Degree of a line bundle

Given a holomorphic line bundle $L \to X$ on a Riemann surface X, we have two different definition of the degree of L.
The goal of these pages is to prove that these two definitions are actually the same.

First definition

$L \to X$ hol line bundle on X. By theorem 29.16 of [1], $\exists$ $\overset{\circ}{\#}$ s meromorphic section of L. If $\mathcal{U} = \{U_i\}$ is an open covering of X s.t. $L|_{U_i} \cong U_i \times \mathbb{C}$ we can identify s with $\{s_i \in \mathcal{M}(U_i)\}_{i \in I}$ and for all $x \in X$

$$ord_x(s) := ord_x(s_i) \quad \text{if } x \in U_i$$

is well-defined. Let $D \in \mathrm{Div}(X)$ be the divisor of s i.e.

$$D(x) := ord_x(s).$$

This defines a map

$$H^1(X, \mathcal{O}_X^*) \xrightarrow{\sim} \mathrm{Div}(X) / \mathrm{Div}_p(X)$$
$$[L] \mapsto [D]$$

$\mathrm{Div}_p(X)$: principal divisors of X

that happens to be an isomorphism of groups (look at Thm 29.19 of [1] to see that this map is well-defined and is an isomorphism of groups)

Def 1 $\deg L := \deg D.$ is the degree of L

well defined by the previous results

Second definition

$L \to X$ hol line bundle on X. Thus we have

$$c_1 : H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z})$$
$$[L] \mapsto c_1([L])$$

Now we have:

$$H^2(X;\mathbb{Z}) \subset H^2(X;\mathbb{C}) \overset{\star}{\cong} H^2_{DR}(X,\mathbb{C}) \xrightarrow{\int_X -} \mathbb{C}$$
$$c_1(L) \longmapsto \int_X c_1(L)$$

↑ De Rham theorem (See page 44 of [2])

Def 2 $\deg(L) := \int_X c_1(L)$.

Proof of the equivalence

Prop $\boxed{\text{def } 1 = \text{def } 2}$

proof

Strategy: we will almost explicitly compute the 2-form $\omega_L := c_1(L)$ via the isomorphism $\star$ and then integrate ω_L ~~ex~~

Let $\{U_i\} = \mathcal{U}$ be an open cover of X s.t.

1. $\forall i$ U_i is contractible
2. $\forall i,j$ $U_{ij} = U_i \cap U_j$ is contractible;
3. $L|_{U_i} = U_i \times \mathbb{C}$ $\forall i$.

(See the proof of thm 5.1 of [3] for the existence of such a cover)

Let $s = \{s_i \in \mathcal{M}(U_i)\}$ be a meromorphic section of L. Then

$$s_i = g_{ij} s_j \quad \text{on } U_{ij}$$

where g_{ij} are the transition functions of L with respect to $\mathcal{U}$.

$$\Rightarrow g_{ij} = s_i / s_j \in \mathcal{O}^*(U_{ij}) \quad \forall i, j$$

<u>Step 1</u> || We compute $c_1(L) \in H^2(X, \mathbb{Z})$:

$\exists$ such f_{ij} because U_{ij} is contractible

Consider

$$\begin{tikzcd}
 & & \{f_{ij}\} \arrow[r, mapsto] \arrow[ddd] & \{g_{ij} = e^{2\pi i f_{ij}}\} & \\
 & & & \cap & \\
0 \arrow[r] & C^1(\mathcal{U}, \mathbb{Z}) \arrow[r] & C^1(\mathcal{U}, \mathcal{O}_X) \arrow[r, "\exp(2\pi i -)"] & C^1(\mathcal{U}, \mathcal{O}_X^*) \arrow[r] & 0 \\
0 \arrow[r] & C^2(\mathcal{U}, \mathbb{Z}) \arrow[r] & C^2(\mathcal{U}, \mathcal{O}_X) \arrow[r] & C^2(\mathcal{U}, \mathcal{O}_X^*) \arrow[r] & 0 \\
 & \cup & \{f_{jh} - f_{ih} + f_{ij}\} & &
\end{tikzcd}$$

$\uparrow$ position ijh

i.e. $c_1(L) = [\{f_{jh} - f_{ih} + f_{ij}\}] \in H^2(X, \mathbb{Z})$

<u>Step 2</u> || We compute the isomorphism ✪ : the image of $c_1(L) \in H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{C})$ under

The isomorphism ✪ is obtained as the composition of two isomorphisms

$$H^2(X, \mathbb{C}) \cong H^1(X, Z^1) \cong H^2_{DR}(X)$$

$\uparrow$ sheaf of closed 1-forms on X

(see page 44 of [2])

Step 2.2 We compute the image of $c_1(L) \in H^2(X,\mathbb{Z})$ under the isomorphism $H^2(X,\mathbb{C}) \cong H^1(X,\mathcal{Z}^1)$:

The isomorphism is given by the map $H^2(X,\mathbb{C}) \leftarrow H^1(X,\mathcal{Z}^1)$ that appears in the long exact sequence of cohomology groups associated to the exact sequence of sheaves

$$0 \to \mathbb{C} \hookrightarrow \underset{\substack{\uparrow \\ \text{sheaf of } \mathcal{C}^\infty \text{ functions with values in } \mathbb{C}}}{\mathcal{E}} \xrightarrow{d} \mathcal{Z}^1 \to 0$$

So consider

the open sets U_{ij} are contractible

$$\{\varphi_{ij}\} \longmapsto \{w_{ij} = d\varphi_{ij}\}$$

$$\begin{array}{ccccccccc}
0 \to & C^1(\mathcal{U},\mathbb{C}) & \to & \mathcal{C}^1(\mathcal{U},\mathcal{E}) & \xrightarrow{d} & C^1(\mathcal{U},\mathcal{Z}^1) & \to 0 \\
 & & & \downarrow & & & \\
0 \to & \mathcal{C}^2(\mathcal{U},\mathbb{C}) & \to & \mathcal{C}^2(\mathcal{U},\mathcal{E}) & \xrightarrow{d} & \mathcal{C}^2(\mathcal{U},\mathcal{Z}^1) & \to 0
\end{array}$$

induces an isomorphism $H^1(X,\mathcal{Z}^1) \cong H^2(X,\mathbb{C})$

$$\{\underset{\substack{\uparrow \\ \text{position } ijh}}{\varphi_{jh} - \varphi_{ih} + \varphi_{ij}}\}$$

and we want that $[\{\varphi_{jh} - \varphi_{ih} + \varphi_{ij}\}] = [\{f_{jh} - f_{ih} + f_{ij}\}] = c_1(L)$

in $H^2(X,\mathbb{C})$.

It is enough to choose $\varphi_{ij} = f_{ij}$.

$$\Rightarrow \quad \begin{array}{ccc} H^2(X,\mathbb{C}) & \cong & H^1(X,\mathcal{Z}^1) \\ \Psi & & \\ c_1(L) & \longleftrightarrow & [\{ d\underbrace{f_{ij}}_{\text{holomorphic}} = \partial f_{ij} \}] \end{array}$$

<u>Step 2.b</u> We compute the image of $[\{df_{ij}\}]$ under the isomorphism

$$H^1(X,\mathcal{Z}^1) \cong H^2_{DR}(X)$$

This isomorphism is given by the map

$$H^2_{DR}(X) \to H^1(X,\mathcal{Z}^1)$$

that appears in the long exact sequence of cohomology groups associated to the exact sequence of sheaves

$$0 \to \underbrace{\mathcal{Z}^1}_{\text{closed 1-forms}} \to \underbrace{\mathcal{E}^1}_{C^\infty \text{ 1-forms}} \xrightarrow{d} \underbrace{\mathcal{Z}^2}_{\text{closed 2 forms}} \to 0$$

here we use the fact that each U_i is contractible

So consider $\{\gamma_i\} \mapsto \{\eta_i = d\gamma_i\}$

$$\begin{tikzcd}
0 \arrow[r] & C^0(\mathcal{U},\mathcal{Z}^1) \arrow[r] & C^0(\mathcal{U},\mathcal{E}^1) \arrow[r] \arrow[d] & C^0(\mathcal{U},\mathcal{Z}^2) \arrow[r] \arrow[dll, dashed] & 0 \\
0 \arrow[r] & C^1(\mathcal{U},\mathcal{Z}^1) \arrow[r] & C^1(\mathcal{U},\mathcal{E}^1) \arrow[r] & C^1(\mathcal{U},\mathcal{Z}^2) \arrow[r] & 0
\end{tikzcd}$$

induces isomorphism

$$\{\gamma_i - \gamma_j\}$$

and we want $\{\gamma_i\}$ s.t. $[\{\gamma_i - \gamma_j\}] = [\{df_{ij}\}] \in H^1(X,\mathcal{Z}^1)$

After that

$$\omega_L|_{U_i} = d\gamma_i$$

<u>Pb</u> How to find $\{\gamma_i\}$ explicitely?

Obs / Summary

Therefore the steps to find w_L are:

$$s=\{s_i\} \rightsquigarrow g_{ij}=\frac{s_i}{s_j}=e^{2\pi i f_{ij}} \rightsquigarrow w_{ij}=df_{ij}=\frac{1}{2\pi i}\frac{s_j}{s_i}\frac{s_i's_j-s_j's_i}{s_j^2}dz$$

$$=\frac{1}{2\pi i}\left(\frac{s_i'}{s_i}-\frac{s_j'}{s_j}\right)dz \quad \text{where } z \text{ are coordinates on } U_{ij}=U_i\cap U_j$$

⚠ We could be tempted to choose

$$\gamma_i=\frac{s_i'}{s_i}dz_i \quad \text{on } U_i$$

but $\frac{s_i'}{s_i}dz_i \notin \mathcal{E}^1(U_i)$ because it has poles of order 1 exactly at the zeros and poles of s.

However this suggests to proceed as follows.

Let $\mu_i := \frac{1}{2\pi i}\frac{s_i'}{s_i}dz_i$ and $\mu := \{\mu_i\} \in C^0(\mathcal{U}, \mathcal{M}^{(1)})$

($\mathcal{M}^{(1)}$: sheaf of meromorphic 1-forms on X)

Then $\mu_i-\mu_j = df_{ij} = \partial f_{ij} \in \Omega(U_{ij})$ (holomorphic 1 forms)

$\Rightarrow$ $\mu:=\{\mu_i\}$ is a Mittag-Leffler distribution of 1-forms.

(See page 133 of [1] for the definition).

$\forall x\in X$ the residue of μ is well defined

$$\mathrm{Res}_x(\mu) := \mathrm{Res}_x(\mu_i) \quad \text{if } x\in U_i$$

Set

$$\mathrm{Res}(\mu) := \sum_{x\in X}\mathrm{Res}_x(\mu).$$

Then $\mathrm{Res}(\mu) = \dfrac{\text{zeros of } s - \text{poles of } s}{2\pi i} = \dfrac{\deg(L)}{2\pi i}$ according to def 1.

To conclude considere Dolbeault's isomorphism

$$[\{df_{ij}\}] \in H^1(X,\Omega) \cong \mathcal{E}^2(X) / d\mathcal{E}^{1,0}(X)$$

2 forms on X

forms of type (1,0) on X

$\| \quad [\delta\mu]$

obtained by considering the exact sequence of sheaves

$$0 \to \Omega \to \mathcal{E}^{10} \xrightarrow{d} \mathcal{E}^2 \to 0$$

where

$\delta : C^0(\mathcal{U}, \mathcal{M}^1) \to C^1(\mathcal{U}, \mathcal{M}^1)$

<u>Claim</u> Under this isomorphism $[\delta\mu]$ corresponds to $[\omega_L]$

<u>proof of the claim</u>

We have the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & & & \{\eta_i\} & & \\
 & & & & & & \cap & & \\
0 \to & C^0(\mathcal{U}, \mathcal{Z}^1) & \to & C^0(\mathcal{U}, \mathcal{E}^1) & \xrightarrow{d} & & C^0(\mathcal{U}, \mathcal{Z}^2) & \to 0 \\
 & \cup & & \cup & & & \cap & \\
0 \to & C^0(\mathcal{U}, \Omega) & \to & C^0(\mathcal{U}, \mathcal{E}^{10}) & \xrightarrow{d} & & C^0(\mathcal{U}, \mathcal{E}^2) & \to 0 \\
 & & & \{\gamma_i\} & & & & \\
 & \{df_{ij}\} & & & & & & \\
 & \cap & & & & & & \\
0 \to & C^1(\mathcal{U}, \Omega) & \to & C^1(\mathcal{U}, \mathcal{E}^{10}) & \to & & C^1(\mathcal{U}, \mathcal{E}^2) & \to 0 \\
 & \cap & & \cap & & & \cup & \\
0 \to & C^1(\mathcal{U}, \mathcal{Z}^1) & \to & C^1(\mathcal{U}, \mathcal{E}^1) & \to & & C^1(\mathcal{U}, \mathcal{Z}^2) & \to 0
\end{array}$$

i.e. we can choose $\{\gamma_i\} \in C^0(\mathcal{U}, \mathcal{E}^{10})$ instead of in $C^0(\mathcal{U}, \mathcal{E}^1)$.

This is because $df_{ij} \in \Omega(U_{ij})$

∎

Now we use Thm 17.3 of [1], that says

$$\frac{\text{zeros of } s - \text{poles of } s}{2\pi i} = \text{Res}(\mu) = \frac{1}{2\pi i}\int_X \omega_L$$

$\Rightarrow$ def 1 = def 2

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References

[1] Forster, Lectures on Riemann surfaces;

[2] P. Griffith and J. Harris, Principles of algebraic geometry;

[3] R. Bott and L.W. Tu, Differential forms in Algebraic Topology.