Cycles on $\mathcal{A}_g$

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joint with

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Photo: Steve Arrington (unsplash)

Compactifications

$$
\begin{tikzcd}
\bar{\mathcal{A}}_g^{\text{Alexeev}} & & \bar{\mathcal{M}}_g \arrow[ll, bend right, "\text{Extended Torelli}"'] \\
\mathcal{A}_g \arrow[u, phantom, "\cup"] & & \mathcal{M}_g \arrow[u, phantom, "\subset"] \arrow[ll, bend right, "\text{Torelli}"'] \\
& Sp(2g, \mathbb{Z}) \arrow[ul, Rightarrow] & \mathrm{Map}(g) \arrow[l, "\text{hom}"'] \arrow[u, Rightarrow] \\
& \mathcal{K}3(g) \arrow[d, Rightarrow] & \\
& \mathcal{M}_g^{K3} \arrow[uuur, bend right, "\text{Correspondence with linear sections as studied by Mukai}"'] &
\end{tikzcd}
$$

Deligne-Mumford

moduli of ppAV

moduli of smooth curves

moduli spaces as classifying spaces

Groups

moduli of quasipolarized K3 surfaces

$\cap$

various compactifications,

perhaps no winner yet,

but Satake is convenient

I. Moduli of abelian varieties

$Sp(2g,\mathbb{Z}) \curvearrowright \mathcal{H}_g$

Siegel upper half space (contractible)

$$\mathcal{A}_g = \mathcal{H}_g \big/ Sp(2g,\mathbb{Z})$$

model for $B\,Sp(2g,\mathbb{Z})$ up to finite stabilizers

we have : $H^*(\mathcal{A}_g) = H^*_{Sp(2g,\mathbb{Z})}$.

All cohomology taken with $\mathbb{Q}$-coefficients .

$\mathbb{E}$ is defined by

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"] \\ A_g \arrow[u, bend left, "s"] \end{tikzcd}$$

zero section

universal Abelian variety of dim g

Then $\mathbb{E} = s^*(\Omega^1_\pi)$

rank g

Lambda classes: $\lambda_i = c_i(\mathbb{E})$,

$$\mathrm{ch}\, R\pi_* \mathcal{O}_{\mathcal{X}} = \mathrm{ch}\left(1 - \mathbb{E}^\vee + \Lambda^2 \mathbb{E}^\vee \cdots\right).$$

Borel 1974: Stable cohomology of $Sp(\mathbb{Z})$ generated by λ_i.

Following van der Geer, define tautological classes:

- $RH^*(A_g) \subset H^*(A_g)$ cohomology

  Subalgebra generated by all $\lambda_i = c_i(\mathbb{E})$,

- $R^*(A_g) \subset CH^*(A_g)$ algebraic cycles

  Subalgebra generated by all $\lambda_i = c_i(\mathbb{E})$.

<u>Theorem</u> (van der Geer 1996)

$RH^*(A_g) = R^*(A_g)$ with presentation

$$\frac{\mathbb{Q}[\lambda_1, \dots, \lambda_g]}{(\lambda_g = 0, \quad c(\mathbb{E} \oplus \mathbb{E}^\vee) = 1)}$$

$$(1+\lambda_1+\lambda_2+\dots+\lambda_g)\cdot(1-\lambda_1+\lambda_2-\dots(-1)^g\lambda_g) = 1 .$$

As a consequence, $R^*(A_g)$ is a Gorenstein ring with socle

$$R^{\binom{g}{2}}(A_g) \cong \mathbb{Q}\cdot\lambda_1^{\binom{g}{2}} \cong \mathbb{Q}\cdot\lambda_1\lambda_2\cdots\lambda_{g-1}.$$

additional argument by van der Geer provided for nonvanishing

Many open questions:

- Calculate $H^*_{Sp(2g,\mathbb{Z})}$ in unstable ranges
- Calculate $CH^*(A_g)$
- Calculate $H^*_{Sp(2g,\mathbb{Z})}$ with $\mathbb{Z}$-coefficients

all very difficult. We will go in a different direction.

II. Noether-Lefschetz loci

Given a PPAV X, we are interested in the Neron-Severi group

$$NS(X) = \mathrm{Pic}(X) / \text{Numerical equivalence}.$$

for a generic $[X] \in \mathcal{A}_g$,

$$NS(X) \cong \mathbb{Z}.$$

The Noether-Lefschetz loci are defined to be where the rank of the Neron-Severi group jumps.

We will be interested in loci

where $NS(X) \cong \mathbb{Z}^2$.

Then there is a quadratic form

$$\langle a, b \rangle = \int_X c_1(a)\, c_1(b) \cdot c_1(L)^{g-2}$$

where $a, b \in NS(X)$

and $L \in NS(X)$ is the polarization

So $(NS(X), L, \langle\, , \rangle)$ is a

polarized lattice.

To each rank 2 polarized lattice

$$(\Lambda, L, \langle,\rangle),$$

we associate the quasi-projective Noether-Lefschetz locus

notation includes $L, \langle,\rangle$

$$NL_\Lambda \subset A_g$$

Could be empty!

of Abelian varieties with polarized Neron-Severi lattice isomorphic to Λ.

The Noether-Lefschetz Cycle is defined by the class of the closure

$$[\overline{NL}_\Lambda] \in CH^*(A_g).$$

Example: The locus

$$\mathcal{A}_{g_1} \times \mathcal{A}_{g_2} \longrightarrow \mathcal{A}_g$$

has generic Neron-Severi lattice

$$\Lambda_{g_1,g_2} = (g-2)! \begin{pmatrix} g_1^2 - g_1 & g_1 g_2 \\ g_1 g_2 & g_2^2 - g_2 \end{pmatrix}, \quad L = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Since $\dim_{\mathbb{C}} \mathcal{A}_g = \binom{g+1}{2}$, the

Codimension of $\mathcal{A}_{g_1} \times \mathcal{A}_{g_2}$ is

$$\binom{g+1}{2} - \binom{g_1+1}{2} - \binom{g_2+1}{2} = g_1 g_2 \qquad \square$$

Question: are there points of $NL_{\Lambda_{g_1,g_2}}$ which do not correspond to products $X_{g_1} \times X_{g_2}$?

The expected codim of a rank 2 NL is given by Hodge theory:

$$\dim H^{0,2}(X) = \binom{g}{2}$$

Non algebraic deformations of curve classes

Every rank 2 NL locus carries a virtual fundamental class

$$[\overline{NL}_\Lambda]^{vir} \in CH^{\binom{g}{2}}(A_g)$$

Construction clearly in $H^{2\binom{g}{2}}(A_g)$ but algebraic by O-min GAGA Bakker, Brunebarbe Tsimerman

III. Three Speculations

Speculation I: For every rank 2 NL locus,

$$[\overline{NL}_{\Lambda}]^{vir} \in R^{\binom{g}{2}}(A_g).$$

Socle isomorphic to $\mathbb{Q}$

Let us examine further the locus

$$A_{g_1} \times A_{g_2} \longrightarrow A_g$$

which may or may not be the whole NL locus, but is at least a connected component.

The virtual class is then well-defined

corresponding to $A_{g_1} \times A_{g_2}$

$$[\overline{NL}_{g_1,g_2}]^{vir} \in CH^{\binom{g}{2}}(A_g).$$

Question: Can we compute $\left[\overline{\mathcal{NL}}_{g_1,g_2}\right]^{vir}$?

Answer: Yes, we can study tangent spaces to find the excess bundle:

$$\mathrm{Tan}_{A_g} = \mathrm{Sym}^2\,\mathbb{E}_g^\vee$$

$$\mathrm{Tan}_{A_{g_1}\times A_{g_2}} = \mathrm{Sym}^2\,\mathbb{E}_{g_1}^\vee \oplus \mathrm{Sym}^2\,\mathbb{E}_{g_2}^\vee$$

$$\mathrm{Nor}_{A_g/A_{g_1}\times A_{g_2}} = \mathbb{E}_{g_1}^\vee \boxtimes \mathbb{E}_{g_2}^\vee$$

All standard identifications.

Excess bundle for the NL intersection on

$A_{g_1} \times A_{g_2}$:

$$0 \to \mathbb{E}_{g_1}^\vee \boxtimes \mathbb{E}_{g_2}^\vee \to \overbrace{\Lambda^2 \mathbb{E}_g^\vee}^{H^{0,2}(X)} \to \mathrm{Obs} \to 0$$

We see $\mathrm{Obs} \cong \Lambda^2 \mathbb{E}_{g_1}^\vee \oplus \Lambda^2 \mathbb{E}_{g_2}^\vee$

Check: $\dim \mathrm{Obs} = \binom{g_1}{2} + \binom{g_2}{2}$

$\operatorname{codim} A_{g_1} \times A_{g_2} = g_1 g_2$

$$\binom{g_1}{2} + \binom{g_2}{2} + g_1 g_2 = \binom{g}{2}$$

Conclusion: $$\left[\overline{NL}_{g_1,g_2}\right]^{vir} = e\Big(\underbrace{\Lambda^2 \mathbb{E}_{g_1}^\vee}_{\text{on } A_{g_1}}\Big) \cdot e\Big(\underbrace{\Lambda^2 \mathbb{E}_{g_2}^\vee}_{\text{on } A_{g_2}}\Big)$$

A nice calculation in the tautological ring yields the formula:

$$e(\Lambda^2 \mathbb{E}_{g_1}^\vee) = (-1)^{\binom{g_1}{2}} \lambda_1 \lambda_2 \cdots \lambda_{g_1-1} \quad \text{on} \quad A_{g_1}$$

$$e(\Lambda^2 \mathbb{E}_{g_2}^\vee) = (-1)^{\binom{g_2}{2}} \lambda_1 \lambda_2 \cdots \lambda_{g_2-1} \quad \text{on} \quad A_{g_2}.$$

We have $\quad e(\Lambda^2 \mathbb{E}_{g_1}^\vee) \in R^{\binom{g_1}{2}}(A_{g_1})$

$$e(\Lambda^2 \mathbb{E}_{g_2}^\vee) \in R^{\binom{g_2}{2}}(A_{g_2}).$$

So Speculation I here amounts to

a compatibility of socles under

$$A_{g_1} \times A_{g_2} \longrightarrow A_g.$$

up to the issue of identifying the full NL locus

Speculation II: The socle $R^{\binom{g}{2}}(A_g)$ is generated by the class of

$$A_1 \times A_1 \times \cdots \times A_1 \longrightarrow A_g$$

abelian varieties which factor fully in g factors

Check: $\dim A_1 \times A_1 \times \cdots \times A_1 = g$

$$\operatorname{Codim} A_1 \times A_1 \times \cdots \times A_1 = \binom{g+1}{2} - g = \binom{g}{2}$$

It is clear that Speculation II implies the compatibility of socles under

$$A_{g_1} \times A_{g_2} \longrightarrow A_g .$$

The third speculation is about proportionalities.

Since, by Speculation I, all

$$\left[\overline{NL}_{\Lambda}\right]^{vir} \in R^{\binom{g}{2}}(A_g) \cong \mathbb{Q}$$

we can compare their sizes.

Speculation III : When arranged properly (perhaps by discriminant and coset as in the K3 case) the numbers $\left[\overline{NL}_{\Lambda}\right]^{vir}$ are related to the fourier coefficients of a modular form.

III. Ideas related to $A_1 \times A_1 \times \cdots \times A_1 \to A_g$

Consider the cycle

$$[A_1 \times A_{g-1}] \in CH^{g-1}(A_g)$$

Question: Is $[A_1 \times A_{g-1}] \in R^{g-1}(A_g)$?

Proposition: If $[A_1 \times A_{g-1}] \in R^{g-1}(A_g)$,

Then $[A_1 \times A_{g-1}] = \dfrac{(-1)^{g+1} g}{6 B_{2g}} \lambda_{g-1}$.

Proof: First show that $[A_1 \times A_{g-1}]$ is proportional λ_{g-1} using the presentation of $R^*(A_g)$. Second, fix the coefficient.

Cases $g = 4, 5$
Grushevsky-Hulek

Can use Faber-P
Hodge integral formulas

always assuming the hypothesis

As a Corollary when used repeatedly:

$$[A_1 \times A_1 \times \dots \times A_1] = \mathrm{Const} \cdot \lambda_{g-1} \lambda_{g-2} \cdots \lambda_1$$

generator of socle

So we obtain

Speculation II.

$$R^{\binom{g}{2}}(A_g)$$

But how can we test whether

$$[A_1 \times A_{g-1}] \in R^{g-1}(A_g) \;?$$

Idea: use the Torelli map.

$\mathrm{Tor} : \mathcal{M}_g^{ct} \to A_g$

If $[A_1 \times A_{g-1}] \in R^{g-1}(A_g)$,

Lambda ring, very small

Then $\mathrm{Tor}^*[A_1 \times A_{g-1}] \in \Lambda^{g-1}(\mathcal{M}_g^{ct})$

$\cap$

$R^{g-1}(\mathcal{M}_g^{ct})$

very large

We know a lot : Pixton's relations,

λ_g pairing,

admcycles.

The difficulty is to compute the

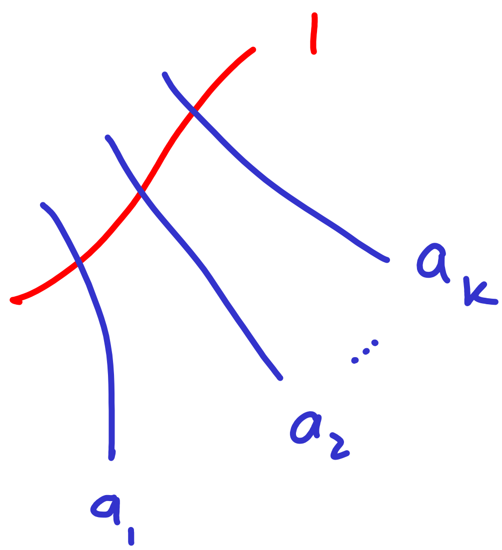
pull-back $\mathrm{Tor}^*[A_1 \times A_{g-1}]$.

$\mathrm{Tor}^{-1}(A_1 \times A_{g-1})$ = union of irreducible components indexed by partitions of $g-1$

$$a_1 + a_2 + \cdots + a_k = g-1$$

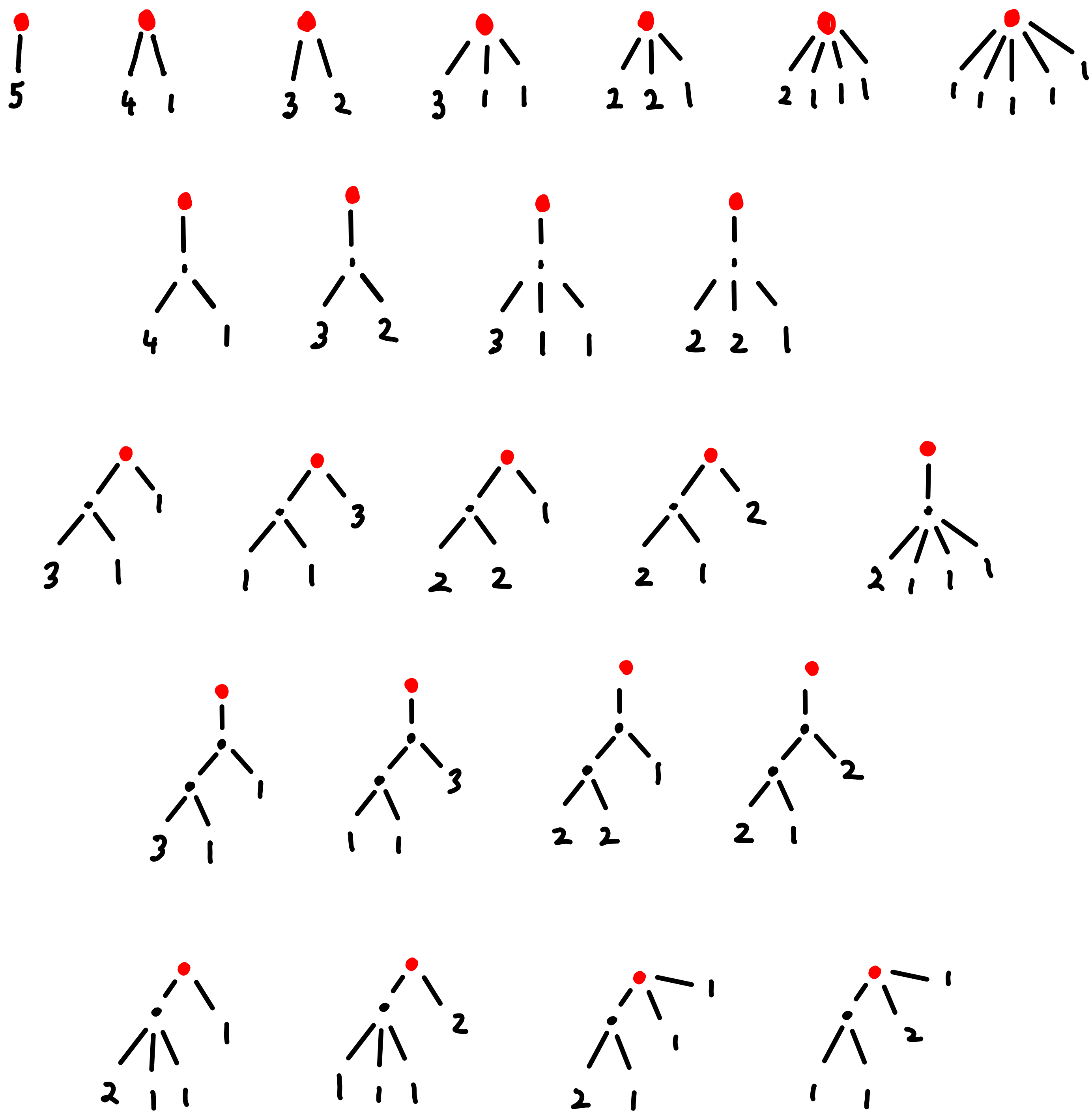
$$\Updownarrow$$

Component



These components are typically excess dimensional and intersect in a complicated configuration of strata

Example: For $g=6$, there are 24 Strata to consider

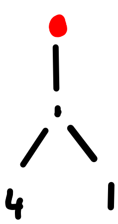


Tautological classes on moduli of curves.

$$Tor^*[A_1 \times A_{g-1}] = \sum_{\substack{\text{all strata } \Gamma \text{ of} \\ Tor^{-1}(A_1 \times A_{g-1})}} Cont(\Gamma)$$

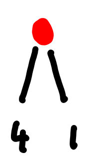
recursive formula for these excess contributions

<u>Corollary</u>: $Tor^*[A_1 \times A_{g-1}] \in R^{g-1}(M_g^{ct})$.

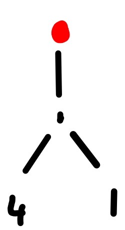
Example: $Cont\Big($  $\Big) = -3\lambda_2 + 4\lambda_1\psi_1 - 5\psi_1^2$

all on the $M^{ct}_{4,1}$ factor

x 

y 

$$\boxed{\begin{aligned} &6\, c_1(E)\, c_1(N_{z,y}) - 10\, c_1(N_{z,y})^2 \\ &+4\, c_1(E)\, c_1(N_{z,x}) - 10\, c_1(N_{z,x})\, c_1(N_{z,y}) \\ &-5\, c_1(N_{z,x})^2 - 3\, c_2(E) + 5\, c_2(N_{z,x}) \end{aligned}}$$

z  $= x \cap y$

N denotes normal bundle

E is the pull back of $N_{A_1 \times A_5,\, A_6}$

Now we can test:

$$\mathrm{Tor}^*[A_1 \times A_{g-1}] \overset{?}{=} \frac{(-1)^{g+1} g}{6\, B_{2g}} \lambda_{g-1} \in R^{g-1}(M_g^{ct})$$

- $g = 1, 2, 3$ the cases are easy to check: equality holds.

  No surprise: all Chow of $A_{g \leq 3}$ is tautological.

- $g = 4$ first nontrivial calculation: equality holds. So maybe

  $$[A_1 \times A_3] \in R^3(A_4)\ ?$$

  It remains an open question.

But we still can use the equality

$$Tor^*[A_1 \times A_3] = 20\,\lambda_3 \in R^3(M_4^{ct})$$

Since $Tor_*[M_4^{ct}] = 8\,\lambda_1$

Iguwa

We see:

$$Tor_*\,Tor^*[A_1 \times A_3] = 8 \cdot 20 \cdot \lambda_1 \lambda_3$$

$$\shortparallel$$

$$8 \cdot \lambda_1 \cdot [A_1 \times A_3]$$

So $\lambda_1^2\,[A_1 \times A_3] = 20\,\lambda_1^2 \lambda_3 \in R^5(A_4)$

$\lambda_1^3\,[A_1 \times A_3] = 20\,\lambda_1^3 \lambda_3 \in R^6(A_4)$

These are proportional to

$$[A_1 \times A_1 \times A_2] \quad \text{and} \quad [A_1 \times A_1 \times A_1 \times A_1]$$

respectively, so both are tautological on A_4,

a new result and the first nontrivial

case of Speculation II.

- $g = 5$ Long calculation: equality holds

$$\mathrm{Tor}^*[A_1 \times A_4] = 11\, \lambda_4 \in R^4(M_5^{ct})$$

The dimension is 19

so unlikely to be an accident

- $g = 6$ Our most interesting calculation yields something surprising.

Pixton has conjectured a complete set of relations among tautological classes on M_6^{ct}.

Pixton's Conjecture predicts:

$$R^4(M_6^{ct}) \times R^5(M_6^{ct}) \xrightarrow{\lambda_6\text{-Pairing}} \mathbb{Q}$$

dim 71 dim 72

rank of the pairing is 71

By our new calculation (Jan 2023):

$$\mathrm{Tor}^*[A_1 \times A_5] - \frac{2730}{691}\lambda_5 \in R^5(M_6^{ct})$$

generates the kernel of the λ_6-pairing.

<u>Theorem</u>: Pixton's Conjecture for M_6^{ct} holds

iff $\Updownarrow$

$$\mathrm{Tor}^*[A_1 \times A_5] \notin \Lambda^5(M_6^{ct}).$$

- $\boxed{\text{Pixton's Conjecture for } M_6^{ct} \Rightarrow [A_1 \times A_5] \notin R^5(A_6)}$

- $\boxed{[A_1 \times A_5] \in R^5(A_6) \Rightarrow \text{Pixton's Conjecture false for } M_6^{ct}}$

Project now by S. Canning, H. Larson, J. Schmitt to settle Pixton's Conjecture for M_6^{ct}.

Reduced to an admcycles calculation for $\overline{\mathcal{M}}_6$ which hopefully will take < 6 Months.

The tautological rings

$$R^*(\mathcal{M}_6^{ct}), \quad R^*(\mathcal{M}_{5,2}^{ct}), \quad R^*(\mathcal{M}_{4,4}^{ct})$$

$$R^*(\mathcal{M}_{3,6}^{ct}), \quad R^*(\mathcal{M}_{2,8}^{ct})$$

are closely related.

D. Petersen proved that the λ_2-pairing

$$R^4 \times R^5 \to \mathbb{Q}$$

is not perfect

So there is a general belief in Pixton's Conjecture for $\mathcal{M}_6^{ct}$,

but we will see in a few months.

IV. Speculation III : modular forms

Can we make Speculation III precise for A_2?

Answer is yes : Aitor Iribar Lopez connects the question to calculations of van der Geer 1982

Step 1 : Define the Noether-Lefschetz divisors exactly as in Maulik-P 2013 for K3 surfaces

following Borcherds, Klemm, ...

$$NL_{d,h} = \left\{ (X,\Theta) \in \mathcal{A}_2 \;\middle|\; \begin{array}{l} \exists \beta \in NS(X) \text{ with} \\ \langle \Theta, \beta \rangle = d, \ \langle \beta, \beta \rangle = 2h-2 \end{array} \right\}$$

$$NL_{d,h} \subset \mathcal{A}_2$$

associated lattice $\Rightarrow$ $\begin{pmatrix} 2 & d \\ d & 2h-2 \end{pmatrix}$, $\Delta = d^2 + 4 - 4h$

data classified by discriminant

Step 2 : The Humbert surfaces H_N in $\mathcal{A}_2$ are defined as the closures of the loci of abelian surfaces (X,Θ) with

$$\operatorname{disc}\big(NS(X,\Theta)\big) = N.$$

We have a formula relating stack classes:

$$[NL_\Delta] = \sum_{f^2 \mid \Delta} v\left(\frac{\Delta}{f^2}\right) [\mathcal{H}_{\frac{\Delta}{f^2}}]$$

$d^2 + 4 - 4h$

$$v(n) = \begin{cases} \frac{1}{2} & \text{if } n = 1, 4 \\ 1 & \text{otherwise} \end{cases}$$

Care is Needed with the treatment of Automorphisms!

Step 3: Modularity

Iribar's interpretation of van der Geer 1982

$$-\frac{1}{12} + \sum_{\Delta > 0} \frac{[NL_\Delta]}{[\mathcal{H}_1]} q^{\Delta} = \frac{1}{12}\left(20\,\Theta F_2 - \Theta^5\right)$$

Fourier expansion of a modular form of weight $\frac{5}{2}$ for $\Gamma_0(4)$ where

$$\Theta(q) = \sum_{n \in \mathbb{Z}} q^{n^2},$$

$$F_2(z) = \frac{1}{4} z^{-3/2} \sum_{n,m>0} \left(\frac{m}{n}\right)\left(\frac{-4}{n}\right)^{-5/2} (nz+m)^{-5/2},$$

$$q = e^{2\pi i z}.$$

Question: How can such an analysis be carried out for $A_{g\geq 3}$?

The End

Update

$\Downarrow$

Update on 15 February 2023

• After the lecture, several people pointed out to me the paper by Debarre and Laszlo on Noether-Lefschetz loci in A_g:

Debarre-Laszlo, *Le lieu de Noether-Lefschetz pour les variétés abéliennes*, C. R. Acad. Sci. Paris (1990).

There are at least two consequences:

(i) The locus $A_{g_1} \times A_{g_2}$ is a full Noether-Lefschetz locus in A_g (not just part of one). Voisin also sketched a different argument for the same conclusion.

(ii) Since the Noether-Lefschetz loci are reduced and essentially nonsingular, straightforward excess intersection theory shows that the virtual classes are in Chow, so o-minimal GAGA arguments are not needed for the foundations.

• I was wrong about how long the admcycles check on $\overline{M}_6$ would take. Johannes Schmitt was able to find a much faster computational strategy and today confirms that

$$\dim R^5(\overline{M}_6) = 988\,,$$

as predicted by Pixton. Via a theoretical argument, Canning-Larson-Schmitt then are able to prove Pixton's conjecture for M_6^{ct}. As a result, the conclusion of the Torelli argument of the lecture for A_6 is:

$$[A_1 \times A_5] \notin R^5(A_6)\,.$$