\beta} = I_{\mu_{\alpha,\beta}} \otimes \mathbb{C}[z^\pm] = \langle x^i y^j : (i,j) \notin \mu_{\alpha,\beta} \rangle \otimes \mathbb{C}[z^\pm].$$

If the vertex $\alpha$ is connected to $\beta_1, \beta_2, \beta_3$ the $\mathbf{T}$ fixed ideal restricted to $U_\alpha$ is spanned by monomials and given by a three dimensional partition $\pi_\alpha \in \mathbb{Z}^3_{\geq 0}$

$$I_\alpha = I_{\pi_\alpha} = \langle x^i y^j z^k : (i,j,k) \notin \pi_\alpha \rangle$$

such that $\pi_\alpha$ grows like $\mu_{\alpha,\beta_1}, \mu_{\alpha,\beta_2}, \mu_{\alpha,\beta_3}$ along the three axis. Ultimately there is a discrete collection of fixed points each one given by a collection $\{\pi_\alpha\}_{\alpha \in \mathcal{V}}$ of three dimensional Young diagrams or stacks of boxes at each vertex.

Finally a torus fixed stable pair must be supported on the torus fixed curves and the zero dimensional cokernel of the section must be supported at the fixed points. So on each $U_{\alpha,\beta}$ the fixed pair $(F,s)$ has surjective section $s$ and can be identified with the structure sheaf of a one dimensional subscheme. So exactly as in the ideal sheaf case above, the fixed pair restricted to $U_{\alpha,\beta}$ it is described by a two dimensional partition $\mu_{\alpha,\beta} \subset \mathbb{Z}^2_{\geq 0}$.

Now if $(F,s)$ is a stable pair supported on $C$ with the cokernel of $s$ supported on the support of $\mathcal{O}_C/\mathfrak{m}$. Then such stable pairs can be classified by elements of $\varinjlim \operatorname{Hom}(\mathfrak{m}^r, \mathcal{O}_C)/\mathcal{O}_C$ [19]. In the case of toric fixed points the ideal $\mathfrak{m}$ must be torus fixed so describing fixed points.

Like with stable maps the $\mathbf{T}$-fixed loci in the space of stable pairs are positive dimensional. Each component $\mathbf{Q} = \prod_{\alpha \in \mathcal{V}} \mathcal{Q}_\alpha$ splits as a product of components coming for the description of the fixed pairs at each vertex.

Let us consider the restriction of a fixed pair to a vertex $\alpha$ where the coordinate ring is given by $\mathbb{C}[x_1,x_2,x_3]$ so that $\mathfrak{m} = (x_1,x_2,x_3)$ is the maximal ideal of the origin. We define first the $\mathbb{C}[x_1,x_2,x_3]$-module

$$M_i = \mathbb{C}[x_i^{\pm}] \otimes \mathbb{C}[x_j,x_k]/I_{\mu_{j,k}}$$

then set $M = M_1 \oplus M_2 \oplus M_3$. By the above classification result the configurations we are interested in are finite dimensional submodules of the quotient $M/\langle(1,1,1)\rangle$. Define the cylinders $Cyl_i \subset \mathbb{Z}^3$ as collections of boxes

$$Cyl_i := \{(a,b,c) \in \mathbb{Z}^3_{\geq 0} : x_1^a x_2^b x_3^c \in \mathbb{C}[x_i^{\pm}] \otimes \mathbb{C}[x_j,x_k]/I_{\mu_{j,k}}\}.$$

There are three kinds of boxes in a finite dimensional submodule of $M/\langle(1,1,1)\rangle$

- Type I: $(r_1,r_2,r_3) \in Cyl_i$ and $r_i < 0$,
- Type II: $(r_1,r_2,r_3) \in Cyl_j \cap Cyl_k \setminus Cyl_i$,

- Type III: $(r_1, r_2, r_3) \in Cyl_1 \cap Cyl_2 \cap Cyl_3$.

A box configuration is a finite collection of such boxes $(\lambda_1, \lambda_2, \lambda_3)$ where the boxes of type III are potential labeled by a line in

$$\mathbb{P}\left(\frac{\mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3}{\mathbb{C} \cdot (1,1,1)}\right)$$

or are unlabeled whence they occur with multiplicity two. For the collection to define a module there are simple conditions on the configuration

1. If $w \in \mathrm{I}$ and any of $w - (1,0,0), w - (0,1,0), w - (0,0,1)$ are in the configuration then so is $w$.
2. If $w \in \mathrm{II} \setminus Cyl_i$ and any of $w - (1,0,0), w - (0,1,0), w - (0,0,1)$ are in the configuration and not equal to a box labeled by the line $\mathbb{P}(\mathbb{C}e_i)$ then $w$ is in the configuration.
3. If $w \in \mathrm{III}$ and any of $w - (1,0,0), w - (0,1,0), w - (0,0,1)$ are in the configuration then so is $w$. The box is either unlabeled or labeled by the span of the three previous boxes.

As such collections of labeled box configurations are now parameterized by continuous families $\mathcal{Q}_\alpha$ of products of $\mathbb{P}^1$'s, one for each (unrestricted) path component in the set of labeled type III boxes [20].

To summarize in all three case the fixed loci have a partition $\mu_{\alpha_1,\alpha_2}$ associated to each fixed line $C_{\alpha_1,\alpha_2}$ describing the degree contribution of that line to the class $\beta \in H_2(X, \mathbb{Z})$. Then the fixed points are given by:

- **Stable maps** Set of graphs with one edge for every part of each $\mu_{\alpha,\beta}$ and one vertex for every contracted component in the stable map labeled by, the fixed point $\alpha$ to which it maps, the genus of the contacted curve, and the marked points on that contracted component.
- **Ideal sheaves** Set of box configurations with one plane partition at each vertex $\pi_\alpha \in \mathcal{PP}(\mu_{\alpha,\beta_1}, \mu_{\alpha,\beta_2}, \mu_{\alpha,\beta_3})$ with profile $\mu_{\alpha,\beta_1}, \mu_{\alpha,\beta_2}, \mu_{\alpha,\beta_3}$ along each of the lines $C_{\alpha,\beta_1}, C_{\alpha,\beta_2}, C_{\alpha,\beta_3}$ passing through $\alpha$.
- **Stable pairs** Set of box configurations with one labeled partition at each vertex $\mathcal{Q}_\alpha \in \mathcal{LP}(\mu_{\alpha,\beta_1}, \mu_{\alpha,\beta_2}, \mu_{\alpha,\beta_3})$ containing boxes of type I, II, and III with profile $\mu_{\alpha,\beta_1}, \mu_{\alpha,\beta_2}, \mu_{\alpha,\beta_3}$ along each of the lines $C_{\alpha,\beta_1}, C_{\alpha,\beta_2}, C_{\alpha,\beta_3}$ passing through $\alpha$.

## 6 The equivariant vertex.

We are interested in the virtual counts of curves $\langle \tau_{a_1}(\gamma_1)\dots\tau_{a_k}(\gamma_k)\rangle_\beta^{GW,\mathbf{T}}(g)$, $\langle \tau_{a_1}(\gamma_1)\dots\tau_{a_k}(\gamma_k)\rangle_\beta^{DT,\mathbf{T}}(n)$, and $\langle \tau_{a_1}(\gamma_1)\dots\tau_{a_k}(\gamma_k)\rangle_\beta^{PT,\mathbf{T}}(n)$. In the previous section we saw how these integrals can be reduced to integrals over the fixed point loci in the moduli spaces $\overline{\mathcal{M}}'_{g,k}(X,\beta), I_n(X,\beta)$, and $P_n(X,\beta)$ via the virtual localization formula.

**Ideal sheaves.** Consider the case of ideal sheaves $I_n(X,\beta)$ with fixed point set given by the finite collection of ideals $I$. By the localization formula above we have

$$[I_n(X,\beta)]^{vir} = \sum_{I\in I_n(X,\beta)^{\mathbf{T}}} \frac{[I]^{vir}}{e^{\mathbf{T}}(N^{vir})}$$

by definition $e^{\mathbf{T}}(N^{vir})^{-1} = e^{\mathbf{T}}(\mathrm{Ext}_0^{2,m}(I,I))/e^{\mathbf{T}}(\mathrm{Ext}_0^{1,m}(I,I))$. On a toric manifold $H^i(\mathcal{O}_X) = 0$ for $i = 1,2$ so all extensions are traceless. It can be shown that $\mathrm{Ext}^1(I,I)$ and $\mathrm{Ext}^2(I,I)$ have no trivial $\mathbf{T}$-reps so we have $[I]^{vir} = [I]$ and $e^{\mathbf{T}}(N^{vir})^{-1} = e^{\mathbf{T}}(\mathrm{Ext}^2(I,I))/e^{\mathbf{T}}(\mathrm{Ext}^1(I,I))$ [15, Lemmas 6 and 8].

We now need to know how the homology operation $\tau_k(\gamma)$ acts on a point $[I]$ this is given by

$$\pi_{I*}\left(\pi_X^*(\gamma)\cdot \mathrm{ch}^{\mathbf{T}}_{2+k}(\mathbb{I})\cap \sum_{\alpha\in\mathcal{V}} \frac{[I\times\alpha]}{e^{\mathbf{T}}(TX_\alpha)}\right) = [I]\cdot \sum_{\alpha\in\mathcal{V}} \frac{\gamma|_\alpha\cdot \mathrm{ch}^{\mathbf{T}}_{2+k}(I|_\alpha)}{e^{\mathbf{T}}(TX_\alpha)}.$$

Putting this together the series $\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_l}(\gamma_l)\rangle_\beta^{DT,\mathbf{T}}$ can be computed as

$$\sum_{n\geq 0} q^n \sum_{I\in I_n(X,\beta)^{\mathbf{T}}} \frac{e^{\mathbf{T}}(\mathrm{Ext}^2(I,I))}{e^{\mathbf{T}}(\mathrm{Ext}^1(I,I))}\cdot \prod_{i=1}^{l}\sum_{\alpha\in\mathcal{V}} \frac{\gamma_i|_\alpha\cdot \mathrm{ch}^{\mathbf{T}}_{2+k_i}(I|_\alpha)}{e^{\mathbf{T}}(TX|_\alpha)}$$

all invariants can be computed by knowing the $\mathbf{T}$-representations of the above vector spaces.

First let us consider the calculation of the Chern character. Recall that each fixed ideal is given by the combinatorial data of a three dimensional partition $\pi_\alpha$ at each vertex $\alpha$. Let $R_\alpha = \mathbb{C}[x_1, x_2, x_3] \cong \Gamma(U_\alpha)$ be the coordinate ring of the local chart at $\alpha$ (with standard torus action) and $I_\alpha = I|_{U_\alpha} \subset R$ the restriction of our fixed ideal. Now there exist an equivariant resolution of $I_\alpha$,

$$0 \to F_s \to \cdots \to F_2 \to F_2 \to I_\alpha \to 0$$

where $F_i = \bigoplus_j R(d_{ij})$ with $d_{ij} \in \mathbb{Z}^3$ and so the Poincare polynomial of $I_\alpha$ equals,

$$P_\alpha = \sum_{i,j} (-1)^i t^{d_{ij}}.$$

This can alternatively be computed directly from the $\mathbf{T}$-character of the quotient representation $R/I_\alpha$ namely

$$Q_\alpha = \sum_{(i,j,k)\in\pi_\alpha} t^{(i,j,k)} = \frac{1 - P_\alpha(t_1, t_2, t_3)}{(1-t_1)(1-t_2)(1-t_3)}.$$

The total equivariant Chern character is then

$$\mathrm{ch}^{\mathbf{T}}(I|_\alpha) = 1 - (1-e^{s_1})(1-e^{s_2})(1-e^{s_3}) \sum_{(i,j,k)\in\pi_\alpha} e^{i\cdot s_1 + j\cdot s_2 + k\cdot s_3} \in \mathbb{Q}[|s_1, s_2, s_3|].$$

Given another fixed point with nonstandard torus action we take $s_1^\alpha, s_2^\alpha, s_3^\alpha$ the three tangent weights of the tours at $\alpha$ in the above formula replacing $s_1, s_2, s_3$.

Next let us consider the contribution from the virtual normal bundle. By the local to global spectral sequence we can rewrite the virtual $\mathbf{T}$-representation $\mathrm{Ext}^1(I,I) - \mathrm{Ext}^2(I,I) = \chi(\mathcal{O}_X, \mathcal{O}_X) - \chi(I,I)$ as

$$\begin{aligned} \chi(I,I) &= \sum_{i,j=1}^{3} (-1)^{i+j} H^i(\mathcal{E}xt^j(I,I)) \\ &= \sum_{i,j=1}^{3} (-1)^{i+j} \mathfrak{C}^i(\mathcal{E}xt^j(I,I)). \end{aligned}$$

In the second line we have replaced the cohomology groups $H^i$ with the Cech complex for the covering $\mathfrak{U} = \{U_\alpha\}_{\alpha\in\mathcal{V}}$ these representations are infinite dimensional but as graded vector spaces each weight space is finites dimensional and we have complete cancelation for all but finitely many weights. Now on triple intersections we have $\mathcal{O}_X = I$ so we only need to consider the single and double intersections in the above difference of Euler characteristics. So the above representation is the difference of the vertex contribution

$$\bigoplus_{\alpha} \left( \Gamma(U_\alpha) - \sum_i (-1)^i \Gamma(U_\alpha, \mathcal{E}xt^j(I, I)) \right)$$

minus the edge contribution

$$\bigoplus_{\alpha,\beta} \left( \Gamma(U_{\alpha,\beta}) - \sum_i (-1)^i \Gamma(U_{\alpha,\beta}, \mathcal{E}xt^j(I, I)) \right).$$

Using our $\mathbf{T}$-equivariant resolution from above again we have

$$\chi(I_\alpha, I_\alpha) = \sum_{i,j,k,l} (-1)^{i+k} \mathrm{Hom}_R(R(d_{ij}), R(d_{kl}))$$

and the $\mathbf{T}$-character coming from the vertex $\alpha$ is

$$\frac{1 - P_\alpha(t_1, t_2, t_3)\overline{P}_\alpha(t_1, t_2, t_3)}{(1 - t_1)(1 - t_2)(1 - t_3)}$$

where we introduce notation $\overline{G}(t_1, t_2, t_3) = G(t_1^{-1}, t_2^{-1}, t_3^{-1})$. Again we can rewrite this simply interms of the quotient representation without reference to the Poincare polynomial of the resolution as

$$\mathrm{tr}_{\chi(R_\alpha, R_\alpha) - \chi(I_\alpha, I_\alpha)} = Q_\alpha - \frac{\overline{Q}_\alpha}{t_1 t_2 t_3} + Q_\alpha \overline{Q}_\alpha \frac{(1 - t_1)(1 - t_2)(1 - t_3)}{t_1 t_2 t_3}.$$

The edge contribution can be computed similarly. We let $R_{\alpha,\beta} = \mathbb{C}[x_2, x_3] \otimes \mathbf{C}[x_1^{\pm}] \cong \Gamma(U_{\alpha,\beta})$ be the co-ordinate ring of the local chart (with standard torus action) and $I_{\alpha,\beta} = I|_{U_{\alpha,\beta}} \subset R_{\alpha,\beta}$ the restriction of our fixed ideal. Define the formal delta function $\delta(t_1) = \sum_{k \in \mathbf{Z}} t_1^k$ and $Q_{\alpha,\beta} = \sum_{(i,j) \in \mu_{\alpha,\beta}} t_2^i t_3^j$ where $\mu_{\alpha,\beta}$ is the profile along the fixed line $C_{\alpha,\beta}$. The $\mathbf{T}$-character that we add from each edge is

$$\delta(t_1) \left( -Q_{\alpha,\beta} - \frac{\overline{Q}_{\alpha,\beta}}{t_2 t_3} + Q_{\alpha,\beta} \overline{Q}_{\alpha,\beta} \frac{(1 - t_2)(1 - t_3)}{t_2 t_3} \right).$$

If the partitions $\mu_{\alpha,\beta}$ are non-empty then the individual expressions for the vertex and edge contributions will be infinite. By redistributing some of the edge terms into the vertex terms we can rewrite all contributions as finite sums. Let

$$F_{\alpha,\beta} = \left( -Q_{\alpha,\beta} - \frac{\overline{Q}_{\alpha,\beta}}{t_2 t_3} + Q_{\alpha,\beta}\overline{Q}_{\alpha,\beta} \frac{(1-t_2)(1-t_3)}{t_2 t_3} \right)$$

and define the new vertex contribution by

$$\mathsf{V}_\alpha = \mathrm{tr}_{\chi(R_\alpha,R_\alpha)-\chi(I_\alpha,I_\alpha)} + \sum_{i=1}^{3} \frac{F_{\alpha,\beta_i}(t_{i'}, t_{i''})}{1-t_i}$$

where $C_{\alpha,\beta_1}, C_{\alpha,\beta_2}, C_{\alpha,\beta_3}$ are the three invariant line through $\alpha$ and $\{t_i, t_{i'}, t_{i''}\} = \{t_1, t_2, t_3\}$ in each summand. After adding these three terms to the vertex contribution the terms remaining in the edge sums are

$$\mathsf{E}_{\alpha,\beta} = t_1^{-1} \frac{F_{\alpha,\beta}}{(1-t_1^{-1})} - \frac{F_{\alpha,\beta}(t_2 t_1^{-m_{\alpha,\beta}}, t_3 t_1^{-m'_{\alpha,\beta}})}{(1-t_1^{-1})}.$$

It is not hard to show that all of $\mathsf{V}_\alpha$ and $\mathsf{E}_{\alpha,\beta}$ are now Laurent polynomials. Note that the above expression was valid at a torus fixed point where we had a standard torus action with weights $s_1, s_2, s_3$. At other vertices $\alpha$ we apply the same formula for the tangent weights $s_1^\alpha, s_2^\alpha, s_3^\alpha$. Now we have rewritten the contribution form the virtual normal bundle as a finite combinatorial expression

$$\frac{1}{e^{\mathbf{T}}(N^{vir})} = \frac{e^{\mathbf{T}}(\mathrm{Ext}^2(I,I))}{e^{\mathbf{T}}(\mathrm{Ext}^1(I,I))} = \prod_{\alpha} e^{\mathbf{T}}(-\mathsf{V}_\alpha) \prod_{\alpha,\beta} e^{\mathbf{T}}(-\mathsf{E}_{\alpha,\beta}) \in \mathbb{Q}(s_1, s_2, s_3).$$

The equivariant vertex measure is a measure on the set $\mathcal{PP}(\mu_1, \mu_2, \mu_3)$ of plane partition with asymptotics $\mu_1, \mu_2, \mu_3$ given by

$$\begin{aligned} \mathsf{w} \ &: \ \mathcal{PP}(\mu_1,\mu_2,\mu_3) \to \mathbb{Q}(s_1,s_2,s_3) \\ &: \ \pi \mapsto \prod_{i,j,k \in \mathbb{Z}} (i \cdot s_1 + j \cdot s_1 + k \cdot s_3)^{-a_{i,j,k}} \end{aligned}$$

where $\mathsf{V}_\alpha = \sum_{i,j,k} a_{i,j,k} t_1^i t_2^j t_3^k$ is the vertex contribution coming from an ideal $I_\alpha \subset R_\alpha = \mathbb{C}[x_1, x_2, x_3]$ is the ideal corresponding to $\pi$ in a neighborhood with the standard $\mathbf{T}$-action. The Chern characters $\mathrm{ch}_{2+r}(I|_\alpha)$ can also be used to define random variables

$$\begin{aligned} \tau_r &: \mathcal{PP}(\mu_1, \mu_2, \mu_3) \to \mathbb{Q}[s_1, s_2, s_3] \\ &: \pi \mapsto \mathrm{ch}^{\mathbf{T}}_{2+r}(I|_\alpha) \end{aligned}$$

where $I_\alpha \subset R_\alpha = \mathbf{C}[x_1, x_2, x_3]$ is the ideal corresponding to $\pi$. This too has a simple combinatorial description in terms of $\pi$ as we saw above. Using this new notation the descendent series $\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_l}(\gamma_l) \rangle^{DT,\mathbf{T}}_\beta$ equals

$$\sum_{\mu_{\alpha\beta}} \prod_{\alpha\beta} e^{\mathbf{T}}(-\mathsf{E}_{\alpha,\beta}) q^{|\mu_{\alpha,\beta}|} \prod_{\alpha} \sum_{\pi_\alpha} q^{|\pi_\alpha|} \mathsf{w}(\pi_\alpha) \cdot \prod_{i=1}^{l} \sum_{\alpha \in \mathcal{V}} \frac{\gamma_i|_\alpha \cdot \tau_{k_i}(\pi_\alpha)}{s_1^\alpha s_2^\alpha s_3^\alpha}$$

Where we have a finite sum over all edge degrees $\mu_{\alpha,\beta}$ representing the class $\beta \in H_2(X, \mathbb{Z})$ and the volumes $|\pi|$ and $|\mu|$ are given by

$$\begin{aligned} |\pi| &= \#(\pi \cap [0, N]^3) - \sum_{i=1}^{3} (N+1)(|\mu_i|) \text{ for } N \gg 0, \\ |\mu| &= \sum_{(i,j) \in \mu} (-m_{\alpha,\beta} - m'_{\alpha,\beta} + 1). \end{aligned}$$

By linearity we see that all the invariants $\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_l}(\gamma_l) \rangle^{DT,\mathbf{T}}_\beta(n)$ for the toric threefold are now expressible as simple polynomials in the universal series called the Donaldson–Thomas 3-leg equivariant descendent vertex:

$$\mathsf{W}^{DT}_{\mu_1,\mu_2,\mu_3}(\tau_{k_1} \cdots \tau_{k_r}) = \sum_{\pi \in \mathcal{PP}(\mu_1,\mu_2,\mu_3)} \tau_{k_1}(\pi) \cdots \tau_{k_r}(\pi) \cdot \mathsf{w}(\pi) \cdot q^{|\pi|}.$$

**Stable Pairs.** Next consider the case of ideal sheaves $P_n(X, \beta)$ with fixed point loci $\mathbf{Q} = \prod_{\alpha \in \mathcal{V}} \mathcal{Q}_\alpha$ given by the moduli spaces $\mathcal{Q}_\alpha$ parameterising box configurations with a given labeling at each vertex $\alpha$. By the virtual localization formula above we have

$$[P_n(X, \beta)]^{vir} = \sum_{\mathbf{Q} \in P_n(X,\beta)^{\mathbf{T}}} \frac{[\mathbf{Q}]^{vir}}{e^{\mathbf{T}}(N^{vir}_{\mathbf{Q}})}.$$

The Euler class of the virtual normal bundle is given by the $\mathbf{T}$-varrying part of the obstruction theory $E^\bullet \to \mathbb{L}_{P_n(X,\beta)}$, i.e. $e^{\mathbf{T}}(N^{vir})^{-1} = \frac{e^{\mathbf{T}}(E^{-1,m\vee})}{e^{\mathbf{T}}(E^{0,m\vee})}$. The virtual fundamental class of $\mathbf{Q}$ is given by the restriction of the $\mathbf{T}$-fixed part of the obstruction theory. Above we mentioned that the labels on type III boxes contribute $\mathbb{P}^1$ factors so that $\mathcal{Q}_\alpha$ is a product of $\mathbf{P}^1$'s. In particular it is true that $\mathbf{Q}$ is smooth [20, Proposition 4] and hence the cotangent complex $\mathbb{L}_{\mathbf{Q}} = \Omega_{\mathbf{Q}}$ concentrated in degree 0. Then the fixed part of the perfect obstruction theory gives an exact sequence

$$E^{-1,f} \to E^{0,f} \to \Omega_{\mathbf{Q}} \to 0$$

so that if $K$ is the kernel on the left then $[\mathbf{Q}]^{vir} = e(K^\vee) \cap [\mathbf{Q}]$ or expressing this in terms of the classes of the other bundles $[\mathbf{Q}]^{vir} = e(T_{\mathbf{Q}}) \cdot \frac{e(E^{-1,f\vee})}{e(E^{0,f\vee})} \cap [\mathbf{Q}] \in H_*(\mathbf{Q})$ combining this the virtual localization formula becomes

$$[P_n(X,\beta)]^{vir} = \sum_{\mathbf{Q} \in P_n(X,\beta)^{\mathbf{T}}} e(T_{\mathbf{Q}}) \cdot \frac{e((E^{-1})^\vee|_{\mathbf{Q}})}{e((E^0)^\vee|_{\mathbf{Q}})} \cap [\mathbf{Q}]$$

in the ring $H_*^{\mathbf{T}}(P_n(X,\beta)) \otimes \mathbb{C}(s_1, s_2, s_3)$. We now need to know how the homology operation $\tau_k(\gamma)$ acts on each of the components $\mathbf{Q}$ this is given by

$$\pi_{P*}\left(\pi_X^*(\gamma) \cdot \mathrm{ch}_{2+k}^{\mathbf{T}}(\mathbb{F}) \cap \sum_{\alpha \in \mathcal{V}} \frac{[\mathbf{Q} \times \{\alpha\}]}{e^{\mathbf{T}}(TX_\alpha)}\right) = \sum_{\alpha \in \mathcal{V}} \frac{\gamma|_\alpha \cdot \mathrm{ch}_{2+k}^{\mathbf{T}}(\mathbb{F}|_\alpha) \cap [\mathbf{Q}]}{e^{\mathbf{T}}(TX_\alpha)}.$$

Putting this all together the series $\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_l}(\gamma_l) \rangle_\beta^{PT,\mathbf{T}}$ can be computed as

$$\sum_{n \geq 0} q^n \sum_{\mathbf{Q} \in P_n(X,\beta)^{\mathbf{T}}} \int_{\mathbf{Q}} e^{\mathbf{T}}(T_{\mathbf{Q}}) \cdot \frac{e^{\mathbf{T}}((E^{-1})^\vee|_{\mathbf{Q}})}{e^{\mathbf{T}}((E^0)^\vee|_{\mathbf{Q}})} \cdot \prod_{i=1}^{l} \sum_{\alpha \in \mathcal{V}} \frac{\gamma_i|_\alpha \cdot \mathrm{ch}_{2+k_i}^{\mathbf{T}}(\mathbb{F}|_\alpha)}{e^{\mathbf{T}}(TX|_\alpha)}.$$

The determination of these series is similar to the ideal sheaves calculation above. First we analyse the classes coming from the Chern character at the vertex $\alpha$. Let us assume that at $\alpha$ the coordinate ring is given by $R = \mathbb{C}[x_1, x_2, x_3] \cong \Gamma(U_\alpha)$ with the standard $\mathbf{T}$ action. Let $\mathbb{I}_\alpha^\bullet$ denote the universal complex restriced to $\mathbf{Q} \times U_\alpha$. Take a $\mathbf{T}$-equivariant free resolution

$$0 \to \mathcal{F}_s \to \cdots \to \mathcal{F}_1 \to \mathbb{I}_\alpha^\bullet$$

where $\mathcal{F}_i = \bigoplus_j \mathcal{L}_{ij} \otimes R(d_{ij})$ with $d_{ij} \in \mathbb{Z}^3$ and $\mathcal{L}_{ij} \in \mathrm{Pic}(\mathcal{Q}_\alpha)$. The complex then has a Poincare polynomial

$$P_\alpha = \sum_{i,j} (-1)^i [\mathcal{L}_{ij}] \otimes t^{d_{ij}} \in K(\mathcal{Q}_\alpha) \otimes_{\mathbb{Z}} \mathbb{Z}[t_1^\pm, t_2^\pm, t_3^\pm].$$

If we denote by $\mathsf{F}_\alpha$ the $\mathbf{T}$-character of $F|_{U_\alpha}$ then from the short exact sequence

$$0 \to \mathcal{O}_C|_{U_\alpha} \to F|_{U_\alpha} \to Q|_{U_\alpha} \to 0$$

we can compute $\mathsf{F}_\alpha$ from the two modules on the sides. In particular we have one $\mathbf{T}$-weight for every monomial in $\mathcal{O}_C|_{U_\alpha}$ and one for every box in the labeled configuration describing $Q|_{U_\alpha}$. The contributions form $Q|_{U_\alpha}$ are, every type I or type II box contributes simply with its weight, an unlabeled type III box contributes twice with its weight, a labeled type III contributes once, and finally for every (unrestricted) component in the type III boxes we tensor with the $K$-class of the associated $\mathcal{O}_{\mathbb{P}^1}(-1)$.

This gives a direct method to compute the Poincare polynomial

$$\mathsf{F}_\alpha = \frac{1 + P_\alpha}{(1-t_1)(1-t_2)(1-t_3)}.$$

Resulting in an explicit formula for the Chern character similar to the case of ideal sheaves

$$\mathrm{ch}^{\mathbf{T}}_{2+k}(\mathbb{F}|_{\mathbf{Q}\times\{\alpha\}}) = \mathrm{ch}^{\mathbf{T}}_{2+k}(\mathsf{F}_\alpha(1-t_1)(1-t_2)(1-t_3)) \in H^*(\mathbf{Q}) \otimes \mathbb{C}(s_1, s_2, s_3).$$

In order to calculate the virtual normal bundle term we again use the local to global principal. The virtual normal bundle again equals the difference of two Euler characteristics

$$\mathrm{Ext}^1(\mathbb{I}^\bullet, \mathbb{I}^\bullet) - \mathrm{Ext}^2(\mathbb{I}^\bullet, \mathbb{I}^\bullet) \in K(\mathbf{Q}) \otimes_{\mathbb{Z}} \mathbb{Z}[t_1^\pm, t_2^\pm, t_3^\pm].$$

As before we replace these finite rank bundles over $\mathbf{Q}$, using the local to global spectral sequence, by infinite rank bundles with fibers given by $\mathbf{T}$-representations

$$\Gamma(U_\alpha) - \sum_i (-1)^i \mathrm{Ext}^i(I^\bullet_\alpha, I^\bullet_\alpha)$$

$$\Gamma(U_{\alpha\beta}) - \sum_i (-1)^i \mathrm{Ext}^i(I^\bullet_{\alpha\beta}, I^\bullet_{\alpha\beta})$$

over the fixed pair $I^\bullet \in D^b(\mathbf{Q})$. Now we can compute the $K$-classes of these bundles at the vertcies and edges. The contribution from the vertex, where we have the standard $\mathbf{T}$-action, is given by

$$\mathrm{tr}_{R-\chi(\mathbb{I}^\bullet_\alpha,\mathbb{I}^\bullet_\alpha)} = \mathsf{F}_\alpha - \frac{\overline{\mathsf{F}}_\alpha}{t_1t_2t_3} + \mathsf{F}_\alpha\overline{\mathsf{F}}_\alpha\frac{(1-t_1)(1-t_2)(1-t_3)}{t_1t_2t_3}.$$

Now the $K$-theory operation $A \mapsto \overline{A}$ must be properly understood as taking the dual of the relevant bundle on $\mathbf{Q}$.

As mentioned earlier the edge contribution is identical to the case of ideal sheaves. In the same way as with ideal sheaves both the edge contribution and the vertex contribution may be infinite. We redistribute the terms, as with ideal sheaves, adding the same series

$$\mathsf{V}_\alpha = \mathrm{tr}_{R-\chi(\mathbb{I}^\bullet_\alpha,\mathbb{I}^\bullet_\alpha)} + \sum_{i=1}^{3}\frac{F_{\alpha\beta_i}(t_i',t_i'')}{1-t_i}.$$

where $C_{\alpha\beta_1}, C_{\alpha\beta_2}, C_{\alpha\beta_3}$ are the three invariant lines through $\alpha$ and $\{t_i, t_i', t_i''\} = \{t_1, t_2, t_3\}$. The terms $\mathsf{E}_{\alpha\beta}$ are the same as in the ideal sheaves case. The virtual normal bundle is now given by a product over contributions form each vertex and edge

$$e^{\mathbf{T}}(N^{vir})^{-1} = \prod_{\alpha\in\mathcal{V}} e^{\mathbf{T}}(T_{\mathcal{Q}})\cdot e^{\mathbf{T}}(-\mathsf{V}_\alpha)\cdot\prod_{\alpha,\beta} e^{\mathbf{T}}(-\mathsf{E}_{\alpha\beta}) \in H^*(\mathbf{Q})\otimes\mathbb{Q}(s_1,s_2,s_3).$$

At a point $\alpha$ where the torus action is standard we define the descendent weight $\mathsf{w}(\tau_{k_1}\cdots\tau_{k_r})_{\mu_1,\mu_2,\mu_3}(\mathcal{Q}_\alpha)$ of the fixed component $\mathcal{Q}_\alpha$ as

$$\int_{\mathcal{Q}_\alpha} e^{\mathbf{T}}(T_{\mathcal{Q}_\alpha})\cdot e^{\mathbf{T}}(-\mathsf{V}_\alpha)\prod_{i=1}^{r}\mathrm{ch}^{\mathbf{T}}_{2+k_i}(\mathsf{F}_\alpha(1-t_1)(1-t_2)(1-t_3)) \in \mathbb{Q}(s_1,s_2,s_3).$$

Let $l(\mathcal{Q}_\alpha)$ be the number of boxes in the labeled configuration, i.e. the length of the cokernel. Then let $|\mathcal{Q}_\alpha|$ be the renormalized number of boxes in module $\mathcal{O}_C(U_\alpha)$ as in the ideal case above. By linearity we again see that all the invariants $\langle\tau_{k_1}\cdots\tau_{k_r}\rangle^{PT,\mathbf{T}}_\beta(n)$ for the toric threefold are expressible as an identical polynomial only now in the Pandharipande–Thomas 3-leg equivariant descendent vertex:

$$\mathsf{W}^{PT}_{\mu_1,\mu_2,\mu_3}(\tau_{k_1}\cdots\tau_{k_r}) = \sum_{\mathcal{Q}\in\mathcal{LP}(\mu_1,\mu_2,\mu_3)} q^{|\mathcal{Q}|+l(\mathcal{Q})}\mathsf{w}(\tau_{k_1}\cdots\tau_{k_r})_{\mu_1,\mu_2,\mu_3}(\mathcal{Q}).$$

**Stable Maps.** Finally in the case of stable maps the torus fixed loci in our moduli spaces were indexed by labeled graphs $\Gamma$ with each component given by a quotient $\overline{\mathcal{M}}_\Gamma/\mathbf{A}_\Gamma$ where $\overline{\mathcal{M}}_\Gamma = \prod_{v\in\Gamma}\overline{\mathcal{M}}_{g(v),\mathrm{val}(v)}$. So that each vertex corresponds to a contracted component $C_v$ and each edge corresponds to a rational curve $C_e$ mapping with degree $d_e$. It is more convenient to compute the residue on the space $\overline{\mathcal{M}}_\Gamma$ and then divide the result by the order of $\mathbf{A}_\Gamma$. By virtual localization we have

$$\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_l}(\gamma_l)\rangle_\beta^{GW,\mathbf{T}} = \sum_\Gamma \frac{u^{2g(\Gamma)-2}}{|\mathbf{A}_\Gamma|}\int_{[\overline{\mathcal{M}}_\Gamma]^{vir}} \frac{\prod_{i=1}^l ev_i^*(\gamma_i)\psi^{a_i}}{e^{\mathbf{T}}(N_\Gamma^{vir})}.$$

where we sum over all graphs $\Gamma$ with edge degrees $d_e$ describing the class $\beta \in H_2(X,\mathbb{Z})$. Again we will see that this series can be rewritten in terms of universal series at each of the vertices of $\Gamma$ together with some edge terms.

A flag in the graph $\Gamma$ is defined to be an edge vertex pair $F = (e,v)$ then $w_F$ equals the weight of the tangent space to the embedded rational curve at the fixed point associated to $v$ denied by the factor $d_e$. At the vertex $v \in \Gamma$ mapping to the fixed point $\alpha(v) \in \mathcal{V}$ we denote the three tangent weights by $s_1^\alpha, s_2^\alpha, s_3^\alpha$. To compute the virtual normal bundle we examine the moving part of the perfect obstruction theory in the long exact sequence

$$0 \longrightarrow \mathrm{Aut}(C,p_1,\ldots,p_k)$$

$$\longrightarrow H^0(f^*TX) \longrightarrow \mathcal{T}_1(\overline{\mathcal{M}}_{g,n}(X,\beta)) \longrightarrow \mathrm{Def}(C,p_1,\ldots,p_k)$$

$$\longrightarrow H^1(f^*TX) \longrightarrow \mathcal{T}_2(\overline{\mathcal{M}}_{g,n}(X,\beta)) \longrightarrow 0$$

so we have

$$\frac{1}{e^{\mathbf{T}}(N^{vir}_{\overline{\mathcal{M}}_\Gamma})} = \frac{e^{\mathbf{T}}(H^1(f^*TX)^m)e^{\mathbf{T}}(\mathrm{Aut}(C,p_1,\ldots,p_k)^m)}{e^{\mathbf{T}}(H^0(f^*TX)^m)e^{\mathbf{T}}(\mathrm{Def}(C,p_1,\ldots,p_k)^m)} \in H^*_{\mathbf{T}}(\overline{\mathcal{M}}_\Gamma).$$

Automorphisms of the curve: First consider the automorphisms of a (prestable) fixed curve $(C,p_1,\ldots,p_k)$. The only components with potential automorphisms are those with non-contracted components. Each such rational curve has a one dimensional family of automorphism fixing 0 and $\infty$.

When one of the vertices is a non-special point we have an additional one dimensional family of automorphisms coming from moving this point. The first collection of automorphisms are canceled by deformations in $H^0(f^*(TX))$ however the automorphisms at the non-special points give a contribution

$$\prod_{\mathrm{val}(F)=1} w_F$$

since at these points the space of deformations is described by the tangent space.

Deformations of the curve: There are two sources of deformations those coming from deforming the smooth contracted curves with marked points and those coming from deforming the nodes. The first collection is just the tangent bundle to $\overline{\mathcal{M}}_\Gamma$ coming from contracting curves since these are invariant such deformations are all $\mathbf{T}$-fixed. The deformation space at a node is given by the tensor product of the two tangent spaces at the curves. If the node is the intersection of a contracted and non-contracted component we have a one dimensional family with equivariant Euler characterisctic $w_F - \psi_F$ giving a total contribution from these notes

$$\prod_{\mathrm{Flags}(\Gamma)} (w_F - \psi_F).$$

Whereas at nodes with two non-contracted components meet we have a total contribution

$$\prod_{\mathrm{val}(F)=2} (w_{F_1} + w_{F_2})$$

over the collection of such nodes.

Deformations/obstructions of maps: Consider the normalization exact sequence

$$0 \to \mathcal{O}_C \to \bigoplus_{\mathrm{Verticies}(\Gamma)} \mathcal{O}_{C_v} \oplus \bigoplus_{\mathrm{Edges}(\Gamma)} \mathcal{O}_{C_e} \to \bigoplus_{\mathrm{Flags}(\Gamma)} \mathcal{O}_{x_F} \to 0.$$

From the associated long exact sequence we have the relation in equivariant $K$-theory

$$\begin{aligned} H^0(C, f^*TX) - H^1(C, f^*TX) &= \bigoplus_{\mathrm{Verticies}(\Gamma)} TX_{\alpha(v)} + \bigoplus_{\mathrm{Edges}(\Gamma)} H^0(C_e, f^*TX) \\ &\quad - \bigoplus_{\mathrm{Flags}(\Gamma)} TX_{\alpha(F)} - \bigoplus_{\mathrm{Verticies}(\Gamma)} H^1(C_v, f^*TX). \end{aligned}$$

Since $f$ contracts the curve $C_v$ to the point $\alpha(v)$ the bundle becomes

$$H^1(C_v, f^*TX) = H^1(C_v, \mathcal{O}_{C_v}) \otimes TX_{\alpha(v)}.$$

We get three copies of the dual Hodge bundle $\mathbb{E}^\vee = H^1(C_v, \mathcal{O}_{C_v})$ at the vertex $v \in \Gamma$ mapping to the fixed point $\alpha(v)$. So the space $H^1(C_v, f^*TX)$ contributes a factor

$$e^{\mathbf{T}}(\mathbb{E}^\vee \otimes TX_{\alpha(v)}) = \prod_{i=1}^{3} c(\mathbb{E}^\vee)((s_i^{\alpha(v)})^{-1}) \cdot (s_i^{\alpha(v)})^{g(v)}.$$

Where here $c(V)(t)$ is the Chern polynomial of the vector bundle $V$. Putting this all together gives

$$\begin{aligned}
\frac{1}{e^{\mathbf{T}}(N^{vir}_{\overline{\mathcal{M}}_\Gamma})} \quad = \quad & \prod_{\mathrm{val}(v)=1} w_F \cdot \prod_{\mathrm{val}(v)=2} \frac{1}{w_{F_1} + w_{F_2}} \\
& \cdot \prod_{\mathrm{Edges}(\Gamma)} e^{\mathbf{T}}(H^0(C_e, f^*TX)) \\
& \cdot \prod_{\mathrm{Flags}(\Gamma)} \frac{e^{\mathbf{T}}(TX_{\alpha(F)})}{w_F - \psi_F} \\
& \cdot \prod_{\mathrm{Verticies}(\Gamma)} \prod_{i=1}^{3} c(\mathbb{E}^\vee)((s_i^{\alpha(v)})^{-1}) \cdot (s_i^{\alpha(v)})^{g(v)-1}.
\end{aligned}$$

The curves $C_e$ are rigid and so the bundle $H^0(C_e, f^*TX)$ is constant and hence the first two lines above give a simple rational function in the equivariant weights. Again the edge terms above are independent of the genus at each vertex. We have actually now computed all the weights in the tangent and obstruction spaces $\mathcal{T}_1, \mathcal{T}_2$ it follows that the fixed part of $\mathcal{T}_2$ equals zero so that the virtual fundamental class of $\overline{\mathcal{M}}_\Gamma$ is just the usual fundamental class. We define the equivariant descedent Gromov–Witten vertex $\mathsf{W}^{GW}_{\mu_1,\mu_2,\mu_3}(\tau_{k_1} \cdots \tau_{k_r})$:

$$\sum_{g \geq 0} u^{2g-2} \int_{\overline{\mathcal{M}}_{g,r_4}} \prod_{j=1}^{k} \psi_j^{a_j} \prod_{i=1}^{3} \prod_{j=1}^{l(\mu_i)} \frac{s_i}{s_i - \psi_{r_i+j}} c(\mathbb{E}^\vee)((s_i)^{-1}) \cdot (s_i)^{g(v)-1}$$

where $r_i = k + \sum_{j=1}^{i-1} l(\mu_j)$.

Finally all the Gromov–Witten invariants of a toric threefold $X$ such as $\langle\tau_{a_1}(\gamma_1)\cdots\tau_{a_k}(\gamma_k)\rangle_\beta^{GW,\mathbf{T}}$ can again be written as polynomials in these universal series of triple hodge integrals.

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