Talk Diableretes

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2 Vertex algebra of Joyce and the conformal element

To explain the Lie bracket that appeared in Miguel's talk, I will describe how it is derived from the Vertex algebra constructed by Joyce(17'). For this, we will begin with the moduli stack of complexes on X

$$\mathcal{M}_X$$

and consider its shifted homology for each component labeled by a class α:

$$\hat{H}_*(\mathcal{M}_\alpha) = H_{*-\chi(\alpha,\alpha)}\big(\mathcal{M}_X, \mathbb{C}\big)\,,$$

where χ is the Euler pairing*. The vertex algebra on

$$V_* = \bigoplus_\alpha \hat{H}_*(\mathcal{M}_\alpha) \tag{2.1}$$

consists mainly of two important ingredients:

- the translation operator
$$T : V_* \longrightarrow V_{*+2}$$
- and the state-to-field correspondence
$$Y(-,z) : V_* \longrightarrow \mathrm{End}(V_*)[\![z, z^{-1}]\!]$$

satisfying additional compatibilities. One of them being the condition of locality

$$(z-w)^N\Big[Y(v,z), Y(u,z)\Big] = 0$$

for any two $u, v \in V$. These are constructed out of their respective ingredients

*We will always work over $\mathbb{C}$ for a reason that will become apparent later.

- the action of $\rho : [*/\mathbb{G}_m] \times \mathcal{M}_X \to \mathcal{M}_X$ which in S-families of complexes on X is given by rescalling by a line bundle L on S pulled back from S:

$$\big(\mathcal{E} \to X \times S\big) \mapsto \big(\mathcal{E} \otimes \pi_S^*(L) \to X \times S\big)\,.$$

- The morphism induced by mapping sheaves to their direct sums

$$\sigma : \mathcal{M}_X \times \mathcal{M}_X \longrightarrow \mathcal{M}_X$$

and the complex

$$\mathcal{E}\mathrm{xt} \longrightarrow \mathcal{M}_X \times \mathcal{M}_X$$

given by $\mathrm{Ext}(E, F)$ at each point $([E], [F]) \in \mathcal{M}_X \times \mathcal{M}_X$ together with the translation operator T itself.

The construction of both of these are then given by

1. letting t be the dual of $c_1(\mathcal{L})$ for the universal line bundle on $[*/\mathbb{G}_m]$ and setting

$$T(u) = \phi_*(t \boxtimes u)\,.$$

2. For $u \in \hat{H}_d(\mathcal{M}_\alpha)$ and $v \in \hat{H}_e(\mathcal{M}_\beta)$, we construct

$$Y(u,z)v = (-1)^{\chi(\alpha,\beta)} z^{\chi(\alpha,\beta)+\chi(\beta,}\sigma_*\Big(e^{zT} \otimes \mathrm{id}\big(u \boxtimes v \cap c_{z^{-1}}(\mathcal{E}\mathrm{xt}^\vee + \sigma^*\mathcal{E}\mathrm{xt}\big)\Big)\,, \quad (2.2)$$

It was the observation of Joyce(17') that these satisfy the necessary axioms of a vertex algebra.

Example 2.1. We will mostly focus here on the case of a curve C. Cheating a little bit this will also include acyclic quivers Q and complexes of representations, because both are homological dimension 1. The construction above still makes sense in this case. The homology (2.1) has now an explicit description in terms of $K^0 := K^0(C)$ (resp. $K^0(Q)$). Fixing a basis B of $K^0 \otimes \mathbb{C}$ constructed as

$$B = \sqcup_{p,q} B^{p,q}\,,$$

where $B^{p,q}$ is a basis of $H^{p,q}(C, \mathbb{C})$, we define the classes

$$\mu_{v,i} = \mathrm{ch}_i\big(\mathbb{G}/v^\vee\big)\,.$$

where $\mathbb{F} \to X \times \mathcal{M}_X$ is the universal sheaf.

To compare with Miguel's notation one sees by a simple computation that

$$\mathrm{ch}_i(w) = \sum_{v \in B} \int_X w \cdot v\mu_{v,i-[p,q]}\,,$$

where $[p,q] := \left\lfloor \frac{p-q}{2} \right\rfloor$. We may write

$$H^*(\mathcal{M}_C) = \mathbb{C}[K^0(C)] \otimes \mathrm{SSym}^\bullet_{\mathbb{C}}[\![\mu_{v,k}, v \in B, k > 0]\!]\,,$$
$$H_*(\mathcal{M}_C) = \mathbb{C}[K^0(C)] \otimes \mathrm{SSym}^\bullet_{\mathbb{C}}[\![u_{v,k}, v \in B, k > 0]\!]\,,$$

where $\mu_{v,k} = \frac{1}{(k-1)!}\frac{\partial}{\partial u_{v,k}}$. The latter has a natural vertex algebra structure called the *lattice vertex algebra* associated to the pairing

$$\chi_{\mathrm{ssym}} : K^0 \times K^0 \longrightarrow \mathbb{Z}\,, \qquad \chi_{\mathrm{ssym}}(v,w) = \chi(v,w) + (-1)^{\deg(v)}\chi(w,v)\,.$$

and by Gross(19')/Joyce(17') it is the one described by (2.2).

Miguel constructed in his talk a map $\xi^{\mathbb{G}} : \mathbb{D}^C \to H^*(\mathcal{M}_C)$ using the universal sheaf $\mathbb{G}$ on $C \times \mathcal{M}_C$ such that after imposing the relations

$$\mathrm{ch}_i(\gamma) = \begin{cases} 0 & \text{if} \quad p-q \geq 2i \quad \text{and} \quad p \neq q\,, \\ \int_Y \mathrm{ch}(\alpha)\cdot\gamma & \text{if} \quad p = q, i = 0\,. \end{cases} \tag{2.3}$$

it becomes an isomorphisms only because we are working with curves. This has to do with $B^{0,q}$ being bounded by $q = 1$.

Continuing to use the notation of the example, recall that Miguel defined the operators

$$T_k^*, R_k^*$$

on the algebra $\mathbb{D}^C$ in terms of $\mathrm{ch}_i(v)$ and their derivations. The condition that these can be formulated on the full $H^*(\mathcal{M}_C)$ is equivalent to

$$\xi^{\mathbb{G}}\big[R_k^*\big(\mathrm{ch}_i(v^{p,q})\big)\big] = 0\,,$$

whenever the degree of $\xi^{\mathbb{G}}\big(\mathrm{ch}_i(v^{p,q})\big)$ is zero. This holds again only for curves C, because in this case degree of $\mathrm{ch}_i(v^{1,0}) = 0$. We then denote by

$$T_k, R_k$$

their duals on the homology $H_*(\mathcal{M}_C)$. The beauty of this is that we have a natural conformal element

$$\omega = \sum_{v\in B} e^0 \otimes u_{v,1}u_{v^\vee,1}\,,$$

† such that its field

$$Y(\omega, z) = \sum_{k\in\mathbb{Z}} L_k z^{-k-2} \tag{2.4}$$

recovers $L_k = T_k + R_k$ whenever $k \geq -1$ and

$$B = \bigsqcup B_{p,p}\,.$$

I will discuss the necessary modification when there are non (p,p)-classes later.

3 Compatibility explained on an Example: Flag and Gr(n,k)

We show the power of rephrasing Virasoro constraints in terms of the vertex algebra proving some new results without having to do many computations.

†Here I used the notation $\{v^\vee\}$ to denote the dual basis of B with respect to χ_{ssym} pretending that it is non-degenerate. This is not true in general, but can be easily fixed.

Gross–Joyce–Tanaka(20') formulated the wall-crossing to take place in the quotient

$$W_* = V_{*+2}/DV_*$$

which carries a Lie algebra structure by

$$[\bar{u}, \bar{v}] = \overline{[z^{-1}]Y(u,z)v}$$

for any lifts $u, v \in V_*$ of $\bar{u}, \bar{v} \in W_*$.

They defined for each reasonable choice of stability condition σ and $\alpha \in K^0$ the homology classes

$$\big[M_\alpha(\sigma)\big]^{\text{in}}$$

satisfying the wall-crossing formulae

$$\big[M_\alpha(\sigma_2)\big]^{\text{in}} = \sum_{\vec{\alpha} \vdash \alpha} \text{coeff.}\Big[\cdots\Big[\big[M_{\alpha_1}(\sigma_1)\big]^{\text{in}}, \big[M_{\alpha_1}(\sigma_1)\big]^{\text{in}}\Big], \cdots, \big[M_{\alpha_k}(\sigma_1)\big]^{\text{in}}\Big]$$

for some pre-described coefficients depending on the two different stability condition σ_1, σ_2. Their main contribution is in showing that when there are no strictly σ-semistables in class α, then these are the pushforwards of the usual virtual fundamental classes

$$\big[M_\alpha(\sigma)\big]^{\text{vir}}$$

under the open embeddings $M_\alpha(\sigma) \hookrightarrow \mathcal{M}_C$. Recall the map

$$\eta : \mathbb{D}_C \longrightarrow \mathbb{D}_C^{\text{in}},$$

from Miguel's talk which was used to define an invariant Virasoro [‡]. Since the dual of R_{-1} is our translation operator T, the dual $\lambda = \eta^*$ induces a well-defined lift

$$\lambda : W_* \longrightarrow V_*.$$

Standard result: If L_k are given by the Virasoro element as in (2.4), then if

$$L_k \circ \lambda\big[M_{\alpha_1(\sigma)}\big]^{\text{in}} = \delta_{k,0}\lambda\big[M_\alpha(\sigma)\big]^{\text{in}}$$

is satisfied for all $i = 1, \cdots, k$, then

$$L_k \circ \lambda\big[M_\alpha(\sigma)\big]^{\text{in}} = \delta_{k,0}\lambda\big[M_\alpha(\sigma)\big]^{\text{in}}. \tag{3.1}$$

The point is to show that the set $P \subset L$ of *physical states* satisfying precisely the condition (3.1) forms a Lie subalgebra of L.

Unfortunately in this raw form this can not be applied in too many interesting geometries. That is why I selected these examples. Start with any acyclic quiver Q, for example just A^k

$$\overset{1}{\circ} \longrightarrow \overset{2}{\circ} \longrightarrow \overset{3}{\circ} \longrightarrow \cdots \longrightarrow \overset{k}{\circ}, \tag{3.2}$$

and consider the μ-stability, a representation $V_\bullet$ is said to be (semi)-stable if for every $W_\bullet \subset V_\bullet$ and the respective dimension vectors $\vec{e}$ and $\vec{d}$ we have

$$\sum_i e_i\mu_i < (\leq) \sum_i d_i\mu_i.$$

We only need two cases

[‡] Here $\mathbb{D}_C^{\text{in}} = \ker(R_{-1})$

1. In the case when $\mu_{i+1} > \mu_i$, where i increases in the direction of the arrows we get that the only non-zero classes
$$\left[M_{\vec{d}},(\mu)\right]^{\mathrm{in}}$$
are those for the dimension vectors δ_i when $i = 1, \cdots, k$ and these are just the point classes in $H_*(\mathcal{M}_{\delta_i})$.

2. In the example (**??**), we choose real numbers
$$1 \gg \mu_1 \gg \mu_2 \gg \cdots \gg \mu_{k-1}$$
together with $\mu_k < -\frac{k(k-1)}{2}$. It is then easy to see that there are no strictly semi-stables in the class
$$\vec{d} = (1, 2, \cdots, k)$$
and
$$[M_{\vec{d}}(\mu)]^{\mathrm{in}} = [\mathrm{Flag}(\mathbb{C}^k)]\,.$$

Similar arguments can be applied to the case
$$\overset{1}{\circ} \longrightarrow \overset{2}{\circ}$$
to get $[\mathrm{Gr}(n, k)] = [M_{(k,n)}(\mu)]^{\mathrm{in}}$. This proves the claim.

Remark 3.1. The statement for flag-varieties was also proved by Miguel purely using geometry.

4 Rank reduction for curves

The first difference that appears when moving on to curves is the issue of extending L_k for $k \geq -1$ from Miguel's talk to a full field
$$\sum_{k\in\mathbb{Z}} L_k z^{-k-2}\,,$$

The next few lines are not strictly necessary in the setting of working with curves, but they are a precursor for things to come when thinking of higher dimensions. To shorten notation it makes sense to introduce the *holomorphic pairing*
$$\chi^{\mathrm{H}}\big(v, w\big) = (-1)^{p(v)} \int_X v \cdot w \cdot \mathrm{td}(X)\,,$$
[§] and its supersymmetrization
$$\chi^{\mathrm{H}}_{\mathrm{ssym}}\big(v, w\big) = \chi^{H}(v, w) + (-1)^{\deg(v)}\chi^{\mathrm{H}}\big(w, v\big)\,,$$
The usual field of the lattice vertex algebra we mentioned before is
$$Y(u_{v,1}, z) = \Big\{\sum_{k>0} u_{v,k} \cdot z^{k-1} + (-1)^{\delta_{\mathrm{odd}}(v)} \sum_{k>0}\sum_{w\in B} \frac{k!}{\big(k - \delta_{\mathrm{odd}}(v)\big)!}\chi_{\mathrm{ssym}}(v, w)\frac{\partial}{\partial u_{w,k}} z^{-k-\delta_{\mathrm{ev}}(v)} + \chi(v, \beta)z^{-1}\Big\}\,,$$

[§]Here H stands for Hodge, because it interacts non-trivially with the Hodge grading.

where we use $\delta_{\text{odd}}(v) \equiv \deg(v) \pmod 2$. Again to make things easier on the eye, we define its modification

$$Y^{\mathrm{H}}(v,z) = \sum_{k>0} u_{v,k} z^k - \sum_{k>0, w\in B} \frac{k!}{(k-\delta_{\text{odd}}(v))!} \chi^H_{\text{ssym}}(v,w) \frac{\partial}{\partial u_{w,k}} z^{-k-\delta_{\text{ev}}(v)} + \chi(v,\beta) z^{-1}\,.$$

To obtain the full field, we may now take the sums of normal ordering products

$$L(z) = \sum_{\substack{\in B^{p,q}\\ 0\le q\le p\le 1}} :Y^H(v,z)\frac{\partial}{\partial z}Y^H(\hat v,z): + \sum_{\substack{\in B^{0,1}\\ 0\le q\le p\le 1}} :\frac{\partial}{\partial z}Y^H(v,z)Y^H(\hat v,z):$$

by a direct computation.¶ This is again very special for curves and is important for the next result.

Lemma 4.1.

$$L_k \circ \lambda\big[M_{\alpha_1}(\sigma)\big]^{\text{in}} = \delta_{k,0}\lambda\big[M_\alpha(\sigma)\big]^{\text{in}} \tag{4.1}$$

is satisfied for all $i = 1, \cdots, k$, then

$$L_k \circ \lambda\big[M_\alpha(\sigma)\big]^{\text{in}} = \delta_{k,0}\lambda\big[M_\alpha(\sigma)\big]^{\text{in}}\,.$$

Proof. The proof is to show again that there is a Lie-subalgebra of physical states. This relies this time on the fact that the *locality condition* of the fields is satisfied:

$$(z-w)^N\big[L(z), Y([M_\alpha(\sigma)]^{\text{in}}, w)\big] = 0\,.$$

This follow, because the field $L(z)$ could be constructed from the usual pairing in the same way instead, which can be observed by showing that the part of the T_k operator coming from $v^{0,1}$, resp. $v^{1,0}$ vanishes in both scenarios and the rest is unchanged. We then use Don's lemma.

Lemma 4.2. *If three fields $a(z)$, $b(z)$ and $c(z)$ are mutually local, then*

$$:a^{(n)}(z)b^{(m)}(z):$$

is local with respect to c(z).

□

It is now a simple observation that if we define the operator

$$\mathcal{L} = \sum_{j\ge -1} \frac{(-1)^j}{(j+1)!} D^j L_j : W_* \longrightarrow V_*$$

which is the dual of the one introduced by Miguel, then (4.1) is equivalent to

$$\mathcal{L}\big[M_{\alpha(\sigma)}\big]^{\text{in}} = 0\,.$$

Furthermore, by Miguel's talk this makes our (3.1) equivalent to the geometric Virasoro constraints observed by van Bree/Moreira/MOOP. We will now use the rank-reduction of

¶This time $\hat v$ is the dual of v with respect to χ^H.

Joyce/Thaddeus/Mochizuki together with the pushforward formula of Miguel from his talk to prove that

$$\mathcal{L}[M_{r,\chi}]^{\mathrm{in}} = 0$$

and more! For this we also need to consider the moduli schemes of Bradlow pairs

$$\mathcal{O}_X(-k) \longrightarrow F$$

which we denote by $P^k_{r,\chi}$ and their classes sit in the homology of a stack of pairs $\mathcal{P}^k_C$.

Theorem 4.3. *We have*

$$\mathcal{L}^{\mathcal{O}_X(-k)}\big[P^k_{r,\chi}\big]^{\mathrm{in}} = 0\,, \quad \mathcal{L}\big[M_{r,\chi}\big]^{\mathrm{in}} = 0\,. \tag{4.2}$$

for any choice of (r,χ).

Proof. The result follows by starting with

$$P_{0,n} = C^{[n]}\,, \qquad M_{1,d} = \mathrm{Jac}^d\,.$$

We then use first that there is a wall-crossing

$$[P^k_{r,\chi}]^{\mathrm{in}} = \sum_{\substack{(\vec{r},\vec{\chi})\vdash(r,\chi)\\ \vec{d}\vdash 1\\ r>0}} \mathrm{coeff.}\Big[\Big[[P^k_{r_1,\chi_1}]^{\mathrm{in}}, [P^k_{r_2,\chi_2}]^{\mathrm{in}}\Big], \cdots, [P^k_{r_k,\chi_k}]^{\mathrm{in}}\Big]\,, \tag{4.3}$$

where additionally $r_i > 0$ to show that

$$\mathcal{L}^{\mathcal{O}_X(-k)}\big[P^k_{r,\chi}\big]$$

holds assuming (4.2) is satisfied for all lower ranks. Then using

$$[M_{r,\chi}]^{\mathrm{in}} = \Omega_{r,\chi} + \sum_{\substack{\vec{r},\vec{\chi})\vdash(r,\chi)\\ r_i>0}} \mathrm{coeff.}\Big[\cdots\Big[[M]^{\mathrm{in}}, [M_{\alpha_2}]^{\mathrm{in}}\Big], \cdots, [M_{\alpha_k}]^{\mathrm{in}}\Big]\,, \tag{4.4}$$

where the first term on the right hand side is defined by

$$\Omega_{r,\chi} = \pi_*\Big([P^k_{r,\chi}]^{\mathrm{in}} \cap e(\mathbb{T}_{\mathcal{P}^k_C/\mathcal{M}_C})\Big) \tag{4.5}$$

using the projection $\pi : \mathcal{P}^k_C \to \mathcal{M}_C$. We therefore only need a stack version of the result discussed by Miguel

Proposition 4.4. *If* $[P^k_{r,\chi}]^{\mathrm{in}}$ *satisfies*

$$\mathcal{L}^{\mathcal{O}_X(-k)}[P^k_{r,\chi}]^{tin} = 0$$

then

$$\mathcal{L}\Omega^k_{r,\chi} = 0\,.$$

This then shows that

$$\mathcal{L}\big[M_{r,\chi}\big]^{\mathrm{in}} = 0\,.$$

¨for all ranks. ☐