GROMOV-WITTEN THEORY OF $Hilb^n(\mathbb{C}^2)$ AND NOETHER-LEFSCHETZ THEORY OF \mathcal{A}_g

AITOR IRIBAR LÓPEZ, RAHUL PANDHARIPANDE, AND HSIAN-HUA TSENG

ABSTRACT. We calculate the genus 1 Gromov-Witten theory of the Hilbert scheme $\text{Hilb}^n(\mathbb{C}^2)$ of points in the plane. The fundamental 1-point invariant (with a divisor insertion) is calculated using a correspondence with the families local curve Gromov-Witten theory over the moduli space $\overline{\mathcal{M}}_{1,1}$. The answer exactly matches a parallel calculation related to the Noether-Lefschetz geometry of the moduli space \mathcal{A}_g of principally polarized abelian varieties. As a consequence, we prove that the associated cycle classes satisfy a homomorphism property for the projection operator on $CH^*(\mathcal{A}_g)$. The fundamental 1-point invariant determines the full genus 1 Gromov-Witten theory of $\text{Hilb}^n(\mathbb{C}^2)$ modulo a nondegeneracy conjecture about the quantum cohomology. A table of calculations is given.

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0. INTRODUCTION

0.1. Hilbert schemes. The Hilbert scheme $\text{Hilb}^n(\mathbb{C}^2)$ of n points in the plane \mathbb{C}^2 is a nonsingular, irreducible, quasi-projective variety of dimension 2n parameterizing ideals $\mathcal{I} \subset \mathbb{C}[x, y]$ of colength n,

$$\dim_{\mathbb{C}} \mathbb{C}[x, y] / \mathcal{I} = n$$

An open dense set of $\text{Hilb}^n(\mathbb{C}^2)$ parameterizes ideals associated to configurations of n distinct unordered points. The geometry of $\text{Hilb}^n(\mathbb{C}^2)$ has been studied from many points of view for several decades now, see [11, 16, 17, 20, 32]. Our perspective here is related to the interactions of $\text{Hilb}^n(\mathbb{C}^2)$ with Gromov-Witten theory and the relative Donaldson-Thomas invariants of threefolds as developed in [3, 27, 28, 29, 30, 37, 38, 39, 43, 44, 45].

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The algebraic torus $T = (\mathbb{C}^*)^2$ acts diagonally on \mathbb{C}^2 by scaling coordinates,

$$(z_1, z_2) \cdot (x, y) = (z_1 x, z_2 y)$$

Let t_1 and t_2 denote the equivariant parameters corresponding to the weights of the T-action on the tangent space $\text{Tan}_0(\mathbb{C}^2)$ at the origin of \mathbb{C}^2 .

There is a canonically induced T-action on $\text{Hilb}^n(\mathbb{C}^2)$. The associated T-equivariant cohomology, $H^*_{\mathsf{T}}(\text{Hilb}^n(\mathbb{C}^2), \mathbb{Q})$, admits a natural basis (as a $\mathbb{Q}[t_1, t_2]$ -module) called the *Nakajima basis*. The Nakajima basis element $|\mu\rangle$ corresponding to the partition μ of n is

$$\frac{1}{\Pi_i \mu_i} [V_\mu]$$

where $[V_{\mu}]$ is (the cohomological dual of) the class of the subvariety of Hilb^{| μ |}(\mathbb{C}^2) with generic element given by a union of schemes of lengths

$$\mu_1,\ldots,\mu_{\ell(\mu)}$$

supported at $\ell(\mu)$ distinct points¹ of \mathbb{C}^2 . The element $|1^n\rangle$ corresponds to the unit

$$1 \in H^*_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2), \mathbb{Q})$$

See [33] for a foundational treatment.

The Hilbert scheme carries a tautological rank n vector bundle,

$$(0.1) \qquad \qquad \mathcal{O}/\mathcal{I} \to \mathsf{Hilb}^n(\mathbb{C}^2)$$

with fiber $\mathbb{C}[x, y]/\mathcal{I}$ over $[\mathcal{I}] \in \text{Hilb}^n(\mathbb{C}^2)$. The T-action on $\text{Hilb}^n(\mathbb{C}^2)$ lifts canonically to the tautological bundle (0.1). Let

$$D = c_1(\mathcal{O}/\mathcal{I}) \in H^2_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2), \mathbb{Q})$$

be the T-equivariant first Chern class. A straightforward calculation² for $n \ge 2$ shows

$$D = -\left|2, 1^{n-2}\right\rangle,$$

see [25].

The T-equivariant quantum cohomology of $\text{Hilb}^n(\mathbb{C}^2)$ has been determined in [37]. The *matrix elements* of the T-equivariant quantum product count³ rational curves meeting three given subvarieties of $\text{Hilb}^n(\mathbb{C}^2)$. The (non-negative) *degree* of an effective⁴ curve class

$$\beta \in H_2(\mathsf{Hilb}^n(\mathbb{C}^2),\mathbb{Z})$$

is defined by pairing with D,

$$d = \int_{\beta} D \, .$$

Curves of degree d are counted with weight q^d , where q is the quantum parameter. The ordinary multiplication in T-equivariant cohomology is recovered by setting q = 0.

¹The points and parts of μ are considered here to be unordered.

²The n = 0, 1 cases are degenerate: D = 0 for both.

³The count is virtual.

⁴The $\beta = 0$ is considered here effective.

Let $M_D^{Hilb^n(\mathbb{C}^2)}$ be the operator of quantum multiplication⁵ by the divisor D,

$$\mathsf{M}_{D}^{\mathsf{Hilb}^{n}(\mathbb{C}^{2})}:QH^{*}_{\mathsf{T}}(\mathsf{Hilb}^{n}(\mathbb{C}^{2}))\to QH^{*}_{\mathsf{T}}(\mathsf{Hilb}^{n}(\mathbb{C}^{2}))\,,\quad \mathsf{M}_{D}^{\mathsf{Hilb}^{n}(\mathbb{C}^{2})}(\gamma)=D\star\gamma\,.$$

The operator $M_D^{\text{Hilb}^n(\mathbb{C}^2)}$ is calculated explicitly in the Nakajima basis for all $\text{Hilb}^n(\mathbb{C}^2)$ in [37]. The matrix coefficients of $M_D^{\text{Hilb}^n(\mathbb{C}^2)}$ lie in the field of rational functions in q (with coefficients⁶ in $\mathbb{Q}(t_1, t_2)$) and determine all genus 0 Gromov-Witten invariants of $\text{Hilb}^n(\mathbb{C}^2)$ by [37, Section 4.2].

We are interested here in the T-equivariant Gromov-Witten invariants of $\text{Hilb}^n(\mathbb{C}^2)$ in genus 1. Let μ^1, \ldots, μ^r be partitions of n. Define

(0.2)
$$\langle \mu^1, \mu^2, \dots, \mu^r \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)} = \sum_{d=0}^{\infty} \langle \mu^1, \mu^2, \dots, \mu^r \rangle_{1,d}^{\mathsf{Hilb}^n(\mathbb{C}^2)} q^d \in \mathbb{Q}(t_1, t_2)[[q]].$$

The series (0.2) is always a rational function in q,

$$\langle \mu^1, \mu^2, \dots, \mu^r \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)} \in \mathbb{Q}(t_1, t_2)(q).$$

The first nontrivial computation in genus 1 appeared in [44]:

$$\big\langle \, D \, \big\rangle_1^{\mathrm{Hilb}^2(\mathbb{C}^2)} = -\frac{1}{24} \frac{(t_1+t_2)^2}{t_1 t_2} \cdot \frac{q+1}{q-1} \, .$$

Our goal here is to provide calculations of all of the series (0.2) starting with the basic case of $\langle D \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)}$.

0.2. Gromov-Witten theory of families of local elliptic curves. Let $\overline{\mathcal{M}}_{1,r}$ be the moduli space of Deligne-Mumford stable curves of genus 1 with *r* markings.⁷ Let

$$\pi: \mathcal{E} \to \overline{\mathcal{M}}_{1,r}$$

be the universal elliptic curve with sections

$$p_1,\ldots,p_r:\overline{\mathcal{M}}_{1,r}\to\mathcal{E}$$

associated to the markings. Let

$$\pi_{\mathbb{C}^2}: \mathcal{E} \times \mathbb{C}^2 \to \overline{\mathcal{M}}_{1,r}$$

be the universal *local curve* over $\overline{\mathcal{M}}_{1,r}$. The torus $\mathsf{T} = (\mathbb{C}^*)^2$ acts on the \mathbb{C}^2 factor as before.

Let $\mu^1, \ldots, \mu^r \in Part(n)$, and let $\overline{\mathcal{M}}_g^{\bullet}(\pi_{\mathbb{C}^2}, \mu^1, \ldots, \mu^r)$ be the moduli space of stable⁸ relative maps to the fibers of $\pi_{\mathbb{C}^2}$,

$$\epsilon: \overline{\mathcal{M}}_{g}^{\bullet}(\pi_{\mathbb{C}^{2}}, \mu^{1}, \dots, \mu^{r}) \to \overline{\mathcal{M}}_{1,r}.$$

The fiber of ϵ over the moduli point

$$(E, p_1, \ldots, p_r) \in \overline{\mathcal{M}}_{1,r}$$

⁵Here, the symbol \star denotes the small quantum product. In Section 4.3, the large quantum product will also play a role (and will be denoted by \star_t).

⁶In the context of quantum cohomology, the definitions require localization, so we will always consider $H^*_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2))$ and $QH^*_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2))$ as modules over $\mathbb{Q}(t_1, t_2)$.

⁷We will always assume r > 0 for stability.

⁸The superscript \bullet indicates possibly *disconnected* domain curves (but no connected component of the domain is contracted to a point). We follow the conventions of [3].

is the moduli space of stable maps of genus g to $E \times \mathbb{C}^2$ relative to the divisors determined by the nodes and the markings of E with boundary⁹ condition μ^i over the divisor $p_i \times \mathbb{C}^2$. Since the degree n is recorded in the size of the partitions μ^i , we omit n from the notation for $\overline{\mathcal{M}}_g^{\bullet}(\pi_{\mathbb{C}^2}, \mu^1, \dots, \mu^r)$.

The moduli space $\overline{\mathcal{M}}_{g}^{\bullet}(\pi_{\mathbb{C}^{2}}, \mu^{1}, \dots, \mu^{r})$ has $\pi_{\mathbb{C}^{2}}$ -relative virtual dimension

$$-nr + \sum_{i=1}^{r} \ell(\mu^i)$$

The Gromov-Witten series of the family $\pi_{\mathbb{C}^2}$ is defined by

$$\left\langle \mu^{1}, \mu^{2}, \dots, \mu^{r} \right\rangle^{\pi_{\mathbb{C}^{2}}, \bullet} = \sum_{b=0}^{\infty} u^{b} \int_{\overline{\mathcal{M}}_{1,r}} \epsilon_{*} \left(\left[\overline{\mathcal{M}}_{g[b]}^{\bullet} (\pi_{\mathbb{C}^{2}}, \mu^{1}, \dots, \mu^{r}) \right]^{vir_{\pi_{\mathbb{C}^{2}}}} \right).$$

Here, the summation index b is the branch point number, so

$$2g[b] - 2 = b + nr - \sum_{i=1}^{r} \ell(\mu^{i})$$

The moduli space of stable maps $\overline{\mathcal{M}}_{g[b]}^{\bullet}(\pi_{\mathbb{C}^2}, \mu^1, \dots, \mu^r)$ is empty unless g[b] is an integer. The virtual class $\left[\overline{\mathcal{M}}_{g[b]}^{\bullet}(\pi_{\mathbb{C}^2}, \mu^1, \dots, \mu^r)\right]^{vir_{\pi_{\mathbb{C}^2}}}$ is the $\pi_{\mathbb{C}^2}$ -relative T-equivariant virtual class of the family of relative stable maps to the fibers of $\pi_{\mathbb{C}^2}$. We define

$$\langle \mu^{1}, \mu^{2}, \dots, \mu^{r} \rangle_{g}^{\pi_{\mathbb{C}^{2}}, \bullet} = \operatorname{Coeff}_{u^{2g-2}} \left[\langle \mu^{1}, \mu^{2}, \dots, \mu^{r} \rangle^{\pi_{\mathbb{C}^{2}}, \bullet} \right]$$
$$= \int_{\overline{\mathcal{M}}_{1,r}} \epsilon_{*} \left(\left[\overline{\mathcal{M}}_{g}^{\bullet} (\pi_{\mathbb{C}^{2}}, \mu^{1}, \dots, \mu^{r}) \right]^{vir_{\pi_{\mathbb{C}^{2}}}} \right)$$

To emphasize the degree n, we will sometimes use the notation

$$\left\langle \mu^1, \mu^2, \dots, \mu^r \right\rangle_{g,n}^{\pi_{\mathbb{C}^2}, \bullet} = \left\langle \mu^1, \mu^2, \dots, \mu^r \right\rangle_g^{\pi_{\mathbb{C}^2}, \bullet}$$

The Gromov-Witten series of $\text{Hilb}^n(\mathbb{C}^2)$ and the Gromov-Witten series of the family $\pi_{\mathbb{C}^2}$ are related by the following result of [44].

Theorem A (Pandharipande-Tseng). For all $\mu^1, \mu^2, \ldots, \mu^r \in Part(n)$, we have

$$\langle \mu^1, \mu^2, \dots, \mu^r \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)} = (-i)^{\sum_{i=1}^r \ell(\mu^i) - |\mu^i|} \langle \mu^1, \mu^2, \dots, \mu^r \rangle^{\pi_{\mathbb{C}^2}, \bullet}$$

after the variable change $-q = e^{iu}$.

The calculation of the Gromov-Witten invariants of $\text{Hilb}^n(\mathbb{C}^2)$ in genus 1 is therefore *equivalent* to the calculation of the Gromov-Witten invariants of the family $\pi_{\mathbb{C}^2}$ of local elliptic curves. For example,

(0.3)
$$\langle D \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)} = -(-i)^{-1} \langle (2, 1^{n-2}) \rangle^{\pi_{\mathbb{C}^2}, \bullet}.$$

⁹The boundary conditions are unordered, and the cohomology weights of the boundary conditions are all the identity class.

0.3. Moduli of abelian varieties. Let \mathcal{A}_g be the moduli space of principally polarized abelian varieties (X, θ) of dimension g. The space \mathcal{A}_g is a nonsingular Deligne-Mumford stack of dimension $\binom{g+1}{2}$. Let

$$\nu:\mathcal{X}_g\to\mathcal{A}_g$$

be the universal principally polarized abelian variety. We refer the reader to [2] for the foundations of the study of the moduli of abelian varieties.

A general abelian variety (X, θ) parameterized by \mathcal{A}_g has Picard number 1. The *Noether-Lefschetz locus* of \mathcal{A}_g parameterizes abelian varieties with Picard number at least 2. The simplest components of the Noether-Lefschetz locus of \mathcal{A}_g are related to the geometry of elliptic curves on abelian varieties. For $g \ge 1$ and $n \ge 1$, let

$$\mathsf{NL}_{g,n} = \left\{ (X,\theta) \in \mathcal{A}_g \, \middle| \begin{array}{c} X \text{ contains a subgroup } E \subset X \\ \text{which is an elliptic curve of degree } \theta \cdot [E] = n \end{array} \right\}.$$

Let $[NL_{g,n}] \in CH^{g-1}(\mathcal{A}_g)$ be the associated cycle class.¹⁰ The following linear combination of components plays a geometrically important role¹¹

$$[\widetilde{\mathsf{NL}}_{g,n}] = \sum_{n'|n} \sigma_1\left(\frac{n}{n'}\right) [\mathsf{NL}_{g,n'}] \in \mathsf{CH}^{g-1}(\mathcal{A}_g),$$

see [19].

The Hodge bundle is the rank g vector bundle

$$\mathbb{E}_q = \nu_*(\Omega_\nu)$$

The Chern classes $\lambda_i = c_i(\mathbb{E}_q) \in \mathsf{CH}^i(\mathcal{A}_q)$ generate the *tautological ring* [48].

 $\mathsf{R}^*(\mathcal{A}_q) \subset \mathsf{CH}^*(\mathcal{A}_q)$.

A canonical Q-linear projection operator

$$\mathsf{taut}: \mathsf{CH}^*(\mathcal{A}_q) \to \mathsf{R}^*(\mathcal{A}_q)$$

has been constructed in [4].

The following result of [22] determines the projections of the simplest components of the Noether-Lefschetz loci.

Theorem B (Iribar-López). For $g \ge 2$ and $n \ge 1$, we have

$$\mathsf{taut}([\mathsf{NL}_{g,n}]) = \frac{n^{2g-1}g}{6|B_{2g}|} \prod_{p|n} (1-p^{2-2g})\lambda_{g-1}$$

*or, equivalently*¹²,

(0.4)
$$\operatorname{taut}\left(\frac{(-1)^g}{24}\lambda_{g-1} + \sum_{n=1}^{\infty} [\widetilde{\mathsf{NL}}_{g,n}] Q^n\right) = \frac{(-1)^g}{24} E_{2g}(Q)\lambda_{g-1},$$

where $E_{2g}(Q)$ is the Eisenstein modular function of weight 2g in the variable $Q = e^{2\pi i \tau}$.

¹⁰The g = 1 case is degenerate: $[NL_{1,n}] = [A_1]$ by definition. All Chow groups are taken with \mathbb{Q} -coefficients. ¹¹As usual, $\sigma_r(n) = \sum_{d|n} d^r$.

¹²Equation (0.4) is correct also for g = 1 with the convention $[NL_{1,n}] = [A_1]$.

A basic open question regarding the structure of the projection operator is whether taut is a homomorphism of \mathbb{Q} -algebras:

$$\mathsf{taut}(\gamma) \cdot \mathsf{taut}(\widehat{\gamma}) \stackrel{?}{=} \mathsf{taut}(\gamma \cdot \widehat{\gamma}) \in \mathsf{R}^*(\mathcal{A}_g)$$

for all $\gamma, \widehat{\gamma} \in \mathsf{CH}^*(\mathcal{A}_g)$.

0.4. A triple equivalence.

0.4.1. *Hilbert schemes of points*. Let Tr_n be the normalized trace of the operator $\operatorname{M}_D^{\operatorname{Hilb}^n(\mathbb{C}^2)}$ of quantum multiplication by the divisor D on $QH^*_{\mathsf{T}}(\operatorname{Hilb}^n(\mathbb{C}^2))$,

$$\mathsf{Tr}_n = \frac{1}{t_1 + t_2} \mathsf{trace}(\mathsf{M}_D^{\mathsf{Hilb}^n(\mathbb{C}^2)}) \,.$$

Using the formula of [37] for $M_D^{Hilb^n(\mathbb{C}^2)}$, we obtain

(0.5)
$$\operatorname{Tr}_{n} = \sum_{\mu \in \operatorname{Part}(n)} \sum_{i} \left(\frac{\mu_{i}^{2}}{2} \frac{(-q)^{\mu_{i}} + 1}{(-q)^{\mu_{i}} - 1} - \frac{\mu_{i}}{2} \frac{(-q) + 1}{(-q) - 1} \right) \in \mathbb{Q}(q),$$

where the first summation is over the set Part(n) of partitions of n and the second summation is over all parts μ_i of the partition $\mu \in Part(n)$.

The first of three equivalent statements is the following calculation of the basic genus 1 Gromov-Witten invariant of the Hilbert scheme of points.

Theorem 1. For $n \ge 1$, we have

$$\langle D \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)} = -\frac{1}{24} \frac{(t_1 + t_2)^2}{t_1 t_2} \left(\mathsf{Tr}_n + \sum_{k=2}^{n-1} \sigma_{-1}(n-k) \mathsf{Tr}_k \right) \,.$$

The evaluations for n = 1, 2 agree with the known results:

$$\begin{split} \left\langle \, D \, \right\rangle_1^{\mathrm{Hilb}^1(\mathbb{C}^2)} &= -\frac{1}{24} \frac{(t_1 + t_2)^2}{t_1 t_2} \cdot \mathrm{Tr}_1 = 0 \,, \\ \left\langle \, D \, \right\rangle_1^{\mathrm{Hilb}^2(\mathbb{C}^2)} &= -\frac{1}{24} \frac{(t_1 + t_2)^2}{t_1 t_2} \cdot \mathrm{Tr}_2 = -\frac{1}{24} \frac{(t_1 + t_2)^2}{t_1 t_2} \cdot \frac{q + 1}{q - 1} \,. \end{split}$$

For n = 3, 4, 5, we obtain

$$\begin{split} \left\langle \left. D \right\rangle_{1}^{\mathsf{Hilb}^{3}(\mathbb{C}^{2})} &= -\frac{1}{24} \cdot \frac{(t_{1}+t_{2})^{2}}{t_{1}t_{2}} \cdot \left(\mathsf{Tr}_{3}+\mathsf{Tr}_{2}\right) \\ &= -\frac{1}{24} \cdot \frac{(t_{1}+t_{2})^{2}}{t_{1}t_{2}} \cdot \left(\frac{5q^{3}-3q^{2}-3q+5}{(q-1)(q^{2}-q+1)}\right) \\ &= -\frac{1}{24} \cdot \frac{(t_{1}+t_{2})^{2}}{t_{1}t_{2}} \cdot \left(-5-7q-q^{2}+2q^{3}-q^{4}-7q^{5}-10q^{6}-7q^{7}+\ldots\right), \end{split}$$

$$\begin{split} \left\langle \left. D \right\rangle_{1}^{\mathsf{Hilb}^{4}(\mathbb{C}^{2})} &= -\frac{1}{24} \cdot \frac{(t_{1}+t_{2})^{2}}{t_{1}t_{2}} \cdot \left(\mathsf{Tr}_{4} + \mathsf{Tr}_{3} + \frac{3}{2}\mathsf{Tr}_{2} \right) \\ &= -\frac{1}{24} \cdot \frac{(t_{1}+t_{2})^{2}}{t_{1}t_{2}} \cdot \left(\frac{35q^{5} - 28q^{4} + 23q^{3} + 23q^{2} - 28q + 35}{2(q-1)(q^{2}+1)(q^{2}-q+1)} \right) \\ &= -\frac{1}{24} \cdot \frac{(t_{1}+t_{2})^{2}}{t_{1}t_{2}} \cdot \left(-\frac{35}{2} - 21q - q^{2} - 3q^{3} - 17q^{4} - 21q^{5} - 19q^{6} - 21q^{7} + \ldots \right) \,, \end{split}$$

$$\begin{split} \langle D \rangle_1^{\mathsf{Hilb}^5(\mathbb{C}^2)} &= -\frac{1}{24} \cdot \frac{(t_1 + t_2)^2}{t_1 t_2} \cdot \left(\mathsf{Tr}_5 + \mathsf{Tr}_4 + \frac{3}{2} \mathsf{Tr}_3 + \frac{4}{3} \mathsf{Tr}_2 \right) \\ &= -\frac{1}{24} \cdot \frac{(t_1 + t_2)^2}{t_1 t_2} \cdot \left(-\frac{272q^9 - 539q^8 + 760q^7 - 629q^6 + 302q^5 + 302q^4 - 629q^3 + 760q^2 - 539q + 272}{6(q - 1)(q^2 + 1)(q^2 - q + 1)(q^4 - q^3 + q^2 - q + 1)} \right) \\ &= -\frac{1}{24} \cdot \frac{(t_1 + t_2)^2}{t_1 t_2} \cdot \left(\frac{136}{3} + \frac{277}{6}q - \frac{41}{6}q^2 + \frac{17}{3}q^3 + \frac{151}{6}q^4 + \frac{127}{6}q^5 + \frac{101}{3}q^6 + \frac{277}{6}q^7 + \ldots \right). \end{split}$$

0.4.2. *Families of local elliptic curves*. It is natural to look for an analogue of Theorem 1 for the Gromov-Witten theory of families of local elliptic curves using the correspondence of Theorem A. The best statement is expressed in terms of the *descendent* Gromov-Witten theory of the family

$$\pi: \mathcal{E} \to \overline{\mathcal{M}}_{1,1}$$

for stable maps with connected domains.

Let $\overline{\mathcal{M}}_{q,1}^{\circ}(\pi, n)$ be the moduli space of stable¹³ relative maps to the fibers of π ,

$$\epsilon: \overline{\mathcal{M}}_{g,1}^{\circ}(\pi, n) \to \overline{\mathcal{M}}_{1,1}.$$

The fiber of ϵ over the moduli point

$$(E, p_1) \in \overline{\mathcal{M}}_{1,1}$$

is the moduli space of stable maps of 1-pointed connected genus g curves to E of degree n relative to the divisors determined by the nodes of C. The moduli space $\overline{\mathcal{M}}_{g,1}^{\circ}(\pi, n)$ has π -relative virtual dimension 2g - 1 and total virtual dimension 2g.

For $g \ge 2$, let

$$\langle \tau_1(\mathsf{p}_1)\lambda_g\lambda_{g-2}\rangle_{g,n}^{\pi,\circ} = \int_{[\overline{\mathcal{M}}_{g,1}^\circ(\pi,n)]^{vir_{\pi}}} \tau_1(\mathsf{p}_1)\lambda_g\lambda_{g-2} \in \mathbb{Q},$$

where p_1 is viewed as a divisor class on \mathcal{E} and the Hodge classes $\lambda_g \lambda_{g-2}$ are pulled-back from the moduli of the domain curves.¹⁴

Theorem 2. For $g \ge 2$, we have

$$\sum_{n=0}^{\infty} \left\langle \tau_1(\mathbf{p}_1) \lambda_g \lambda_{g-2} \right\rangle_{g,n}^{\pi,\circ} Q^n = \frac{(-1)^g}{24} \frac{|B_{2g}|}{4g} \frac{|B_{2g-2}|}{(2g-2)!} E_{2g}(Q) \,.$$

¹³The superscript \circ indicates *connected* domain curves.

¹⁴The λ_q class plays an important role in the theory Hodge integrals, see [10, 13, 31].

The g = 1 case is degenerate. The corresponding evaluation is

(0.6)
$$\sum_{n=0}^{\infty} \left\langle \tau_1(\mathbf{p}_1) \right\rangle_{1,n}^{\pi,\circ} Q^n = \frac{(-1)}{24} \frac{|B_2|}{4} \frac{|B_0|}{0!} E_2(Q) = -\frac{1}{576} E_2(Q) \, .$$

Theorem A together with a families relative/descendent correspondence will be used to show that Theorem 1 and 2 are equivalent.

0.4.3. *The homomorphism question*. The third equivalence involves the homomorphism question for the projection operator

$$\mathsf{taut}: \mathsf{CH}^*(\mathcal{A}_a) \to \mathsf{R}^*(\mathcal{A}_a)$$
 .

Let \mathcal{M}_{q}^{ct} be the 3g-3 dimensional moduli space of genus g curves of compact type, and let

$$\mathsf{Tor}:\mathcal{M}^{\mathsf{ct}}_{g} o\mathcal{A}_{g}$$

be the Torelli map. Since Tor is proper, we can push forward the fundamental class¹⁵:

$$\operatorname{Tor}_{*}[\mathcal{M}_{g}^{\operatorname{ct}}] \in \operatorname{CH}^{\binom{g+1}{2}-3g+3}(\mathcal{A}_{g}).$$

The intersection theory of the Torelli cycle has been studied in [5, 7, 22].

Theorem 3. For $g \ge 1$ and $n \ge 1$, we have

$$\mathsf{taut}(\mathsf{Tor}_*[\mathcal{M}_q^{\mathsf{ct}}]) \cdot \mathsf{taut}([\mathsf{NL}_{g,n}]) = \mathsf{taut}(\mathsf{Tor}_*[\mathcal{M}_q^{\mathsf{ct}}] \cdot [\mathsf{NL}_{g,n}]) \in \mathsf{R}^*(\mathcal{A}_g)$$

In other words, the classes $\text{Tor}_*[\mathcal{M}_g^{\text{ct}}], [\mathsf{NL}_{g,n}] \in \mathsf{CH}^*(\mathcal{A}_g)$ satisfy the multiplicative property with respect to taut. Theorem B together with a study of the geometry of the Torelli map was used to show that Theorems 2 and 3 are equivalent in [22].

0.4.4. *Proof strategy*. A central result of the paper is the proof of Theorem 2. As a consequence of the equivalences, Theorems 1 and 3 will then also be proven.

While Gromov-Witten theory is well-developed for the study of a fixed target variety, the main difficulty in proving Theorem 2 is that there are very few techniques which are useful for the calculations of Gromov-Witten invariants in families. Our proof of Theorem 2 uses a mix of Hodge integrals, formulas for the Gromov-Witten theory of a fixed elliptic target, and new constraints on family invariants. The methods yields calculations of more general families Hodge integrals in Section 2.6.

0.5. Reduction of multi-point series to 1-point series. By [37, Section 4.2], the quantum powers of D generate the quantum cohomology $QH^*_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2))$. More precisely, the set

$$\{1, D, D^{*2}, D^{*3}, \dots, D^{*(|\mathsf{Part}(n)|-1)}\}$$

is a basis of $QH^*_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2))$ as a $\mathbb{Q}(t_1, t_2)(q)$ -vector space. Our first reduction result is that the genus 1 Gromov-Witten theory (0.2) of $\mathsf{Hilb}^n(\mathbb{C}^2)$ can be effectively reduced to 1-point series in the basis (0.7).

¹⁵The g = 1 case is degenerate: Tor_{*} $[\mathcal{M}_1^{\text{ct}}] = [\mathcal{A}_1]$ by definition.

Theorem 4. To every genus 1 series $\langle D^{*k_1}, ..., D^{*k_\ell} \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)}$, there are canonically associated functions

(0.8)
$$\left\{C_{k,m}\right\}_{0 \le k \le |\mathsf{Part}(n)|-1, 0 \le m \le \ell-1} \subset \mathbb{Q}(t_1, t_2)(q)$$

for which the following equation holds:

$$\left\langle D^{*k_1}, ..., D^{*k_\ell} \right\rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)} = \sum_{k=0}^{|\mathsf{Part}(n)|-1} \sum_{m=0}^{\ell-1} C_{k,m} \cdot \left(q \frac{d}{dq}\right)^m \left\langle D^{*k} \right\rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)}$$

The functions (0.8) are constructed as rational functions of the matrix coefficients of $M_D^{Hilb^n(\mathbb{C}^2)}$. The proof of Theorem 4 uses the WDVV relations in genus 0 and Getzler's relation in genus 1.

0.6. **Reduction to** $\langle D \rangle_1^{\text{Hilb}^n(\mathbb{C}^2)}$. The reduction of the genus 1 Gromov-Witten theory of $\text{Hilb}^n(\mathbb{C}^2)$ to the basic series $\langle D \rangle_1^{\text{Hilb}^n(\mathbb{C}^2)}$ is more subtle. Since the quantum cohomology of $\text{Hilb}^n(\mathbb{C}^2)$ is semisimple, the Givental-Teleman reconstruction result [15, 47] reduces the full higher genus Gromov-Witten theory of $\text{Hilb}^n(\mathbb{C}^2)$ to the genus 0 theory together with the calculation of the R-matrix. The R-matrix was proven to be determined¹⁶ by constant map invariants in [44]. The resulting control of the full Gromov-Witten theory of $\text{Hilb}^n(\mathbb{C}^2)$ was used in [44] to prove both the crepant resolution and the GW/DT correspondences associated to the geometry, but was not sufficient to obtain closed form evaluations of *any* higher genus series (because of the difficulty in the determination and the lack of closed forms for the R-matrix).

In genus 1, the Givental-Teleman reconstruction formula reduces to an earlier equation due to Givental [14] on the associated Frobenius manifold:

(0.9)
$$d\mathcal{F}_1^{\mathsf{Hilb}^n(\mathbb{C}^2)} = \frac{1}{2} \sum_{i=1}^{|\mathsf{Part}(n)|} \mathsf{R}_{ii} \, du_i + \frac{1}{48} \sum_{i=1}^{|\mathsf{Part}(n)|} d\log \Delta_i \, .$$

Here, $\mathcal{F}_1^{\mathsf{Hilb}^n(\mathbb{C}^2)}$ is the g = 1 Gromov-Witten potential of $\mathsf{Hilb}^n(\mathbb{C}^2)$, the u_i are the canonical coordinates, and

$$\Delta_i = \left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i} \right\rangle^{-1}$$

From Theorem 1, we know explicitly the functions

(0.10)
$$\left\langle \underbrace{D, \dots, D}_{\ell} \right\rangle_{1}^{\mathsf{Hilb}^{n}(\mathbb{C}^{2})} = \left(q \frac{d}{dq} \right)^{\ell-1} \left\langle D \right\rangle_{1}^{\mathsf{Hilb}^{n}(\mathbb{C}^{2})}$$

¹⁶Since the Gromov-Witten theory of $\mathsf{Hilb}^n(\mathbb{C}^2)$ is equivariant, the associated CohFT is *not* conformal.

for $\ell = 1, ..., |Part(n)|$. We construct a $|Part(n)| \times |Part(n)|$ matrix which expresses the functions (0.10) in terms of the functions $R_{ii}|_{t=0}$, where t is the coordinate on the Frobenius manifold. The non-degeneracy of the Wronskian

$$\mathsf{W} = \begin{pmatrix} \nabla_D u_1 \big|_{\mathbf{t}=0} & \nabla_D u_2 \big|_{\mathbf{t}=0} & \cdots & \nabla_D u_{|\mathsf{Part}(n)|} \big|_{\mathbf{t}=0} \\ q \frac{d}{dq} \nabla_D u_1 \big|_{\mathbf{t}=0} & q \frac{d}{dq} \nabla_D u_2 \big|_{\mathbf{t}=0} & \cdots & q \frac{d}{dq} \nabla_D u_{|\mathsf{Part}(n)|} \big|_{\mathbf{t}=0} \\ \vdots & \vdots & \vdots & \vdots \\ (q \frac{d}{dq})^{|\mathsf{Part}(n)|-1} \nabla_D u_1 \big|_{\mathbf{t}=0} & (q \frac{d}{dq})^{|\mathsf{Part}(n)|-1} \nabla_D u_2 \big|_{\mathbf{t}=0} & \cdots & (q \frac{d}{dq})^{|\mathsf{Part}(n)|-1} \nabla_D u_{|\mathsf{Part}(n)|} \big|_{\mathbf{t}=0} \end{pmatrix}$$

implies the invertibility of the system.

Theorem 5. If det(W) $\neq 0$, the diagonal entries $\mathsf{R}_{ii}\Big|_{\mathsf{t}=0}$ are determined by $\langle D \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)}$.

By Givental's equation (0.9), the entire genus 1 Gromov-Witten theory of $\text{Hilb}^n(\mathbb{C}^2)$ is then determined by $\langle D \rangle_1^{\text{Hilb}^n(\mathbb{C}^2)}$ together with genus 0 data. We have checked the non-degeneracy of the Wronskian for $\text{Hilb}^n(\mathbb{C}^2)$ for $n \leq 7$ and conjecture the nondegeneracy for all n.

Theorem 6. If det(W) $\neq 0$, the full genus 1 Gromov-Witten theory of $\operatorname{Hilb}^{n}(\mathbb{C}^{2})$ can be effectively reconstructed from $\langle D \rangle_{1}^{\operatorname{Hilb}^{n}(\mathbb{C}^{2})}$ and $\operatorname{M}_{D}^{\operatorname{Hilb}^{n}(\mathbb{C}^{2})}$.

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1. Degree 0 invariants of $\mathsf{Hilb}^n(\mathbb{C}^2)$

We expand here explicitly the formula of Theorem 1 for the degree 0 invariants of $\text{Hilb}^n(\mathbb{C}^2)$ to show the connections to Hilbert scheme calculations of Carlsson and Okounkov [6].

Consider first the generating series over the Hilbert schemes of points:

$$\sum_{n=0}^{\infty} \langle D \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)} \, Q^n = \langle D \rangle_1^{\mathsf{Hilb}^2(\mathbb{C}^2)} \, Q^2 + \langle D \rangle_1^{\mathsf{Hilb}^3(\mathbb{C}^2)} \, Q^3 + \langle D \rangle_1^{\mathsf{Hilb}^4(\mathbb{C}^2)} \, Q^4 + \dots,$$

where the n = 0, 1 terms vanish since D = 0 for n = 0, 1. Theorem 1 can be written as

(1.1)
$$\sum_{n=0}^{\infty} \langle D \rangle_{1}^{\mathsf{Hilb}^{n}(\mathbb{C}^{2})} Q^{n} = -\frac{1}{24} \frac{(t_{1}+t_{2})^{2}}{t_{1}t_{2}} \cdot \left(1 + \sum_{n=1}^{\infty} \sigma_{-1}(n) Q^{n}\right) \cdot \left(\sum_{n\geq 2}^{\infty} \mathsf{Tr}_{n} Q^{n}\right) \,.$$

The only q-dependence on the right side of (1.1) is via Tr_n . After restricting (0.5) to q = 0, we obtain

(1.2)
$$\sum_{n\geq 2}^{\infty} \operatorname{Tr}_{n}|_{q=0} Q^{n} = -\sum_{n=1}^{\infty} Q^{n} \sum_{\mu\in\operatorname{Part}(n)} \sum_{i} \left(\frac{\mu_{i}^{2} - \mu_{i}}{2}\right).$$

Let $\mathcal{P}(Q) = \prod_{k \ge 1} \frac{1}{1-Q^k}$, $\mathcal{E}_2(Q) = \sum_{k \ge 1} \frac{kQ^k}{1-Q^k}$, and $\mathcal{E}_3(Q) = \sum_{k \ge 1} \frac{k^2Q^k}{1-Q^k}$. The following identities follow easily from the generating function of T(n, a), the number of times the part *a* occurs in all partitions of *n*, see [40]:

$$\sum_{k=1}^{\infty} Q^k \left(\sum_{\mu \in \mathsf{Part}(k)} \sum_i \mu_i \right) = \mathcal{P}(Q) \cdot \mathcal{E}_2(Q) ,$$
$$\sum_{k=1}^{\infty} Q^k \left(\sum_{\mu \in \mathsf{Part}(k)} \sum_i \mu_i^2 \right) = \mathcal{P}(Q) \cdot \mathcal{E}_3(Q) .$$

After combining with (1.2), we obtain

$$\sum_{n=0}^{\infty} \operatorname{Tr}_{n}|_{q=0} Q^{n} = \frac{1}{2} \mathcal{P}(Q) (\mathcal{E}_{2}(Q) - \mathcal{E}_{3}(Q)) .$$

The geometric formula for the q = 0 term of $\langle D \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)}$ is obtained via identification of the virtual fundamental class:

$$\int_{[\overline{\mathcal{M}}_{1,1}(\mathsf{Hilb}^{n}(\mathbb{C}^{2}),0)]^{vir}} \operatorname{ev}_{1}^{*}(D) = \int_{\overline{\mathcal{M}}_{1,1}\times\mathsf{Hilb}^{n}(\mathbb{C}^{2})} D \cdot c_{\mathsf{top}}(\mathbb{E}_{1}^{*}\otimes T_{\mathsf{Hilb}^{n}(\mathbb{C}^{2})})$$
$$= -\frac{1}{24} \int_{\mathsf{Hilb}^{n}(\mathbb{C}^{2})} D \cdot c_{\mathsf{top}-1}(T_{\mathsf{Hilb}^{n}(\mathbb{C}^{2})}) .$$

Therefore, the q = 0 part of Theorem 1 is equivalent to the following identity of generating functions:

(1.3)
$$\sum_{n=0}^{\infty} Q^n \int_{\mathsf{Hilb}^n(\mathbb{C}^2)} D \cdot c_{\mathsf{top}-1}(T_{\mathsf{Hilb}^n(\mathbb{C}^2)}) = \frac{(t_1+t_2)^2}{t_1 t_2} (1+\log \mathcal{P}(Q)) \cdot \frac{1}{2} \mathcal{P}(Q)(\mathcal{E}_2(Q)-\mathcal{E}_3(Q)).$$

Consider now the series $\langle c_1(\mathcal{O}/\mathcal{I}) \rangle$ of [6, Corollary 3] where we take the line bundle \mathcal{L} to be \mathcal{O} with equivariant weight m as in [6, Section 2.1.1]:

$$\begin{aligned} \left\langle c_1(\mathcal{O}/\mathcal{I}) \right\rangle &= \sum_{n=0}^{\infty} Q^n \int_{\mathsf{Hilb}^n(\mathbb{C}^2)} D \cdot c(T_{\mathsf{Hilb}^n(\mathbb{C}^2)}, m) \\ &= \sum_{n=0}^{\infty} Q^n \int_{\mathsf{Hilb}^n(\mathbb{C}^2)} D \cdot \left(c_{\mathsf{top}}(T_{\mathsf{Hilb}^n(\mathbb{C}^2)}) + m \cdot c_{\mathsf{top}-1}(T_{\mathsf{Hilb}^n(\mathbb{C}^2)}) + m^2 \cdot c_{\mathsf{top}-2}(T_{\mathsf{Hilb}^n(\mathbb{C}^2)}) \dots \right) \end{aligned}$$

The left side of (1.3) is the coefficient of m^1 of $\langle c_1(\mathcal{O}/\mathcal{I}) \rangle$. By [6, Corollary 3] together with evaluations from [6, Section 2.2.2], we have

$$\frac{\left\langle c_1(\mathcal{O}/\mathcal{I})\right\rangle}{\left\langle 1\right\rangle} = \frac{1}{2}(\mathcal{E}_2(Q) - \mathcal{E}_3(Q)) \cdot \frac{(t_1 + t_2)(t_1 + m)(t_2 + m)}{t_1 t_2},$$

where the series $\left<1\right>$ is

$$\begin{aligned} \left\langle 1 \right\rangle &= \sum_{n=0}^{\infty} Q^n \int_{\mathsf{Hilb}^n(\mathbb{C}^2)} c(T_{\mathsf{Hilb}^n(\mathbb{C}^2)}, m) \\ &= \prod_{n \ge 1} (1 - Q^n)^{\left(\frac{m(-t_1 - t_2 - m)}{t_1 t_2} - 1\right)} \end{aligned}$$

by [6, Corollary 1]. The matching (1.3) then follows from a simple algebraic expansion of the m^1 coefficient of $\langle c_1(\mathcal{O}/\mathcal{I}) \rangle$.

The degree 0 term of Gromov-Witten series $\langle 1 \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)}$ can be studied similarly. The geometric formula for the q = 0 term of $\langle 1 \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)}$ is

$$\int_{[\overline{\mathcal{M}}_{1,1}(\mathsf{Hilb}^{n}(\mathbb{C}^{2}),0)]^{vir}} \operatorname{ev}_{1}^{*}(1) = \int_{\overline{\mathcal{M}}_{1,1}\times\mathsf{Hilb}^{n}(\mathbb{C}^{2})} c_{\mathsf{top}}(\mathbb{E}_{1}^{*}\otimes T_{\mathsf{Hilb}^{n}(\mathbb{C}^{2})})$$
$$= -\frac{1}{24} \int_{\mathsf{Hilb}^{n}(\mathbb{C}^{2})} c_{\mathsf{top}-1}(T_{\mathsf{Hilb}^{n}(\mathbb{C}^{2})}),$$

which equals the m^1 coefficient of $-\frac{1}{24}\langle 1 \rangle$. A calculation then yields

$$\operatorname{Coeff}_{m^1}\left[\langle 1 \rangle\right] = \frac{t_1 + t_2}{t_1 t_2} \cdot \mathcal{P}(Q) \log \mathcal{P}(Q) \,.$$

We obtain the evaluation

$$\sum_{n=0}^{\infty} \langle 1 \rangle_{1}^{\mathsf{Hilb}^{n}(\mathbb{C}^{2})} |_{q=0} Q^{n} = -\frac{1}{24} \frac{t_{1} + t_{2}}{t_{1} t_{2}} \cdot \mathcal{P}(Q) \log \mathcal{P}(Q) \,.$$

The Gromov-Witten series $\langle 1 \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)}$ is much simpler than $\langle D \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)}$. Because of the axiom of the fundamental class, *all* positive degree terms of $\langle 1 \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)}$ vanish. Therefore,

$$\langle 1 \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)} = \mathsf{Coeff}_{Q^n} \left[-\frac{1}{24} \frac{t_1 + t_2}{t_1 t_2} \cdot \mathcal{P}(Q) \log \mathcal{P}(Q) \right].$$

2. FAMILIES HODGE INTEGRALS

2.1. **Overview.** Our first result is the calculation of the families Hodge integral of Theorem 2 over the moduli spaces of stable maps with connected domains,

$$\epsilon: \overline{\mathcal{M}}_{g,1}^{\circ}(\pi, n) \to \overline{\mathcal{M}}_{1,1}, \quad \pi: \mathcal{E} \to \overline{\mathcal{M}}_{1,1}.$$

After evaluation of the special n = 1 and g = 1 cases by hand, the main motivation is to use the vanishing of the virtual class of the moduli space of stable maps to a K3 surface to prove Theorem 2.

2.2. The n = 0 case. The n = 0 invariant concerns the moduli space of degree 0 stable maps $\overline{\mathcal{M}}_{g,1}^{\circ}(\pi, 0)$. After imposing the evaluation condition on the marking, the moduli space is

$$\overline{\mathcal{M}}_{g,1}^{\circ}(\pi,0)\supset \mathrm{ev}_{1}^{-1}(\mathsf{p}_{1})=\overline{\mathcal{M}}_{g,1}\times\overline{\mathcal{M}}_{1,1}$$

with obstruction bundle $\mathbb{E}_g^{\vee} \otimes \mathsf{T}$, where T is the tangent line on $\overline{\mathcal{M}}_{1,1}$ associated to the marking.

• For $g \ge 2$, we evaluate the n = 0 terms by:

$$\langle \tau_1(\mathbf{p}_1)\lambda_g\lambda_{g-2}\rangle_{g,0}^{\pi,\circ} = \int_{[\overline{\mathcal{M}}_{g,1}\times\overline{\mathcal{M}}_{1,1}]^{vir}} \psi_1\lambda_g\lambda_{g-2}$$

$$= \int_{\overline{\mathcal{M}}_{g,1}\times\overline{\mathcal{M}}_{1,1}} \psi_1\lambda_g\lambda_{g-2} \cdot \mathbf{e}(\mathbb{E}_g^{\vee}\otimes\mathsf{T})$$

$$= (-1)^g \int_{\overline{\mathcal{M}}_{g,1}\times\overline{\mathcal{M}}_{1,1}} \psi_1\lambda_g\lambda_{g-2}(\lambda_g+\lambda_{g-1}\boxtimes\psi)$$

$$= \frac{(-1)^g(2g-2)}{24} \frac{|B_{2g}||B_{2g-2}|}{4g(2g-2)(2g-2)!}$$

$$= \frac{(-1)^g}{24} \frac{|B_{2g}|}{4g} \frac{|B_{2g-2}|}{(2g-2)!} .$$

Here, ψ_1 denotes the cotangent line at the marking of $\overline{\mathcal{M}}_{g,1}$. In the third equality, we have denoted the dual of $c_1(\mathsf{T})$ by ψ , the cotangent line at the marking of $\overline{\mathcal{M}}_{1,1}$. In the fourth equality, we have used the dilaton equation and the Hodge integral evaluated in [9, Theorem 4]. The final evaluation agrees with Theorem 2 since the Eisenstein series start with 1,

(2.1)
$$E_{2g}(Q) = 1 - \frac{4g}{B_{2g}} \sum_{n=1}^{\infty} \sigma_{2g-1}(n) Q^n.$$

• For g = 1, we evaluate the n = 0 term by:

$$\begin{aligned} \langle \tau_{1}(\mathbf{p}_{1}) \rangle_{1,0}^{\pi,\circ} &= \int_{[\overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{1,1}]^{vir}} \psi_{1} \\ &= \int_{\overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{1,1}} \psi_{1} \cdot \mathbf{e}(\mathbb{E}_{1}^{\vee} \otimes \mathsf{T}) \\ &= (-1) \int_{\overline{\mathcal{M}}_{1,1} \times \overline{\mathcal{M}}_{1,1}} \psi_{1}(\lambda_{1} + \lambda_{0} \boxtimes \psi) \\ &= -\frac{1}{576} \,, \end{aligned}$$

which agrees with the constant term on the right side of (0.6).

2.3. The g = 1 case. For g = 1 and $n \ge 1$, we can evaluate the integral $\langle \tau_1(\mathsf{p}_1) \rangle_{1,n}^{\pi,\circ}$ by hand. There are no branch points, so the cotangent line on the domain is pulled-back from the cotangent line of $\overline{\mathcal{M}}_{1,1}$. Using the latter cotangent line class, we can express $\langle \tau_1(\mathsf{p}_1) \rangle_{1,n}^{\pi,\circ}$ as $\frac{1}{24}$ times a Gromov-Witten invariant

of maps to a fixed elliptic target (E, p_1) :

$$\langle \tau_1(\mathbf{p}_1) \rangle_{1,n}^{\pi,\circ} = \frac{1}{24} \langle \tau_0(p_1) \rangle_{1,n}^{E,\circ}.$$

Using the well-known evaluation

$$\sum_{n=0}^{\infty} \left\langle \tau_0(p_1) \right\rangle_{1,n}^{E,\circ} = -\frac{1}{24} E_2(Q) \,,$$

we deduce (0.6) for all¹⁷ $n \ge 1$.

2.4. Proof of Theorem 2 for $g \ge 2$ and n > 0. Let S be an elliptically fibered K3 surface S with a section¹⁸ p₁,

$$S \xrightarrow[\mathbf{p}_1]{\pi_S} \mathbb{P}^1,$$

and 24 nodal fibers $R_1, \ldots, R_{24} \subset S$. The fibers of π_S are 1-pointed genus 1 stable curves. The map

$$\mathbb{P}^1 \longrightarrow \overline{\mathcal{M}}_{1,1}$$

induced by π_S is of degree 48. Therefore,

$$\int_{[\overline{\mathcal{M}}_{g,1}^{\circ}(\pi,n)]^{vir}} \tau_1(\mathsf{p}_1)\lambda_g\lambda_{g-2} = \frac{1}{48} \int_{[\overline{\mathcal{M}}_{g,1}^{\circ}(\pi_S,n)]^{vir}} \tau_1(\mathsf{p}_1)\lambda_g\lambda_{g-2}.$$

The moduli space of stable maps to the fibers of $\pi_S : S \to \mathbb{P}^1$ lies over \mathbb{P}^1 ,

 $\epsilon_S: \overline{\mathcal{M}}_{g,1}^{\circ}(\pi_S, n) \to \mathbb{P}^1.$

Proposition 7. The following vanishing holds for $g \ge 2$ and n > 0:

$$\int_{[\overline{\mathcal{M}}_{g,1}^{\circ}(\pi_{S},n)]^{vir}} \tau_{1}(\mathsf{p}_{1})\lambda_{g-2} \cdot \mathsf{e}\left(\mathbb{E}_{g}^{\vee} \otimes \epsilon_{S}^{*}(\mathsf{Tan}_{\mathbb{P}^{1}})\right) = 0.$$

The motivation for Proposition 7 comes from the vanishing of Gromov-Witten invariants for K3 surfaces in non-zero curve classes. The Euler class $e\left(\mathbb{E}_{g}^{\vee} \otimes \epsilon_{S}^{*}(\operatorname{Tan}_{\mathbb{P}^{1}})\right)$ in the integrand relates the families virtual class for π_{S} to the virtual class for maps to S. A proof using deformation to the normal cones of the nodal fibers $R_{i} \subset S$ requires a subtle study of the logarithmic degeneration formula. We will prove the vanishing of Proposition 7 via another path in Section 2.5.

Proposition 7 allow us to exchange the families Hodge integral

$$\int_{[\overline{\mathcal{M}}_{g,1}^{\circ}(\pi_{S},n)]^{vir}}\tau_{1}(\mathsf{p}_{1})\lambda_{g}\lambda_{g-2}$$

for a fixed target Gromov-Witten invariant. We first expand the Euler class as

(2.2)
$$\mathsf{e}\left(\mathbb{E}_{g}^{\vee}\otimes\epsilon_{S}^{*}(\mathsf{Tan}_{\mathbb{P}^{1}})\right) = (-1)^{g}\lambda_{g} + (-1)^{g-1}\lambda_{g-1}\cdot\epsilon_{S}^{*}(2[\mathsf{pt}]).$$

¹⁷The n = 0 term $\langle \tau_1(\mathbf{p}_1) \rangle_{1,0}^{\pi,\circ}$ also matches $\frac{1}{24} \langle \tau_0(p_1) \rangle_{1,0}^{E,\circ}$.

¹⁸The section is denoted by p_1 following the conventions related to families of points curves. The class of p_1 is a *divisor* class in the total space S of the family.

where $[\mathsf{pt}]\in\mathsf{CH}_0(\mathbb{P}^1)$ is the point class. By Proposition 7, we obtain

$$\int_{[\overline{\mathcal{M}}_{g,1}^{\circ}(\pi_{S},n)]^{vir}} \tau_{1}(\mathsf{p}_{1})\lambda_{g-2} \cdot \left((-1)^{g} \lambda_{g} + (-1)^{g-1} \lambda_{g-1} \cdot 2\epsilon_{S}^{*}([\mathsf{pt}])) \right) = 0$$

Hence,

$$\int_{[\overline{\mathcal{M}}_{g,1}^{\circ}(\pi_{S},n)]^{vir}} \tau_{1}(\mathbf{p}_{1})\lambda_{g}\lambda_{g-2} = 2 \int_{[\overline{\mathcal{M}}_{g,1}^{\circ}(\pi_{S},n)]^{vir}} \tau_{1}(\mathbf{p}_{1})\lambda_{g-1}\lambda_{g-2} \cdot [\mathsf{pt}]$$
$$= 2 \int_{[\overline{\mathcal{M}}_{g,1}^{\circ}(E,n)]^{vir}} \tau_{1}(p_{1})\lambda_{g-1}\lambda_{g-2},$$

where the last integral is the Gromov-Witten invariant with a fixed elliptic curve target (E, p_1) .

The evaluation of the required integral for (E, p_1) follows from the methods of [34],

(2.3)
$$\int_{[\overline{\mathcal{M}}_{g,1}^{\circ}(E,n)]^{vir}} \tau_1(p_1)\lambda_{g-1}\lambda_{g-2} = \frac{|B_{2g-2}|\sigma_{2g-1}(n)}{(2g-2)!}$$

as we will explain in Section 2.6. The integral can also be obtained using the study of the Gromov-Witten theory of target curves [35] and was first calculated by Pixton in [46].

,

Theorem 2 then follows from

$$\int_{[\overline{\mathcal{M}}_{g,1}^{\circ}(\pi,n)]^{vir}} \tau_1(\mathbf{p}_1) \lambda_g \lambda_{g-2} = \frac{1}{24} \frac{|B_{2g-2}|\sigma_{2g-1}(n)}{(2g-2)!}$$

and the definition of the Eisenstein series (2.1).

2.5. **Proof of Proposition 7.** The moduli space of stable maps to the fibers of $\pi_S : S \to \mathbb{P}^1$ lies over \mathbb{P}^1 ,

$$\epsilon_S: \overline{\mathcal{M}}_{g,1}^{\circ}(\pi_S, n) \to \mathbb{P}^1$$

The universal curve $\mu: \mathcal{C} \to \overline{\mathcal{M}}_{g,1}^{\circ}(\pi_S, n)$ carries a universal evaluation map



to the universally expanded target



The target \mathcal{T} is a family of elliptic curves over $\overline{\mathcal{M}}_{g,1}^{\circ}(\pi_S, n)$ with possible expansion over the 24 nodal fibers R_i . The only permitted expansions over R_i are simple circuits of rational curves.

The relative dualizing sheaf ω_{π_S} of the of elliptic fibration $\pi_S: S \to \mathbb{P}^1$ is pulled-back from the base,

$$\omega_{\pi_S} \cong \pi_S^*(\mathsf{Tan}_{\mathbb{P}^1})$$
 .

Since g is an isomorphism (except for collapsing chains of unstable rational curves to nodes),

$$\omega_{\nu} \cong h^* \omega_{\pi_M} \cong \nu^* \epsilon_S^* (\operatorname{Tan}_{\mathbb{P}^1}).$$

We therefore obtain $\nu_*\omega_{\nu} \cong \epsilon_S^*(\mathsf{Tan}_{\mathbb{P}^1})$.

Consider next the stabilization map



By the geometry of relative stable maps, the contraction st is an isomorphism (except for collapsing chains of unstable rational curves to nodes). Therefore,

$$\omega_{\mu} \cong \mathrm{st}^* \omega_{\mu_{\mathrm{st}}},$$

and we obtain $\mu_*\omega_{\mu} \cong \mathbb{E}_g$.

Since the moduli space $\overline{\mathcal{M}}_{g,1}^{\circ}(\pi_S, n)$ parametrizes relative maps to the fibers of π_S , we have a pull-back map

(2.4)
$$\epsilon_{S}^{*}(\operatorname{Tan}_{\mathbb{P}^{1}}) \cong \nu_{*}\omega_{\nu} \xrightarrow{f^{*}} \mu_{*}\omega_{\mu} \cong \mathbb{E}_{g}$$

over $\overline{\mathcal{M}}_{g,1}(\pi_S, n)$. Since n > 0, the map (2.4) is injective, so we obtain an exact sequence

(2.5)
$$0 \to \epsilon_S^*(\operatorname{Tan}_{\mathbb{P}^1}) \to \mathbb{E}_g \to \mathbb{F} \to 0$$

where \mathbb{F} is a rank g-1 vector bundle on $\overline{\mathcal{M}}_{g,1}^{\circ}(\pi_S, n)$. We therefore have a factorization

(2.6)
$$\lambda_g = c_1(\epsilon_S^*(\operatorname{Tan})_{\mathbb{P}^1}) \cdot c_{g-1}(\mathbb{F}) = 2\epsilon_S^*(\operatorname{pt}) \cdot \lambda_{g-1}(\mathbb{F})$$

on $\overline{\mathcal{M}}_{g,1}(\pi_S, n)$. For the second equality, the restriction of (2.5) to a fiber of ϵ_S is used to obtain

$$c_{g-1}(\mathbb{F})|_{\epsilon_S^{-1}(\mathsf{pt})} = \lambda_{g-1} \,.$$

The factorization (2.6) of λ_g implies the vanishing

$$\int_{[\overline{\mathcal{M}}_{g,1}^{\circ}(\pi_{S},n)]^{vir}} \tau_{1}(\mathsf{p}_{1})\lambda_{g-2} \cdot \mathsf{e}(\mathbb{E}_{g}^{\vee} \otimes \mathsf{Tan}_{\mathbb{P}^{1}}) = 0$$

by (2.2).

2.6. Hodge integrals and Gromov-Witten theory for a fixed elliptic target. The proof of Proposition 7 yields a stronger statement for n > 0:

$$\mathsf{e}(\mathbb{E}_g^{\vee}\otimes\mathsf{Tan}_{\mathbb{P}^1})\cap[\overline{\mathcal{M}}_{g,r}^{\circ}(\pi_S,n)]^{vir}=0$$

in $CH^*(\overline{\mathcal{M}}_{g,r})$. If $F(\lambda)$ is a homogeneous¹⁹ polynomial in Hodge classes λ_i satisfying

$$\sum_{i=1}^{r} k_i + \deg(\mathsf{F}) = g - 1$$

we obtain (as in Section 2.4):

(2.7)
$$\left\langle \prod_{i=1}^{m} \tau_{k_i}(\mathbf{p}_1) \prod_{j=m+1}^{r} \tau_{k_j+1}(1) \cdot \lambda_g \mathsf{F}(\lambda) \right\rangle_{g,n}^{\pi,\circ} = \frac{1}{24} \left\langle \prod_{i=1}^{m} \tau_{k_i}(p_1) \cdot \prod_{j=m+1}^{r} \tau_{k_j+1}(1) \cdot \lambda_{g-1} \mathsf{F}(\lambda) \right\rangle_{g,n}^{E,\circ},$$

where the n = 0 case follows from an application of Mumford's formula.

The Gromov-Witten Hodge integral of a fixed elliptic curve target (E, p_1) on the right side can be effectively computed. By [34], we have an equality of cycles

$$\sum_{n=0}^{\infty} Q^{n} [\overline{\mathcal{M}}_{g,m}^{\circ}(E,n)]^{vir} \cdot \prod_{i=1}^{m} \tau_{0}(\mathsf{p}_{1}) \cdot \lambda_{g-1} = \frac{(-1)^{g}(2g-1)!}{(2g-2+m)!} \lambda_{g} \lambda_{g-1} \sum_{i=1}^{m} \prod_{j \neq i} \psi_{j} \left(Q \frac{d}{dQ}\right)^{m-1} E_{2g}(Q)$$

in $H^*(\overline{\mathcal{M}}_{g,m})$. Therefore, the right hand side of (2.7) is equal to the coefficient of Q^n in

$$\frac{(-1)^g (2g-1)!}{24(2g+m-2)!} \left(Q\frac{d}{dQ}\right)^{m-1} E_{2g}(Q) \sum_{i=1}^m \int_{\overline{\mathcal{M}}_{g,r}} \prod_{j=1}^r \psi_j^{k_j+1-\delta_{ij}} \cdot \mathsf{F}(\lambda) \lambda_g \lambda_{g-1}$$

The integrals over $\overline{\mathcal{M}}_{g,r}$ can be evaluated effectively via Hodge integral techniques.

• For F = 1, we have the exact evaluation

$$\int_{\overline{\mathcal{M}}_{g,r}} \prod_{j=1}^{r} \psi_{j}^{k_{j}+1-\delta_{ij}} \cdot \lambda_{g} \lambda_{g-1} = \frac{|B_{2g}|}{2^{2g-1}(2g)!} \frac{(2g+r-3)!(2k_{i}+1)}{(2k_{1}+1)!! \dots (2k_{r}+1)!!},$$

given by the Virasoro constraints of \mathbb{P}^2 [13]. We arrive at the following:

where

$$C = \frac{(2g+r-3)! \sum_{i=1}^{m} (2k_i+1)}{2^{2g-2}(2g+m-2)!(2k_1+1)!! \dots (2k_r+1)!!}$$

• For $F = \lambda_{g-2}$ and r = m = 1, we obtain (2.3):

$$\langle \tau_1(p_1)\lambda_{g-1}\lambda_{g-2}\rangle_{g,n}^{E,\circ} = (-1)^{g-1} \frac{4g}{B_{2g}} \sigma_{2g-1}(n) \int_{\overline{\mathcal{M}}_{g,1}} \psi_1 \lambda_g \lambda_{g-1} \lambda_{g-2} = \frac{|B_{2g-2}|}{(2g-2)!} \sigma_{2g-1}(n) \,.$$

¹⁹The class λ_i has degree *i*.

Further development of these ideas will appear in [23].

3. The invariant
$$\langle D \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)}$$

3.1. **Overview.** We present here the proof of Theorem 1:

$$\langle D \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)} = -\frac{1}{24} \frac{(t_1 + t_2)^2}{t_1 t_2} \left(\mathsf{Tr}_n + \sum_{k=2}^{n-1} \sigma_{-1}(n-k) \mathsf{Tr}_k \right) \,.$$

Our strategy is to convert the invariant $\langle D \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)}$ to a families Hodge integral. After several steps related to the connected/disconnected calculus and descendent/relative correspondence, we will show that Theorem 1 follows from Theorem 2.

3.2. Connected/disconnected calculus. The genus 1 Gromov-Witten invariants of $\text{Hilb}^n(\mathbb{C}^2)$ are expressed in terms of families invariants with possibly *disconnected*²⁰ domain curves by by (0.3):

$$\langle D \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)} = -\langle (2, 1^{n-2}) \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)} = \frac{1}{i} \langle (2, 1^{n-2}) \rangle^{\pi_{\mathbb{C}^2}, \bullet}$$

We will transform $\langle (2, 1^{n-2}) \rangle^{\pi_{\mathbb{C}^2}, \bullet}$ to invariants with *connected* domain curves. The connected/disconnected correspondence is well-known for the Gromov-Witten theory of a fixed target. For families invariants, new aspects appear.

There are two connected Gromov-Witten invariants ((i) and (iii)) and two partition counts ((ii) and (iv)) which occur in the connected/disconnect calculus for $\langle (2, 1^{n-2}) \rangle^{\pi_{\mathbb{C}^2}, \bullet}$:

(i) Let $\overline{\mathcal{M}}_{g}^{\circ}(\pi, (2, 1^{n-2}))$ be the moduli space of degree n stable relative maps to the fibers of the universal elliptic curve

$$\pi: \mathcal{E} \to \overline{\mathcal{M}}_{1,1}$$

with connected²¹ domains of genus g and relative condition at the marking given by $(2, 1^{n-2})$. Let

$$\langle \lambda_{g-2}\lambda_g | (2, 1^{n-2}) \rangle_{g,n}^{\pi, \circ} = \int_{[\overline{\mathcal{M}}_g^{\circ}(\pi, (2, 1^{n-2}))]^{vir\pi}} \lambda_{g-2}\lambda_g \, .$$

The descendent/relative correspondence via the degeneration to the normal cone of the section p_1 yields the following result proven in Section 3.5.

Proposition 8. For $g \ge 2$ and $n \ge 2$, we have

$$\langle \tau_1(\mathbf{p}_1)\lambda_{g-2}\lambda_g \rangle_{g,n}^{\pi,\circ} = \frac{\sigma_1(n)}{24} \frac{|B_{2g-2}|}{(2g-2)!} + \langle \lambda_{g-2}\lambda_g | (2, 1^{n-2}) \rangle_{g,n}^{\pi,\circ}.$$

Since $\langle \tau_1(\mathbf{p}_1)\lambda_{g-2}\lambda_g \rangle_{g,n}^{\pi,\circ}$ was calculated in Theorem 2, Proposition 8 completely determines the invariant $\langle \lambda_{g-2}\lambda_g | (2, 1^{n-2}) \rangle_{g,n}^{\pi,\circ}$.

 $^{^{20}}$ The superscript • indicates possibly *disconnected* domain curves (but no connected component of the domain is contracted to a point).

²¹The superscript \circ denotes connected domains.

The g = 1 case takes a special form. For $n \ge 2$, we easily obtain

$$\langle \tau_1(\mathbf{p}_1) \rangle_{1,n}^{\pi,\circ} = \frac{\sigma_1(n)}{24} \frac{|B_0|}{0!} + \langle (2, 1^{n-2}) \rangle_{1,n}^{\pi,\circ}.$$

Using the evaluation (0.6) of $\langle \tau_1(p_1) \rangle_{1,n}^{\pi,\circ}$, we see

$$\langle (2, 1^{n-2}) \rangle_{1,n}^{\pi, \circ} = 0$$

for $n \geq 2$.

(ii) For an integer $l \ge 1$, let Part(l) be the number of partitions of l. Partitions arise naturally in the Gromov-Witten theory of E: Part(l) is the count of possibly disconnected unramified covers of E of degree l where each cover is weighted by the order of the automorphism group. The corresponding generating series is

$$\mathcal{P}(x) = 1 + \sum_{l=1}^{\infty} \mathsf{Part}(l) \, x^l = \prod_{l=1}^{\infty} \frac{1}{1 - x^l} \, .$$

(iii) Let E be a fixed elliptic curve. We denote by $\langle (2, 1^{n-2}) \rangle_{g,n}^{E \times \mathbb{C}^2, \circ}$ the connected genus g, degree n, T-equivariant Gromov-Witten invariant of $E \times \mathbb{C}^2$ with relative condition at the divisor $\{p_1\} \times \mathbb{C}^2$ given by $(2, 1^{n-2})$. The connected/disconnected correspondence here is

(3.1)
$$\langle (2,1^{n-2}) \rangle_{g,n}^{E \times \mathbb{C}^2, \bullet} = \sum_{2 \le m \le n} \langle (2,1^{m-2}) \rangle_{g,m}^{E \times \mathbb{C}^2, \circ} \cdot \mathsf{Part}(n-m) \, .$$

~

The disconnected invariants $\langle (2, 1^{n-2}) \rangle_{g,n}^{E \times \mathbb{C}^2, \bullet}$ can be calculated by degenerating E to a curve of arithmetic genus 1 with a unique node and applying the correspondence between local Gromov-Witten theory of \mathbb{P}^1 and quantum cohomology ring of $\text{Hilb}^n(\mathbb{C}^2)$ [3, 37]:

(3.2)
$$-\sum_{g\in\mathbb{Z}} u^{2g-3} \langle (2,1^{n-2}) \rangle_{g,n}^{E\times\mathbb{C}^2,\bullet} = (-i) \cdot \operatorname{trace}\left(\mathsf{M}_D^{\operatorname{Hilb}^n(\mathbb{C}^2)}(q)\right) = (-i) \cdot \operatorname{Tr}_n \cdot (t_1 + t_2),$$

after $-q = e^{iu}$.

(iv) For an integer $l \ge 1$, let Part(l) be the count of possibly disconnected unramified covers of E of degree l where each cover is weighted by the order of automorphism group *and* the number of connected components. For example, Part(1) = 1 and

$$Part(2) = 1 + 3/2 = 5/2$$

where $1 = (1/2) \cdot 1 \cdot 2$ is the contribution of the disconnected cover and $3/2 = (1/2) \cdot 3 \cdot 1$ is the contribution of the connected covers. The generating series is

$$\widetilde{\mathcal{P}}(x) = \sum_{l=1}^{\infty} \widetilde{\mathsf{Part}}(l) \, x^l \, .$$

Lemma 9. The series $\widetilde{\mathcal{P}}$ is determined by the equation $\widetilde{\mathcal{P}} = \mathcal{P} \log \mathcal{P}$.

Proof. Let Hur(l, k) be the automorphism-weighted count of possibly disconnected unramified covers of an elliptic curve of degree l with exactly k connected component, and let

$$\mathcal{F}(x,y) = 1 + \sum_{l=1}^{\infty} \sum_{k=1}^{\infty} \operatorname{Hur}(l,k) \, x^l y^k \, .$$

Since $\log(\mathcal{P}(x))$ is the generating series of connected unramified covers of an elliptic curve,

$$\mathcal{F}(x,y) = \exp(y \log(\mathcal{P}(x))).$$

We then obtain $\widetilde{\mathcal{P}}(x) = \partial_y \mathcal{F}(x, y)|_{y=1} = \mathcal{P}(x) \log \mathcal{P}(x)$ by the definition of $\operatorname{Hur}(l, k)$.

We can now state the main connected/disconnect equation which will play an essential role in the proof of Theorem 1. The proof will be presented in Section 3.4.

Proposition 10 (Connected/disconnected calculus). For $g \ge 2$ and $n \ge 2$, we have

$$\begin{split} \langle (2, 1^{n-2}) \rangle_{g,n}^{\pi_{\mathbb{C}^2}, \bullet} &= \frac{(t_1 + t_2)^2}{t_1 t_2} \sum_{2 \le m \le n} \langle \lambda_{g-2} \lambda_g | (2, 1^{m-2}) \rangle_{g,m}^{\pi, \circ} \cdot \mathsf{Part}(n-m) \\ &- \frac{1}{24} \frac{t_1 + t_2}{t_1 t_2} \sum_{2 \le m \le n} \langle (2, 1^{m-2}) \rangle_{g,m}^{E \times \mathbb{C}^2, \circ} \cdot \widetilde{\mathsf{Part}}(n-m) \,. \end{split}$$

The g = 1 case of the connected/disconnected calculus takes a special form. For $n \ge 2$,

(3.3)
$$\langle (2, 1^{n-2}) \rangle_{1,n}^{\pi_{\mathbb{C}^2}, \bullet} = -\frac{1}{24} \frac{t_1 + t_2}{t_1 t_2} \sum_{2 \le m \le n} \langle (2, 1^{m-2}) \rangle_{1,m}^{E \times \mathbb{C}^2, \circ} \cdot \widetilde{\mathsf{Part}}(n-m)$$

The genus g = 1 case will be discussed in Section 3.4. In fact, both sides of (3.3) vanish for g = 1.

3.3. **Proof of Theorem 1.** Let $\mathcal{B}(u, Q)$ denote the power series

$$\mathcal{B}(u,Q) = \sum_{g=1}^{\infty} \sum_{m=1}^{\infty} \frac{|B_{2g-2}|}{(2g-2)!} (\sigma_{2g-1}(m) - \sigma_1(m)) Q^m u^{2g-3}.$$

The u^{-1} term of $\mathcal{B}(u, y)$ corresponding to g = 1 vanishes. Recall the definition

$$\mathsf{Tr}_m = \frac{1}{t_1 + t_2} \mathsf{trace}(\mathsf{M}_D^{\mathsf{Hilb}^m(\mathbb{C}^2)})$$

of Section 0.4.1.

Lemma 11. Under the variable change $-q = e^{iu}$, we have

$$(-i)\sum_{m=1}^{\infty}\operatorname{Tr}_{m}(q)Q^{m}=\mathcal{P}(Q)\mathcal{B}(u,Q).$$

Proof. The diagonal terms of $M_D^{Hilb^m(\mathbb{C}^2)}$ have been described in [37, Section 2]:

$$\mathsf{M}_{D}^{\mathsf{Hilb}^{m}(\mathbb{C}^{2})} = \sum_{r=1}^{\infty} \left(\frac{r}{2} \frac{(-q)^{r} + 1}{(-q)^{r} - 1} - \frac{1}{2} \frac{(-q) + 1}{(-q) - 1} \right) \alpha_{-r} \alpha_{r} + \dots,$$

where α_{-r} and α_r are the standard creation and annihilation operators on the subspace $\mathcal{F}_m \subset \mathcal{F}$ of Fock space

$$\mathcal{F} = \sum_{m=0}^{\infty} \mathcal{F}_m$$
 .

Using the diagonal elements, we compute

$$(3.4) \qquad (-i)\sum_{m=1}^{\infty} \operatorname{Tr}_{m}(q) Q^{m} = \sum_{r\geq 1} \left(\frac{-ir}{2} \frac{(-q)^{r}+1}{(-q)^{r}-1} - \frac{-i}{2} \frac{(-q)+1}{(-q)-1} \right) \sum_{m=1}^{\infty} \operatorname{trace}(\alpha_{-r}\alpha_{r\mid\mathcal{F}_{m}}) \cdot Q^{m}.$$

Since $\alpha_{-r}\alpha_r(|\mu\rangle) = r \cdot |\{i \mid \mu_i = r\}| \cdot |\mu\rangle$,

(3.5)
$$\sum_{m=1}^{\infty} \operatorname{trace}(\alpha_{-r}\alpha_{r\mid\mathcal{F}_{m}}) \cdot Q^{m} = r \frac{Q^{r}}{1 - Q^{r}} \prod_{m=1}^{\infty} \frac{1}{1 - Q^{m}}$$

The expansion of the cotangent function yields

(3.6)
$$\frac{-ir}{2}\frac{(-q)^{r}+1}{(-q)^{r}-1} = \frac{-ir}{2}\frac{e^{iur/2}+e^{-iur/2}}{e^{iur/2}-e^{iur/2}}$$
$$= -\frac{r}{2}\cot\left(\frac{ur}{2}\right)$$
$$= -\frac{1}{u} + \sum_{h=0}^{\infty}\frac{|B_{2h}|r^{2h}}{(2h)!}u^{2h-1}.$$

After putting together (3.4), (3.5) and (3.6) and shifting h = g - 1, we obtain:

$$\begin{split} (-i)\sum_{m=1}^{\infty} \operatorname{Tr}_{m}(q) Q^{m} &= \mathcal{P}(Q) \left(\sum_{g=1}^{\infty} \sum_{r=1}^{\infty} \frac{|B_{2g-2}|}{(2g-2)!} (r^{2g-1} - r) \frac{Q^{r}}{1 - Q^{r}} u^{2g-3} \right) \\ &= \mathcal{P}(Q) \left(\sum_{g=1}^{\infty} \frac{|B_{2g-2}|}{(2g-2)!} u^{2g-3} \sum_{r=1}^{\infty} (r^{2g-1} - r) (Q^{r} + Q^{2r} + \ldots) \right) \\ &= \mathcal{P}(Q) \left(\sum_{g=1}^{\infty} \frac{|B_{2g-2}|}{(2g-2)!} u^{2g-3} \sum_{m=1}^{\infty} Q^{m} \sum_{k|m} (k^{2g-1} - k) \right) \\ &= \mathcal{P}(Q) \mathcal{B}(u, Q) \,. \end{split}$$

By Theorem 2 and Proposition 8, we obtain

(3.7)
$$\sum_{g=1}^{\infty} \sum_{n=1}^{\infty} u^{2g-3} Q^n \langle \lambda_{g-2} \lambda_g | (2, 1^{n-2}) \rangle_{g,n}^{\pi, \circ} = \frac{1}{24} \mathcal{B}(u, Q)$$

The g = 1 terms on the left are degenerate and interpreted as

$$\langle (2, 1^{n-2}) \rangle_{1,n}^{\pi, \circ} = 0,$$

so all u^{-1} vanish on both sides of (3.7).

By Lemma 11 and (3.2), the connected/disconnected equation for a fixed elliptic curve target (3.1) can be written as

$$\mathcal{P}(Q)\left(\sum_{g=1}^{\infty}\sum_{n=1}^{\infty}\langle (2,1^{n-2})\rangle_{g,n}^{E\times\mathbb{C}^{2},\circ}u^{2g-3}Q^{n}\right) = i(t_{1}+t_{2})\sum_{n=1}^{\infty}\operatorname{Tr}_{n}(q)Q^{n} = -(t_{1}+t_{2})\mathcal{P}(Q)\mathcal{B}(u,Q),$$

after $-q = e^{iu}$. The connected/disconnected calculus of Proposition 10 then yields

$$\begin{split} \sum_{g=1}^{\infty} \sum_{n=1}^{\infty} \langle (2, 1^{n-2}) \rangle_{g,n}^{\pi_{\mathbb{C}^2}, \bullet} Q^n u^{2g-3} &= -\frac{1}{24} \frac{t_1 + t_2}{t_1 t_2} (-(t_1 + t_2) \mathcal{B}(u, Q)) \cdot \widetilde{\mathcal{P}}(Q) + \frac{(t_1 + t_2)^2}{t_1 t_2} \frac{1}{24} \mathcal{B}(u, Q) \mathcal{P}(Q) \\ &= \frac{1}{24} \frac{(t_1 + t_2)^2}{t_1 t_2} \mathcal{B}(u, Q) \mathcal{P}(Q) (1 + \log \mathcal{P}(Q)) \\ &= \frac{1}{i} \frac{1}{24} \frac{(t_1 + t_2)^2}{t_1 t_2} \left(\sum_{n=1}^{\infty} \operatorname{Tr}_n(q) Q^n \right) \left(1 + \sum_{k=1}^{\infty} \sigma_{-1}(k) Q^k \right) . \end{split}$$

In the second equality, we have applied Lemma 9. In the third equality, we have used

$$\log \mathcal{P}(Q) = -\sum_{n=1}^{\infty} \log(1-Q^n) = \sum_{k=1}^{\infty} \sigma_{-1}(k)Q^k.$$

Since, by Theorem A,

$$\langle D \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)} = \frac{1}{i} \langle (2, 1^{n-2}) \rangle^{\pi_{\mathbb{C}^2}, \bullet}$$

 $\langle D \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)}$ is $\frac{-1}{24} \frac{(t_1+t_2)^2}{t_1t_2}$ times the Q^n coefficient of

$$\left(\sum_{n=1}^{\infty} \operatorname{Tr}_{n}(q) Q^{n}\right) \left(1 + \sum_{k=1}^{\infty} \sigma_{-1}(k) Q^{k}\right) \,.$$

We have proven Theorem 1.

3.4. **Proof of Proposition 10.** We study here the connected/disconnected calculus for the family $\pi_{\mathbb{C}^2}$. To start, we consider the universal elliptic curve

$$\pi: \mathcal{E} \to \mathcal{M}_{1,1}$$

with section p_1 . Let $(m, \mathbf{k}, \mathbf{g})$ be a triple,

- $2 \le m \le n$,
- $\mathbf{k} = (k_1, \dots, k_s)$ is a partition of n m,
- $\mathbf{g} = (g_0, \dots, g_s)$ is a partition of g + s 1,

where the parts of k and g are positive integers. We define

$$\overline{\mathcal{M}}_{m,\mathbf{k},\mathbf{g}}^{\mathsf{fiber}} = \overline{\mathcal{M}}_{g_0}^{\circ}(\pi, (2, 1^{m-2})) \times_{\overline{\mathcal{M}}_{1,1}} \overline{\mathcal{M}}_{g_1}^{\circ}(\pi, (1^{k_1})) \times_{\overline{\mathcal{M}}_{1,1}} \ldots \times_{\overline{\mathcal{M}}_{1,1}} \overline{\mathcal{M}}_{g_s}^{\circ}(\pi, (1^{k_s})), \\
\overline{\mathcal{M}}_{m,\mathbf{k},\mathbf{g}} = \overline{\mathcal{M}}_{g_0}^{\circ}(\pi, (2, 1^{m-2})) \times \overline{\mathcal{M}}_{g_1}^{\circ}(\pi, (1^{k_1})) \times \ldots \times \overline{\mathcal{M}}_{g_s}^{\circ}(\pi, (1^{k_s})),$$

where $\overline{\mathcal{M}}_{m,\mathbf{k},\mathbf{g}}^{\text{fiber}}$ is the base change of $\overline{\mathcal{M}}_{m,\mathbf{k},\mathbf{g}}$ along the diagonal $\Delta: \overline{\mathcal{M}}_{1,1} \to \overline{\mathcal{M}}_{1,1} \times \ldots \times \overline{\mathcal{M}}_{1,1}$.

The fixed locus of $\overline{\mathcal{M}}_{g}^{\bullet}(\pi_{\mathbb{C}^{2}}, n, (2, 1^{n-2}))$ for the action of the torus $(\mathbb{C}^{*})^{2}$ is the disjoint union of the stacks $\overline{\mathcal{M}}_{m,\mathbf{k},\mathbf{g}}^{\mathsf{fiber}}$, up to an automorphism factor which we denote by $\operatorname{Aut}(\mathbf{k}, \mathbf{g})$, and the restriction of the virtual class $[\overline{\mathcal{M}}_{g}^{\bullet}(\pi_{\mathbb{C}^{2}}, (2, 1^{n-2}))]^{vir}$ to $\overline{\mathcal{M}}_{m,\mathbf{k},\mathbf{g}}^{\mathsf{fiber}}$ agrees with $\Delta^{!}[\overline{\mathcal{M}}_{m,\mathbf{k},\mathbf{g}}]^{vir}$. By the localization formula of [18], we have

$$\overline{\mathcal{M}}_{g}^{\bullet}(\pi_{\mathbb{C}^{2}}, (2, 1^{n-2}))]^{vir} = \sum_{(m, \mathbf{k}, \mathbf{g})} \frac{1}{\operatorname{Aut}(\mathbf{k}, \mathbf{g})} \frac{\Delta^{!} [\mathcal{M}_{m, \mathbf{k}, \mathbf{g}}]^{vir}}{e_{(\mathbb{C}^{*})^{2}}(N^{vir})},$$

where the contribution of the virtual normal bundle on each factor of $\overline{\mathcal{M}}_{m,\mathbf{k},\mathbf{g}}$ is given by

(3.8)
$$\frac{1}{\mathsf{e}_{(\mathbb{C}^*)^2}(N^{vir})} = \frac{c(\mathbb{E}^{\vee} \otimes t_1)c(\mathbb{E}^{\vee} \otimes t_2)}{t_1 t_2} \\ = -\frac{t_1 + t_2}{t_1 t_2} \lambda_{g_i} \lambda_{g_i-1} + \frac{(t_1 + t_2)^2}{t_1 t_2} \lambda_{g_i} \lambda_{g_i-2} - (t_1 + t_2) \lambda_{g_i-1} \lambda_{g_i-2} + \dots$$

if $g_i > 1$, and $-\frac{t_1+t_2}{t_1t_2}\lambda_1 + 1$ if $g_i = 1$. We denote the class (3.8) by N_{g_i} , and note that the lower order terms in the Hodge classes will not contribute for dimensional reasons. After integration, we obtain

$$\langle (2, 1^{n-2}) \rangle_{g,n}^{\pi_{\mathbb{C}^{2}}, \bullet} = \sum_{(m, \mathbf{k}, \mathbf{g})} \frac{1}{\operatorname{Aut}(\mathbf{k}, \mathbf{g})} \int_{[\overline{\mathcal{M}}_{(m, \mathbf{k}, \mathbf{g})}^{\text{fiber}}]^{vir}} \prod_{i=0}^{s} N_{g_{i}}$$

$$= \sum_{(m, \mathbf{k}, \mathbf{g})} \frac{1}{\operatorname{Aut}(\mathbf{k}, \mathbf{g})} \sum_{j=0}^{s} \int_{[\overline{\mathcal{M}}_{(m, \mathbf{k}, \mathbf{g})}]^{vir}} \prod_{i=0}^{s} N_{g_{i}} \prod_{i \neq j} \operatorname{ev}_{i}^{*}([\mathsf{pt}])$$

$$= \sum_{(m, \mathbf{k}, \mathbf{g})} \frac{1}{\operatorname{Aut}(\mathbf{k}, \mathbf{g})} \left(\langle N_{g_{0}} | (2, 1^{m-2}) \rangle_{g_{0}, m}^{\pi, \circ} \prod_{i=1}^{s} \langle N_{g_{i}} | (1^{k_{i}}) \rangle_{g_{i}, k_{i}}^{E, \circ}$$

$$+ \langle N_{g_{0}} | (2, 1^{m-2}) \rangle_{g_{0}, m}^{E, \circ} \sum_{j=1}^{s} - \frac{t_{1} + t_{2}}{t_{1} t_{2}} \langle \lambda_{g_{j}} \lambda_{g_{j}-1} | (1^{k_{j}}) \rangle_{g_{j}, k_{j}}^{\pi, \circ} \prod_{i \neq j} \langle N_{g_{i}} | (1) \rangle_{g_{i}, k_{i}}^{E, \circ}$$

where $ev_i : \overline{\mathcal{M}}_{(m,\mathbf{k},\mathbf{g})} \to \overline{\mathcal{M}}_{1,1}$ recovers the isomorphism class of the target curve of the *i*-th map. The second equality follows from the decomposition of the diagonal of $\overline{\mathcal{M}}_{1,1}$:

$$[\Delta] = \sum_{j=0}^{s} [\mathsf{pt}] \boxtimes \ldots \boxtimes [\mathsf{pt}] \boxtimes 1 \boxtimes [\mathsf{pt}] \boxtimes \ldots \boxtimes [\mathsf{pt}] \in \mathsf{CH}^{s}(\overline{\mathcal{M}}_{1,1} \times \cdots \times \overline{\mathcal{M}}_{1,1})$$

If $g_i > 1$, the class N_{g_i} contains a λ_{g_i} factor. Hence, $\langle N_{g_i} | (1^{k_i}) \rangle_{g_i,k_i}^{E,\circ} = 0$ unless $g_i = 1$, in which case

$$\langle N_1 | (1^{k_i}) \rangle_{1,k_i}^{E,\circ} = \langle (1^{k_i}) \rangle_{1,k_i}^{E,\circ} = \frac{\sigma(k_i)}{k_i}$$

Lemma 12. If $g_i > 1$, we have $\langle \lambda_{g_i} \lambda_{g_i-1} | (1^{k_i}) \rangle_{g_i,k_i}^{\pi,\circ} = 0$. If $g_i = 1$,

$$\langle \lambda_1 \lambda_0 | (1^{k_i}) \rangle_{1,k_i}^{\pi,\circ} = \frac{1}{24} \langle (1^{k_i}) \rangle_{1,k_i}^{E,\circ} = \frac{\sigma_1(k_i)}{24k_i}$$

Proof. The first step of the argument is the equality

$$\langle \lambda_{g_i} \lambda_{g_i-1} | (1^{k_i}) \rangle_{g_i,k_i}^{\pi,\circ} = \langle \lambda_{g_i} \lambda_{g_i-1} \rangle_{g_i,k_i}^{\pi,\circ}$$

obtained from the standard degeneration to the normal cone of the section p_1 .

We now follow the geometric notation and analysis of Section 2.5. The moduli space of stable maps to the fibers of π lies over $\overline{\mathcal{M}}_{1,1}$,

$$\epsilon: \overline{\mathcal{M}}_{g_i}^{\circ}(\pi, k_i) \to \overline{\mathcal{M}}_{1,1}$$

The universal curve $\mu : \mathcal{C} \to \overline{\mathcal{M}}_{g_i}^{\circ}(\pi, k_i)$ carries a universal evaluation map



to the universally expanded target



The target \mathcal{T} is a family of elliptic curves over $\overline{\mathcal{M}}_{g_i}^{\circ}(\pi, k_i)$ with possible expansion over the nodal fibers. We have a pull-back map²²

(3.10)
$$\epsilon^* \mathbb{E}_1 \cong \nu_* \omega_\mu \xrightarrow{f^*} \mu_* \omega_\mu \cong \mathbb{E}_g$$

over $\overline{\mathcal{M}}_{q_i}(\pi, k_i)$. Since $k_i > 0$, the map (3.10) is injective, so we obtain an exact sequence

$$0 \to \epsilon^* \mathbb{E}_1 \to \mathbb{E}_g \to \mathbb{F} \to 0 \,,$$

where \mathbb{F} is a rank g-1 vector bundle on $\overline{\mathcal{M}}_{g_i}^{\circ}(\pi, k_i)$. We therefore have a factorization

(3.11)
$$\lambda_g = c_1(\epsilon^* \mathbb{E}_1) \cdot c_{g-1}(\mathbb{F}) = \frac{1}{24} \epsilon^*([E]) \cdot \lambda_{g-1}(\mathbb{F})$$

on $\overline{\mathcal{M}}_{g_i}(\pi, k_i)$, where $[E] \in \overline{\mathcal{M}}_{1,1}$ is the moduli point of a fixed nonsingular elliptic curve E.

The factorization (3.11) of λ_q implies

(3.12)
$$\int_{[\overline{\mathcal{M}}_{g_i}^{\circ}(\pi,k_i)]^{vir}} \lambda_{g_i} \lambda_{g_i-1} = \frac{1}{24} \int_{[\overline{\mathcal{M}}_{g_i}^{\circ}(E,k_i)]^{vir}} \lambda_{g_i-1}^2.$$

If $g_i > 1$, then $\lambda_{g_i-1}^2 = 2\lambda_{g_i}\lambda_{g_i-2}$. Hence, the integral (3.12) is 0 by λ_g -vanishing for elliptic targets. If $g_i = 1$, the integral is easily evaluated as claimed.

²² \mathbb{E}_1 is the Hodge bundle on $\overline{\mathcal{M}}_{1,1}$.

Therefore, the only positive contributions to the right hand side of (3.9) come from triplets $(e, \mathbf{k}, \mathbf{g})$ where $\mathbf{g} = (g, 1, \dots, 1)$.

• For g > 1, we have

$$\langle N_g | (2, 1^{m-2}) \rangle_{g,m}^{\pi, \circ} = \frac{(t_1 + t_2)^2}{t_1 t_2} \langle \lambda_g \lambda_{g-2} | (2, 1^{m-2}) \rangle_{g,m}^{\pi, \circ}$$

By a similar localization analysis, we have

$$\langle N_g | (2, 1^{m-2}) \rangle_{g,m}^{E, \circ} = -(t_1 + t_2) \langle \lambda_{g-1} \lambda_{g-2} | (2, 1^{m-2}) \rangle_{g,m}^{E, \circ} = \langle (2, 1^{m-2}) \rangle_{g,m}^{E \times \mathbb{C}^2}$$

Therefore, the formula in (3.9) simplifies to

$$\langle (2, 1^{n-2}) \rangle_{g,n}^{\pi_{\mathbb{C}^{2}}, \bullet} = \sum_{m=2}^{n} \sum_{\mathbf{k} \vdash n-m} \left(\frac{(t_{1} + t_{2})^{2}}{t_{1}t_{2}} \langle \lambda_{g} \lambda_{g-2} | (2, 1^{m-2}) \rangle_{g,m}^{\pi, \circ} \frac{\prod_{i} \langle (1^{k_{i}}) \rangle_{1,k_{i}}^{E, \circ}}{\operatorname{Aut}(\mathbf{k})} - \frac{1}{24} \frac{t_{1} + t_{2}}{t_{1}t_{2}} \langle (2, 1^{m-2}) \rangle_{g,m}^{E \times \mathbb{C}^{2}, \circ} \frac{l(\mathbf{k}) \prod_{i} \langle (1^{k_{i}}) \rangle_{1,k_{i}}^{E, \circ}}{\operatorname{Aut}(\mathbf{k})} \right).$$

Summing over all possible partitions, we obtain the coefficients Part(n-m) and $\widetilde{Part}(n-m)$ of Proposition 10 as defined in Section 3.2.

• For g = 1, $\langle N_1 | (2, 1^{m-2}) \rangle_{1,m}^{\pi,\circ} = \langle (2, 1^{m-2}) \rangle_{1,m}^{\pi,\circ} = 0$ by Proposition 8, so both sides of formula (3.3) after Proposition 10 vanish.

3.5. **Proof of Proposition 8.** Let $g \ge 2$. Let $Bl_{p_1 \times \{0\}}(\mathcal{E} \times \mathbb{A}^1)$ be the degeneration to the normal cone of the section p_1 . The special fiber over $0 \in \mathbb{A}^1$ is the family

$$\pi \cup \pi_P : \mathcal{E} \cup \mathbb{P}(T_{\mathbf{p}_1} \mathcal{E} \oplus \mathbb{C}) \longrightarrow \overline{\mathcal{M}}_{1,1},$$

where the gluing identifies the section p_1 of \mathcal{E} with the 0-section of $\mathbb{P}(T_{p_1}\mathcal{E} \oplus \mathbb{C})$. The universal section p_1 is now the ∞ -section of π_P . We will use the degeneration formula

(3.13)
$$\langle \tau_1(\mathsf{p}_1)\lambda_g\lambda_{g-2}\rangle_{g,n}^{\pi,\circ} = \sum_{(\Gamma_1,\Gamma_2,\mu)} \langle \Lambda_1|\mu\rangle_{\Gamma_1}^{\pi,\bullet} \langle \Lambda_2\tau_1(\infty)|\mu^*\rangle_{\Gamma_2}^{\pi_P,\bullet} ,$$

with notation:

- Γ_1 and Γ_2 are possibly disconnected topological types corresponding to the domains mapping to the components \mathcal{E} and $\mathbb{P}(T_p \mathcal{E} \oplus \mathbb{C})$,
- μ is a partition of d decorated with elements of a basis of $H^*(p) = H^*(\overline{\mathcal{M}}_{1,1})$,
- Λ_1 and Λ_2 reflect the distribution of the Hodge insertions.

The Hodge insertion $\lambda_g \lambda_{g-2} = 2\lambda_{g-1}^2$ annihilates all contributions of graphs which are not of compact type or have more than two vertices with positive genus. If there are exactly two vertices of genera $0 < g_1, g_2 < g$, then the Hodge insertion distributes as

(3.14)
$$\lambda_g \lambda_{g-2} |_{\overline{\mathcal{M}}_{g_1} \times \overline{\mathcal{M}}_{g_2}} = \lambda_{g_1} \lambda_{g_1-1} \boxtimes \lambda_{g_2} \lambda_{g_2-1}.$$

An analysis of the virtual dimension of each vertex shows that all vertex contributions vanish for dimension reasons except for the following list:

(3.15)
$$\langle \lambda_g \lambda_{g-2} | (2[\overline{\mathcal{M}}_{1,1}], 1[\overline{\mathcal{M}}_{1,1}], \dots, 1[\overline{\mathcal{M}}_{1,1}]) \rangle_g^{\pi, \epsilon}$$

(3.16)
$$\langle \lambda_g \lambda_{g-2} | (1[\mathsf{pt}], 1[\overline{\mathcal{M}}_{1,1}], \dots, 1[\overline{\mathcal{M}}_{1,1}]) \rangle_g^{\pi,\circ}$$

(3.17)
$$\langle \lambda_{g_1} \lambda_{g_1-1} | (1[\overline{\mathcal{M}}_{1,1}], \dots, 1[\overline{\mathcal{M}}_{1,1}]) \rangle_{g_1}^{\pi,\circ}$$

 $(3.18) \qquad \langle (1[\mathsf{pt}]) \rangle_0^{\pi_P,\circ}$

(3.19)
$$\langle \tau_1(\infty) | (1[\overline{\mathcal{M}}_{1,1}]) \rangle_0^{\pi_P,\circ}$$

(3.20)
$$\langle \tau_1(\infty) | (2[\mathsf{pt}]) \rangle_0^{\pi_P, c}$$

(3.21)
$$\langle \tau_1(\infty) | (1[\mathsf{pt}], 1[\mathsf{pt}]) \rangle_0^{\pi_P, \circ}$$

(3.22)
$$\langle \tau_1(\infty)\lambda_{g_2}\lambda_{g_2-1}|(1[\mathsf{pt}])\rangle_{g_2}^{\pi_P,\circ}$$

Three further vanishings hold for other reasons:

• Vertex (3.16) vanishes because

$$[\overline{\mathcal{M}}_g(\pi/\mathsf{p}_1)]^{vir} \cap \operatorname{ev}^*_{\overline{\mathcal{M}}_{1,1}}(\mathsf{pt}) = [\overline{\mathcal{M}}_g(E/p_1)]^{vir}$$

pairs to zero with λ_g .

- Vertex (3.21) vanishes because of the two point conditions over $\overline{\mathcal{M}}_{1,1}$.
- Vertex (3.17) vanishes unless $g_1 = 1$ by Lemma 12.

It follows that the only possible combinatorial types that contribute to the right hand side of (3.13) are given by the two configurations of Figure 1, where the first configuration is counted n times.



FIGURE 1. Non-zero contributions to (3.13)

The vertices (3.18) and (3.20) contribution factors 1 and 1/2 respectively (the latter is cancelled by the multiplicity which occurs in the degeneration formula).

The above vertex analysis show that (3.13) specializes to:

$$\langle \tau_1(\mathbf{p}_1)\lambda_g\lambda_{g-2}\rangle_{g,n}^{\pi,\circ} = \frac{\sigma_1(n)}{24}\langle \tau_1(\infty)\lambda_{g-1}\lambda_{g-2}|(1[\mathsf{pt}])\rangle_{g-1,1}^{\pi_P} + \langle \lambda_g\lambda_{g-2}|(2,1^{n-2})\rangle_{g,n}^{\pi,\circ}.$$

To complete the proof of Proposition 8, we must evaluate

$$\langle \tau_1(\infty)\lambda_{g-1}\lambda_{g-2}|(1[\mathsf{pt}])\rangle_{g-1,1}^{\pi_P} = \langle \tau_1(\infty)\lambda_{g-1}\lambda_{g-2}\rangle_{g-1,1}^{\mathbb{P}^1},$$

where the equality with the absolute invariant is proven by the standard degeneration method. To calculate, we localize with respect to the \mathbb{C}^* - action on \mathbb{P}^1 . Since the integrand has a $\lambda_g \lambda_{g-1}$ insertion, there are only two components of the fixed locus that contribute, corresponding to the graphs in Figure 2, see [18].



FIGURE 2. Localization contributions, where half edges correspond to marked points, vertices are contracted components, and edges are non-contracted components.

After expanding the localization formula, we conclude

$$\langle \tau_1(\infty)\lambda_{g-1}\lambda_{g-2}\rangle_{g-1,1}^{\mathbb{P}^1} = (2g-2)\int_{\overline{\mathcal{M}}_{g-1,1}} \frac{\lambda_{g-1}\lambda_{g-2}c(\mathbb{E}^{\vee})}{1-\psi} = \frac{|B_{2g-2}|}{(2g-2)!}$$

where the Hodge integral on the right is calculated in [41].

4. RECONSTRUCTION OF MULTI-POINT INVARIANTS

4.1. **Reduction to** 1-**point series.** We recall the notation introduced in Section 0.1:

(4.1)
$$\left\langle D^{*k_1}, ..., D^{*k_\ell} \right\rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)} = \sum_{d=0}^{\infty} \left\langle D^{*k_1}, ..., D^{*k_\ell} \right\rangle_{1,d}^{\mathsf{Hilb}^n(\mathbb{C}^2)} q^d.$$

In order to emphasize the number of insertions, we also write

$$\left\langle D^{*k_1},...,D^{*k_\ell}\right\rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)} = \left\langle D^{*k_1},...,D^{*k_\ell}\right\rangle_{1,\left\lceil \ell \right\rceil}^{\mathsf{Hilb}^n(\mathbb{C}^2)}$$

where the boxed subscript indicates insertion number (not the curve degree as in (4.1)).

We prove here the reduction of the genus 1 Gromov-Witten theory of $Hilb^n(\mathbb{C}^2)$ to 1-point series as stated in Theorem 4:

To every genus 1 series
$$\langle D^{\star k_1}, ..., D^{\star k_\ell} \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)}$$
, there are canonically associated functions
 $\{C_{k,m}\}_{0 \le k \le |\mathsf{Part}(n)|-1, 0 \le m \le \ell-1} \subset \mathbb{Q}(t_1, t_2)(q)$

for which the following equation holds:

$$\left\langle D^{*k_1}, \dots, D^{*k_\ell} \right\rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)} = \sum_{k=0}^{|\mathsf{Part}(n)|-1} \sum_{m=0}^{\ell-1} C_{k,m} \cdot \left(q\frac{d}{dq}\right)^m \left\langle D^{*k} \right\rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)}.$$

4.2. **Proof of Theorem 4.** We adapt the reconstruction strategy of [50, Section 6] to prove the result. Let $m \ge 0$, and fix classes

$$v_1, v_2, v_3, v_4, \phi_1, ..., \phi_m \in H^*_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2))$$
.

Pulling back Getzler's relation from $\overline{\mathcal{M}}_{1,4}$ to $\overline{\mathcal{M}}_{1,4+m}(\mathsf{Hilb}^n(\mathbb{C}^2), d)$ and summing over the curve degree d, we obtain the following equation (see [26, Equation (27)])

$$3\sum_{h\in S_4} \langle v_{h(1)} \star v_{h(2)}, v_{h(3)} \star v_{h(4)}, \phi_1, ..., \phi_m \rangle_{1, \boxed{2+m}}^{\mathsf{Hilb}^n(\mathbb{C}^2)} \sim 4\sum_{h\in S_4} \langle v_{h(1)} \star v_{h(2)} \star v_{h(3)}, v_{h(4)}, \phi_1, ..., \phi_m \rangle_{1, \boxed{2+m}}^{\mathsf{Hilb}^n(\mathbb{C}^2)},$$

where the relation $X \sim Y$ means that the difference X - Y is an explicit sum of arbitrary genus 0 invariants and genus 1 invariants with at most m + 1 insertions. Getzler's relation can be equivalently cast in the following form:

$$\begin{split} \Psi(v_1, v_2, v_3, v_4) &= \langle v_1 \star v_2, v_3 \star v_4, \phi_1, ..., \phi_m \rangle_{1, \boxed{2 + m}}^{\mathsf{Hilb}^n(\mathbb{C}^2)} + \langle v_1 \star v_3, v_2 \star v_4, \phi_1, ..., \phi_m \rangle_{1, \boxed{2 + m}}^{\mathsf{Hilb}^n(\mathbb{C}^2)} \\ &+ \langle v_1 \star v_4, v_2 \star v_4, \phi_1, ..., \phi_m \rangle_{1, \boxed{2 + m}}^{\mathsf{Hilb}^n(\mathbb{C}^2)} \\ &- \langle v_1, v_2 \star v_3 \star v_4, \phi_1, ..., \phi_m \rangle_{1, \boxed{2 + m}}^{\mathsf{Hilb}^n(\mathbb{C}^2)} - \langle v_2, v_1 \star v_3 \star v_4, \phi_1, ..., \phi_m \rangle_{1, \boxed{2 + m}}^{\mathsf{Hilb}^n(\mathbb{C}^2)} \\ &- \langle v_3, v_1 \star v_2 \star v_4, \phi_1, ..., \phi_m \rangle_{1, \boxed{2 + m}}^{\mathsf{Hilb}^n(\mathbb{C}^2)} - \langle v_4, v_1 \star v_2 \star v_3, \phi_1, ..., \phi_m \rangle_{1, \boxed{2 + m}}^{\mathsf{Hilb}^n(\mathbb{C}^2)} \end{split}$$

satisfies the relation

(4.2)
$$\Psi(v_1, v_2, v_3, v_4) \sim 0$$

Let $a, b \in H^{\leq 2}_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2))$, and let $\phi_1, ..., \phi_m \in H^*_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2))$. For integers $l \geq 2$ and $0 \leq i \leq l$ define the following series:

$$\begin{split} h(l) &= \langle a \star b \star D^{*(l-2)}, D^{*2}, \phi_1, ..., \phi_m \rangle_{1, \boxed{2+m}}^{\mathsf{Hilb}^n(\mathbb{C}^2)}, \\ f(i) &= \langle a \star D^{*i}, b \star D^{*(l-i)}, \phi_1, ..., \phi_m \rangle_{1, \boxed{2+m}}^{\mathsf{Hilb}^n(\mathbb{C}^2)}. \end{split}$$

From the string and divisor equations, we immediately obtain

$$f(0) \sim 0, \quad f(l) \sim 0$$

By expanding the definitions, we see, for $0 \le k \le l-2$,

(4.3)
$$-\Psi(a \star D^{*k}, b \star D^{*(l-2-k)}, D, D) = f(k+2) - 2f(k+1) + f(k) - h(l) + 2\langle D, a \star b \star D^{*(l-1)}, \phi_1, ..., \phi_m \rangle_{1, \boxed{2+m}}^{\mathsf{Hilb}^n(\mathbb{C}^2)}.$$

By applying (4.2) and the divisor equation, we obtain a *difference* equation,

$$f(k+2) - 2f(k+1) + f(k) - h(l) \sim 0$$

from (4.3). A linear algebraic result [50, Lemma 6.2] about solutions to the difference equation then yields

(4.4)
$$f(i) \sim -\frac{i(l-i)}{2}h(l)$$

for $0 \le i \le l$.

We can write the relation (4.4) in the following form. Let $l \ge 2$, and let i + j = l for $i, j \ge 0$. Then,

(4.5)
$$f(i) \sim -\frac{ij}{2}h(l).$$

The case j = 2 of (4.5) with b = 1 is

$$\langle a \star D^{*i}, D^{*2}, \phi_1, ..., \phi_m \rangle_{1, \boxed{2+\mathbf{m}}}^{\mathsf{Hilb}^n(\mathbb{C}^2)} \sim -i \langle a \star D^{*i}, D^{*2}, \phi_1, ..., \phi_m \rangle_{1, \boxed{2+\mathbf{m}}}^{\mathsf{Hilb}^n(\mathbb{C}^2)}$$

The case j = 1 of (4.5) with b = D is

$$\langle a \star D^{*i}, D \star D^{*1}, \phi_1, ..., \phi_m \rangle_{1, \boxed{2+\mathbf{m}}}^{\mathsf{Hilb}^n(\mathbb{C}^2)} \sim -\frac{i}{2} \langle a \star D \star D^{*(i-1)}, D^{*2}, \phi_1, ..., \phi_m \rangle_{1, \boxed{2+\mathbf{m}}}^{\mathsf{Hilb}^n(\mathbb{C}^2)}$$

These two relations imply

(4.6)
$$\langle D^{*i}, D^{*2}, \phi_1, ..., \phi_m \rangle_{1, [2+m]}^{\mathsf{Hilb}^n(\mathbb{C}^2)} \sim 0$$

for all $i \ge 1$. The i = 0 case, $\langle D^{*0}, D^{*2}, \phi_1, ..., \phi_m \rangle_{1, \boxed{2+m}}^{\mathsf{Hilb}^n(\mathbb{C}^2)} \sim 0$ follows from the string equation.

As a consequence of (4.6), we have $h(l) \sim 0$ for all $l \geq 2$. Then, relation (4.5) yields

$$\langle a \star D^{*i}, b \star D^{*j}, \phi_1, ..., \phi_m \rangle_{1, \boxed{2 + \mathbf{m}}}^{\mathsf{Hib}^n(\mathbb{C}^2)} \sim 0$$

for all $i, j \ge 0$ with $i + j \ge 2$. Again the i + j = 0 and i + j = 1 cases follow from the string equation. After setting a = b = 1, we conclude

(4.7)
$$\langle D^{*i}, D^{*j}, \phi_1, ..., \phi_m \rangle_{1, [2+m]}^{\mathsf{Hib}^n(\mathbb{C}^2)} \sim 0$$

for all $i, j \ge 0$.

Repeated use of relation (4.7) shows that $\langle D^{*i}, D^{*j}, \phi_1, ..., \phi_m \rangle_{1, [2+m]}^{\mathsf{Hilb}^n(\mathbb{C}^2)}$ can be expressed in terms of genus 0 invariants and 1-*point* genus 1-functions. Moreover, since we use the divisor equation in the argument, the resulting expressions are linear in 1-point genus 1 functions *and* their $q \frac{d}{dq}$ derivatives. By the genus 0 reconstruction result in [37], the genus 0 series can be written as rational functions in matrix coefficients of $\mathsf{M}_D^{\mathsf{Hilb}^n(\mathbb{C}^2)}(q)$. The proof of Theorem 4 is complete.

4.3. Givental's formula in genus 1. Consider the (full) genus 1 Gromov-Witten potential

$$\mathcal{F}_{1}^{\mathsf{Hilb}^{n}(\mathbb{C}^{2})}(\mathsf{t},q) = \sum_{d \ge 0} \sum_{\ell \ge 0} \frac{q^{d}}{\ell!} \langle \mathsf{t},...,\mathsf{t} \rangle_{1,\lfloor \ell \rfloor,d}^{\mathsf{Hilb}^{n}(\mathbb{C}^{2})}, \quad \mathsf{t} \in H^{*}_{\mathsf{T}}(\mathsf{Hilb}^{n}(\mathbb{C}^{2})).$$

We have, for $v_1, ..., v_k \in H^*_{\mathsf{T}}(\mathsf{Hilb}^n(\mathbb{C}^2))$,

$$\langle v_1, ..., v_k \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)} = \nabla_{v_1} \nabla_{v_2} ... \nabla_{v_k} \mathcal{F}_1^{\mathsf{Hilb}^n(\mathbb{C}^2)}(\mathsf{t}, q) \big|_{\mathsf{t}=0}$$

Since the quantum cohomology of $Hilb^n(\mathbb{C}^2)$ is semisimple, we can apply Givental's formula [14]:

$$\nabla_{v} \mathcal{F}_{1}^{\mathsf{Hilb}^{n}(\mathbb{C}^{2})}(\mathbf{t},q) = \frac{1}{2} \sum_{i=1}^{|\mathsf{Part}(n)|} \mathsf{R}_{ii}^{1} \cdot \nabla_{v} u_{i} + \frac{1}{48} \nabla_{v} \log \left(\prod_{i=1}^{|\mathsf{Part}(n)|} \Delta_{i}\right)$$

We follow here the notation of [14] and refer the reader to [14, 24] for an exposition:

• R_{ii}^1 are matrix coefficients of the first term of the classifying R-matrix,

$$\mathsf{R} = \mathsf{Id} + \mathsf{R}^1 \cdot z + \mathsf{R}^2 \cdot z^2 + ... \,,$$

- $u_1, \ldots, u_{|\mathsf{Part}(n)|}$ are the canonical coordinates,
- $\Delta_1, \ldots, \Delta_{|\mathsf{Part}(n)|}$ are the inverses of the squares of the lengths of the corresponding idempotents $\epsilon_1, \ldots, \epsilon_{|\mathsf{Part}(n)|}$.

By Theorem 4, we need only consider the 1-point invariants $\langle D^{\star k} \rangle_1^{\text{Hilb}^n(\mathbb{C}^2)}$. Givental's formula then can be written as:

(4.8)
$$\langle D^{\star k} \rangle_{1}^{\mathsf{Hilb}^{n}(\mathbb{C}^{2})} = \frac{1}{2} \sum_{i=1}^{|\mathsf{Part}(n)|} \mathsf{R}_{ii}^{1} \big|_{\mathsf{t}=0} \cdot \nabla_{D^{\star k}} u_{i} \big|_{\mathsf{t}=0} + \frac{1}{48} \nabla_{D^{\star k}} \log \left(\prod_{i=1}^{|\mathsf{Part}(n)|} \Delta_{i} \right) \big|_{\mathsf{t}=0}$$

We explain first how to explicitly compute the functions Δ_i . Denote by $\{e_1, ..., e_{|\mathsf{Part}(n)|}\}$ the distinct eigenvalues [37] of $D\star_t$ and let $\{v_1, ..., v_{\mathsf{Part}(n)}\}$ be the corresponding eigenvectors.²³ We have

$$v_i \star_{\mathsf{t}} v_j = \delta_{ij} c_i v_i$$

so the idempotents are $\epsilon_i = v_i/c_i$. By the Frobenius property,

$$c_i \langle v_i, 1 \rangle = \langle v_i \star_{\mathsf{t}} v_i, 1 \rangle = \langle v_i, v_i \rangle.$$

We then compute:

$$\Delta_i = \frac{1}{\langle \epsilon_i, \epsilon_i \rangle} = \frac{c_i^2}{\langle v_i, v_i \rangle} = \frac{1}{\langle v_i, v_i \rangle} \cdot \frac{\langle v_i, v_i \rangle^2}{\langle v_i, 1 \rangle^2} = \frac{\langle v_i, v_i \rangle}{\langle v_i, 1 \rangle^2}.$$

Since the eigenvectors v_i are found by solving the equation $D \star_t v_i = e_i v_i$, the components of v_i are rational functions of e_i with coefficients in the matrix coefficients of $D\star_t$. Therefore, the Δ_i are rational functions in the eigenvalues e_i with coefficients in the field $\mathbb{Q}(t_1, t_2)(q)[[t]]$, where the matrix coefficients of $D\star_t$ lie.

To evaluate the term $\nabla_{D^{\star k}} \log(\prod_i \Delta_i)$, we need only evaluate symmetric rational functions in the derivatives of the eigenvalues e_i with coefficients in $\mathbb{Q}(t_1, t_2)(q)[[t]]$. By Proposition 14 of Appendix A, these expressions lie in the field of rational functions of derivatives of the symmetric functions of e_i with coefficients in $\mathbb{Q}(t_1, t_2)(q)[[t]]$. The outcome is an explicit calculation of

$$\nabla_{D^{\star k}} \log(\prod_i \Delta_i) \Big|_{\mathbf{t}=0} \in \mathbb{Q}(t_1, t_2)(q).$$

The same argument can be used to calculate derivatives

$$\nabla_{v_1} \nabla_{v_2} \cdots \nabla_{v_k} \log(\prod_i \Delta_i) \Big|_{\mathbf{t}=0} \in \mathbb{Q}(t_1, t_2)(q).$$

Next, we consider the term $\nabla_{D^{\star k}} u_i |_{t=0}$ of (4.8). Since the eigenvalues of $D^{\star tk} \star_t$ are simply the k^{th} powers of the eigenvalues of $D \star_t$, we have

$$\nabla_{D^{\star k}} u_i \big|_{\mathsf{t}=0} = \left(\nabla_D u_i \big|_{\mathsf{t}=0} \right)^k = e_i^k \big|_{\mathsf{t}=0}$$

Hence, $\nabla_{D^{\star k}} u_i |_{t=0}$ is also determined by the genus 0 theory of $\mathsf{Hilb}^n(\mathbb{C}^2)$.

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²³Here, the symbol \star_t denotes the big quantum product.

The difficulty in applying formula (4.8) to calculate $\langle D^{\star k} \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)}$ lies in controlling the R-matrix terms. We will use our calculation of $\langle D \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)}$ to determine $\mathsf{R}^1_{ii}|_{t=0}$ up to the nondegeneracy of the Wronskian.

4.4. Proof of Theorem 5. By the divisor equation,

$$\langle D, ..., D \rangle_{1, [\ell]}^{\mathsf{Hilb}^{n}(\mathbb{C}^{2})} = \left(q \frac{d}{dq} \right)^{\ell-1} \langle D \rangle_{1}^{\mathsf{Hilb}^{n}(\mathbb{C}^{2})}$$

We define

$$\delta_l = \langle D, .., D \rangle_{1, [\ell]} - \left(q \frac{d}{dq} \right)^{l-1} \frac{1}{48} \nabla_D \log \left(\prod_{i=1}^{|\mathsf{Part}(\mathsf{n})|} \Delta_i \right) \Big|_{\mathsf{t}=0}.$$

By Theorem 1 and the discussion in Section 4.3, $\delta_l \in \mathbb{Q}(t_1, t_2)(q)$ can be explicitly calculated.

By (4.8), we have, for $\ell \geq 1$,

(4.9)
$$2\delta_{\ell} = \left(q\frac{d}{dq}\right)^{\ell-1} \left(\sum_{i=1}^{|\mathsf{Part}(n)|} \mathsf{R}_{ii}^{1} \big|_{\mathsf{t}=0} \cdot \nabla_{D} u_{i} \big|_{t=0}\right)$$
$$= \sum_{i=1}^{|\mathsf{Part}(n)|} \sum_{k=0}^{\ell-1} \binom{\ell-1}{k} \left(q\frac{d}{dq}\right)^{k} \mathsf{R}_{ii}^{1} \big|_{\mathsf{t}=0} \cdot \left(q\frac{d}{dq}\right)^{\ell-1-k} \nabla_{D} u_{i} \big|_{\mathsf{t}=0}.$$

By the construction of the R-matrix [14], the derivatives $q \frac{d}{dq} R_{ii}^1 \Big|_{t=0}$ are given by

$$q\frac{d}{dq}\mathsf{R}_{ii}^{1}\big|_{\mathsf{t}=0} = \sum_{l}\mathsf{R}_{il}^{1}\big|_{\mathsf{t}=0} \cdot \left(\nabla_{D}u_{l}\big|_{\mathsf{t}=0} - \nabla_{D}u_{i}\big|_{\mathsf{t}=0}\right) \cdot \mathsf{R}_{li}^{1}\big|_{\mathsf{t}=0},$$

where the off-diagonal terms $\mathsf{R}^{1}_{il}\Big|_{t=0}$ are computed by the equation

$$\Psi^{-1} \cdot q \frac{d}{dq} \Psi = \left[\nabla_D U, \mathsf{R}^1 \right],$$

where $\nabla_D U$ is the diagonal matrix with diagonal entries $\nabla_D u_i |_{t=0}$ and Ψ is the matrix whose columns are normalized eigenvectors. The functions $q \frac{d}{dq} R_{ii}^1 |_{t=0}$ are therefore rational functions of the $q \frac{d}{dq}$ derivatives of the eigenvalues $e_i|_{t=0}$ with coefficients in the field $\mathbb{Q}(t_1, t_2)(q)$, where the matrix coefficients of $\mathsf{M}_D^{\mathsf{Hilb}^n(\mathbb{C}^2)}(q)$ lie. The higher order derivatives

$$\left(q\frac{d}{dq}\right)^{k>1}\mathsf{R}^{1}_{ii}\big|_{\mathsf{t}=0}$$

are rational functions of the $q\frac{d}{dq}$ derivatives of the eigenvalues $e_i|_{t=0}$ with coefficients in $\mathbb{Q}(t_1, t_2)(q)$.

We would like to calculate the diagonal terms $\mathsf{R}^1_{ii}\Big|_{t=0}$. Define the column vector

$$2\vec{\delta} = (2\delta_1, 2\delta_2, \dots, 2\delta_{|\mathsf{Part}(n)|})^T$$

of length $|\mathsf{Part}(n)|$ and the column vector

$$\begin{split} \vec{r} &= \left(\mathsf{R}_{11}^{1} \left|_{\mathsf{t}=0}, ..., \mathsf{R}_{|\mathsf{Part}(n)||\mathsf{Part}(n)|}^{1} \left|_{\mathsf{t}=0}, q \frac{d}{dq} \mathsf{R}_{11}^{1} \right|_{\mathsf{t}=0}, ..., q \frac{d}{dq} \mathsf{R}_{|\mathsf{Part}(n)||\mathsf{Part}(n)|}^{1} \left|_{\mathsf{t}=0}, ..., \left(q \frac{d}{dq}\right)^{|\mathsf{Part}(n)|-1} \mathsf{R}_{11}^{1} \right|_{\mathsf{t}=0}, ..., \left(q \frac{d}{dq}\right)^{|\mathsf{Part}(n)|-1} \mathsf{R}_{|\mathsf{Part}(n)||\mathsf{Part}(n)|}^{1} \left|_{\mathsf{t}=0}\right)^{T} \end{split}$$

of length $|Part(n)|^2$. Equation (4.9) can be written as a matrix equation

Here, A is a $|\mathsf{Part}(n)| \times |\mathsf{Part}(n)|^2$ matrix of the shape

$$\mathsf{A} = (\mathsf{W} \,|\, \mathsf{W})$$

where W is the $|\mathsf{Part}(n)| \times |\mathsf{Part}(n)|$ matrix given by

$$\mathsf{W} = \begin{pmatrix} \nabla_D u_1 \big|_{\mathsf{t}=0} & \nabla_D u_2 \big|_{\mathsf{t}=0} & \dots & \nabla_D u_{|\mathsf{Part}(n)|} \big|_{\mathsf{t}=0} \\ q \frac{d}{dq} \nabla_D u_1 \big|_{\mathsf{t}=0} & q \frac{d}{dq} \nabla_D u_2 \big|_{\mathsf{t}=0} & \dots & q \frac{d}{dq} \nabla_D u_{|\mathsf{Part}(n)|} \big|_{\mathsf{t}=0} \\ \vdots & \vdots & \vdots & \vdots \\ (q \frac{d}{dq})^{|\mathsf{Part}(n)|-1} \nabla_D u_1 \big|_{\mathsf{t}=0} & (q \frac{d}{dq})^{|\mathsf{Part}(n)|-1} \nabla_D u_2 \big|_{\mathsf{t}=0} & \dots & (q \frac{d}{dq})^{|\mathsf{Part}(n)|-1} \nabla_D u_{|\mathsf{Part}(n)|} \big|_{\mathsf{t}=0} \end{pmatrix}$$

and \widehat{W} is an explicit $|\mathsf{Part}(n)| \times (|\mathsf{Part}(n)|^2 - |\mathsf{Part}(n)|)$ matrix with entries given by

$$\binom{\ell-1}{k} \left(q \frac{d}{dq} \right)^{\ell-1-k} \nabla_D u_i \Big|_{\mathbf{t}=0}$$

for k > 0. Since $\nabla_D u_i \Big|_{t=0} = e_i \Big|_{t=0}$, the matrix coefficients of A are $q \frac{d}{dq}$ derivatives of the eigenvalues $e_i \Big|_{t=0}$.

The matrix W is the *Wronskian* of the eigenvalues $e_i |_{t=0}$ as functions of $\log(q)$. Since the eigenvalues are analytic functions of $\log(q)$, a result of Boecher (see [49, Lemma 1.12]) implies that the eigenvalues are linearly independent over $\mathbb{Q}(t_1, t_2)$ if and only if det(W) is not identically 0.

If det(W) is nonzero, then W is invertible. Equation (4.10) then implies

(4.11)
$$W^{-1} 2\vec{\delta} = W^{-1} A \vec{r}.$$

The $|\mathsf{Part}(n)| \times |\mathsf{Part}(n)|^2$ matrix $\mathsf{W}^{-1}\mathsf{A}$ is of the form

$$\mathsf{W}^{-1}\mathsf{A} = \left(\mathsf{I}_{|\mathsf{Part}(n)| \times |\mathsf{Part}(n)|} \,|\, \mathsf{W}^{-1}\mathsf{W}\right).$$

Via the identity matrix $I_{|Part(n)| \times |Part(n)|}$, equation (4.11) yields equations for

$$\mathsf{R}_{11}^{\mathsf{I}} \Big|_{\mathsf{t}=0}, \, \dots, \, \mathsf{R}_{|\mathsf{Part}(n)||\mathsf{Part}(n)|}^{\mathsf{I}} \Big|_{\mathsf{t}=0}$$

in terms of the functions $\delta_1, ..., \delta_{|\mathsf{Part}(n)|}$, the $q\frac{d}{dq}$ derivatives of the eigenvalues $e_i|_{t=0}$, and the higher $q\frac{d}{dq}$ derivatives of the functions $\mathsf{R}^1_{ii}|_{t=0}$.

We have proven that $\mathsf{R}_{11}^1 \Big|_{\mathsf{t}=0}, ..., \mathsf{R}_{|\mathsf{Part}(n)||\mathsf{Part}(n)|}^1 \Big|_{\mathsf{t}=0}$ can be computed explicitly from

$$\left\{\left.\left\langle D,...,D\right\rangle_{1,\left[\ell\right]}^{\mathsf{Hilb}^{n}(\mathbb{C}^{2})}=\left(q\frac{d}{dq}\right)^{\ell-1}\left\langle D\right\rangle_{1}^{\mathsf{Hilb}^{n}(\mathbb{C}^{2})}\right.\right\}_{k=1}^{|\mathsf{Part}(n)|}$$

and rational functions of the $q \frac{d}{dq}$ derivatives of the eigenvalues $e_i |_{t=0}$ with coefficients in $\mathbb{Q}(t_1, t_2)(q)$. \Box

4.5. Proof of Theorem 6. By construction, the final expressions for

$$\mathsf{R}_{11}^{1} \Big|_{\mathsf{t}=0}, \, \dots, \, \mathsf{R}_{|\mathsf{Part}(n)||\mathsf{Part}(n)|}^{1} \Big|_{\mathsf{t}=0}$$

in terms of the $q \frac{d}{dq}$ derivatives of the eigenvalues

$$e_1 \Big|_{t=0}, \ldots, e_{\mathsf{Part}(n)} \Big|_{t=0}$$

with coefficients in $\mathbb{Q}(t_1, t_2)(q)$ are equivariant under permutations of the indices. Therefore, after substitution in

$$\langle D^{\star k} \rangle_{1}^{\mathsf{Hilb}^{n}(\mathbb{C}^{2})} = \frac{1}{2} \sum_{i=1}^{|\mathsf{Part}(n)|} \mathsf{R}_{ii}^{1} \big|_{\mathsf{t}=0} \cdot \nabla_{D^{\star k}} u_{i} \big|_{\mathsf{t}=0} + \frac{1}{48} \nabla_{D^{\star k}} \log \left(\prod_{i=1}^{|\mathsf{Part}(n)|} \Delta_{i} \right) \big|_{\mathsf{t}=0}$$

the series $\langle D^{\star k} \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)}$ are symmetric rational functions of the $q \frac{d}{dq}$ derivatives of the eigenvalues $e_i \big|_{t=0}$ with coefficients in $\mathbb{Q}(t_1, t_2)(q)$. After an application of Proposition 14 of Appendix A, the series $\langle D^{\star k} \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)}$ can be effectively reconstructed from $\langle D \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)}$ and $\mathsf{M}_D^{\mathsf{Hilb}^n(\mathbb{C}^2)}(q)$.

4.6. The Wronskian. We formulate the following nondegeneracy conjecture for the quantum cohomology of $\text{Hilb}^n(\mathbb{C}^2)$.

Conjecture 13. For all $n \ge 1$, the Wronskian matrix W associated to $\text{Hilb}^n(\mathbb{C}^2)$ is nondegenerate:

$$\det(\mathsf{W}) \neq 0.$$

We have verified Conjecture 13 for $n \le 7$ by computer calculations. As discussed in Section 4.4, Conjecture 13 can reformulated as the assertion that the eigenvalues

$$e_1 \Big|_{t=0}, \ldots, e_{\mathsf{Part}(n)} \Big|_{t=0}$$

are *linearly independent* over $\mathbb{Q}(t_1, t_2)$.

5. CALCULATIONS

• For all n, the series $\langle (1^n) \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)}$ has only a constant term in q. The calculation was already discussed in Section 1:

(5.1)
$$\langle (1^n) \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)} = \mathsf{Coeff}_{Q^n} \left[-\frac{1}{24} \frac{t_1 + t_2}{t_1 t_2} \cdot \mathcal{P}(Q) \log \mathcal{P}(Q) \right].$$

• For all n, the series $\langle (2, 1^{n-2}) \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)} = - \langle D \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)}$ is evaluated by Theorem 1,

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(5.2)
$$\langle (2, 1^{n-2}) \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)} = \frac{1}{24} \frac{(t_1 + t_2)^2}{t_1 t_2} \left(\mathsf{Tr}_n + \sum_{k=2}^{n-1} \sigma_{-1}(n-k) \mathsf{Tr}_k \right) \,.$$

For $2 \le n \le 5$ and every partition μ of n, we present here closed formulas for the 1-point series

$$\langle \mu \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)} \in \mathbb{Q}(t_1, t_2)(q)$$

We use a combination of inputs: the full genus 0 theory, the evaluations (5.1) and (5.2), and Getzler's equation. Once the 1-point series are known, Theorem 4 effectively determines the full genus 1 Gromov-Witten theory of $\text{Hilb}^n(\mathbb{C}^2)$.

While Givental's formula was used in Section 4.4 to prove a structural reconstruction result, calculations are more efficiently obtained from the known series by Getzler's equation.

• For n = 2, we have:

$$\begin{split} \langle (1,1) \rangle_1^{\mathsf{Hilb}^2(\mathbb{C}^2)} &= -\frac{1}{24} \cdot \frac{t_1 + t_2}{t_1 t_2} \cdot \frac{5}{2} \,, \\ \langle (2) \rangle_1^{\mathsf{Hilb}^2(\mathbb{C}^2)} &= \frac{1}{24} \frac{(t_1 + t_2)^2}{t_1 t_2} \cdot \frac{q+1}{q-1} \,. \end{split}$$

• For n = 3, we have:

$$\begin{split} \langle (1,1,1) \rangle_1^{\mathsf{Hilb}^3(\mathbb{C}^2)} &= -\frac{1}{24} \cdot \frac{t_1 + t_2}{t_1 t_2} \cdot \frac{29}{6} \,, \\ \langle (2,1) \rangle_1^{\mathsf{Hilb}^3(\mathbb{C}^2)} &= \frac{1}{24} \cdot \frac{(t_1 + t_2)^2}{t_1 t_2} \cdot \frac{5q^3 - 3q^2 - 3q + 5}{(q - 1)(q^2 - q + 1)} \,, \\ \langle (3) \rangle_1^{\mathsf{Hilb}^3(\mathbb{C}^2)} &= \frac{-1}{12} \frac{(t_1 + t_2)}{t_1 t_2} \frac{(t_1^2 + \frac{1}{3} t_1 t_2 + t_2^2)(q^4 + 1) - \frac{1}{2}(t_1^2 - \frac{17}{3} t_1 t_2 + t_2^2)(q^3 + q) - (3t_1^2 + 13t_1 t_2 + 3t_2^2)q^2}{(q^2 - q + 1)^2} \,. \end{split}$$

• For n = 4, we have:

$$\begin{split} \langle (1,1,1,1)\rangle_1^{\mathsf{Hilb}^4(\mathbb{C}^2)} &= -\frac{1}{24} \cdot \frac{t_1 + t_2}{t_1 t_2} \cdot \frac{109}{12} \,, \\ \langle (2,1,1)\rangle_1^{\mathsf{Hilb}^4(\mathbb{C}^2)} &= \frac{1}{24} \cdot \frac{(t_1 + t_2)^2}{t_1 t_2} \cdot \frac{35q^5 - 28q^4 + 23q^3 + 23q^2 - 28q + 35}{2(q-1)(q^2+1)(q^2-q+1)} \,, \end{split}$$

$$\begin{split} \langle (2,2) \rangle_1^{\mathsf{Hilb}^4(\mathbb{C}^2)} &= \frac{-(t_1+t_2)}{t_1t_2} \cdot \frac{3}{16(q^2+1)^2(q-1)^2} \cdot \\ & \left((t_1^2 + \frac{1}{18}t_1t_2 + t_2^2)(q^6+1) + (\frac{2}{9}t_1^2 + 5t_1t_2 + \frac{2}{9}t_2^2)(q^5+q) \right. \\ & \left. + (\frac{11}{9}t_1^2 - \frac{77}{18}t_1t_2 + \frac{11}{9}t_2^2)(q^4+q^2) + (4t_1^2 + \frac{170}{9}t_1t_2 + 4t_2^2)q^3 \right) \,, \end{split}$$

• For n = 5, we have:

$$\langle 1 \rangle_1^{\mathsf{Hilb}^5(\mathbb{C}^2)} = -\frac{1}{24} \cdot \frac{t_1 + t_2}{t_1 t_2} \cdot \frac{907}{60}$$

We will write remaining series in terms of the traces²⁴ of quantum multiplication,

 $\mathsf{Tr}_m^{\mu} = \mathsf{trace}\left(\,\mu \star : QH^*_\mathsf{T}(\mathsf{Hilb}^m(\mathbb{C}^2)) \to QH^*_\mathsf{T}(\mathsf{Hilb}^m(\mathbb{C}^2))\,\right)\,.$

The trace which appears in Theorem 1 can be written as

$$\mathsf{Tr}_m = -\frac{1}{t_1 + t_2}\mathsf{Tr}_m^{(2,1^{m-2})}$$

Then, we have:

$$\begin{split} \langle (2,1,1,1) \rangle_1^{\mathsf{Hilb}^5(\mathbb{C}^2)} &= -\frac{t_1 + t_2}{18t_1t_2} \mathsf{Tr}_2^{(2)} - \frac{t_1 + t_2}{16t_1t_2} \mathsf{Tr}_3^{(2,1)} - \frac{t_1 + t_2}{24t_1t_2} \mathsf{Tr}_4^{(2,1,1)} - \frac{t_1 + t_2}{24t_1t_2} \mathsf{Tr}_5^{(2,1,1)} \\ &= \frac{1}{24} \cdot \frac{(t_1 + t_2)^2}{t_1t_2} \cdot \left(\frac{272q^9 - 539q^8 + 760q^7 - 629q^6 + 302q^5 + 302q^4 - 629q^3 + 760q^2 - 539q + 272}{6(q - 1)(q^2 + 1)(q^2 - q + 1)(q^4 - q^3 + q^2 - q + 1)} \right), \end{split}$$

$$\begin{split} \langle (2,2,1) \rangle_{1}^{\text{Hilb}^{5}(\mathbb{C}^{2})} &= \frac{775t_{1}^{2} + 733t_{1}t_{2} + 775t_{2}^{2}}{1200(t_{1} + t_{2})} + \frac{1}{200(t_{1} + t_{2})} \operatorname{Tr}_{3}^{(3)} + \frac{-10t_{1}^{2} - 13t_{1}t_{2} - 10t_{2}^{2}}{240t_{1}t_{2}(t_{1} + t_{2})} \operatorname{Tr}_{4}^{(2,2)} \\ &+ \frac{-3}{200(t_{1} + t_{2})} \operatorname{Tr}_{4}^{(3,1)} + \frac{-25t_{1}^{2} - 68t_{1}t_{2} - 25t_{2}^{2}}{600t_{1}t_{2}(t_{1} + t_{2})} \operatorname{Tr}_{5}^{(2,2,1)} + \frac{1}{200(t_{1} + t_{2})} \operatorname{Tr}_{5}^{(3,1,1)} \\ &+ \frac{50t_{1}^{2} - 139t_{1}t_{2} + 50t_{2}^{2}}{2400t_{1}t_{2}(t_{1} + t_{2})} \operatorname{Tr}_{2}^{(2)} \cdot \operatorname{Tr}_{2}^{(2)} + \frac{-25t_{1}^{2} - 44t_{1}t_{2} - 25t_{2}^{2}}{600t_{1}t_{2}(t_{1} + t_{2})} \operatorname{Tr}_{3}^{(2)} \cdot \operatorname{Tr}_{3}^{(2,1)} + \frac{1}{100(t_{1} + t_{2})} \operatorname{Tr}_{2}^{(2)} \cdot \operatorname{Tr}_{5}^{(2,1,1,1)}, \end{split}$$

$$\begin{split} \langle (3,1,1) \rangle_{1}^{\mathsf{Hilb}^{5}(\mathbb{C}^{2})} &= \frac{175t_{1}^{2} + 77t_{1}t_{2} + 175t_{2}^{2}}{300(t_{1} + t_{2})} + \frac{-225t_{1}^{2} - 532t_{1}t_{2} - 225t_{2}^{2}}{3600t_{1}t_{2}(t_{1} + t_{2})} \mathsf{Tr}_{3}^{(3)} + \frac{1}{120(t_{1} + t_{2})} \mathsf{Tr}_{4}^{(2,2)} + \frac{-25t_{1}^{2} - 59t_{1}t_{2} - 25t_{2}^{2}}{600t_{1}t_{2}(t_{1} + t_{2})} \mathsf{Tr}_{3}^{(3)} \\ &+ \frac{-3}{100(t_{1} + t_{2})} \mathsf{Tr}_{5}^{(2,2,1)} + \frac{-25t_{1}^{2} - 47t_{1}t_{2} - 25t_{2}^{2}}{600t_{1}t_{2}(t_{1} + t_{2})} \mathsf{Tr}_{5}^{(3,1,1)} + \frac{-2}{75(t_{1} + t_{2})} \mathsf{Tr}_{2}^{(2)} \cdot \mathsf{Tr}_{2}^{(2)} \\ &+ \frac{1}{100(t_{1} + t_{2})} \mathsf{Tr}_{2}^{(2)} \cdot \mathsf{Tr}_{3}^{(2,1)} + \frac{1}{100(t_{1} + t_{2})} \mathsf{Tr}_{2}^{(2)} \cdot \mathsf{Tr}_{5}^{(2,1,1,1)} \,, \end{split}$$

²⁴The subscript m of Tr_m^μ is redundant since $m = |\mu|$, but is included for clarity.

$$\begin{split} \langle (3,2) \rangle_{1}^{\text{Hib}^{5}(\mathbb{C}^{2})} &= \frac{865t_{1}^{2} + 1556t_{1}t_{2} + 865t_{2}^{2}}{1800(t_{1} + t_{2})} \operatorname{Tr}_{2}^{(2)} + \frac{-158t_{1}^{2} - 91t_{1}t_{2} - 158t_{2}^{2}}{900(t_{1} + t_{2})} \operatorname{Tr}_{3}^{(2,1)} + \frac{20t_{1}^{2} + 79t_{1}t_{2} + 20t_{2}^{2}}{450(t_{1} + t_{2})} \operatorname{Tr}_{4}^{(2,1,1)} + \frac{-1}{300(t_{1} + t_{2})} \operatorname{Tr}_{4}^{(2)} \\ &+ \frac{35t_{1}^{2} - 77t_{1}t_{2} + 35t_{2}^{2}}{900(t_{1} + t_{2})} \operatorname{Tr}_{5}^{(2,1,1,1)} + \frac{-25t_{1}^{2} - 77t_{1}t_{2} - 25t_{2}^{2}}{600t_{1}t_{2}(t_{1} + t_{2})} \operatorname{Tr}_{5}^{(3,2)} + \frac{1}{300(t_{1} + t_{2})} \operatorname{Tr}_{5}^{(4,1)} + \frac{-25t_{1}^{2} - 39t_{1}t_{2} - 25t_{2}^{2}}{600t_{1}t_{2}(t_{1} + t_{2})} \operatorname{Tr}_{5}^{(2)} \cdot \operatorname{Tr}_{4}^{(3,2)} \\ &+ \frac{-23}{1800(t_{1} + t_{2})} \operatorname{Tr}_{2}^{(2)} \cdot \operatorname{Tr}_{4}^{(3,1)} + \frac{4}{225(t_{1} + t_{2})} \operatorname{Tr}_{2}^{(2)} \cdot \operatorname{Tr}_{4}^{(2,2)} + \frac{-4}{225(t_{1} + t_{2})} \operatorname{Tr}_{2}^{(2)} \cdot \operatorname{Tr}_{5}^{(2,2,1)} + \frac{23}{1800(t_{1} + t_{2})} \operatorname{Tr}_{2}^{(2)} \cdot \operatorname{Tr}_{5}^{(3,1,1)} , \\ &+ \frac{-23}{1800(t_{1} + t_{2})} \operatorname{Tr}_{2}^{(2)} \cdot \operatorname{Tr}_{4}^{(3,1)} + \frac{4}{225(t_{1} + t_{2})} \operatorname{Tr}_{2}^{(2)} \cdot \operatorname{Tr}_{4}^{(2,2)} + \frac{-4}{225(t_{1} + t_{2})} \operatorname{Tr}_{2}^{(2)} \cdot \operatorname{Tr}_{5}^{(2,2,1)} + \frac{23}{1800(t_{1} + t_{2})} \operatorname{Tr}_{2}^{(2)} \cdot \operatorname{Tr}_{5}^{(3,1,1)} , \\ &+ \frac{-23}{1800(t_{1} + t_{2})} \operatorname{Tr}_{2}^{(2)} \cdot \operatorname{Tr}_{4}^{(3,1)} + \frac{4}{225(t_{1} + t_{2})} \operatorname{Tr}_{2}^{(2)} \cdot \operatorname{Tr}_{4}^{(2,2)} + \frac{-4}{225(t_{1} + t_{2})} \operatorname{Tr}_{2}^{(2)} \cdot \operatorname{Tr}_{5}^{(2,2,1)} + \frac{23}{1800(t_{1} + t_{2})} \operatorname{Tr}_{2}^{(2)} \cdot \operatorname{Tr}_{5}^{(3,1,1)} , \\ &+ \frac{-100t_{1}^{2} - 223t_{1}t_{2} - 400t_{2}^{2}}{2400(t_{1} + t_{2})} \operatorname{Tr}_{2}^{(2)} + \frac{-461t_{1}^{2} + 503t_{1}t_{2} - 461t_{2}^{2}}{4800(t_{1} + t_{2})} \operatorname{Tr}_{5}^{(2,1)} + \frac{-100t_{1}^{2} - 223t_{1}t_{2} - 100t_{2}^{2}}{2400(t_{1} + t_{2})} \operatorname{Tr}_{5}^{(2)} + \frac{-1}{600(t_{1} + t_{2})} \operatorname{Tr}_{5}^{(2)} \operatorname{Tr}_{5}^{(3,1,1)} + \frac{-1}{600(t_{1} + t_{2})} \operatorname{Tr}_{5}^{(2)} \operatorname{Tr}_{5}^{(3,1)} + \frac{-1}{600(t_{1} + t_{2})} \operatorname{Tr}_{5}^{(2)} \cdot \operatorname{Tr}_{5}^{(3,1,1)} + \frac{-1}{600(t_{1} + t_{2})} \operatorname{Tr}_{5}^{(2,1,1)} + \frac{-1}{600(t_{1} + t_{2})} \operatorname{Tr}_{5}^{(2)} \cdot \operatorname{T$$

$$\begin{split} &(5)\rangle_{1}^{\mathsf{Hib}^{5}(\mathbb{C}^{2})} = \frac{t_{1}t_{2}(110t_{1}^{2} - 1391t_{1}t_{2} + 110t_{2}^{2})}{600(t_{1} + t_{2})} + \frac{-15t_{1}^{2} - 53t_{1}t_{2} - 15t_{2}^{2}}{300(t_{1} + t_{2})} \operatorname{Tr}_{3}^{(3)} + \frac{-50t_{1}^{2} + 11t_{1}t_{2} - 50t_{2}^{2}}{600(t_{1} + t_{2})} \operatorname{Tr}_{4}^{(2,2)} \\ &+ \frac{-25t_{1}^{2} - 59t_{1}t_{2} - 25t_{2}^{2}}{300(t_{1} + t_{2})} \operatorname{Tr}_{5}^{(2,2,1)} + \frac{75t_{1}^{2} + 53t_{1}t_{2} + 75t_{2}^{2}}{600(t_{1} + t_{2})} \operatorname{Tr}_{5}^{(3,1,1)} + \frac{-5t_{1}^{2} - 16t_{1}t_{2} - 5t_{2}^{2}}{120t_{1}t_{2}(t_{1} + t_{2})} \operatorname{Tr}_{5}^{(5)} \\ &+ \frac{175t_{1}^{2} + 94t_{1}t_{2} + 175t_{2}^{2}}{150(t_{1} + t_{2})} \operatorname{Tr}_{2}^{(2)} \cdot \operatorname{Tr}_{2}^{(2)} + \frac{44t_{1}^{2} + 45t_{1}t_{2} + 44t_{2}^{2}}{120(t_{1} + t_{2})} \operatorname{Tr}_{3}^{(2)} \cdot \operatorname{Tr}_{3}^{(2,1)} + \frac{-60t_{1}^{2} - 53t_{1}t_{2} - 60t_{2}^{2}}{200(t_{1} + t_{2})} \operatorname{Tr}_{2}^{(2)} \cdot \operatorname{Tr}_{4}^{(2,1,1)} \\ &+ \frac{15t_{1}^{2} + 23t_{1}t_{2} + 15t_{2}^{2}}{300(t_{1} + t_{2})} \operatorname{Tr}_{2}^{(2)} \cdot \operatorname{Tr}_{5}^{(2,1,1,1)} . \end{split}$$

• There is no obstruction (apart from expected nondegeneracies) to extending the above tables to higher n. Whether further structures can be found in the 1-points series $\langle \mu \rangle_1^{\mathsf{Hilb}^n(\mathbb{C}^2)}$ is an interesting open question.

APPENDIX A. ON SYMMETRIC FUNCTIONS

Let \mathbb{K} be a field of characteristic 0. Let $\mathbf{z} = (z_1, \dots, z_m)$ be vector of m variables. Let $f_1(\mathbf{z}), \dots, f_n(\mathbf{z})$ be n abstract functions, and let

$$s_k(\mathbf{z}) = (-1)^k \sum_{I \subseteq [n], |I| = k} \prod_{i \in I} f_i$$

be the elementary symmetric polynomials in $f_1, ..., f_n$. The *discriminant* is defined by

$$\Delta(\mathbf{z}) = \prod_{i \neq j} (f_i - f_j) \,.$$

We consider the following algebras:

• The standard algebra of symmetric functions,

$$\mathsf{Sym} = \mathbb{K}[f_1, \dots, f_n]^{S_n} = \mathbb{K}[s_1, \dots, s_n],$$

is defined by taking invariants of the S_n -action on $\mathbb{K}[f_1, \ldots, f_n]$. The S_n -action is defined by permuting the indices of f_i . By construction, $\Delta \in Sym$.

• Let
$$\mathsf{Df} = \left\{ \frac{\partial^{a_1 + \dots + a_m}}{\partial z_1^{a_1} \cdots \partial z_m^{a_m}} f_i \right\}_{(a_1, \dots, a_m) \in (\mathbb{Z}_{\geq 0})^m}$$
 be the set of all partial derivatives of the functions f_1, \dots, f_n ,

$$\mathsf{Df} = \left\{ f_1, \dots, f_n, \dots, \frac{\partial f_i}{\partial z_j}, \dots, \frac{\partial^2 f_i}{\partial z_{j_1} \partial z_{j_2}}, \dots \right\}$$

The algebra $\mathbb{K}[Df]$ of polynomials²⁵ in the functions Df carries an S_n -action defined by permuting²⁶ the indices of f_i . Let SymDf be the algebra of S_n -invariants:

$$\mathsf{SymDf} = \mathbb{K}[\mathsf{Df}]^{S_n}$$
 .

• Let $\mathsf{Ds} = \left\{ \frac{\partial^{a_1 + \dots + a_m}}{\partial z_1^{a_1} \cdots \partial z_m^{a_m}} s_i \right\}_{(a_1, \dots, a_m) \in (\mathbb{Z}_{\geq 0})^m}$ be the set of all partial derivatives of the elementary symmetric functions s_1, \dots, s_n . Let $\mathbb{K}[\mathsf{Ds}]$ be the algebra of polynomials in the functions Ds .

We present a proof of the following result (which is likely known to experts, but we were unable to find a reference).

Proposition 14. SymDf $\subseteq \mathbb{K}[\mathsf{Ds}][1/\Delta]$.

Proof. Let $\mathbf{a} = (a_1, \ldots, b_m)$ and $\mathbf{b} = (b_1, \ldots, b_m)$ both be elements of $(\mathbb{Z}_{\geq 0})^m$. We define

- (i) $\mathbf{a} \leq \mathbf{b}$ if $a_j \leq b_j$ for all $1 \leq j \leq m$,
- (ii) $\mathbf{a} < \mathbf{b}$ if $\mathbf{a} \le \mathbf{b}$ and $a_j < b_j$ for some j.

Let $\mathsf{Ds}_{\leq \mathsf{b}} = \left\{ \frac{\partial^{a_1 + \dots + a_m}}{\partial z_1^{a_1} \dots \partial z_m^{a_m}} s_i \right\}_{\mathsf{a} \leq \mathsf{b}}$ be a finite set of partial derivatives of the elementary symmetric functions s_1, \dots, s_n . Similarly, let $\mathsf{DY}_{<\mathsf{b}} = \left\{ \frac{\partial^{a_1 + \dots + a_m}}{\partial z_1^{a_1} \dots \partial z_m^{a_m}} Y \right\}_{\mathsf{a} < \mathsf{b}}$ be the finite set of partial derivatives of a single abstract function $Y(\mathbf{z})$.

Consider the polynomial

$$P(x) = x^n + s_1 x^{n-1} + \ldots + s_n = \prod_i (x - f_i).$$

We can take the $\frac{\partial}{\partial z_i}$ derivative of the relation $P(f_i) = 0$:

(A.1)
$$\frac{\partial f_i}{\partial z_j} P_x(f_i) + \frac{\partial s_1}{\partial z_j} f_i^{n-1} + \ldots + \frac{\partial s_{n-1}}{\partial z_j} f_i + \frac{\partial s_n}{\partial z_j} = 0,$$

where P_x is the derivative of P as a polynomial in x. By a simple calculation,

$$\Delta = \prod_{i=1}^{n} P_x(f_i) \, .$$

Let $\mathbf{b} \in (\mathbb{Z}_{\geq 0})^m$. By repeatedly taking derivatives of (A.1), we find that there is a universal polynomial $\Phi_{\mathbf{b}}(\mathsf{Ds}_{\leq \mathbf{b}}, \mathsf{DY}_{< \mathbf{b}}) \in \mathbb{K}[\mathsf{Ds}_{\leq \mathbf{b}}, \mathsf{DY}_{< \mathbf{b}}]$

²⁵While Df is an infinite set of functions, only finitely many appear in any given polynomial.

 $^{{}^{26}}S_n$ does *not* act on the variables z_j nor on the operators $\frac{\partial}{\partial z_i}$.

which satisfies the following property: for every f_i ,

(A.2)
$$\frac{\partial^{b_1+\ldots+b_m}f_i}{\partial z_1^{b_1}\cdots\partial z_m^{b_m}}P_x(f_i) = \Phi_{\mathbf{b}}|_{Y=f_i}$$

Therefore, there is a universal polynomial $\Omega_{\mathbf{b}}(\mathsf{Ds}_{\leq \mathbf{b}}, Y) \in \mathbb{K}[\mathsf{Ds}_{\leq \mathbf{b}}, Y]$ for which

$$\frac{\partial^{b_1+\ldots+b_m}f_i}{\partial z_1^{b_1}\cdots\partial z_m^{b_m}}P_x(f_i)^{N_{\mathbf{b}}} = \Omega_{\mathbf{b}}(\mathsf{Ds}_{\leq \mathbf{b}}, f_i)$$

for a (possibly large) integer $N_{\mathbf{b}}$.

We now take an arbitrary monomial in the functions of Df:

$$M = \prod_{i=1}^{n} \prod_{u=1}^{v_i} \partial_{\mathbf{b}(i,u)} f_i \,,$$

where we have used the notation

$$\partial_{\mathbf{b}(i,u)} f_i = \frac{\partial^{b_1(i,u)+\ldots+b_m(i,u)} f_i}{\partial z_1^{b_1(i,u)} \cdots \partial z_m^{b_m(i,u)}}.$$

By (A.2), we have

(A.3)
$$\Delta^{\sum_{i=1}^{n} \sum_{u=1}^{v_i} N_{\mathbf{b}(i,u)}} \cdot M = \prod_{i=1}^{n} \prod_{u=1}^{v_i} \Omega_{\mathbf{b}(i,u)} (\mathsf{Ds}_{\leq \mathbf{b}(i,u)}, f_i) \cdot \Delta_i^{N_{\mathbf{b}(i,u)}},$$

where $\Delta_i = \Delta/P_x(f_i)$.

Consider next the S_n -invariant element

$$\mathrm{sym}M = \sum_{\sigma \in S_n} \sigma(M) \in \mathrm{Sym}\mathsf{Df}$$

Using (A.3), we obtain

(A.4)
$$\mathsf{sym}M = \Delta^{-\sum_{i=1}^{n}\sum_{u=1}^{v_i}N_{\mathbf{b}(i,u)}} \sum_{\sigma \in S_n} \sigma \left(\prod_{i=1}^{n}\prod_{u=1}^{v_i}\Omega_{\mathbf{b}(i,u)}(\mathsf{Ds}_{\leq \mathbf{b}(i,u)}, f_i) \cdot \Delta_i^{N_{\mathbf{b}(i,u)}}\right)$$

where the right side of (A.4) lies in $\mathbb{K}[Ds][1/\Delta]$. As M varies over all monomials in $\mathbb{K}[Df]$, the elements symM generate SymDf.

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DEPARTMENT OF MATHEMATICS, ETH ZÜRICH, RÄMISTRASSE 101, 8092 ZÜRICH, SWITZERLAND

Email address: aitor.iribarlopez@math.ethz.ch

DEPARTMENT OF MATHEMATICS, ETH ZÜRICH, RÄMISTRASSE 101, 8092 ZÜRICH, SWITZERLAND

Email address: rahul@math.ethz.ch

DEPARTMENT OF MATHEMATICS, OHIO STATE UNIVERSITY, 100 MATH TOWER, 231 WEST 18TH AVE., COLUM-BUS, OH 43210, USA

Email address: hhtseng@math.ohio-state.edu