TAUTOLOGICAL AND NON-TAUTOLOGICAL CYCLES ON THE MODULI SPACE OF ABELIAN VARIETIES

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ABSTRACT. The tautological Chow ring of the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g was defined and calculated by van der Geer in 1999. By studying the Torelli pullback of algebraic cycles classes from $\mathcal{A}_g$ to the moduli space $\mathcal{M}_g^{\mathrm{ct}}$ of genus g of curves of compact type, we prove that the product class $[\mathcal{A}_1 \times \mathcal{A}_5] \in \mathsf{CH}^5(\mathcal{A}_6)$ is non-tautological, the first construction of an interesting non-tautological algebraic class on the moduli spaces of abelian varieties. For our proof, we use the complete description of the the tautological ring $\mathsf{R}^*(\mathcal{M}_6^{\mathrm{ct}})$ in genus 6 conjectured by Pixton and recently proven by Canning-Larson-Schmitt. The tautological ring $\mathsf{R}^*(\mathcal{M}_6^{\mathrm{ct}})$ has a 1-dimensional Gorenstein kernel, which is geometrically explained by the Torelli pullback of $[\mathcal{A}_1 \times \mathcal{A}_5]$. More generally, the Torelli pullback of the difference between $[\mathcal{A}_1 \times \mathcal{A}_{g-1}]$ and its tautological projection always lies in the Gorenstein kernel of $\mathsf{R}^*(\mathcal{M}_g^{\mathrm{ct}})$.

The product map $\mathcal{A}_1 \times \mathcal{A}_{g-1} \to \mathcal{A}_g$ is a Noether-Lefschetz locus with general Neron-Severi rank 2. A natural extension of van der Geer's tautological ring is obtained by including more general Noether-Lefschetz loci. Results and conjectures related to cycle classes of Noether-Lefschetz loci for all g are presented.

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1. INTRODUCTION

1.1. **Moduli of abelian varieties.** Let $g \geq 1$ be an integer, and let $\mathfrak{H}_g$ denote the Siegel upper half-space

$$\mathfrak{H}_g = \{\Omega \in \mathrm{Mat}_{g\times g}(\mathbb{C}) : \;\; \Omega^T = \Omega, \;\; \mathrm{Im}(\Omega) > 0\} .$$

To each $\Omega \in \mathfrak{H}_g$, we associate the abelian variety

$$X_\Omega = \mathbb{C}^g/(\Omega\mathbb{Z}^g + \mathbb{Z}^g) ,$$

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which is naturally principally polarized by the matrix $\mathrm{Im}\,(\Omega)^{-1}$. There is an action of the symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathfrak{H}_g$ given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \Omega = (A\Omega + B)(C\Omega + D)^{-1}\,.$$

Two principally polarized abelian varieties X_Ω and $X_{\Omega'}$ are isomorphic if and only if Ω and Ω' are in the same $\mathrm{Sp}_{2g}(\mathbb{Z})$-orbit:

$$X_\Omega \simeq X_{\Omega'} \iff \exists M \in \mathrm{Sp}_{2g}(\mathbb{Z}) \text{ such that } \Omega' = M\Omega\,.$$

The quotient space

$$\mathcal{A}_g = \left[\mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathfrak{H}_g\right] \tag{1}$$

is the moduli of principally polarized abelian varieties. The action of $\mathrm{Sp}_{2g}(\mathbb{Z})$ on $\mathfrak{H}_g$ has finite stabilizers. The space $\mathcal{A}_g$ is a nonsingular Deligne-Mumford stack of dimension $\binom{g+1}{2}$. We refer the reader to [2] for the foundations of the study of the moduli of abelian varieties.

Since $\mathfrak{H}_g$ is contractible, the rational cohomology[1] of $\mathcal{A}_g$ can be identified with the rational cohomology of the group $\mathrm{Sp}_{2g}(\mathbb{Z})$,

$$\mathsf{H}^*(\mathcal{A}_g) = \mathsf{H}^*_{\mathrm{Sp}_{2g}(\mathbb{Z})}(\bullet)\,,$$

via the presentation (1). By a fundamental result of Borel [3], the stable cohomology of $\mathrm{Sp}_{2g}(\mathbb{Z})$ as g increases is the free polynomial algebra

$$\lim_{g\to\infty} \mathsf{H}^*_{\mathrm{Sp}_{2g}(\mathbb{Z})}(\bullet) = \mathbb{Q}[\lambda_1, \lambda_3, \lambda_5, \ldots] \tag{2}$$

in variables λ_k of degree $2k$, where k is an odd positive integer.

Let $\pi : \mathcal{X}_g \to \mathcal{A}_g$ denote the universal principally polarized abelian variety. The Hodge bundle is the rank g vector bundle

$$\mathbb{E}_g = \pi_*(\Omega_\pi).$$

The λ classes in Borel's stability result (2) are the Chern classes of the Hodge bundle,

$$\lambda_i = c_i(\mathbb{E}_g)\,.$$

Only the odd Chern classes of $\mathbb{E}$ appear in the stability result.

For fixed g, complete calculations of the cohomology of $\mathcal{A}_g$ have so far been restricted to low dimensions. Complete results are available for $g \leq 3$, see [30]. For $g = 4$, partial results can be found in [31]. Further studies of the cohomology of $\mathcal{A}_g$ (together with the cohomology of various compactifications) can be found in [4–6, 29]. Other related results are surveyed in [32].

[1] All cohomology and Chow theories in the paper will be taken with $\mathbb{Q}$-coefficients.

1.2. **The tautological ring.** For all $g \geq 1$, van der Geer [50] proved that the Chern classes of the Hodge bundle satisfy two basic relations in $\mathsf{CH}^*(\mathcal{A}_g)$:

$$\lambda_g = 0\,, \tag{3}$$

$$(1+\lambda_1+\lambda_2+\ldots+\lambda_g)(1-\lambda_1+\lambda_2-\ldots+(-1)^g\lambda_g) = 1\,. \tag{4}$$

Esnault and Viewheg [16] showed that relation (4) also extends to toroidal compactifications of $\mathcal{A}_g$. As a consequence of (4), usually called Mumford's relation, the even degree λ classes can be expressed in terms of the λ classes of odd degree (which explains the omission of even λ classes in Borel's result (2)).

Motivated by stability, van der Geer [50] defined the tautological ring

$$\mathsf{R}^*(\mathcal{A}_g) \subset \mathsf{CH}^*(\mathcal{A}_g)$$

to be the $\mathbb{Q}$-subalgebra generated by the odd λ classes. The definition of van der Geer is entirely parallel to Mumford's definition [40] of the tautological ring

$$\mathsf{R}^*(\mathcal{M}_g) \subset \mathsf{CH}^*(\mathcal{M}_g)$$

of the moduli space of curves as the $\mathbb{Q}$-subalgebra generated by the κ classes (the free generators of the stable cohomology of the mapping class group [36]). A central result of [50] is the complete determination of $\mathsf{R}^*(\mathcal{A}_g)$.

Theorem 1 (van der Geer)**.** *The following properties hold:*

(i) *The kernel of the quotient*

$$\mathbb{Q}[\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_g] \to \mathsf{R}^*(\mathcal{A}_g) \to 0$$

is generated as an ideal by the relations (3) *and* (4)*.*

(ii) $\mathsf{R}^*(\mathcal{A}_g)$ *is a Gorenstein local ring with socle in codimension* $\binom{g}{2}$*,*

$$\mathsf{R}^{\binom{g}{2}}(\mathcal{A}_g) \cong \mathbb{Q}\,.$$

The class $\lambda_1\lambda_2\lambda_3\cdots\lambda_{g-1}$ *is a generator of the socle.*

(iii) *For* $g \leq 3$*,* $\mathsf{R}^*(\mathcal{A}_g) = \mathsf{CH}^*(\mathcal{A}_g)$*.*

Statements (i) and (ii) are found in [50]. The presentation (i) implies

$$\mathsf{R}^*(\mathcal{A}_g) \cong \mathsf{CH}^*(\mathsf{LG}_{g-1})$$

where LG_{g-1} denotes the Lagrangian Grassmannian of $(g-1)$-dimensional Lagrangian subspaces of $\mathbb{C}^{2g-2}$. Statement (ii) is consistent with this isomorphism since $\dim \mathsf{LG}_{g-1} = \binom{g}{2}$. Statement (iii) is established in [49].

Many interesting cycle classes on $\mathcal{A}_g$ admit explicit expressions in the tautological ring, see [51] for a survey.

1.3. **Curves of compact type.** For $g \geq 2$, let $\mathcal{M}_g^{\mathrm{ct}}$ denote the moduli space of curves of compact type. The moduli space $\mathcal{M}_g^{\mathrm{ct}}$ also carries a Hodge bundle

$$\mathbb{E}_g = \pi_*(\omega_\pi)\,,$$

where $\pi : \mathcal{C}_g \to \mathcal{M}_g^{\mathrm{ct}}$ is the universal curve. The Torelli map

$$\mathsf{Tor} : \mathcal{M}_g^{\mathrm{ct}} \to \mathcal{A}_g\,, \quad \mathsf{Tor}([C]) = [\mathsf{Jac}(C), \Theta]$$

sends a curve C to the Jacobian $\mathsf{Jac}(C)$ parameterizing line bundles over C of degree 0 on every irreducible component. The Jacobian has a canonical principal polarization given by the theta divisor Θ. A simple check shows that the Torelli map respects the two Hodge bundles,

$$\mathsf{Tor}^* \, \mathbb{E}_g = \mathbb{E}_g\,.$$

Let $\mathsf{R}^*(\overline{\mathcal{M}}_g)$ denote the tautological ring of $\overline{\mathcal{M}}_g$. The tautological ring $\mathsf{R}^*(\mathcal{M}_g^{\mathrm{ct}})$ is defined by restriction, as the image

$$\mathsf{R}^*(\overline{\mathcal{M}}_g) \subset \mathsf{CH}^*(\overline{\mathcal{M}}_g) \to \mathsf{CH}^*(\mathcal{M}_g^{\mathrm{ct}})\,.$$

A survey of definitions, results, and conjectures about the tautological rings of the moduli spaces of cuves can be found in [20, 42].

We can also consider the smaller $\mathbb{Q}$-subalgebra generated by λ classes

$$\Lambda^*(\mathcal{M}_g^{\mathrm{ct}}) \subset \mathsf{R}^*(\mathcal{M}_g^{\mathrm{ct}})\,.$$

Since the Torelli map respects the Hodge bundles, the image of

$$\mathsf{Tor}^* : \mathsf{R}^*(\mathcal{A}_g) \to \mathsf{CH}^*(\mathcal{M}_g^{\mathrm{ct}})$$

is contained in $\Lambda^*(\mathcal{M}_g^{\mathrm{ct}})$.

1.4. **The λ_g-pairing.** By [25, Section 5.6] and [19, Proposition 3], we have

$$\mathsf{R}^{2g-3}(\mathcal{M}_g^{\mathrm{ct}}) = \mathbb{Q}\,, \quad \mathsf{R}^{>2g-3}(\mathcal{M}_g^{\mathrm{ct}}) = 0\,. \tag{5}$$

Furthermore, as noted in [19], there exists a canonical evaluation

$$\epsilon^{\mathrm{ct}} : \mathsf{R}^{2g-3}(\mathcal{M}_g^{\mathrm{ct}}) \to \mathbb{Q}\,, \quad \alpha \mapsto \int_{\overline{\mathcal{M}}_g} \overline{\alpha} \cdot \lambda_g\,. \tag{6}$$

The integration requires a lift $\overline{\alpha}$ of α to the compactification. The answer is well-defined (independent of lift) since λ_g vanishes on the complement $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g^{\mathrm{ct}}$. The evaluation ϵ^{ct} induces a pairing between classes of complementary degrees,

$$\mathsf{R}^k(\mathcal{M}_g^{\mathrm{ct}}) \times \mathsf{R}^{2g-3-k}(\mathcal{M}_g^{\mathrm{ct}}) \to \mathsf{R}^{2g-3}(\mathcal{M}_g^{\mathrm{ct}}) \cong \mathbb{Q}\,, \quad (\alpha,\beta) \mapsto \int_{\overline{\mathcal{M}}_g} \overline{\alpha} \cdot \overline{\beta} \cdot \lambda_g\,,$$

which is called the *λ_g-pairing.*

The λ_g-pairing arises naturally in the Gromov-Witten theory of curves [23]. See [18, 35, 41] for explicit formulas and structures related to the λ_g-pairing.

1.5. **The product locus $\mathcal{A}_1 \times \mathcal{A}_{g-1}$.** Via the product of principally polarized abelian varieties, there is a proper morphism

$$\mathcal{A}_1 \times \mathcal{A}_{g-1} \to \mathcal{A}_g \, .$$

By the dimension formula, the image is of codimension $g-1$ in $\mathcal{A}_g$. For $g \geq 1$, let

$$[\mathcal{A}_1 \times \mathcal{A}_{g-1}] \in \mathsf{CH}^{g-1}(\mathcal{A}_g)$$

be the pushforward of the fundamental class. In the $g = 1$ case,

$$[\mathcal{A}_1 \times \mathcal{A}_0] = [\mathcal{A}_1] \in \mathsf{CH}^0(\mathcal{A}_1)\, .$$

Proposition 2. *For $g \geq 1$, if $[\mathcal{A}_1 \times \mathcal{A}_{g-1}] \in \mathsf{CH}^{g-1}(\mathcal{A}_g)$ is a tautological class, then*[2]

$$[\mathcal{A}_1 \times \mathcal{A}_{g-1}] = \frac{g}{6|B_{2g}|}\lambda_{g-1} \in \mathsf{CH}^{g-1}(\mathcal{A}_g)\, .$$

To show Proposition 2, we use properties of the tautological ring $\mathsf{R}^*(\mathcal{A}_g)$ together with a study of the pullback to $\mathcal{M}_g^{\mathrm{ct}}$ via the Torelli map and the λ_g-pairing.[3] A version of Proposition 2 for $g \leq 5$ was proven earlier by Grushevsky and Hulek, see [28, Lemma 8.1, Proposition 9.3]. The formula of Proposition 2 was also found independently by Faber in unpublished work.

1.6. **Main results.** But is $[\mathcal{A}_1 \times \mathcal{A}_{g-1}]$ tautological? Proposition 2 provides no answer to the latter question. Motivated by Proposition 2, we define

$$\Delta_g = [\mathcal{A}_1 \times \mathcal{A}_{g-1}] - \frac{g}{6|B_{2g}|}\lambda_{g-1} \ \in \mathsf{CH}^{g-1}(\mathcal{A}_g)$$

for $g \geq 1$.

The class Δ_g detects whether $[\mathcal{A}_1 \times \mathcal{A}_{g-1}] \in \mathsf{CH}^{g-1}(\mathcal{A}_g)$ is tautological:

$$[\mathcal{A}_1 \times \mathcal{A}_{g-1}] \in \mathsf{R}^{g-1}(\mathcal{A}_g) \iff \Delta_g = 0 \in \mathsf{CH}^{g-1}(\mathcal{A}_g)\, .$$

The vanishing $\Delta_1 = 0 \in \mathsf{CH}^0(\mathcal{A}_1)$ is trivial. For $g = 2$ and $g = 3$, the classes $[\mathcal{A}_1 \times \mathcal{A}_1] \in \mathsf{CH}^1(\mathcal{A}_2)$ and $[\mathcal{A}_1 \times \mathcal{A}_2] \in \mathsf{CH}^2(\mathcal{A}_3)$ are tautological by Theorem 1(iii). The vanishings

$$\Delta_2 = 0\, , \quad \Delta_3 = 0$$

were also noted in [49, Lemma 2.2, Proposition 2.1].

For higher g, we will use the Torelli map to study the class Δ_g. While *a priori*, we know only that $\mathsf{Tor}^*\Delta_g \in \mathsf{CH}^{g-1}(\mathcal{M}_g^{\mathrm{ct}})$, we prove the following stronger result by an explicit analysis of Fulton's excess intersection class [21] for the fiber product

$$\begin{array}{ccc} \mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1}) & \longrightarrow & \mathcal{M}_g^{\mathrm{ct}} \\ \downarrow & & \downarrow \mathsf{Tor} \\ \mathcal{A}_1 \times \mathcal{A}_{g-1} & \longrightarrow & \mathcal{A}_g\, . \end{array}$$

[2] B_{2g} is the Bernoulli number.

[3] An alternative proof can be found in [8], where the tautological projection of every cycle of the form

$$[\mathcal{A}_{g_1} \times \ldots \mathcal{A}_{g_\ell}] \in \mathsf{CH}^*(\mathcal{A}_g)\, , \quad g_1 + \ldots + g_\ell = g$$

is defined and explicitly calculated. The answer is given as a Schur determinant in the Hodge classes.

Theorem 3. *We have* $\mathsf{Tor}^*\Delta_g \in \mathsf{R}^{g-1}(\mathcal{M}_g^{\mathrm{ct}})$.

Our proof yields a formula for $\mathsf{Tor}^*\Delta_g$ in tautological classes on $\mathcal{M}_g^{\mathrm{ct}}$. By evaluating[4] the formula for $g = 4$ and $g = 5$ and using Pixton's relations [34, 43, 46], we obtain the vanishings

$$\mathsf{Tor}^*\Delta_4 = 0\,, \quad \mathsf{Tor}^*\Delta_5 = 0\,. \tag{7}$$

Further vanishing is established in the following result related to the geometry of the moduli space of curves of compact type.

Theorem 4. *For all* g, *the class* $\mathsf{Tor}^*\Delta_g \in \mathsf{R}^{g-1}(\mathcal{M}_g^{\mathrm{ct}})$ *lies in the kernel of the* λ_g*-pairing on* $\mathsf{R}^*(\mathcal{M}_g^{\mathrm{ct}})$.

As a consequence of Theorem 4, if $\mathsf{R}^*(\mathcal{M}_g^{\mathrm{ct}})$ is Gorenstein ring, then

$$\mathsf{Tor}^*\Delta_g = 0\,.$$

The first case for which $\mathsf{R}^*(\mathcal{M}_g^{\mathrm{ct}})$ is *not* Gorenstein is $g = 6$. The full structure of $\mathsf{R}^*(\mathcal{M}_g^{\mathrm{ct}})$ has been conjectured by Pixton [46] and has been proven by Canning-Larson-Schmitt [7] for $g \leq 7$. The kernel of the λ_6-pairing (called the *Gorenstein kernel*) is 1-dimensional and lies in $\mathsf{R}^5(\mathcal{M}_6^{\mathrm{ct}})$. More precisely, the λ_6-pairing

$$\mathsf{R}^4(\mathcal{M}_6^{\mathrm{ct}}) \times \mathsf{R}^5(\mathcal{M}_6^{\mathrm{ct}}) \to \mathbb{Q}$$

has rank 71 while we have

$$\dim_{\mathbb{Q}} \mathsf{R}^4(\mathcal{M}_6^{\mathrm{ct}}) = 71\,, \quad \dim_{\mathbb{Q}} \mathsf{R}^5(\mathcal{M}_6^{\mathrm{ct}}) = 72\,.$$

Theorem 5. *The class* $\mathsf{Tor}^*\Delta_6 = \mathsf{Tor}^*[\mathcal{A}_1 \times \mathcal{A}_5] - \frac{2370}{691}\lambda_5$ *generates the 1-dimensional kernel of the* λ_6*-pairing*

$$\mathsf{R}^4(\mathcal{M}_6^{\mathrm{ct}}) \times \mathsf{R}^5(\mathcal{M}_6^{\mathrm{ct}}) \to \mathbb{Q}\,.$$

Therefore, $\mathsf{Tor}^*\Delta_6 \neq 0 \in \mathsf{R}^5(\mathcal{M}_6^{\mathrm{ct}})$ *and* $[\mathcal{A}_1 \times \mathcal{A}_5] \notin \mathsf{R}^5(\mathcal{A}_6)$.

The class $[\mathcal{A}_1 \times \mathcal{A}_5] \in \mathsf{CH}^5(\mathcal{A}_6)$ is the first interesting non-tautological algebraic cycle class constructed on the moduli of abelian varieties. While the idea of using the intersection theory of the Torelli map is basic, there are reasons the study had not been undertaken before. The first is that the fiber product $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ consists of many intersecting components of excess dimension. The calculation of Fulton's excess class here is subtle and requires, in particular, knowledge of the precise scheme structure of $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$. The second, and perhaps more fundamental reason, is that, until recently, the structure of $\mathsf{R}^*(\mathcal{M}_g^{\mathrm{ct}})$ was completely unknown. Pixton's conjecture [46] offers a framework for understanding $\mathsf{R}^*(\mathcal{M}_g^{\mathrm{ct}})$ and plays a crucial role in our work.

In genus $g = 7$, the tautological ring $\mathsf{R}^*(\mathcal{M}_7^{\mathrm{ct}})$ has a 1-dimensional Gorenstein kernel [7] as predicted by Pixton. We have

$$\dim_{\mathbb{Q}} \mathsf{R}^5(\mathcal{M}_7^{\mathrm{ct}}) = 277\,, \quad \dim_{\mathbb{Q}} \mathsf{R}^6(\mathcal{M}_7^{\mathrm{ct}}) = 278\,.$$

[4] The evaluations are presented in Propositions 36 and 38 of Section 6.

The λ_7-pairing

$$\mathsf{R}^5(\mathcal{M}_7^{\mathrm{ct}}) \times \mathsf{R}^6(\mathcal{M}_7^{\mathrm{ct}}) \to \mathbb{Q}$$

has rank 277. But a surprise occurs: the class $\mathsf{Tor}^*\Delta_7 \in \mathsf{R}^6(\mathcal{M}_7^{\mathrm{ct}})$ does *not* generate the kernel of the pairing!

Proposition 6. *We have* $\mathsf{Tor}^*\Delta_7 = \mathsf{Tor}^*[\mathcal{A}_1 \times \mathcal{A}_6] - \lambda_6 = 0 \in \mathsf{R}^6(\mathcal{M}_7^{\mathrm{ct}})$.

The generator of the kernel in $\mathsf{R}^6(\mathcal{M}_7^{\mathrm{ct}})$ of the λ_7-pairing is constructed from $\mathsf{Tor}^*\Delta_6$ in Proposition 40 of Section 6.3.

For the moduli space of curves and abelian varieties, let

$$\mathsf{RH}^*(\mathcal{M}_g^{\mathrm{ct}}) \subset \mathsf{H}^*(\mathcal{M}_g^{\mathrm{ct}}) \ \text{ and } \ \mathsf{RH}^*(\mathcal{A}_g) \subset \mathsf{H}^*(\mathcal{A}_g)$$

denote the images of $\mathsf{R}^*(\mathcal{M}_g^{\mathrm{ct}})$ and $\mathsf{R}^*(\mathcal{A}_g)$ under the cycle class map (which doubles the degree index). For $g = 6$, the cycle class map is an isomorphism

$$\mathsf{R}^*(\mathcal{M}_6^{\mathrm{ct}}) \simeq \mathsf{RH}^*(\mathcal{M}_6^{\mathrm{ct}})$$

by [7]. Hence, $\mathsf{Tor}^*\Delta_6 \neq 0 \in \mathsf{RH}^{10}(\mathcal{M}_6^{\mathrm{ct}})$ and

$$[\mathcal{A}_1 \times \mathcal{A}_5] \notin \mathsf{RH}^{10}(\mathcal{A}_6)\,.$$

In fact, $g = 6$ is the first genus where algebraic classes can be non-tautological in cohomology.

Proposition 7. *All algebraic cycles are tautological in cohomology for* $g \leq 5$.

Proof. By [32, Theorem 17, Theorem 32], the intersection cohomology $\mathsf{IH}^*(\mathcal{A}_g^{\mathrm{Sat}})$ of the Satake compactification is tautological when $g \leq 5$. Since $\mathsf{IH}^*(\mathcal{A}_g^{\mathrm{Sat}})$ surjects onto the pure weight cohomology of $\mathcal{A}_g$, see [15, Lemma 2], and algebraic cycles are of pure weight, the Proposition follows. □

Taïbi [32, Theorem 33] has furthermore shown that $\mathsf{IH}^k(\mathcal{A}_g^{\mathrm{Sat}})$ is tautological for $k < 2g - 2$. Therefore, all algebraic cycles of codimension less than $g - 1$ are tautological in cohomology.

Based on Theorem 5, Proposition 6, and Proposition 7, our expectation is

$$\begin{aligned} &\Delta_g = 0 \in \mathsf{CH}^{g-1}(\mathcal{A}_g) \qquad \text{for } 2 \leq g \leq 5 \text{ and } g = 7\,, \\ &\Delta_g \neq 0 \in \mathsf{CH}^{g-1}(\mathcal{A}_g) \qquad \text{for } g \geq 6,\ g \neq 7\,. \end{aligned}$$

Iribar López [33] has subsequently found a proof of the non-vanishing of Δ_g in Chow for $g = 12$ and even $g \geq 16$. So for even g, only the cases $g = 4, 8, 10, 14$ are open.

1.7. **Product extension.** Since basic classes such as product loci should be included in a tautological calculus for $\mathcal{A}_g$, proposals to enlarge the tautological ring are natural to consider. The simplest extension of $\mathsf{R}^*(\mathcal{A}_g)$ is obtained by considering the closure

$$\mathsf{R}^*_{\mathrm{pr}}(\mathcal{A}_g) \subset \mathsf{CH}^*(\mathcal{A}_g)$$

of $\mathsf{R}^*(\mathcal{A}_g)$ under all product maps.

Definition 8. Define $\mathsf{R}^*_{\mathrm{pr}}(\mathcal{A}_g) \subset \mathsf{CH}^*(\mathcal{A}_g)$ to be the $\mathbb{Q}$-vector subspace generated by all classes

$$[\mathcal{A}_{g_1} \times \mathcal{A}_{g_2} \times \cdots \times \mathcal{A}_{g_\ell}, \mathsf{P}(\lambda^1, \lambda^2, \ldots, \lambda^\ell)] \in \mathsf{CH}^*(\mathcal{A}_g)$$

with $g = \sum_{i=1}^{\ell} g_i$ and all $g_i \geq 1$. Here, λ^i denotes the set of all λ classes on the factor $\mathcal{A}_{g_i}$,

$$\lambda_1, \ldots, \lambda_{g_i} \in \mathsf{CH}^*(\mathcal{A}_{g_i}),$$

and $\mathsf{P} \in \mathbb{Q}[\lambda^1, \ldots, \lambda^\ell]$ is an arbitrary polynomial.

While the definition of $\mathsf{R}^*_{\mathrm{pr}}(\mathcal{A}_g)$ leaves behind the connection to the stable cohomology of $\mathrm{Sp}_{2g}(\mathbb{Z})$, the closure under products is natural from the perspective of the tautological ring of $\overline{\mathcal{M}}_{g,n}$ with respect to the boundary gluing maps.

Proposition 9. *The subspace $\mathsf{R}^*_{\mathrm{pr}}(\mathcal{A}_g)$ satisfies the following properties:*

(i) $\mathsf{R}^*_{\mathrm{pr}}(\mathcal{A}_g)$ *is closed under multiplication, so is a $\mathbb{Q}$-algebra.*
(ii) *There is a product pushforward*

$$\mathsf{R}^*_{\mathrm{pr}}(\mathcal{A}_{g_1}) \times \mathsf{R}^*_{\mathrm{pr}}(\mathcal{A}_{g_2}) \to \mathsf{R}^*_{\mathrm{pr}}(\mathcal{A}_{g_1+g_2}).$$

(iii) $\mathsf{R}^{>\binom{g}{2}}_{\mathrm{pr}}(\mathcal{A}_g) = 0$.
(iv) $\mathsf{R}^*(\mathcal{A}_6) \subsetneq \mathsf{R}^*_{\mathrm{pr}}(\mathcal{A}_6)$.

Part (ii) holds by definition, and part (iv) is consequence of Theorem 5. Parts (i) and (iii) will be proven in Section 2. A natural conjecture concerns the codimension $\binom{g}{2}$ classes.

Conjecture 10. *For all $g \geq 1$, $\mathsf{R}^{\binom{g}{2}}_{\mathrm{pr}}(\mathcal{A}_g) \cong \mathbb{Q}$.*

The class of the locus of abelian varieties that factor completely, $[\mathcal{A}_1 \times \cdots \times \mathcal{A}_1] \in \mathsf{R}^{\binom{g}{2}}_{\mathrm{pr}}(\mathcal{A}_g)$, provides a candidate for the generator of $\mathsf{R}^{\binom{g}{2}}_{\mathrm{pr}}(\mathcal{A}_g)$. Conjecture 10 is equivalent to the following assertion: *for all $g \geq 1$,*

$$[\underbrace{\mathcal{A}_1 \times \cdots \times \mathcal{A}_1}_{g}] \in \mathsf{R}^{\binom{g}{2}}(\mathcal{A}_g). \tag{8}$$

In fact, a sharper claim can be made [8, Theorem 6]: *if* (8) *holds, then*

$$[\underbrace{\mathcal{A}_1 \times \cdots \times \mathcal{A}_1}_{g}] = \left(\prod_{k=1}^{g} \frac{k}{6|B_{2k}|}\right) \lambda_1 \cdots \lambda_{g-1} \in \mathsf{R}^{\binom{g}{2}}(\mathcal{A}_g).$$

For $g \leq 3$, we have $\mathsf{R}^*(\mathcal{A}_g) = \mathsf{R}^*_{\mathrm{pr}}(\mathcal{A}_g)$ since both are the full Chow ring by Theorem 1(iii). Therefore, Conjecture 10 is true for $g \leq 3$. For $g = 4$, Conjecture 10 is proven in Proposition 37 in Section 5. For $g \geq 5$, the question is open.

1.8. **Noether–Lefschetz loci.** A further expansion of $\mathsf{R}^*(\mathcal{A}_g)$ via Noether-Lefschetz loci is motivated by the study of tautological classes [37, 44] on the the moduli space of quasi-polarized $K3$ surfaces.

The very general principally polarized abelian variety (X, Θ) has Néron-Severi group

$$\mathsf{NS}(X) \cong \mathbb{Z}\,.$$

However, the Néron-Severi rank can jump on special subvarieties of $\mathcal{A}_g$. For each r, let

$$\mathsf{NL}_g^r \subset \mathcal{A}_g$$

be the *Noether-Lefschetz locus* of abelian varieties with

$$\mathsf{NS}(X) \cong \mathbb{Z}^r\,.$$

The locus NL_g^r is a countable union of irreducible locally closed substacks of $\mathcal{A}_g$.

Let $\mathsf{NL}_g^r \subset \overline{\mathsf{NL}}_g^r$ denote the Zariski closure in $\mathcal{A}_g$. A *marked irreducible component of* $\overline{\mathsf{NL}}_g^r$ is a moduli space $\mathcal{S}$ of principally polarized abelian varieties (X, Θ, ϕ) with the data of a *marking*

$$\phi : \mathbb{Z}^r \hookrightarrow \mathsf{NS}(X)$$

satisfying two properties:

(i) the polarization lies in the image of ϕ,

$$\phi(1, 0, \ldots, 0) = \Theta\,,$$

(ii) the induced map $\iota_{\mathsf{S}} : \mathcal{S} \to \overline{\mathsf{NL}}_g^r \subset \mathcal{A}_g$ surjects onto an irreducible component of $\overline{\mathsf{NL}}_g^r$.

Two marked abelian varieties (X, Θ, ϕ) and (X', Θ', ϕ') are isomorphic if there are isomorphisms

$$\alpha : X \to X'\,, \quad \beta : \mathbb{Z}^r \to \mathbb{Z}^r$$

satisfying $\alpha^*\Theta' = \Theta$ and $\alpha^* \circ \phi' = \phi \circ \beta$.

Marked irreducible components $\mathcal{S}$ are algebraic, see [11] or [13, Remarque 1]. Moreover, $\mathcal{S}$ admits a canonical quotient presentation with respect to a subgroup $\mathsf{G}_{\mathcal{S}} \subset \mathsf{Sp}(2g, \mathbb{Z})$ and carries automorphic algebraic vector bundles.

Definition 11. Define $\mathsf{R}^*_{\mathrm{NL}}(\mathcal{A}_g) \subset \mathsf{CH}^*(\mathcal{A}_g)$ to be the $\mathbb{Q}$-subalgebra generated by all classes

$$\iota_{\mathcal{S}*}(\mathsf{P}) \,\in\, \mathsf{CH}^*(\mathcal{A}_g)\,,$$

where $\mathcal{S}$ is a marked irreducible component of $\overline{\mathsf{NL}}_g^r$ and P is a polynomial in the Chern classes of automorphic algebraic vector bundles on $\mathcal{S}$.

Further extensions of the tautological ring of $\mathcal{A}_g$ would more generally include Hodge loci corresponding to arbitrary Hodge types, see [39, Section 3] or [26, 53] for definitions. We will not pursue these directions here. Related constructions regarding the tautological rings of Shimura varieties via Chern classes of automorphic bundles are discussed in [54].

By definition[5], we have inclusions of tautological rings

$$\mathsf{R}^*(\mathcal{A}_g) \subset \mathsf{R}^*_{\mathrm{pr}}(\mathcal{A}_g) \subset \mathsf{R}^*_{\mathrm{NL}}(\mathcal{A}_g)\,. \tag{9}$$

Both inclusions are equalities for $g \leq 3$. We have seen that the first inclusion in (9) is strict for $g = 6$. Iribar López observes in [33] that the Torelli pullback $\mathrm{NL}_{2,g}$ to $\mathcal{M}_g$ is exactly the bielliptic locus. Here, $\mathrm{NL}_{2,g}$ is the locus of abelian g-folds containing an elliptic curve such that the induced polarization on the elliptic curve is of degree 2. As the bielliptic locus is non-tautological in $\mathsf{CH}^{g-1}(\mathcal{M}_g)$ for $g = 12$ and $g \geq 16$ even [1, 52], Iribar López concludes

$$[\mathrm{NL}_{2,g}] \notin \mathsf{R}^{g-1}_{\mathrm{pr}}(\mathcal{A}_g)\,.$$

Therefore, the second inclusion of (9) is also strict.

Conjecture 12. *The ring $\mathsf{R}^*_{\mathrm{NL}}(\mathcal{A}_g)$ satisfies the following socle and vanishing properties:*

(i) $\mathsf{R}^{\binom{g}{2}}_{\mathrm{NL}}(\mathcal{A}_g) \cong \mathbb{Q}\,.$

(ii) $\mathsf{R}^{k}_{\mathrm{NL}}(\mathcal{A}_g) = 0$ *for* $k > \binom{g}{2}\,.$

By Proposition 7 and Theorem 1, the conjecture is true in cohomology for $g \leq 5$. In fact, the stronger vanishing

$$\mathsf{H}^k(\mathcal{A}_g, \mathbb{Q}) = 0, \quad k > 2\binom{g}{2}$$

is expected, see [6, Question 1.1], as well as equations (1) and (2) there for supporting results.

1.9. **Noether-Lefschetz loci of rank 2 and virtual fundamental classes.** The Noether–Lefschetz locus NL^2_g plays a special role in geometry of $\mathcal{A}_g$. Debarre and Laszlo [13] have classified the irreducible components of the Noether–Lefschetz locus of rank 2.

Theorem 13 (Debarre-Laszlo)**.** *The irreducible components of the closure of the Noether–Lefschetz locus $\mathsf{NL}^2_g \subset \mathcal{A}_g$ are:*

(i) *For each integer $1 \leq k \leq \frac{g}{2}$, the locus of principally polarized abelian varieties containing an abelian subvariety of dimension k such that the induced polarization is of a fixed degree.*

(ii) *For every divisor n of g, $n \neq g$, the irreducible components of the locus of Shimura–Hilbert–Blumenthal varieties.*

The Shimura–Hilbert–Blumenthal varieties parametrize abelian varieties X whose endomorphism algebra $\mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ contains a totally real subfield. The components in Theorem 13(i) that arise when the induced polarization is principal are exactly the product loci $\mathcal{A}_k \times \mathcal{A}_{g-k}$.

The expected codimension of the Noether–Lefschetz locus of rank 2 is

$$\binom{g}{2} = \dim H^{2,0}(X)\,,$$

[5]We easily see that the classes λ^i arise from automorphic bundles on the marked irreducible component of $\overline{\mathsf{NL}}^{\ell}_g$ determined by $\mathcal{A}_{g_1} \times \mathcal{A}_{g_2} \times \cdots \times \mathcal{A}_{g_\ell} \to \overline{\mathsf{NL}}^{\ell}_g \subset \mathcal{A}_g$.

for any abelian variety X, while the actual dimension can be different. Every marked irreducible component of NL_g^2 carries a virtual fundamental class

$$[\mathcal{S}]^{\mathrm{vir}} \in \mathsf{CH}^{\binom{g}{2}}(\mathcal{A}_g)\,,$$

as constructed in Section 7.

Proposition 14. *The virtual class of the locus $\mathcal{A}_k \times \mathcal{A}_{g-k}$ of products is given by*

$$[\mathcal{A}_k \times \mathcal{A}_{g-k}]^{\mathrm{vir}} = (-1)^{\binom{k}{2}} \prod_{i=1}^{k-1} \lambda_i \otimes (-1)^{\binom{g-k}{2}} \prod_{j=1}^{g-k-1} \lambda_j\,.$$

If Conjecture 10 is correct, then we have the following consequence:

$$[\mathcal{A}_k \times \mathcal{A}_{g-k}]^{\mathrm{vir}} \in \mathsf{R}^{\binom{g}{2}}(\mathcal{A}_g)\,.$$

Perhaps the virtual fundamental classes are always in van der Geer's tautological ring?

Speculation 15. For all $g \geq 1$ and all marked irreducible components $\mathcal{S}$ of $\mathsf{NL}_g^2 \subset \mathcal{A}_g$, we have

$$[\mathcal{S}]^{\mathrm{vir}} \in \mathsf{R}^{\binom{g}{2}}(\mathcal{A}_g)\,.$$

Whenever Speculation 15 is true, the structure of the proportionalities

$$[\mathcal{S}]^{\mathrm{vir}} \in \mathsf{R}^{\binom{g}{2}}(\mathcal{A}_g) \cong \mathbb{Q}$$

as $\mathcal{S}$ varies among irreducible components is an interesting question. The $g = 2$ case, where the virtual and fundamental classes coincide, has been solved by van der Geer [48] in terms of a Fourier expansion of a modular form.[6]

1.10. **Plan of the paper.** We start in Section 2 by studying intersections of product loci and properties of the product tautological rings. In particular, Proposition 9 is established. In Section 3, we compute the class of the product $[\mathcal{A}_1 \times \mathcal{A}_{g-1}]$ when tautological, thus proving Proposition 2. Theorem 4 is also proven in Section 3. In Sections 4 and 5, we calculate the class $\mathsf{Tor}^*\Delta_g$ via excess intersection theory, and establish Theorem 3. In Section 6, we present low genus examples, and prove Theorem 5 and Proposition 6. In Section 7, we discuss the virtual fundamental classes of the Noether-Lefschetz loci and prove Proposition 14.

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[6]See [33] for a discussion.

The paper started with a discussion at the piano bar *Vincent* overlooking the Spree during the conference *Resonance, topological invariants of groups, and moduli* at HU Berlin in November 2022 organized by Gavril Farkas. The research here was motivated in part by earlier work with Davesh Maulik on the moduli space of $K3$ surfaces [38].

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2. Intersection theory of product loci

2.1. **Overview.** We prove here that product loci in $\mathcal{A}_g$ always intersect each other trivially in $\mathsf{CH}^*(\mathcal{A}_g)$. As a consequence, we give a proof of Proposition 9.

2.2. **Unique decomposition for abelian varieties.** A principally polarized abelian variety $(A, \Theta) \in \mathcal{A}_g$ is *decomposable* if

$$(A, \Theta) \in \mathcal{A}_{g_1} \times \mathcal{A}_{g_2}$$

for some $g_1 + g_2 = g$ with $g_i \neq 0, g$. If (A, Θ) is decomposable, then the theta divisor Θ is reducible. Conversely, if Θ is reducible, then (A, Θ) is decomposable by a result of Shimura, see [12, Lemma 3.20]. The following result is [12, Corollary 3.23].

Proposition 16. *A principally polarized abelian variety (A, Θ) decomposes uniquely, up to reordering, as a product of indecomposable principally polarized abelian varieties.*

2.3. **The intersection product.** Associated to the partition $g = g_1 + \ldots + g_\ell$ is the finite morphism

$$\mathcal{A}_{g_1} \times \ldots \times \mathcal{A}_{g_\ell} \to \mathcal{A}_g \,.$$

The pushforward of the fundamental class is the cycle $[\mathcal{A}_{g_1} \times \ldots \times \mathcal{A}_{g_\ell}] \in \mathsf{CH}^*(\mathcal{A}_g)$.

Proposition 17. *The intersection product vanishes,*

$$[\mathcal{A}_{g_1} \times \ldots \times \mathcal{A}_{g_\ell}] \cdot [\mathcal{A}_{h_1} \times \ldots \times \mathcal{A}_{h_k}] = 0 \in \mathsf{CH}^*(\mathcal{A}_g)\,,$$

for all partitions $g_1 + \ldots + g_\ell = h_1 + \ldots + h_k = g$ with $\ell \geq 2$ and $k \geq 2$.

Proof. First, we establish that

$$[\mathcal{A}_{g_1} \times \ldots \times \mathcal{A}_{g_\ell}]^2 = 0\,. \tag{10}$$

Strictly speaking, the case (10) does not require a separate discussion, but the simpler analysis illustrates the main point. Using the self-intersection formula, it suffices to prove that the normal bundle of the morphism

$$p : \mathcal{A}_{g_1} \times \ldots \times \mathcal{A}_{g_\ell} \to \mathcal{A}_g$$

has vanishing Euler class. The tangent bundle to the moduli stack of principally polarized abelian varieties is $T\mathcal{A}_g = \operatorname{Sym}^2 \mathbb{E}_g^\vee$. Furthermore, we have the splitting

$$p^*\mathbb{E}_g = \mathbb{E}_{g_1} \boxplus \ldots \boxplus \mathbb{E}_{g_\ell}. \tag{11}$$

Therefore, the normal bundle of the morphism p equals

$$\mathcal{N} = \operatorname{Sym}^2(\mathbb{E}_{g_1} \boxplus \ldots \boxplus \mathbb{E}_{g_\ell})^\vee - \operatorname{Sym}^2 \mathbb{E}_{g_1}^\vee - \ldots - \operatorname{Sym}^2 \mathbb{E}_{g_\ell}^\vee = \bigoplus_{\{i,j\}} \mathbb{E}_{g_i}^\vee \boxtimes \mathbb{E}_{g_j}^\vee. \tag{12}$$

The sum is taken over the 2-element sets $\{i,j\} \subset \{1,\ldots,\ell\}$. We will repeatedly use the following remark concerning the Euler classes of two vector bundles $\mathcal{V}, \mathcal{W}$ and their tensor product $\mathcal{V} \otimes \mathcal{W}$:

$$\mathsf{e}(\mathcal{V}) = 0 \ \text{ and } \ \mathsf{e}(\mathcal{W}) = 0 \implies \mathsf{e}(\mathcal{V} \otimes \mathcal{W}) = 0\,. \tag{13}$$

This assertion is clear if $\mathcal{V}, \mathcal{W}$ are both line bundles, while the general case follows by the splitting principle. In our case, the Hodge bundles $\mathbb{E}_{g_i}$ have trivial Euler classes, so (13) implies that the same is true about the normal bundle $\mathcal{N}$. We conclude the vanishing (10).

Before going to the general case, we consider another simpler situation,

$$[\mathcal{A}_1 \times \mathcal{A}_{g-1}] \cdot [\mathcal{A}_k \times \mathcal{A}_{g-k}] = 0\,. \tag{14}$$

We may assume $k \neq 1$ since the case $k = 1$ was considered above. Let $Z = \mathcal{A}_1 \times \mathcal{A}_{g-1}$, and let $W = \mathcal{A}_k \times \mathcal{A}_{g-k}$. Consider the fiber product diagram:

$$\begin{array}{ccc} X \sqcup Y & \longrightarrow & W \\ \downarrow & & \downarrow \\ Z & \longrightarrow & \mathcal{A}_g\,. \end{array}$$

By Proposition 16, there are 2 disjoint components of the fiber product of Z and W, corresponding to whether the elliptic factor in Z belongs to the dimension k or dimension $g-k$ factor in W. Therefore,

$$X = \mathcal{A}_1 \times \mathcal{A}_{k-1} \times \mathcal{A}_{g-k} \ \text{ and } \ Y = \mathcal{A}_1 \times \mathcal{A}_k \times \mathcal{A}_{g-k-1}$$

of codimension $k-1$ and $g-k-1$ in W respectively. The case $g = 2k$ is special: the fiber product admits a single component $X = Y$.

The contributions of X and Y to the intersection product (14) are found by an excess bundle calculation. For X, we compute the excess bundle with the aid of (12). We find

$$\mathcal{N}_{Z/\mathcal{A}_g}\Big|_X - \mathcal{N}_{X/W} = \mathbb{E}_1^\vee \boxtimes (\mathbb{E}_{k-1}^\vee \boxplus \mathbb{E}_{g-k}^\vee) - \mathbb{E}_1^\vee \boxtimes \mathbb{E}_{k-1}^\vee = \mathbb{E}_1^\vee \boxtimes \mathbb{E}_{g-k}^\vee\,.$$

Since Z has codimension $g-1$ in $\mathcal{A}_g$, we must select the Chern class of degree

$$g - k = (g-1) - (k-1)\,,$$

which is Euler class of the tensor product $\mathbb{E}_1^\vee \boxtimes \mathbb{E}_{g-k}^\vee$. The Euler class vanishes by (13). The analysis for Y is similar.

For the general case, we form the fiber product diagram

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{A}_{h_1} \times \ldots \times \mathcal{A}_{h_k} \\ \big\downarrow & & \big\downarrow \\ \mathcal{A}_{g_1} \times \ldots \times \mathcal{A}_{g_\ell} & \longrightarrow & \mathcal{A}_g \, . \end{array}$$

For simplicity, we write

$$Z = \mathcal{A}_{g_1} \times \ldots \times \mathcal{A}_{g_\ell} \quad \text{and} \quad W = \mathcal{A}_{h_1} \times \ldots \times \mathcal{A}_{h_k} \, .$$

To identify the components of the fiber product $\mathcal{F}$, we follow an argument similar to [24, Proposition 9] in the context of the moduli of curves. A partition $\sigma_1 + \ldots + \sigma_p = g$ *refines* the partition $\tau_1 + \ldots + \tau_n = g$ if there exists a decomposition into disjoint sets

$$\{1, \ldots, p\} = I_1 \sqcup \ldots \sqcup I_n$$

such that for all $1 \leq j \leq n$, we have

$$\sum_{i \in I_j} \sigma_i = \tau_j \, .$$

Each refinement σ of τ determines the tuple $(I_1, \ldots, I_n)$ inducing a morphism

$$\mathcal{A}_{\sigma_1} \times \ldots \times \mathcal{A}_{\sigma_p} \to \mathcal{A}_{\tau_1} \times \ldots \times \mathcal{A}_{\tau_n}.$$

We write $\sigma \to \tau$ to indicate refinement, with the sets $(I_1, \ldots, I_n)$ being understood (though not explicitly recorded by the notation).

Let us abbreviate $\vec{g}, \vec{h}$ for the two partitions $g = g_1 + \ldots + g_\ell$, and $g = h_1 + \ldots + h_k$. Let Σ denote the set of all partitions σ that refine both $\vec{g}$ and $\vec{h}$, or more precisely triples

$$(\sigma, \ \sigma \to \vec{g}, \ \sigma \to \vec{h}) \, .$$

Each $\sigma \in \Sigma$ induces morphisms

$$\mathcal{A}_{\sigma_1} \times \ldots \times \mathcal{A}_{\sigma_p} \to \mathcal{A}_{g_1} \times \ldots \times \mathcal{A}_{g_\ell} \, , \quad \mathcal{A}_{\sigma_1} \times \ldots \times \mathcal{A}_{\sigma_p} \to \mathcal{A}_{h_1} \times \ldots \times \mathcal{A}_{h_k} \, ,$$

and thus a morphism to the fiber product $\mathcal{A}_{\sigma_1} \times \ldots \times \mathcal{A}_{\sigma_p} \to \mathcal{F}$. The set Σ can be ordered by (further) refinement. We consider the extremal refinements σ which do not arise as further refinements of other members of Σ. Then

$$\mathcal{F} = \bigsqcup_{\sigma \text{ extremal}} \mathcal{A}_{\sigma_1} \times \ldots \times \mathcal{A}_{\sigma_p}.$$

For each component $X = \mathcal{A}_{\sigma_1} \times \ldots \times \mathcal{A}_{\sigma_p}$, the excess bundle equals

$$\mathsf{V}_X = \mathcal{N}_{Z/\mathcal{A}_g}\Big|_X - \mathcal{N}_{X/W} \, .$$

Using (12), we have

$$\mathcal{N}_{Z/\mathcal{A}_g} = \bigoplus_{\{i,j\}} \mathbb{E}_{g_i}^\vee \boxtimes \mathbb{E}_{g_j}^\vee \implies \mathcal{N}_{Z/\mathcal{A}_g}\Big|_X = \bigoplus_{\{i,j\}} \bigoplus_{\alpha \in I_i, \, \beta \in I_j} \mathbb{E}_{\sigma_\alpha}^\vee \boxtimes \mathbb{E}_{\sigma_\beta}^\vee.$$

Here $\{i,j\} \subset \{1,\ldots,\ell\}$ is any set with 2 distinct elements, and $(I_1,\ldots,I_\ell)$ correspond to the refinement $\sigma \to \vec{g}$. Similarly, let $(J_1,\ldots,J_k)$ denote the sets corresponding to the refinement $\sigma \to \vec{h}$. Then, using (12) again, we find

$$\mathcal{N}_{X/W} = \bigoplus_s \bigoplus_{\{\alpha,\beta\}\subset J_s} \mathbb{E}^\vee_{\sigma_\alpha} \boxtimes \mathbb{E}^\vee_{\sigma_\beta}\,.$$

Of course, V_X is an actual bundle. Indeed, for each set $\{\alpha,\beta\}\subset J_s$ with two elements, we let $\alpha \in I_i$ and $\beta \in I_j$ for some $\{i,j\}\subset\{1,\ldots,\ell\}$. We only need to show $i \neq j$. Assuming $i = j$, we can form the partition τ replacing the parts $(\sigma_\alpha,\sigma_\beta)$ of σ by the sum $\sigma_\alpha+\sigma_\beta$. Furthermore, we place the sum in the sets I_i and J_s. The new partition τ thus remains a common refinement of $\vec{g}$ and $\vec{h}$, so $\tau \in \Sigma$. Furthermore σ is a refinement of τ, which contradicts the extremality of σ in Σ. As a result, V_X is sum of various tensor products of Hodge bundles $\mathbb{E}^\vee_{\sigma_\alpha}\boxtimes\mathbb{E}^\vee_{\sigma_\beta}$, so the Euler class of V_X vanishes by (13). ◻

2.4. **Proof of Proposition 9.** Part (i) follows from Proposition 17 which shows more generally that the product of two classes supported on product loci vanishes. Part (ii) is clear by definition, while part (iv) is a consequence of Theorem 5, which will be established below.

To establish part (iii), consider a nonzero class of the form

$$[\mathcal{A}_{g_1}\times\mathcal{A}_{g_2}\times\cdots\times\mathcal{A}_{g_\ell}, \mathsf{P}(\lambda^1,\lambda^2,\ldots,\lambda^\ell)] \in \mathsf{CH}^*(\mathcal{A}_g)\,.$$

Since $\mathsf{R}^*(\mathcal{A}_{g_i})$ vanishes in degree $> \binom{g_i}{2}$, the above class has degree at most

$$\sum_{i=1}^k \binom{g_i}{2} + \operatorname{codim}(\mathcal{A}_{g_1}\times\ldots\times\mathcal{A}_{g_k}/\mathcal{A}_g) = \binom{g}{2},$$

as claimed. ◻

3. Products with an elliptic factor

3.1. **Overview.** We prove here Proposition 2, which determines the class $[\mathcal{A}_1\times\mathcal{A}_{g-1}]\in\mathsf{CH}^*(\mathcal{A}_g)$ in the tautological case. See [8, Theorem 6] for a different argument. The proof below gives slightly more and will be used to establish Theorem 4.

3.2. **Proof of Proposition 2.** By [50], the monomials $\lambda_J = \prod_{j\in J}\lambda_j$ with $J\subset\{1,2,\ldots,g-1\}$ determine a basis for the $\mathbb{Q}$-vector space $\mathsf{R}^*(\mathcal{A}_g)$. If $[\mathcal{A}_1\times\mathcal{A}_{g-1}]$ is tautological, we can write

$$[\mathcal{A}_1\times\mathcal{A}_{g-1}] = \sum_J c_J\lambda_J\,, \quad c_J\in\mathbb{Q}\,.$$

The summation here runs over subsets $J\subset\{1,\ldots,g-1\}$ such that the sum of all elements in J is $g-1$.

As seen in (11), the Hodge bundle splits as a sum over the factors

$$\mathbb{E}_g\Big|_{\mathcal{A}_1\times\mathcal{A}_{g-1}} = \mathbb{E}_1\boxplus\mathbb{E}_{g-1}\,.$$

Using the vanishing (3) applied to $\mathcal{A}_1$ and $\mathcal{A}_{g-1}$, we find

$$\lambda_{g-1}[\mathcal{A}_1 \times \mathcal{A}_{g-1}] = 0 \implies \sum_J c_J \lambda_{g-1}\lambda_J = 0\,.$$

In the sum on the right, the term corresponding to $J = \{g-1\}$ vanishes by the Mumford relation

$$\lambda_{g-1}^2 = 0\,.$$

For the remaining terms, we must have $g-1 \notin J$ since the sum of elements of J is $g-1$. Then, the monomials $\lambda_{g-1}\lambda_J$ are part of the basis for $\mathsf{R}^*(\mathcal{A}_g)$, and therefore $c_J = 0$. We conclude that

$$[\mathcal{A}_1 \times \mathcal{A}_{g-1}] = c\,\lambda_{g-1}\,.$$

for some constant c.

To determine the constant c, we pull back to $\mathcal{M}_g^{\mathrm{ct}}$ under the Torelli map Tor, and we intersect both sides with λ_{g-2}. In $\mathsf{CH}^{2g-3}(\mathcal{M}_g^{\mathrm{ct}})$, we obtain

$$\lambda_{g-2} \cdot \mathsf{Tor}^*[\mathcal{A}_1 \times \mathcal{A}_{g-1}] = c\,\lambda_{g-2}\lambda_{g-1}\,.$$

On the left hand side, $\mathsf{Tor}^*[\mathcal{A}_1 \times \mathcal{A}_{g-1}]$ will be computed in Section 5 via excess intersection theory. As we will see in (44) below, the resulting expression takes the form

$$\mathsf{Tor}^*[\mathcal{A}_1 \times \mathcal{A}_{g-1}] = \sum_{\mathsf{T}} \frac{1}{|\mathrm{Aut}_{\mathsf{T}}|}\iota_{\mathsf{T}*}\,\mathsf{Cont}_{\mathsf{T}}\,.$$

Here, T is a tree whose vertices carry genus decorations. The tree possesses a genus 1 root, and the remaining genera sum up to $g-1$. In addition, all genus 0 vertices must have valence at least 3. The contribution $\mathsf{Cont}_{\mathsf{T}}$ corresponding to the tree T is supported on the boundary stratum in $\mathcal{M}_g^{\mathrm{ct}}$ of curves with dual graph T. The map ι_{T} denotes the inclusion of this stratum.

Multiplying $\mathsf{Tor}^*[\mathcal{A}_1 \times \mathcal{A}_{g-1}]$ by λ_{g-2} sends all but one of the contributions to zero. Indeed, using (3), we see that λ_{g-2} vanishes on all trees whose vertices have genera at most $g-2$. The remaining contribution comes from the divisor

$$\iota : \mathcal{M}_{1,1}^{\mathrm{ct}} \times \mathcal{M}_{g-1,1}^{\mathrm{ct}} \to \mathcal{M}_g^{\mathrm{ct}}\,.$$

We will see in equation (43) that the excess contribution equals $\left[\frac{c(\mathbb{E}^\vee)}{1-\psi_1}\right]_{g-2}$, where the subscript denotes selecting the indicated degree. The Hodge bundle and the ψ-class here are over the second factor. Therefore,

$$\lambda_{g-2} \cdot \mathsf{Tor}^*[\mathcal{A}_1 \times \mathcal{A}_{g-1}] = \lambda_{g-2} \cdot \iota_* \left[\frac{c(\mathbb{E}^\vee)}{1-\psi_1}\right]_{g-2} = c\,\lambda_{g-2}\lambda_{g-1}\,.$$

Next, we apply the canonical evaluation ϵ^{ct} introduced in (6) to both sides of the above identity. Both sides extend naturally to the compactification $\overline{\mathcal{M}}_g$. Therefore

$$\int_{\overline{\mathcal{M}}_g} \lambda_{g-2}\lambda_g \cdot \iota_* \left[\frac{c(\mathbb{E}^\vee)}{1-\psi_1}\right]_{g-2} = c \int_{\overline{\mathcal{M}}_g} \lambda_{g-1}\lambda_{g-2}\lambda_g\,.$$

Using the splitting of the Hodge bundle (11), we see that

$$\iota^*(\lambda_{g-2}\lambda_g) = \lambda_1 \boxtimes \lambda_{g-1}\lambda_{g-2}$$

over $\overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{g-1,1}$. We have $\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1 = \frac{1}{24}$. Furthermore,

$$\int_{\overline{\mathcal{M}}_g} \lambda_g \lambda_{g-1} \lambda_{g-2} = \frac{1}{2} \int_{\overline{\mathcal{M}}_g} \lambda_{g-1}^3 = \frac{1}{2(2g-2)!} \cdot \frac{|B_{2g}|}{2g} \cdot \frac{|B_{2g-2}|}{2g-2}\,.$$

The first equality follows from Mumford's relations, while the second integral was calculated in [17, Theorem 4]. We therefore obtain

$$\frac{1}{24} \int_{\overline{\mathcal{M}}_{g-1,1}} \frac{c(\mathbb{E}^\vee)}{1-\psi_1} \lambda_{g-1} \lambda_{g-2} = \frac{c}{2(2g-2)!} \cdot \frac{|B_{2g}|}{2g} \cdot \frac{|B_{2g-2}|}{2g-2}\,.$$

To confirm the value of the constant $c = \frac{g}{6|B_{2g}|}$, we must show

$$\int_{\overline{\mathcal{M}}_{g,1}} \frac{c(\mathbb{E}^\vee)}{1-\psi_1} \lambda_g \lambda_{g-1} = \frac{|B_{2g}|}{2g \cdot (2g)!}\,, \tag{15}$$

where we have shifted from $g-1$ to g.

The integral (15) can also be extracted from [17]. Set

$$\Lambda(z) = \sum_{i=0}^{g} z^i \lambda_{g-i}\,.$$

The series

$$g(z,t) = 1 + \sum_{g=1}^{\infty} t^{2g} \int_{\overline{\mathcal{M}}_{g,1}} \frac{\Lambda(-1)\Lambda(0)\Lambda(z)}{1-\psi_1} = \left(\frac{\sin(t/2)}{t/2}\right)^{-z}$$

is computed by [19, Propositions 3 and 4]. Differentiating with respect to z, we find

$$\sum_{g=1}^{2g} t^{2g} \int_{\overline{M}_{g,1}} \frac{c(\mathbb{E}^\vee)}{1-\psi_1} \lambda_g \lambda_{g-1} = \frac{\partial}{\partial z} \left(\frac{\sin(t/2)}{t/2}\right)^{-z} \Bigg|_{z=0} = -\log\left(\frac{\sin(t/2)}{t/2}\right).$$

Finally, the identity

$$\sum_{g=1}^{\infty} \frac{|B_{2g}|}{2g \cdot (2g)!} \cdot t^{2g} = -\log\left(\frac{\sin(t/2)}{t/2}\right)$$

is established in [19, Lemma 3]. Equation (15) follows. □

3.3. **Proof of Theorem 4.** The strategy of the proof is due to Aaron Pixton. By Theorem 3, which will be proven in Section 5, the class $\mathsf{Tor}^*\Delta_g$ is tautological on $\mathcal{M}_g^{\mathrm{ct}}$. We wish to show that $\mathsf{Tor}^*\Delta_g$ is in the kernel of the pairing

$$\mathsf{R}^{g-2}(\mathcal{M}_g^{\mathrm{ct}}) \times \mathsf{R}^{g-1}(\mathcal{M}_g^{\mathrm{ct}}) \to \mathsf{R}^{2g-3}(\mathcal{M}_g^{\mathrm{ct}}) \cong \mathbb{Q}\,.$$

Let $j : \mathcal{M}_{g_1,1}^{\mathrm{ct}} \times \mathcal{M}_{g_2,1}^{\mathrm{ct}} \to \mathcal{M}_g^{\mathrm{ct}}$ be a boundary divisor, where $g_1 + g_2 = g$. Then

$$j^*\mathsf{Tor}^*\Delta_g = 0\,. \tag{16}$$

Indeed, we easily see that

$$j^*\lambda_{g-1} = \lambda_{g_1-1} \boxtimes \lambda_{g_2} + \lambda_{g_1} \boxtimes \lambda_{g_2-1} = 0$$

using that the top Hodge class vanishes on curves of compact type. Furthermore, the morphism $\mathsf{Tor} \circ j$ factors as

$$\mathcal{M}_{g_1,1}^{\mathrm{ct}} \times \mathcal{M}_{g_2,1}^{\mathrm{ct}} \xrightarrow{\mathsf{Tor}\times\mathsf{Tor}} \mathcal{A}_{g_1} \times \mathcal{A}_{g_2} \xrightarrow{p} \mathcal{A}_g\,.$$

By Proposition 17, $p^*[\mathcal{A}_1 \times \mathcal{A}_{g-1}] = 0$. Therefore,

$$j^*\mathsf{Tor}^*[\mathcal{A}_1 \times \mathcal{A}_{g-1}] = 0\,,$$

establishing (16).

On the other hand, the proof of Proposition 2 shows that

$$\lambda_{g-2} \cdot \mathsf{Tor}^*\Delta_g = 0\,. \tag{17}$$

To finish the argument, we note that $\mathsf{R}^{g-2}(\mathcal{M}_g)$ is generated by λ_{g-2}, so all classes in $\mathsf{R}^{g-2}(\mathcal{M}_g^{\mathrm{ct}})$ can be written as

$$c\lambda_{g-2} + \text{classes supported on the boundary.}$$

By (17), $\mathsf{Tor}^*\Delta_g$ pairs trivially with λ_{g-2}, while from (16), $\mathsf{Tor}^*\Delta_g$ pairs trivially with all classes supported on the boundary. Thus, $\mathsf{Tor}^*\Delta_g$ is in the Gorenstein kernel. □

4. Local equations for the Torelli pullback

4.1. **Overview.** In Sections 5 and 6, we will compute the class

$$\mathsf{Tor}^*[\mathcal{A}_1 \times \mathcal{A}_{g-1}] \in \mathsf{CH}^{g-1}(\mathcal{M}_g^{\mathrm{ct}})$$

using Fulton's intersection theory [21]. Consider the fiber product diagram

$$\begin{array}{ccc} \mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1}) & \longrightarrow & \mathcal{M}_g^{\mathrm{ct}} \\ \downarrow & & \downarrow \mathsf{Tor} \\ \mathcal{A}_1 \times \mathcal{A}_{g-1} & \longrightarrow & \mathcal{A}_g\,. \end{array}$$

The class $\mathsf{Tor}^*[\mathcal{A}_1 \times \mathcal{A}_{g-1}]$ is the pushforward to $\mathcal{M}_g^{\mathrm{ct}}$ of a refined intersection class on the fiber product $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$. The intersection calculation is subtle because $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ has many excess components that meet each other. Knowledge of the scheme structure of the fiber product $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ is required for the excess analysis. We will find local equations for $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ and prove that the scheme structure is reduced.

While we use the superscript -1 in the notation, the stack $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ is *not* a substack of $\mathcal{M}_g^{\mathrm{ct}}$. This is due to the fact that the morphism

$$\mathcal{A}_1 \times \mathcal{A}_{g-1} \to \mathcal{A}_g \tag{18}$$

is *not* an embedding because it is not injective. However, since (18) induces an injection on tangent spaces,

$$\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1}) \to \mathcal{M}_g^{\mathrm{ct}}$$

is étale locally (on the domain) an embedding.

4.2. **Extremal trees and the strata of $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$.** The points of the fiber product $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ are simple to understand. Using Proposition 16, we have the following result.

Corollary 18. *If C is a genus g curve of compact type with Jacobian isomorphic (as a principally polarized abelian variety) to a product,*

$$J(C) \cong X_1 \times X_{g-1} \textit{ with } X_1 \in \mathcal{A}_1 \textit{ and } X_{g-1} \in \mathcal{A}_{g-1}\,,$$

then C has an irreducible component C_1 of genus 1 satisfying $J(C_1) \cong X_1$.

Corollary 18 leads to a natural stratification of the fiber product $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ indexed by *extremal trees.*

Definition 19. Let $g \geq 2$ be an integer. An *extremal tree* T of genus g is a connected rooted tree with a genus assignment on the vertices,

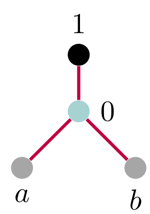
$$\mathsf{g} : \mathsf{V}(\mathsf{T}) \to \mathbb{Z}_{\geq 0}\,,$$

which satisfies the following properties:

(i) T is stable[7] with respect to g,
(ii) the root vertex has genus 1,
(iii) all vertices of T, except for the root and the leaves[8], have genus 0,
(iv) the genus condition $g = \sum_{v \in \mathsf{V}(\mathsf{T})} \mathsf{g}(v)$ holds.

Stability (i) implies that the genus of every leaf vertex is positive. An extremal tree I such that every vertex is a root or leaf is called *irreducible.*

The figure below shows an extremal tree. The root is shown as a black dot, while the leaves of genera a, b are shown as gray dots. The remaining internal vertex has genus 0. Because of the internal vertex, the extremal tree is not irreducible.



An automorphism of an extremal tree T is an automorphism of the underlying tree that fixes the root and respects g. Given an extremal tree T of genus g, we define

$$\mathcal{M}^{\mathrm{ct}}_{\mathsf{T}} = \prod_{v \in \mathsf{V}(\mathsf{T})} \mathcal{M}^{\mathrm{ct}}_{\mathsf{g}(v),\mathsf{n}(v)}\,,$$

where $\mathsf{n}(v)$ is valence of v.

[7]Stability of the tree is equivalent here to the condition that all vertices of genus 0 to have valence at least 3.
[8]A *leaf* vertex is a non-root vertex of valence 1. An *internal vertex* is a vertex that is neither a root nor a leaf.

We denote the canonical Torelli map from $\mathcal{M}_{\mathsf{T}}^{\mathrm{ct}}$ to $\mathcal{A}_1 \times \mathcal{A}_{g-1}$ by

$$\mathsf{Tor}_{1,g-1}^{\mathsf{T}} : \mathcal{M}_{\mathsf{T}}^{\mathrm{ct}} \to \mathcal{A}_1 \times \mathcal{A}_{g-1}\,,$$

where the root of T corresponds to the $\mathcal{A}_1$ factor. Let

$$\iota_{\mathsf{T}} : \mathcal{M}_{\mathsf{T}}^{\mathrm{ct}} \to \mathcal{M}_g^{\mathrm{ct}}$$

be the gluing morphism associated to T. Since $\mathsf{Tor}_{1,g-1}^{\mathsf{T}}$ and ι_{T} are equal after mapping to $\mathcal{A}_g$, we obtain a canonical map

$$\epsilon_{\mathsf{T}} : \mathcal{M}_{\mathsf{T}}^{\mathrm{ct}} \to \mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})\,.$$

Moreover, because ι_{T} and the map

$$\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1}) \to \mathcal{M}_g^{\mathrm{ct}}$$

are proper, so is ϵ_{T}. By definition, the image of ϵ_{T} is the *closed stratum* determined by T. The irreducible components of $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ are the closed strata determined by irreducible extremal trees I.

The *strict stratum* determined by T is the open subset of points of $\mathcal{M}_{\mathsf{T}}^{\mathrm{ct}}$ which do not lie in any closed strata for extremal trees T' which are nontrivial degenerations[9] of T. Let

$$\mathcal{M}_{\mathsf{T}}^{\circ\,\mathrm{ct}} = \prod_{v \in V(\mathsf{T})} \mathcal{M}_{\mathsf{g}(v),\mathsf{n}(v)}^{\circ\,\mathrm{ct}} \subset \mathcal{M}_{\mathsf{T}}^{\mathrm{ct}}, \tag{19}$$

where we define

- $\mathcal{M}_{\mathsf{g}(v),\mathsf{n}(v)}^{\circ\,\mathrm{ct}} = \mathcal{M}_{0,\mathsf{n}(v)}$ if $v \in \mathsf{V}(\mathsf{T})$ is an internal vertex,
- $\mathcal{M}_{\mathsf{g}(v),\mathsf{n}(v)}^{\circ\,\mathrm{ct}} = \mathcal{M}_{1,\mathsf{n}(v)}$ if $v \in \mathsf{V}(\mathsf{T})$ is the root,
- $\mathcal{M}_{\mathsf{g}(v),\mathsf{n}(v)}^{\circ\,\mathrm{ct}} \subset \mathcal{M}_{\mathsf{g}(v),\mathsf{n}(v)}^{\mathrm{ct}}$ is the open locus where the marking[10] lies on a component of positive genus if $v \in \mathsf{V}(\mathsf{T})$ is a leaf.

Then, the strict stratum determined by T is

$$\epsilon_{\mathsf{T}}(\mathcal{M}_{\mathsf{T}}^{\circ\,\mathrm{ct}}) \subset \mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})\,.$$

For notational convenience, we will refer to the closed and strict strata of $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ determined by T by $\mathcal{M}_{\mathsf{T}}^{\mathrm{ct}}$ and $\mathcal{M}_{\mathsf{T}}^{\circ\,\mathrm{ct}}$ respectively.

4.3. **Irreducible components of the fiber product.** We now show that the fiber product is nonsingular away from the intersections of the components. Let I be an irreducible extremal tree, $\mathcal{M}_{\mathsf{I}}^{\circ\,\mathrm{ct}}$ the associated strict stratum, and

$$\epsilon_{\mathsf{I}}^{\circ} : \mathcal{M}_{\mathsf{I}}^{\circ\,\mathrm{ct}} \to \mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$$

the restriction of ϵ_{I} to the strict stratum.

[9]Degenerations of extremal trees will be defined in Section 4.5.3 below. The definition of $\mathcal{M}_{\mathsf{T}}^{\circ\,\mathrm{ct}}$ as the complement in $\mathcal{M}_{\mathsf{T}}^{\mathrm{ct}}$ of the closed strata of nontrivial degenerations will be proven there. The definition of $\mathcal{M}_{\mathsf{T}}^{\circ\,\mathrm{ct}}$ by (19) is explicit.

[10]For a leaf v, $\mathsf{n}(v) = 1$.

Proposition 20. *The stack theoretic image of $\epsilon_{\mathsf{I}}^{\circ}$ is nonsingular. In particular, $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ is nonsingular away from the intersection of its components.*

Proof. The tangent space of $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ at a point $(C,(E,B))$ is the fiber product of the tangent space $\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)$ to $\mathcal{M}_g^{\mathrm{ct}}$ at C with the tangent space $\mathrm{Sym}^2 H^0(\Omega_E)^\vee \oplus \mathrm{Sym}^2 H^0(\Omega_B)^\vee$ to $\mathcal{A}_1 \times \mathcal{A}_{g-1}$ at (E,B) over the tangent space

$$\mathrm{Sym}^2 H^0(\Omega_E)^\vee \oplus (H^0(\Omega_E)^\vee \otimes H^0(\Omega_B)^\vee) \oplus \mathrm{Sym}^2 H^0(\Omega_B)^\vee$$

at $J(C) \cong E \times B$ of $\mathcal{A}_g$.

Let k denote the number of leaves of the extremal tree I. Assume first $k = 1$, for simplicity. A general point of $\epsilon_{\mathsf{I}}^{\circ}(\mathcal{M}_{\mathsf{I}}^{\circ\,\mathrm{ct}})$ is the Jacobian of a curve $C = E \cup D$, where E is nonsingular of genus 1 and D is nonsingular of genus $g-1$. There is a tangent vector $v \in \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)$ corresponding to the smoothing of the node p of C, hence $v \in T_pE \otimes T_pD$. Under the differential of the Torelli map

$$\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C) \to \mathrm{Sym}^2 H^0(\omega_C)^\vee \cong \mathrm{Sym}^2 H^0(\Omega_E)^\vee \oplus (H^0(\Omega_E)^\vee \otimes H^0(\Omega_D)^\vee) \oplus \mathrm{Sym}^2 H^0(\Omega_D)^\vee\,,$$

v maps to a nonzero vector in $(H^0(\Omega_E)^\vee \otimes H^0(\Omega_D)^\vee)$. Hence, v does not lie in the tangent space to $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ at $(C,(E,J(D)))$, which thus has codimension at least 1 in $\mathrm{Ext}^1(\Omega_C, \mathcal{O}_C)$. Because $\mathcal{M}_{\mathsf{I}}^{\circ\,\mathrm{ct}}$ is of codimension 1 in $\mathcal{M}_g^{\mathrm{ct}}$, we see that $\epsilon_{\mathsf{I}}^{\circ}(\mathcal{M}_{\mathsf{I}}^{\circ\,\mathrm{ct}})$ is nonsingular at $(C,(E,J(D)))$.

Next, we suppose $C = E \cup D$, where $D = \cup_{i=0}^n D_i$ is a compact type curve of genus $g-1$ glued to E at a exactly one point on D_0, where D_0 is nonsingular of genus $0 < h < g-1$. Again, we consider the tangent vector v corresponding to a family of curves smoothing the node $E \cap D_0$. Under the Torelli map, this family maps to $\mathcal{A}_{h+1} \times \mathcal{A}_{g-h-1} \subset \mathcal{A}_g$. Therefore, we can view the codomain of the differential of the Torelli map as

$$\mathrm{Sym}^2 H^0(\Omega_E)^\vee \oplus (H^0(\Omega_E)^\vee \otimes H^0(\Omega_{D_0})^\vee) \oplus \mathrm{Sym}^2 H^0(\Omega_{D_0})^\vee \oplus \mathrm{Sym}^2 \left(\bigoplus_{i=1}^n H^0(\Omega_{D_i})^\vee \right)\,,$$

where the first three summands correspond to the $\mathcal{A}_{h+1}$ factor and the latter summands correspond to the $\mathcal{A}_{g-h-1}$ factor. As above, the tangent vector v has nonzero image in the $(H^0(\Omega_E)^\vee \otimes H^0(\Omega_{D_0})^\vee)$ summand. Hence, the vector v does not lie in the tangent space to $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ at $(C,(E,J(D)))$, and the conclusion follows as in the previous paragraph.

The cases when $k > 1$ are proved analogously by analyzing the image of the tangent vectors corresponding to smoothings of the nodes represented by edges in I. ☐

4.4. **Correspondence with stable maps.** To apply excess intersection theory in Section 5 below, we will require local equations for $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$. Our analysis will show that the scheme structure of $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ is reduced.

For the study of the scheme structure, we will use a fundamental correspondence which relates $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ to a moduli space of stable maps to the universal elliptic curve. For the correspondence, a marked point is required. Consider the fiber product diagram

$$\begin{array}{ccc} \mathsf{Tor}_1^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1}) & \longrightarrow & \mathcal{M}_{g,1}^{\mathrm{ct}} \\ \big\downarrow & & \big\downarrow \mathsf{Tor}_1 \\ \mathcal{A}_1 \times \mathcal{A}_{g-1} & \longrightarrow & \mathcal{A}_g\,. \end{array} \tag{20}$$

The map Tor_1 is defined as the composition of the forgetful map $\mathcal{M}_{g,1}^{\mathrm{ct}} \to \mathcal{M}_g^{\mathrm{ct}}$ and Tor. The set-theoretic description in Section 4.3 generalizes directly to $\mathsf{Tor}_1^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$: the components and their intersections correspond to extremal trees, and the additional marked point is allowed to lie on any vertex.

Let $u : \mathcal{E} \to \mathcal{M}_{1,1}$ denote the universal elliptic curve, $s : \mathcal{M}_{1,1} \to \mathcal{E}$ the universal section, and $\mathcal{M}_{g,1}^{\mathrm{ct}}(u,1)$ the moduli space of u-relative stable maps of fiber degree 1 from compact type curves of genus g. There is a forgetful morphism

$$v : \mathcal{M}_{g,1}^{\mathrm{ct}}(u,1) \to \mathcal{M}_{g,1}^{\mathrm{ct}}$$

and an evaluation morphism

$$\mathrm{ev} : \mathcal{M}_{g,1}^{\mathrm{ct}}(u,1) \to \mathcal{E}\,.$$

Define $Q_{g,1}$ by the fiber product diagram

$$\begin{array}{ccc} Q_{g,1} & \longrightarrow & \mathcal{M}_{1,1} \\ \big\downarrow & & \big\downarrow s \\ \mathcal{M}_{g,1}^{\mathrm{ct}}(u,1) & \xrightarrow[\mathrm{ev}]{} & \mathcal{E}\,. \end{array} \tag{21}$$

The space $Q_{g,1}$ is the closed substack of $\mathcal{M}_{g,1}^{\mathrm{ct}}(u,1)$ parametrizing stable maps that send the marked point to the origin in each fiber of u.

Proposition 21. *There is a natural isomorphism*

$$F : \mathsf{Tor}_1^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1}) \to Q_{g,1}\,.$$

Proof. We begin by constructing the morphism F. There is a universal pointed curve

$$\begin{array}{c} \mathcal{C} \\ \big\downarrow \big\uparrow \sigma \\ \mathsf{Tor}_1^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1}) \end{array}$$

pulled back from $\mathcal{M}_{g,1}^{\mathrm{ct}}$. Let $\mathcal{X}_g \to \mathcal{A}_g$ be the universal abelian variety. Using the section σ, we obtain a well-defined Abel–Jacobi map

$$\mathcal{C} \to \mathcal{X}_g\,.$$

The map factors through $\mathcal{E} \times \mathcal{X}_{g-1}$. Projecting to $\mathcal{E}$, we obtain a map $\mathcal{C} \to \mathcal{E}$ sending the section σ to the origin in each fiber of $\mathcal{E}$. The construction defines the morphism F.

To show F is an isomorphism, we construct an inverse. We have a map $Q_{g,1} \to \mathcal{M}^{\mathrm{ct}}_{g,1}$ defined by the composition

$$Q_{g,1} \to \mathcal{M}^{\mathrm{ct}}_{g,1}(u,1) \to \mathcal{M}^{\mathrm{ct}}_{g,1}\,.$$

Further composing with Tor_1 defines a map $Q_{g,1} \to \mathcal{A}_g$. We show that this map factors through a map $Q_{g,1} \to \mathcal{A}_1 \times \mathcal{A}_{g-1}$, and thus induces a morphism

$$G : Q_{g,1} \to \mathsf{Tor}_1^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})\,.$$

By pulling back from $\mathcal{M}^{\mathrm{ct}}_{g,1}(u,1)$, we see that the universal curve $\mathcal{C}'$ over $Q_{g,1}$ admits a universal evaluation morphism $\mathcal{C}' \to \mathcal{E}$ sending the universal section of $\mathcal{C}'$ to the origin of $\mathcal{E}$ in the fibers. Taking Jacobians shows that the universal Jacobian $J(\mathcal{C}')$ has an elliptic curve factor. Therefore, $Q_{g,1} \to \mathcal{A}_g$ factors through $\mathcal{A}_1 \times \mathcal{A}_{g-1}$. After unwinding the definitions, F and G are easily seen to be inverses to each other. $\square$

A more general version of Proposition 21 (showing also the compatibility of virtual classes) has been recently proven by Greer and Lian [27].

4.5. **Scheme structure.**

4.5.1. *Reducedness.* The central result that controls the scheme structure of the fiber product $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ is reducedness.

Theorem 22. *The fiber product* $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ *has reduced scheme structure.*

Our proof of Theorem 22 uses several special properties of the locus $\mathcal{A}_1 \times \mathcal{A}_{g-1}$ including the connection between the fiber product $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ with the moduli space of stable maps to a moving elliptic curve provided by Proposition 21.

Whether reducedness is special for the fiber product with the Noether-Lefschetz locus $\mathcal{A}_1 \times \mathcal{A}_{g-1}$ or a property that holds for more general Torelli fiber products of Noether-Lefschetz loci is an interesting question. While preliminary calculations suggest the fiber product $\mathsf{Tor}^{-1}(\mathcal{A}_2 \times \mathcal{A}_{g-2})$ is also reduced, we do not have a proof.

4.5.2. *Strategy of proof.* The proof of Theorem 22 will be given in several steps. Consider first the strict stratum

$$\mathcal{M}^{\circ\,\mathrm{ct}}_{\mathsf{I}} \subset \mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$$

determined by an *irreducible* extremal tree I. In the irreducible case, $\mathcal{M}^{\circ\,\mathrm{ct}}_{\mathsf{I}}$ is a Zariski open set of $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$. By Proposition 20, $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ is reduced (and, in fact, nonsingular) on the Zariski open disjoint union

$$\coprod_{\mathsf{I}\,\mathrm{irr}} \mathcal{M}^{\circ\,\mathrm{ct}}_{\mathsf{I}} \subset \mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})\,. \tag{22}$$

We will prove Theorem 22 by adding strict strata

$$\mathcal{M}^{\circ\,\mathrm{ct}}_{\mathsf{T}} \subset \mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$$

for non-irreducible extremal trees T to (22) one at a time until all of $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ is covered. At each stage, we must ensure that we have a Zariski open set of $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ and that the scheme structure on the Zariski open is reduced.

4.5.3. T-*structures and degenerations.* To define the order of addition of $\mathcal{M}^{\circ\,\mathrm{ct}}_{\mathsf{T}}$ to (22), we introduce T-structures and degenerations of extremal trees.

Definition 23. Let T be an extremal tree of genus g with ℓ vertices, and let T' be a extremal tree of genus g. A T-structure on T' is given by a set partition[11] of the vertex set of T',

$$\mathcal{V} = \{\mathcal{V}_1, \ldots, \mathcal{V}_\ell\}\,, \quad \mathcal{V}_1 \cup \ldots \cup \mathcal{V}_\ell = \mathsf{V}(\mathsf{T}')\,,$$

together with a bijection

$$\phi : \mathsf{V}(\mathsf{T}) \to \{1, \ldots, \ell\}$$

satisfying the following properties:

(i) The bijection ϕ respects the root structure, $\mathsf{root} \in \mathcal{V}_{\phi(\mathsf{root})}$.

(ii) For all $v \in \mathsf{V}(\mathsf{T})$, the vertex subset $\mathcal{V}_{\phi(v)} \subset \mathsf{V}(\mathsf{T}')$ determines a connected subtree of T' with

$$\mathsf{g}(v) = \sum_{v' \in \mathcal{V}_{\phi(v)}} \mathsf{g}(v')\,.$$

(iii) An edge $e \in \mathsf{E}(\mathsf{T})$ connects the vertices $v, w \in \mathsf{V}(\mathsf{T})$ if and only if there exists an edge $e' \in \mathsf{E}(\mathsf{T}')$ which connects a vertex of $\mathcal{V}_{\phi(v)}$ to a vertex of $\mathcal{V}_{\phi(w)}$.

For an extremal tree T' to carry a T-structure, we must have

$$|\mathsf{V}(\mathsf{T})| \leq |\mathsf{V}(\mathsf{T}')|\,. \tag{23}$$

Moreover, if equality holds for (23), then a T-structure on T' is equivalent to an isomorphism of T and T' as extremal trees. We define T' to be a *nontrivial degeneration* of T if T' carries a nontrivial T-structure. We denote nontrivial degenerations by

$$\mathsf{T} \rightsquigarrow \mathsf{T}'\,.$$

We also refer to T as a *smoothing* of T'.

Lemma 24. *The strict stratum $\mathcal{M}^{\circ\,\mathrm{ct}}_{\mathsf{T}}$ is the complement in $\mathcal{M}^{\mathrm{ct}}_{\mathsf{T}}$ of the union of closed strata corresponding to nontrivial degenerations T' of T,*

$$\mathcal{M}^{\circ\,\mathrm{ct}}_{\mathsf{T}} = \mathcal{M}^{\mathrm{ct}}_{\mathsf{T}} \setminus \bigcup_{\mathsf{T} \rightsquigarrow \mathsf{T}'} \mathcal{M}^{\mathrm{ct}}_{\mathsf{T}'}\,.$$

Proof. From the definitions. □

A chain of nontrivial degenerations of *length d* is a sequence of extremal trees of genus g

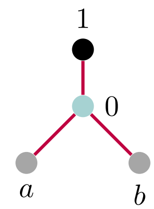
$$\mathsf{T}_0 \rightsquigarrow \mathsf{T}_1 \rightsquigarrow \ldots \rightsquigarrow \mathsf{T}_d$$

where T_{i+1} is nontrivial degeneration of T_i for $0 \leq i \leq d-1$.

[11] Every element of a set partition here is required to be non-empty.

Definition 25. An extremal tree T of genus g is has *depth* d if the maximal chain of nontrivial degenerations of extremal trees ending with T has length d.

For example, an irreducible tree I admits no nontrivial degenerations. Hence, the depth of I is 0 in the irreducible case. On the other hand, the tree below has depth 1.



We will add the strict strata $\mathcal{M}_{\mathsf{T}}^{\circ\,\mathrm{ct}}$ to (22) in order of increasing depth. We start with all the depth 0 extremal trees to obtain (22). We then add all the strict strata corresponding to the trees of depth exactly 1. Next, we add all the strict strata corresponding to the extremal trees of depth exactly 2, and so on. The resulting subsets, indexed by depth, are

$$\coprod_{\mathsf{I}\ \mathrm{irr}} \mathcal{M}_{\mathsf{I}}^{\circ\,\mathrm{ct}} = U_0 \subset U_1 \subset U_2 \subset \ldots \subset \mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})\,.$$

Lemma 26. *The subsets* $U_i \subset \mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ *constructed by the increasing depth procedure are Zariski open and cover* $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ *after finitely many steps.*

Proof. From the definitions. □

4.5.4. *Induction step: set up.* By Proposition 20, $U_0 \subset \mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ is a reduced open set. Let $d \geq 1$ and assume that

$$U_{d-1} \subset \mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$$

is a reduced open set. We will show then that

$$U_d \subset \mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$$

is also a reduced open set.

Let T be an extremal tree of genus g and of depth exactly d. Let

$$\pi : (C, p) \to (E, 0) \tag{24}$$

be stable map with $[\pi] \in Q_{g,1}$. Such a stable map has a unique irreducible component of the domain $\widehat{E}$ which maps isomorphically to the target

$$\pi|_{\widehat{E}} : \widehat{E} \cong E\,.$$

Our first assumption is:

(i) the marking p lies on $\widehat{E}$ (and is mapped to 0 under π by the definition of $Q_{g,1}$).

Via Proposition 21, $[\pi] \in Q_{g,1}$ corresponds to the point

$$F^{-1}([\pi]) \in \mathsf{Tor}_1^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})\,.$$

After forgetting the marking p, we obtain a point

$$A_\pi \in \mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})\,.$$

Our second assumption is:

(ii) A_π is an element of $\mathcal{M}_{\mathsf{T}}^{\circ\mathrm{ct}}$.

The construction can be reversed. Given $A_\pi \in \mathcal{M}_{\mathsf{T}}^{\circ\mathrm{ct}}$, we can find a stable map (24) satisfying conditions (i) and (ii). By the isomorphism of Proposition 21 and the smoothness of

$$\mathsf{Tor}_1^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1}) \to \mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$$

at $F^{-1}([\pi]) \in \mathsf{Tor}_1^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$, we can study the scheme structure of $Q_{g,1}$ near $[\pi]$ to prove the reducedness of $\mathsf{Tor}_1^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ near A_π.

4.5.5. *Local equations for the reduced scheme structure.* Our next goal is to find local equations for the reduced scheme structure

$$Q_{g,1}^{\mathsf{red}} \subset Q_{g,1}$$

near the point $[\pi : (C,p) \to (E,0)]$ satisfying conditions (i) and (ii) of Section 4.5.4. Since

$$\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1}) \to \mathcal{A}_g$$

is an immersion, $Q_{g,1}$ is a closed subscheme of $\mathcal{M}_{g,1}^{\mathrm{ct}}$ locally at $[\pi] \in Q_{g,1}$ in the analytic topology. In particular, there exists an analytic open set W,

$$[\pi] \in W \subset \mathcal{M}_{g,1}^{\mathrm{ct}}\,,$$

such that $Q_{g,1}$ is, locally at $[\pi]$, cut out by equations in W. We may take W to be a versal deformation space of $[\pi]$. We have a map from W to the deformation spaces of the nodes of C,

$$\mu : W \to \prod_{e \in \mathsf{E}(\mathsf{T})} \mathbb{C}_e\,,$$

where $\mathbb{C}_e$ is the 1-dimensional versal deformation space of the node of C corresponding to the edge e of T.

Let x_e be the standard coordinate on $\mathbb{C}_e$. Let $v \in \mathsf{V}(\mathsf{T})$ be a leaf. To v, we associate a monomial $\mathsf{Mon}(v)$ in the variables $\{x_e\}_{e\in\mathsf{E}(\mathsf{T})}$ by the following equation:

$$\mathsf{Mon}(v) = \prod_{e \in \mathsf{path}(v)} x_e\,,$$

where the product is over all edges $e \in \mathsf{E}(\mathsf{T})$ that lie on the minimal path from the leaf v to the root of T.

Proposition 27. *The reduced subscheme $Q_{g,1}^{\mathsf{red}}$ is defined in W by the pullback from $\prod_{e\in\mathsf{E}(\mathsf{T})}\mathbb{C}_e$ of the monomial set*

$$\{\,\mathsf{Mon}(v)\,|\, v \text{ is a leaf of } \mathsf{T}\,\} \subset \mathbb{C}\big[\{x_e\}_{e\in\mathsf{E}(\mathsf{T})}\big]\,.$$

Proof. The vanishing of the monomial set $\mathsf{Mon}(v)$ defines a reduced scheme, locally cut out by union of linear subspaces. Every monomial ideal with generators given by products of distinct variables is reduced (as can be proven by induction on the number of variables). Since μ is formally smooth, reducedness still holds after pullback. □

4.5.6. *Induction step: deformation theory.* Let T be an extremal tree of genus g and of depth exactly d. Let

$$\pi : (C, p) \to (E, 0) \tag{25}$$

be stable map with $[\pi] \in Q_{g,1}$ satisfying conditions (i) and (ii) of Section 4.5.4.

Near points of $Q_{g,1} \cap W$ not in the strict stratum associated to T, $Q_{g,1}$ is reduced by the induction hypothesis (via Proposition 21 and the smoothness of the point choice). To complete the induction step, we need only check that the pullbacks via μ of the monomials

$$\{\, \mathsf{Mon}(v) \,|\, v \text{ is a leaf of } \mathsf{T} \,\} \tag{26}$$

vanish on all flat deformations of the stable map (25) over Artinian bases. Since these monomials generate the reduced structure by Proposition 27, we then conclude that $Q_{g,1} \cap W$ is reduced. Therefore,

$$U_d \subset \mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$$

is also reduced.

Consider a flat deformation of the stable map (25) over $(A, \mathfrak{m})$, where $\mathfrak{m}$ is the maximal ideal in a local Artin ring A. We have a diagram

$$\begin{array}{ccc} (\mathcal{C}, \mathcal{P}) & \xrightarrow{\ \widetilde{\pi}\ } & (\mathcal{E}, 0) \\ \downarrow & & \downarrow \\ \mathsf{Spec}(A) & \xrightarrow{\ \sim\ } & \mathsf{Spec}(A)\,. \end{array} \tag{27}$$

Here, $\mathcal{P}$ is a section of $\mathcal{C} \to \mathsf{Spec}(A)$. Such a deformation maps to W and then (via μ) to $\prod_{e \in \mathsf{E}(\mathsf{T})} \mathbb{C}_e$, so we can pull back the monomials (26).

The first simplification is that we can assume the deformation of the target $(\mathcal{E}, 0) \to \mathsf{Spec}(A)$ is trivial. The moduli of stable maps $Q_{g,1}$ has locally trivial structure over the moduli space of elliptic targets

$$Q_{g,1} \to \mathcal{M}^{\mathrm{ct}}_{1,1}\,.$$

Locally analytically near $[\pi] \in Q_{g,1}$ the moduli space of stable maps is isomorphic to the moduli space of maps to a fixed elliptic target,

$$Q^E_{g,1} = \mathrm{ev}_p^{-1}(0) \subset \mathcal{M}^{\mathrm{ct}}_{g,1}(E, 1)\,,$$

times an open set of $\mathbb{C}$. Moreover, the pullbacks of the monomials (26) factor through the projection to $Q^E_{g,1}$. Therefore, we can restrict our attention to the simpler deformation:

$$(\mathcal{C},\mathcal{P}) \xrightarrow{\widetilde{\pi}} (E,0) \quad \downarrow \quad \mathsf{Spec}(A)\,. \tag{28}$$

In addition, we may assume that the root of T has valence 1, since in the argument below we can treat the connected components of the curve $C \setminus E$ one at a time.

Let v be an arbitrary leaf of T. We will show that the monomial $\mathsf{Mon}(v)$ vanishes when pulled back to $\mathsf{Spec}(A)$ via the family (28). Let

$$v - v_1 - v_2 - \ldots - v_k - \mathsf{root} \tag{29}$$

be the minimal path from the leaf v to the root of T, and let

$$D - P_1 - P_2 - \ldots - P_k - E$$

be the corresponding closed subcurves of C. The curve D is of compact type, but may be reducible. By definition,

$$[D,q] \in \mathcal{M}^{\circ\,\mathrm{ct}}_{\mathsf{g}(v),1}$$

where q is the point where D meets P_1, and the intermediate subcurves $P_1,\ldots,P_k$ are all isomorphic to $\mathbb{P}^1$ by assumption (ii) of Section 4.5.4.

Let $s \in E$ denote the nodal point corresponding to the intersection of E with P_k. In the target E, we choose a local parameter $z \in \mathcal{O}_{E,s}$ which we represent by a regular function

$$z : \Delta \to \mathbb{C}$$

in a neighborhood $s \in \Delta \subset E$. The function

$$F = \widetilde{\pi}^* z$$

is regular on the open subcurve $\widetilde{\pi}^{-1}(\Delta) \subset \mathcal{C}$. Let $\mathcal{E}^-$ be the open subcurve of $\mathcal{C}$ obtained by removing [12] all the components of C other than E. Then $\mathcal{E}^-$ is a flat deformation over $\mathsf{Spec}(A)$ of the smooth affine curve $E^- = E \setminus \{s\}$. Such a deformation is necessarily trivial by [47, Theorem 1.2.4]. Consequently, the regular functions on $\mathcal{E}^-$ are of the form $A \otimes_{\mathbb{C}} \mathcal{O}(E^-)$. The restriction of F to $\mathcal{E}^- \cap \widetilde{\pi}^{-1}(\Delta)$ is the function $1 \otimes z$.

We will construct a different function G on a subcurve of $\mathcal{C}$ (the domain will be specified below), which agrees with $1 \otimes z$ on $\mathcal{E}^- \cap \widetilde{\pi}^{-1}(\Delta)$. The strategy is then to compare F and G on the common domain and use the comparison to show the vanishing of the monomial $\mathsf{Mon}(v)$.

To specify the domain of G, we require a few preliminary constructions. Let $\mathcal{L}$ denote the set of leaves of the tree T. Each leaf in T determines a positive genus subcurve of C, not necessarily

[12] Since A is Artinian, the Zariski topologies of C and $\mathcal{C}$ are the same.

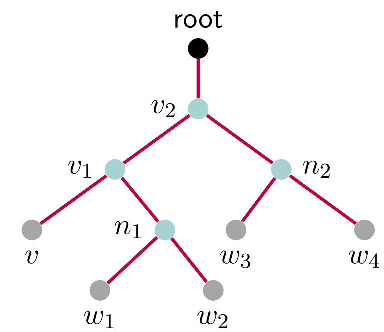
irreducible. Removing from $\mathcal{C}$ the closed subcurves corresponding to the leaves in $\mathcal{L}$ yields an open curve $\mathcal{C}^-$. The function G will be defined on $\mathcal{C}^- \cap \widetilde{\pi}^{-1}(\Delta)$.

Before constructing G explicitly, we need to single out two more curves.

- The subcurve $\mathcal{C}_v^\circ \subset \mathcal{C}$ is obtained by removing from $\mathcal{C}$ the Zariski closed set consisting of all components *not contained* on the path (29).
- We define $\mathcal{C}_v^- = \mathcal{C}_v^\circ \smallsetminus D$.

Clearly, $\mathcal{C}_v^-$ is a subcurve of $\mathcal{C}^-$. In general, the difference between these two curves is due to the internal nodes of T lying on minimal paths from the leaves in $\mathcal{L}$ to the root which are *not* on the path (29).

For example, consider the following extremal tree:



We have $\mathcal{L} = \{v, w_1, w_2, w_3, w_4\}$. Furthermore,

- the curve $\mathcal{C}^-$ is obtained by removing from $\mathcal{C}$ the components corresponding to the leaves v, w_1, w_2, w_3, w_4,
- the curve $\mathcal{C}_v^\circ$ is obtained by removing from $\mathcal{C}$ the components corresponding to the vertices $w_1, w_2, w_3, w_4, n_1, n_2$,
- the curve $\mathcal{C}_v^-$ is obtained by removing from $\mathcal{C}_v^\circ$ the component corresponding to v.

The difference between the two curves $\mathcal{C}^-$ and $\mathcal{C}_v^-$ is due to the nodes n_1 and n_2.

For general T, the map $\mathcal{C}_v^- \to \mathsf{Spec}(A)$ is a flat deformation. The fiber C_v^- over the closed point of $\mathsf{Spec(A)}$ is the chain of punctured rational curves

$$P_1^- - P_2^- - \ldots - P_k^- - E\,.$$

The punctures in P_j^- correspond to the removal of various components. For the i^{th} rational curve P_i, we fix standard coordinates:

$$[x_i : 1]\,, \quad [1 : y_i]\,, \quad x_i = \frac{1}{y_i}\,.$$

Our conventions are:

- P_k is attached to E at $[0 : 1]$,
- $[1 : 0] \in P_j$ is identified with $[0 : 1] \in P_{j-1}$, for all j,
- the curve D is attached to the point $[1 : 0] \in P_1$.

For simplicity, assume first that via the versal deformation space, the equations of the deformation of C_v^- are given by the deformations at the nodes

$$zx_k - a_k\,, \quad y_k x_{k-1} - a_{k-1}\,, \quad y_{k-1}x_{k-2} - a_{k-2}\,, \quad \dots\,, \quad y_2 x_1 - a_1\,, \tag{30}$$

where $a_k, a_{k-1}, a_{k-2}, \dots, a_1 \in A$. The general case will be considered shortly.

We define G on $\mathcal{E}^- \cap \widetilde{\pi}^{-1}(\Delta)$ as $1 \otimes z$. We first extend G to $\mathcal{C}_v^- \cap \widetilde{\pi}^{-1}(\Delta)$ as follows:

- Using the first equation $zx_k - a_k = 0$ of (30), on P_k we set
$$G = \frac{a_k}{x_k} = a_k y_k\,.$$
- Using the second equation $y_k x_{k-1} - a_{k-1} = 0$ of (30), on P_{k-1} we set
$$G = \frac{a_k a_{k-1}}{x_{k-1}} = a_k a_{k-1} y_{k-1}\,.$$
- By repeatedly applying the equations of (30), we see that G can be extended to $\mathcal{C}_v^- \cap \widetilde{\pi}^{-1}(\Delta)$.

In fact, a stronger statement can be made. Let q denote the node on P_1 corresponding to the attaching point of D, and suppose that the deformation of the node q corresponds to the local equation

$$y_1 u = a_0, \quad a_0 \in \mathfrak{m}\,,$$

where u is a local coordinate on D and $\mathfrak{m}$ is the maximal ideal of A. Then the extension G satisfies

$$G = \frac{a_k \cdots a_0}{u}$$

in an analytic neighborhood of $q \in D$.

The deformation (30) may not be the most general. In fact, the general deformation is given by

$$zx_k - a_k\,, \quad f_{k-1}(y_k)x_{k-1} - a_{k-1}\,, \quad f_{k-2}(y_{k-1})x_{k-2} - a_{k-2}\,, \quad \dots\,, \quad f_1(y_2)x_1 - a_1\,, \tag{31}$$

where $a_k, a_{k-1}, a_{k-2}, \dots, a_1 \in \mathfrak{m}$. Furthermore, $f_1, \dots, f_{k-1}$ are formal changes of coordinates centered at the origin. We may assume $f_j(0) = 0$ and $f_j'(0) = 1$, after normalization of the a's. In this case, the extension can be constructed as follows. On P_k, no changes are necessary, and $G = a_k y_k$ is still valid. We consider the inverse change of coordinate g_{k-1} such that

$$y = g_{k-1}(f_{k-1}(y))\,, \quad g_{k-1}(0) = 0\,, \quad g_{k-1}'(0) = 1\,.$$

On P_{k-1}, we set

$$G = a_k g_{k-1}\left(\frac{a_{k-1}}{x_{k-1}}\right) = a_k g_{k-1}(a_{k-1}y_{k-1}) = a_k a_{k-1} y_{k-1} + \dots\,,$$

where the higher order terms contain coefficients divisible by $a_k a_{k-1}^2$. Since the maximal ideal $\mathfrak{m}$ is nilpotent, $a_{k-1} \in \mathfrak{m}$ is nilpotent as well, and the last expression consists only in finitely many terms. We can continue in the same fashion over the remaining components $P_{k-2}, \dots, P_1$, and then to an analytic neighborhood of the node q in D. Near q, we then obtain

$$G = \frac{a_k \cdots a_0}{u} + \text{finitely many higher order terms in } \frac{1}{u}\,. \tag{32}$$

The coefficients of the higher order terms necessarily belong to the ideal spanned by $a_k \cdots a_0 \mathfrak{m}$. The extra factor of $\mathfrak{m}$ comes from the fact that the higher powers of $\frac{1}{u}$ contribute extra a's, and all $a_i \in \mathfrak{m}$. We will use these facts in Lemma 28 below.

The above procedure defines G over $\mathcal{C}_v^- \cap \widetilde{\pi}^{-1}(\Delta)$. The curve $\mathcal{C}^-$ contains other rational components. These correspond to internal vertices lying on minimal paths that join a leaf w in $\mathcal{L} \setminus \{v\}$ to one of the vertices v_j. (The case we just did corresponds to the leaf v in $\mathcal{L}$.) For those components, the argument is similar: we can extend along genus 0 components with the aid of the equations of the nodes. Since T possesses no cycles, the extension is a well-defined regular function G on $\mathcal{C}^- \cap \widetilde{\pi}^{-1}(\Delta)$.

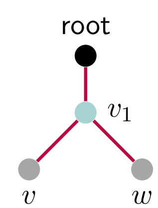
Let $h = F - G$. Since F, G are both regular on $\mathcal{C}^- \cap \widetilde{\pi}^{-1}(\Delta)$, so is h. Studying h will be crucial for the proof of the following result.

Lemma 28. *For the family* (27)*, the pullback from the versal deformation space of the nodes of* $\mathsf{Mon}(w)$ *vanishes on* $\mathsf{Spec}(A)$ *for all leaves* w *in* $\mathcal{L}$.

Lemma 28 is exactly the vanishing of $\mathsf{Mon}(v)$ we claimed, and completes the proof of the induction step and of Theorem 22. The main additional point is that the vanishing of $\mathsf{Mon}(v)$ has to be proven simultaneously with the vanishings coming from all leaves in $\mathcal{L}$.

Proof. We present a detailed argument in a representative case. For the general case, no new ideas are needed, but the notation is more complicated.

For simplicity, we assume v and root are separated by a single genus 0 vertex v_1 and v_1 is incident to only one other leaf w. We have $\mathcal{L} = \{v, w\}$, and we seek to prove the vanishing of the monomials $\mathsf{Mon}(v)$ and $\mathsf{Mon}(w)$.



The curve C here has following components:

- a genus 1 component E corresponding to the root,
- a genus zero component $P \simeq \mathbb{P}^1$ corresponding to v_1 with coordinates x, $y = 1/x$ on P,
- a positive genus curve D_v corresponding to the leaf v attached to $[1:0] \in P$ at the node q
- a positive genus curve D_w corresponding to the leaf w attached to the point $[r:1] \in P$.

The curves of compact type D_v and D_w may not be irreducible. However, the nodes corresponding to the intersections with P do not lie on genus 0 components of D_v and D_w by assumption (ii) of Section 4.5.4.

Assume first that the local equations of the nodes take the form

$$zx = a\,, \quad yu = b\,, \quad (x - r)t = c$$

where $a, b, c \in \mathfrak{m}$, and u, t are local coordinates on D_v, D_w near the respective nodes. We seek to show

$$ab = 0\,, \quad ac = 0\,.$$

The function h is regular on

$$\mathcal{C}^- \cap \widetilde{\pi}^{-1}(\Delta) = \widetilde{\pi}^{-1}(\Delta) \smallsetminus (D_v \cup D_w)\,.$$

In particular, h is regular on $\mathcal{P}^-$, the open set above $P^- = P \smallsetminus \{[r:1],[1:0]\}$. Since P^- is affine and nonsingular, the deformation $\mathcal{P}^-$ is trivial. Thus, the regular functions on $\mathcal{P}^-$ are of the form $A \otimes_{\mathbb{C}} \mathcal{O}(P^-)$. We can write

$$h = \alpha + \sum_{i=1}^{M} s_i \otimes \frac{1}{y^i} + \sum_{i=1}^{M} t_i \otimes \frac{1}{(x-r)^i}\,, \tag{33}$$

for $\alpha \in A$, $s_i \in A$, and $t_i \in A$. Let $\mathfrak{s}$ and $\mathfrak{t}$ be the two ideals in A spanned by the s_i and t_i respectively. By definition, $F = h + G$ is regular over $\widetilde{\pi}^{-1}(\Delta)$.

We inspect next the curves D_v and D_w, which we assume for now to be irreducible. Let D_v^-, D_w^- be the two affine curves obtained from D_v, D_w by removing the nodes corresponding to the intersections with P. The induced deformations $\mathcal{D}_v^-, \mathcal{D}_w^-$ of D_v^-, D_w^- are trivial. Recall that $G = ab \otimes \frac{1}{u}$ over $\mathcal{D}_v^-$. Furthermore, over the trivial deformation $\mathcal{D}_v^-$ we have

$$F = \sum_{i=1}^{d} a_i \otimes f_i$$

where $a_i \in A$ and f_i are regular on D_v^-. Therefore, we can write locally

$$h = \sum_{i=1}^{d} a_i \otimes f_i - ab \otimes \frac{1}{u} = \sum_{i=1}^{d} a_i \otimes \left(\sum_j f_{ij} u^j \right) - ab \otimes \frac{1}{u}\,, \tag{34}$$

where $f_{ij} \in \mathbb{C}$. On the other hand, we examine expression (33). Expanding near the node q (with coordinate $y = 0$):

$$\frac{1}{x-r} = \sum_{i \geq 0} c_i y^i\,,$$

we obtain[13]

$$h = \alpha + \sum_{i>0} \frac{s_i}{b^i} \otimes u^i + \sum_{i>0} \widetilde{t}_i b^i \otimes \frac{1}{u^i}\,,$$

where $\widetilde{t}_i \in \mathfrak{t}$ are combinations of the t_i's and c_i's. The second sum only requires finitely many terms since $b \in \mathfrak{m}$ is nilpotent. Comparing with (34), we conclude $s_i/b^i \in A$ for all $i > 0$. Now, we analyze the expression h over $\mathsf{Spec}(A/\mathfrak{t})$. Reducing mod $\mathfrak{t}$, we obtain

$$h = \alpha + \sum_{i>0} \frac{s_i}{b^i} \otimes u^i$$

[13]Let $x, y \in A$. By $x/y \in A$, we mean an element of A satisfying the property $y \cdot (x/y) = x \in A$.

over $D_v^- \times \mathsf{Spec}(A/\mathfrak{t})$. On the other hand, $G = ab \otimes \frac{1}{u}$ and we can write the reduction

$$F \mod \mathfrak{t} = \sum_{i=1}^{N} \widetilde{a}_i \otimes \widetilde{f}_i \tag{35}$$

where $\widetilde{a}_i$'s are a basis for the Artinian ring $A/\mathfrak{t}$ as a $\mathbb{C}$-vector space, and $\widetilde{f}_i$ are regular[14] on D_v^-. Therefore,

$$\sum_{i=1}^{N} \widetilde{a}_i \otimes \widetilde{f}_i = ab \otimes \frac{1}{u} + \alpha + \sum_{i>0} \frac{s_i}{b^i} \otimes u^i \,.$$

Expressing the images of α, ab, $\frac{s_i}{b^i}$ under the map $A \to A/\mathfrak{t}$ in terms of the basis $\widetilde{a}_i$, we see that one of the $\widetilde{f}_i$ on D_v^- must admit at worst a simple pole at $u = 0$ (and only there). No such function exists on a positive genus curve. In fact, all functions with at worst simple pole only at $u = 0$ are constant, and hence their expansion contains no nonnegative powers of u. We thus obtain

$$ab = 0 \in A/\mathfrak{t} \quad \text{and} \quad s_i/b^i = 0 \text{ in } A/\mathfrak{t}\,. \tag{36}$$

Therefore, for $i > 0$ we have

$$s_i/b^i \in \mathfrak{t} \implies s_i \in b^i\mathfrak{t} \subset \mathfrak{m}\mathfrak{t}\,.$$

We conclude

$$ab \in \mathfrak{t}\,, \quad \mathfrak{s} \subset \mathfrak{m}\mathfrak{t}\,.$$

The parallel analysis for D_w^- yields

$$ac \in \mathfrak{s}\,, \quad \mathfrak{t} \subset \mathfrak{m}\mathfrak{s}\,.$$

Let $\mathfrak{i} = \mathfrak{s} + \mathfrak{t}$. The above conclusions show that

$$ab \in \mathfrak{i}\,, \quad ac \in \mathfrak{i}\,, \quad \mathfrak{i} \subset \mathfrak{m}\mathfrak{i}\,.$$

Since $\mathfrak{m}$ is nilpotent in A, we find $\mathfrak{i} = 0$ and hence $ab = ac = 0$, as required.

To address the most general deformation, consider the equations of the nodes of the form

$$zx = a\,, \quad f(y)u = b\,, \quad g(x-r)t = c$$

where f, g are normalized changes of coordinates with $f(0) = g(0) = 0, f'(0) = g'(0) = 1$. Let $\widetilde{y} = f(y)$, so that

$$\widetilde{y}u = b\,.$$

Since $\widetilde{y}$ is a local coordinate near the node q, we have

$$\frac{1}{y} = \frac{1}{\widetilde{y}} + \sum_{i \geq 0} \epsilon_i \widetilde{y}^i, \quad \epsilon_i \in \mathbb{C}\,.$$

Similarly, we can expand near q:

$$\frac{1}{x-r} = \frac{y}{1-yr} = \sum_{i \geq 0} \tau_i \widetilde{y}^i, \quad \tau_i \in \mathbb{C}\,.$$

[14] We are *not* claiming that a_i, f_i project to $\widetilde{a}_i$, $\widetilde{f}_i$ under $A \to A/\mathfrak{t}$.

Using $\widetilde{y}u = b$, and substituting into the expression (33), we can write

$$h = \alpha' + \sum_{i=1}^{M} \frac{s_i'}{b^i} \otimes u^i + \sum_{i>0} \widetilde{s}_i b^i \otimes \frac{1}{u^i} + \sum_{i>0} \widetilde{t}_i b^i \otimes \frac{1}{u^i}$$

on $\mathcal{D}_v^-$. Since b is nilpotent, all sums are finite. As before $s_i'/b^i \in A$ for $1 \le i \le M$. It is not hard to write down the expressions for the new coefficients $s_i' \in A$. In fact, for $1 \le i \le M$, we find

$$s_i' = s_i + \text{terms involving } s_{i+1}, \ldots, s_M \text{ with coefficients that depend on } \epsilon\,.$$

Thus, we have

$$\mathfrak{s} = \langle s_1', \ldots, s_M' \rangle\,, \tag{37}$$

where as before $\mathfrak{s}$, $\mathfrak{t}$ are the two ideals generated by s_i and t_i. Furthermore,

$$\widetilde{s}_i \in \mathfrak{s}, \quad \widetilde{t}_i \in \mathfrak{t}\,.$$

We also have by (32):

$$G = ab \otimes \frac{1}{u} + \text{higher order terms in } \frac{1}{u} \text{ with coefficients in } ab \cdot \mathfrak{m}\,.$$

The next step is to reduce modulo the ideal $\mathfrak{a} = \mathfrak{m}(\mathfrak{s} + \mathfrak{t} + \langle ab \rangle)$. This reduction kills many terms in h (using $b \in \mathfrak{m}$) and G:

$$h \mod \mathfrak{a} = \alpha' + \sum_{i=1}^{M} \frac{s_i'}{b^i} \otimes u^i, \quad G \mod \mathfrak{a} = ab \otimes \frac{1}{u}\,.$$

Writing

$$F = \sum_{i=1}^{N} \widetilde{a}_i \otimes \widetilde{f}_i$$

over $\mathsf{Spec}(A/\mathfrak{a}) \times D_v^-$, with $\widetilde{a}_i$ giving a basis for $A/\mathfrak{a}$, we find that one of the functions $\widetilde{f}_i$ has at worst simple pole only at $u = 0$. As before, this implies

$$ab = 0 \mod \mathfrak{a}, \quad s_i'/b^i = 0 \mod \mathfrak{a}\,.$$

Consequently, $ab \in \mathfrak{a}$. Moreover, $s_i' \in \mathfrak{a}$, and hence by (37) we have $\mathfrak{s} \subset \mathfrak{a}$. Therefore, we established

$$\langle ab \rangle + \mathfrak{s} \subset \mathfrak{m}(\mathfrak{s} + \mathfrak{t} + \langle ab \rangle)\,.$$

A similar argument shows

$$\langle ac \rangle + \mathfrak{t} \subset \mathfrak{m}(\mathfrak{s} + \mathfrak{t} + \langle ac \rangle)\,.$$

Let $\mathfrak{i} = \mathfrak{s} + \mathfrak{t} + \langle ab, ac \rangle$. Adding the two inclusions above gives

$$\mathfrak{i} \subset \mathfrak{m}\mathfrak{i}\,.$$

Since $\mathfrak{m}$ is nilpotent, it follows $\mathfrak{i} = 0$, hence $ab = 0$, $ac = 0$.

When the curves D_v or D_w are not irreducible, the argument is parallel. In the irreducible case, a key step in the argument is (36). This relied upon the fact that there are no nonconstant functions over nonsingular projective curves of positive genus possessing at worst one simple pole. The same is true over curves of compact type (X, q), provided $q \in X$ is a nonsingular point of an irreducible component T of positive genus:

$$H^0(X, \mathcal{O}_X(q)) = \mathbb{C}\,. \tag{38}$$

To see this, let $T_1, \ldots, T_\ell$ denote the connected components of the closure of $X \smallsetminus T$ in X. Let $q_1, \ldots, q_\ell$ denote the corresponding nodes. The claim follows from the exact sequence

$$0 \to \bigoplus_{i=1}^{\ell} \mathcal{O}_{T_i}(-q_i) \to \mathcal{O}_X(q) \to \mathcal{O}_T(q) \to 0\,.$$

Let us give more details on how the proof is completed from here. By cohomology and base change, we first promote (38) to the following family version. Write for simplicity $\mathcal{Y} = \mathsf{Spec}(A)$. Assume $\pi : \mathcal{X} \to \mathcal{Y}$ is a flat proper family with a section $\mathcal{Q} : \mathcal{Y} \to \mathcal{X}$, such that basechanging to $A/\mathfrak{m} \simeq \mathbb{C}$, the pair (X, q) is a pointed curve of compact type satisfying the above conditions. Then

$$\pi_*(\mathcal{O}_{\mathcal{X}}(\mathcal{Q})) = \mathcal{O}_{\mathcal{Y}}\,. \tag{39}$$

The argument is standard. We first form the commutative diagram

$$\begin{array}{ccccc} \mathcal{O}_{\mathcal{Y}} \otimes \mathbb{C}_y & \longrightarrow & \pi_*\mathcal{O}_{\mathcal{X}} \otimes \mathbb{C}_y & \longrightarrow & H^0(X, \mathcal{O}_X) = \mathbb{C}_y \\ \Vert & & \downarrow & & \downarrow \simeq \\ \mathcal{O}_{\mathcal{Y}} \otimes \mathbb{C}_y & \longrightarrow & \pi_*(\mathcal{O}_{\mathcal{X}}(\mathcal{Q})) \otimes \mathbb{C}_y & \longrightarrow & H^0(X, \mathcal{O}_X(q)) = \mathbb{C}_y\,. \end{array}$$

The composition of the arrows on the second row is surjective (because the same is true for the top row). By cohomology and base change, it follows that the second map on the second row is in fact an isomorphism, and furthermore $\pi_*(\mathcal{O}_{\mathcal{X}}(\mathcal{Q}))$ is locally free of rank 1. Thus, the first map on the second row is surjective

$$\mathcal{O}_{\mathcal{Y}} \otimes \mathbb{C}_y \to \pi_*(\mathcal{O}_{\mathcal{X}}(\mathcal{Q})) \otimes \mathbb{C}_y\,.$$

By Nakayama's Lemma, this implies

$$\mathcal{O}_{\mathcal{Y}} \to \pi_*(\mathcal{O}_{\mathcal{X}}(\mathcal{Q}))$$

is a surjective morphism of vector bundles of the same rank, hence an isomorphism.

Returning to the original proof, let us assume D_v is reducible, and let T be the irreducible component intersecting the genus zero curve P at the node q. Let T^- be the smooth affine curve obtained by removing from T all nodes, and $\mathcal{T}^- \to \mathsf{Spec}(A)$ be the deformation obtained by restricting the flat family $\mathcal{C} \to \mathsf{Spec}(A)$. The deformation $\mathcal{T}^-$ is necessarily trivial. On the other hand, removing from $\mathcal{C}$ the components E and P, we obtain a flat curve $\mathcal{Z} \to \mathsf{Spec}(A)$. We glue $\mathcal{Z}$ to the trivial deformation of T over $\mathsf{Spec}(A)$ along $\mathcal{T}^-$, yielding a flat curve $\mathcal{X} \to \mathsf{Spec}(A)$ with a section $\mathcal{Q}$ corresponding to the node q. Now, keeping the same notation as in the proof of (36), the function F has the property that in a neighborhood of q, we have

$$F = \frac{ab}{u} + \text{positive powers of } u \mod \mathsf{t}\,.$$

Thus, F is a section of $\mathcal{O}_{\mathcal{X}}(\mathcal{Q})$, of course after basechanging to $\mathsf{Spec}(A/\mathsf{t})$. Therefore, by (39), we have $F \in A/\mathsf{t}$. We thus obtain assertion (36), and the proof is completed as before. $\square$

4.6. **Local equations.** As a consequence of the proof of Theorem 22, we have constructed canonical equations for $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ at every point (expressed in the versal deformation space of the corresponding compact type curve). Since we will require these equations for the excess intersection calculation, we record the result as follows.

Proposition 29. *The local equations of* $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ *near a point in the strict stratum indexed by* T *are given by the pullback from* $\prod_{e \in \mathsf{E}(\mathsf{T})} \mathbb{C}_e$ *of the monomial set*

$$\{\, \mathsf{Mon}(v) \,|\, v \text{ is a leaf of } \mathsf{T} \,\} \subset \mathbb{C}\big[\{x_e\}_{e \in \mathsf{E}(\mathsf{T})}\big]\,.$$

5. Excess intersection theory

5.1. **Overview.** We have the fiber product diagram

$$\begin{array}{ccc} \mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1}) & \longrightarrow & \mathcal{M}_g^{\mathrm{ct}} \\ \downarrow & & \downarrow \mathsf{Tor} \\ \mathcal{A}_1 \times \mathcal{A}_{g-1} & \longrightarrow & \mathcal{A}_g\,. \end{array}$$

By Fulton's intersection theory [21], the class $\mathsf{Tor}^*[\mathcal{A}_1 \times \mathcal{A}_{g-1}]$ is the pushforward to $\mathcal{M}_g^{\mathrm{ct}}$ of a refined intersection class on the fiber product $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$. We give an inductive method to compute the refined class based on the local equations of Section 4 for the strata of $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$. We illustrate the method with several examples that will be used later to prove Theorems 3, 4, and 5.

5.2. **Inductive method for the excess calculation.** The fiber product $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ is stratified with strata indexed by extremal trees of genus g. The partial ordering on the strata corresponds to smoothing of the extremal trees: an extremal tree T' is a smoothing of an extremal tree T if T has a nontrivial T' structure, see Section 4.5.3.

By repeated application of the excision sequence, $\mathsf{Tor}^*[\mathcal{A}_1 \times \mathcal{A}_{g-1}]$ can be expressed as a sum of contributions $\mathsf{Cont}_{\mathsf{T}}$ supported on $\mathcal{M}_{\mathsf{T}}^{\mathrm{ct}}$ for each extremal tree T of genus g. Because the degree of $\mathsf{Tor}^*[\mathcal{A}_1 \times \mathcal{A}_g]$ is $g-1$, only extremal trees with at most $g-1$ edges contribute: if $|\mathsf{E}(\mathsf{T})| \geq g$, then $\mathsf{Cont}_{\mathsf{T}} = 0$. The contributions will be computed inductively. The base cases for the induction are the extremal trees that admit no smoothings. These are the irreducible extremal trees, which correspond to the irreducible components of $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$.

The formula for the contributions is in terms of the Chern classes of the normal bundle to $\mathcal{A}_1 \times \mathcal{A}_{g-1} \subset \mathcal{A}_g$ and the Chern classes of the normal bundles of the substacks in the stratification of $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ by extremal trees. These contributions can be found using excess residual intersections as in [21, Chapter IX]. When one of the components is divisorial and the residual scheme is a regular embedding, [21, Corollary 9.2.1] gives a formula for the residual contribution in terms of the Chern classes of the normal bundles (of the residual scheme and its intersection with the divisorial part). The arbitrary case is reduced to this situation using suitable blowups and is treated in [21, Corollary 9.2.3]. Crucially for us, the exact residual contributions are *universal expressions* depending only on the normal bundle data. We can therefore compute these contributions in a suitable local model. The local equations in Proposition 29 will be used for the local model calculations.

Let T be an extremal tree with n edges and k leaves. The local model near the stratum $\mathcal{M}^{\mathrm{ct}}_{\mathsf{T}}$ of $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ is constructed as follows. We start with torus equivariant space $\mathbb{C}^n$. The coordinates on $\mathbb{C}^n$ are placed in bijection with edges e of T, and so we label the coordinates by $\{z_e\}_{e\in \mathsf{E}(\mathsf{T})}$. The variable z_e corresponds to the weight of the torus in the local model and to the normal bundles of the smoothing of the node corresponding to the edge e in the moduli of curves.

Let $v \in \mathsf{V}(\mathsf{T})$ be a leaf and $\mathsf{path}(v) \subset \mathsf{E}(\mathsf{T})$ be the set of edges on the minimal path from v to the root of T. We set

$$\mathsf{Mon}(v) = \prod_{e \in \mathsf{path}(v)} z_e \,.$$

Let $\mathcal{N}$ be a rank $g-1$ vector bundle on $\mathbb{C}^n$ of the form

$$\mathcal{N} = \mathcal{O}^{\oplus k} \oplus L_1 \oplus \cdots \oplus L_{g-1-k}\,,$$

where the L_i are arbitrary torus equivariant line bundles. Consider the section

$$s = (\mathsf{Mon}(v_1), \ldots, \mathsf{Mon}(v_k), 0, \ldots, 0) \in H^0(\mathcal{N})\,,$$

where $v_1, \ldots, v_k$ are the leaves of T. The local model for the excess intersection geometry of $\mathsf{Tor}^*[\mathcal{A}_1 \times \mathcal{A}_g]$ near the stratum $\mathcal{M}^{\mathrm{ct}}_{\mathsf{T}}$ is the excess calculation of $c_{g-1}(\mathcal{N})$ determined by the zero locus of s.

In the local model, we have

$$c_{g-1}(\mathcal{N}) = (\ell_1 \cdot \ldots \cdot \ell_{g-1-k}) \prod_{i=1}^{k} \Big(\sum_{e \in \mathsf{path}(v_i)} z_e \Big)\,,$$

where the ℓ_i are the equivariant Chern classes of L_i.

(i) First consider the case where T is an irreducible extremal tree. Then, $n = k$. The contribution $\mathsf{Cont}_{\mathsf{T}}$ can be computed by the usual excess intersection formula:

$$\mathsf{Cont}_{\mathsf{T}} = \left[\frac{c(\mathcal{N})}{\prod_{e\in \mathsf{E}(\mathsf{T})} (1+z_e)} \right]_{g-1-k} . \tag{40}$$

The subscript indicates that only the part of degree $g-1-k$ is considered. The pushforward to the ambient torus equivariant $\mathbb{C}^k$ is computed by multiplying by the top Chern class of the normal bundle which equals $z_1 \cdot \ldots \cdot z_k$. Thus

$$\iota_{\mathsf{T}*}\mathsf{Cont}_{\mathsf{T}} = z_1 \cdot \ldots \cdot z_k \left[\frac{c(\mathcal{N})}{\prod_{e\in \mathsf{E}(\mathsf{T})} (1+z_e)} \right]_{g-1-k} .$$

(ii) Next, let T be an arbitrary extremal tree. By induction, we can assume we have computed $\mathsf{Cont}_{\mathsf{T}'}$ for all smoothings T' of T. [15] We set

$$\mathsf{Cont}_{\mathsf{T}} \cdot \prod_{e \in \mathsf{E}(\mathsf{T})} z_e = c_{g-1}(\mathcal{N}) - \sum_{\mathsf{T}'} \iota_{\mathsf{T}'*}\mathsf{Cont}_{\mathsf{T}'} \,. \tag{41}$$

Solving equation (41) gives a formula for $\mathsf{Cont}_{\mathsf{T}}$.

The expression for $\mathsf{Cont}_{\mathsf{T}}$ thus obtained depends on the variables z_e and ℓ_i. Since $\mathsf{Cont}_{\mathsf{T}}$ is symmetric in the ℓ_i, we can write

$$\mathsf{Cont}_{\mathsf{T}} = \mathsf{P}_{\mathsf{T}}(Z, \mathcal{N})\,,$$

where P_{T} is a uniquely determined polynomial in the variables $Z = \{z_e\}_{e \in \mathsf{E}(\mathsf{T})}$ and the Chern classes of $\mathcal{N}$.

The formula for $\mathsf{Cont}_{\mathsf{T}}$ in terms of tautological classes is then obtained via substitution of variables:

- we replace each edge variable z_e by the normal factor corresponding to the smoothing of the edge e (the sum of tangent lines corresponding to the two half-edges of e),
- we replace the Chern classes of $\mathcal{N}$ by the Chern classes of the normal bundle of the immersion

$$\mathcal{A}_1 \times \mathcal{A}_{g-1} \to \mathcal{A}_g \,.$$

In the end, $\mathsf{Cont}_{\mathsf{T}}$ is expressed in terms of tautological ψ and λ classes obtained from the moduli of curves.

5.3. **Excess contributions of the irreducible components.** We continue to work with the fiber diagram

$$\begin{array}{ccc} \mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1}) & \longrightarrow & \mathcal{M}_g^{\mathrm{ct}} \\ \big\downarrow & & \big\downarrow \mathsf{Tor} \\ \mathcal{A}_1 \times \mathcal{A}_{g-1} & \longrightarrow & \mathcal{A}_g \,. \end{array}$$

Recall from Section 4.2 that the irreducible components of the fiber product are indexed by irreducible extremal trees I,

$$\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_g) = \bigcup_{\mathsf{I}} \mathcal{M}_{\mathsf{I}}^{\mathrm{ct}} \,.$$

We let k denote the number of leaves in I, and we let $g_1, \ldots, g_k$ denote the genus assignment for each of the k leaves, so that

$$g_1 + \ldots + g_k = g - 1\,.$$

Thus $\mathcal{M}_{\mathsf{I}}^{\mathrm{ct}}$ is covered by the product

$$\mathcal{M}_{1,k}^{\mathrm{ct}} \times \mathcal{M}_{g_1,1}^{\mathrm{ct}} \times \ldots \times \mathcal{M}_{g_k,1}^{\mathrm{ct}} \,.$$

The irreducible component $\mathcal{M}_{\mathsf{I}}^{\mathrm{ct}}$ has codimension k in $\mathcal{M}_g^{\mathrm{ct}}$. On the other hand, the expected codimension of $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_{g-1})$ in $\mathcal{M}_g^{\mathrm{ct}}$ is $g-1$. Thus, the only component $\mathcal{M}_{\mathsf{I}}^{\mathrm{ct}}$ of the expected

[15] The contributions $\mathsf{Cont}_{\mathsf{T}'}$ depend on the variables $\{z_e\}$, for $e \in \mathsf{E}(\mathsf{T}') \subset \mathsf{E}(\mathsf{T})$. The latter inclusion holds since each edge of T' corresponds to a unique edge of T thanks to Definition 23 (iii).

codimension corresponds to the tree with $g-1$ leaves attached to the root and genus distribution $(1,\dots,1)$.

The excess contribution of the locus $\mathcal{M}_{\mathsf{I}}^{\mathrm{ct}}$ is given by (40). By (12), the normal bundle of the immersion $\mathcal{A}_1 \times \mathcal{A}_{g-1} \to \mathcal{A}_g$ is

$$\mathrm{Sym}^2(\mathbb{E}_1^\vee \boxplus \mathbb{E}_{g-1}^\vee) - \mathrm{Sym}^2\,\mathbb{E}_1^\vee - \mathrm{Sym}^2\,\mathbb{E}_{g-1}^\vee = \mathbb{E}_1^\vee \boxtimes \mathbb{E}_{g-1}^\vee\,.$$

When pulled back to $\mathcal{M}_{\mathsf{I}}^{\mathrm{ct}}$, the normal bundle splits as

$$\mathbb{E}_1^\vee \boxtimes (\mathbb{E}_{g_1}^\vee \boxplus \ldots \boxplus \mathbb{E}_{g_k}^\vee)\,. \tag{42}$$

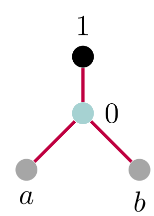
The normal bundle of $\mathcal{M}_{\mathsf{I}}^{\mathrm{ct}}$ in $\mathcal{M}_g^{\mathrm{ct}}$ is the sum of contributions corresponding to the smoothings of each of the k nodes of the curve. Therefore, the excess contribution for $\mathcal{M}_{\mathsf{I}}^{\mathrm{ct}}$ equals

$$\left[\frac{\prod_{i=1}^k c(\mathbb{E}_{g_i}^\vee)}{\prod_{e\in\mathsf{E}(\mathsf{I})}(1-\psi'_e-\psi''_e)}\right]_{g-1-k}, \tag{43}$$

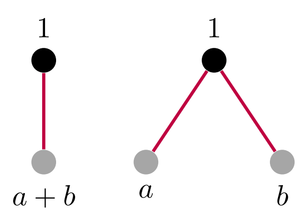
where ψ'_e, ψ''_e are the cotangent classes at the node associated to e. The Hodge bundle over $\mathbb{E}_1 \to \mathcal{A}_1$ does not enter the expression since $c_1(\mathbb{E}_1)$ vanishes. The subscript indicates that only the part of degree $g-1-k$ is considered.

5.4. **Examples.** We work out a few explicit examples that will play a role in Section 6. The examples can all be calculated by hand. For the reader's convenience, we provide code for the computations in [9].

Example 30. Consider the following extremal tree T with 4 vertices: the root shown as a black dot, an internal vertex of genus 0, and two leaves of genera a, b.



The extremal tree T has two nontrivial smoothings, R and S.



The extremal trees R, S have no further smoothings, and their contributions can be computed using equation (40) for the irreducible case:

$$\mathsf{Cont}_{\mathsf{R}} = \left[\frac{c(\mathcal{N})}{1+z_1}\right]_{a+b-1}, \quad \mathsf{Cont}_{\mathsf{S}} = \left[\frac{c(\mathcal{N})}{(1+z_2)(1+z_3)}\right]_{a+b-2}.$$

Here, we label the edge of T incident to root by z_1, while the remaining edges are labelled by z_2, z_3. From equation (41), we have

$$\mathsf{Cont}_{\mathsf{T}} \cdot z_1 z_2 z_3 = c_{a+b}(\mathcal{N}) - z_1 \left[\frac{c(\mathcal{N})}{1+z_1}\right]_{a+b-1} - z_2 z_3 \left[\frac{c(\mathcal{N})}{(1+z_2)(1+z_3)}\right]_{a+b-2} .$$

For later use, we explicitly record a few special cases.

(i) Assume first $g-1 = a+b = 3$. Then, the total Chern class of $\mathcal{N}$ is

$$c(\mathcal{N}) = (1+z_1+z_2)(1+z_1+z_3)(1+\ell_1) .$$

Expanding the two power series and dividing through by $z_1 z_2 z_3$, we obtain

$$\mathsf{Cont}_{\mathsf{T}} = -3 .$$

(ii) Asume now that $a+b = g-1 = 4$. Then, the total Chern class of $\mathcal{N}$ is

$$c(\mathcal{N}) = (1+z_1+z_2)(1+z_1+z_3)(1+\ell_1)(1+\ell_2) .$$

Expanding the two power series and dividing through by $z_1 z_2 z_3$, we obtain

$$\mathsf{Cont}_{\mathsf{T}} = z_2 + z_3 - 3\ell_1 - 3\ell_2 = -3c_1(\mathcal{N}) + 6z_1 + 4z_2 + 4z_3 .$$

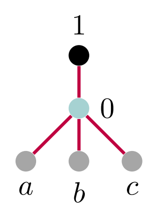
(iii) In the same scenario as above, except with $g-1 = a+b = 5$, we write

$$c(\mathcal{N}) = (1+z_1+z_2)(1+z_1+z_3)(1+\ell_1)(1+\ell_2)(1+\ell_3) .$$

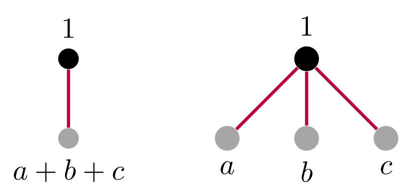
Solving the recursion, we find

$$\mathsf{Cont}_{\mathsf{T}} = -3c_2(\mathcal{N}) + c_1(\mathcal{N}) \cdot (6z_1 + 4z_2 + 4z_3) - 10z_1^2 - 10z_1 \cdot (z_2+z_3) - 5(z_2+z_3)^2 + 5z_2 z_3 .$$

Example 31. Next, consider the extremal tree T shown below.



There are two nontrivial smoothings, R and S.



We label the edge incident to the root by z_1, while the remaining edges are labelled by z_2, z_3, z_4. The contributions of R and S are obtained from (40):

$$\mathsf{Cont}_{\mathsf{R}} = \left[\frac{c(\mathcal{N})}{1+z_1}\right]_{a+b+c-1} ,$$

$$\mathsf{Cont}_{\mathsf{S}} = \left[\frac{c(\mathcal{N})}{(1+z_2)(1+z_3)(1+z_4)}\right]_{a+b+c-3} .$$

From equation (41), we have

$$\mathsf{Cont}_\mathsf{T}\cdot z_1z_2z_3z_4 = c_{a+b+c}(\mathcal{N}) - z_1\left[\frac{c(\mathcal{N})}{1+z_1}\right]_{a+b+c-1} - z_2z_3z_4\left[\frac{c(\mathcal{N})}{(1+z_2)(1+z_3)(1+z_4)}\right]_{a+b+c-3}.$$

(i) When $a+b+c=g-1=4$, we have

$$c(\mathcal{N}) = (1+z_1+z_2)(1+z_1+z_3)(1+z_1+z_4)(1+\ell_1).$$

Solving for Cont_T, we obtain

$$\mathsf{Cont}_\mathsf{T} = -4.$$

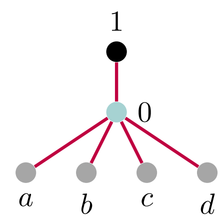
(ii) Assuming $a+b+c=g-1=5$, we have

$$c(\mathcal{N}) = (1+z_1+z_2)(1+z_1+z_3)(1+z_1+z_4)(1+\ell_1)(1+\ell_2).$$

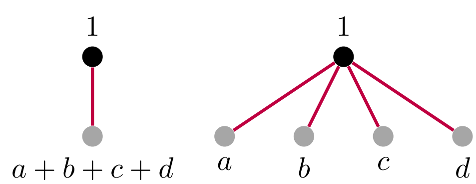
Solving for Cont_T, we obtain

$$\mathsf{Cont}_\mathsf{T} = -4\ell_1 - 4\ell_2 - 2z_1 + z_2 + z_3 + z_4 = -4c_1(\mathcal{N}) + 10z_1 + 5(z_2+z_3+z_4).$$

Example 32. Consider the extremal tree T



with smoothings



We label the edge incident to the root by z_1, while the remaining edges are labelled by z_2, z_3, z_4, z_5. We find

$$\mathsf{Cont}_\mathsf{T} = -5$$

when $a+b+c+d=g-1=5$. The contribution is computed from the recursion

$$\mathsf{Cont}_\mathsf{T}\cdot z_1z_2z_3z_4z_5 = c_5(\mathcal{N}) - z_1\left[\frac{c_1(\mathcal{N})}{1+z_1}\right]_4 - z_2z_3z_4z_5\left[\frac{c(\mathcal{N})}{(1+z_2)(1+z_3)(1+z_4)(1+z_5)}\right]_1,$$

where

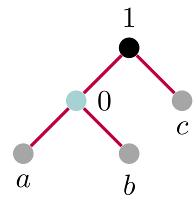
$$c(\mathcal{N}) = (1+z_1+z_2)(1+z_1+z_3)(1+z_1+z_4)(1+z_1+z_5)(1+\ell_1).$$

Remark 33. In general, for an extremal tree T with a single genus 0 vertex attached to the root, and with adjacent leaves of genera $g_1,\ldots,g_k$ with $g_1+\ldots+g_k=k+1=g-1$, the solution of the above recursion yields

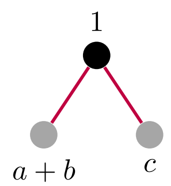
$$\mathsf{Cont}_\mathsf{T} = -(k+1),$$

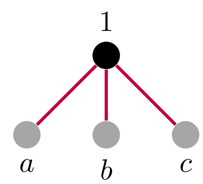
which is consistent with Examples 30(i), 31(i) and 32.

Example 34. Next, we consider the more complicated extremal tree T shown below.



There are two nontrivial smoothings, R and S.





The contributions of R and S are obtained from (40),

$$\mathsf{Cont}_{\mathsf{R}} = \left[\frac{c(\mathcal{N})}{(1+z_1)(1+z_2)}\right]_{a+b+c-2},$$

$$\mathsf{Cont}_{\mathsf{S}} = \left[\frac{c(\mathcal{N})}{(1+z_2)(1+z_3)(1+z_4)}\right]_{a+b+c-3}.$$

Here, z_1 corresponds to the edge joining the genus 0 vertex to the root, z_2 corresponds to the edge joining the genus c vertex to the root, while z_3, z_4 correspond to the remaining edges.

We only consider the case $a+b+c = g-1 = 5$. From equation (41), we obtain

$$\mathsf{Cont}_{\mathsf{T}} \cdot z_1z_2z_3z_4 = c_5(\mathcal{N}) - z_1z_2\left[\frac{c(\mathcal{N})}{(1+z_1)(1+z_2)}\right]_3 - z_2z_3z_4\left[\frac{c(\mathcal{N})}{(1+z_2)(1+z_3)(1+z_4)}\right]_2,$$

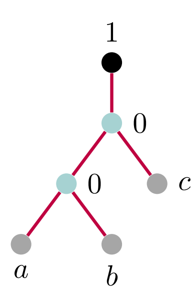
where

$$c(\mathcal{N}) = (1+z_2)(1+z_1+z_3)(1+z_1+z_4)(1+\ell_1)(1+\ell_2).$$

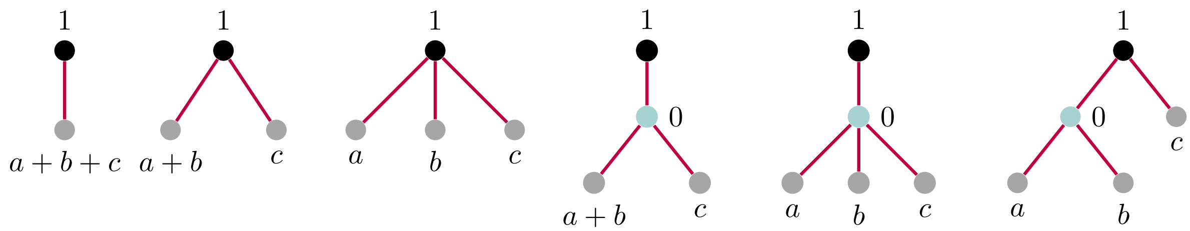
Therefore

$$\mathsf{Cont}_{\mathsf{T}} = -3c_1(\mathcal{N}) + 6z_1 + 3z_2 + 4(z_3+z_4).$$

Example 35. Similarly, we compute the contribution of the extremal tree T shown below.



There are 6 smoothings indexed by the following trees $\mathsf{R}_1 - \mathsf{R}_6$.



We have

$$c(\mathcal{N}) = (1 + z_1 + z_2 + z_4)(1 + z_1 + z_2 + z_5)(1 + z_1 + z_3)(1 + \ell_1)(1 + \ell_2),$$

where the edge emanating from the root is labelled z_1, and the edges at the adjacent genus 0 vertex are z_2, z_3 from left to right, and z_4, z_5 are the remaining edges, again labeled from left to right. We assume $a + b + c = 5 = g - 1$.

The contributions of the first 3 irreducible trees are

$$\begin{aligned}
\mathsf{Cont}_{\mathsf{R}_1} &= \left[\frac{c(\mathcal{N})}{1 + z_1}\right]_4, \\
\mathsf{Cont}_{\mathsf{R}_2} &= \left[\frac{c(\mathcal{N})}{(1 + z_2)(1 + z_3)}\right]_3, \\
\mathsf{Cont}_{\mathsf{R}_3} &= \left[\frac{c(\mathcal{N})}{(1 + z_3)(1 + z_4)(1 + z_5)}\right]_2.
\end{aligned}$$

The contributions of R_4, R_5, and R_6 can be calculated using Examples 30 (iii), 31(ii), and 34, respectively. We find

$$\begin{aligned}
\mathsf{Cont}_{\mathsf{R}_4} &= -3c_2(\mathcal{N}) + c_1(\mathcal{N}) \cdot (6z_1 + 4z_2 + 4z_3) - 10z_1^2 - 10z_1 \cdot (z_2 + z_3) - 5(z_2 + z_3)^2 + 5z_2z_3, \\
\mathsf{Cont}_{\mathsf{R}_5} &= -4c_1(\mathcal{N}) + 10z_1 + 5(z_3 + z_4 + z_5), \\
\mathsf{Cont}_{\mathsf{R}_6} &= -3c_1(\mathcal{N}) + 6z_2 + 3z_3 + 4(z_4 + z_5).
\end{aligned}$$

The recursion to be solved is

$$\begin{aligned}
c_5(\mathcal{N}) = z_1z_2z_3z_4z_5\mathsf{Cont}_{\mathsf{T}} + z_1\mathsf{Cont}_{\mathsf{R}_1} + z_2z_3\mathsf{Cont}_{\mathsf{R}_2} + z_3z_4z_5\mathsf{Cont}_{\mathsf{R}_3} + z_1z_2z_3\mathsf{Cont}_{\mathsf{R}_4} \\
+ z_1z_3z_4z_5\mathsf{Cont}_{\mathsf{R}_5} + z_2z_3z_4z_5\mathsf{Cont}_{\mathsf{R}_6},
\end{aligned}$$

which gives $\mathsf{Cont}_{\mathsf{T}} = 15$.

Proof of Theorem 3. The algorithm in Section 5.2 yields the equation

$$\mathsf{Tor}^*[\mathcal{A}_1 \times \mathcal{A}_{g-1}] = \sum_{\mathsf{T}} \frac{1}{|\operatorname{Aut}\mathsf{T}|} \iota_{\mathsf{T}*}\mathsf{Cont}_{\mathsf{T}}, \tag{44}$$

where $\mathsf{Cont}_{\mathsf{T}}$ is a polynomial in λ and ψ classes. The contributions can be computed recursively one tree at a time, with (43) providing the base case of the recursion. In particular, formula (44) shows that

$$\mathsf{Tor}^*[\mathcal{A}_1 \times \mathcal{A}_{g-1}] \in \mathsf{R}^{g-1}(\mathcal{M}_g^{\mathrm{ct}}).$$

□

5.5. **Pixton's formula.** Pixton has solved our recursion to provide a beautiful and concise expression for $\mathsf{Cont}_{\mathsf{T}}$. Though not needed for the results of our paper, we present his formula here. The proof will appear in [45].

Let T be an extremal tree of genus g with n edges and k leaves. Let $Z = \{z_e\}_{e \in \mathsf{E}(\mathsf{T})}$ be the set of edge variables as before. Consider first the expression

$$(-1)^k \frac{\prod_{v \in \mathsf{V}(\mathsf{T})} \left(1 + \sum_{e \in \mathsf{path}(v)} z_e\right)^{\mathsf{val}(v)-2}}{\prod_{e \in \mathsf{E}(\mathsf{T})} z_e} . \tag{45}$$

After expanding the numerator in (45), we obtain a Laurent series in the variables Z. Let

$$\left[(-1)^k \frac{\prod_{v \in \mathsf{V}(\mathsf{T})} \left(1 + \sum_{e \in \mathsf{path}(v)} z_e\right)^{\mathsf{val}(v)-2}}{\prod_{e \in \mathsf{E}(\mathsf{T})} z_e} \right]_{Z \geq 0}$$

denote the *Taylor part*: the power series in Z obtained by removing all the strictly polar parts of the Laurent series (45).

When considering power series in the variables Z and the Chern classes $c_i(\mathcal{N})$, we will use the standard Chow degree: z_e has degree 1, $c_i(\mathcal{N})$ has degree i.

Theorem [Pixton's formula]**.** *The polynomial* $\mathsf{P}_{\mathsf{T}}(Z, \mathcal{N})$ *determining* $\mathsf{Cont}_{\mathsf{T}}$ *is the degree* $g - 1 - n$ *part of the power series*

$$\left[(-1)^k \frac{\prod_{v \in \mathsf{V}(\mathsf{T})} \left(1 + \sum_{e \in \mathsf{path}(v)} z_e\right)^{\mathsf{val}(v)-2}}{\prod_{e \in \mathsf{E}(\mathsf{T})} z_e} \right]_{Z \geq 0} \cdot c(\mathcal{N}) ,$$

where $c(\mathcal{N})$ *denotes the total Chern class.*

6. Calculations for $g \leq 7$

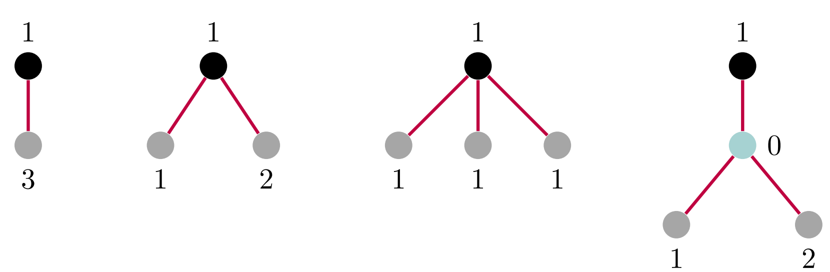
6.1. **Genus 4 and 5.** We implement here the excess intersection theory developed in Section 5 to calculate the Torelli pullback of

$$\Delta_g = [\mathcal{A}_1 \times \mathcal{A}_{g-1}] - \frac{g}{6|B_{2g}|} \lambda_{g-1} .$$

As discussed in Section 1.6, $\Delta_g = 0 \in \mathsf{CH}^{g-1}(\mathcal{A}_g)$ for $1 \leq g \leq 3$.

Proposition 36. *For* $g = 4$, *we have* $\mathsf{Tor}^* \Delta_4 = \mathsf{Tor}^*[\mathcal{A}_1 \times \mathcal{A}_3] - 20\lambda_3 = 0 \in \mathsf{R}^3(\mathcal{M}_4^{\mathrm{ct}})$.

Proof. In genus 4, there are four extremal trees with at most 3 edges: $\mathsf{A}, \mathsf{B}, \mathsf{C}$, and D, drawn below:



The first three, A, B, and C, correspond to the irreducible components of $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_3)$, and the fourth D is the intersection of the first two components. Other extremal trees corresponding to the remaining intersections occur in higher codimension, and thus do not contribute to the calculation. The contribution from A is computed via (43):

$$\left[\frac{c(\mathbb{E}^\vee)}{1-\psi_1}\right]_2 = [1, \lambda_2 - \lambda_1\psi_1 + \psi_1^2]\,.$$

Here, the component A is $\mathcal{M}_{1,1}^{\mathrm{ct}} \times \mathcal{M}_{3,1}^{\mathrm{ct}}$ and the notation $[,]$ indicates the contribution from each factor, respecting the order of the factors in the product. Similarly, B corresponds to the product $\mathcal{M}_{1,2}^{\mathrm{ct}} \times \mathcal{M}_{1,1}^{\mathrm{ct}} \times \mathcal{M}_{2,1}^{\mathrm{ct}}$. The contribution of B can also be found via (43) yielding

$$[\psi_1 + \psi_2, 1, 1] + [1, 1, \psi_1 - \lambda_1]\,.$$

The extremal tree C occurs in the correct codimension and has an automorphism group of order 6. Finally, by Example 30(i) in Section 5, the contribution of D is -3 times the fundamental class.

We push forward the contributions of $\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}$ to $\mathcal{M}_4^{\mathrm{ct}}$, dividing by the orders of the respective automorphism groups, and subtract $20\lambda_3$. Using `admcycles` [14], we verify

$$\mathsf{Tor}^*[\mathcal{A}_1 \times \mathcal{A}_3] - 20\lambda_3 = 0 \in \mathsf{R}^3(\mathcal{M}_4^{\mathrm{ct}})\,.$$

The code for the calculation can be found in [9]. □

Proposition 37. *The classes* $[\mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_2]$ *and* $[\mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_1]$ *are tautological in* $\mathsf{CH}^*(\mathcal{A}_4)$*:*

$$[\mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_2] = 420\lambda_2\lambda_3\,, \quad [\mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_1] = 4200\lambda_1\lambda_2\lambda_3\,.$$

Proposition 37 implies Conjecture 10 in genus 4. Whether the class $[\mathcal{A}_1 \times \mathcal{A}_3] \in \mathsf{CH}^3(\mathcal{A}_4)$ is tautological remains an open question. By Proposition 7, $[\mathcal{A}_1 \times \mathcal{A}_3]$ is tautological in cohomology.

Proof. In genus 4, the Schottky locus is a divisor in $\mathcal{A}_4$, hence the class $\mathsf{Tor}_*[\mathcal{M}_4^{\mathrm{ct}}] \in \mathsf{CH}^1(\mathcal{A}_4)$ is a multiple[16] of λ_1 since the Picard rank of $\mathcal{A}_4$ equals 1. By Proposition 36 and the projection formula, we find

$$\mathsf{Tor}^*\Delta_4 = 0 \implies \mathsf{Tor}_*\mathsf{Tor}^*([\mathcal{A}_1 \times \mathcal{A}_3] - 20\lambda_3) = 0 \implies \lambda_1([\mathcal{A}_1 \times \mathcal{A}_3] - 20\lambda_3) = 0\,.$$

Intersecting with λ_1, we obtain

$$\lambda_1^2\,[\mathcal{A}_1 \times \mathcal{A}_3] = 20\lambda_1^2\lambda_3 = 40\lambda_2\lambda_3 \implies [\mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_2] = 420\lambda_2\lambda_3\,,$$

where the Mumford relation $\lambda_1^2 = 2\lambda_2$ was used in the first equation, and the relation

$$[\mathcal{A}_1 \times \mathcal{A}_2] = \frac{21}{2}\lambda_1^2 \in \mathsf{CH}^2(\mathcal{A}_3)$$

of [49, Proposition 3.2] is used for the second equation. Intersecting with λ_1 one more time, and using $[\mathcal{A}_1 \times \mathcal{A}_1] = 10\lambda_1 \in \mathsf{CH}^1(\mathcal{A}_2)$ by [49, Lemma 2.2], we obtain

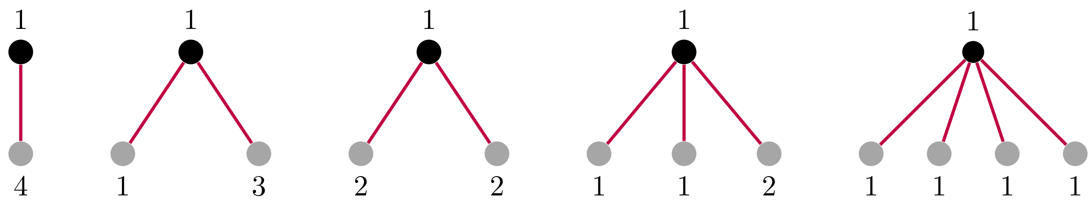
$$[\mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_1] = 10\lambda_1[\mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_2] = 4200\,\lambda_1\lambda_2\lambda_3\,.$$ □

[16] By results of Igusa the multiple equals 8, but our argument does not require knowledge of the multiple.

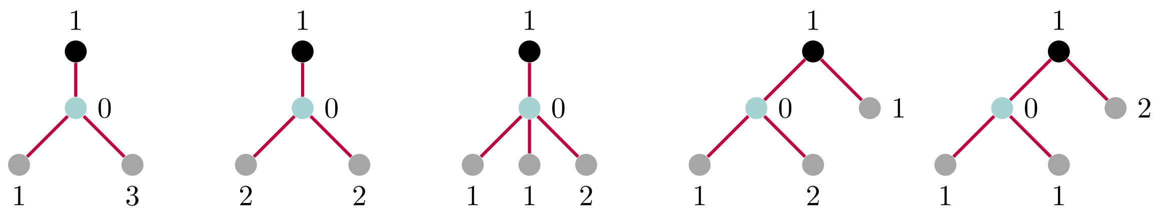
Proposition 38. *We have* $\mathsf{Tor}^*\Delta_5 = \mathsf{Tor}^*[\mathcal{A}_1 \times \mathcal{A}_4] - 11\lambda_4 = 0 \in \mathsf{R}^4(\mathcal{M}_5^{\mathrm{ct}})$.

Proof. We calculate as in the proof of Proposition 36, but that there are more trees to consider.

The irreducible components of $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_4)$ are indexed by the following extremal trees labelled A, B, C, D, E respectively:



We also list the intersections that have codimension at most 4:



The above 5 extremal trees correspond to the intersections $\mathsf{A} \cap \mathsf{B}$, $\mathsf{A} \cap \mathsf{C}$, $\mathsf{A} \cap \mathsf{D}$, $\mathsf{B} \cap \mathsf{D}$, $\mathsf{C} \cap \mathsf{D}$ respectively.

We compute the contributions of each of the 10 trees above as follows:

(i) The contributions of the extremal trees $\mathsf{A} - \mathsf{E}$ are computed using (43). The trees C and D have 2 automorphisms each, while the tree E has 24 automorphisms. There are no automorphisms for A, B. We obtain

$$\frac{1}{|\operatorname{Aut}(\mathsf{A})|}\mathsf{Cont}_{\mathsf{A}} = [1, -\lambda_3 + \lambda_2\psi_1 - \lambda_1\psi_1^2 + \psi_1^3]\,,$$

$$\frac{1}{|\operatorname{Aut}(\mathsf{B})|}\mathsf{Cont}_{\mathsf{B}} = [1,1,\lambda_2] - [\psi_1+\psi_2, 1, \lambda_1] - [1,1,\lambda_1\psi_1] + [\psi_1, 1, \psi_1] + 2[\psi_2, 1, \psi_1] + [1,1,\psi_1^2]\,,$$

$$\begin{aligned}\frac{1}{|\operatorname{Aut}(\mathsf{C})|}\mathsf{Cont}_{\mathsf{C}} = \frac{1}{2}\big(&[1,\lambda_1,\lambda_1] - [\psi_1+\psi_2,\lambda_1,1] - [\psi_1+\psi_2,1,\lambda_1] - [1,\psi_1\lambda_1,1] - [1,\psi_1,\lambda_1]\\ &+ [2\psi_1+\psi_2,\psi_1,1] + [1,\psi_1^2,1] - [1,\lambda_1,\psi_1] - [1,1,\lambda_1\psi_1] + [\psi_1+2\psi_2,1,\psi_1]\\ &+ [1,\psi_1,\psi_1] + [1,1,\psi_1^2]\big)\,,\end{aligned}$$

$$\frac{1}{|\operatorname{Aut}(\mathsf{D})|}\mathsf{Cont}_{\mathsf{D}} = \frac{1}{2}\left([\psi_1+\psi_2+\psi_3,1,1,1] + [1,1,1,\psi_1-\lambda_1]\right)$$

$$\frac{1}{|\operatorname{Aut}(\mathsf{E})|}\mathsf{Cont}_{\mathsf{E}} = \frac{1}{24}[1,1,1,1,1]\,.$$

The order of the terms in the brackets [] places the root contribution on the first position, followed by the contribution of the non-root vertices from left to right in increasing order of the genus. We also ignore the terms of degree $> 2g-3+n$ on any component $\mathcal{M}_{g,n}^{\mathrm{ct}}$ since such terms vanish by (5).

(ii) Moving on to the intersections, we consider the first extremal tree which represents $\mathsf{A}\cap\mathsf{B}$. The corresponding locus is $\mathcal{M}^{\mathrm{ct}}_{1,1}\times\mathcal{M}^{\mathrm{ct}}_{0,3}\times\mathcal{M}^{\mathrm{ct}}_{1,1}\times\mathcal{M}^{\mathrm{ct}}_{3,1}$. By Example 30(ii) the contribution equals

$$-3c_1(\mathcal{N})+6z_1+4z_2+4z_3\,.$$

Here, $\mathcal{N}$ is the restriction of the normal bundle of

$$\mathcal{A}_1\times\mathcal{A}_4\to\mathcal{A}_5$$

to $\mathsf{A}\cap\mathsf{B}$. By (42) and using that the Hodge bundle in genus $0,1$ has trivial Chern classes, we find

$$c_1(\mathcal{N})=[1,1,1,-\lambda_1]\,.$$

Next, as explained in Section 5.2, we substitute the edge variables by the negative sums of cotangent classes:

$$z_1\mapsto 0,\quad z_2\mapsto 0,\quad z_3\mapsto[1,1,1,-\psi_1]\,,$$

where we have used that the ψ-classes on the factors $\mathcal{M}^{\mathrm{ct}}_{1,1}$ and $\mathcal{M}^{\mathrm{ct}}_{0,3}$ vanish. Collecting terms, we see

$$\frac{1}{|\operatorname{Aut}(\mathsf{AB})|}\mathsf{Cont}_{\mathsf{AB}}=[1,1,1,3\lambda_1-4\psi_1]\,.$$

For the second tree corresponding to $\mathsf{A}\cap\mathsf{C}$ the calculation is similar. The contribution equals

$$-3c_1(\mathcal{N})+6z_1+4z_2+4z_3$$

over $\mathcal{M}^{\mathrm{ct}}_{1,1}\times\mathcal{M}^{\mathrm{ct}}_{0,3}\times\mathcal{M}^{\mathrm{ct}}_{2,1}\times\mathcal{M}^{\mathrm{ct}}_{2,1}$. We must divide by 2 because of automorphisms. We obtain

$$\frac{1}{|\operatorname{Aut}(\mathsf{AC})|}\mathsf{Cont}_{\mathsf{AC}}=\frac{1}{2}\left([1,1,3\lambda_1-4\psi_1,1]+[1,1,1,3\lambda_1-4\psi_1]\right).$$

(iii) For the last 3 extremal trees, the corresponding loci have codimension 4. The excess contributions are computed by Example 31(i) and Example 30(i) and they equal $-4,-3,-3$. The number of automorphisms are $2,1,2$. The contributions of these loci divided by the order of the automorphism group are $-\frac{4}{2},-3,-\frac{3}{2}$ times their fundamental classes, respectively.

We collect all terms in (i)-(iii), push forward the weighted contributions to $\mathcal{M}^{\mathrm{ct}}_5$ and subtract $11\lambda_4$. Using `admcycles` [14], we verify

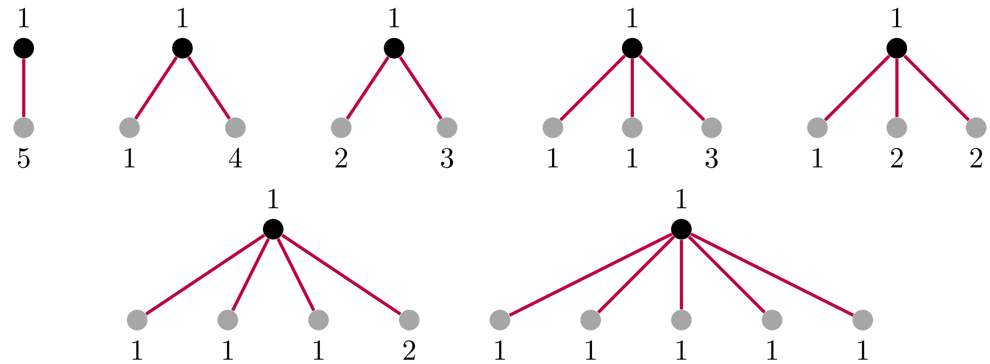
$$\mathsf{Tor}^*[\mathcal{A}_1\times\mathcal{A}_4]-11\lambda_4=0\in\mathsf{R}^4(\mathcal{M}^{\mathrm{ct}}_5)\,.$$

The code for the calculation can be found in [9]. □

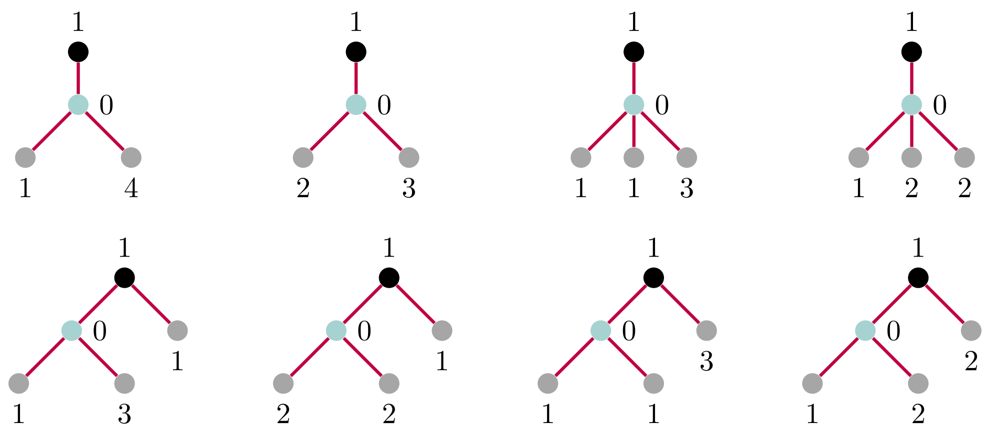
Remark 39. An alternate proof of Propositions 36 and 38 is as follows. By Theorem 4, $\mathsf{Tor}^*\Delta_g$ is in the kernel of the λ_g-pairing, but the λ_4 and λ_5-pairings are perfect [7].

6.2. **Genus 6: Proof of Theorem 5.** As explained in Section 1.6, the last assertion of Theorem 5 follows from Proposition 2. The kernel of the λ_6-pairing was computed in [7]: it is an explicit 1-dimensional subspace of $\mathsf{R}^5(\mathcal{M}_6^{\mathrm{ct}})$. We will compute $\mathsf{Tor}^*\Delta_6$ using the excess calculus. Then, we will show that $\mathsf{Tor}^*\Delta_6$ generates the kernel of the λ_6-pairing using `admcycles` [14].

There are 24 extremal trees contributing to the calculation of $\mathsf{Tor}^*\Delta_6$. The irreducible components of $\mathsf{Tor}^{-1}(\mathcal{A}_1 \times \mathcal{A}_5)$ are indexed by the following 7 extremal trees denoted $\mathsf{A} - \mathsf{G}$:



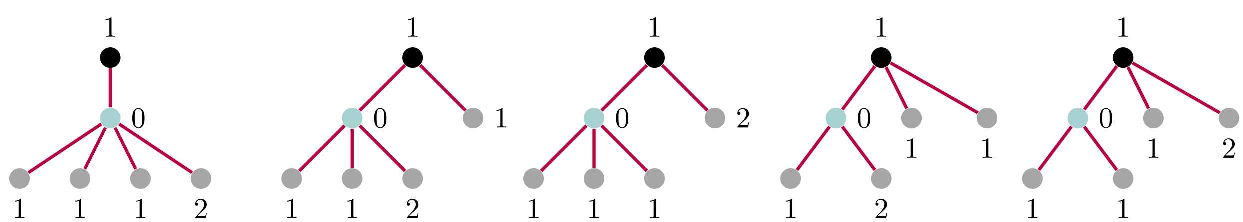
Additionally, there are 8 extremal trees with at most 4 edges which arise from intersections of the components:

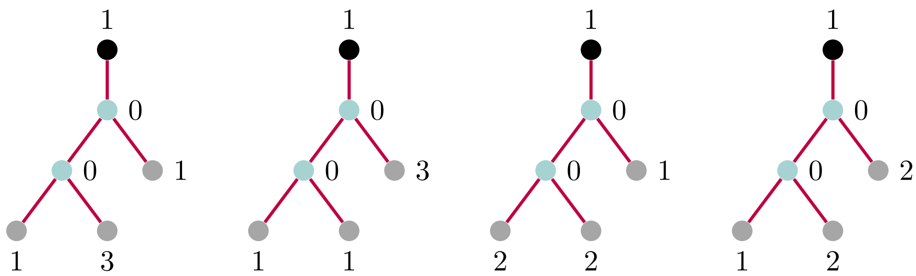


In order, these correspond to the intersections

$$\mathsf{A} \cap \mathsf{B},\ \mathsf{A} \cap \mathsf{C},\ \mathsf{A} \cap \mathsf{D},\ \mathsf{A} \cap \mathsf{E},\ \mathsf{B} \cap \mathsf{D},\ \mathsf{B} \cap \mathsf{E},\ \mathsf{C} \cap \mathsf{D},\ \mathsf{C} \cap \mathsf{E}.$$

Finally, the remaining 9 extremal trees have 5 edges:





The extremal trees on the first row correspond to the intersections

$$\mathsf{A} \cap \mathsf{F}\,,\ \mathsf{B} \cap \mathsf{F}\,,\ \mathsf{C} \cap \mathsf{F}\,,\ \mathsf{D} \cap \mathsf{F}\,,\ \mathsf{E} \cap \mathsf{F}\,.$$

The extremal trees on the second row correspond to the triple intersections

$$\mathsf{A} \cap \mathsf{B} \cap \mathsf{D}\,,\ \mathsf{A} \cap \mathsf{C} \cap \mathsf{D}\,,\ \mathsf{A} \cap \mathsf{B} \cap \mathsf{E}\,,\ \mathsf{A} \cap \mathsf{C} \cap \mathsf{E}\,.$$

We compute the contributions of the 24 extremal trees above.

(i) For the trees $\mathsf{A}-\mathsf{G}$, the contributions can be found from (43), weighted by the number of automorphisms $1, 1, 1, 2, 2, 6, 120$ respectively. We obtain

$$\frac{1}{|\operatorname{Aut}(\mathsf{A})|}\mathsf{Cont}_{\mathsf{A}} = [1, \lambda_4 - \psi_1\lambda_3 + \psi_1^2\lambda_2 - \psi_1^3\lambda_1 + \psi_1^4]\,,$$

$$\frac{1}{|\operatorname{Aut}(\mathsf{B})|}\mathsf{Cont}_{\mathsf{B}} = [1, 1, -\lambda_3 + \psi_1\lambda_2 - \psi_1^2\lambda_1 + \psi_1^3] + [\psi_1 + \psi_2, 1, \lambda_2] + [-\psi_1 - 2\psi_2, 1, \lambda_1\psi_1] + [\psi_1 + 3\psi_2, 1, \psi_1^2]\,,$$

$$\begin{aligned}\frac{1}{|\operatorname{Aut}(\mathsf{C})|}\mathsf{Cont}_{\mathsf{C}} = {} & [1, 1, \psi_1\lambda_2 - \psi_1^2\lambda_1 + \psi_1^3] + [1, \lambda_1, -\lambda_2 + \psi_1\lambda_1 - \psi_1^2] + [1, \psi_1^2 - \psi_1\lambda_1, \psi_1 - \lambda_1] \\ & + [\psi_1 + \psi_2, 1, \lambda_2] + [\psi_1 + \psi_2, \lambda_1, \lambda_1] + [3\psi_1 + \psi_2, \psi_1^2, 1] + [\psi_1 + 3\psi_2, 1, \psi_1^2] \\ & + [-\psi_1 - 2\psi_2, 1, \psi_1\lambda_1] + [-2\psi_1 - \psi_2, \psi_1\lambda_1, 1] + [-\psi_1 - 2\psi_2, \lambda_1, \psi_1] \\ & + [-2\psi_1 - \psi_2, \psi_1, \lambda_1] + [1, \psi_1, \lambda_2 - \lambda_1\psi_1 + \psi_1^2] + [2\psi_1 + 2\psi_2, \psi_1, \psi_1]\,,\end{aligned}$$

$$\begin{aligned}\frac{1}{|\operatorname{Aut}(\mathsf{D})|}\mathsf{Cont}_{\mathsf{D}} = {} & \frac{1}{2}([1, 1, 1, \lambda_2 - \psi_1\lambda_1 + \psi_1^2] + [\psi_1^2 + \psi_2^2 + \psi_3^2 + \psi_1\psi_2 + \psi_1\psi_3 + \psi_2\psi_3, 1, 1, 1] \\ & + [-\psi_1 - \psi_2 - \psi_3, 1, 1, \lambda_1] + [\psi_1 + \psi_2 + 2\psi_3, 1, 1, \psi_1])\,,\end{aligned}$$

$$\begin{aligned}\frac{1}{|\operatorname{Aut}(\mathsf{E})|}\mathsf{Cont}_{\mathsf{E}} = {} & \frac{1}{2}([\psi_1^2 + \psi_2^2 + \psi_3^2 + \psi_1\psi_2 + \psi_1\psi_3 + \psi_2\psi_3, 1, 1, 1] + [-\psi_1 - \psi_2 - \psi_3, 1, 1, \lambda_1] \\ & + [-\psi_1 - \psi_2 - \psi_3, 1, \lambda_1, 1] + [1, 1, 1, \psi_1^2 - \psi_1\lambda_1] + [1, 1, \psi_1^2 - \psi_1\lambda_1, 1] \\ & + [\psi_1 + \psi_2 + 2\psi_3, 1, 1, \psi_1] + [\psi_1 + 2\psi_2 + \psi_3, 1, \psi_1, 1] + [1, 1, \lambda_1 - \psi_1, \lambda_1 - \psi_1])\,,\end{aligned}$$

$$\frac{1}{|\operatorname{Aut}(\mathsf{F})|}\mathsf{Cont}_{\mathsf{F}} = \frac{1}{6}([\psi_1 + \psi_2 + \psi_3 + \psi_4, 1, 1, 1, 1] + [1, 1, 1, 1, \psi_1 - \lambda_1])\,,$$

$$\frac{1}{|\operatorname{Aut}(\mathsf{G})|}\mathsf{Cont}_{\mathsf{G}} = \frac{1}{120}[1, 1, 1, 1, 1, 1]\,.$$

As before, the first position in the bracket records the contribution of the root, while the next entries correspond to the remaining vertices, listed in increasing order by genus (from left to right in the picture). We slightly simplified the answer by ignoring terms of degree $> 2g-3+n$ for each vertex of genus g with n markings, due to (5).

(ii) We next consider the intersection of strata. The first extremal tree on the list corresponds to $\mathsf{A}\cap\mathsf{B}$. The locus A has codimension 1, B has codimension 2, the intersection has codimension 3, while the expected codimension is 5. By Example 30(iii), the excess contribution is given by

$$-3c_2(\mathcal{N}) + c_1(\mathcal{N}) \cdot (6z_1 + 4z_2 + 4z_3) - 10z_1^2 - 10z_1 \cdot (z_2 + z_3) - 5(z_2 + z_3)^2 + 5z_2 z_3 \,.$$

Here, $\mathcal{N}$ is the restriction of the normal bundle of $\mathcal{A}_1 \times \mathcal{A}_5 \to \mathcal{A}_6$ to $\mathsf{A}\cap\mathsf{B}$. We express the answer in terms of the standard tautological classes over $\mathcal{M}^{\mathrm{ct}}_{1,1} \times \mathcal{M}^{\mathrm{ct}}_{0,3} \times \mathcal{M}^{\mathrm{ct}}_{1,1} \times \mathcal{M}^{\mathrm{ct}}_{4,1}$, with the bracket entries reflecting the ordering of the factors. Using (42), we obtain

$$c_1(\mathcal{N}) = [1,1,1,-\lambda_1]\,, \quad c_2(\mathcal{N}) = [1,1,1,\lambda_2]\,.$$

Next, as explained in Section 5.2, we substitute the edge variables in terms of the cotangent classes at the nodes:

$$z_1 \mapsto 0\,, \quad z_2 \mapsto 0\,, \quad z_3 \mapsto [0,0,0,-\psi_1]\,.$$

We have used here the vanishing of the ψ classes on $\mathcal{M}^{\mathrm{ct}}_{0,3}$ and $\mathcal{M}^{\mathrm{ct}}_{1,1}$. We obtain

$$\frac{1}{|\operatorname{Aut}(\mathsf{AB})|}\mathsf{Cont}_{\mathsf{AB}} = [1,1,1,-3\lambda_2 + 4\lambda_1\psi_1 - 5\psi_1^2]\,.$$

The contribution of the intersection $\mathsf{A}\cap\mathsf{C}$ corresponding to the locus

$$\mathcal{M}^{\mathrm{ct}}_{1,1} \times \mathcal{M}^{\mathrm{ct}}_{0,3} \times \mathcal{M}^{\mathrm{ct}}_{2,1} \times \mathcal{M}^{\mathrm{ct}}_{3,1}$$

is found using the same method. We obtain

$$\begin{aligned}\frac{1}{|\operatorname{Aut}(\mathsf{AC})|}\mathsf{Cont}_{\mathsf{AC}} &= [1,1,1,-3\lambda_2 + 4\lambda_1\psi_1 - 5\psi_1^2] + [1,1,4\lambda_1\psi_1 - 5\psi_1^2,1] \\ &\quad + [1,1,-3\lambda_1 + 4\psi_1,\lambda_1] + [1,1,4\lambda_1 - 5\psi_1,\psi_1]\,.\end{aligned}$$

(iii) We consider the codimension 1 locus A and the codimension 3 locus D intersecting the codimension 4 locus $\mathsf{A}\cap\mathsf{D}$. The excess contribution is found by Example 31(ii):

$$-4c_1(\mathcal{N}) + 10z_1 + 5(z_2 + z_3 + z_4)\,,$$

where $\mathcal{N}$ is the restriction of the normal bundle of $\mathcal{A}_1 \times \mathcal{A}_5 \to \mathcal{A}_6$ to $\mathsf{A}\cap\mathsf{D}$. Expressing in terms of tautological classes over the product

$$\mathcal{M}^{\mathrm{ct}}_{1,1} \times \mathcal{M}^{\mathrm{ct}}_{0,4} \times \mathcal{M}^{\mathrm{ct}}_{1,1} \times \mathcal{M}^{\mathrm{ct}}_{1,1} \times \mathcal{M}^{\mathrm{ct}}_{3,1}\,,$$

and accounting for automorphisms, we obtain

$$\frac{1}{|\operatorname{Aut}(\mathsf{AD})|}\mathsf{Cont}_{\mathsf{AD}} = \frac{1}{2}([1,1,1,1,4\lambda_1 - 5\psi_1] + [1,-10\psi_1 - 5\psi_2 - 5\psi_3 - 5\psi_4,1,1,1])\,.$$

Over the genus 0 vertex, the markings and the ψ classes are numbered starting from the edge connecting to the root. The convention is necessary to make precise the second term above.

The intersection $\mathsf{A} \cap \mathsf{E}$ corresponds to the product

$$\mathcal{M}_{1,1}^{\mathrm{ct}} \times \mathcal{M}_{0,4}^{\mathrm{ct}} \times \mathcal{M}_{1,1}^{\mathrm{ct}} \times \mathcal{M}_{2,1}^{\mathrm{ct}} \times \mathcal{M}_{2,1}^{\mathrm{ct}} ,$$

and the associated contribution is computed by the same formula. We find

$$\frac{1}{|\operatorname{Aut}(\mathsf{AE})|}\mathsf{Cont}_{\mathsf{AE}} = \frac{1}{2}([1,1,1,4\lambda_1 - 5\psi_1, 1] + [1,1,1,1,4\lambda_1 - 5\psi_1] \\ + [1, -10\psi_1 - 5\psi_2 - 5\psi_3 - 5\psi_4, 1,1,1]) .$$

(iv) Next, we consider the codimension 2 locus B and the codimension 3 locus D intersecting in the codimension 4 locus $\mathsf{B} \cap \mathsf{D}$. The contribution is found from Example 34:

$$-3c_1(\mathcal{N}) + 6z_1 + 3z_2 + 4(z_3 + z_4) ,$$

where as usual $\mathcal{N}$ is the restriction of the normal bundle of $\mathcal{A}_1 \times \mathcal{A}_5 \to \mathcal{A}_6$ to $\mathsf{B} \cap \mathsf{D}$. Simple geometry yields

$$c_1(\mathcal{N}) = [1,1,1,-\lambda_1,1] , \quad z_1 \mapsto [-\psi_1,1,1,1,1] , \quad z_2 \mapsto [-\psi_2,1,1,1,1] , \quad z_3 \mapsto 0 , \\ z_4 \mapsto [1,1,1,-\psi_1,1] .$$

We obtain

$$\frac{1}{|\operatorname{Aut}(\mathsf{BD})|}\mathsf{Cont}_{\mathsf{BD}} = [1,1,1,3\lambda_1 - 4\psi_1, 1] + [-6\psi_1 - 3\psi_2, 1,1,1,1] .$$

Here, the ordering in the bracket corresponds to the natural ordering in the product

$$\mathcal{M}_{1,2}^{\mathrm{ct}} \times \mathcal{M}_{0,3}^{\mathrm{ct}} \times \mathcal{M}_{1,1}^{\mathrm{ct}} \times \mathcal{M}_{3,1}^{\mathrm{ct}} \times \mathcal{M}_{1,1}^{\mathrm{ct}} .$$

We read the tree from the root down, and from left to right.

The intersections $\mathsf{B} \cap \mathsf{E}$, $\mathsf{C} \cap \mathsf{D}$ and $\mathsf{C} \cap \mathsf{E}$ are computed in the same manner. These loci correspond to the products

$$\mathcal{M}_{1,2}^{\mathrm{ct}} \times \mathcal{M}_{0,3}^{\mathrm{ct}} \times \mathcal{M}_{2,1}^{\mathrm{ct}} \times \mathcal{M}_{2,1}^{\mathrm{ct}} \times \mathcal{M}_{1,1}^{\mathrm{ct}} , \quad \mathcal{M}_{1,2}^{\mathrm{ct}} \times \mathcal{M}_{0,3}^{\mathrm{ct}} \times \mathcal{M}_{1,1}^{\mathrm{ct}} \times \mathcal{M}_{1,1}^{\mathrm{ct}} \times \mathcal{M}_{3,1}^{\mathrm{ct}} , \\ \mathcal{M}_{1,2}^{\mathrm{ct}} \times \mathcal{M}_{0,3}^{\mathrm{ct}} \times \mathcal{M}_{1,1}^{\mathrm{ct}} \times \mathcal{M}_{2,1}^{\mathrm{ct}} \times \mathcal{M}_{2,1}^{\mathrm{ct}} ,$$

respectively. After accounting for automorphisms, we obtain

$$\frac{1}{|\operatorname{Aut}(\mathsf{BE})|}\mathsf{Cont}_{\mathsf{BE}} = \frac{1}{2}([1,1,1,3\lambda_1 - 4\psi_1,1] + [1,1,3\lambda_1 - 4\psi_1,1,1] + [-6\psi_1 - 3\psi_2,1,1,1,1]) ,$$

$$\frac{1}{|\operatorname{Aut}(\mathsf{CD})|}\mathsf{Cont}_{\mathsf{CD}} = \frac{1}{2}([1,1,1,1,3\lambda_1 - 3\psi_1] + [-6\psi_1 - 3\psi_2,1,1,1,1]) ,$$

$$\frac{1}{|\operatorname{Aut}(\mathsf{CE})|}\mathsf{Cont}_{\mathsf{CE}} = [1,1,1,1,3\lambda_1 - 3\psi_1] + [-6\psi_1 - 3\psi_2,1,1,1,1] + [1,1,1,3\lambda_1 - 4\psi_1,1] .$$

(v) For the 9 extremal trees with 5 edges, there are $6,2,6,2,2$ automorphisms, respectively, for the 5 trees on the first row, and the excess contributions are $-5,-4,-4,-3,-3$, respectively, see Examples 32, 31(i) and 30(i). The contributions of these loci equal the fundamental class multiplied by

$$-\frac{5}{6}, \ -\frac{4}{2}, \ -\frac{4}{6}, \ -\frac{3}{2}, \ -\frac{3}{2} .$$

For the remaining 4 trees on the second row, the number of automorphisms is $1, 2, 2, 1$ and the excess contribution is 15 for each of these extremal trees, see Example 35. The contributions of these loci equal the fundamental class times

$$15,\ \frac{15}{2},\ \frac{15}{2},\ 15\,.$$

To complete the proof, we collect the terms (i)-(v), push forward to $\mathcal{M}_6^{\mathrm{ct}}$, and subtract $\frac{2370}{691}\lambda_5$. We verify using `admcycles` [14] that the resulting class pairs trivially with all elements in $\mathsf{R}^4(\mathcal{M}_6^{\mathrm{ct}})$, as expected from Theorem 4. Furthermore, we see

$$\mathsf{Tor}^*\Delta_6 \neq 0 \in \mathsf{R}^5(\mathcal{M}_6^{\mathrm{ct}})$$

using completeness of Pixton's relations in $\mathsf{R}^*(\mathcal{M}_6^{\mathrm{ct}})$ proven in [7]. The implementation can be found in [9]. □

6.3. **Genus 7: Proof of Proposition 6.** Proposition 6 can be proven by analyzing the extremal trees and their contributions (as in the proof of Theorem 5). We instead give a simpler proof based on the structure of the Gorenstein kernel of $\mathsf{R}^*(\mathcal{M}_7^{\mathrm{ct}})$, which was suggested to us by Aaron Pixton. The methods here are developed systematically in [7] to study the Gorenstein kernel of $\mathsf{R}^*(\mathcal{M}_{g,n}^{\mathrm{ct}})$ for general g and n.

By [7], the kernel of the λ_7-pairing on $\mathsf{R}^*(\mathcal{M}_7^{\mathrm{ct}})$ is a 1-dimensional subspace of $\mathsf{R}^6(\mathcal{M}_7^{\mathrm{ct}})$ in the graded piece

$$\mathsf{R}^5(\mathcal{M}_7^{\mathrm{ct}}) \times \mathsf{R}^6(\mathcal{M}_7^{\mathrm{ct}}) \to \mathsf{R}^{11}(\mathcal{M}_7^{\mathrm{ct}})\,. \tag{46}$$

We define a class $\alpha \in \mathsf{R}^6(\mathcal{M}_7^{\mathrm{ct}})$ by pulling back $\mathsf{Tor}^*\Delta_6$ along the forgetful map

$$\pi : \mathcal{M}_{6,1}^{\mathrm{ct}} \to \mathcal{M}_6^{\mathrm{ct}}$$

and attaching an elliptic tail via

$$j : \mathcal{M}_{6,1}^{\mathrm{ct}} \times \mathcal{M}_{1,1}^{\mathrm{ct}} \to \mathcal{M}_7^{\mathrm{ct}}\,.$$

In other words, $\alpha = j_*(\pi^*\mathsf{Tor}^*\Delta_6 \times [\mathcal{M}_{1,1}^{\mathrm{ct}}]) \in \mathsf{R}^6(\mathcal{M}_7^{\mathrm{ct}})$.

Proposition 40. *The class* $\alpha = j_*(\pi^*\mathsf{Tor}^*\Delta_6 \times [\mathcal{M}_{1,1}^{\mathrm{ct}}]) \in \mathsf{R}^6(\mathcal{M}_7^{\mathrm{ct}})$ *spans the 1-dimension Gorenstein kernel of* $\mathsf{R}^*(\mathcal{M}_7^{\mathrm{ct}})$.

Proof. We first show α does not vanish. Consider the pull back

$$j^*(\alpha) = (-\psi_1 \cdot \pi^*\mathsf{Tor}^*\Delta_6) \times [\mathcal{M}_{1,1}^{\mathrm{ct}}]\,.$$

The right side is nonzero. Indeed, the class $-\psi_1 \cdot \pi^*\mathsf{Tor}^*\Delta_6 \neq 0$ since its pushforward under π is a nonzero multiple of $\mathsf{Tor}^*\Delta_6 \neq 0$, using Theorem 5. Hence $\alpha \neq 0 \in \mathsf{R}^6(\mathcal{M}_7^{\mathrm{ct}})$.

We prove next that α lies in the Gorenstein kernel of (46). For every $\beta \in \mathsf{R}^5(\mathcal{M}_7^{\mathrm{ct}})$, we must show that

$$\alpha \cdot \beta = j_*(\mathsf{Tor}^*\Delta_6 \times [\mathcal{M}_{1,1}^{\mathrm{ct}}]) \cdot \beta = j_*((\pi^*\mathsf{Tor}^*\Delta_6 \times [\mathcal{M}_{1,1}^{\mathrm{ct}}]) \,\cdot\, j^*\beta) = 0 \in \mathsf{R}^{11}(\mathcal{M}_7^{\mathrm{ct}})\,.$$

It suffices to show

$$(\pi^*\mathsf{Tor}^*\Delta_6 \times [\mathcal{M}_{1,1}^{\mathrm{ct}}]) \cdot j^*\beta = 0$$

on $\mathcal{M}_{6,1}^{\mathrm{ct}} \times \mathcal{M}_{1,1}^{\mathrm{ct}}$. By [24, Propositon 12], the Künneth components of $j^*\beta$ are tautological,

$$j^*\beta \in \mathsf{R}^*(\mathcal{M}_{6,1}^{\mathrm{ct}}) \otimes \mathsf{R}^*(\mathcal{M}_{1,1}^{\mathrm{ct}})\,.$$

Since $\mathsf{R}^*(\mathcal{M}_{1,1}^{\mathrm{ct}}) = \mathbb{Q}$, we need only show that

$$\pi^*\mathsf{Tor}^*\Delta_6 \cdot \gamma = 0 \in \mathsf{R}^{10}(\mathcal{M}_{6,1}^{\mathrm{ct}})$$

for any class $\gamma \in \mathsf{R}^5(\mathcal{M}_{6,1}^{\mathrm{ct}})$. We have

$$\mathsf{R}^{10}(\mathcal{M}_{6,1}^{\mathrm{ct}}) = \mathsf{R}^9(\mathcal{M}_6^{\mathrm{ct}}) = \mathbb{Q}\,.$$

Furthermore, using the description of the socle generator in [19, Section 4.1.2] or [25, Section 5.6], we know

$$\pi_* : \mathsf{R}^{10}(\mathcal{M}_{6,1}^{\mathrm{ct}}) \to \mathsf{R}^9(\mathcal{M}_6^{\mathrm{ct}})$$

is an isomorphism. Therefore, it remains to prove

$$\pi_*(\pi^*\mathsf{Tor}^*\Delta_6 \cdot \gamma) = 0 \text{ or } \mathsf{Tor}^*\Delta_6 \cdot \pi_*\gamma = 0\,,$$

which is clear since $\mathsf{Tor}^*\Delta_6 \in \mathsf{R}^5(\mathcal{M}_6^{\mathrm{ct}})$ is in the Gorenstein kernel and $\pi_*\gamma \in \mathsf{R}^4(\mathcal{M}_6^{\mathrm{ct}})$. ☐

So $\mathsf{Tor}^*\Delta_6 \in \mathsf{R}^5(\mathcal{M}_6^{\mathrm{ct}})$ explains not only the the Gorenstein kernel of $\mathsf{R}^*(\mathcal{M}_6^{\mathrm{ct}})$, but also the Gorenstein kernel of $\mathsf{R}^*(\mathcal{M}_7^{\mathrm{ct}})$!

Proof of Proposition 6. Since $\alpha \in \mathsf{R}^6(\mathcal{M}_7^{\mathrm{ct}})$ is a generator of the Gorenstein kernel of (46) and $\mathsf{Tor}^*\Delta_7$ also lies in the Gorenstein kernel by Theorem 4, there exists a constant $c \in \mathbb{Q}$ for which

$$\mathsf{Tor}^*\Delta_7 = c \cdot \alpha\,. \tag{47}$$

The pullback $j^*(\mathsf{Tor}^*\Delta_7)$ vanishes by the proof of Theorem 4. Since we have seen $j^*\alpha$ does not vanish, we must have $c = 0$. ☐

6.4. **Outlook in higher genus.** For $g \geq 8$, the full structure of $\mathsf{R}^*(\mathcal{M}_g^{\mathrm{ct}})$ is not yet understood, but a complete proposal is provided by Pixton's conjecture [46].

Assuming Pixton's relations are complete for $\mathsf{R}^*(\mathcal{M}_g^{\mathrm{ct}})$, we have shown that $\mathsf{Tor}^*\Delta_8 \in \mathsf{R}^7(\mathcal{M}_8^{\mathrm{ct}})$ and $\mathsf{Tor}^*\Delta_9 \in \mathsf{R}^8(\mathcal{M}_9^{\mathrm{ct}})$ are nonzero using Pixton's formula in Section 5.5 (and computing with `admcycles` [14]). Because of the computational complexity, higher genus calculations using these methods remain out of reach. On the other hand, Iribar López has shown that

$$\Delta_g \neq 0 \in \mathsf{CH}^{g-1}(\mathcal{A}_g)$$

for $g = 12$ and even $g \geq 16$ [33].

Using the methods of [32, Theorem 33], Taïbi has shown that $\mathsf{IH}^{2g-2}(\mathcal{A}_g^{\mathrm{Sat}})$ is not generated by λ classes when $g \geq 8$. We view his calculations as evidence that Δ_g is nonzero for $g \geq 8$.

7. Virtual fundamental classes on the Noether-Lefschetz loci

We study the virtual geometry of the Noether-Leschetz loci. The components of the Noether-Lefschetz locus NL_g^2 have been classified by Debarre and Lazslo [13], see Theorem 13. We will follow the notation of Theorem 13. All irreducible components are nonsingular [13]. The components of type (i) have codimension $k(g-k)$, while the components of type (ii) have codimension $g(n+1)/2$. However, the expected codimension of each Noether-Lefschetz component is the larger number

$$\binom{g}{2} = \dim H^{2,0}(A)\,,$$

where A is an abelian variety, see for instance [10, 3.a.25].

Let $j : \mathcal{S} \to \mathcal{A}_g$ denote a Noether-Lefschetz component. Consider the universal family

$$\pi : \mathcal{X}_g \to \mathcal{A}_g\,,$$

and the variation of Hodge structure on the second cohomology

$$\mathcal{F}^2 \subset \mathcal{F}^1 \subset \mathcal{F}^0 = \mathbf{R}^2\pi_*\mathbb{C} \otimes \mathcal{O}_{\mathcal{A}_g}.$$

Griffiths transversality yields a map

$$\nabla : \mathcal{F}^1/\mathcal{F}^2 \to \mathcal{F}^0/\mathcal{F}^1 \otimes \Omega_{\mathcal{A}_g}.$$

Over the marked Noether-Lefschetz locus $\mathcal{S}$, the additional generator of the Picard group furnishes a section of $\mathcal{F}^1/\mathcal{F}^2$, while $\mathcal{F}^0/\mathcal{F}^1 = \wedge^2\mathbb{E}^\vee$. Thus, over $\mathcal{S}$, we dually have a natural map $T_{\mathcal{A}_g} \to j^* \wedge^2 \mathbb{E}$ whose kernel is the tangent space to the Noether-Lefschetz locus $\mathcal{S}$, see [53, Lemma 5.16]. Writing $\mathcal{N}$ for the normal bundle of $\mathcal{S}$, we find

$$0 \to \mathcal{N} \to j^* \wedge^2 \mathbb{E}^\vee.$$

The simplest example of a Noether-Lefschetz locus is the product $j : \mathcal{A}_k \times \mathcal{A}_{g-k} \to \mathcal{A}_g$ with normal bundle $\mathcal{N} = \mathbb{E}_k^\vee \boxtimes \mathbb{E}_{g-k}^\vee$. Over the product locus, we consider the obstruction bundle

$$0 \to \mathcal{N} \to j^* \wedge^2 \mathbb{E}_g^\vee \to \mathrm{Obs} \to 0\,.$$

Since $j^*\mathbb{E}_g^\vee = \mathbb{E}_k^\vee \boxplus \mathbb{E}_{g-k}^\vee$, we find

$$\mathrm{Obs} = \wedge^2\mathbb{E}_k^\vee \boxplus \wedge^2\mathbb{E}_{g-k}^\vee\,.$$

The product locus carries a virtual fundamental class of the expected dimension:

$$[\mathcal{A}_k \times \mathcal{A}_{g-k}]^{\mathrm{vir}} = \mathsf{e}(\mathrm{Obs}).$$

Of course, the same construction makes sense over all the Noether-Lefschetz loci of Theorem 13.

In the case of products, an explicit description of the virtual fundamental class is possible. The following result proves Proposition 14 and implies that $j_*[\mathcal{A}_k \times \mathcal{A}_{g-k}]^{\mathrm{vir}} \in \mathsf{R}^*_{\mathrm{pr}}(\mathcal{A}_g)$.

Lemma 41. *We have* $\mathsf{e}(\wedge^2\mathbb{E}) = \lambda_1 \cdots \lambda_{g-1} \in \mathsf{R}^*(\mathcal{A}_g)$.

Proof. For any vector bundle V of rank r, we have

$$\mathsf{e}(\wedge^2 V) = s_\delta(x_1, ..., x_r)\,, \tag{48}$$

where $x_1, \ldots, x_r$ are the Chern roots of V and s_δ is the Schur polynomial corresponding to

$$\delta = (r-1, r-2, \ldots, 0)\,.$$

The equality (48) follows from the formula for Schur polynomials using alternants [22]:

$$s_\lambda = \frac{a_{\lambda+\delta}}{a_\delta}\,,$$

where $a_\mu = \det(x_j^{\mu_i + r - i})$. If $\lambda = \delta$, the numerator and denominator are both Vandermonde determinants with values $\prod_{i<j}(x_i^2 - x_j^2)$ and $\prod_{i<j}(x_i - x_j)$ respectively. As a consequence, we have

$$\mathsf{e}(\wedge^2 V) = \prod_{i<j}(x_i + x_j) = s_\delta(x_1, \ldots, x_r)\,.$$

We apply (48) to the Hodge bundle. We use the second Jacobi-Trudi formula to compute the Schur polynomial in terms of the elementary symmetric functions, which correspond to the Chern classes of $\mathbb{E}$. The Jacobi-Trudi determinant has the following shape

$$\begin{vmatrix} \lambda_{g-1} & \lambda_g & 0 & 0 & \ldots & 0 & 0 \\ \lambda_{g-3} & \lambda_{g-2} & \lambda_{g-1} & \lambda_g & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 \end{vmatrix}$$

In other words, the determinant has $\lambda_{g-1}, \lambda_{g-2}, \ldots, \lambda_1, 1$ on the diagonal and the indices increase in the rows.

Write D_g for the Jacobi-Trudi determinant, which is a polynomial in $\lambda_1, \ldots, \lambda_g$. We let R_g be the polynomial ring generated by classes $\lambda_1, \ldots, \lambda_g$ subject to Mumford's relations. We have

$$\mathsf{R}^*(\mathcal{A}_g) = \mathsf{R}_g/(\lambda_g) = \mathsf{R}_{g-1}\,.$$

We seek to show that

$$\mathsf{D}_g = \lambda_1 \cdots \lambda_{g-1}$$

in $\mathsf{R}^*(\mathcal{A}_g) = \mathsf{R}_{g-1}$. We proceed by induction on g, the base case being clear. By induction, we have

$$\mathsf{D}_{g-1} = \lambda_1 \cdots \lambda_{g-2}$$

in $\mathsf{R}_{g-2} = \mathsf{R}_{g-1}/(\lambda_{g-1})$. Therefore, we must have

$$\mathsf{D}_{g-1} = \lambda_1 \cdots \lambda_{g-2} + \lambda_{g-1} \cdot \mathsf{P}$$

in R_{g-1}, for some polynomial P in the λ-classes. We expand D_g on the first row. Since $\lambda_g = 0$ in R_{g-1}, we obtain that in R_{g-1} we have

$$\mathsf{D}_g = \lambda_{g-1}\mathsf{D}_{g-1} = \lambda_{g-1}(\lambda_1 \cdots \lambda_{g-2} + \lambda_{g-1} \cdot \mathsf{P}) = \lambda_1 \cdots \lambda_{g-1}.$$

Here, we used that $\lambda_{g-1}^2 = 0$ in R_{g-1} by Mumford's relation. We have completed the inductive step. $\square$

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