The birational geometry of moduli spaces of level curves

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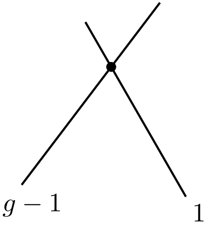
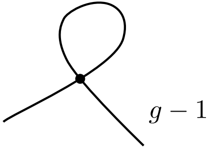
Section 1

$\overline{\mathcal{M}}_g$

Moduli space of stable curves

- $\mathcal{M}_g$ is the moduli space of smooth curves of genus g.
- Deligne–Mumford compactification $\overline{\mathcal{M}}_g$ of stable curves of genus g.





Severi's theorem and conjecture

Theorem (Severi, 1915)

$\overline{\mathcal{M}}_g$ *is unirational for* $g \leq 10$*.*

Conjecture (Severi)

$\overline{\mathcal{M}}_g$ is unirational for all g.

. . . some years passed . . .

Theorem (Sernesi, 1981)

$\overline{\mathcal{M}}_{12}$ *is unirational.*

The result of Mumford–Harris and Eisenbud–Harris

Theorem (Harris–Mumford, 1982; Eisenbud–Harris, 1987)

$\overline{\mathcal{M}}_g$ is of general type for $g \geq 24$.

Geometric consequences

- General type implies non-uniruledness.
- In particular no general curve of high genus is a hyperplane section of a non-ruled surface.
- Only “special” curves of high genus can be obtained by constructions involving free parameters.

How to prove unirationality

- Approach: write down simultaneous equations for almost all the curves of a given genus.
- Elliptic curves: Weierstraß equations.
- Works similarly for other low values of g.

Proofs of general type results

- Let λ be the Hodge class on $\overline{\mathcal{M}}_g$, note that λ is big.
- Formula for the canonical class on $\overline{\mathcal{M}}_g$ explicitly known:

$$K_{\overline{\mathcal{M}}_g} = 13\lambda - 2\delta$$

- Want to find an effective divisor D such that we can write

$$K_{\overline{\mathcal{M}}_g} = D + \varepsilon\lambda$$

  for some rational number $\varepsilon > 0$.
- This then implies $K_{\overline{\mathcal{M}}_g}$ is big as well.

Singularities are a problem ... or are they?

- $\overline{\mathcal{M}}_g$ has *non-canonical* singularities, i.e., canonical differentials do not locally extend when resolving the singularities.

Theorem (Harris–Mumford)

If $g \geq 4$, then for all m, every m-canonical form on $\overline{\mathcal{M}}_g^{\mathrm{reg}}$ extends to an m-canonical form on $\overline{\mathcal{M}}_g$. More precisely:

$$H^0\left(\overline{\mathcal{M}}_g^{\mathrm{reg}}, K_{\overline{\mathcal{M}}_g^{\mathrm{reg}}}^{\otimes m}\right) \cong H^0\left(\widehat{\mathcal{M}}_g, K_{\widehat{\mathcal{M}}_g}^{\otimes m}\right)$$

for every desingularization $\widehat{\mathcal{M}}_g$ of $\overline{\mathcal{M}}_g$.

Overview of known results for $\mathcal{M}_g$

g	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
																							≥ 2	

Overview of known results for $\mathcal{M}_g$

g	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24

Overview of known results for $\mathcal{M}_{g,n}$

$n \backslash g$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
0																								
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15	1																							
16	1																							

Section 2

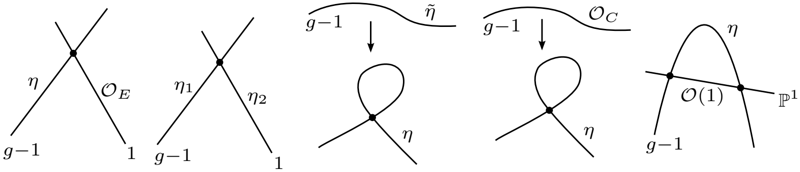
Moduli spaces of level curves

Moduli spaces of level curves

- Choose a prime number ℓ.
- Let $\mathcal{R}_{g,\ell}$ be the moduli space of pairs $[C, \eta]$ where $C \in \mathcal{M}_g$ and η is a line bundle of order ℓ.
- Such an η induces an isomorphism class of a cyclic étale cover $\widetilde{C} \to C$ of order ℓ.
- $\mathcal{R}_{g,\ell}$ generalizes the modular curve $Y_1(\ell)$ to higher genus.

Modular compactification of $\mathcal{R}_{g,\ell}$

- Various compactifications available.
- Most useful for us: based on Deligne–Mumford $\overline{\mathcal{M}}_g$ with new types of *quasistable* curves lying over points in Δ_0.



Prym varieties

Classical case: $\ell = 2$

- From $[C, \eta]$ get a double cover $\pi \colon \widetilde{C} \to C$.
- The cover π induces an endomorphism γ of $\mathrm{Jac}(\widetilde{C})$.
- $P(C, \eta) = \mathrm{Image}(1 - \gamma)$ is a ppav of dimension $g - 1$.
- $P(C, \eta)$ is called a *Prym variety*.
- Prym map $\mathcal{R}_{g,2} \to \mathcal{A}_{g-1}$ dominant for $g \leq 6$, used to study ppavs of low dimension.

Singularities are getting worse

- Again, $\overline{\mathcal{R}}_{g,\ell}$ has non-canonical singularities.
- For $\ell \geq 5$ there exists a difficult new type of singularity.

Theorem (Farkas–Ludwig, Chiodo–Farkas)

Fix $g \geq 4$ and $\ell = 2$ or $\ell = 3$. Let $\widehat{\mathcal{R}}_{g,\ell} \to \overline{\mathcal{R}}_{g,\ell}$ be any desingularization. Then every pluricanonical form defined on the smooth locus $\overline{\mathcal{R}}_{g,\ell}^{\mathrm{reg}}$ of $\overline{\mathcal{R}}_{g,\ell}$ extends holomorphically to $\widehat{\mathcal{R}}_{g,\ell}$, that is, for all integers $m \geq 0$ we have isomorphisms

$$H^0\left(\overline{\mathcal{R}}_{g,\ell}^{\mathrm{reg}}, K_{\overline{\mathcal{R}}_{g,\ell}^{\mathrm{reg}}}^{\otimes m}\right) \cong H^0\left(\widehat{\mathcal{R}}_{g,\ell}, K_{\widehat{\mathcal{R}}_{g,\ell}}^{\otimes m}\right)$$

Overview of known results

ℓ \ g	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1																						
2												≥ 0			≥ 1							
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Section 3

Prym varieties of genus 15

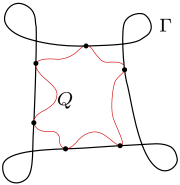
The case of $g = 15$ and $\ell = 2$

Theorem (—, 2015)

$\overline{\mathcal{R}}_{15,2}$ is of general type.

Which divisor works for $g = 15$ and $\ell = 2$? I

- Motivation from genus 6.
- General curve of genus 6 has a finite number of (base point free) $L \in W^2_6(C)$.
- L induces a 4-nodal plane sextic model Γ.
- Ask for a conic Q that is tangent to Γ at every point of intersection.



- Equivalently:

$$\operatorname{Sym}^2 H^0(C, L \otimes \eta) \to \frac{H^0(C, L^{\otimes 2})}{\operatorname{Sym}^2 H^0(C, L)}$$

not injective, where η is a 2-torsion line bundle.

Which divisor works for $g = 15$ and $\ell = 2$? II

Now in genus 15:

- General curve has a smooth degree 16 model in $\mathbb{P}^4$ induced by a line bundle L.
- Ask for

$$\operatorname{Sym}^2 H^0(C, L \otimes \eta) \to \frac{H^0(C, L^{\otimes 2})}{\operatorname{Sym}^2 H^0(C, L)}$$

  not injective.
- Pairs $[C, \eta]$ with such an L form a virtual divisor $\mathcal{D}_{15}$ in $\mathcal{R}_{15,2}$.

Which divisor works for $g = 15$ and $\ell = 2$? III

$$\operatorname{Sym}^2 H^0(C, L \otimes \eta) \to \frac{H^0(C, L^{\otimes 2})}{\operatorname{Sym}^2 H^0(C, L)}$$

- Have to construct a pair $[C, \eta]$ where the above map is injective.
- A curve with a theta characteristic in $W^4_{14}(C)$ works.
- Also need to prove that the moduli space of triples $[C, \eta, L]$ is irreducible.
- Use a globalized version of the map to calculate the divisor class.

Which divisor works for $g = 15$ and $\ell = 2$? IV

Theorem (—,2015)

The class

$$[\overline{\mathcal{D}}_{15}]' \equiv 31020\left(\frac{3127}{470}\lambda - (\delta_0' + 4\delta_0'') - \frac{3487}{1880}\delta_0^{(1)}\right)$$

in $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{R}'_{15,2})$ *is effective. Here* $\mathcal{R}'_{15,2}$ *is a partial compactification of* $\mathcal{R}_{15,2}$ *including only curves lying over general points in* $\Delta_0 \subset \mathcal{M}_{15}$.

Section 4

Mukai's geometry of low genus curves

Unirationality of $\overline{\mathcal{M}}_g$ for $g = 3, 4, 5$

General canonical curve of genus

- $g = 3$ is a plane quartic.
- $g = 4$ is a $(2, 3)$-complete intersection in $\mathbb{P}^3$.
- $g = 5$ is a $(2, 2, 2)$-complete intersection in $\mathbb{P}^4$.

Genus 6 and beyond

- The general curve of genus 6 (and above) is *not* a complete intersection in projective space.
- Mukai's insight: If we consider curves embedded in *homogeneous spaces*, not just projective space, then we can continue up to genus 9.
- Concretely: Curves of genus $6 \leq g \leq 9$ arise as complete intersections in Grassmannian varieties.

Definition

Restricting the universal quotient bundle of a Grassmannian to an embedded curve C gives a vector bundle E_C, the *Mukai bundle* of C.

Mukai's geometry in concrete numbers

g	X_g	$\mathrm{rk}(E_C)$	$h^0(C, E_C)$	BN condition
6	$\mathrm{G}(5,2)$	2	5	$\#W^1_4(C) < \infty$
7	$\mathrm{OG}(10,5)$	5	10	$W^1_4(C) = \emptyset$
8	$\mathrm{G}(6,2)$	2	6	$W^2_7(C) = \emptyset$
9	$\mathrm{SpG}(6,3)$	3	6	$W^1_5(C) = \emptyset$

Section 5

Using the Mukai bundle

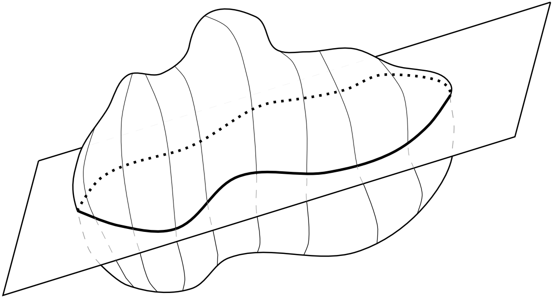
General type for $g = 8$ and $\ell = 3$

Theorem (—, 2016)

$\overline{\mathcal{R}}_{8,3}$ *is of general type.*

Recap on Mukai geometry of genus 8 curves

- General canonical curve C is an intersection of the Grassmannian $\mathrm{G}(6,2) \subseteq \mathbb{P}^{14}$ and a 7-dimensional plane.



- Mukai bundle E_C is the restriction of the universal rank 2 quotient bundle on $\mathrm{G}(6,2)$ to C.

Which divisor works for $g = 8$ and $\ell = 3$? I

- Consider the locus

$$\mathcal{D}_{8,3} = \left\{ [C, \eta] \;\middle|\; H^0(C, E_C \otimes \eta) \neq 0 \right\}.$$

- E_C is (locally) an extension

$$0 \to A \to E_C \to L \to 0$$

with $A \in W^1_5(C)$ and $L = K_C - A \in W^3_9(C)$.

- Description in terms of the map

$$H^0(C, L \otimes \eta) \otimes H^0(C, L \otimes \eta^{-1}) \to \frac{H^0(C, L^{\otimes 2})}{\operatorname{Sym}^2 H^0(C, L)}$$

Which divisor works for $g = 8$ and $\ell = 3$? II

- $\mathcal{D}_{8,3}$ is a divisor: construct one example $[C, \eta]$ where we have $H^0(C, E_C \otimes \eta) = 0$.
- By semi-continuity and irreducibility of $\mathcal{R}_{8,3}$ this is then true for the general pair.
- Proof by specialization first to plane nodal septics: there $E_C = M \oplus M'$.
- Further specialization to hyperelliptic curves necessary.
- Need irreducibility of some moduli spaces of linear series.

Which divisor works for $g = 8$ and $\ell = 3$? III

- Compactification $\overline{\mathcal{D}}_{8,3}$ has a useful class.

Theorem (—,2016)

The class

$$[\overline{\mathcal{D}}_{8,3}]' = 196\lambda - 28(\delta_0' + 2\delta_0'') - \frac{308}{3}\delta_0^{(1)}$$

in $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{R}'_{8,3})$ *is effective. Here* $\mathcal{R}'_{8,3}$ *is a partial compactification of* $\mathcal{R}_{8,3}$ *including only curves lying over general points in* $\Delta_0 \subset \mathcal{M}_8$.

- We have a similar theorem for other ℓ, as well as for $g = 6$.

Overview of known results, now

ℓ \ g	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1																						
2												≥ 0			≥ 1							
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Overview of known results, now

$\ell \backslash g$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
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Section 6

Idle speculation

Implications for other genera?

ℓ \ g	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
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2												≥ 0										
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- Nothing is known about $\mathcal{R}_{9,3}$ and $\mathcal{R}_{10,3}$.
- We have $\kappa(\overline{\mathcal{R}}_{11,3}) \geq 19$. Observe $\dim(\overline{\mathcal{R}}_{11,3}) = 30$.
- Theorem suggests these three spaces could be of general type as well.

Relation between the gaps?

ℓ \ g	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1																						
2												≥ 0										
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- Strips almost map to each other under

$$[C,\eta] \mapsto [\widetilde{C} \to C] \mapsto [\widetilde{C}] \in \mathcal{M}_{\ell g - \ell + 1}$$

- Coincidence?
- In what respect are the curves $\widetilde{C}$ general?

Comparison with spin moduli spaces

g	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
												≥ 0										

g	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$+$								0														
$-$																						

- Why do spin curves seem to be easier?
- What about $\overline{\mathcal{R}}_{12,2}$?
- Program for higher order spin curves?

The end

Thank you!